ONE SMART AGENT*

John Sutton
London School of Economics and Political Science

The Toyota Centre
Suntory and Toyota International Centres for
Economics and Related Disciplines
London School of Economics and Political Science
Houghton Street
London WC2A 2AE
Tel.: (020) 7955 6674

Discussion Paper
No. EI/8
December 1995

---

* I would like to thank Alison Hole, Reinout Koopmans and Michael Raith for their very helpful comments on an earlier draft. The financial support of the Economic and Social Research Council and the Leverhulme Trust is gratefully acknowledged.

Abstract

This paper proposes an equilibrium concept for a class of games in which players make irreversible costly decisions; these games have been widely used in the recent I.O. literature. The equilibrium concept is defined, not in the space of strategies, but in the space of (observable) outcomes. It is weaker than perfect Nash equilibrium, and involves combining a form of 'survivor principle' with an assumption regarding entry. This assumption involves only a very weak rationality requirement: if a profitable opportunity exists in the market, there is 'one smart agent' who will fill it. This weak equilibrium concept is sufficient to imply some empirically interesting regularities in the area of market structure.

Keywords: equilibrium concept; one smart agent; market structure; games; survivor principle; rationality requirement; regularities.

Contact address: Professor John Sutton, STICERD, London School of Economics and Political Science, Houghton Street, London WC2A 2AE. email: j.sutton@lse.ac.uk

I. INTRODUCTION

The game theoretic literature in Industrial Organisation has had a mixed response in recent years. It has been noted that a rich variety of plausible models can be constructed, so that a huge range of observed phenomena can be rationalised within the literature. But this begs an obvious question: is this class of models so broad that it excludes nothing? Is the literature empirically empty? (Fisher (1989), Sutton (1990), Pelzman (1991)).

One response to this problem is offered by the 'bounds' approach proposed in Sutton (1991). This approach begins from the notion that there is usually no one 'true model' which can adequately represent any interestingly broad class of industries. Instead, some class of 'admissible models' may be defined. The space of feasible outcomes is then partitioned into those which may be equilibria under *some* admissible model, and those which can not be supported as equilibria in *any* admissible model. This partitioning is carried out by defining a number of facets (constraints) in the space of outcomes. Each facet is derived by reference to the availability of some particular 'profitable deviation'. Any outcome lying beyond this facet will be broken by a particular kind of deviation in *any* admissible model. The empirical content of the theory consists of the claim that outcomes must lie within these facets.

This is the first of three papers which set out a more fully developed version of the 'bounds' approach of Sutton (1991). In this first paper, the aim is to define a set of outcomes bounded by two facets, and to show that certain empirically relevant results of the standard game-theoretic (Nash equilibrium) models follow directly from the properties of these two facets.

The class of games with which this paper is concerned are the finite horizon multi-stage games on which much of the recent literature on market structure rests. In these games, firms first take a sequence of costly irreversible actions, and these actions then lead to some final 'configuration' of the market from which agents' payoffs can be deduced. This class of games provides models of capacity choice, plant location, product differentiation, advertising, and R&D outlays. Such games can be described as consisting of two elements. The first element is a profit function $\mathbf{\Pi}(\cdot)$ obtained by 'solving out' the final stage subgame; this function specifies the profit of each firm in terms of the configuration of plants, product specifications, and so on, which firms have inherited as a result of earlier investments. The second element is an extensive form specifying the entry stage(s) of the game. In analysing (perfect) equilibrium in the entry stage, the function $\mathbf{\Pi}(\cdot)$ serves as the payoff function of the game, in accordance with the standard 'backward induction' procedure.

In applying these models, we can sometimes regard the parameters entering the function $\mathbf{\Pi}(\cdot)$ as observable outcomes (number of plants, products, etc.), and then finesse the presence of unobservables in the 'final stage game' from which $\mathbf{\Pi}(\cdot)$ is derived by making some (weak) assumptions directly on the function $\mathbf{\Pi}(\cdot)$ itself (see for example, Sutton (1991), Chapters 2,3). The design of the extensive form poses more serious problems. Only in rare circumstances is it possible to offer any convincing argument for the choice of any one of a number of widely used representations. Issues such as the existence or otherwise of strategic asymmetries (first mover advantages), and so on, raise questions which can rarely be addressed by any appeal to data available to the economist. It is of great interest, therefore, to ask what restrictions can be placed on outcomes while making the weakest possible assumptions about the extensive form. This is the aim of the present paper.

The argument is this: by reference to the function $\Pi(\cdot)$ alone, it can be shown that some outcomes can not be supported as perfect Nash equilibria for *any* extensive form (chosen from a very broad class of admissible forms). This allows us to separate out some restrictions on equilibrium market structure which emanate from the properties of the $\Pi(\cdot)$ function alone; these form the subject of the present paper. (Those further restrictions which depend both on $\Pi(\cdot)$ and on the extensive form are developed in later papers in this sequence.)

There is a second point of interest in the present approach: the results of this paper can be obtained using only very weak rationality requirements on agents. The equilibrium concept defined here involves two restrictions. The first is a (weak) form of the 'survivor principle', which states that loss making strategies will be avoided. The second assumption states that no configuration can survive in which some profitable opportunity remains unexploited. There is always 'One smart agent' who will take up such an opportunity. This assumption is analogous to the central principle of the economics of finance: that there are no profitable opportunities for arbitrage. Like that principle, it derives its force from the fact that it assumes nothing of the general run of agents in the market. It merely assumes that there is some agent in the market who will take advantage of rivals' failures to spot good opportunities. In modelling situations where agents take costly and irreversible actions in a series of novel or unique market environments, the attractiveness of appealing to this very weak rationality requirement is evident.

What is of interest is that this weak assumption together with a version of the survivor principle proves to be sufficient to imply certain important regularities in the area of market structure. In particular, it is sufficient to generate the basic limit theorems for the 'Exogenous sunk costs' and 'Endogenous sunk costs' models of Sutton (1991).

II. AN EQUILIBRIUM CONCEPT

The class of games which concern us here have the following structure:

There are N players (firms). Firms take actions at certain specified stages. An action involves occupying some subset, possibly empty, of 'locations' in some abstract 'space of locations'. At the end of the game, each firm will occupy some set of locations.

The notation is as follows: a location is an element a of the set of locations $\mathbf{A}$. The set of locations occupied by firm i at the end of the game is denoted $\mathbf{a}_i$, where $\mathbf{a}_i$ is a subset of $\mathbf{A}$. If firm i has not entered at any location then $\mathbf{a}_i = \emptyset$.

Associated with any set of locations is a fixed and sunk cost incurred in entering at these locations. This cost is strictly positive and bounded away from zero, viz. for any $\mathbf{a}_i \neq \varnothing$, $F(\mathbf{a}_i) \geq \varepsilon > 0$. The N-tuple of all locations occupied by all agents at the end of the game is written as

$$\{\mathbf{a}_i\} \equiv \{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N \}$$

The payoff (profit) of firm i, if it occupies locations $\mathbf{a}_i$, is written

$$\Pi(\mathbf{a}_i | \{\mathbf{a}_{-i}\}) \equiv \pi(\mathbf{a}_i | \{\mathbf{a}_{-i}\}) - F(\mathbf{a}_i)$$

where $\{\mathbf{a}_{-i}\}$ denotes $\{\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1} \ldots, \mathbf{a}_N\}$. In most applications of the theory, the function $\pi(\mathbf{a}_i | \{\mathbf{a}_{-i}\})$ is computed as the payoff function in some subsequent game, usually called the 'price competition sub-game', in which the $\mathbf{a}_i$ enter as parameters in the firms' payoff functions. A firm taking no action at any stage incurs zero cost and receives payoff zero. Assumption 1 introduces two restrictions. Restriction (a) excludes 'non-viable' markets in which no product can cover its entry cost. Restriction (b) ensures that the number of potential entrants N is large (at equilibrium, we will have at least one inactive player).

Assumption 1: (a) There is some set of locations $\mathbf{a}_o$ such that $\Pi(\mathbf{a}_o | \{\varnothing\}) > F(\mathbf{a}_o)$.

(b) The final stage payoff received by all agents is bounded above by $N\varepsilon$, where N denotes the number of players and ε is the minimum setup cost (entry fee).

Examples of this structure include:

- capacity choice games (here $\mathbf{a}_i$ collapses to a scalar representing firm i's level of capacity).

- location games (here $\mathbf{a}_i$ is a set of locations at which firm i establishes outlets).

- horizontal product differentiation models. In Hotelling-type models $\mathbf{a}_i$ is a set of locations. In 'symmetric' models of the Dixit-Stiglitz kind, $\mathbf{a}_i$ collapses to an integer denoting the 'number of varieties entered'.

- vertical product differentiation models. Here $\mathbf{a}_i$ is a set of locations in the space of products.

The equilibrium concept proposed here is defined not on the space of strategies, but directly on the space of outcomes, i.e. on the configuration $\{\mathbf{a}_i\}$. It depends, therefore, only on the function $\mathbf{\Pi}(\cdot)$ and not on the entry game itself, which we have not yet specified:

Definition: $\{\mathbf{a}_i\}$ is an Equilibrium Configuration if:

(i) Viability ('survivor principle'): For all agents i,

$$\Pi_i \; (\mathbf{a}_i \mid \{ \mathbf{a}_{-i} \}) \geq 0$$

(ii) Stability ('one smart agent'): There is no set of actions $\mathbf{a}_{N+1}$ such that entry is profitable, viz. for all sets of actions $\mathbf{a}_{N+1}$,

$$\Pi_{N+1} \ (\mathbf{a}_{N+1} \mid \{ \mathbf{a}_i \}) < 0$$

Condition (i) requires something weaker than Nash equilibrium. Nonetheless, it is a substantial restriction. It is reasonable in the present context only because the extensive forms considered in this paper incorporate complete information, and do not allow exit. (If the class of extensive forms is broadened to allow for exit, then it is appropriate to rewrite condition (i) as a requirement that profit net of the *avoidable* cost which can be saved by exiting should be non-negative. The effect of relaxing condition (i) in this way is noted in Section VI(p.20).

