Semiparametric Estimation of Fractional Cointegration

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Abstract

A semiparametric bivariate fractionally cointegrated system is considered, integration orders possibly being unknown and I (0) unobservable inputs having nonparametric spectral density. Two kinds of estimate of the cointegrating parameter ν are considered, one involving inverse spectral weighting and the other, unweighted statistics with a spectral estimate at frequency zero. We establish under quite general conditions the asymptotic distributional properties of the estimates of ν, both in case of "strong cointegration" (when the difference between integration orders of observables and cointegrating errors exceeds 1/2) and in case of "weak cointegration" (when that difference is less than 1/2), which includes the case of (asymptotically) stationary observables. Across both cases, the same Wald test statistic has the same standard null χ^2 limit distribution, irrespective of whether integration orders are known or estimated. The regularity conditions include unprimitive ones on the integration orders and spectral density estimates, but we check these under more primitive conditions on particular estimates. Finite-sample properties are examined in a Monte Carlo study.

JEL Classification: C32.

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1. Introduction

Semiparametric modelling has become popular in cointegration analysis of $I(1)$ time series with $I(0)$ cointegrating errors. In the simplest parametric setting, observables follow a random walk and cointegrating errors are serially uncorrelated. Autoregressive (AR) extensions have been developed (e.g. Johansen, 1991), but optimal inference on the unknown cointegrating relations loses validity if the AR order is under-specified, or if the process lies outside the AR class. Phillips and Hansen (1990), Phillips (1991a) and others showed that one can do as well allowing the $I(0)$ inputs to have nonparametric autocorrelation, under suitable conditions on the bandwidth employed in the smoothed nonparametric spectrum estimate.

Another source of possible misspecification is the basic $I(1)/I(0)$ framework itself. Recently, optimal inference has developed in a fractional setting (see e.g. Jeganathan, 1999, Robinson and Hualde, 2003). Here, integration orders were allowed to be unknown, which is a non-trivial generalization of the $I(1)/I(0)$ assumption, but theory was developed only in a fully parametric setting, incurring the familiar concern about misspecification. On the other hand, Dolado and Marmol (1996), Kim and Phillips (2000) have allowed for nonparametric autocorrelation in $I(0)$ inputs. However these authors have either assumed knowledge of integration orders, or proposed sub-optimal estimates.

The present paper develops optimal inference on cointegrating relations in a semiparametric fractional setting, with unknown integration orders. To describe our model, we introduce the following definitions corresponding to ones in Robinson and Hualde (2003) (hereafter RH). For any scalar or vector sequence v_t, $t = 0, \pm 1, ...$, we denote

$$v_t^{\#} = v_t 1(t > 0), \tag{1.1}$$

where $1(\cdot)$ is the indicator function. Defining the difference operator $\Delta = 1 - L$, where L is the lag operator, the fractional difference operator is given formally, for any real α, $\alpha \neq -1, -2, ..$, by

$$\Delta^{-\alpha} = \sum_{j=0}^{\infty} a_j(\alpha) L^j, \quad a_j(\alpha) = \frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+1)}, \tag{1.2}$$

with Γ denoting the gamma function. With the prime denoting transposition, $u_t = (u_{1t}, u_{2t})'$ is a bivariate covariance stationary unobservable process with zero mean and nonparametric spectral density matrix $f(\lambda)$, given by

$$E(u_0 u_j') = \int_{-\pi}^{\pi} e^{ij\lambda} f(\lambda) d\lambda, \tag{1.3}$$

that is at least nonsingular and continuous at all frequencies. For real-valued ν, β, δ satisfying

$$\nu \neq 0, \tag{1.4}$$

$$\delta \geq \beta > 0, \tag{1.5}$$

where at least ν is unknown, a cointegrating relation between the scalar observable sequences x_t, y_t is given by

$$y_t = \nu x_t + \Delta^{\beta-\delta} u_{1t}^{\#}, \tag{1.6}$$

$$x_t = \Delta^{-\delta} u_{2t}^{\#}, \tag{1.7}$$

for $t = 0, \pm 1, ...$

When $\beta = \delta = 1$, this is just the usual bivariate cointegrated $I(1)/I(0)$ system. However to cope with fractional systems when δ falls in the nonstationary region $\delta > 1/2$, the truncations in (1.6), (1.7) ensure that $x_t = y_t = 0$, for all $t \leq 0$. Under (1.6), (1.7), x_t and y_t are said to have integration order δ and are called $I(\delta)$ processes, while the cointegrating error $\Delta^{\beta-\delta} u_{1t}^{\#}$ has integration order $\delta - \beta < \delta$ and is called an $I(\delta - \beta)$ process. This version of fractional integration ("Type II" process) and cointegration, and terminology, accords with that in RH. Alternative ones (based on "Type I" fractional processes), for which the procedures developed below nevertheless apply, are in Dolado and Marmol (1996), Jeganathan (1999), Kim and Phillips (2000). None of these references covers δ within the stationary region, $\delta \in (0, 1/2)$, which is permitted by our (1.5); we call this "stationary cointegration". A larger subset of (1.5) (where $\delta > 1/2$ is possible), consists of $\beta \in (0, 1/2)$, which we call "weak cointegration", because β is the gap between the integration order of observables and cointegrating error. The case $\beta > 1/2$, which includes the usual $I(1)/I(0)$ one, is called "strong cointegration". Stationary cointegration was discussed by Robinson (1994a) and "weak cointegration" by Hualde and Robinson (2001). The main contribution of the present paper is to not only extend the method of estimating ν in RH (under $\beta > 1/2$) to allow a nonparametric f, but to simultaneously cover also $\beta < 1/2$, including $\delta < 1/2$, unlike in any previous paper. Asymptotic theory for point estimation differs significantly across these cases, but we find that the same rules of inference prevail throughout, with a Wald statistic having a null limit χ^2 distribution. However, while the estimates have optimal properties when $\beta > 1/2$, and indicate no loss in the lack of parametric assumptions on f, or of knowledge of β, δ, they are not when $\beta < 1/2$, indeed having slower convergence rate than is optimal here.

We find it convenient to treat our case of nonparametric autocorrelation in the frequency domain. This prompts consideration of two alternative methods of estimating ν. One involves a ratio of weighted periodogram averages either across all frequencies in the Nyquist band, or only over those within a shrinking neighbourhood of zero frequency. The weighting is inverse with respect to smoothed estimates of f. Because of the concentration of spectral mass around zero frequency, where f changes little, computationally simpler estimates, with the same asymptotic properties, replace the weights by multiplicative factors based on an estimate of $f(0)$. Both types of estimate are described in the following section. Regularity conditions and asymptotic properties are presented in Section 3. The conditions include some unprimitive ones on the estimates of β, δ and f, and these are checked in Section 4 for particular estimates; this is an especially delicate issue in our semiparametric setting. Section 5 contains a Monte Carlo study of finite-sample behaviour. All proofs are relegated to Appendices.

2. Estimation of ν

Using again notation from RH, we define for real c, d

$$z_t(c, d) = (y_t(c), x_t(d))', \tag{2.1}$$

where for any sequence $\{w_t\}$, and real c, $w_t(c) = \Delta^c w_t^{\#}$. Thus (1.6), (1.7) can be written as

$$z_t(\gamma, \delta) = \zeta x_t(\gamma)\nu + u_t^{\#}, \tag{2.2}$$

introducing $\zeta = (1, 0)'$ and

$$\gamma = \delta - \beta, \tag{2.3}$$

the integration order of the cointegrating error. Note that we allow γ to lie in the nonstationary region when δ does.

As discussed by RH, the filtering of x_t, y_t in (2.2) provides the orthogonality that justifies a form of generalized least squares estimation. However they treated autocorrelation in u_t parametrically, whereas we require a smoothed nonparametric estimate of $f(\lambda)$. Given an estimate, $\widehat{f}(\lambda)$, define

$$\widehat{p}(\lambda) = \zeta' \widehat{f}(\lambda)^{-1}, \quad \widehat{q}(\lambda) = \zeta' \widehat{f}(\lambda)^{-1} \zeta. \tag{2.4}$$

For generic sequences ξ_t, χ_t, define the discrete Fourier transform, cross-periodogram and periodogram

$$w_\xi(\lambda) = \frac{1}{(2\pi n)^{\frac{1}{2}}} \sum_{t=1}^{n} \xi_t e^{it\lambda}, \;\; I_{\xi\chi}(\lambda) = w_\xi(\lambda) w_\chi(-\lambda)', \;\; I_\xi(\lambda) = I_{\xi\xi}(\lambda). \tag{2.5}$$

Denote by $\lambda_j = 2\pi j/n$, $j = 0, ..., [n/2]$, the Fourier frequencies, where $[\cdot]$ means integer part.

Given observations x_t, y_t, $t = 1, ..., n$, define the statistics

$$\widehat{a}_m(c,d) = \mathrm{Re}\left\{ \sum_{j=0}^{m} s_j \widehat{p}(\lambda_j) I_{z(c,d)x(c)}(\lambda_j) \right\}, \quad \widehat{b}_m(c) = \mathrm{Re}\left\{ \sum_{j=0}^{m} s_j \widehat{q}(\lambda_j) I_{x(c)}(\lambda_j) \right\}, \tag{2.6}$$

$$\widehat{a}_m^{\mathrm{o}}(c,d) = \mathrm{Re}\left\{ \widehat{p}(0) \sum_{j=0}^{m} s_j I_{z(c,d)x(c)}(\lambda_j) \right\}, \quad \widehat{b}_m^{\mathrm{o}}(c) = \widehat{q}(0) \sum_{j=0}^{m} s_j I_{x(c)}(\lambda_j), \tag{2.7}$$

for an integer m such that

$$m \to \infty \text{ as } n \to \infty, \quad 1 \le m \le n/2, \tag{2.8}$$

and $s_j = 1$, $j = 0, n/2$, $s_j = 2$, otherwise. Defining

$$\widehat{\nu}_m(c,d) = \frac{\widehat{a}_m(c,d)}{\widehat{b}_m(c)}, \quad \widehat{\nu}_m^{\mathrm{o}}(c,d) = \frac{\widehat{a}_m^{\mathrm{o}}(c,d)}{\widehat{b}_m^{\mathrm{o}}(c)}, \tag{2.9}$$

we consider the two sets of estimates

$$\text{W (“weighted”)} \; : \; \widehat{\nu}_m(\gamma,\delta), \; \widehat{\nu}_m(\widehat{\gamma},\delta), \; \widehat{\nu}_m(\gamma,\widehat{\delta}), \; \widehat{\nu}_m(\widehat{\gamma},\widehat{\delta}), \tag{2.10}$$

$$\text{Z (“zero-frequency”)} \; : \; \widehat{\nu}_m^{\mathrm{o}}(\gamma,\delta), \; \widehat{\nu}_m^{\mathrm{o}}(\widehat{\gamma},\delta), \; \widehat{\nu}_m^{\mathrm{o}}(\gamma,\widehat{\delta}), \; \widehat{\nu}_m^{\mathrm{o}}(\widehat{\gamma},\widehat{\delta}). \tag{2.11}$$

Both, (2.10) and (2.11) cover cases when both, one or neither of γ, δ is known, the former including the traditional one in which $\gamma = 0$, $\delta = 1$ is known. When $m = [n/2]$, (2.10), (2.11) are semiparametric counterparts of the parametric estimates in (2.18) and (3.13) of RH, because the real operators and s_j can be dropped and summations over $[0, [n/2]]$ replaced by ones over $[1, n]$, due to symmetry properties. As noted there, the computational simplicity of the Z estimates (2.11) over the W estimates (2.10) is not only due to having to estimate f at only frequency zero, but to

$$\widehat{a}_{[n/2]}^{\mathrm{o}}(c,d) = \frac{\widehat{p}(0)}{2\pi} \sum_{t=1}^{n} z_t(c,d) x_t(c), \; \widehat{b}_{[n/2]}^{\mathrm{o}}(c) = \frac{\widehat{q}(0)}{2\pi} \sum_{t=1}^{n} x_t^2(c). \tag{2.12}$$

However, RH found, in their parametric setting with $\beta > 1/2$, that “zero-frequency” estimates only do as well as “weighted” ones when $\beta > 1$; for $\beta = 1$ a “second-order bias” appears and for $1/2 < \beta < 1$

the convergence rate is inferior due to the lack of optimal weighting, and in each case the mixed-normal asymptotics which underlies the desirable limit null χ^2 distribution of Wald test statistics, is lost. Requiring m to satisfy

$$m/n^{\beta} \to 0, \text{ as } n \to \infty \tag{2.13}$$

in (2.11), repairs this defect. On the other hand for $\beta < 1/2$ an alternative condition limiting the increase of m is imposed,

$$m^{1+2\vartheta}/n^{2\vartheta} \to 0, \text{ as } n \to \infty, \text{ some } \vartheta > 0, \tag{2.14}$$

with respect to both W and Z estimates. ϑ relates to the smoothness of f at frequency 0.

Essentially, (2.13) and (2.14) correct for simultaneity bias due to correlation between x_t and u_{1t} in (1.6), as in earlier work of Robinson (1994a), Robinson and Marinucci (2001, 2003) on the simple estimate based on unfiltered data

$$\overline{\nu}_m = \frac{\operatorname{Re}\left\{\sum_{j=0}^{m} I_{xy}(\lambda_j)\right\}}{\sum_{j=0}^{m} I_x(\lambda_j)}. \tag{2.15}$$

For $m = [n/2]$, $\overline{\nu}_m$ is the ordinary least squares (OLS) estimate in the regression of y_t on x_t, and under (2.14) $\overline{\nu}_m$ is a narrow-band least squares (NBLS). Robinson (1994a) found that $m/n \to 0$ is necessary for consistency of (2.15) when $\delta < 1/2$ (see also Christensen and Nielsen, 2001), and Robinson and Marinucci (2001, 2003) found that $m/n \to 0$ reduces the bias of (2.15) when $\delta > 1/2$ but $\delta + \gamma \leq 1$. For similar reasons, (2.13) is needed for our result on the "filtered" estimates (2.11) when $1/2 < \beta \leq 1$, whereas (2.14) is needed for both W and Z estimates when $\beta < 1/2$. Phillips (1991a) considered similar estimates to $\widehat{\nu}_m(0,1)$, $\widehat{\nu}_m^{o}(0,1)$ assuming $\gamma = 0$, $\delta = 1$ is known.

3. Regularity conditions and asymptotic theory

We introduce first a series of regularity conditions. Let I_2 be the 2-rowed identity matrix. For the W estimates (2.10) we introduce

Assumption 1. *The process u_t, $t = 0, \pm 1, \ldots$, has representation*

$$u_t = A(L)\varepsilon_t, \quad A(z) = I_2 + \sum_{j=1}^{\infty} A_j z^j, \tag{3.1}$$

where

$$\det\{A(z)\} \neq 0, \ \ |z| = 1, \tag{3.2}$$

$A(e^{i\lambda})$ is differentiable in $\lambda \in [-\pi, \pi]$ with derivative in $Lip(\eta)$, $\eta > 1/2$, and with $\|\cdot\|$ denoting the Euclidean norm, the ε_t are independent and identically distributed vectors with mean zero, positive definite covariance matrix Ω, and $E\|\varepsilon_t\|^p < \infty$, $p \geq 4$, $p > 2/(2\beta - 1)$.

This is Assumption 1 of RH and is easily satisfied if u_t is a stationary autoregressive-moving average (ARMA) process, imposing a global smoothness condition on $f(\lambda)$ which implies that

$$\sum_{j=1}^{\infty} j\|A_j\| < \infty, \ \sum_{r=-\infty}^{\infty} |r|\,\|\Gamma(r)\| < \infty, \tag{3.3}$$

where $\Gamma(r) = E\left(u_t u_{t-r}'\right)$. It is imposed even under (2.14) because it enables the use of the functional limit theorem of Marinucci and Robinson (2000). However, for the Z estimates (2.11) we can slightly relax it to

Assumption 1°. *Assumption 1 holds with the condition* $\det\{A(1)\} \neq 0$*, replacing (3.2).*

Both sets of estimates use:

Assumption 2. *There exists* $K < \infty$ *such that*

$$|\widehat{\gamma}| + \left|\widehat{\delta}\right| \leq K, \tag{3.4}$$

and $\kappa > 0$ *such that*

$$\widehat{\gamma} = \gamma + O_p\left(n^{-\kappa}\right), \ \widehat{\delta} = \delta + O_p\left(n^{-\kappa}\right), \tag{3.5}$$

where, as $n \to \infty$

$$n^{-\kappa} m^{1-\max\{\min\{\beta,1\},1/2\}} \log m \to 0. \tag{3.6}$$

On $\widehat{f}$ we impose either of the following two assumptions, for the W and Z estimates respectively.

Assumption 3. *Uniformly in* j*, there exist* $\varkappa > 0$*,* $\phi > 0$*, such that*

$$\begin{aligned} \widehat{f}(\lambda_j) - f(\lambda_j) &= O_p\left(n^{-\varkappa}\right), && (3.7) \\ \widehat{f}(\lambda_{j+1}) - f(\lambda_{j+1}) - \left(\widehat{f}(\lambda_j) - f(\lambda_j)\right) &= O_p\left(n^{-\phi}\right), && (3.8) \end{aligned}$$

where, as $n \to \infty$

$$\begin{aligned} n^{-\varkappa} m^{1-\max\{\min\{\beta,1\},1/2\}} &\to 0, && (3.9) \\ n^{-\phi} m^{2-\max\{\min\{\beta,1\},1/2\}} &\to 0. && (3.10) \end{aligned}$$

Assumption 3°. *There exists* $\varkappa > 0$ *such that*

$$\widehat{f}(0) - f(0) = O_p\left(n^{-\varkappa}\right), \tag{3.11}$$

for which (3.9) is satisfied.

Assumptions 2, 3 and 3° are unprimitive, and it is not always straightforward to see how they can be satisfied. The most familiar semiparametric estimates of integration orders and smooth spectral densities have convergence rates no better than $n^{2/5}$, so for example (3.6) and (3.9) cannot hold when $m = [n/2]$ and $\beta \leq 3/5$. To cover all situations some bias-reducing device is required. For smooth spectrum estimation, Parzen (1957) proposed a method corresponding to the use of higher-order kernels in the frequency domain, and recently Robinson and Henry (2003) employed higher-order kernels to improve the convergence rate of semiparametric estimates of stationary integration orders. We thus pursue a higher-order kernel approach to check Assumptions 2, 3 and 3° in Section 4. Alternative approaches due to Moulines and Soulier (1999), Hurvich and Brodsky (2001), Andrews and Sun (2004), could be developed.

Finally Assumptions 4 and 4° below are imposed on the bandwidth m in case of the W and Z estimates respectively.

Assumption 4. *When $\beta < 1/2$, for η in Assumption 1,*

$$m^{\beta-1/2}\log^{1/2} n + n^{3+2\eta}/n^{2+2\eta} \to 0, \text{ as } n \to \infty. \tag{3.12}$$

Assumption 4$^{\mathrm{o}}$. *Assumption 4 holds and, for $\beta > 1/2$ (2.13) holds.*

The first part of (3.12) holds whenever m increases with n at algebraic rate, and the second is equivalent to (2.14) with $\vartheta = 1 + \eta$. The role of (2.13) was discussed in Section 2.

