

Inequality, Welfare and Monotonicity

by

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Abstract

We establish a general relationship between the standard form of the individualistic social-welfare function and the "reduced-form" version that is expressed in terms of inequality and mean income. This shows the relationship between the property of monotonicity and the slope of the equity-efficient trade-off. Particularly simple results are available for a large class of inequality measures that includes the Gini. These results do not require differentiability of the social-welfare function.

Keywords: Inequality, social welfare, monotonicity.

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1 Introduction

Inequality indices embody explicitly or implicitly social values on income distribution. So it is common for those who are concerned with comparing distributional changes over time or between economies to draw conclusions from comparisons of estimates of average income and inequality. However the relationship between intuitive comparisons in terms of say, mean income and the Gini coefficient, and formal principles of social welfare is not always made clear. The present paper focuses upon this relationship by characterising the shape of the equity-efficiency trade-off.

This trade-off is at the heart of much theoretical and applied research. For example, it is become standard practice to use the obvious two-way relationship between certain classes of inequality indices and their associated social-welfare functions to infer propositions about distributional rankings and the ethical meaning of inequality measures.¹ In addition there is an interest in constructing social-welfare functions for inequality measures with principally pragmatic appeal: for example, there have been a number of attempts to formalise a social-welfare function on the basis that the Gini coefficient is the appropriate index of inequality.² The precise method of doing this formalisation does not concern us here, except as illustrations of our approach. We want to examine in a general way what may

¹See for example Atkinson (1970), Blackorby and Donaldson (1978, 1980), Kolm (1976a, 1976b).

²See for example Chipman (1974), Katz (1972), Kondor (1975), Lambert (1985), Newbery (1970), Sheshinski (1972).

be said in terms of standard welfare criteria if we use well-known practical indices as basic tools of distributional analysis.

2 The Approach

First, some notation and definitions.

Definition 1 *The set of income distributions X is the set of all ordered finite-dimensional non-negative vectors, excluding the zero vector.*

In implementing this definition we adopt the convention that for any $\mathbf{x} \in X$: $x_1 \leq x_2 \leq x_3 \leq \dots$. Also, for any $\mathbf{x} \in X$, denote the number of persons in the distribution as $n(\mathbf{x})$ and the arithmetic mean of the distribution as $\mu(\mathbf{x})$. Social welfare and inequality are given, respectively, by functions $W : X \rightarrow \mathfrak{R}$, $I : X \rightarrow \mathfrak{R}$. Economic meaning is given to W and I by a number of assumptions about their mathematical properties, many of which have come to be accepted as standard. Foremost among these are

Definition 2 *The social welfare function is monotonic if, for any $\mathbf{x}, \mathbf{x}' \in X$, $\mathbf{x}' > \mathbf{x}$ implies $W(\mathbf{x}') > W(\mathbf{x})$.³*

Definition 3 *The social welfare function (resp. an inequality measure) satisfies the principle of transfers if, for any $\mathbf{x}, \mathbf{x}' \in X$ and $\delta > 0$ such that $x'_i = x_i + \delta$,*

³“ $\mathbf{x}' > \mathbf{x}$ ” means “ $\mathbf{x}' \geq \mathbf{x}$, $\mathbf{x}' \neq \mathbf{x}$ ”.

$x'_j = x_j - \delta$, $x'_k = x_k$, $k \neq i, j$ and $x_i < x'_i \leq x'_j < x_j$ $W(\mathbf{x}') \geq W(\mathbf{x})$ (resp. $I(\mathbf{x}') \leq I(\mathbf{x})$).

The property of monotonicity is of particular interest in view of its relationship to the Pareto principle - see Amiel and Cowell (1994b) - although it does not enjoy universal support as a welfare principle - see Amiel and Cowell (1994a), McClelland and Rohrbaugh (1978).