It is condition (ii) which is central to what follows. It states that there is no profitable gap in the market. Can such gaps exist in standard models of perfect Nash equilibrium? The answer, in general, is 'yes'. However this is only possible for some rather special combinations of payoff functions and extensive forms of the entry game. (An example is given in the Section V.) In the next Section, it is shown that, for a very broad class of extensive forms, condition (ii) will hold for all outcomes supportable as perfect Nash equilibria.

III. NASH EQUILIBRIA: A CLASS OF EXTENSIVE FORMS

The recent literature on market structure has relied heavily on three kinds of extensive form in describing the entry/investment process:

(a) Simultaneous entry, where there is just one entry stage (date), and all firms are free to make investments at that stage;

(b) Sequential entry, where firms are ranked and each firm is assigned a different single stage (date) at which it is free to make investments;

(c) Each firm is free to make investments at any stage (time).

Models (a) and (b) are widely used in 'capacity choice' games, and in the product differentiation literature. Model (c) is sometimes used in the patent race literature.

While the simultaneous entry and sequential entry models provide many useful simple examples, it is probably fair to say that their main attraction lies in analytical simplicity rather than any a priori 'reasonableness' of these forms. Indeed, the simultaneous entry form is often said to be 'unrealistic' in that firms 'enter over a period of time' in practice, so that it may be inappropriate to impose 'strategic symmetry' on firms; firm 1 may have a 'first mover advantage' in that it may, in choosing action a_1, know that firm 2 will condition its action a_2 on a_1. It is sometimes said that sequential entry is a more appropriate representation, in that it permits such stories to be told. Yet authors who have used sequential entry models have pointed out that the equilibria of such models have an unattractive feature: if a firm was permitted to delay its turn in the sequence, it might be profitable to do so. This is true, for example, in the simplest 'capacity choice' games (where firms producing a homogeneous good take turns in building plant capacity; see Eaton and Ware (1987)).

It might seem at first glance that the obvious choice is to let all firms move at any time (option (c) above). Even this, however, is open to the objection that it excludes the kind of strategic asymmetry (first mover advantage) just noted. Different firms may in practice be formed at different times, and may first consider entry to a given market at different times. To allow for this, it seems appropriate to define a more flexible setup, as follows:

Assumption 2: (Extensive Form): We define vectors $\{t_i\}$ and $\mathbf{A}^t$. Firm i is free to enter any subset of the set of products $\mathbf{A}^t$ at any stage t, such that $t_i \leq t \leq T$. The set $\mathbf{A}^t$ satisfies $\mathbf{A} \equiv \mathbf{A}^T \supseteq \mathbf{A}^{t+1} \supseteq \mathbf{A}^t, \forall\, t < T$.

The date t_i is the date of arrival of firm i. Following its arrival, it can enter a set of products at any stage. The set of products which it is feasible to produce may expand over time (as a result of exogenous changes in the available technology, or otherwise), but will not contract. The number of firms is finite. T denotes a last stage at which entry can occur. We exclude infinite horizon games.

Of the three standard entry processes listed above, this setup includes both case (a), obtained by setting $t_i = T = 1$, and case (c) obtained by setting $t_i = 1$ and $T > 1$, for all i. Case (b) is more complicated: for many 'well behaved' models, the outcomes available under the Sequential Entry setup will coincide with those obtained by setting $t_i = i$, and $\mathbf{A}^t = \mathbf{A}^T$ on $t_i \leq t \leq T$ in the above scheme. This is not always the case, however. Counterexamples arise in those sequential entry models where firm i would find it optimal to postpone entry beyond its allocated time. It seems desirable to allow such a postponement.

Assumption 2 ensures that any outcome that can be supported as a perfect Nash equilibrium generates an outcome which is an 'equilibrium configuration' in the sense defined above:

Proposition 1 (Inclusion): Any outcome that can be supported as a perfect Nash equilibrium is an equilibrium configuration.

Proof: (Viability): a Nash equilibrium outcome satisfies (i) since 'Don't Enter' is an available action. Playing $\{\emptyset\}$ at every stage strictly dominates any strategy violating (i).

(Stability): Assumption 2 implies that all firms are free to enter any subset of $\mathbf{A}^T$ at date T, *taking as given* the set of products entered by rivals at that date. Assumption 1 ensures that there is at least one firm which has not entered any product prior to date

T. Hence, if (ii) is violated, a profitable deviation is available to that firm.

IV. AN ILLUSTRATION: HOTELLING'S SIMPLE LOCATION GAME

In this section, we illustrate the inclusion property by looking at the 'simple location game' of Hotelling (1929). Here, n firms each choose a single location on the line segment [0,1]. Consumers are assumed to be distributed uniformly with density S. A firm's payoff is simply the number of consumers closer to it than to any rival; where firms' locations coincide, they share consumers equally.[2] It is well known that the Nash equilibria of this game have the following form (Lipsey and Eaton (1975)):

n = 1: any location choice is a Nash equilibrium.

n = 2: the only Nash equilibrium is where both firms locate at the midpoint (½).

n = 3: there is no Nash equilibrium in pure strategies (a symmetric mixed strategy equilibrium is described by Shaked (1982)).

[2]This 'simple location game' should be distinguished from the Hotelling model proper, in which firms compete in prices in a post-entry stage.

n = 4: the only pure strategy Nash equilibrium is where two firms locate at ¼ and two firms at ¾.

n > 4: the only pure strategy Nash equilibria are of the following form: two firms locate at some point a, two firms locate at (1-a), and the remaining firms occupy locations strictly between these points. The Nash equilibrium is not unique for $n \geq 6$.

In this simple Hotelling game, the number of firms is taken as a parameter. What is of interest here is the related one-shot 'entry' game, specified as follows. A firm's strategy takes one of two forms: either 'Don't Enter' (payoff = 0), or 'Enter at some location', in which case the payoff is the payoff in the game defined above less some entry fee ε. We normalize by setting $\varepsilon = 1$. The following analysis is confined to pure strategy equilibria. It is clear that a configuration involving n active firms is supported as a Nash equilibrium in the entry game if and only if it is an equilibrium configuration *and* the vector of locations forms a Nash equilibrium in the n-firm Hotelling game. It is also obvious that an equilibrium configuration involving n firms exists if and only if S lies in the interval [n,2n].[3] Bearing this in mind, it is easy to identify the range of market sizes $[S_m, S_M]$ over which an n-firm configuration can be supported. The results are shown in Table 1; the derivation is given in Appendix 1.

[3]To construct such a configuration, place firm i at (2i - 1)/(2n) for i = 1, ..., n.

Figure 1 illustrates the inclusion relationship in (n,S) space between the Nash equilibrium outcomes and the equilibrium configurations.

n	S_m	S_M
1	1	2
2	2	2
3	-	-
4	4	4
5	6	6
6	6	8

Table 1. The entry game for Hotelling's simple location model (single product firm/simultaneous entry). The table shows, for each n, the range $[S_m, S_M]$ of market size for which there is a pure strategy Nash equilibrium involving n products (firms).

Now this model can be extended in a number of ways, by varying the extensive form (changing simultaneous entry to sequential entry, say) or by otherwise altering the strategy space (permitting multiproduct firms, say). Rather than explore different variations, we here comment on one case which is of particular interest in the present context. This is where we use sequential entry, and allow multiproduct firms. In this setting, it is easy to show that for any S, there is always a (perfect) Nash equilibrium in which the first firm pre-empts in the sense that it enters the smallest number of products such that no later firm can enter profitably (n is the smallest integer satisfying $n \geq S/2$. This outcome can also be obtained using an extensive form satisfying Assumption 2 above: set $t_i = i$ and let $\mathbf{A}^t = \mathbf{A}$ be the set of all sets of locations in [0,1]. Again, there is a (perfect) Nash equilibrium of this game in which firm 1 preempts by entering at these locations.