To describe limit distribution theory we introduce the following notation. Denote by $N(0, \theta)$ a normal random variable with mean 0 and variance θ. Denote by $W(r)$ the 2×1 vector Brownian motion with covariance matrix Ω, and define

$$W(r;\beta) = \int_0^r \frac{(r-s)^{\beta-1}}{\Gamma(\beta)} dW(s), \quad \widetilde{W}(r;\beta) = \xi' B(1)^{-1} W(r;\beta), \; \xi = (0,1)'. \tag{3.13}$$

Denote by $f_{ij}(\lambda)$, $f^{ij}(\lambda)$ the (i,j)th components of $f(\lambda)$, $f^{-1}(\lambda)$ respectively. Denoting by $\to_d$ convergence in distribution and by $\Rightarrow$ convergence in the Skorohod J_1 topology of $D[0,1]$, we say that an estimate $\widehat{\nu}$ of ν has Property M if, as $n \to \infty$,

$$m^{\frac{1}{2}} \lambda_m^{-\beta} (\widehat{\nu} - \nu) \to_d N\left(0, \frac{1-2\beta}{2f^{11}(0) f_{22}(0)}\right) \tag{3.14}$$

when $\beta < 1/2$, and

$$n^{\beta}(\widehat{\nu} - \nu) \Rightarrow \left\{ q(0) \int_0^1 \widetilde{W}(r;\beta)^2 dr \right\}^{-1} 2\pi\zeta' A(1)^{-1\prime} \Omega^{-1} \int_0^1 \widetilde{W}(r;\beta) dW(r) \tag{3.15}$$

when $\beta > 1/2$.

The following theorem is proved in Appendix A.

Theorem

(i) If Assumptions 1, 2, 3 and 4 hold, the W estimates (2.10) have Property M;

(ii) If Assumptions 1$^{\mathrm{o}}$, 2, 3$^{\mathrm{o}}$ and 4$^{\mathrm{o}}$ hold, the Z estimates (2.11) have Property M.

Property M is so designated because it indicates Mixed normal asymptotics; the mixed normal nature of the limit in (3.15) was discussed by RH, while it is trivially satisfied in (3.14). Introducing the Wald statistics

$$W = b_m (\nu_m - \nu)^2, \quad W^{\mathrm{o}} = b_m^{\mathrm{o}} (\nu_m^{\mathrm{o}} - \nu)^2, \tag{3.16}$$

where b_m and b_m^{o} denote respectively $b_m(c)$ and $b_m^{\mathrm{o}}(c)$ for $c = \gamma$ or $\widehat{\gamma}$ and ν_m and ν_m^{o} respectively denote any of (2.10) and (2.11). Then we can deduce for both $\beta < 1/2$ and $\beta > 1/2$,

$$W \to_d \chi_1^2, \quad W^{\mathrm{o}} \to_d \chi_1^2. \tag{3.17}$$

For $\beta > 1/2$ this follows from the Theorem as indicated by RH. For $\beta < 1/2$ it follows from the Theorem and the fact that

$$\frac{\lambda_m^{2\beta}}{2m} b_m,\ \frac{\lambda_m^{2\beta}}{2m} b_m^{o} \to_p \ \frac{f^{11}(0) f_{22}(0)}{1-2\beta}, \text{ as } n \to \infty, \tag{3.18}$$

from the proof of the Theorem. Thus, the standard limit theory of Wald tests, familiar in many classical situations in econometrics and associated with optimal procedures in the $I(1)/I(0)$ cointegration literature (see e.g. Johansen, 1991, Phillips, 1991a,b) is shown to hold here simultaneously for weak (including stationary) and strong cointegration, and in the possible presence of unknown integration orders of observables and/or cointegrating errors.

4. Estimation of integration orders and spectral density

This section presents estimates of γ, δ and f for which Assumptions 2, 3 (or alternatively 3^{o}) hold under primitive conditions. A similar objective was achieved by Robinson (2002), who justified the unprimitive conditions required in RH in a fully parametric framework. In our semiparametric situation, bias-reduction techniques seem unavoidable, and in particular, we use higher-order kernels.

We first justify the existence of estimates of γ, δ, satisfying Assumption 2 under primitive conditions, for which we extend a case of the general class of estimates presented in Robinson and Henry (2003), given there for invertible covariance stationary time series, to (possibly unobservable) Type II fractional processes allowing for the possibility of arbitrarily large memory. We focus on estimating γ, which, since u_{1t} is unobservable, is a harder problem than estimating δ. Noting (1.6), for a preliminary estimate of ν, say $\widehat{\nu}$, we define the processes

$$\widetilde{v}_t = y_t - \nu x_t = u_{1t}(-\gamma), \quad \widehat{v}_t = y_t - \widehat{\nu} x_t, \tag{4.1}$$

and v_t, where setting $r = [\gamma + 1/2]$,

$$v_t = \psi_t(-\gamma), \quad \psi_t = \Delta^{r-\gamma} u_{1t} = \sum_{j=0}^{\infty} a_j(\gamma - r) u_{1,t-j}, \tag{4.2}$$

noting that $-1/2 \leq \gamma - r < 1/2$, so that ψ_t is well defined in mean square, v_t and $\widetilde{v}_t$ being Type I and II fractionally integrated processes of order γ respectively (see Marinucci and Robinson, 1999).

The procedure of Robinson and Henry (2003) applies to a generic invertible covariance stationary process, which covers v_t in case $r = 0$. In Proposition 1 below, we show that after tapering similar results to theirs apply for Type I or II processes with arbitrarily large γ. In Proposition 2, we show that for $\beta > 1/2$ the same result as in Proposition 1 holds if we base estimates on $\widehat{v}_t$, whereas for $\beta < 1/2$, we deduce a rate of convergence for the estimate of γ based on $\widehat{v}_t$. The reason for this lack of uniformity is that under weak cointegration $\widehat{\nu}$ converges relatively slowly, severely affecting estimation of γ.

Defining a taper $\{g_t\}_{t=1}^{n}$ of order p as in Velasco (1999a,b), and a sequence ξ_t, the discrete Fourier transform and periodogram of the tapered sequence $g_t \xi_t$ are

$$w_\xi^p(\lambda) = \left(2\pi \sum_{t=1}^{n} g_t^2\right)^{-1/2} \sum_{t=1}^{n} g_t \xi_t e^{it\lambda},\ I_\xi^p(\lambda) = \left|w_\xi^p(\lambda)\right|^2. \tag{4.3}$$

For an integer $q \geq 1$ to be discussed subsequently, introduce a real function $k_q(u)$, $0 \leq u \leq 1$, satisfying

Assumption P1. $k_q(u)$, $0 \le u \le 1$ *is a boundedly differentiable function such that* $\int_0^1 k_q(u)\,du = 1$, *and defining* $U_{iq} = \int_0^1 (\log u + 1)\, u^{2i} k_q(u)\,du$, *we have*

$$U_{iq} = 0,\ 0 \le i \le q-1; \quad U_{qq} \ne 0. \tag{4.4}$$

Robinson and Henry (2003) described $k_q(u)$ as a higher-order kernel and proposed an example. Following Robinson and Henry (2003), for an integer l to be described subsequently such that l/p is integer, for suitable q, $k_q(u)$, we define

$$q_\xi^p(c) = \frac{p}{l} \sum{}' b_{q,j} \left(I_\xi^p(\lambda_j)\, \lambda_j^{2c} - 1 \right), \tag{4.5}$$

where $\sum' = \sum_{j=p,2p,..}^{l}$ and

$$b_{q,j} = k_{q,j} v_{q,j},\ k_{q,j} = k_q(j/l),\ v_{q,j} = \log \lambda_j - \frac{\sum' k_{q,j} \log \lambda_j}{\sum' k_{q,j}}. \tag{4.6}$$

We present now our estimates of γ. Denoting by $\overline{\gamma}_G$, $\widetilde{\gamma}_G$, $\widehat{\gamma}_G$, the tapered local Whittle or Gaussian semiparametric estimates based on processes v_t, $\widetilde{v}_t$, $\widehat{v}_t$, respectively, which optimize over the interval $\Theta = [\nabla_1, \nabla_2]$ the loss function of Velasco (1999a), we define our estimates $\overline{\gamma}$, $\widetilde{\gamma}$, $\widehat{\gamma}$ of γ based on v_t, $\widetilde{v}_t$, $\widehat{v}_t$, as the zeroes of $q_v^p(c)$, $q_{\widetilde{v}}^p(c)$, $q_{\widehat{v}}^p(c)$, which are closest to $\overline{\gamma}_G$, $\widetilde{\gamma}_G$, $\widehat{\gamma}_G$, respectively. Our estimates correspond to the qth-order kernel M-estimate proposed by Robinson and Henry (2003) for the choices $J = 1$, $g(\lambda) = \lambda$, $\psi(z) = \psi_1(z)$, so that they are higher-order kernel versions of the local Whittle estimates of Künsch (1987) and Robinson (1995a), with corresponding loss functions $Q_v^p(c)$, $Q_{\widetilde{v}}^p(c)$, $Q_{\widehat{v}}^p(c)$, where

$$Q_\xi^p(c) = l\left(\log G_\xi^p(c) - 2c \frac{\sum' k_{q,j} \log \lambda_j}{\sum' k_{q,j}} \right), \quad G_\xi^p(c) = \frac{\sum' k_{q,j} \lambda_j^{2c} I_\xi^p(\lambda_j)}{\sum' k_{q,j}}, \tag{4.7}$$

assuming the estimates do not fall on the boundary of Θ. Before presenting our results, we introduce a couple of additional regularity conditions.

Assumption P2. $f_{11}(\lambda)$ *is* s*-times continuously differentiable at* $\lambda = 0$, $s \ge 1$.

Defining $h(\lambda) = \left(2 \sin(\lambda/2)\, \lambda^{-1}\right)^{-2\gamma} f_{11}(\lambda)$, for $s \ge 2$ and setting $q = [s/2]$, Assumption P2 is equivalent to

$$h(\lambda) = f_{11}(0) + \sum_{i=1}^{q} \frac{h_i \lambda^{2i}}{(2i)!} + O\left(\lambda^{2(q+1)}\right) \text{ as } \lambda \to 0, \tag{4.8}$$

where h_i represents the $2i$th derivative of $h(\lambda)$ at $\lambda = 0$. As established in Robinson and Henry (2003), Assumption P2 can be exploited by use of a $k_q(u)$ for suitable q to reduce asymptotic bias when $q \ge 2$ (or equivalently $s \ge 4$). If $q = 1$, we are in the situation covered by Robinson (1995a) and Velasco (1999a), where the maximum rate of convergence is $n^{2/5}$. For $s = 1$, following these references, only the slower rate $n^{1/3}$ is achievable, whereas our Assumption 1 permits the rate $n^{(1+\eta)/(3+2\eta)}$.

Assumption P3. *For any* $\epsilon > 0$, *as* $n \to \infty$,

$$l \to \infty, \quad l = O\left(n^{4q/(4q+1)}\right), \quad l^\epsilon / \log n \to \infty. \tag{4.9}$$

The second condition in (4.9) (taken from Robinson and Henry, 2003) imposes the maximum rate at which the bandwidth l can grow, while the third is innocuous if l is a power root of n.

Defining

$$\Phi = \lim_{n\to\infty}\left(\sum\nolimits_{t=1}^{n} h_t^2\right)^{-2}\sum\nolimits_{k=0,p,2p,..}^{n-p}\left(\sum\nolimits_{t=1}^{n} h_t^2\cos\left(t\lambda_k\right)\right)^2, \tag{4.10}$$

$$V_q = \int_0^1 \left(\log u+1\right)^2 k_q\left(u\right)du, \quad W_q = \int_0^1 \left(\log u+1\right)^2 k_q^2\left(u\right)du, \tag{4.11}$$

and denoting by v_t^* either processes v_t or $\widetilde{v}_t$, and by γ^* either $\overline{\gamma}$ or $\widetilde{\gamma}$, we establish the following results.

Proposition 1. *Under Assumptions 1, P1-P3, $\gamma\in(\nabla_1,\nabla_2)$, $\nabla_1>-1/2$, $p\geq\max\{r,[\nabla_2+1/2]+1\}$, γ^* satisfies $q_v^p(\gamma^*)=0$ with probability approaching one as $n\to\infty$, and furthermore*

$$l^{1/2}\left(\gamma^*-\gamma\right)+\frac{(2\pi)^q U_{qq}h_q}{2\,(2q)!f_{11}\left(0\right)V_q}\frac{l^{2q+1/2}}{n^{2q}}\to_d N\left(0,\frac{p\Phi W_q}{4V_q^2}\right). \tag{4.12}$$

Proposition 2. *Under Assumptions 1, P1-P3, $\gamma\in(\nabla_1,\nabla_2)$, $\nabla_1>-1/2$,*

$$p\geq\max\left\{r,[\nabla_2+1/2]+1,[\delta+1/2]+1\right\}, \tag{4.13}$$

and

$$\widehat{\nu}-\nu=O_p\left(n^{\gamma-\delta+\psi}\right), \tag{4.14}$$

for any $\psi>0$, $\widehat{\gamma}$ satisfies $q_{\widehat{v}}^p(\widehat{\gamma})=0$ with probability approaching one as $n\to\infty$. Furthermore, if $\beta>1/2$

$$l^{1/2}\left(\widehat{\gamma}-\gamma\right)+\frac{(2\pi)^q U_{qq}h_q}{2\,(2q)!f_{11}\left(0\right)V_q}\frac{l^{2q+1/2}}{n^{2q}}\to_d N\left(0,\frac{p\Phi W_q}{4V_q^2}\right), \tag{4.15}$$

whereas if $\beta<1/2$, for any $\varphi>0$

$$\widehat{\gamma}-\gamma=O_p\left(l^{-(\beta-\varphi)}\right). \tag{4.16}$$

Propositions 1 and 2 are justified in Appendix B. For our purposes, the main implication (when $\beta>1/2$) is that on choosing $l\sim n^{4q/(4q+1)}$ (where "$\sim$" denotes here exact rate) and $s\geq 2$, the convergence rate of our estimates is $n^{2q/(4q+1)}$, which can be made arbitrarily close to the parametric rate $n^{1/2}$ for q (and thus s) large enough. The same rate clearly applies to the corresponding estimate of δ under equivalent conditions. Note also that for the suggested choice of l the bias term in (4.12) has exact rate $O(1)$, while (4.9) prevents this from dominating. Related to the $\beta<1/2$ situation, for the previous choice of l, the rate of convergence of the feasible estimate of γ is $n^{4q(\beta-\varphi)/(4q+1)}$. Though (4.14) is strong, it is satisfied by OLS (cf. (2.15)) for $\beta\geq 1$. OLS does not satisfy (4.14) for every (γ,δ) combination if $\beta<1$, but the NBLS (see (2.15) again) suffices. As in Assumption 2, we denote the rate of convergence of these estimates of γ (and also the corresponding ones of δ) for the proposed choice of l by n^κ with $\kappa=\kappa(s)$ and

$$\kappa\left(s\right) = 2q/\left(1+4q\right), \text{ for estimates of } \delta \text{ and } \gamma \text{ (when } \beta>1/2\text{)}, \tag{4.17}$$

$$= 4q\left(\beta-\varphi\right)/\left(1+4q\right), \text{ for estimates of } \gamma \text{ when } \beta<1/2. \tag{4.18}$$

Thus, for large enough s, $\kappa(s)$ can be arbitrarily close to 1/2 and β when $\beta > 1/2$ and $\beta < 1/2$ respectively. This, in view of (3.6), implies that in strong cointegration the choice $m \sim n$ is feasible, but if β is close to (but above) 1/2, the existence of a large number of derivatives and use of an appropriate higher order kernel $k_q(u)$ are necessary. A drawback of our approach is that as β is unknown, so one can never be sure that the q employed is adequate, even if $f(\lambda)$ is analytic. This problem is similar to the choice of taper order, p. Under weak cointegration, (3.6) restricts further the growth of m, which is already constrained in (3.12), as the inequality $\beta > (1+\eta)/(3+2\eta)$, $\eta \in (1/2, 1)$, does not hold for every $\beta \in (0, 1/2)$. In any case, in view of (3.5), (3.6), a more slowly converging estimate of γ (based for example on OLS residuals) would further restrict m, affecting the rate of convergence of estimates of ν.

Next we propose a nonparametric estimate of $f(\lambda)$ based on the residual vector $\widehat{u}_t = (y_t(\widehat{\gamma}) - \widehat{\nu} x_t(\widehat{\gamma}), x_t(\widehat{\delta}))'$, for an estimate $\widehat{\nu}$ of ν, satisfying (4.14) and using our estimates $\widehat{\gamma}$, $\widehat{\delta}$, of the orders. We again need to exploit smoothness, and introduce the following assumption, which strengthens Assumption P2.

Assumption P4. *$A(e^{i\lambda})$ is s times differentiable in $\lambda \in [-\pi, \pi]$ with sth derivative in Lip(η), $\eta > 1/2$, $s \geq 1$.*

This strengthens Assumption P2 and ensures certain rates of convergence for our estimates of the spectral density at all Fourier frequencies.

We introduce the weighted periodogram estimate of f,

$$\widehat{f}(\lambda) = \frac{2\pi b}{n} \sum_{j=-\infty}^{\infty} \overline{G}(b(\lambda - \lambda_j)) I_{\widehat{u}}(\lambda_j) = \frac{2\pi}{n} \sum_{j=1}^{n} \overline{G}_b(\lambda - \lambda_j) I_{\widehat{u}}(\lambda_j), \tag{4.19}$$

where

$$\overline{G}_b(\lambda) = b \sum_{j=-\infty}^{\infty} \overline{G}(b(\lambda + 2\pi j)), \tag{4.20}$$

for a user-chosen integrable function $\overline{G}$ and bandwidth sequence $b = b_n$.

Define

$$g(x) = \int_{\Re} \overline{G}(\lambda) e^{ix\lambda} d\lambda, \ x \in \Re, \tag{4.21}$$

and introduce

Assumption P5. *b is a sequence of positive real numbers such that $b^{-1} + b/n \to 0$ as $n \to \infty$; $\overline{G}(\lambda)$ is a real, even function such that*

$$\int_{\Re} \left|\overline{G}(\lambda)\right| d\lambda < \infty, \quad \int_{\Re} \overline{G}(\lambda) d\lambda = 1, \quad \int_0^{\infty} \left(1 + x^2\right) |g(x)| dx < \infty, \tag{4.22}$$

and in a neighbourhood of the origin of radius $\epsilon > 0$,

$$|1 - g(x)| < K |x|^h \text{ for some } h \geq s. \tag{4.23}$$

Because $\overline{G}(\lambda)$ is even, so is g, and $\sup_x |g(x)| \leq \int_{\Re} |G(\lambda)| \, d\lambda < \infty$, which implies along with (4.22) that $g(x)$ is square integrable. (4.23) implies that $g(x)$ is locally (in a neighbourhood of 0) $Lip(h)$. If $h > 1$, this implies that $d^c g(x) / dx^c = 0$ for any $c < h$, so bias reduction is possible provided $f(\lambda)$ is smooth enough. Indeed in view of (4.21), $\int_{-\pi}^{\pi} \mu^c \overline{G}_b(\mu) \, d\mu = 0$, so that (4.23), introduced by Parzen (1957), corresponds when $h > 2$ to a higher-order kernel property of $\overline{G}_b$. The larger h is chosen, the faster the rate of convergence of our estimates will be. As Robinson (1991) mentions, condition (4.23) holds for $h = 1, 2$ for many of the usual kernels, but in case the h required is very large, a careful choice of the covariance averaging kernel g is required.

Denoting by Assumption P6 the set of all conditions needed in order to obtain rates (4.17), (4.18) for our estimates of γ, δ, we show in two propositions the results for $\widehat{f}(\lambda)$.

Proposition 3. *Under Assumptions P4-P6, uniformly in j,*

$$\widehat{f}(\lambda_j) - f(\lambda_j) = O_p\left(b^{-s} + \left(\frac{b}{n}\right)^{\frac{1}{2}} + bn^{-\kappa(s)}\right) = O_p\left(n^{-\chi}\right), \tag{4.24}$$

where

$$\chi = \chi(s) = s\kappa(s) / (1 + s), \tag{4.25}$$

if $b = b^* \sim n^{\kappa(s)/(1+s)}$.