Definition 4 *The class \mathfrak{W}_1 of social welfare functions consists of functions $W : X \rightarrow \mathfrak{R}$ that satisfy the principles of monotonicity and transfers. If, for some $\mathbf{x}, \mathbf{x}' \in X$, it is the case that $W(\mathbf{x}) > W(\mathbf{x}')$ for all $W \in \mathfrak{W}_1$ we shall say that distribution \mathbf{x} is \mathfrak{W}_1 -preferable to \mathbf{x}' .*

We shall establish results for a fairly broad class of inequality indices. To introduce this we need the following two definitions:

Definition 5 *For any $\mathbf{x} \in X$, the income share of person i is:*

$$s_i(\mathbf{x}) := \frac{x_i}{n(\mathbf{x})\mu(\mathbf{x})} \quad (1)$$

Definition 6 *An additive distributional-shares inequality index can be written as*

$$\sum_{i=1}^{n(\mathbf{x})} w_i \phi(s_i(\mathbf{x})) \quad (2)$$

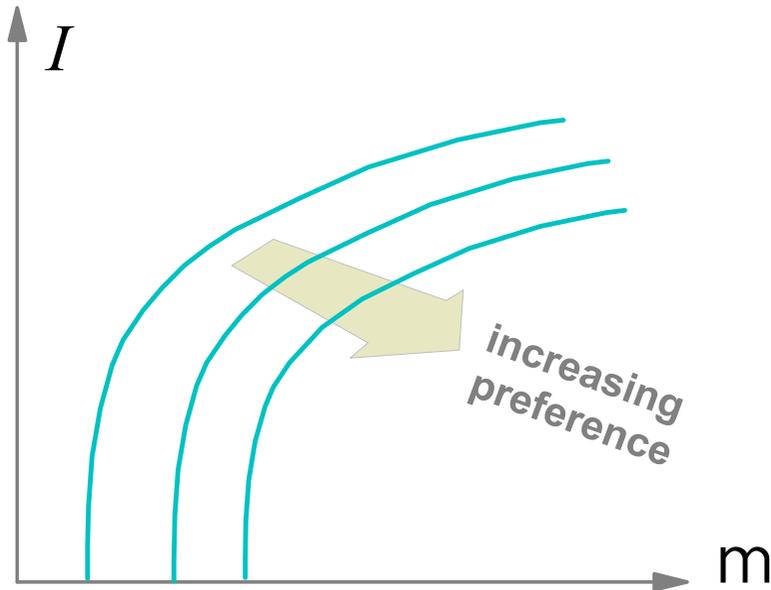


Figure 1: The Reduced-Form Social Welfare Function

where w_i is a weight and $\phi : [0, 1] \mapsto \mathfrak{R}$ is monotonic and convex.⁴

Examples of the ADSI class of indices include all the generalised-entropy indices, the Gini index, the relative mean deviation and the logarithmic variance.⁵

The results which follow will also apply to monotonic affine transforms of (2).

⁴The approach is based upon the so-called relative inequality aggregation property Blackorby and Donaldson (1984). Notice the essential difference from the standard welfare-theoretic approach to the definition of an inequality measure as in Atkinson (1970), Kolm (1969), Sen (1973) where the specification of the welfare function or of equally-distributed equivalent income usually forces monotonicity on the part of the social welfare function - see also Ebert (1988).

⁵This is distinct from the variance of logarithms - see Cowell (1995).

2.1 The Reduced-form SWF

It is common to find the social welfare function written in a “reduced form” - as a function of mean income and inequality.⁶ To formalise this approach let us introduce the concept of a *distributional profile* $z := (\mu, I)$, a mean-income and inequality pair. In this interpretation the admissible set of profiles $Z \subseteq \mathfrak{R}_+^2$ may be written

$$Z := \{(\mu, I) : \mu \geq 0, 0 \leq I \leq \bar{I}\} \quad (3)$$

where \bar{I} is the least upper bound on I . Then the reduced-form SWF $H^{(W)} : Z \rightarrow \mathfrak{R}$ is given by

$$W(\mathbf{x}) = H^{(W)}(\mu(\mathbf{x}), I(\mathbf{x})) \quad (4)$$

where $H^{(W)}$ is non-decreasing in its first argument and non-increasing in its second argument.