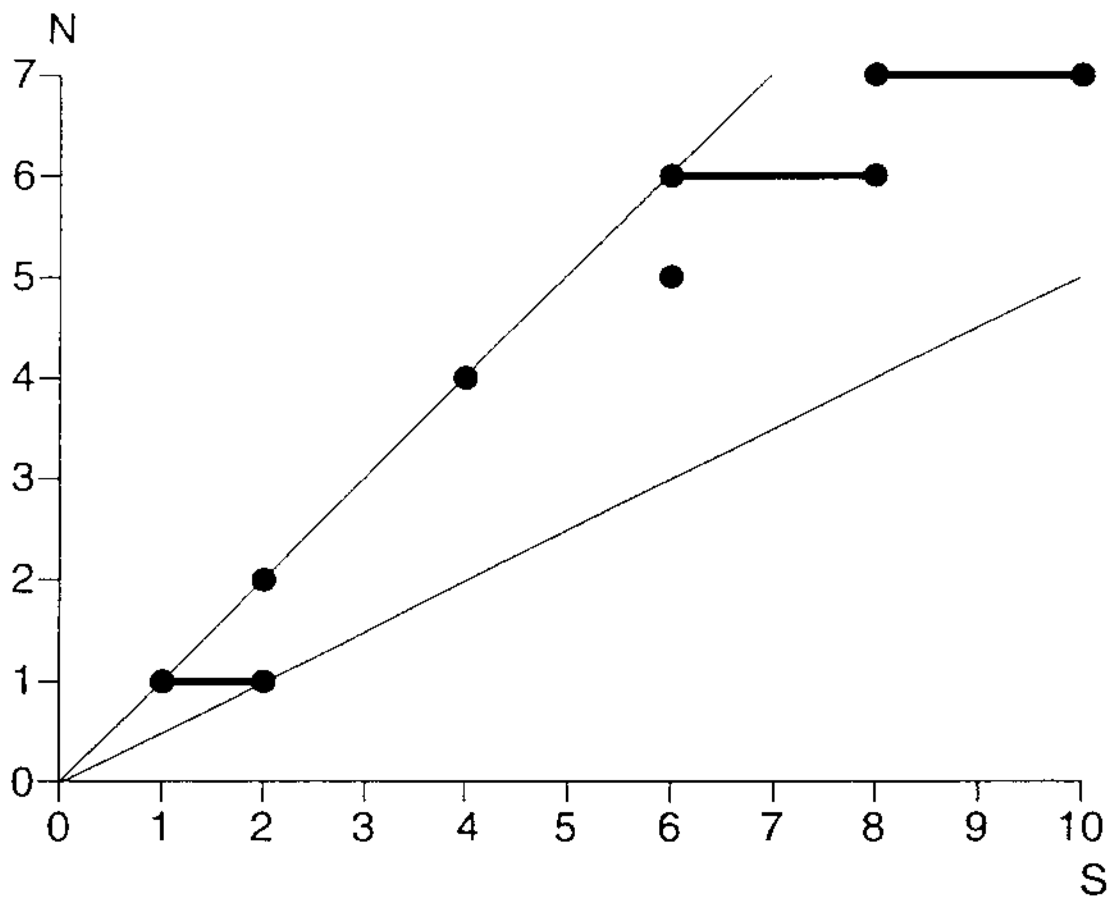

Figure 1. The range of market sizes $[S_m, S_M]$ for which a pure strategy Nash equilibrium exists in which n products are entered, for simultaneous entry. All outcomes in the cone are equilibrium configurations.

In this case, we obtain a sequence of equilibria which lie along the *lower* ray in Figure 1. Combining this with the simultaneous entry example, we see that if we pool the equilibria of *all* 'admissible models' then the cone containing the equilibrium configurations is 'filled' in the following sense: for any n, however large, there exist (perfect) Nash equilibria of 'admissible models' which lie on each boundary of this cone.

V. EXISTENCE

The extension of the simple Hotelling model to 2-dimensions provides a familiar source of examples of non-existence of Nash equilibria (in pure strategies). It is well known, for example, that with three firms present on the plane, there is no pure strategy Nash equilibrium under simultaneous entry (Shaked (1975)).

The non-existence of a simultaneous entry Nash equilibrium does not necessarily imply the non-existence of an equilibrium configuration (as the 3-firm case in 1-dimension shows). It may, however, be the case that no equilibrium configuration exists, as the following example illustrates:

Let the distribution of consumers on the plane consist of three atoms, each of weight 3/5, placed on the corners of an equilateral triangle (Figure 2).

Note that for $\varepsilon = 1$, any configuration involving two or more firms violates the viability condition (i), while any configuration involving no entry violates the stability condition (ii). We now show that any configuration involving exactly one active firm also violates condition (ii), from which it follows that there is no equilibrium configuration.

Let a single firm be active at location h. We show by construction that there is some point g(h) at which a rival firm may enter profitably. Identify a vertex which is closest, or equal closest, to h; label this vertex A. Drop a perpendicular from h to the opposite side of the triangle, BC. Let it meet BC, or its projection, at D. If D lies between B and C, choose g(h) = D. Otherwise, choose g(h) = B or C, whichever is closer to D. The entrant at g(h) earns profit $^6/_5 > 1$, for it captures the customers at B and C. Hence there is no equilibrium configuration.

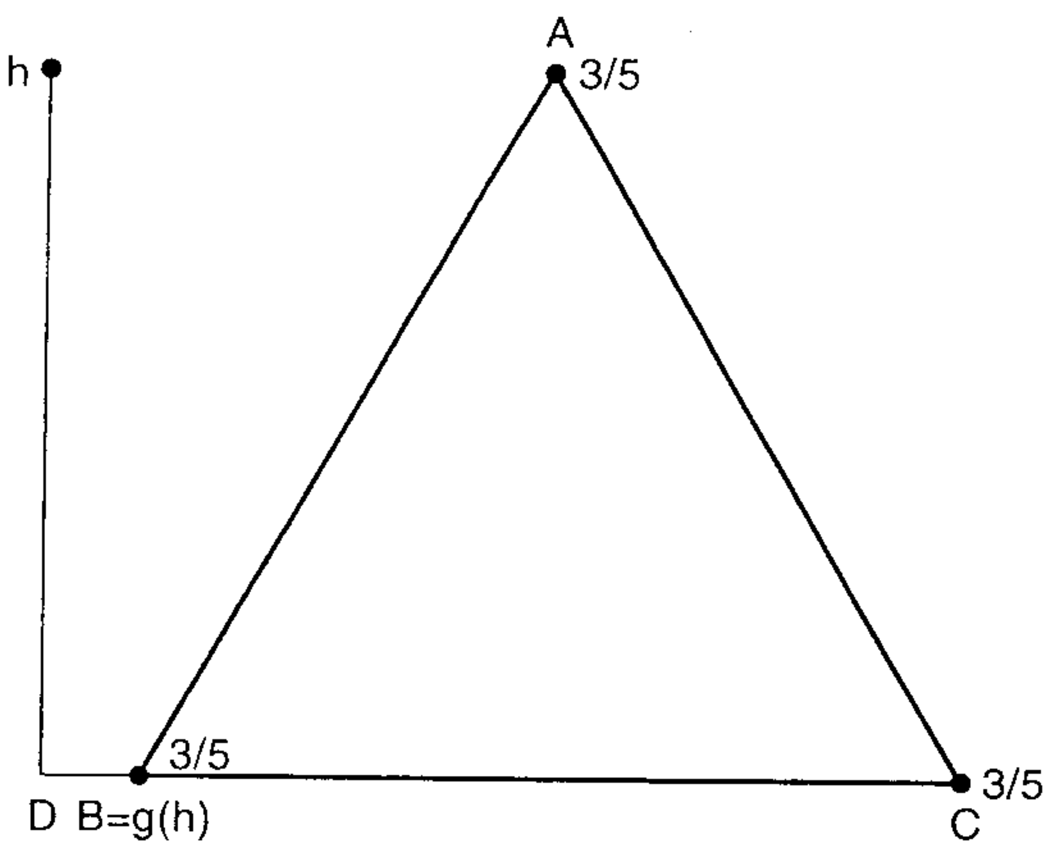

Figure 2. Nonexistence of an equilibrium configuration in a 2-dimensional Hotelling model.

It follows that, for any extensive form satisfying Assumption 2, there is no Nash equilibrium in pure strategies. However, under sequential entry, a Nash equilibrium *does* exist in this example. Let n firms enter products sequentially. It is clear that the last firm in the sequence enters one product, and no other firm enters a product. (This illustrates precisely the 'unattractive' property of sequential equilibria described by Eaton and Ware, which was noted in Section III. Any firm but the last would prefer, given the stage assigned to each rival and the rivals' equilibrium strategies, to delay its move - in fact, to move last in the sequence.)

VI. AN APPLICATION: THE BOUNDS APPROACH

We now turn to the application of the equilibrium configuration concept within the 'bounds' approach to the analysis of market structure. The idea is to use the viability condition (i) and the stability condition (ii) to define two facets of a set in an appropriately chosen space of outcomes. It is then shown that the properties of this set induce a lower bound to concentration as a function of market size. The properties of the lower bound to concentration as a function of market size are as follows (Shaked and Sutton (1987), Sutton (1991)):

(i) if setup costs are fixed exogenously, i.e. each product can be entered for some given cost , then there is a lower bound to concentration as a function of market size S, which converges to zero as S increases.

This 'exogenous sunk cost' setup arises as a special limiting case of a more general model in which a firm is free to spend more on fixed outlays in some early stage(s) of the game with a view to enhancing consumers' willingness-to-pay for the product it offers (Shaked and Sutton (1987), p.140).

Suppose that by spending more fixed outlays $F(\cdot)$ than rivals - on R&D, advertising, etc. - a firm can guarantee itself some minimal level of gross profit $\pi(\cdot)$ in the final stage subgame:

(ii) if there exists some $\alpha > 0$ and $K > 1$ such that a firm spending K times the fixed outlay of its highest spending rival can thereby achieve gross profit $\pi \geq \alpha S$, then there exists a positive lower bound, independent of S, to the level of concentration that can be attained in any perfect Nash equilibrium.

Properties (i) and (ii) are 'robust' in the sense that they have been shown to hold good as a description of perfect Nash equilibria over a broad class of models[4].

Now these properties have been established for Nash equilibria (Shaked and Sutton (1987)), but the proofs extend immediately to the case of equilibrium configurations. In the case of property (i), this extension is

[4]One reason why these bounds properties are of interest is that they imply as a corollary (i.e. they encompass) certain long-established correlations between concentration, scale economies, advertising intensity, etc., (see Sutton (1991), p.123ff.)

trivial; given the inclusion property (Proposition 1), if the lowest level of concentration supportable as a Nash equilibrium converges to zero with S, then the lowest level achieved in any equilibrium configuration certainly converges to zero with S.

The case of property (ii) is less obvious; but reference to the proof shows that it is couched directly in terms of conditions (i) and (ii), and can therefore be applied without modification to the equilibrium configurations. (See Sutton (1991) p.73-4. It is this observation which motivated the present paper.)

In what follows, we illustrate how the viability and stability conditions each define a facet in a suitably chosen space of outcomes, and how the properties of these facets in turn induce the 'lower bound' properties on concentration. This can be illustrated within the simplest model of (horizontal) product differentiation which has been used in the recent literature: the 'linear demand schedule' model (Shubik and Levitan (1980), Deneckere and Davidson (1985), Shaked and Sutton (1987)).