The proof of Proposition 3 is given in Appendix B, where b^* is referred to as the "optimal" choice. For s large enough, as for the estimates of the orders, arbitrarily close rates to $n^{1/2}$ and n^{β} for the estimates of the spectral density are achievable under strong and weak cointegration respectively.

Proposition 4. *Under Assumptions P4-P6, uniformly in j,*

$$\widehat{f}(\lambda_{j+1}) - f(\lambda_{j+1}) - \left(\widehat{f}(\lambda_j) - f(\lambda_j)\right) = O_p\left(n^{-1}b^2 n^{-\kappa(s)}\right) + o_p\left(n^{-1}b^{1-s}\right) \tag{4.26}$$

$$= O_p\left(n^{-\phi}\right), \tag{4.27}$$

where

$$\phi = \phi(s) = 1 + \frac{\kappa(s)(s-1)}{1+s}, \tag{4.28}$$

if $b = b^* \sim n^{\kappa(s)/(1+s)}$.

When $\beta > 1/2$, the left side of (4.26) is of order arbitrarily close to $n^{-3/2}$ for s large enough, which in view of (3.10) enables the choice $m \sim n$.

Finally, it is important to note that (3.6), (3.9), (3.10) reflect the trade-off between smoothness of $f(\lambda)$ and rate of growth of m: a higher s implies higher $\kappa(s)$, $\chi(s)$ and $\phi(s)$, so that m is allowed to increase faster. In all cases, (3.6), (3.9) hold for $\beta \geq 1$ for any s, m. For $\beta < 1$, arbitrarily small $\kappa(s)$, $\chi(s)$ also suffice, but the growth of m has to be heavily restricted.

5. Monte Carlo evidence

A Monte Carlo study of finite sample behaviour was carried out, comparing some of our estimates with the simple one $\overline{\nu}_m$, given in (2.15), in terms of bias and dispersion, and also examining the goodness of the χ_1^2 approximation for Wald test statistics. We take $A(z) = I_2 \{(1+\psi z)/(1-\phi z)\}$ in cases where u_t is: white noise (WN), with $\phi = \psi = 0$; AR(1), with $\phi = 0.5, 0.9$, $\psi = 0$; MA(1), with $\psi = 0.5, 0.9$, $\phi = 0$. We generated Gaussian ε_t with covariance matrix Ω having ijth element ω_{ij}, the correlation $\rho = \omega_{12}/(\omega_{11}\,\omega_{22})^{1/2}$ taking values 0, 0.5, -0.5, 0.75 and fixing $\nu = \omega_{11} = \omega_{22} = 1$. For $\beta > 1/2$, we consider the combinations $(\gamma, \delta) = (0, 0.6)$, $(0, 1.2)$, $(0.4, 1.2)$, $(0.4, 2)$. For $\beta < 1/2$, we consider $(\gamma, \delta) = (0, 0.4)$, $(0.2, 0.4)$, $(0.4, 0.8)$, $(0.7, 1)$.

Table 1 presents convergence rates of our W, Z estimates and, for both $\rho \neq 0$ and $\rho = 0$, of $\overline{\nu}_m$, denoted a U (Unfiltered) estimate. These rates are derived from our Theorem and Robinson and Marinucci (2001, 2003). For strong cointegration, the U estimate rates apply for any $m \leq [n/2]$, $m \to \infty$, and the rates of W, Z are optimal in this case. For weak cointegration, we only consider the NBLS version of $\overline{\nu}_m$ with $m/n \to 0$ as $n \to \infty$, noting that (3.12) needs to be satisfied.

TABLE 1
CONVERGENCE RATES: U WITH $\rho \neq 0$, $\rho = 0$ AND W, Z

(γ, δ)	$(0, .6)$	$(0, 1.2)$	$(.4, 1.2)$	$(.4, 2)$	$(0, .4)$	$(.2, .4)$	$(.4, .8)$	$(.7, 1)$
U, $\rho \neq 0$	$n^{.6}m^{-.4}$	$n^{1.2}$	$n^{.8}$	$n^{1.6}$	$n^{.4}m^{-.4}$	$n^{.2}m^{-.2}$	$n^{.4}$	$n^{.3}$
U, $\rho = 0$	$n^{.6}$	$n^{1.2}$	$n^{.8}$	$n^{1.6}$	$n^{.4}m^{.1}$	$n^{.2}m^{.3}$	$n^{.4}$	$n^{.3}$
W, Z	$n^{.6}$	$n^{1.2}$	$n^{.8}$	$n^{1.6}$	$n^{.4}m^{.1}$	$n^{.2}m^{.3}$	$n^{.4}m^{.1}$	$n^{.3}m^{.2}$

We generated 1000 series of lengths $n = 64, 128, 256$, and choosing different bandwidths b (taking values 15, 25, 45, depending on whether n is 64, 128, 256 respectively), computed the unweighted version of (4.19)

$$\widehat{f}(\lambda_j) = \frac{1}{2b+1} \sum_{k=j-b}^{j+b} I_{\widetilde{u}}(\lambda_k), \quad \widetilde{u}_t(c, d, a) = (y_t(c) - a x_t(c), x_t(d))', \tag{5.1}$$

where in all cases $a = \overline{\nu}_b$ and $(c, d) = (\gamma, \delta)$ or $(\widehat{\gamma}, \widehat{\delta})$. The estimates $\widehat{\gamma}$, $\widehat{\delta}$, are Robinson's (1995b) version of the log-periodogram estimates of Geweke and Porter-Hudak (1983) without trimming or pooling applied to the untapered series $y_t - \overline{\nu}_b x_t$ and $\overline{x}_t$, where $\overline{x}_t = x_t$ for $\delta < 1$, $\overline{x}_t = \Delta x_t$ for $\delta \geq 1$, adding back one to the estimate of the order of $\overline{x}_t$ in this case to compute the final estimate of δ. b is also the bandwidth for the semiparametric estimates of γ and δ. Our estimates of f, γ and δ do not make the provision for rate improvement of Section 4, partly because practitioners are likely to use standard estimates in view of readily available software. However, (3.6) is not satisfied for $m = [n/2]$ when $\beta \leq 0.6$ and the Monte Carlo will illustrate the effect.

For $\beta > 1/2$, we computed W and Z Infeasible estimates $\overline{\nu}_I = \widehat{\nu}_m(\gamma, \delta)$, $\overline{\nu}_I^{o} = \widehat{\nu}_m^{o}(\gamma, \delta)$, Feasible estimates $\overline{\nu}_F = \widehat{\nu}_m(\widehat{\gamma}, \widehat{\delta})$, $\overline{\nu}_F^{o} = \widehat{\nu}_m^{o}(\widehat{\gamma}, \widehat{\delta})$ and the U estimates, $\overline{\nu}_U = \overline{\nu}_m$, for three different sets of bandwidths m, given by $(I, II, III) = (10, 20, 32), (20, 40, 64), (40, 80, 128)$, depending on whether $n = 64, 128, 256$ respectively. The largest bandwidth ($m = III$) for each n corresponds to the full band case. For weak cointegration, we only present results in the simplest case $\phi = \psi = 0$, for $\overline{\nu}_I$, $\overline{\nu}_F$, $\overline{\nu}_U$, with $m = (I, II, III) = (2, 8, 15), (2, 12, 20), (3, 15, 25)$ and $n = 64, 128, 256$ respectively, (I, II, III), representing in all cases narrow band situations; instead of Z estimates, we computed infeasible and feasible two-step estimates, given by $\overline{\nu}_{2I}$ and $\overline{\nu}_{2F}$ respectively, where these used (5.1) with $\widetilde{u}_t(\gamma, \delta, \overline{\nu}_I)$ and $\widetilde{u}_t(\widehat{\gamma}_2, \widehat{\delta}, \overline{\nu}_F)$ respectively, $\widehat{\gamma}_2$ being the estimate of γ calculated from residuals $y_t - \overline{\nu}_F x_t$.

While our conclusions and comments refer to the whole experiment, to conserve on space we only report a few representative tables. For the case $\beta > 1/2$, we only present results for $(\rho, \phi, \psi) = (0, 0, 0)$, $(0.5, 0.5, 0)$, $(-0.5, 0, 0.5)$, $m = I, III$, whereas for $\beta < 1/2$, we only report results for $\rho = 0.5$.

We deal first with strong cointegration. Table 2 reports comparative figures corresponding to Monte Carlo bias (defined as the estimate minus ν) and standard deviation (S.D.). The first five rows list the number of times that the corresponding estimate is no worse than any of the others. The last four rows compare *vis a vis* different estimates and reflect the number of times that the corresponding estimate in the relation is strictly better than the other. All figures relate to the total number (180) of cases (for all n, m, ρ and (γ, δ) combinations).

TABLE 2
COMPARISONS BETWEEN ESTIMATES

	BIAS					S.D.				
	WN	AR(.5)	AR(.9)	MA(.5)	MA(.9)	WN	AR(.5)	AR(.9)	MA(.5)	MA(.9)
$\overline{\nu}_I$	144	126	136	130	128	145	148	147	127	124
$\overline{\nu}_I^o$	167	167	146	176	176	144	165	139	164	165
$\overline{\nu}_F$	104	90	90	101	103	58	46	46	57	55
$\overline{\nu}_F^o$	99	87	81	94	89	56	45	36	51	49
$\overline{\nu}_U$	81	89	76	84	83	82	83	92	76	86
$\overline{\nu}_F : \overline{\nu}_U$	89:7	83:15	88:30	87:10	88:10	57:56	29:77	30:92	53:61	53:61
$\overline{\nu}_F^o : \overline{\nu}_U$	91:13	79:18	73:44	84:14	84:15	32:75	14:97	9:120	36:76	35:79
$\overline{\nu}_I : \overline{\nu}_I^o$	10:33	1:42	4:13	1:47	0:48	29:27	6:26	16:2	6:43	6:44
$\overline{\nu}_F : \overline{\nu}_F^o$	53:12	30:28	58:4	22:31	18:32	68:15	85:2	100:1	78:7	80:10

Detailed results for bias are presented in Tables 3, 4. Overall, infeasible (I) estimates dominate, with feasible (F) ones clearly superior to $\overline{\nu}_U$. For the WN situation, this difference is most noticeable when $\beta < 1$. Especially for $\rho \neq 0$, $\overline{\nu}_I^o$ is slightly superior to $\overline{\nu}_I$ with the exception of the full band situation with $\beta = 0.6$, where as the theory predicts $\overline{\nu}_I$ beats $\overline{\nu}_I^o$; on the contrary, $\overline{\nu}_F$ outperforms $\overline{\nu}_F^o$, differences being most noticeable for the full band situation. The only estimates which enjoy large improvements in the AR framework are $\overline{\nu}_U$ and the I estimates, especially for $\beta = 0.6$, this effect being stronger the larger the AR parameter ϕ. Under the AR structure with $\phi = 0.5$, the predominance of the feasible estimates over $\overline{\nu}_U$ is still clear, and more noticeable as β decreases, with $\overline{\nu}_U$ competitive only when n is small and β large. $\overline{\nu}_I^o$ outperforms $\overline{\nu}_I$, although in general the differences are very small, whereas both F estimates behave rather similarly. When $\phi = 0.9$, F estimates are still preferred to $\overline{\nu}_U$, although this is competitive for β large. $\overline{\nu}_I^o$'s superiority over $\overline{\nu}_I$ is less clear now, whereas that of $\overline{\nu}_F$ over $\overline{\nu}_F^o$ is accentuated. In the MA situation, results are mainly unaffected by the value of ψ, and apart from a relative improvement of the Z estimates, results are very similar to those of the WN situation. In general, bias decreases as n, β, increase, and m (with the exception of the case $\rho = 0$) and $|\rho|$ decrease. The bias is signed by ρ, except in case of F estimates when $\beta > 1$, where it is reversed.

Results for S.D. are presented in Tables 5, 6. Overall, I estimates dominate, but now $\overline{\nu}_U$ emerges as competitive relative to $\overline{\nu}_F$ and $\overline{\nu}_F^o$, especially in the AR situation with $\phi = 0.9$, although increases in n generally favour F estimates, whereas large β is more favourable to $\overline{\nu}_U$. In the WN situation, both I estimates behave similarly, but less so in the AR case. Here, when $\phi = 0.5$, $\overline{\nu}_I^o$ beats $\overline{\nu}_I$, the opposite happening when $\phi = 0.9$ (the superiority of $\overline{\nu}_I$ based here mainly on the full band case). In the MA framework, although in general values are close to those of the WN situation, $\overline{\nu}_I^o$ outperforms $\overline{\nu}_I$. For this case, S.D. is quite unaffected by ψ. In general, $\overline{\nu}_F$ beats $\overline{\nu}_F^o$, which deteriorates more when $\beta < 1$. As expected, S.D. decreases when n, β increase, but in general is not much affected by variations in bandwidth.

We next studied the Wald statistics

$$W_I = \widehat{b}_m(\gamma)(\overline{\nu}_I - 1)^2, W_F = \widehat{b}_m(\widehat{\gamma})(\overline{\nu}_F - 1)^2, W_I^{\mathrm{o}} = \widehat{b}_m^{\mathrm{o}}(\gamma)(\overline{\nu}_I^{\mathrm{o}} - 1)^2, W_F^{\mathrm{o}} = \widehat{b}_m^{\mathrm{o}}(\widehat{\gamma})(\overline{\nu}_F^{\mathrm{o}} - 1)^2. \quad (5.2)$$

Tables 7-10 contain empirical sizes corresponding to nominal $\alpha = 0.05, 0.10$. When u_t is WN, results corresponding to the infeasible Wald statistic W_I are on average too large, but certainly close to the nominal sizes, even when $n = 64$, for all values of ρ and m when $\beta > 1$, empirical sizes reacting as theory predicts when n increases. For $\beta = 0.8$, empirical sizes of W_I are worse than in the previous situation when $n = 64$, but react quickly in the appropriate direction, when they are comparable to those for larger β. For this case, sizes are not much affected by changes in ρ, but the combination of simultaneous increase in $|\rho|$ and m leads to deterioration. This is much more evident when $\beta = 0.6$, where in general empirical sizes are substantially larger than for all the previous β cases. Empirical sizes for W_F are substantially larger than for W_I, but in almost all cases react appropriately when n increases, the worst case being $\beta = 0.6$ with $\rho = 0.75$, where for $m = III$ sizes are unacceptably large. The results are better for $\alpha = 0.10$ than for $\alpha = 0.05$. When $\beta > 1$, empirical sizes of W_I^{o} and W_F^{o} are very similar to those of W_I, W_F, for all ρ, m, n and β. For $\beta < 1$, both W_I^{o} and, especially, W_F^{o} are worse than W_I and W_F respectively, and more so as m increases, as is predicted by the theory. AR autocorrelation, especially $\phi = 0.9$, severely damages W_I. When $\beta > 1$, W_I is relatively unaffected by ρ, m and β, decreasing in all cases when n increases, quite slowly for $\phi = 0.9$ though. The behaviour of W_F is striking. For $\phi = 0.5$ and $\rho \leq 0.5$, empirical sizes are substantially smaller than those of W_I, especially when β is large. Again, when $\beta > 1$, sizes are relatively unaffected by m, with small increments as $|\rho|$ increases (especially for $\beta = 1.2$), and always decrease as n increases, with empirical sizes very often smaller than nominal ones when $n = 256$. In fact, when $\phi = 0.9$, empirical sizes when $\beta > 1$ behave qualitatively in a similar way to the $\phi = 0.5$ case, but are significantly reduced, so that when $n = 256$ they are much smaller than nominal ones. Their behaviour when $\beta < 1$ is interesting. When $\phi = 0.5$ and $\rho = 0$, they are substantially smaller than those of W_I, being very close to nominal sizes when $n = 256$. As $|\rho|$ increases, this pattern is less clear, and while when $|\rho| = 0.5$ sizes are still better for W_F (only slightly when $\beta = 0.6$ though), they are clearly worse for $\rho = 0.75$, with significant deterioration as $|\rho|$ increases, the effect being more evident as m increases, especially for $\beta = 0.6$. This is also observed when $\phi = 0.9$, but here, even for the most adverse situation where $\beta = 0.6$ and $\rho = 0.75$, empirical sizes of W_F are better than those of W_I for any m, as now sizes corresponding to W_F decrease when ϕ increases. Generally, W_I^{o}, W_F^{o} perform very similarly but slightly better than W_I, W_F, except when $\beta = 0.8$ or $\beta = 0.6$ and $\phi = 0.9$, for which W_F tends to behave better than W_F^{o}. In the MA framework, W_I behaves similarly to the WN situation (with sizes slightly larger), and is quite unaffected by the value of ψ. Sizes of W_F, although still worse than those of W_I, are closer now than for $\phi = \psi = 0$. Again, the effect of increasing the MA parameter does not have any important effect. Also, W_I^{o} and W_F^{o} perform relatively better than W_I and W_F respectively, the clearest improvement appearing when $\beta = 0.6$.

We consider now the weak cointegration case. Results for the bias are presented in Table 11. The overall ranking is $\overline{\nu}_{2I}$, $\overline{\nu}_I$, $\overline{\nu}_U$, $\overline{\nu}_{2F}$, $\overline{\nu}_F$, which are no worse than any of the other estimates in 134, 10, 9, 8 and 3 out of 144 cases respectively. This indicates an overwhelming dominance of the two-step infeasible estimate. Bias differs substantially depending on whether $\rho = 0$ or $\rho \neq 0$. In the former case, although $\overline{\nu}_{2I}$ is clearly best, for example dominating $\overline{\nu}_I$ in ratio 22:4 out of 36 cases, the same does not happen for the F two-step estimate which is inferior to $\overline{\nu}_I$ and $\overline{\nu}_F$ in ratios 21:10 and 13:11 respectively, smaller bandwidths clearly benefitting one-step estimates. $\overline{\nu}_F$ and $\overline{\nu}_{2F}$ perform better than $\overline{\nu}_U$, in ratios 18:12 and 16:13 respectively, the U estimate being superior only when m and n are small. As theory predicts, biases decrease in absolute value when β and n increase, and, unexpectedly, tend to decrease as m increases. This picture changes dramatically when $\rho \neq 0$. Here, in all cases, biases share the sign of ρ, increase in absolute value when m and $|\rho|$ increase, and show the same pattern as when $\rho = 0$ with respect to β and n. There are two important features to note when $\rho \neq 0$. First, both F estimates are better than the U estimate in all cases. Second, $\overline{\nu}_{2F}$, whose corresponding biases are in almost all cases slightly bigger in absolute value than those of $\overline{\nu}_{2I}$, performs much better not only than $\overline{\nu}_F$, but also, and more importantly, than $\overline{\nu}_I$. The ratio with respect to the one-step infeasible estimate is 91:16 out of 108 cases in favour of $\overline{\nu}_{2F}$. This is an encouraging result, demonstrating significant bias reduction by iteration. Possibly there would be further benefit in continuing the iterations.

Results for S.D. are presented in Table 12. Over the four values of ρ, $\overline{\nu}_U$ is clearly superior, completely predominating for the two cases where $\gamma + \delta < 1$, i.e. $(\gamma, \delta) = (0, 0.4)$, $(0.2, 0.4)$ for all ρ, m and n. This fact is reflected in the overall ranking, which is $\overline{\nu}_U$, $\overline{\nu}_I$, $\overline{\nu}_{2I}$, $\overline{\nu}_F$, $\overline{\nu}_{2F}$, which are no worse than any of the rest in 98, 23, 22, 4 and 0 out of 144 cases respectively. For all estimates, S.D. decreases as β, n, ρ and m increase. $\overline{\nu}_U$ is least affected (although still noticeably) by increments in m, so the gap between this estimate and the rest tends to shrink as m increases. $\overline{\nu}_U$ beats $\overline{\nu}_F$ in ratio 108:34, $\overline{\nu}_F$ predominating only when (γ, δ)=$(0.4, 0.8)$ for the largest m, and $(\gamma, \delta) = (0.7, 1)$ for the two largest m. Similarly, $\overline{\nu}_U$ beats $\overline{\nu}_{2F}$ in ratio 124:20. Also, $\overline{\nu}_{2F}$ is superior to $\overline{\nu}_U$ when $(\gamma, \delta) = (0.7, 1)$ for the two largest m. Contrary to the experience with bias, two-step estimates were clearly worse than one-steps. $\overline{\nu}_I$ dominates $\overline{\nu}_{2I}$ in ratio 122:22, $\overline{\nu}_{2I}$ only being superior to $\overline{\nu}_I$ (with small differences) when $(\gamma, \delta) = (0.7, 1)$ for the two largest m. Even more striking is the difference between F estimates, since $\overline{\nu}_F$ outperforms $\overline{\nu}_{2F}$ in ratio 137:6, $\overline{\nu}_{2F}$ being only superior for some cases of $(\gamma, \delta) = (0.7, 1)$ for the largest m.