Where there is no problem of ambiguity we will drop the superscript on H . Notice that the adoption of this approach implies that two distributions with same z -profile will be regarded as equivalent in welfare terms, and that the trade-off implicit in the use of H is only well-defined for a particular cardinalisation of the inequality index I . We shall assume that the issue of the appropriate cardinalisation has been settled by criteria which are not of concern to the present study.

⁶This is the term used by Champernowne and Cowell (1997). See also the term “abbreviated social-welfare function” used in Lambert (1993), Chapter 5.

3 Differentiable Social-welfare Functions

The ADSI class of measures contains a number of well known indices and yields some readily interpretable results. To see this let us endow I with the elementary cardinalisation of (2). Substituting from (2) into (4) we have

$$W(\mathbf{x}) = H \left(\mu(\mathbf{x}), \sum_{i=1}^{n(\mathbf{x})} w_i \phi(s_i(\mathbf{x})) \right) \quad (5)$$

Assume further that W is differentiable. Then differentiating (5) with respect to x_i we have:

$$n(\mathbf{x}) \frac{\partial W(\mathbf{x})}{\partial x_i} = H_\mu + H_I \left[\frac{w_i \phi_s(s_i(\mathbf{x}))}{\mu(\mathbf{x})} - \frac{\sum_{j=1}^{n(\mathbf{x})} w_j s_j(\mathbf{x}) \phi_s(s_j(\mathbf{x}))}{\mu(\mathbf{x})} \right] \quad (6)$$

where the relevant partial derivatives are denoted by subscripts. From (6) the following result is immediate

Theorem 1 *Given a differentiable reduced form social-welfare function H the underlying individualistic welfare function satisfies monotonicity if and only if the following condition on the slope of the welfare-contour is satisfied:*

$$-\frac{H_\mu}{H_I} > \max \left\{ \frac{w_i \phi_s(s_i(\mathbf{x})) - \sum_{j=1}^{n(\mathbf{x})} w_j s_j(\mathbf{x}) \phi_s(s_j(\mathbf{x}))}{\mu(\mathbf{x})} \right\} \quad (7)$$

3.1 Applications

The condition (7) can be easily interpreted in the case of the Generalised Entropy index:

$$-\frac{H_\mu}{H_I} > \max \frac{\frac{[n(\mathbf{x})s_i(\mathbf{x})]^{\alpha-1}-1}{\alpha-1} - \alpha I(\mathbf{x})}{n(\mathbf{x})\mu(\mathbf{x})} \quad (8)$$

and the relative mean deviation:

$$-\frac{H_\mu}{H_I} > \frac{\frac{2m(\mathbf{x})+1}{n(\mathbf{x})} - I(\mathbf{x})}{n(\mathbf{x})\mu(\mathbf{x})} \quad (9)$$

where $m(\mathbf{x}) = \#\{x_i \leq \mu(\mathbf{x})\}$. The interpretation of these results will be discussed below.

3.2 The Gini Index

The Gini index $G(\mathbf{x})$ can also be obtained as a special case of (7) by rewriting it as:⁷

$$G(\mathbf{x}) = \sum_{i=1}^{n(\mathbf{x})} w_i s_i \quad (10)$$

where

$$w_i = \frac{2i - n(\mathbf{x}) - 1}{n(\mathbf{x})} \quad (11)$$

We will call the *extended Gini* subclass those inequality measures of the form (10) but with the restriction (11) replaced by the mild condition that w_i be increasing

⁷See, for example Cowell (1995), Sen (1973).

with i .⁸ Notice that for the extended Gini and the true Gini $\phi_s = 1$ wherever the differential is defined. Also note that for the true Gini $\max w_i = \frac{n(\mathbf{x})-1}{n(\mathbf{x})}$ which for large populations becomes 1; in this limiting case the right-hand side of (7) becomes:

$$\frac{1 - G(\mathbf{x})}{\mu(\mathbf{x})} \tag{12}$$

However in the case of the Gini index the differentiability assumption used in Theorem 1 will be inappropriate. For this reason and for the reason that - apart from mathematical tractability - differentiability has little to recommend it in social-welfare analysis, we seek a more general characterisation of the main point.