Let each consumer have the same quadratic utility function defined over n varieties of some good,

$$U\ (x_1, x_2, \ldots, x_n)$$

$$= \sum_k (x_k - x_k^2) - \sigma \sum_k \sum_{\ell<k} x_k x_\ell + M \tag{1}$$

where M denotes money spent on outside goods, i.e.

$$M = Y - \sum_k p_k x_k$$

This expression defines utility over the domain of $\{x_k\}$ for which all the marginal utilities U_k are nonnegative. The consumer's optimal purchases will be interior to this domain for all positive price vectors. It is assumed that the consumer's income Y is sufficiently large to ensure that the solution to the optimization problem is the interior solution defined by the set of first order conditions $U_k = p_k$, $\forall k$. The parameter σ, $0 \leq \sigma \leq 2$, is a measure of the degree of substitution between the goods.

If $\sigma = 0$, the cross-product term in the utility function vanishes, while if $\sigma = 2$, the goods are perfect substitutes and

$$U = \Sigma x_k - (\sum_k x_k)^2 + M$$

It follows from (1) that the consumer's inverse demand curve for good k is

$$p_k = 1 - 2x_k - \sigma \sum_{\ell \neq k} x_\ell \qquad (2)$$

Suppose n product varieties are offered, the number offered by firm i being n_i. Assuming zero cost for simplicity, we seek a Nash Equilibrium in quantities. Let S denote the number of consumers in the economy.

Firm i offers quantity Sx_i of each of its varieties[5], the equilibrium price for these varieties being p_i, and it earns profit $n_i\, p_i\, x_i\, S$. The first order condition defining firm i's optimal reply x_i then becomes

$$2\ [2 + \sigma(n_i - 1)]\ x_i + \sigma \sum_{j\neq i} n_j\, x_j = 1 \qquad (3)$$

where the summation Σ is over the N-1 rival firms). Solving this system of equations yields:

$$x_i = \frac{1}{[2(2-\sigma) + \sigma n_i]\{1 + \sum_j \frac{\sigma n_j}{2(2-\sigma) + \sigma n_j}\}} \qquad (4)$$

Substituting in (2) to obtain p_i, the stage 2 profit of firm i in a market of size S = 1 (which equals $n_i\, p_i\, x_i$) becomes

$$\pi(n_i \mid \{n_{-i}\}) = \frac{\frac{(2-\sigma) + \sigma n_i}{[2(2-\sigma) + \sigma n_i]^2} \cdot n_i}{\left\{1 + \sum_j \frac{\sigma n_j}{2(2-\sigma) + \sigma n_j}\right\}^2} \qquad (5)$$

The 'solved out' profit function with respect to which we define the equilibrium configurations is then

$$\Pi(n_i \mid \{n_{-i}\}) = S\pi(n_i \mid \{n_{-i}\}) - n_i\, \varepsilon \qquad (6)$$

[5]It is easy to check that any given firm i will set the same quantity x_i of each of its n_i varieties; we therefore ease notation by simply writing its output per variety as x_i.

where ε is the setup cost per product.

Using equation (4), we can now define the 'viability' and 'stability' conditions in the space of outcomes. We define outcomes in terms of the pair $(\tilde{C}_1, n)$ where $n = \sum_i n_i$ is the total number of varieties entered and

$$\tilde{C}_1 = \frac{\max_i n_i}{\sum_i n_i} = \frac{n_{max}}{n}$$

is the one-firm asset concentration ratio[6].

Now the viability condition states that

$$\Pi \ (n_i \mid \{ n_{-i}\}) \geq 0$$

$$\text{i.e.} \qquad \frac{1}{n_i} S\pi(n_i \mid \{ n_{-i}\}) \geq \varepsilon \qquad (7)$$

From (5) it is readily checked that, for any vector $\{n_i\}$ the profit per product $S\pi(n_i \mid \{n_{-i}\})/n_i$ attains its minimum for the largest n_i.

[6]This is the most natural concentration index to use within this example, and it keeps the algebra relatively simple. Extending the analysis to a sales-concentration ratio C_1 is straightforward, but the algebraic expressions are less transparent. It should be noted that, for any n, the minimum value of both $\tilde{C}_1$ and C_1 are attained at the symmetric solution where all firms have one variety; here $\tilde{C}_1$ and C_1 coincide. The lower bound to C_1 as a function of market size likewise coincides with the lower bound to $\tilde{C}_1$.

Label the firms so that the sequence $\{n_i\}$ is nonincreasing; now from the definition of $\tilde{C}_1$, it follows that $n_1 = n\tilde{C}_1$. It is then easy to show (see Appendix 2) that the viability condition (i) can be written as follows: for any n, the corresponding value of $\tilde{C}_1$ is determined by the equation

$$\frac{1}{n\tilde{C}_1} S\pi(n\tilde{C}_1 \mid n\tilde{C}_1, n\tilde{C}_1, \ldots) = \varepsilon$$

where the number of firms, N, equals $(1/\tilde{C}_1)$. This defines the viability schedule in $(\tilde{C}_1, n)$ space. This can be written explicitly (using equation (5)) as

$$\frac{(2-\sigma) + \sigma n\tilde{C}_1}{[2(2-\sigma) + \sigma n(1 + \tilde{C}_1)]^2} = \varepsilon / S \qquad (8)$$

and this represents an upward sloping schedule in $(\tilde{C}_1, n)$ space.

A second facet in $(\tilde{C}_1, n)$ space may now be defined by combining the viability condition (i) with the stability condition (ii) (see Appendix 2): Fix n and define $\tilde{C}_1$ implicitly via the equation

$$S\pi(1 \mid n\tilde{C}_1, 1, 1 \ldots 1) = \varepsilon$$

where the number of incumbent firms is $n(1-\tilde{C}_1)$. This can be written explicitly as

$$\frac{2}{(4-\sigma)^2} \Big/ \left\{ 1 + \frac{\sigma n \tilde{C}_1}{2(2-\sigma)+\sigma n \tilde{C}_1} + \frac{\sigma[1+n(1-\tilde{C}_1)]}{4-\sigma} \right\}^2 = \varepsilon/S \qquad (9)$$

This defines an upward sloping schedule in $(\tilde{C}_1, n)$ space, which interescts the curve $\tilde{C}_1 = 1/n$ to the left of (8).

<u>An Illustration</u>

The viability and stability facets are illustrated in Figure 3 for a particular value of ε/S. In each case, the feasible values lying within the set are shown. Those values which form Nash equilibria in the simultaneous move game are indicated, as are those values which form equilibria under sequential entry.

As market size S increases, the zone between the two schedules shifts to the right. The minimum level of concentration which can be supported as an equilibrium configuration is defined by the intersection of the viability schedule with the curve $\tilde{C}_1 = 1/n$. As market size goes to infinity, this minimal level of concentration converges to zero. This is the first of the two limit theorems noted above.

The viability schedule was defined above in terms of the cost ε of entering a new plant. In some applications, it will be more appropriate to define it in terms of the avoidable cost $\varepsilon' < \varepsilon$ associated with the closing down of a plant. The replacement of ε by ε' shifts the viability schedule to the right.

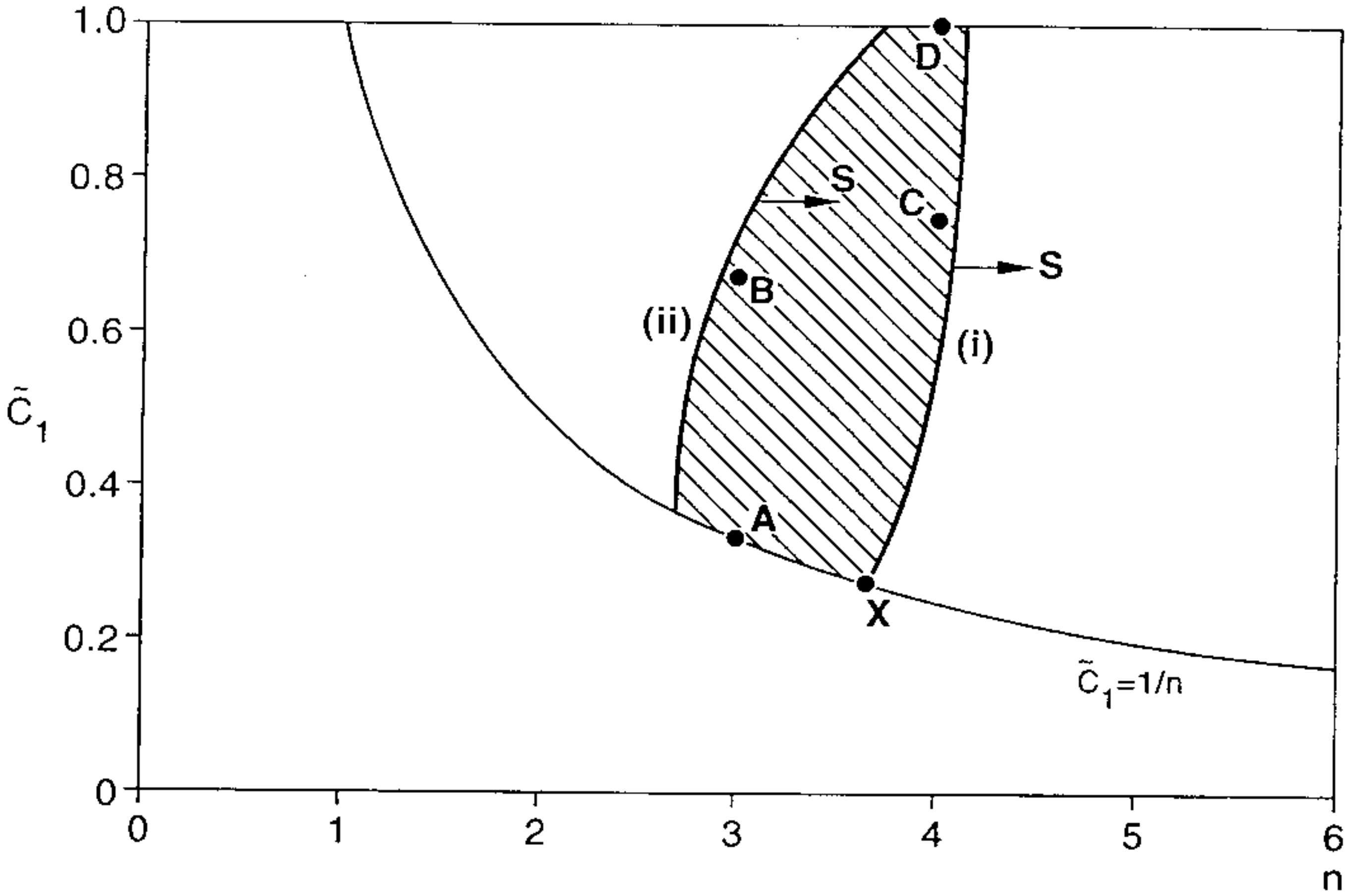

Figure 3. The 'Exogenous Sunk Cost' Case: A numerical example ($\sigma = 0.5$, $\varepsilon = 0.07$, $S = 1$). Schedules (i) and (ii) are the facets defined by the viability and stability conditions respectively. All four points (A, B, C, D) of the form $(\tilde{C}_1, n) = \left(\frac{k}{n}, n\right)$ lying in the shaded area are equilibrium configurations. The only Nash equilibrium under simultaneous entry is A. The only Nash equilibrium under sequential entry is D. As $S \to \infty$, the minimum level of concentration (point X) converges to zero.