We next consider the Wald statistics W_I, W_F, $W_{2I} = \overline{b}_{2I} \left(\overline{\nu}_{2I} - 1\right)^2$, $W_{2F} = \overline{b}_{2F} \left(\overline{\nu}_{2F} - 1\right)^2$, where $\overline{b}_{2I}$ and $\overline{b}_{2F}$ differ from their respective one-step counterparts in the same way as $\overline{\nu}_{2I}$ and $\overline{\nu}_{2F}$ differ from $\overline{\nu}_I$ and $\overline{\nu}_F$. Empirical sizes are given in Tables 13, 14. In all cases sizes are too large, mostly being very far from nominal ones. In some cases there is convergence as n increases, although this is usually very slow. As expected, sizes increase as β decreases. Overall, the results are not encouraging. When $\rho = 0$, empirical sizes corresponding to W_I are too large, if acceptable. For the smallest m, they fall as n increases, though this is less clear for the other two, except for $(\gamma, \delta) = (0.4, 0.8)$. For $(\gamma, \delta) = (0.2, 0.4)$ sizes tend to be smaller as m increases, the opposite clearly happening with $(\gamma, \delta) = (0.7, 1)$, and in a less evident way with $(\gamma, \delta) = (0.4, 0.8)$. Sizes corresponding to the two-step I estimate for this $\rho = 0$ situation are clearly larger than those of W_I, with the exception of some cases for $(\gamma, \delta) = (0.7, 1)$ for the two largest m. These sizes behave in a qualitatively similar way to those of W_I, with significant deterioration as n increases for $(\gamma, \delta) = (0.2, 0.4)$ associated with the largest m. As $|\rho|$ increases, sizes are further affected, especially for $(\gamma, \delta) = (0.2, 0.4)$, $(0.7, 1)$. Also, there is now a substantial deterioration as m increases for all β, without improvement for $(\gamma, \delta) = (0.2, 0.4)$ for the two largest m as n increases. For the smallest m and $|\rho| = 0.5$, sizes of W_{2I} are still larger than those of W_I, but although they also deteriorate as m increases, W_{2I} is less damaged than W_I. Also, W_{2I} deteriorates less than W_I as ρ increases, so that when $\rho = 0.75$, in almost all cases, W_{2I} is better than W_I (especially for $(\gamma, \delta) = (0, 0.4)$). This relative performance is also evident for $|\rho| = 0.5$, but only for the two largest m. When $\rho \neq 0$, W_{2I} is also better than W_I when n increases. Sizes corresponding to W_F and W_{2F} follow in general the same pattern as their I counterparts, but are in almost all cases larger, the gap increasing as $|\rho|$ increases.

A. Appendix A: Proof of Theorem

Denote $l_j = l(\lambda_j)$ for any function $l(\lambda)$, and let K be an arbitrary positive finite constant. We first give the proof for the infeasible estimate $\overline{\nu}_m(\gamma,\delta) = a_m(\gamma,\delta)/b_m(\delta)$ when $\beta > 1/2$, where

$$a_m(\gamma,\delta) = \mathrm{Re}\left\{\sum\nolimits_{j=0}^{m} s_j p_j I_{z(\gamma,\delta)x(\gamma)}(\lambda_j)\right\}, \quad b_m(\gamma) = \mathrm{Re}\left\{\sum\nolimits_{j=0}^{m} s_j q_j I_{x(\gamma)}(\lambda_j)\right\}, \tag{A.1}$$

writing $p_j = \zeta' f_j^{-1}$, $q_j = \zeta' f_j^{-1}\zeta$. Clearly

$$\overline{\nu}_m(\gamma,\delta) - \nu = \frac{e_m(\gamma)}{b_m(\gamma)}, \quad e_m(\gamma) = \mathrm{Re}\left\{\sum_{j=0}^{m} s_j p_j I_{ux(\gamma)}(\lambda_j)\right\}. \tag{A.2}$$

First, we show that

$$E(e_m(\gamma)) = o(n^{\beta}). \tag{A.3}$$

We can write the left side of (A.3) as the real part of

$$\frac{1}{2\pi n}\sum_{j=0}^{m} s_j p_j \int_{-\pi}^{\pi} D_n(\lambda_j - \mu)\sum_{t=1}^{n} a_{n-t}e^{-i(n-t)\lambda_j} D_t(\mu - \lambda_j) f(\mu)\,\xi d\mu, \tag{A.4}$$

where $a_t = a_t(\beta)$ and $D_t(\lambda) = \sum_{k=1}^{t} e^{ik\lambda}$ is the Dirichlet kernel, where for $0 < \lambda < \pi$,

$$|D_t(\lambda)| < K \min\left\{|\lambda|^{-1}, t\right\}. \tag{A.5}$$

Noting that for any λ,

$$p(\lambda) f(\lambda)\xi = 0, \tag{A.6}$$

by periodicity, we can write (A.4) as

$$\frac{1}{2\pi n}\sum_{j=0}^{m} s_j p_j \int_{-\pi}^{\pi} D_n(-\mu)\sum_{t=0}^{n-1} a_t e^{-it\lambda_j} D_{n-t}(\mu)\left[f(\mu+\lambda_j) - f(\lambda_j)\right]\xi d\mu. \tag{A.7}$$

Next, by summation by parts, (A.7) is

$$\begin{aligned}\frac{1}{2\pi n}\sum_{j=0}^{m} s_j p_j \int_{-\pi}^{\pi} D_n(-\mu)\Bigg\{ & a_{n-1}D_1(\mu)\left[f(\mu+\lambda_j) - f(\lambda_j)\right]\xi\sum_{t=0}^{n-1} e^{-it\lambda_j} d\mu \\ & - \left[f(\mu+\lambda_j) - f(\lambda_j)\right]\xi\sum_{t=0}^{n-2}\left(a_{t+1}D_{n-t-1}(\mu) - a_t D_{n-t}(\mu)\right)\sum_{h=0}^{t} e^{-ih\lambda_j} d\mu.\Bigg\}.\end{aligned} \tag{A.8}$$

Because

$$\sum_{t=0}^{n-1} e^{-it\lambda_j} = n,\ j = 0,\ \mathrm{mod}\ n; \quad = 0,\ \text{otherwise}, \tag{A.9}$$

the contribution of the first term in braces in (A.8) is bounded in modulus by

$$K\left|a_{n-1}\right|\int_{-\pi}^{\pi}\left|D_n(\mu)\right|d\mu=O\left(n^{\beta-1}\log n\right), \tag{A.10}$$

since f is boundedly differentiable, by Stirling's approximation $\left|a_s\left(c\right)\right|\leq K\left(1+s\right)^{c-1}$, for $c>0$, $s\geq 0$, and

$$\int_{-\pi}^{\pi}\left|D_n(\mu)\right|d\mu=O\left(\log n\right), \tag{A.11}$$

(see e.g. Zygmund, 1977). Regarding the second term in (A.8), note that

$$a_{t+1}D_{n-t-1}(\mu)-a_tD_{n-t}(\mu)=\left(a_{t+1}-a_t\right)D_{n-t-1}(\mu)-e^{i(n-t)\mu}a_t. \tag{A.12}$$

The contribution of the first term on the right of (A.12) to the second term of (A.8) is 0 for $\beta=1$, as in this case $a_{t+1}=a_t$, $t=0,...,n-2$. For $\beta\neq 1$, this contribution is bounded in modulus by

$$\begin{aligned}&Kn^{-1}\left\{\sum_{j=0}^{m}\int_{-\pi}^{\pi}\left|D_n\left(\mu\right)\right|^2\left\|f\left(\mu+\lambda_j\right)-f\left(\lambda_j\right)\right\|d\mu\right\}^{\frac{1}{2}}\\&\times\left\{\sum_{j=0}^{m}\int_{-\pi}^{\pi}\left|\sum_{t=0}^{n-2}\left(a_{t+1}-a_t\right)D_{n-t-1}(\mu)\left(D_t(-\lambda_j)+1\right)\right|^2\left\|f\left(\mu+\lambda_j\right)-f\left(\lambda_j\right)\right\|d\mu\right\}^{\frac{1}{2}}.\end{aligned} \tag{A.13}$$

The term in the first braces is bounded by

$$Km\int_{-\pi}^{\pi}\left|\mu\right|\left|D_n\left(\mu\right)\right|^2d\mu=O\left(m\log n\right), \tag{A.14}$$

by (A.5) and (A.11), since f is boundedly differentiable. Next, the term in the second braces is bounded by

$$\begin{aligned}&K\sum_{j=0}^{m}\int_{-\pi}^{\pi}\left|\mu\right|\sum_{t=0}^{n-2}\sum_{s=0}^{n-2}\left(a_{t+1}-a_t\right)D_{n-t-1}(\mu)\left(D_t(-\lambda_j)+1\right)\left(a_{s+1}-a_s\right)D_{n-s-1}(-\mu)\left(D_s(\lambda_j)+1\right)d\mu\\=\;&O\left(n^2\log n\sum_{j=1}^{m}j^{-2}\left(\sum_{t=1}^{n}t^{\beta-2}\right)^2\right),\end{aligned} \tag{A.15}$$

by Lemma C.1 of RH and (A.5), which is $O\left(n^2\log n1\left(\beta<1\right)+n^{2\beta}\log n1\left(\beta>1\right)\right)$, implying that (A.13) is $O\left(m^{1/2}\log n1\left(\beta<1\right)+n^{\beta-1}m^{1/2}\log n1\left(\beta>1\right)\right)$. Finally, the contribution of the second term on the right of (A.12) to the second term of (A.8) is bounded in modulus by

$$Kn^{-1}\sum_{j=0}^{m}\left\{\int_{-\pi}^{\pi}\left|\mu D_n\left(\mu\right)\right|^2d\mu\int_{-\pi}^{\pi}\left|\sum_{t=0}^{n-2}e^{i(n-t)\mu}a_t\left(D_t(-\lambda_j)+1\right)\right|^2d\mu\right\}^{\frac{1}{2}}. \tag{A.16}$$

Now, the first integral inside braces is $O(1)$ by (A.5), whereas noting that

$$\int_{-\pi}^{\pi} e^{-i(s-t)\mu} d\mu = 2\pi,\ s = t;\ \ = 0, \text{ otherwise}, \tag{A.17}$$

the second is bounded by $K\sum_{t=1}^{n} a_t^2 |D_t(\lambda_j)|^2$, so that (A.16) is bounded by $Kn^{-1}\sum_{j=1}^{m}\left\{n^{2\beta+1}j^{-2}\right\}^{1/2}$, which is $O\left(n^{\beta-1/2}\log m\right)$, to conclude the proof of (A.3).

Next, we prove that as $n \to \infty$,

$$n^{-\beta}(e_m(\gamma) - E(e_m(\gamma))) \Rightarrow \zeta' A(1)^{-1\prime}\,\Omega^{-1}\int_0^1 \widetilde{W}(r;\beta)\,dW(r)\,. \tag{A.18}$$

This proof just consists of showing that as $n \to \infty$,

$$e_m(\gamma) - E\{e_m(\gamma)\} = \frac{p_0}{2\pi}\sum_{t=1}^{n} x_{t-1}(\gamma)A(1)\,\varepsilon_t + o_p(n^{\beta}), \tag{A.19}$$

because, normalized by n^{β}, the first term on the right of (A.19) weakly converges to the right of (A.18) by Proposition 3 of RH. Now, in view of Propositions 1, 2 of RH, (A.19) holds on showing $Var\left\{\mathrm{Re}\left\{\sum_{j=m+1}^{[n/2]} s_j p_j I_{ux(\gamma)}(\lambda_j)\right\}\right\} = o\left(n^{2\beta}\right)$,but, as mentioned in Robinson and Marinucci (2001), this follows by a simple modification of their Theorem 5.1, as $p(\lambda)$ is a well-behaved function without poles.

Finally, to complete the proof for $\overline{\nu}_m(\gamma,\delta)$ when $\beta > 1/2$, we show that as $n \to \infty$,

$$n^{-2\beta} b_m(\gamma) \Rightarrow \frac{q_0}{2\pi}\int_0^1 \widetilde{W}(r;\beta)^2\,dr, \tag{A.20}$$

where the right side is almost surely positive. This result follows in view of Propositions 4, 5, 6 of RH, as by Theorem 4.4 and simple modification of Theorem 5.1 of Robinson and Marinucci (2001) and Assumption 1,

$$\mathrm{Re}\left\{\sum_{j=m+1}^{[n/2]} s_j q_j I_{x(\gamma)}(\lambda_j)\right\} = o_p(n^{2\beta}). \tag{A.21}$$

We now prove the result for $\overline{\nu}_m(\gamma,\delta)$ when $\beta < 1/2$. First, defining $\widetilde{x}_t(\gamma) = \sum_{j=0}^{\infty} a_j u_{2,t-j}$, this follows on showing

$$\sum_{j=0}^{m}\mathrm{Re}\left\{s_j p_j I_{ux(\gamma)}(\lambda_j)\right\} \;=\; 2\sum_{j=1}^{m}\mathrm{Re}\left\{p_j I_{u\widetilde{x}(\gamma)}(\lambda_j)\right\} + o_p\left(n^{\beta} m^{\frac{1}{2}-\beta}\right), \tag{A.22}$$

$$\sum_{j=0}^{m}\mathrm{Re}\left\{s_j q_j I_{x(\gamma)}(\lambda_j)\right\} \;=\; 2\sum_{j=1}^{m}\mathrm{Re}\left\{q_j I_{\widetilde{x}(\gamma)}(\lambda_j)\right\} + o_p\left(n^{2\beta} m^{1-2\beta}\right), \tag{A.23}$$

$$m^{\frac{1}{2}}\lambda_m^{\beta-1}\frac{2\pi}{n}\sum_{j=1}^{m}\operatorname{Re}\left\{p_j I_{u\widetilde{x}(\gamma)}(\lambda_j)\right\} \rightarrow_d N\left(0,\frac{f^{11}(0)\, f_{22}(0)}{2(1-2\beta)}\right), \tag{A.24}$$

$$\lambda_m^{2\beta-1}\frac{2\pi}{n}\sum_{j=1}^{m}\operatorname{Re}\left\{q_j I_{\widetilde{x}(\gamma)}(\lambda_j)\right\} \rightarrow_p \frac{f^{11}(0)\, f_{22}(0)}{1-2\beta}, \tag{A.25}$$

by simple application of Cramer's Theorem. First, we show (A.22). The left side of (A.22) is

$$2\sum_{j=1}^{m}\operatorname{Re}\left\{p_j I_{u\widetilde{x}(\gamma)}(\lambda_j)\right\} + p_0 I_{u\widetilde{x}(\gamma)}(0) + p_0\left(I_{ux(\gamma)}(0) - I_{u\widetilde{x}(\gamma)}(0)\right)$$
$$+2\sum_{j=1}^{m}\operatorname{Re}\left\{p_j\left(I_{ux(\gamma)}(\lambda_j) - I_{u\widetilde{x}(\gamma)}(\lambda_j)\right)\right\}. \tag{A.26}$$

Clearly, the second term in (A.26) is $O_p\left(n^{\beta}\right) = o_p\left(n^{\beta}m^{1/2-\beta}\right)$, as under Assumption 1, $\sum_{t=1}^{n} u_t = O_p\left(n^{1/2}\right)$, $\sum_{t=1}^{n}\widetilde{x}_t(\gamma) = O_p\left(n^{1/2+\beta}\right)$ (see e.g. Robinson, 1994b). The third term in (A.26) is

$$\frac{p_0}{2\pi n}\sum_{t=1}^{n} u_t \sum_{s=1}^{n}\left(x_s(\gamma) - \widetilde{x}_s(\gamma)\right), \tag{A.27}$$

where the second summation in (A.27) has mean 0 and variance

$$\begin{aligned} Var\left(\sum_{s=1}^{n}\sum_{l=0}^{\infty} a_{s+l}u_{2,-l}\right) &\leq K\int_{-\pi}^{\pi}\left|\sum_{s=1}^{n}\sum_{l=0}^{\infty} a_{s+l}e^{il\mu}\right|^2 d\mu \leq K\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{l=0}^{\infty}(t+l)^{\beta-1}(s+l)^{\beta-1} \\ &\leq K\sum_{t=1}^{n}\sum_{l=0}^{\infty}(t+l)^{2\beta-2} + K\sum_{t=2}^{n}\sum_{s=1}^{t-1}\sum_{l=0}^{\infty}(t+l)^{\beta-1}(s+l)^{\beta-1} \\ &\leq K\sum_{t=1}^{n}\sum_{l=t}^{\infty} l^{2\beta-2} + K\sum_{t=2}^{n}\sum_{s=1}^{t-1}\sum_{l=0}^{\infty}(s+l)^{2\beta-2} \leq Kn^{2\beta+1}, \end{aligned} \tag{A.28}$$

implying that $\sum_{s=1}^{n}\left(x_s(\gamma) - \widetilde{x}_s(\gamma)\right) = O_p\left(n^{\beta+1/2}\right)$, and the same conclusion as for the second term. The expectation of the fourth is the real part of

$$\frac{1}{\pi n}\int_{-\pi}^{\pi}\sum_{j=1}^{m} p_j D_n(\lambda_j - \mu)\sum_{s=0}^{\infty}\sum_{k=1}^{n} a_{k+s}e^{-ik\lambda_j} f(\mu)\,\xi e^{-is\mu} d\mu, \tag{A.29}$$

which by (A.6) and periodicity equals

$$\frac{1}{\pi n}\int_{-\pi}^{\pi}\sum_{j=1}^{m}p_jD_n(-\mu)\sum_{s=0}^{\infty}\sum_{k=s+1}^{n+s}a_ke^{-ik\lambda_j}\left(f(\mu+\lambda_j)-f(\lambda_j)\right)\xi e^{-is(\mu-\lambda_j)}d\mu$$

$$\leq \frac{K}{n}\left\{\int_{-\pi}^{\pi}\sum_{j=1}^{m}|D_n(-\mu)|\,\|f(\mu+\lambda_j)-f(\lambda_j)\|^2d\mu\int_{-\pi}^{\pi}\sum_{j=1}^{m}\left|\sum_{s=0}^{\infty}\sum_{k=s+1}^{n+s}a_ke^{-ik\lambda_j}e^{-is(\theta-\lambda_j)}\right|^2d\theta\right\}^{\frac{1}{2}}. \tag{A.30}$$