4 A General Result

To avoid the over-simplification associated with the differentiability assumption consider the general shape properties of the reduced-form social-welfare function at any arbitrary point in the set of all profiles Z .

For any distributional profile $z_0 \in Z$ it is clear that any point in

$$A(z_0) := \{z \in Z : \mu \geq \mu_0, I \leq I_0, z \neq z_0\} \tag{13}$$

⁸See for example the contributions by Donaldson and Weymark (1980, 1983), Weymark (1981), , Yitzhaki (1983). However note that Chakravarty's extended Gini does not fit within this category (Chakravarty 1988).

will be regarded as \mathfrak{W}_1 -preferable to z_0 . However, we can be more precise about the set of \mathfrak{W}_1 -preferable profiles. First define the set of income distributions consistent with any profile $z_0 \in Z$:

$$X(z_0) := \{\mathbf{x} : \mathbf{x} \in X, \mu(\mathbf{x}) = \mu_0, I(\mathbf{x}) = I_0\} \quad (14)$$

and a conditional upper-bound value of inequality, of a given population size $n = n(\mathbf{x})$:

$$I^*(\mu; z_0) := \max_{\mathbf{x} \in X(z_0)} I(\mathbf{x} + \boldsymbol{\iota}_n n [\mu - \mu_0]) \quad (15)$$

where $\boldsymbol{\iota}_n := (0, 0, \dots, 0, 1)$. These constructs then permit the definition of the following subset of $A(z_0)$:

$$P(z_0) := \{z \in Z : \mu \geq \mu_0, I \leq I^*(\mu; z_0), z \neq z_0\} \quad (16)$$

Theorem 2 $\forall W \in \mathfrak{W}_1, \forall z_1 \in P(z_0) :$

$$H^{(W)}(z_1) > H^{(W)}(z_0). \quad (17)$$

Proof. For some $z_0 \in Z$ and some $z_1 := (\mu_1, I_1) \in P(z_0)$, given that $\mu_0 < \mu_1$ by construction, if $I_0 \geq I_1$ then it is trivial to show that (17) holds. So take the

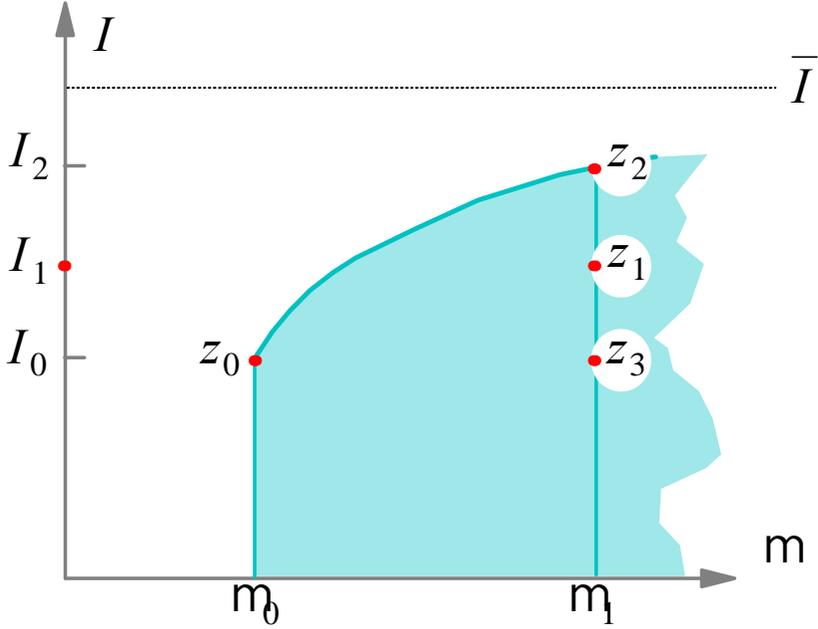


Figure 2: The set $P(z_0)$

more interesting subcase where $I_0 < I_1$, and define

$$\mathbf{x}_2 := \arg \max_{\mathbf{x} \in X(z_0)} I(\mathbf{x} + \mathbf{t}_n n[\mu_1 - \mu_0]) \quad (18)$$

$$\mathbf{x}_3 := \frac{\mu_1}{\mu_0} \mathbf{x}_0 \quad (19)$$

The profiles z_2 and z_3 corresponding to \mathbf{x}_2 and \mathbf{x}_3 are illustrated in Figure 2.