Endogenous Sunk Costs

This example can be extended to illustrate the case of 'Endogenous sunk costs'. To do this, we introduce the notion that each good i has a 'quality' level $u_i \geq 0$, and a rise in u_i shifts the demand curve for that good outwards. The simplest setup is one in which the utility function (1) is modified as follows:

$$U = \sum_k \left(x_k - \frac{x_k^2}{u_k^2} \right) - \sigma \sum_{k>\ell} \frac{x_k}{u_k} \cdot \frac{x_\ell}{u_\ell} + M \tag{10}$$

so that the individual consumer's inverse demand schedule becomes

$$p_k = 1 - \frac{2x_k}{u_k^2} - \frac{\sigma}{u_k} \sum_{\ell \neq k} \frac{x_\ell}{u_\ell} \tag{11}$$

For $u_k = 0$, demand is zero for any $p_k \geq 0$. An increase in u_k swivels the (linear) demand schedule outwards, about its vertical intercept. (In what follows, subscripts k and ℓ refer to products; firms are labelled by i and j.)

We assume zero marginal cost, and seek a Nash equilibrium in quantities. It is convenient to express the solution in terms of the output variables x_ℓ/u_ℓ which we write as y_ℓ. It is also convenient to express the average value of y_ℓ for the n_i products owned by firm i as $\bar{y}_i$. It is shown in Appendix 2 that the Nash equilibrium is defined by a set of equations describing the output levels of all products/firms that have positive output at equilibrium, viz.

$$\bar{y}_i = \frac{1}{2(2-\sigma) + \sigma n_i} \left\{ \bar{u}_i - \frac{\sum_j a_j \bar{u}_j}{1 + \sum_j a_j} \right\}, \tag{12}$$

$$\text{where } a_j = \frac{\sigma n_j}{2(2-\sigma) + \sigma n_j}$$

and for any individual product k owned by firm i,

$$y_k = \bar{y}_i + \frac{u_k - \bar{u}_i}{4-2\sigma} \tag{13}$$

Two elementary properties follow. (The proofs are included in Appendix 2).

Proposition 2: (i) Let the quality of any good increase (decrease). Then the equilibrium profit earned on each other good decreases (increases).

(ii) Deleting any good raises the equilibrium profit of each remaining good.

Proposition 3: Let u_M denote the highest quality on offer. Then a necessary condition for a product of quality v to have positive output at equilibrium is $v > (\sigma/4)\, u_M$.

In the case of single product firms the solution reduces to

$$\frac{x_i}{u_i} = \frac{1}{4 + (N-1)\sigma} \left\{ u_i + \frac{N\sigma}{4-\sigma}(u_i - \overline{u}) \right\} \qquad (14)$$

for $u_i \geq [N\sigma \;/\; (4+(N-1)\sigma)\bar{u}]$ and zero otherwise, where N denotes the number of firms (= number of products). In the former case, the equilibrium profit of firm i equals

$$S\pi_i = Sp_i x_i = \frac{2}{[4+(N-1)\sigma]^2} \left\{ u_i + \frac{N\sigma}{4-\sigma} (u_i - \overline{u}) \right\}^2 S \quad (15)$$

In particular, if all N firms have the same quality u then each firm earns profit

$$S\pi(u|N) = \frac{2u^2}{[4+(N-1)\sigma]^2} S \qquad (15)'$$

and if there is only one firm, then profit equals the monopoly profit,

$$S\pi(u|1) = (u^2/8)S \qquad (15)''$$

Note from Proposition 3 that this last expression also represents the profit of a firm offering quality u when all its rivals offer qualities of $(\sigma/4)u$ or less.

If we set all the qualities equal to unity in this model, then the model collapses to the basic model (equation (1)). In what follows, we restrict the qualities to the interval $[1,\infty)$ and we introduce a fixed and sunk cost $F(u)$ of entering a product of quality u. We set

$$F(u) = \varepsilon u^{\beta}$$

where $\beta > 2$. (This restriction on β ensures $F(u)$ rises at least as rapidly as profit, as $u \to \infty$; see equation (15)'.) If all qualities equal unity, then the setup cost equals ε, as in the basic (exogenous sunk costs) model.

We now turn to the viability and stability conditions for this model. In this section we define these conditions in terms of the standard (sales) concentration ratio. The viability condition takes the same form as in the basic (exogenous sunk cost) model, and is obtained by setting all the qualities equal to unity. The form of the stability condition now changes. An explicit calculation of the related facet is now difficult, but its qualitative properties are easily characterised.

The facet shifts left relative to the facet for the exogenous sunk cost model, at high values of concentration. The intuition for this is seen by considering the minimum number of products which a monopolist needs to offer in order to deter entry. The menu of possibilities open to the monopolist is now extended, relative to the exogenous sunk cost model, in that he may choose either to offer a large number of products of quality 1, or a smaller number of products quality greater than 1. So long as σ is strictly positive, so that there is some degree of substitution among products, a single product monopolist may be able to deter entry by setting u sufficiently high. The following property is established in Appendix 2.

Property I: For any given $\sigma > 0$, then for β sufficiently low (close to 2) the stability facet passes through the monopoly solution $(C_1,N) = (1,1)$.

The second property states that the lower part of the schedule shifts upwards, reflecting the fact that at low levels of concentration, entry by a high quality producer breaks the configuration. This property leads to the basic 'nonconvergence' result for the 'Endogenous sunk cost' model (see Figure 4).

Property II: For any $\beta > 2$, the 1-firm sales concentration ratio C_1 is bounded below by

$$\underline{C}_1 > \frac{1}{1 + \frac{4}{\sigma}\left[\left(\frac{4}{\sigma}\right)^{\beta-2} - 1\right]}$$

in any equilibrium configuration. In particular, the stability facet lies above this critical value of $\underline{C}_1$ for all N, independently of S.

Proof: To establish this result, we calculate an upper bound on the number of firms that can have positive sales in any equilibrium configuration.

Consider any configuration in which N firms have positive sales, the highest and lowest qualities with positive sales being denoted u_M and u_m respectively. Proposition 3 implies $u_M \geq (4/\sigma)\, u_m$.

Since this configuration is viable, it follows from Proposition 2 that there exists a viable configuration of N single product firms offering qualities in the same range $[u_m\, ,\, u_M]$. (The viability condition ensures that each firm covers its fixed costs, and so it must have at least one product that covers its fixed costs. Assign one such product to each firm and delete all other products. Proposition 2 ensures that this new N firm configuration satisfies the viability condition)

Denote the lowest quality offered in this single-product firm configuration as v. Proposition 2 implies that the profits earned by this firm cannot exceed the profits it would earn if all firms had quality v. Hence viability implies, using equation (15)′, that

$$F(v) = \varepsilon v^{\beta} \leq \frac{2v^2}{[4 + (N-1)\sigma]^2} S$$

whence (since all qualities are defined on $[1,\infty)$ and $\beta > 2$),

$$v^{\beta-2} \leq \frac{2}{[4 + (N-1)\sigma]^2} \cdot \frac{S}{\varepsilon} \tag{16}$$

Now consider an entrant offering a single product of quality $u = (4/\sigma)u_M$. All rival products have zero sales, by Proposition 2, and the entrant's net profit equals (using (15)″),

$$\frac{u^2}{8}S - F(u) = \frac{u^2}{8}S - \varepsilon u^{\beta}$$

The stability condition requires that this be non-positive, whence

$$u^{\beta-2} \geq \frac{1}{8} \cdot \frac{S}{\varepsilon}$$

But $u = (4/\sigma)u_M \geq (4/\sigma)^2 v$, whence

$$\left(\frac{4}{\sigma}\right)^{2(\beta-2)} v^{\beta-2} > \frac{1}{8}\ \frac{S}{\varepsilon} \tag{17}$$

Combining (16) and (17) we have

$$\left(\frac{4}{\sigma}\right)^{2(\beta-2)} \cdot \frac{2}{[4 + (N-1)\sigma]^2} > \frac{1}{8}$$

whence

$$N \leq 1 + \frac{4}{\sigma}\left[\left(\frac{4}{\sigma}\right)^{\beta-2} - 1\right]$$

But since $C_1 \geq 1/N$, this implies Property II.