By Assumption 1 and (A.5) the first integral inside braces in (A.30) is $O(m)$. The second integral is

$$2\pi\sum_{j=1}^{m}\sum_{s=0}^{\infty}\sum_{k=s+1}^{n+s}\sum_{l=s+1}^{n+s}a_ka_le^{i(l-k)\lambda_j}\leq K\sum_{j=1}^{m}\sum_{s=0}^{\infty}\frac{(s+1)^{2\beta-2}}{|\lambda_j|^2}=O\left(n^2\right), \tag{A.31}$$

by Lemma 3.2 in Robinson and Marinucci (2001), to conclude that the expectation is $O\left(m^{1/2}\right)$. The variance of the fourth term in (A.26) is bounded by the real part of

$$\frac{1}{\pi^2n^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{t=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}a_{s+l}a_{q+p}e^{i\lambda_j(t-s)-i\lambda_k(r-q)}$$
$$\times p(\lambda_j)\left\{E(u_tu_r')E(u_{2,-l}u_{2,-p})+E(u_tu_{2,-p})E(u_r'u_{2,-l})+\kappa\right\}p'(-\lambda_k), \tag{A.32}$$

where κ is the fourth cumulant matrix of u_t, u_r, $u_{2,-l}$, $u_{2,-p}$. We just detail the contribution to the variance of the first term in braces in (A.32). By simple application of the Cauchy inequality the second and third terms have the same order as the first term. This contribution is bounded by

$$Kn^{-2}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=0}^{\infty}\sum_{s=1}^{n}a_{s+l}e^{-i\lambda_js}\sum_{q=1}^{n}a_{q+l}e^{i\lambda_kq}\sum_{t=1}^{n}e^{it(\lambda_j-\lambda_k)}\leq Kn^{-1}\sum_{j=1}^{m}\sum_{l=0}^{\infty}\sum_{s=1}^{n}a_{s+l}e^{-i\lambda_js}\sum_{q=1}^{n}a_{q+l}e^{i\lambda_jq}, \tag{A.33}$$

by (A.9), while (A.33) is bounded by

$$Kn^{-1}m\sum_{l=0}^{\infty}\sum_{s=1}^{n}a_{s+l}^2+Kn^{-1}\sum_{j=1}^{m}\sum_{l=0}^{\infty}\sum_{s\neq q}^{n}\sum a_{s+l}a_{q+l}e^{i\lambda_j(q-s)}. \tag{A.34}$$

Clearly, the first term in (A.34) is $O\left(mn^{2\beta-1}\right)$, and by (A.5) the second is bounded by

$$\begin{aligned}Kn^{-1}\sum_{l=0}^{\infty}\sum_{s\neq q}^{n}\sum a_{s+l}a_{q+l}\frac{1}{|\lambda_{q-s}|} &\leq K\sum_{l=0}^{\infty}\sum_{q=2}^{n}\sum_{s=1}^{q-1}\frac{(s+l)^{\beta-1}(q+l)^{\beta-1}}{q-s}\leq K\sum_{q=2}^{n}\sum_{s=1}^{q-1}\frac{1}{q-s}\sum_{l=s}^{\infty}l^{2\beta-2}\\ &\leq K\sum_{q=2}^{n}\sum_{s=1}^{q-1}\frac{s^{2\beta-1}}{q-s}=K\sum_{q=1}^{n-1}q^{-1}\sum_{s=1}^{n-q}s^{2\beta-1}\leq Kn^{2\beta}\log n.\end{aligned} \tag{A.35}$$

Thus, the fourth term in (A.26) is $O_p\left(m^{1/2}+n^{\beta}\log^{1/2}n\right)$, which is $o_p\left(n^{\beta}m^{1/2-\beta}\right)$, by (3.12), to conclude the proof of (A.22). Next, we show (A.23). First, noting that from previous arguments

$$\frac{q_0}{2\pi n}\left(\sum_{t=1}^{n}x_t\left(\gamma\right)\right)^2=O_p\left(n^{2\beta}\right)=o_p\left(n^{2\beta}m^{1-2\beta}\right), \tag{A.36}$$

(A.23) follows on showing

$$\sum_{j=1}^{m}\operatorname{Re}\left\{q_j w_{x(\gamma)}(\lambda_j)\left(w_{x(\gamma)}(-\lambda_j)-w_{\widetilde{x}(\gamma)}(-\lambda_j)\right)\right\}=o_p\left(n^{2\beta}m^{1-2\beta}\right). \tag{A.37}$$

First the expectation of the left side of (A.37) is the real part of

$$\begin{aligned}&\frac{1}{2\pi n}\sum_{j=1}^{m}q_j\sum_{t=1}^{n}\sum_{q=0}^{n-t}\sum_{s=1}^{n}\sum_{l=0}^{\infty}a_q e^{iq\lambda_j}a_{s+l}e^{i\lambda_j(t-s)}\int_{-\pi}^{\pi}f_{22}\left(\mu\right)e^{-i(l+t)\mu}d\mu\\ =\;&\frac{1}{2\pi n}\int_{-\pi}^{\pi}\sum_{j=1}^{m}q_j\sum_{t=1}^{n}a_{n-t}e^{i(n-t)\lambda_j}D_t\left(\lambda_j-\mu\right)\sum_{s=1}^{n}\sum_{l=0}^{\infty}a_{s+l}e^{-i\lambda_j s}e^{-il\mu}\left(f_{22}\left(\mu\right)-f_{22}\left(\lambda_j\right)\right)d\mu,\end{aligned} \tag{A.38}$$

since $\int_{-\pi}^{\pi}e^{-i(l+t)\mu}d\mu=0$, for all $t\geq 1$, $l\geq 0$. Then, (A.38) is bounded by

$$\frac{K}{n}\left\{\int_{-\pi}^{\pi}\sum_{j=1}^{m}\left|\sum_{t=1}^{n}a_{n-t}e^{i(n-t)\lambda_j}D_t\left(\lambda_j-\mu\right)\left(f_{22}\left(\mu\right)-f_{22}\left(\lambda_j\right)\right)\right|^2d\mu\int_{-\pi}^{\pi}\sum_{j=1}^{m}\left|\sum_{s=1}^{n}\sum_{l=0}^{\infty}a_{s+l}e^{-i\lambda_j s}e^{-il\theta}\right|^2d\theta\right\}^{\frac{1}{2}}. \tag{A.39}$$

Now, the first integral in braces in (A.39) is bounded by

$$K\sum_{j=1}^{m}\sum_{t=1}^{n}\sum_{q=1}^{n}a_{n-t}a_{n-q}\left|D_t\left(\lambda_j-\mu\right)\right|\left|D_q\left(\mu-\lambda_j\right)\right|\left(f_{22}\left(\mu\right)-f_{22}\left(\lambda_j\right)\right)^2\leq Kmn^{2\beta}, \tag{A.40}$$

by (A.5). The second integral in braces is

$$2\pi\sum_{j=1}^{m}\sum_{s=1}^{n}\sum_{p=1}^{n}\sum_{l=0}^{\infty}a_{s+l}a_{p+l}e^{i(p-s)\lambda_j}\leq Km\sum_{s=1}^{n}\sum_{l=s}^{\infty}a_l^2+Kn\sum_{s\neq p}^{n}\sum\sum_{l=0}^{\infty}\frac{a_{s+l}a_{p+l}}{|s-p|}, \tag{A.41}$$

which is $O\left(mn^{2\beta}+n^{2\beta+1}\log n\right)$, where the order corresponding to the second term in the right of the inequality in (A.41) is calculated as in (A.35). Thus, the expectation of the left side of (A.37) is $O\left(n^{2\beta-1/2}m^{1/2}\log^{1/2}n\right)$. Next, the variance of the left side of (A.37) is bounded by the real part of

$$\begin{aligned}&\frac{1}{4\pi^2n^2}\sum_{j=1}^{m}\sum_{k=1}^{m}q_jq_{-k}\sum_{t=1}^{n}\sum_{r=1}^{n}\sum_{q=0}^{n-t}\sum_{p=0}^{n-r}\sum_{s=1}^{n}\sum_{u=1}^{n}\sum_{l=0}^{\infty}\sum_{v=0}^{\infty}a_q e^{iq\lambda_j}a_p e^{-ip\lambda_k}a_{s+l}a_{u+v}e^{i\lambda_j(t-s)}\\ &\times e^{-i\lambda_k(r-u)}\left\{E\left(u_{2t}u_{2r}\right)E\left(u_{2,-l}u_{2,-v}\right)+E\left(u_{2t}u_{2,-v}\right)E\left(u_{2r}u_{2,-l}\right)+\kappa\right\},\end{aligned} \tag{A.42}$$

where κ is now the fourth cumulant of u_{2t}, u_{2r}, $u_{2,-l}$, $u_{2,-v}$. As before, we just consider the contribution of the first term in braces, the treatment of remaining terms being very similar. This contribution is bounded by

$$Kn^{-2}\sum_{t=1}^{n}\left\{\sum_{l=0}^{n}+\sum_{l=n+1}^{\infty}\right\}\left|\sum_{j=1}^{m}\sum_{q=0}^{n-t}a_q e^{iq\lambda_j}\sum_{s=l+1}^{n+l}a_s e^{-is\lambda_j}e^{i(t+l)\lambda_j}\right|^2. \tag{A.43}$$

Now, noting that by Lemma 3.2 in Robinson and Marinucci (2001)

$$\left|\sum_{s=l+1}^{n+l}a_s e^{-is\lambda_j}\right|\leq K\frac{(l+1)^{\beta/2-1/2}}{|\lambda_j|^{\beta/2+1/2}}, \tag{A.44}$$

the contribution of the summation in l from 0 to n to (A.43) is bounded by

$$Kn^{-1}\sum_{l=1}^{n}\left(l^{\frac{\beta-1}{2}}\sum_{j=1}^{m}\frac{1}{\lambda_j^{\frac{3\beta+1}{2}}}\right)^2\leq K\left(n^{4\beta}1(\beta>\frac{1}{3})+n^{4\beta}\log^2 m1(\beta=\frac{1}{3})+n^{4\beta}m^{1-3\beta}1(\beta<\frac{1}{3})\right). \tag{A.45}$$

Next, by Lemma 3.2 in Robinson and Marinucci (2001), the contribution of the second summation in l in (A.43) is bounded by

$$Kn^{-1}\sum_{l=n+1}^{\infty}\left(l^{\beta-1}n^{1+\beta}\sum_{j=1}^{m}j^{-1-\beta}\right)^2\leq Kn^{2\beta+1}\sum_{l=n+1}^{\infty}l^{2\beta-2}\leq Kn^{4\beta}. \tag{A.46}$$

We conclude that the left side of (A.37) is

$$O_p\left(n^{2\beta-\frac{1}{2}}m^{\frac{1}{2}}\log^{\frac{1}{2}}n+n^{2\beta}\left(1(\beta>\frac{1}{3})+\log m1(\beta=\frac{1}{3})+m^{\frac{1-3\beta}{2}}1(\beta<\frac{1}{3})\right)\right), \tag{A.47}$$

in all cases $o_p\left(n^{2\beta}m^{1-2\beta}\right)$. Finally, (A.24), (A.25) follow as in the proof of Theorem 2 in Christensen and Nielsen (2001) who adapted the steps in Lobato (1999) to a somewhat different situation. From these references, it can be easily shown that

$$m^{\frac{1}{2}}\lambda_m^{\beta-1}\frac{2\pi}{n}\sum_{j=1}^{m}\mathrm{Re}\left\{p_j I_{u\widetilde{x}(\gamma)}(\lambda_j)\right\}=\sum_{t=2}^{n}\zeta_t\sum_{s=1}^{t-1}c_{t-s}\zeta_s+o_p\left(n^{\beta}m^{1/2-\beta}\right), \tag{A.48}$$

where $\zeta_t=\Omega^{-1/2}\varepsilon_t$,

$$c_t=\frac{1}{2\pi nm^{1/2}}\sum_{j=1}^{m}\varrho_j\cos(t\lambda_j), \tag{A.49}$$

and

$$\varrho(\lambda)=\lambda_m^{\beta}\left[B'(\lambda)p'(\lambda)\xi'\left(1-e^{-i\lambda}\right)^{-\beta}B(-\lambda)+\left(1-e^{i\lambda}\right)^{-\beta}B'(\lambda)\xi p(-\lambda)B(-\lambda)\right], \tag{A.50}$$

with $B(\lambda) = A\left(e^{i\lambda}\right)\Omega^{1/2}$. The only point worth mentioning is that

$$\frac{1}{4\pi^2 n^2 m}\sum_{j=1}^{m} tr\left\{\varrho_{-j}'\varrho_j\right\}\sum_{t=1}^{n}\sum_{s=1}^{t-1}\cos^2\left((t-s)\lambda_j\right) = \frac{(n-1)^2}{16\pi^2 n^2 m}\sum_{j=1}^{m} tr\left\{\varrho_{-j}'\varrho_j\right\}$$

$$= \frac{(n-1)^2\lambda_m^{2\beta}}{8\pi^2 n^2 m}\sum_{j=1}^{m} tr\left\{B'(\lambda_j)\,p_j'\xi'\left(1-e^{-i\lambda_j}\right)^{-\beta}B(-\lambda_j)\left(1-e^{i\lambda_j}\right)^{-\beta}B'(\lambda_j)\,\xi p_{-j}B(-\lambda_j)\right\}, \tag{A.51}$$

cancellations taking place due to (A.6), so that (A.51) equals

$$\frac{(n-1)^2\lambda_m^{2\beta}}{2n^2 m}\sum_{j=1}^{m}\left|1-e^{i\lambda_j}\right|^{-2\beta} f_{22}(\lambda_j)\,f^{11}(-\lambda_j)\to\frac{f_{22}(0)\,f^{11}(0)}{2(1-2\beta)}, \tag{A.52}$$

as $n\to\infty$, by (2.8).

We have shown that $\overline{\nu}_m(\gamma,\delta)$ has property M. We now show that as $n\to\infty$

$$\widehat{\nu}_m(\gamma,\delta) - \overline{\nu}_m(\gamma,\delta) = o_p\left(n^{\beta}m^{1/2-\min\{\beta,1/2\}}\right), \tag{A.53}$$

$$\widehat{\nu}_m(\widehat{\gamma},\widehat{\delta}) - \widehat{\nu}_m(\gamma,\delta) = o_p\left(n^{\beta}m^{1/2-\min\{\beta,1/2\}}\right), \tag{A.54}$$

noting that the proof for $\widehat{\nu}_m(\widehat{\gamma},\delta)$ and $\widehat{\nu}_m(\gamma,\widehat{\delta})$ is implied by the proof of (A.54). First, (A.53) follows on showing

$$\widehat{e}_m(\gamma) - e_m(\gamma) = o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}}), \tag{A.55}$$

$$\widehat{b}_m(\gamma) - b_m(\gamma) = o_p(n^{2\beta}m^{1-2\min\{1/2,\beta\}}). \tag{A.56}$$

We just prove (A.55), the proof for (A.56) being significantly simpler. The left side of (A.55) is

$$\mathrm{Re}\left\{\sum_{j=0}^{m} s_j(\widehat{p}_j - p_j)I_{ux(\gamma)}(\lambda_j)\right\}. \tag{A.57}$$

Noting that

$$\widehat{p}(\lambda) - p(\lambda) = \zeta' f(\lambda)^{-1}\left[f(\lambda) - \widehat{f}(\lambda)\right]\widehat{f}(\lambda)^{-1}, \tag{A.58}$$

the two possible terms for which $s_j = 1$ are $O_p(n^{\beta-\varkappa}) = o_p(n^{\beta}m^{1/2-\min\{1/2,\beta\}})$ by (3.9), as by Assumption 1, $\sum_{t=1}^{n}u_t = O_p(n^{1/2})$, and by results in Robinson and Marinucci (2001) and previous arguments, $\sum_{t=1}^{n}x_t(\gamma) = O_p(n^{\beta+1/2})$. By summation by parts, the remaining terms in (A.57) equal

$$2\,\mathrm{Re}\left\{(\widehat{p}_{m^*} - p_{m^*})\sum_{j=1}^{m^*}I_{ux(\gamma)}(\lambda_j) - \sum_{j=1}^{m^*-1}(\widehat{p}_{j+1} - p_{j+1} - (\widehat{p}_j - p_j))\sum_{h=1}^{j}I_{ux(\gamma)}(\lambda_h)\right\}, \tag{A.59}$$

where $m^* = m-1$ if $m = n/2$ or $m^* = m$, otherwise. Using techniques in Robinson and Marinucci (2003) is not difficult to show that

$$\sum_{h=1}^{j} I_{ux(\gamma)}(\lambda_h) = O_p\left(n^{\beta} m^{1-\beta} 1(\beta < 1) + n^{\beta} 1(\beta \geq 1)\right), \tag{A.60}$$

uniformly in $j \in [1, m]$. Thus, by Assumption 3, the first term of (A.59) is

$$O_p\left(n^{\beta-\varkappa} m^{1-\beta} 1(\beta < 1) + n^{\beta-\varkappa} 1(\beta \geq 1)\right), \tag{A.61}$$

so the first term of (A.59) is $o_p(n^{\beta} m^{1/2-\min\{1/2,\beta\}})$ noting (3.9). Similarly, by (A.60) and Assumptions 1, 3, the second term of (A.59) is

$$O_p\left(\left(n^{\beta-1-\varkappa} + n^{\beta-\phi}\right) m^{2-\beta} 1(\beta < 1) + \left(n^{\beta-1-\varkappa} + n^{\beta-\phi}\right) m 1(\beta \geq 1)\right), \tag{A.62}$$

which is $o_p(n^{\beta} m^{1/2-\min\{1/2,\beta\}})$ by (3.10), to conclude the proof of (A.53).

Next, noting that

$$\widehat{\nu}_m(\widehat{\gamma}, \widehat{\delta}) - \nu = \frac{\widehat{e}_m(\widehat{\gamma}, \widehat{\delta})}{\widehat{b}_m(\widehat{\gamma})}, \quad \widehat{e}_m(\widehat{\gamma}, \widehat{\delta}) = \mathrm{Re}\left\{\sum_{j=0}^{m} s_j \widehat{p}_j I_{v(\widehat{\gamma},\widehat{\delta})x(\widehat{\gamma})}(\lambda_j)\right\}, \tag{A.63}$$

where $v(\widehat{\gamma}, \widehat{\delta}) = (u_{1t}(\widehat{\gamma} - \gamma), x_t(\widehat{\delta}))'$, (A.54) follows on establishing

$$\begin{aligned} e_m(\widehat{\gamma}, \widehat{\delta}) - e_m(\gamma) &= o_p(n^{\beta} m^{1/2-\min\{1/2,\beta\}}), & \text{(A.64)} \\ \widehat{e}_m(\widehat{\gamma}, \widehat{\delta}) - e_m(\widehat{\gamma}, \widehat{\delta}) - \widehat{e}_m(\gamma) + e_m(\gamma) &= o_p(n^{\beta} m^{1/2-\min\{1/2,\beta\}}), & \text{(A.65)} \\ b_m(\widehat{\gamma}) - b_m(\gamma) &= o_p(n^{2\beta} m^{1-2\min\{1/2,\beta\}}), & \text{(A.66)} \\ \widehat{b}_m(\widehat{\gamma}) - b_m(\widehat{\gamma}) - \widehat{b}_m(\gamma) + b_m(\gamma) &= o_p(n^{2\beta} m^{1-2\min\{1/2,\beta\}}), & \text{(A.67)} \end{aligned}$$

where $e_m(\widehat{\gamma}, \widehat{\delta})$ is like $\widehat{e}_m(\widehat{\gamma}, \widehat{\delta})$ but with $p(\lambda)$ replacing $\widehat{p}(\lambda)$ in (A.63). We just prove (A.64), (A.65), the proofs for (A.66), (A.67) being similar but simpler.