Noting that $W(\mathbf{x}_2) > W(\mathbf{x}_0)$ we have

$$\forall \alpha \in [0, 1] : W(\alpha \mathbf{x}_2 + [1 - \alpha] \mathbf{x}_3) > W(\mathbf{x}_0), \quad (20)$$

$$\forall \alpha \in [0, 1] : \mu(\alpha \mathbf{x}_2 + [1 - \alpha] \mathbf{x}_3) = \mu_1 > \mu(\mathbf{x}_0) = \mu_0, \quad (21)$$

and by the principle of transfers $I^*(\mu; z_0) = I(\mathbf{x}_2) \geq I_1 \geq I(\mathbf{x}_3) = I_0$. By continuity there must be some $\alpha \in [0, 1]$ such that⁹

$$I(\alpha \mathbf{x}_2 + [1 - \alpha] \mathbf{x}_3) = I_1 \quad (22)$$

in which case it is immediate from (20) that (17) holds. ■

Notice that, by construction, $P(z_0)$ is the largest set of profiles which are regarded as \mathfrak{W}_1 -preferable to z_0 , and that it is bounded by the graph of I^* : the set of points of the form $\bar{P}(z_0) := \{z : z = (\mu, I^*(\mu; z_0)), \mu \geq \mu_0\}$ provides a boundary conditional on a particular distributional profile z_0 . For the ADSI class we have

$$I^*(\mu; z_0) := \sum_{i=1}^{n-1} w_i \left[\phi \left(\frac{x_i}{n\mu} \right) \right] + w_n \left[\phi \left(\frac{x_n + n[\mu - \mu_0]}{n\mu} \right) \right] \quad (23)$$

which, in the case of the subclass of extended Gini measures, reduces to

$$I^*(\mu; z_0) := \bar{I} - [\bar{I} - I_0] \frac{\mu_0}{\mu} \quad (24)$$

Notice that through each point (profile) $z_0 \in Z$, there is a frontier \bar{P} and a contour of the reduced-form social welfare function $H^{(W)}$. We may then state:

Theorem 3 *For any inequality-averse monotonic SWF and any $z_0 \in Z$, when-*

⁹See the profile z_1 in Figure 2. Notice that I must be continuous in view of the convexity of ϕ .

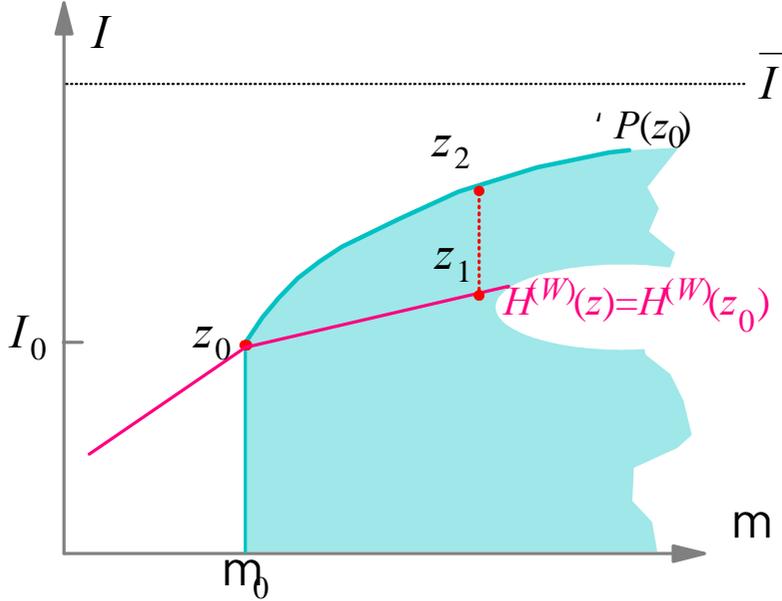


Figure 3: A violation of monotonicity

ever the contour of $H^{(W)}$ intersects $\bar{P}(z_0)$, it must do so from below.