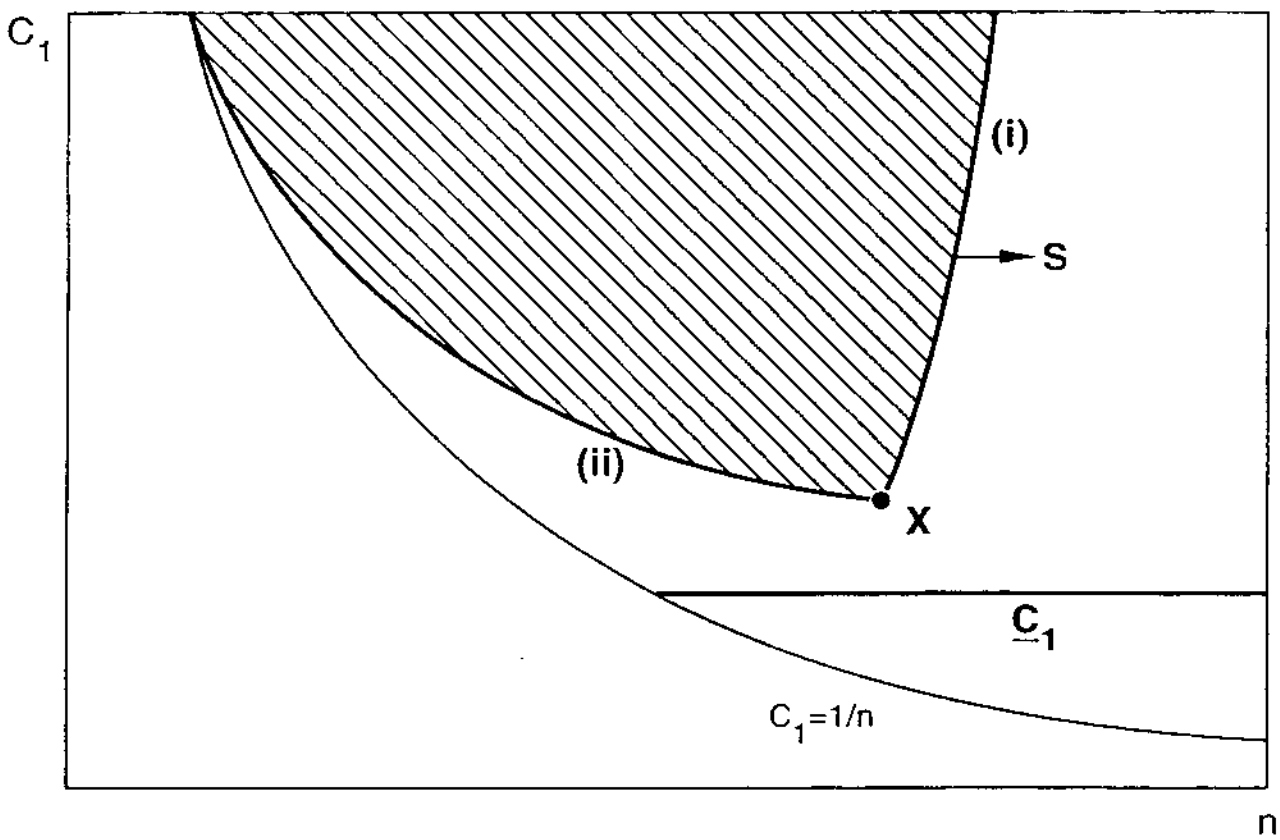

Figure 4. The general (Endogenous sunk cost) case. The equilibrium configurations lie in the shaded area. Schedules (i) and (ii) are the facets defined by the viability and stability conditions respectively. Schedule (ii) always lies above the horizontal line labelled $\underline{C}_1$ (Appendix 2). The minimum level of concentration (point X) is bounded away from zero as market size $S \to \infty$.

VII. CONCLUDING REMARKS

There are two separate lines of argument which can be used to motivate the present approach.

On the one hand, it may seem reasonable to place a fairly weak requirement on agents' maximizing behaviour in a context where firms face situations that are in some way novel, and where it is unattractive to assume that each firm 'knows the true model'. This is a point of view that has been emphasised in the 'Schumpeterian' literature on technology and market structure. But even if such arguments are set aside, so that all firms are assumed to share the same correct beliefs as to the 'true model' and to choose optimal strategies relative to those beliefs, it many for some purposes be better to model the situation by invoking only the present weak requirement. In analysing outcomes we cannot observe firms' beliefs, and so we cannot assess the *ex ante* optimality of their actions. We may rationalize many outcomes by positing some particular extensive form representation and/or by attributing particular beliefs to agents at past dates. In the present approach, where restrictions are imposed directly on the space of observed outcomes, we avoid the temptation of offering excessively detailed 'explanations' of events 'within the theory' by postulating some possibly quite complex strategy space and/or structure of beliefs which is not directly observable.

This does not imply a denial of the value of adding a richer strategic structure where we know enough about the market to justify additional assumptions (For an example, see Sutton (1991), Chapter 9). Rather, it involves the idea that it is better to first determine what restrictions can be placed on observed outcomes using assumptions that can be justified over a very broad range of situations, before proceeding to add assumptions that can be justified only in a narrower set of circumstances.

Once we define the set of equilibrium configurations in the way proposed here, the testing of the theory revolves around two questions:

(a) do all observed outcomes lie 'within this set'?

(b) do the observed outcomes 'fill this set'?

In two companion papers, these issues are explored. It can be argued on the basis of various empirical studies that observed outcomes do indeed tend to lie 'within this set'[7]. On the other hand, they do not typically 'fill the set'. Moreover, the reason the bounds defined by theory fail to be 'tight' can not be attributed to our relaxation of the Nash equilibrium notion; rather, it reflects the fact that some configurations, though 'possible' in the sense that they are (perfect) Nash equilibria, are 'unlikely' to occur. To make this notion precise, we need to go beyond the analysis of the present paper. This theme is taken up in Sutton (1995 a,b).

---

[7]See for example Sutton (1991), Matraves (1993), Robinson (1994).

APPENDIX 1

Hotelling's Simple Location Model: The Entry Game

Note that the only candidate n-firm equilibria of the Entry game correspond to the equilibria of the simple Hotelling game in which the number of firms is fixed at n, for otherwise some firm is not optimally located relative to its rivals. The following results then follow:

$n = 1$: $[S_m, S_M] = [1,2]$

$n = 2$: $S_m = S_M = 2$

The only candidate equilibrium is $A = B = \frac{1}{2}$. A deviant can enter at $\frac{1}{2} - \varepsilon$, obtaining payoff $(\frac{1}{2} - \frac{1}{2}\varepsilon)S$. Hence if $S > 2$, the two firm configuration is not a Nash equilibrium, since $(\frac{1}{2} - \frac{1}{2}\varepsilon)S > \sigma = 1$ for ε sufficiently small.

$n = 3$: No Nash equilibrium in pure strategies exists.

$n = 4$: $S_m = S_M = 4$

The only candidate equilibrium is $A,B = \frac{1}{4}$; $C,D = \frac{3}{4}$. A deviant can enter at $\frac{1}{4} - \varepsilon$, obtaining payoff $(\frac{1}{4} - \frac{1}{2}\varepsilon)S$. Hence if $S > 4$, the four firm configuration is not a Nash equilibrium, since $(\frac{1}{4} - \frac{1}{2}\varepsilon)S > \sigma = 1$ for ε sufficiently small.

n = 5: $S_m = S_M = 6$

The only candidate equilibrium is $A = B = {}^1/_6$, $C = ½$, $D = E = {}^5/_6$. At equilibrium, $\pi_A = \pi_B = \pi_D = \pi_E = (S/6)$ and $\pi_C = S/3$. This configuration is viable only if $S \geq 6$. However, if $S > 6$, then a deviant can enter profitably at ½ (whence C and the deviant both earn more than S/6).

n= 6: $[S_m, S_M] = [6,8]$

Any configuration of the form shown in Figure 5 in which $1/8 \leq d \leq {}^1/_6$ is a Nash equilibrium of the Hotelling model (Eaton and Lipsey (1975)).

For any $S \in [6,8]$, a 6-firm equilibrium of the location game exists. To show this, set $d = 1/S$. Note that all firms earn at least profit $S \cdot d = 1$ (firms C and D earn more). The highest profit that can be earned by an entrant is $S \cdot d = 1$. Hence this is a Nash equilibrium of the entry game.

For any S outside this range, there is no 6-firm equilibrium of the entry game. (For $S < 6$ at least one firm makes profit $S/6 < 1$. For $S > 8$ entry - just to the left of A, for example - is profitable.)

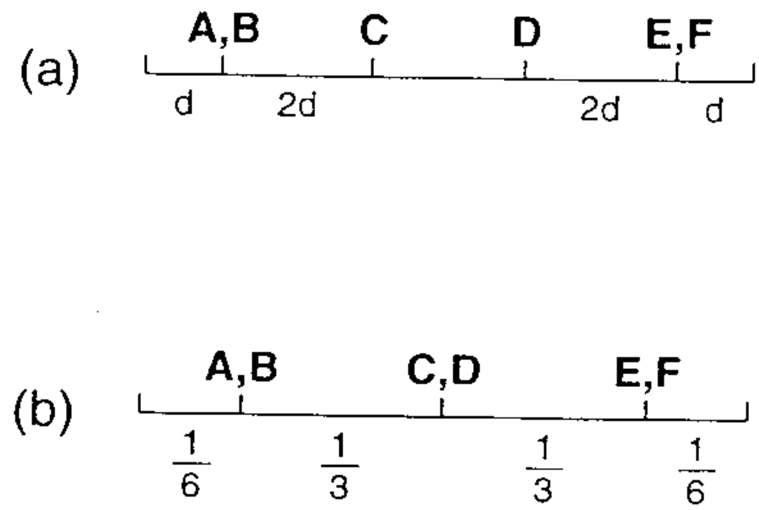

Figure 5. Equilibrium configurations in the Simple Hotelling Model for n = 6. In panel (a) $1/8 \le d \le 1/6$. If $d = 1/6$ firms C and D coincide (panel (b)).