The left side of (A.64) is the real part of

$$\sum_{j=0}^{m} s_j p_j \left\{\left[w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j)\right]\left[w_{v(\widehat{\gamma},\widehat{\delta})}(\lambda_j) - w_u(\lambda_j)\right] + w_{x(\gamma)}(-\lambda_j)\left[w_{v(\widehat{\gamma},\widehat{\delta})}(\lambda_j) - w_u(\lambda_j)\right]\right\}$$
$$+ \sum_{j=0}^{m} s_j p_j \left[w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j)\right] w_u(\lambda_j). \tag{A.68}$$

We just consider the third term of (A.68), as, following similar techniques to those of RH, one could easily show that the same order of magnitude obtained applies also to the whole of (A.68). By Taylor's theorem, the third term of (A.68) is the real part of

$$\sum_{r=1}^{R-1} \frac{(\gamma - \widehat{\gamma})^r}{r!} \sum_{j=0}^{m} s_j p_j w_{u_2}^{(r)}(-\lambda_j; \beta) w_u(\lambda_j) + \frac{(\gamma - \widehat{\gamma})^R}{R!} \sum_{j=0}^{m} s_j p_j w_{u_2}^{(R)}(-\lambda_j; \delta - \overline{\gamma}) w_u(\lambda_j), \tag{A.69}$$

where for a vector or scalar sequence φ_t, and real b, $w_{\varphi}^{(r)}(\lambda; b) = (2\pi n)^{-1/2} \sum_{t=2}^{n} \sum_{s=1}^{t-1} a_s^{(r)}(b)\varphi_{t-s}e^{it\lambda}$, $a_s^{(r)}(b) = d^r a_s(b)/db^r$, and $|\overline{\gamma} - \gamma| \leq |\widehat{\gamma} - \gamma|$. By a straightforward extension of results in Robinson and Marinucci (2001, 2003)

$$\sum_{j=0}^{m} s_j p_j w_{u_2}^{(r)}(-\lambda_j; \beta) w_u(\lambda_j) = O_p\left(n^{\beta} (\log m)^r \left(m^{1-\beta} 1(\beta < 1) + 1(\beta \geq 1)\right)\right), \tag{A.70}$$

the only differences being that the weights $a_s^{(r)}(\beta)$ that are involved (see Lemma C.1 of RH), are not covered by the weights of Robinson and Marinucci (2001) (but it can be easily shown that they just contribute the $(\log m)^r$ factors), and the smooth weighting factor $s_j p(\lambda_j)$, which, as mentioned before, can be handled by simple modification of the proofs of Robinson and Marinucci (2001, 2003). Next, the summation in the second term of (A.69) is bounded by

$$K \sum_{j=0}^{m} \left| w_{u_2}^{(R)}(-\lambda_j; \delta - \overline{\gamma}) \right| \|w_u(\lambda_j)\| \leq K n^2 \sum_{j=1}^{m} \left| a_j^{(R)}(\delta - \overline{\gamma}) \right| = O_p\left(n^{\beta+\epsilon+2}\right), \tag{A.71}$$

for any $\epsilon > 0$ in view of Lemma C.5 of RH. Thus, by Assumption 2, choosing $R > (\kappa + 2)/\kappa$, the third term of (A.68) is

$$O_p\left(n^{\beta-\kappa} \log m \left(m^{1-\beta} 1(\beta < 1) + 1(\beta \geq 1)\right)\right) = o_p(n^{\beta} m^{1/2-\min\{1/2,\beta\}}), \tag{A.72}$$

in view of (3.6).

Next, noting that the left side of (A.65) is the real part of

$$\sum_{j=0}^{m} s_j (\widehat{p}_j - p_j) \left\{ w_{x(\widehat{\gamma})}(-\lambda_j) \left[w_{v(\widehat{\gamma},\widehat{\delta})}(\lambda_j) - w_u(\lambda_j) \right] + \left[w_{x(\widehat{\gamma})}(-\lambda_j) - w_{x(\gamma)}(-\lambda_j) \right] w_u(\lambda_j) \right\}, \tag{A.73}$$

by summation by parts, similar analysis to that of (A.68) and a straightforward extension of (A.60), it can be easily shown that by Assumptions 2, 3, (A.73) is

$$O_p\left(n^{\beta-\kappa} \log m (n^{-\varkappa} + n^{-\phi} m) \left(m^{1-\beta} 1(\beta < 1) + 1(\beta \geq 1)\right)\right), \tag{A.74}$$

which is $o_p(n^{\beta} m^{1/2-\min\{1/2,\beta\}})$ by (3.6), (3.9), (3.10). This proves the Theorem for the W estimates (2.10).

Finally, we give the proof for the Z estimates (2.11). Define the infeasible estimate $\overline{\nu}_m^{\mathrm{o}}(\gamma, \delta) = a_m^{\mathrm{o}}(\gamma, \delta)/b_m^{\mathrm{o}}(\delta)$, where

$$a_m^{\mathrm{o}}(\gamma, \delta) = \mathrm{Re}\left\{ p_0 \sum\nolimits_{j=0}^{m} s_j I_{z(\gamma,\delta)x(\gamma)}(\lambda_j) \right\}, \quad b_m^{\mathrm{o}}(\gamma) = q_0 \sum_{j=0}^{m} s_j I_{x(\gamma)}(\lambda_j). \tag{A.75}$$

We just show that $\overline{\nu}_m^{\mathrm{o}}(\gamma, \delta)$ has property M, then this follows immediately for the estimates (2.11) from the proof for the W estimates. Clearly

$$\overline{\nu}_m^{\mathrm{o}}(\gamma, \delta) - \nu = \frac{e_m^{\mathrm{o}}(\gamma)}{b_m^{\mathrm{o}}(\gamma)}, \quad e_m^{\mathrm{o}}(\gamma) = \mathrm{Re}\left\{ p_0 \sum_{j=0}^{m} s_j I_{ux(\gamma)}(\lambda_j) \right\}. \tag{A.76}$$

For $\beta > 1$, the result follows in view of Theorem 2 of RH when $m = [n/2]$. For $m < [n/2]$,

$$\mathrm{Re}\left\{\sum_{j=0}^{m} s_j I_{ux(\gamma)}(\lambda_j)\right\} = \sum_{j=1}^{n} I_{ux(\gamma)}(\lambda_j) + o_p(n^{\beta}), \quad \mathrm{Re}\left\{\sum_{j=0}^{m} s_j I_{x(\gamma)}(\lambda_j)\right\} = \sum_{j=1}^{n} I_{x(\gamma)}(\lambda_j) + o_p(n^{2\beta}), \tag{A.77}$$

by Propositions 4.1, 4.2 of Robinson and Marinucci (2003); we then conclude as in the case $m = [n/2]$. For $\beta = 1$, as mentioned in RH, the result follows by Theorem 4.3 of Robinson and Marinucci (2001) and (2.13). For $1/2 < \beta < 1$ we first prove that

$$E(e_m^{\mathrm{o}}(\gamma)) = o(n^{\beta}). \tag{A.78}$$

By the orthogonality property (A.6), we can write the left side of (A.78) as the real part of

$$\frac{1}{2\pi n}\sum_{j=0}^{m}\int_{-\pi}^{\pi} D_n(\lambda_j - \mu)\sum_{t=1}^{n} a_{n-t}e^{-i(n-t)\lambda_j} D_t(\mu - \lambda_j)\left\{\Xi(\mu, \lambda_j) + \Xi(\lambda_j, 0)\right\} d\mu, \tag{A.79}$$

where $\Xi\left(a, b\right) = p_0\left\{f\left(a\right) - f\left(b\right)\right\}\xi$. The contribution of the second term in braces in (A.79) is

$$n^{-1}\sum_{j=1}^{m}\Xi(\lambda_j, 0)\sum_{t=0}^{n-1} a_t\left(n - t\right)e^{-it\lambda_j}. \tag{A.80}$$

By summation by parts, (A.80) is bounded in modulus by

$$n^{-1}\sum_{t=0}^{n-1}|a_t|\left(n - t\right)\left|\sum_{j=1}^{m-1}\left[\Xi(\lambda_j, 0) - \Xi(\lambda_{j+1}, 0)\right]D_j\left(-\lambda_t\right) + \Xi(\lambda_m, 0)D_m\left(-\lambda_t\right)\right| \leq Km\sum_{t=1}^{n} t^{\beta-2} \leq Km, \tag{A.81}$$

as we only consider $\beta < 1$, to conclude by (2.13). Finally, the proof of (A.3) readily implies that the contribution of the first term in braces in (A.79) is $o(n^{\beta})$.

Next, we show that, as $n \to \infty$,

$$n^{-\beta}(e_m^{\mathrm{o}}(\gamma) - E(e_m^{\mathrm{o}}(\gamma))) \Rightarrow \zeta' A\left(1\right)^{-1\prime}\Omega^{-1}\int_0^1 \widetilde{W}\left(r; \beta\right) dW\left(r\right). \tag{A.82}$$

By Theorem 5.1 of Robinson and Marinucci (2001), as $n \to \infty$,

$$Var(e_m^{\mathrm{o}}(\gamma)) = Var\left(p_0\sum_{j=1}^{n} I_{ux(\gamma)}(\lambda_j)\right) + o(n^{2\beta}), \tag{A.83}$$

implying that

$$e_m^{\mathrm{o}}(\gamma) - E(e_m^{\mathrm{o}}(\gamma)) = \frac{p_0}{2\pi}\sum_{t=1}^{n}\left\{x_t(\gamma)u_t - E\left[x_t(\gamma)u_t\right]\right\} + o_p(n^{\beta}). \tag{A.84}$$

Thus, in view of previous steps, it just remain to prove that

$$\frac{p_0}{2\pi}\sum_{t=1}^{n}\left\{x_t(\gamma)u_t - E\left[x_t(\gamma)u_t\right]\right\} - \frac{p_0}{2\pi}\sum_{t=2}^{n}x_{t-1}(\gamma)A\left(1\right)\varepsilon_t = o_p(n^{\beta}). \tag{A.85}$$

Note that

$$\sum_{t=1}^{n}\left\{x_t(\gamma)u_t - E\left[x_t(\gamma)u_t\right]\right\} - \sum_{t=1}^{n}\left\{x_t(\gamma)A\left(1\right)\varepsilon_t - E\left[x_t(\gamma)A\left(1\right)\varepsilon_t\right]\right\}$$

$$= \sum_{t=1}^{n}\left\{x_t(\gamma)\left(w_{t-1}-w_t\right) - E\left[x_t(\gamma)\left(w_{t-1}-w_t\right)\right]\right\}, \tag{A.86}$$

where $w_t = \sum_{j=0}^{\infty}\widetilde{A}_j\varepsilon_{t-j}$, $\widetilde{A}_j = \sum_{k=j+1}^{\infty}A_k$, and

$$\sum_{t=1}^{n}x_t(\gamma)\left(w_{t-1}-w_t\right) = \sum_{t=2}^{n}\left\{x_t(\gamma)-x_{t-1}(\gamma)\right\}w_{t-1} + x_1(\gamma)w_0 - x_n(\gamma)w_n. \tag{A.87}$$

As in the proof of Theorem 5.1 of Robinson and Marinucci (2001), because (3.3) ensures boundedness of the spectrum of w_t and the cross-spectrum of w_t with u_{2t}, it can be easily shown that

$$Var\left\{\sum\nolimits_{t=2}^{n}\left\{x_t(\gamma)-x_{t-1}(\gamma)\right\}w_{t-1}\right\} = O\left(n\right). \tag{A.88}$$

Next, $E\left|x_1(\gamma)w_0\right| \leq \left\{Ex_1(\gamma)^2Ew_0^2\right\}^{1/2} \leq \infty$, due to the truncation in (1.7) and (3.3). Similarly, by Robinson and Marinucci (2001, 2003),

$$E\left|x_n(\gamma)w_n\right| \leq \left\{Ex_n(\gamma)^2Ew_n^2\right\}^{\frac{1}{2}} \leq Kn^{\beta-\frac{1}{2}}, \tag{A.89}$$

to conclude that (A.86) is $o_p(n^{\beta})$. Finally, we have to prove that

$$\sum_{t=2}^{n}x_{t-1}(\gamma)A\left(1\right)\varepsilon_t - \sum_{t=2}^{n}\left\{x_t(\gamma)A\left(1\right)\varepsilon_t - E\left[x_t(\gamma)A\left(1\right)\varepsilon_t\right]\right\} = o_p(n^{\beta}), \tag{A.90}$$

but this immediately follows, as $Var\left\{\sum_{t=2}^{n}\left[x_{t-1}(\gamma)-x_t(\gamma)\right]A\left(1\right)\varepsilon_t\right\} = O\left(n\right)$, by similar arguments to the ones in the proof of Theorem 5.1 of Robinson and Marinucci (2001), to complete the proof for $\beta > 1/2$.

Finally, the proof for $\beta < 1/2$ follows on showing that

$$e_m^{o}\left(\gamma\right) - e_m\left(\gamma\right) = o_p(n^{\beta}m^{1/2-\beta}), \tag{A.91}$$

$$b_m^{o}\left(\gamma\right) - b_m\left(\gamma\right) = o_p(n^{2\beta}m^{1-2\beta}). \tag{A.92}$$

By the bounds for periodograms given in Robinson (1995b), Robinson (2002) and Assumption 1, the left side of (A.91) is bounded by

$$K\left\{\sum_{j=1}^{m}\left\|p_j-p_0\right\|\left\|I_u\left(\lambda_j\right)\right\|\sum_{k=1}^{m}\left\|p_k-p_0\right\|I_{x(\gamma)}\left(\lambda_k\right)\right\}^{\frac{1}{2}}$$

$$\leq K\left\{n^{2\beta-2-2\eta}\sum_{j=1}^{m}j^{1+\eta}\sum_{k=1}^{m}k^{1+\eta-2\beta}\right\}^{\frac{1}{2}} \leq Kn^{\beta-1-\eta}m^{2+\eta-\beta}, \tag{A.93}$$

so that (A.91) holds as $m^{3/2+\eta}/n^{1+\eta} \to 0$ as $n \to \infty$, by (3.12). Finally, by the same arguments, the left side of (A.92) is bounded by

$$K \sum_{j=1}^{m} \lambda_j^{1+\eta} I_{x(\gamma)}(\lambda_j) \leq K n^{2\beta-1-\eta} m^{2+\eta-2\beta}, \tag{A.94}$$

so that (A.92) holds as $m^{1+\eta}/n^{1+\eta} \to 0$ as $n \to \infty$, again by (3.12), to conclude the proof.

B. Appendix B: Proofs of Propositions

Proof of Proposition 1

First, we show the result for $\overline{\gamma}$. The proof strategy is similar to one employed in Andrews and Sun (2004). First, by checking conditions in Theorem 8.1 of Wooldridge (1994), we show that there exists a zero of $q_v^p(c)$, say $\overline{\gamma}_S$, for which the same result as that of Proposition 1 holds. Next, noting that by Velasco (1999a) $\overline{\gamma}_G$ is $d^{1/2}$-consistent, where d is a sequence such that for $\theta = (1+\eta)\,1(s=1) + 21(s>1)$,

$$\frac{1}{d} + \frac{d^{1+2\theta}(\log d)^2}{n^{2\theta}} \to 0 \text{ as } n \to \infty, \tag{B.1}$$

setting d equal to a multiple of a power root of n so that (B.1) is satisfied, $\overline{\gamma}_G$ is also $\log^5 l$-consistent. Thus, noting that by our definition of $\overline{\gamma}$, $|\overline{\gamma}_G - \overline{\gamma}| \leq |\overline{\gamma}_G - \overline{\gamma}_S|$, $\overline{\gamma}$ will be also $\log^5 l$-consistent as $\overline{\gamma}_G - \overline{\gamma}_S = O_p(\log^{-5} l)$. The proof is completed by showing that any zero of $q_v^p(c)$ which is $\log^5 l$-consistent is also $l^{1/2}$-consistent with asymptotic distribution given in Proposition 1, but this immediately follows from Robinson (1995a), Velasco (1999a), noting that the function $k_q(u)$ is bounded.

The score and Hessian corresponding to the objective function $Q_v^p(c)$ are

$$S_v^p(c) = \frac{2l\sum' b_{q,j}\lambda_j^{2c} I_v^p(\lambda_j)}{\sum' k_{q,j}\lambda_j^{2c} I_v^p(\lambda_j)}, \quad H_v^p(c) = \frac{4l\left(G_{2,v}^p(c)\,G_{0,v}^p(c) - \left(G_{1,v}^p(c)\right)^2\right)}{\left(G_{0,v}^p(c)\right)^2}, \tag{B.2}$$

respectively, where

$$G_{g,v}^p(c) = \frac{p}{l}\sum{}' k_{q,j}(\log\lambda_j)^g \lambda_j^{2c} I_v^p(\lambda_j),\ g = 0,1,2. \tag{B.3}$$

We first check that a condition equivalent to (iv) (b) in Wooldridge (1994) holds in our case. More precisely we show that

$$l^{-\frac{1}{2}} S_v^p(\gamma) - \frac{2(2\pi)^q U_{qq} h_q}{(2q)! f_{11}(0)} \frac{l^{2q+1/2}}{n^{2q}} \to_d N(0, 4p\Phi W_q). \tag{B.4}$$

Now,

$$l^{-\frac{1}{2}} S_v^p(\gamma) = 2p^{\frac{1}{2}}\frac{A}{B}, \tag{B.5}$$

where

$$A = \left(\frac{p}{l}\right)^{\frac{1}{2}} \sum{}' b_{q,j}\left(\lambda_j^{2\gamma} I_v^p(\lambda_j) - 1\right), \quad B = \frac{p}{l}\sum{}' k_{q,j}\lambda_j^{2\gamma} I_v^p(\lambda_j), \tag{B.6}$$

noting that $\sum' b_{q,j} = 0$. From (3.1), $u_{1t} = \sum_{j=0}^{\infty} \zeta' A_j \varepsilon_{t-j}$, so there exist sequences $\{b_j\}_{j=0}^{\infty}$ and $\xi_t \sim iid\,(0,1)$ such that $u_{1t} = \sum_{j=0}^{\infty} b_j \xi_{t-j}$. Then, we could set $A = \sum_{i=1}^{4} A_i$ where

$$A_1 = 2\pi \left(\frac{p}{l}\right)^{\frac{1}{2}} \sum{}' b_{q,j} \left\{ E\left(I_\xi^p(\lambda_j)\, h(\lambda_j)\right) - E\left(f_{11}(0)\, I_\xi^p(\lambda_j)\right)\right\}, \tag{B.7}$$

$$A_2 = 2\pi f_{11}(0) \left(\frac{p}{l}\right)^{\frac{1}{2}} \sum{}' b_{q,j} \left\{ I_\xi^p(\lambda_j) - E\left(I_\xi^p(\lambda_j)\right)\right\}, \tag{B.8}$$

$$A_3 = 2\pi \left(\frac{p}{l}\right)^{\frac{1}{2}} \sum{}' b_{q,j} \left\{ I_\xi^p(\lambda_j)\, h(\lambda_j) - f_{11}(0)\, I_\xi^p(\lambda_j) - E\left(I_\xi^p(\lambda_j)\, h(\lambda_j) - f_{11}(0)\, I_\xi^p(\lambda_j)\right)\right\}, \tag{B.9}$$

$$A_4 = \left(\frac{p}{l}\right)^{\frac{1}{2}} \sum{}' b_{q,j} \left\{ I_v^p(\lambda_j)\, \lambda_j^{2\gamma} - 2\pi h(\lambda_j)\, I_\xi^p(\lambda_j)\right\}. \tag{B.10}$$

We have

$$A_1 = \left(\frac{p}{l}\right)^{\frac{1}{2}} \sum{}' b_{q,j} \left(h(\lambda_j) - f_{11}(0)\right) = \left(\frac{p}{l}\right)^{\frac{1}{2}} \sum{}' b_{q,j} \sum_{i=1}^{q} \frac{h_i \lambda_j^{2i}}{(2i)!} + O\left(l^{-\frac{1}{2}} \sum{}' |b_{q,j}|\, \lambda_j^{2(q+1)}\right). \tag{B.11}$$

The first term in (B.11) equals

$$\left(\frac{l}{p}\right)^{\frac{1}{2}} \left\{ \sum_{i=1}^{q} \frac{h_i \lambda_l^{2i}}{(2i)!} U_{iq} + \sum_{i=1}^{q} \frac{h_i \lambda_l^{2i}}{(2i)!} \left[\frac{p}{l} \sum{}' b_{q,j} \left(\frac{j}{l}\right)^{2i} - \int_0^1 (\log u + 1)\, u^{2i} k_q(u)\, du \right]\right\}, \tag{B.12}$$

which, noting Assumption P1, equals

$$\left(\frac{l}{p}\right)^{\frac{1}{2}} \frac{h_q \lambda_l^{2q}}{(2q)!} U_{qq} + O\left(\frac{l^{\frac{3}{2}} \log l}{n^2}\right), \tag{B.13}$$

because proceeding as in Lemma 5 of Velasco (1999a),

$$\frac{p}{l} \sum{}' b_{q,j} \left(\frac{j}{l}\right)^{2i} - \int_0^1 (\log u + 1)\, u^{2i} k_q(u)\, du = O\left(l^{-1} \log l\right), \tag{B.14}$$

implying that

$$A_1 = \left(\frac{l}{p}\right)^{\frac{1}{2}} \frac{h_q \lambda_l^{2q}}{(2q)!} U_{qq} + O\left(\frac{l^{\frac{3}{2}} \log l}{n^2}\right) + O\left(\frac{l^{2q+5/2} \log l}{n^{2q+2}}\right), \tag{B.15}$$

where by (4.9), the third term of (B.15) is of smaller order than the first, while the second is $o(1)$. For A_2, in view of the proof of Lemma 6 of Velasco (1999a), it is straightforward to show that

$$\left(\frac{p}{l}\right)^{\frac{1}{2}} \sum{}' b_{q,j} \left\{ 2\pi I_\xi^p(\lambda_j) - 1\right\} \to_d N(0, W_q \Phi), \tag{B.16}$$

simply noting that as in (B.12)

$$\frac{p}{l} \sum{}' b_{q,j}^2 = W_q + O\left(\frac{\log l}{l}\right). \tag{B.17}$$

Next, by Velasco (1999a) and some of our previous arguments $Var(A_3) = o(1)$, while

$$\begin{aligned} A_4 &= \left(\frac{p}{l}\right)^{\frac{1}{2}} \sum{}' b_{q,j} h(\lambda_j) \left\{ \frac{I_v^p(\lambda_j)}{h(\lambda_j)\lambda_j^{-2\gamma}} - 2\pi I_\xi^p(\lambda_j) \right\} \\ &= O_p\left(l^{(1-\theta)/2} \log l 1(\theta < 2) + l^{-1/2} \log^2 l 1(\theta = 2) + l^{\gamma - p + 1/2} \log^{3/2} l \right) = o_p(1), \quad \text{(B.18)} \end{aligned}$$

by the condition we set on the tapering order p.