Proof. Suppose to the contrary that $H^{(W)}$ intersects $\bar{P}(z_0)$ from above as in Figure 3.¹⁰ By construction $H^{(W)}(z_0) = H^{(W)}(z_1)$, z_2 is \mathfrak{W}_1 -preferable to z_0 . and profiles z_1, z_2 have the same mean. If W is inequality-averse then $H^{(W)}(z_2) > H^{(W)}(z_1) = H^{(W)}(z_0)$ which means that W must violate monotonicity - a contradiction. ■

This theorem may then be used to give a more general interpretation to Theorem 1 which was obtained under the assumption of uniform differentiability:

¹⁰The $H^{(W)}$ -contour has been drawn with a kink at z_0 to illustrate the generality of the result. Of course it need not be kinked, nor piecewise linear, and it may be convex rather than concave.

Corollary 4 *If W satisfies monotonicity, the limit of the slope of $I^*(\mu; z_0)$ as $\mu \downarrow \mu_0$ must be less than the limit of the slope of the contour of $H^{(W)}(z) = H^{(W)}(z_0)$ as $\mu \downarrow \mu_0$.*

Substitution from (23) or from (24) will confirm the result for the general ADSI class and for the special case of the Gini index respectively. In the latter case we find that the condition for monotonicity amounts to:

$$-\frac{H_\mu^{(W)}}{H_I^{(W)}} > \frac{1 - I}{\mu} \quad (25)$$

where the slope is to be interpreted in the manner of Corollary 4. This can be re-expressed as a requirement on the reduced-form SWF in terms of the associated equality index $1 - I$: it should have the property that the elasticity of equality with respect to mean income be less than unity. The condition (25) can then be used to check the properties of reduced-form Gini-SWFs presented in the literature. Examples of measures satisfying the elasticity condition include $n^2[1 - I]\mu$ (Sheshinski 1972)¹¹, $\log \mu - I$ (Katz 1972) and $\frac{\mu}{1+I}$ (Kakwani 1986). However $\frac{1-I}{1+I}\mu$ (Chipman 1974, Dagum 1990) does *not* satisfy the condition and it is easy to see that this index violates monotonicity: Table 1 shows an example where distribution B has a higher mean income and higher inequality than A. B also first-order dominates A. However the Chipman-Dagum social welfare function

¹¹A generalisation of this form is to be found in Lambert (1985). See also Ben Porath and Gilboa (1994)

	A	B
x_1	4500	4600
x_2	4000	4000
x_3	3500	3500
$\mu(\mathbf{x})$	4000	4033
$I(\mathbf{x})$	0.056	0.061
$W(\mathbf{x}) = \frac{1-I}{1+I}\mu$	3579	3572

Table 1: The Gini coefficient I and the Chipman-Dagum Welfare Index W

indicates that society is worse-off in B than in A.

Similar interpretations can be provided in terms of other members of the ADSI class (2).

5 Conclusions

The use of *ad hoc* inequality measures begs the question of whether the empirical judgements made using them will be in accordance with conventional welfare properties. The reduced-form SWF is the appropriate tool to clarify this issue. The transfer principle is inherited automatically by the SWF from an inequality measure; now we have shown that the condition on the reduced-form SWF that ensures consistency with the monotonicity criterion is surprisingly simple. In the case of the Gini index this condition is particularly appealing, and does not rely on arbitrary assumptions of differentiability of the SWF.

Finally, a word of caution. Cardinalisations of inequality measures are sometimes viewed as irrelevant, but in the analysis of the so-called “equity-efficiency

trade-off” this is not quite so. As the discussion of the elasticity of the $H^{(W)}$ -contours makes clear, the results we have obtained are specifically for the “natural” cardinalisation of the index (2). A modified version of Theorem 1 and related results will be applicable under continuous monotonic transformations of I .

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