APPENDIX 2

The 'Linear Demand Schedule' Example

Derivation of the Viability facet (Equation (8))

Note that $\pi(n_i|n_{-i})/n_i$ attains its minimum for the largest n_i. Label firms so that $n_1 \geq n_2 \geq n_3 \geq \ldots$ Fix a value of n. We aim to find the lowest value of n_1 such that

$$\frac{1}{n_1} S\pi \ (n_1|\{n_{-1}\}) \geq \varepsilon$$

For any given n_1, consider the minimum of $\pi(n_i|\{n_j\})$ over all vectors n_j satisfying $\sum_j n_j = n - n_1$ and $n_j \leq n_1$, $\forall_j$. Inspection of the profit function (5) indicates that this minimum is attained when $n_j \leq n_1$, $\forall_j$. To see this, notice that merging any two firms of size n_k, n_ℓ into a firm of size $(n_k + n_\ell)$ reduces the value of the term $\Sigma(\cdot)$ in (5).

Hence the condition (i) becomes

$$\frac{1}{n\tilde{C}_1} S\pi \ (n\tilde{C}_1|n\tilde{C}_1, \ldots, n\tilde{C}_1) \geq \varepsilon$$

Treating n and $\tilde{C}_1$ as continuous variables, the $\Sigma(\cdot)$ term in equation (5) becomes $\frac{\sigma n}{2(2-\sigma) + \sigma n\tilde{C}_1}$ whence the viability schedule becomes[1]

$$\frac{(2-\sigma) + \sigma n\tilde{C}_1}{[2(2-\sigma) + \sigma n\tilde{C}_1]^2} \Bigg/ \left\{ 1 + \frac{\sigma n}{2(2-\sigma) + \sigma n\tilde{C}_1} \right\}^2 = \varepsilon/S$$

from which (8) follows.

Derivation of the Stability facet (Equation (9))

Consider an entrant who enters k products; entering these products is profitable if and only if

$$\frac{1}{k} S\pi \ (k|\{n_i\}) \geq \varepsilon$$

Inspection of the profit function indicates that the expression on the left hand side attains its maximum when k = 1. Hence we many confine attention to the case

$$S\pi(1|\{n_i\}) \geq \varepsilon$$

Fix a pair of values n and n_1. Consider the minimum of the function $\pi(1|\{n_i\})$ over all vectors $\{n_i\}$ satisfying $\sum_{j\neq 1} n_j = n - n_1$,

[1]Each term in Σ is the same, and the number of terms equals the number of firms, which is $1/\tilde{C}_1$

where firms are labelled in descending order of size, as before. Inspection of the profit function (5) indicates that the minimum is attained when $n_j = 1, \forall_j \neq 1$. Hence the stability condition may be defined as follows: fix n and define $\tilde{C}_1$ implicitly via the equation

$$S\pi(1|n\tilde{C}_1, 1, 1, \ldots 1) = \varepsilon$$

where the number of incumbent firms is $n(1-\tilde{C}_1)$. This can be written explicitly by noting that we have one firm with $n\tilde{C}_1$ products, and the remaining $n-n\tilde{C}_1$ products are each owned by single-product firms. So the number of firms, prior to entry, equals $1 + n(1-\tilde{C}_1)$.

Hence the $\Sigma(\cdot)$ term in the profit function (5) becomes

$$\frac{\sigma n\tilde{C}_1}{2(2-\sigma) + \sigma n\tilde{C}_1} + \frac{\sigma[1 + n(1-\tilde{C}_1)]}{2(2-\sigma) + \sigma}$$

(The first term corresponds to firm 1, while the second corresponds to the remaining $n(1-\tilde{C}_1)$ firms, plus the entrant.) Equation (9) now follows.

Solving the Quality Model

The inverse demand schedule for product k is

$$p_k = 1 - \frac{2x_k}{u_k^2} - \frac{\sigma}{u_k} \sum_{\ell \neq k} \frac{x_\ell}{u_\ell} \tag{A1}$$

The profit of firm i equals

$$\pi_i = \sum_{k \in I} p_k \, x_k$$

where I denotes the set of products owned by firm i. Firm i's optimal reply (reaction function) is implicitly defined by the set of equations,

$$\forall k \in I \qquad \frac{\delta \pi_i}{\delta x_k} = p_k + \sum_{\ell \in I} x_\ell \frac{\delta p_\ell}{\delta x_k} = 0 \tag{A2}$$

where $\dfrac{\delta p_k}{\delta x_k} = -\dfrac{2}{u_k^2}$ and $\dfrac{\delta p_\ell}{\delta x_k} = -\dfrac{\sigma}{u_k u_\ell}$ for $\ell \neq k$.

Substituting (A1) into (A2), re-arranging, and subtracting $2\sigma \dfrac{x_k}{u_k^2}$ from each side of the re-arranged equation yields

$$(4 - 2\sigma) \frac{x_k}{u_k^2} = 1 - \sum_{\ell} \frac{\sigma}{u_k u_\ell} x_\ell - \sum_{\ell \in I} \frac{\sigma}{u_k u_\ell} x_\ell \tag{A3}$$

where $\sum_{\ell}$ denotes a sum over all products.

The notation can be simplified by defining the new set of output variables, $y_\ell = x_\ell / u_\ell$, so that (A3) becomes

$$\forall k \epsilon I, \qquad (4 - 2\sigma)\ y_k = u_k - \sigma \sum_{\ell \epsilon I} y_\ell - \sigma \sum_{\ell} y_\ell \tag{A4}$$

Equation (A4) implicitly defines the reaction function of firm i. Denote the number of products owned by firm i as n_i. Denote the average quality of these products as $\overline{u}_i = (\sum_{\ell \epsilon I} u_\ell)/n_i$, and the average value of the output variable as $\overline{y}_i = (\sum_{\ell \epsilon I} y_\ell)/n_i$. Using this notation, we obtain on summing the set of equations (A4) over the set of products I owned by firm i, and dividing by n_i:

$$(4 - 2\sigma)\ \overline{y}_i = \overline{u}_i - \sigma \sum_{\ell \epsilon I} y_\ell - \sigma \sum_{\ell} y_\ell$$

Writing $\sum_{\ell \epsilon I} y_\ell$ as $n_i \overline{y}_i$ and $\sum_{\ell} y_\ell$ as $\sum_j n_j \overline{y}_j$ where $\sum_j$ denotes a sum over all firms, this becomes

$$[2(2-\sigma) + \sigma n_i]\ \overline{y}_i = \overline{u}_i - \sigma \sum_j n_j \overline{y}_j \tag{A5}$$

whence we have:

$$\forall_i, \qquad \overline{y}_i = \frac{\overline{u}_i - \sigma \sum_j n_j \overline{y}_j}{2(2-\sigma) + \sigma n_i} \tag{A6}$$

This set of linear equations for the $\overline{y}$'s can now be solved routinely, as follows: Multiply both sides of (A5) by σn_i and sum the resulting equation over all firms, to obtain:

$$\sigma\sum_j n_j\bar{y}_j = \frac{\sum_j \dfrac{\sigma n_j\bar{u}_j}{2(2-\sigma) + \sigma n_j}}{1 + \sum_j \dfrac{\sigma n_j}{2(2-\sigma) + \sigma n_j}} \tag{A7}$$

Substituting this into (A6) and simplifying yields

$$\bar{y}_i = \frac{1}{2(2 - \sigma) + \sigma n_i}\left\{\frac{\bar{u}_i + \sum_j \dfrac{\sigma n_j(\bar{u}_i - \bar{u}_j)}{2(2 - \sigma) + \sigma n_j}}{1 + \sum_j \dfrac{\sigma n_j}{2(2 - \sigma) + \sigma n_j}}\right\} \tag{A8}$$

which is equivalent to equation (12) of the text.

Finally, the solution for each individual product owned by firm i can be expressed in terms of $\bar{y}_i$ as follows. From (A5) it follows that

$$(4 - 2\sigma)\ \bar{y}_i = \bar{u}_i - \sigma n_i\bar{y}_i - \sigma\sum_j n_j\bar{y}_j$$

while (A4) implies

$$(4 - 2\sigma)\ y_k = u_k - \sigma n_i\bar{y}_i - \sigma\sum_j n_j\bar{y}_j \tag{A9}$$

Subtracting this equation from the preceding equation, and simplifying, yields

$$y_k = \overline{y}_i + \frac{u_k - \overline{u}_i}{4 - 2\sigma} \qquad \text{(A9)}'$$

which is equation (13) of the text.

Lemma 1: Let the quality of some good increase, the other qualities being held constant. Then the total weighted output $\sum_j n_j \overline{y}_j$ increases.

Proof: By inspection of equation (A7).

Corollary: Introducing any good l which has positive output at equilibrium raises $\sum_j n_j \overline{y}_j$. To see this, define a 'threshold' quality at which the newly introduced good has equilibrium output zero (since a good of quality zero has equilibrium output zero, it follows that there is such a nonnegative threshold quality).

Introduce a new good at this threshold quality. The vector of equilibrium outputs for all other goods is unchanged by the introduction of this new good. (This follows by inspection of the system of reaction functions (A4)). Now raise the quality of the new good, and apply Lemma 1.

Lemma 2: Let the quality of some good ℓ increase. Then the equilibrium output of each other good $k \neq \ell$ falls.

Proof: Label the owner of good k as firm i, and consider first the case where ℓ belongs to one of i's rivals, i.e. $\ell \notin I$.

Lemma 1 implies that raising u_ℓ increases $\sum_j n_j \bar{y}_j$, whence (A6) implies that $\bar{y}_i$ falls. It follows from (A9)′ that y_k falls.