Expanding B in a similar way to A, we get $B = \sum_{i=0}^4 B_i$, where

$$B_0 = \frac{2\pi f_{11}(0)p}{l} \sum{}' k_{q,j} E\left(I_\xi^p(\lambda_j)\right), \quad \text{(B.19)}$$

$$B_1 = \frac{2\pi p}{l} \sum{}' k_{q,j} \left\{ E\left(I_\xi^p(\lambda_j) h(\lambda_j)\right) - E\left(f_{11}(0) I_\xi^p(\lambda_j)\right) \right\}, \quad \text{(B.20)}$$

$$B_2 = \frac{2\pi f_{11}(0)p}{l} \sum{}' k_{q,j} \left\{ I_\xi^p(\lambda_j) - E\left(I_\xi^p(\lambda_j)\right) \right\}, \quad \text{(B.21)}$$

$$B_3 = \frac{2\pi p}{l} \sum{}' k_{q,j} \left\{ I_\xi^p(\lambda_j) h(\lambda_j) - f_{11}(0) I_\xi^p(\lambda_j) - E\left(I_\xi^p(\lambda_j) h(\lambda_j) - f_{11}(0) I_\xi^p(\lambda_j)\right) \right\}, \quad \text{(B.22)}$$

$$B_4 = \frac{p}{l} \sum{}' k_{q,j} \left\{ I_v^p(\lambda_j) \lambda_j^{2\gamma} - 2\pi h(\lambda_j) I_\xi^p(\lambda_j) \right\}, \quad \text{(B.23)}$$

where by previous results

$$B_1 = O\left(\left(\frac{l}{n}\right)^{2q}\right), \quad B_2 = O_p\left(l^{-\frac{1}{2}}\right), \quad B_3 = o_p(1), \quad \text{(B.24)}$$

$$B_4 = O_p\left(l^{-\theta/2} 1(\theta < 2) + l^{-1} \log l 1(\theta = 2) + l^{\gamma - p} \log^{1/2} l\right), \quad \text{(B.25)}$$

whereas

$$B_0 = \frac{f_{11}(0)p}{l} \sum{}' k_{q,j} = f_{11}(0) + O\left(l^{-1} \log l\right), \quad \text{(B.26)}$$

to complete the proof of (B.4).

Next, we check condition (iv) (a) of Wooldridge (1994), which in our framework is

$$l^{-1} H_v^p(\gamma) \to_p 4V_q > 0. \quad \text{(B.27)}$$

Clearly,

$$l^{-1} H_v^p(\gamma) = \frac{4\left(F_{2,v}^p(c) F_{0,v}^p(c) - \left(F_{1,v}^p(c)\right)^2\right)}{\left(F_{0,v}^p(c)\right)^2}, \quad \text{(B.28)}$$

where

$$F_{g,v}^p(c) = \frac{p}{l} \sum{}' k_{q,j} \left(\log \frac{j}{l}\right)^g \lambda_j^{2c} I_v^p(\lambda_j), \ g = 0, 1, 2, \quad \text{(B.29)}$$

and by the same decomposition as in the treatment of B, it is easy to show that

$$F_{g,v}^p(\gamma) \to_p f_{11}(0) \int_0^1 k_q(u) (\log u)^g du, \ g = 0, 1, 2, \quad \text{(B.30)}$$

so that (B.27) follows immediately.

Finally, the proof for $\overline{\gamma}$ follows on showing that condition (iii) (b) in Wooldridge (1994) holds, that is

$$\sup_{c\in N_\gamma} c_n^{-1} \left|H_v^p(c) - H_v^p(\gamma)\right| = o_p(1), \tag{B.31}$$

where $N_\gamma = \left\{c \in \Theta, c_n^{1/2}|c-\gamma| \le 1\right\}$, for a sequence of positive numbers c_n increasing with n, such that $c_n/l \to 0$ as $n \to \infty$. We specify this sequence later. Now (B.31) holds on showing

$$\sup_{c\in N_\gamma} c_n^{-1} \left|\sum{}' k_{q,j}\left(\log\frac{j}{l}\right)^g \left(\lambda_j^{2c} - \lambda_j^{2\gamma}\right) I_v^p(\lambda_j)\right| = o_p(1), \tag{B.32}$$

for $g = 0, 1, 2$. By the mean value theorem, the term inside the modulus in (B.32) equals

$$2(c-\gamma)\sum{}' k_{q,j}\left(\log\frac{j}{l}\right)^g \log\lambda_j \lambda_j^{2(\overline{c}-\gamma)}\lambda_j^{2\gamma} I_v^p(\lambda_j), \tag{B.33}$$

where $|\overline{c}-\gamma| \le |c-\gamma|$. By Theorem 6 of Velasco (1999b), under our conditions $E\left|\lambda_j^{2\gamma} I_v^p(\lambda_j)\right| \le K$, so the expectation of the absolute value of (B.33) is bounded by

$$K|c-\gamma|\sum{}'\left|\log\frac{j}{l}\right|^g |\log\lambda_j|\,\lambda_j^{-2|\overline{c}-\gamma|}, \tag{B.34}$$

so the left side of (B.32) is bounded by

$$\sup_{c\in N_\gamma}\frac{|c-\gamma|}{c_n}(\log n)^3\sum{}'|\lambda_j|^{-\frac{2}{\sqrt{c_n}}} \le c_n^{-3/2}(\log n)^3 n^{\frac{2}{\sqrt{c_n}}}\sum{}' j^{-\frac{2}{\sqrt{c_n}}} = O\left((\log n)^3 l c_n^{-3/2}\right), \tag{B.35}$$

to conclude the proof of the Proposition for $\overline{\gamma}$ on setting c_n such that

$$\frac{(\log n)^3 l}{c_n^{3/2}} \to 0 \text{ as } n\to\infty. \tag{B.36}$$

The proof for $\widetilde{\gamma}$ holds on showing that

$$l^{-\frac{1}{2}}\left(S_v^p(\gamma) - S_{\widetilde{v}}^p(\gamma)\right) = o_p(1), \tag{B.37}$$

$$l^{-1}\left(H_v^p(\gamma) - H_{\widetilde{v}}^p(\gamma)\right) = o_p(1), \tag{B.38}$$

$$\sup_{c\in N_\gamma} c_n^{-1}\left|H_{\widetilde{v}}^p(c) - H_{\widetilde{v}}^p(\gamma)\right| = o_p(1), \tag{B.39}$$

$$\widetilde{\gamma}_G - \gamma = o_p\left(\log^{-5} l\right). \tag{B.40}$$

First, (B.37) follows if

$$l^{-\frac{1}{2}}\sum{}' b_{q,j}\lambda_j^{2\gamma}\left(I_v^p(\lambda_j) - I_{\widetilde{v}}^p(\lambda_j)\right) = o_p(1). \tag{B.41}$$

The expectation of the absolute value of the left side of (B.41) is bounded by

$$Kl^{-\frac{1}{2}}\log n\sum{}'\left\{E\left(\lambda_j^{2\gamma}\left|w_v^p(\lambda_j) - w_{\widetilde{v}}^p(\lambda_j)\right|^2\right)E\left(\lambda_j^{2\gamma}\left|w_v^p(\lambda_j) + w_{\widetilde{v}}^p(\lambda_j)\right|^2\right)\right\}^{\frac{1}{2}}, \tag{B.42}$$

which, by the Theorem of Robinson (2002) and results in Velasco (1999b) is

$$O\left(l^{-\frac{1}{2}}\log n\sum{}' j^{\gamma-r-1}\right)=O\left(l^{-\frac{1}{2}}\log n\left(l^{\gamma-r}1\left(\gamma>r\right)+\log l1\left(\gamma=r\right)+1\left(\gamma<r\right)\right)\right), \tag{B.43}$$

which is $o\left(1\right)$ by (4.9), since $\gamma-r<1/2$. Similarly, (B.38) holds because

$$\begin{aligned}&l^{-1}\sum{}' k_{q,j}\left(\log\lambda_j\right)^g\lambda_j^{2\gamma}\left(I_v^p\left(\lambda_j\right)-I_{\widetilde{v}}^p\left(\lambda_j\right)\right)\\=\;&O\left(l^{-1}\log^2 n\left(l^{\gamma-r}1\left(\gamma>r\right)+\log l1\left(\gamma=r\right)+1\left(\gamma<r\right)\right)\right)=o\left(1\right),\end{aligned} \tag{B.44}$$

by (4.9). Next, (B.39) holds if

$$\sup_{c\in N_\gamma}c_n^{-1}\left|\sum{}' k_{q,j}\left(\log\frac{j}{l}\right)^g\left(\lambda_j^{2c}-\lambda_j^{2\gamma}\right)I_{\widetilde{v}}^p\left(\lambda_j\right)\right|=o_p\left(1\right), \tag{B.45}$$

which by the treatment of (B.32) is implied by

$$\sup_{c\in N_\gamma}c_n^{-1}\left|c-\gamma\right|\sum{}'\left|\log\frac{j}{l}\right|^g\left|\log\lambda_j\right|\lambda_j^{-2\left|\overline{c}-\gamma\right|}\lambda_j^{2\gamma}\left|I_v^p\left(\lambda_j\right)-I_{\widetilde{v}}^p\left(\lambda_j\right)\right|=o_p\left(1\right), \tag{B.46}$$

where $\left|\overline{c}-\gamma\right|\leq\left|c-\gamma\right|$. Proceeding as in (B.34), the left side of (B.46) is

$$O\left(c_n^{-3/2}\left(\log n\right)^3\left(l^{\gamma-r}1\left(\gamma>r\right)+\log l1\left(\gamma=r\right)+1\left(\gamma<r\right)\right)\right)=o_p\left(1\right), \tag{B.47}$$

on setting c_n as in (B.36).

Following Robinson (1995a) and Velasco (1999a), we set $\Theta=\Theta_1\cup\Theta_2$, with

$$\Theta_1=\left\{c:\gamma-1/2+\epsilon\leq c\leq\nabla_2\right\},\quad\Theta_2=\left\{c:\nabla_1\leq c<\gamma-1/2+\epsilon\right\}, \tag{B.48}$$

for $\epsilon\in(0,1/4)$ (taking Θ_2 to be empty in case $\nabla_1\geq\gamma-1/2+\epsilon$), in order to show (B.40). Considering that the bandwidth associated with $\widetilde{\gamma}_G$ is d, we show first that $\widetilde{\gamma}_G-\gamma=o_p\left(d^{-1/2}\right)$, so that (B.40) follows on setting d as a multiple of a power root of n. The main steps consist of establishing

$$\sup_{c\in\Theta_1}\left|\frac{G_v^p\left(c\right)-G_{\widetilde{v}}^p\left(c\right)}{G^p\left(c\right)}\right|=o_p\left(\log^{-10}d\right), \tag{B.49}$$

where

$$G_\xi^p\left(c\right)=\frac{p}{d}\sum{}''\lambda_j^{2c}I_\xi^p\left(\lambda_j\right),\;G^p\left(c\right)=f_{11}\left(0\right)\frac{p}{d}\sum{}''\lambda_j^{2\left(c-\gamma\right)}, \tag{B.50}$$

where throughout $\sum{}''=\sum_{j=p,2p,..}^d$, and also

$$\Pr\left(\inf_{\Theta_2}S\left(c\right)\leq0\right)\to0\text{ as }n\to\infty, \tag{B.51}$$

where

$$S\left(c\right)=\log\frac{G_{\widetilde{v}}^p\left(c\right)}{G_{\widetilde{v}}^p\left(\gamma\right)}-2\left(c-\gamma\right)\frac{p}{d}\sum{}''\log\lambda_j. \tag{B.52}$$

We first show (B.49). Now

$$\frac{G_v^p(c)-G_{\widetilde{v}}^p(c)}{G^p(c)}=\frac{\frac{p}{d}\sum''\left(\frac{j}{d}\right)^{2(c-\gamma)}\lambda_j^{2\gamma}\left(I_v^p(\lambda_j)-I_{\widetilde{v}}^p(\lambda_j)\right)}{f_{11}(0)\frac{p}{d}\sum''\left(\frac{j}{d}\right)^{2(c-\gamma)}}, \tag{B.53}$$

so that as in the proof of Theorem 5 of Velasco (1999a), the left side of (B.49) is bounded by

$$\begin{aligned}&K\sup_{c\in\Theta_1}\frac{p}{d}\sum''\left(\frac{j}{d}\right)^{2(c-\gamma)}\lambda_j^{2\gamma}\left|I_v^p(\lambda_j)-I_{\widetilde{v}}^p(\lambda_j)\right|\leq K\frac{p}{d}\sum''\left(\frac{j}{d}\right)^{-1+2\epsilon}\lambda_j^{2\gamma}\left|I_v^p(\lambda_j)-I_{\widetilde{v}}^p(\lambda_j)\right|\\ =\;&O_p\left(d^{-1}\sum''\left(\frac{j}{d}\right)^{-1+2\epsilon}j^{(\gamma-r)-1}\right)=O_p\left(d^{-2\epsilon}\right),\end{aligned} \tag{B.54}$$

by Robinson (2002), since $\gamma-r<1/2$ and $\epsilon<1/4$. Next, we show (B.51). Setting $z=\exp\left(pd^{-1}\sum''\log j\right)$, we have

$$\begin{aligned}\Pr\left(\inf_{\Theta_2}S(c)\leq 0\right)&=\Pr\left(\inf_{\Theta_2}\frac{p}{d}\sum''\left[\left(\frac{j}{z}\right)^{2(c-\gamma)}-1\right]\lambda_j^{2\gamma}I_{\widetilde{v}}^p(\lambda_j)\leq 0\right)\\&\leq\Pr\left(\frac{p}{d}\sum''\left[a_j-1\right]\lambda_j^{2\gamma}I_{\widetilde{v}}^p(\lambda_j)\leq 0\right),\end{aligned} \tag{B.55}$$

where

$$a_j=\begin{cases}\left(\frac{j}{z}\right)^{-1+2\epsilon}, & 1\leq j\leq z,\\ \left(\frac{j}{z}\right)^{2(\triangledown_1-\gamma)}, & z<j\leq d.\end{cases} \tag{B.56}$$

Now (B.55) is $o(1)$ by showing

$$\frac{p}{d}\sum''(a_j-1)\lambda_j^{2\gamma}\left(I_{\widetilde{v}}^p(\lambda_j)-I_v^p(\lambda_j)\right)=o_p(1). \tag{B.57}$$

By the Theorem of Robinson (2002) the left side of (B.57) is bounded by

$$Kd^{-1}\sum_{j=p,2p,..}^{z}\left(\frac{j}{z}\right)^{-1+2\epsilon}j^{\gamma-r-1}+Kd^{-1}\sum_{j=z+p,z+2p,..}^{d}\left(\frac{j}{z}\right)^{2(\triangledown_1-\gamma)}j^{\gamma-r-1}+Kd^{-1}\sum''j^{\gamma-r-1}, \tag{B.58}$$

which is $O\left(d^{-2\epsilon}+d^{-1+\gamma-r}+d^{-1}\log d\right)=o(1)$, on setting $\epsilon<(r-\gamma+1)/2$, to conclude the proof.