Now consider the case $\ell \in I$. In this case, the increase in u_ℓ raises $\bar{u}_i$. It follows from (A7) that

$$\frac{d}{d\bar{u}_i}\left\{ \sigma \sum_j n_j \bar{y}_j \right\} < 1$$

Hence the expression on the r.h.s. of (A5) increases, and so $\bar{y}_i$ increases. This, together with Lemma 1, implies so that y_k decreases; this can be seen by inspection of (A9). (The expression on the r.h.s. of (A9) decreases, since $\bar{y}_i$ increases, while Lemma 1 implies that $\sum_j n_j \bar{y}_j$ increases.)

Remark: When the quality of good ℓ increases, the output of good ℓ increases, while the output of all other goods falls, but the combined fall is not enough to offset the rise in y_ℓ so that $\sum_j n_j \bar{y}_j$ increases (Lemma 1).

Proposition 2: (i) Let the quality of any good increase (decrease). Then the equilibrium profit earned on each other good decreases (increases).

(ii) Deleting any good raises the equilibrium profit of each remaining good.

Proof: From the demand schedule (A1), the profit on good k can be written as

$$\pi_k(x_k) = p_k x_k = x_k \left[1 - \frac{\sigma}{u_k} \sum_{\ell \neq k} y_\ell \right] - \frac{2}{u_k^2} x_k^2$$

This function is strictly increasing on $[0, x_k^*)$, where x_k^* is the equilibrium output defined by (A2).

If the quality of any other good rises, then it follows from Lemma 2 that y_k, and so x_k, falls. It also follows from Lemma 2 that the term $\sum_{\ell \neq k} y_\ell$ rises (Lemma 1 implies that $\sum_j n_j \overline{y}_j \equiv \sum_\ell y_\ell$ rises, while Lemma 2 implies that y_k falls). Hence π_k falls. This establishes part (i).

The proof of part (ii) follows by first considering the entry of a good at a quality such that its output is exactly zero, and then applying part (i).

Lemma 3: Let the set of goods offered by firm i that have positive sales at equilibrium be labelled $u_n \geq \ldots \geq u_2 \geq u_1$. Then $u_1 \geq (\sigma/2)\, u_n$.

Proof: The first order conditions for maximization of $(\pi_1 + \pi_2 + \ldots + \pi_n)$ imply that if $x_n, x_1 > 0$ then

$$\frac{\delta \pi_n}{\delta x_n} = 1 - \frac{4}{u_n^2} x_n - \frac{2\sigma}{u_n u_1} x_1 - \frac{2\sigma}{u_n} \sum_{j \neq 1,n} \frac{x_j}{u_j} = 0$$

$$\frac{\delta \pi_1}{\delta x_1} = 1 - \frac{2\sigma}{u_n u_1} x_n - \frac{4}{u_1^2} x_1 - \frac{2\sigma}{u_1} \sum_{j \neq 1,n} \frac{x_j}{u_j} = 0$$

Comparing coefficients, we have that if $u_1 < (\sigma/2)\, u_n$ then $\frac{\delta \pi_1}{\delta x_1} < \frac{\delta \pi_n}{\delta x_n}$ for all output vectors, which implies a contradiction.

Proposition 3: Let u_M be the highest quality on offer. Then a necessary condition for a good of quality v to have positive output at equilibrium is that $v > (\sigma/4)u_M$.

Proof: Equation (A2) implies that a necessary condition for $y_i > 0$ is that

$$\overline{u}_i > \sum_j \frac{\sigma n_j (\overline{u}_j - \overline{u}_i)}{2(2 - \sigma) + \sigma n_j} \tag{A10}$$

Identify the firm offerring the highest quality, which we label u_M. Denote the number of products offered by this firm (that have positive sales at equilibrium) as n_M and the average quality of these products as $\bar{u}_M$. Note that Lemma 3 implies that each of the n_M -1 other products offered by this firm must have quality $(\sigma/2)u_M$ or higher. Hence:

$$\overline{u}_M \geq \frac{1}{n_M}\left[1 + (n_M - 1)\frac{\sigma}{2} \right] u_M \tag{A11}$$

Consider the firm i with the lowest average quality, $\bar{u}_i$. For this firm, all terms in the sum on the r.h.s. of (A10) are nonnegative, whence

$$\overline{u}_i > \frac{\sigma n_M (\overline{u}_M - \overline{u}_i)}{2(2 - \sigma) + \sigma n_M} \tag{A12}$$

Combining (A11) and (A12) it follows that

$$\bar{u}_i \geq (\sigma/4)\, u_M \tag{A13}$$

Now suppose firm i offers a range of qualities, the lowest being v. We now show that if $v < (\sigma/4)u_M$, then v has zero sales at equilibrium. For suppose the contrary. Then delete all other goods offered by firm i. By Proposition 2, the output of v now increases, and so is still positive. But now $v \equiv \bar{u}_i$, whence (A13) is violated. It follows that $v > (\sigma/4)u_M$.

Equilibrium Configurations

In this section, the qualities are defined on the interval $[1,\infty)$.

Property I (Leftward Shift): For any given $\sigma > 0$, then for β sufficiently low (close to 2) the stability facet passes through the monopoly solution $(C_1,N) = (1,1)$.

To show this, note that the monopolist's profit is $(u^2/8)S$. Choose the level of u such that the monopolist earns just enough profit to cover his fixed costs,

$$\frac{u^2}{8}S = F(u) = \varepsilon u^{\beta} \tag{A14}$$

whence

$$u^{\beta-2} = \frac{S}{8\varepsilon} \tag{A15}$$

We aim to show that no entrant may profitably enter with any quality $v \geq 1$. (Recall that quality is defined on the domain $[1,\infty)$.) To see this, note firstly that if $v > u$, he earns profit no greater than $\frac{v^2}{8}S - \varepsilon v^{\beta}$. Since $\beta > 2$, it follows from the definition of u (equation (A14)) that this is negative. If $1 \leq v \leq \frac{\sigma}{4}u$, the entrant has output zero, and so earns negative profit. Hence we may confine attention to the case where the quality ratio u/v lies in the interval $1 \leq \frac{u}{v} \leq \frac{4}{\sigma}$.

Denote the ratio u/v by k, $1 \leq k \leq \frac{4}{\sigma}$, whence (A15) implies:

$$v^{\beta-2} = \frac{S}{8\varepsilon} \cdot \frac{1}{k^{\beta-2}} \tag{A16}$$

From the profit function (15), we have (on writing $u_i = v$, $\bar{u} = (u + v)/2$ and $N = 2$),

$$\Pi(v|u) = \frac{2}{(4+\sigma)^2 (4-\sigma)^2} (4 - k\sigma)^2 v^2 S - \varepsilon v^{\beta}$$

and this is negative if

$$v^{\beta-2} \geq \frac{2}{(4+\sigma)^2 (4-\sigma)^2} (4 - k\sigma)^2 \frac{S}{\varepsilon} \qquad \text{(A17)}$$

Combining (A16) and (A17), entry is unprofitable if

$$\frac{1}{k^{\beta-2}} \geq \left\{ \frac{4}{(4+\sigma)(4-\sigma)} \right\}^2 (4 - k\sigma)^2$$

or

$$(4 - k\sigma)^2 k^{\beta-2} \leq \left\{ \frac{(4+\sigma)(4-\sigma)}{4} \right\}^2$$

Recall that $1 \leq k \leq 4/\sigma$. Note that the expression on the l.h.s. is decreasing in k on this domain if $\beta < 2 + \frac{2\sigma}{4-\sigma}$, whence the expression on the l.h.s. takes its maximum at k = 1, where it equals $(4-\sigma)^2$. It follows that for all $\beta < 2 + \frac{2\sigma}{4-\sigma}$, the inequality is satisfied.

REFERENCES

Deneckere, R. and Davidson, C., (1985), 'Incentives to Form Coalitions with Bertrand Competition,' Rand Journal of Economics, 16, pp.473-486.

Eaton, B.C. and Lipsey, R.G., (1975), 'The Principle of Minimum Differentiation Reconsidered: Some New Developments in the Theory of Spatial Competition,' Review of Economic Studies, 42(1), pp. 27-49.

Eaton, B.C. and Ware, R., (1987), 'A Theory of Market Structure with Sequential Entry,' Rand Journal of Economics, 18, pp. 1-16.

Fisher, F.M., (1989), 'Games Economists Play: A Noncooperative View, Rand Journal of Economics, 20, pp. 113-24.

Pelzman, S. (1991), 'The Handbook of Industrial Organisation: A Review Article,' Journal of Political Economy, 99, pp. 201-217.

Matraves, C., (1992), 'Endogenous Sunk Costs, Industry Size and Market Structure: A Four Country Test,' Discussion Paper No. 9224, Norwich: The Economics Research Centre.

Robinson, W. (1993), 'Are Sutton's Predictions Robust?: Empirical Evidence on Price Competition, Advertising and the Evolution of Concentration,' unpublished.

Shaked, A, (1975), 'Non-Existence of Equilibrium for the 2-Dimensional 3-Firms Location Problem,' Review of Economic Studies, 42(1), pp. 51-56.

Shaked, A., (1982), 'Existence and Computation of Mixed Strategy Equilibrium for 3-Firms Location Problem', Journal of Industrial Economics, vol 31, pp. 93-96.

Shaked, A., and Sutton, J., (1987), 'Product Differentiation and Industrial Structure,' Journal of Industrial Economics, 36, pp. 131-146.

Shaked, A., and Sutton, J., (1990), 'Multiproduct Firms and Market Structure', Rand Journal of Economics, 21, pp. 45-62.

Shubik, M. and Levitan, R., (1980), Market Structure and Behaviour, Cambridge, MA: Harvard University Press.

Sutton, J., (1990), 'Explaining Everything, Explaining Nothing? Game Theoretic Models in Industrial Organisation' European Economic Review, vol 34, pp. 505-12.

Sutton, J., (1991), Sunk Costs and Market Structure, Cambridge, MA: MIT Press, 1991.

Sutton, J., (1995a,b), 'The Size Distribution of Businesses, Parts I and II', unpublished.