Proof of Proposition 2

For $\beta>1/2$, the proof is very similar to that for $\widetilde{\gamma}$. In fact, (4.15) follows on showing

$$l^{-\frac{1}{2}}\left(S_{\widetilde{v}}^p(\gamma)-S_{\widehat{v}}^p(\gamma)\right)=o_p(1), \tag{B.59}$$

$$l^{-1}\left(H_{\widetilde{v}}^p(\gamma)-H_{\widehat{v}}^p(\gamma)\right)=o_p(1), \tag{B.60}$$

$$\sup_{c\in N_\gamma}c_n^{-1}\left|H_{\widehat{v}}^p(c)-H_{\widehat{v}}^p(\gamma)\right|=o_p(1), \tag{B.61}$$

$$\widehat{\gamma}_G-\gamma=o_p\left(\log^{-5}l\right). \tag{B.62}$$

These results follow as easily as corresponding ones in Proposition 1, noting that in the present case

$$\left|I^p_{\widehat{v}}(\lambda_j) - I^p_{\bar{v}}(\lambda_j)\right| \le |\widehat{\nu} - \nu| \left|w^p_x(\lambda_j)\, w^p_{\bar{v}}(-\lambda_j)\right| + (\widehat{\nu} - \nu)^2 \left|w^p_x(\lambda_j)\right|^2. \quad \text{(B.63)}$$

The only point worth stressing is the proof of (B.59). By (B.63), the left side of (B.59) is bounded by

$$Kl^{-\frac{1}{2}} \log n \left(|\widehat{\nu} - \nu| \sum{}' \lambda_j^{\gamma-\delta} + (\widehat{\nu} - \nu)^2 \sum{}' \lambda_j^{2(\gamma-\delta)} \right), \quad \text{(B.64)}$$

so that the left side of (B.59) is $O_p\left(l^{-1/2}\log n\left(\left(n^{\psi}l^{1-\beta} + n^{2\psi}\right)1\left(1/2 < \beta < 1\right) + n^{2\psi}1\left(\beta \ge 1\right)\right)\right) = o_p(1)$ on setting ψ and l such that

$$\frac{n^{2\psi}\log n}{l^{1/2}} + \frac{n^{\psi}\log n}{l^{\beta-1/2}} \to 0 \text{ as } n \to \infty. \quad \text{(B.65)}$$

For $\beta < 1/2$ the proof is also very similar. The main difference is that now (B.59) does not hold, because its left side is $O_p\left(l^{-1/2}\log n\left(n^{\psi}l^{1-\beta} + n^{2\psi}\right)\right) = O_p\left(l^{1/2-\beta+\varphi}\right)$, on setting l, ψ, φ, such that

$$\frac{n^{\psi}\log n}{l^{\varphi}} + \frac{n^{2\psi}\log n}{l^{1/2}} \to 0 \text{ as } n \to \infty. \quad \text{(B.66)}$$

Proof of Proposition 3

First,

$$\widehat{f}(\lambda) = \frac{1}{2\pi}\sum_{s=1-n}^{n-1} g\left(\frac{s}{b}\right)\widehat{\Gamma}(s)\, e^{-is\lambda} + \widehat{Q}(\lambda), \quad \text{(B.67)}$$

where

$$\widehat{\Gamma}(s) = \frac{1}{n}\sum_{t=1}^{n-s} \widehat{u}_t \widehat{u}'_{t+s},\ s \ge 0; \quad = \widehat{\Gamma}'(-s),\ s < 0. \quad \text{(B.68)}$$

Now $\widehat{u}_t = u_t + r_t$, where

$$r_t = \left[\left(u_{1t}\left(\widehat{\gamma} - \gamma\right) - u_{1t}\right) - \left(\widehat{\nu} - \nu\right) x_t(\widehat{\gamma}), \left(u_{2t}\left(\widehat{\delta} - \delta\right) - u_{2t}\right)\right]'. \quad \text{(B.69)}$$

Define $\widehat{\Gamma}(s) = \overline{\Gamma}(s) + \widehat{R}(s)$, where

$$\overline{\Gamma}(s) = \frac{1}{n}\sum_{t=1}^{n-s} u_t u'_{t+s},\ s \ge 0; \quad = \overline{\Gamma}'(-s),\ s < 0, \quad \text{(B.70)}$$

and

$$\widehat{R}(s) = \frac{1}{n}\sum_{t=1}^{n-s}\left\{r_t r'_{t+s} + u_t r'_{t+s} + r_t u'_{t+s}\right\},\ s \ge 0; \quad = \widehat{R}'(-s),\ s < 0, \quad \text{(B.71)}$$

and split the first term on the right of (B.67) as $\overline{f}(\lambda) + \widehat{g}(\lambda)$ where

$$\overline{f}(\lambda) = \frac{1}{2\pi}\sum_{s=1-n}^{n-1} g\left(\frac{s}{b}\right)\overline{\Gamma}(s)\, e^{-is\lambda},\ \ \widehat{g}(\lambda) = \frac{1}{2\pi}\sum_{s=1-n}^{n-1} g\left(\frac{s}{b}\right)\widehat{R}(s)\, e^{-is\lambda}. \quad \text{(B.72)}$$

By arguments similar to those of the proof of Theorem, $r_t = r_{1t} + r_{2t} + d_t$, where

$$r_{1t} = \begin{pmatrix} (\gamma - \widehat{\gamma}) & 0 \\ 0 & (\delta - \widehat{\delta}) \end{pmatrix} \sum_{j=1}^{t-1} a_j'(0) u_{t-j}; \quad r_{2t} = (\widehat{\nu} - \nu) \begin{pmatrix} x_t(\gamma) & 0 \end{pmatrix}', \tag{B.73}$$

and d_t involves terms of smaller order. Hence, the order of magnitude of $\widehat{R}(s)$ is given by the order of $n^{-1} \sum_{t=1}^{n-s} a_t b_{t+s}'$, for the different combinations of $a_t, b_t = r_{it}, u_t$, $i = 1, 2$. Thus, in view of the conditions specified in Proposition 4 and Lemmas B.1, B.2, B.3 of RH, uniformly in s,

$$\widehat{R}(s) = O_p\left(n^{-\kappa(s)} + n^{-\min\{\beta,1\}+\psi}\right) = O_p\left(n^{-\kappa(s)}\right), \tag{B.74}$$

for $\psi < \min\{\beta, 1\} - \kappa(s)$. Then, as $b^{-1} \sum_{s=1-n}^{n-1} |g(s/b)| = O(1)$ by (4.22) and $\kappa(s) < 1/2$, (B.74) readily implies that uniformly in λ, $\widehat{g}(\lambda) = O_p\left(bn^{-\kappa(s)}\right)$. By Theorem 5A and a straightforward modification of Theorem 5B of Parzen (1957),

$$\overline{f}(\lambda) - f(\lambda) = O_p\left(b^{-\min\{h,s\}} + (b/n)^{\frac{1}{2}}\right), \tag{B.75}$$

for covariance averaging kernels g satisfying (4.23), so that the first term on the right of (B.67) is

$$f(\lambda) + O_p\left(b^{-s} + \left(\frac{b}{n}\right)^{\frac{1}{2}} + bn^{-\kappa(s)}\right), \tag{B.76}$$

as the kernel is chosen such that $h \geq s$.

Finally, as in the proof of Theorem 2.1 of Robinson (1991), $\widehat{Q}(\lambda)$ is bounded in norm by

$$K \sum_{s=1-n}^{n-1} \left| g\left(\frac{s}{b}\right) \right| \left\| \widehat{\Gamma}(n-s) \right\| \leq K \sum_{s=1-n}^{n-1} \left| g\left(\frac{s}{b}\right) \right| \left\| \overline{\Gamma}(n-s) \right\| + K \sum_{s=1-n}^{n-1} \left| g\left(\frac{s}{b}\right) \right| \left\| \widehat{R}(n-s) \right\|, \tag{B.77}$$

uniformly in $\lambda \in [-\pi, \pi]$. The second term on the right side of (B.77) can be treated as in the analysis of $\widehat{g}(\lambda)$, whereas the first term is

$$O_p\left(n^{-1} \sum_{s=1-n}^{n-1} \left| sg\left(\frac{s}{b}\right) \right| \right) = O_p\left(n^{-1} b^2\right), \tag{B.78}$$

by (4.22), since $\overline{\Gamma}(n-s)$ is a sum of s terms whose mean exists and is uniformly bounded, implying by (B.76) that

$$\widehat{f}(\lambda) - f(\lambda) = O_p\left(b^{-s} + \left(\frac{b}{n}\right)^{\frac{1}{2}} + bn^{-\kappa(s)} + n^{-1} b^2\right). \tag{B.79}$$

In order to find the "optimal" rate for b, let $b \sim n^{\alpha}$ for some $\alpha > 0$. Clearly, $n^{\alpha - \kappa(s)}$ dominates $n^{(\alpha-1)/2}$ for any s, whereas $n^{\alpha-\kappa(s)}$ and $n^{2\alpha-1}$ share the same rate for $\alpha = (1+s)/(1+2s)$, which is not a sensible choice as we seek estimates of $f(\lambda)$ which could approach the parametric rate $n^{-1/2}$ for s large enough. For $\alpha < (1+s)/(1+2s)$, again $n^{\alpha-\kappa(s)}$ dominates $n^{2\alpha-1}$. Finally, noting that $n^{-\alpha s}$ and

$n^{\alpha-\kappa(s)}$ move in opposite directions when α changes, it is clear that the value of α which maximizes the rate of convergence should satisfy -$\alpha s = \alpha - \kappa(s)$, i.e. $\alpha^* = \kappa(s)/(1+s)$, to conclude the proof.

Proof of Proposition 4

Considering first the contribution of the first term on the right of (B.67), we have

$$\widehat{g}_{j+1} - \widehat{g}_j = \frac{1}{2\pi} \sum_{r=1-n}^{n-1} g\left(\frac{r}{b}\right) \widehat{R}(r) \left(e^{-ir\lambda_{j+1}} - e^{-ir\lambda_j}\right), \tag{B.80}$$

which is bounded in norm, uniformly in $j \in [1, n]$, by

$$Kn^{-1} \sum_{r=1-n}^{n-1} \left| rg\left(\frac{r}{b}\right)\right| \left\|\widehat{R}(r)\right\| \leq Kn^{-1}b^2 \sum_{r=1-n}^{n-1} \frac{|r|}{b^2} \left|g\left(\frac{r}{b}\right)\right| \left\|\widehat{R}(r)\right\|, \tag{B.81}$$

as

$$\max_j |\exp(ir\lambda_j) - \exp(ir\lambda_{j+1})| \leq |r| n^{-1}. \tag{B.82}$$

Thus, by (B.74), (4.22), uniformly in $j \in [1, n]$,

$$\widehat{g}_{j+1} - \widehat{g}_j = O_p\left(n^{-1}b^2 n^{-\kappa(s)}\right). \tag{B.83}$$

Next, defining $a_{1j} = \overline{f}_j - E\overline{f}_j$, $a_{2j} = E\overline{f}_j - f_j$, $a_{1,j+1} - a_{1j}$ is bounded in norm by

$$K \left\| \sum_{r=1-n}^{n-1} g\left(\frac{r}{b}\right) \left(\overline{\Gamma}(r) - E\overline{\Gamma}(r)\right) \left(e^{-ir\lambda_{j+1}} - e^{-ir\lambda_j}\right) \right\|, \tag{B.84}$$

which is, uniformly in j, $O_p(b^2 n^{-3/2})$ by (B.82) since (4.22) implies that $b^{-2}\sum_{r=1-n}^{n-1} |rg(r/b)| = O(1)$, and ε_t being an iid sequence with finite fourth moment and $f(\lambda)$ being continuous readily implies that, uniformly in r, $\overline{\Gamma}(r) - E\overline{\Gamma}(r) = O_p\left(n^{-1/2}\right)$. Next, by (B.82), $a_{2,j+1} - a_{2j}$ is, uniformly in j, bounded in norm by

$$Kn^{-1} \sum_{r=1-n}^{n-1} \left|1 - g\left(\frac{r}{b}\right)\right| |r| \|\Gamma(r)\| + Kn^{-2} \sum_{r=1-n}^{n-1} r^2 \left|g\left(\frac{r}{b}\right)\right| \|\Gamma(r)\| + Kn^{-1} \sum_{|r|\geq n} |r| \|\Gamma(r)\|. \tag{B.85}$$

The third term is bounded by

$$Kn^{-s} \sum_{|r|\geq n} |r|^s \|\Gamma(r)\| = o\left(n^{-s}\right), \tag{B.86}$$

as Assumption P5 implies $\sum_{r=-\infty}^{\infty} |r|^s \|\Gamma(r)\| < \infty$. As again by Assumption P5, $\sup_r |r| \|\Gamma(r)\| \leq K$, the second term is bounded by

$$Kn^{-2} \sum_{r=1-n}^{n-1} |r| \left|g\left(\frac{r}{b}\right)\right| = O\left(b^2 n^{-2}\right), \tag{B.87}$$

from (4.22). Finally, the first term in (B.85) is bounded by

$$Kn^{-1}\sum_{|r|<\varepsilon b}\left|1-g\left(\frac{r}{b}\right)\right||r|\left\|\Gamma\left(r\right)\right\|+Kn^{-1}\sum_{|r|\geq\varepsilon b}\left|1-g\left(\frac{r}{b}\right)\right||r|\left\|\Gamma\left(r\right)\right\|, \tag{B.88}$$

for $\varepsilon\in(0,\epsilon)$, where ϵ is given in Assumption P5. Letting $i=h-s$, the first term in (B.88) is bounded by

$$Kn^{-1}b^{1-s}b^{-(i+1)}\sum_{|r|<\varepsilon b}|r|^{1+s+i}\left\|\Gamma\left(r\right)\right\|=o\left(n^{-1}b^{1-s}\right), \tag{B.89}$$

from Lemma 4 of Parzen (1957). The second term in (B.88) is bounded by

$$Kn^{-1}\sum_{|r|\geq\varepsilon b}\frac{|r|^{s}}{(b\varepsilon)^{s-1}}\left\|\Gamma\left(r\right)\right\|=o\left(n^{-1}b^{1-s}\right), \tag{B.90}$$

to conclude as in (B.89). Finally, $\widehat{Q}_{j+1}-\widehat{Q}_{j}$ is bounded in norm by

$$Kn^{-1}\sum_{r=1-n}^{n-1}\left|rg\left(\frac{r}{b}\right)\right|\left\|\overline{\Gamma}\left(n-r\right)\right\|+Kn^{-1}\sum_{r=1-n}^{n-1}\left|rg\left(\frac{r}{b}\right)\right|\left\|\widehat{R}\left(n-r\right)\right\|, \tag{B.91}$$

uniformly in j. The second term in (B.91) can be treated as (B.81), whereas by the previous analysis, the first term is

$$O_{p}\left(n^{-2}\sum_{r=1-n}^{n-1}r^{2}\left|g\left(\frac{r}{b}\right)\right|\right)=O_{p}\left(n^{-2}b^{3}\right), \tag{B.92}$$

by (4.22). Clearly, $b^{2}n^{-1-\kappa(s)}$ dominates $b^{2}n^{-3/2}$, $b^{2}n^{-2}$ and also $b^{3}n^{-2}$, since this last rate and $b^{2}n^{-1-\kappa(s)}$ are only equal for $b\sim n^{(1+s)/(1+2s)}$, when $\beta<1/2$, $b\sim n^{(1+2s(1-(\beta-\varphi)))/(1+2s)}$. Also, $n^{-1}b^{1-s}$ dominates n^{-s}, equating it only when $s=1$.

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TABLE 3

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_I^{\mathrm{o}}, \overline{\nu}_F, \overline{\nu}_F^{\mathrm{o}}, \overline{\nu}_U$ FOR $m = I$

	n	64				128				256			
	γ, δ	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$
WN	$\overline{\nu}_I$	-.003	-.001	-.002	.000	-.001	.000	-.001	.000	.000	.000	.000	.000
	$\overline{\nu}_I^{\mathrm{o}}$	-.003	-.001	-.003	.000	-.001	.000	-.001	.000	.000	.000	.000	.000
	$\overline{\nu}_F$	-.005	-.001	-.004	.000	-.002	.000	-.001	.000	.000	.000	.000	.000
	$\overline{\nu}_F^{\mathrm{o}}$	-.006	-.001	-.004	.000	-.001	.000	-.001	.000	.000	.000	.000	.000
	$\overline{\nu}_U$	-.005	-.001	-.007	.000	-.002	.000	-.003	.000	.000	.000	-.001	.000
AR	$\overline{\nu}_I$	.041	.001	.007	.000	.029	.000	.004	.000	.019	.000	.001	.000
	$\overline{\nu}_I^{\mathrm{o}}$	.039	.001	.007	.000	.027	.000	.003	.000	.018	.000	.001	.000
	$\overline{\nu}_F$	.063	-.002	.019	-.001	.045	-.001	.010	.000	.029	.000	.004	.000
	$\overline{\nu}_F^{\mathrm{o}}$	.061	-.002	.019	-.001	.042	-.001	.010	.000	.026	.000	.004	.000
	$\overline{\nu}_U$	.101	.002	.032	.000	.082	.001	.019	.000	.066	.000	.011	.000
MA	$\overline{\nu}_I$	-.045	-.001	-.006	.000	-.031	.000	-.003	.000	-.022	.000	-.001	.000
	$\overline{\nu}_I^{\mathrm{o}}$	-.041	-.001	-.006	.000	-.027	.000	-.002	.000	-.020	.000	-.001	.000
	$\overline{\nu}_F$	-.065	.001	-.017	.001	-.043	.001	-.007	.000	-.028	.001	-.002	.000
	$\overline{\nu}_F^{\mathrm{o}}$	-.063	.001	-.018	.001	-.040	.001	-.007	.000	-.025	.001	-.002	.000
	$\overline{\nu}_U$	-.112	-.003	-.034	.000	-.090	-.001	-.020	.000	-.074	.000	-.011	.000

TABLE 4

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_I^{\mathrm{o}}, \overline{\nu}_F, \overline{\nu}_F^{\mathrm{o}}, \overline{\nu}_U$ FOR $m = III$

	n	64				128				256			
	γ, δ	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$
WN	$\overline{\nu}_I$	-.003	-.001	-.003	.000	-.001	.000	-.001	.000	.000	.000	.000	.000
	$\overline{\nu}_I^{\mathrm{o}}$	-.003	-.001	-.003	.000	.000	.000	-.001	.000	.000	.000	.000	.000
	$\overline{\nu}_F$	-.005	-.001	-.005	.000	-.002	.000	-.002	.000	.000	.000	.000	.000
	$\overline{\nu}_F^{\mathrm{o}}$	-.006	-.001	-.006	.000	-.001	-.001	-.002	.000	.001	.000	.000	.000
	$\overline{\nu}_U$	-.005	-.001	-.007	.000	-.002	.000	-.003	.000	.000	.000	-.001	.000
AR	$\overline{\nu}_I$	.052	.001	.009	.000	.037	.000	.004	.000	.025	.000	.002	.000
	$\overline{\nu}_I^{\mathrm{o}}$	.046	.001	.007	.000	.032	.000	.004	.000	.021	.000	.001	.000
	$\overline{\nu}_F$	.082	-.001	.024	-.001	.057	-.001	.012	.000	.038	.000	.005	.000
	$\overline{\nu}_F^{\mathrm{o}}$	.089	.000	.029	-.001	.058	-.001	.013	.000	.037	.000	.005	.000
	$\overline{\nu}_U$	.121	.003	.033	.000	.097	.001	.020	.000	.078	.000	.011	.000
MA	$\overline{\nu}_I$	-.062	-.001	-.009	.000	-.043	.000	-.004	.000	-.030	.000	-.001	.000
	$\overline{\nu}_I^{\mathrm{o}}$	-.053	-.001	-.007	.000	-.036	.000	-.003	.000	-.026	.000	-.001	.000
	$\overline{\nu}_F$	-.087	.000	-.023	.001	-.058	.001	-.010	.000	-.038	.000	-.003	.000
	$\overline{\nu}_F^{\mathrm{o}}$	-.088	.000	-.025	.001	-.057	.001	-.011	.000	-.037	.000	-.003	.000
	$\overline{\nu}_U$	-.145	-.004	-.036	.000	-.118	-.001	-.020	.000	-.099	.000	-.011	.000

TABLE 5

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_I^o, \overline{\nu}_F, \overline{\nu}_F^o, \overline{\nu}_U$ FOR $m = I$

	n	64				128				256			
	γ, δ	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$
WN	$\overline{\nu}_I$	.111	.026	.072	.009	.065	.011	.037	.003	.040	.004	.020	.001
	$\overline{\nu}_I^o$	.110	.025	.070	.009	.065	.010	.037	.003	.039	.004	.020	.001
	$\overline{\nu}_F$	.115	.030	.077	.010	.068	.012	.041	.003	.042	.005	.021	.001
	$\overline{\nu}_F^o$	.114	.030	.077	.010	.068	.011	.041	.003	.041	.005	.021	.001
	$\overline{\nu}_U$	.098	.025	.080	.011	.058	.010	.046	.003	.036	.004	.025	.001
AR	$\overline{\nu}_I$	.099	.021	.060	.008	.060	.009	.032	.002	.036	.004	.017	.001
	$\overline{\nu}_I^o$	.098	.021	.060	.008	.059	.009	.032	.002	.036	.004	.017	.001
	$\overline{\nu}_F$	.108	.030	.076	.010	.065	.013	.042	.003	.041	.005	.023	.001
	$\overline{\nu}_F^o$	.110	.031	.078	.011	.065	.013	.043	.003	.041	.005	.023	.001
	$\overline{\nu}_U$	.093	.022	.070	.010	.060	.009	.041	.003	.039	.004	.022	.001
MA	$\overline{\nu}_I$	.093	.020	.058	.007	.058	.009	.031	.003	.037	.004	.017	.001
	$\overline{\nu}_I^o$	.092	.020	.057	.007	.058	.009	.031	.003	.036	.004	.017	.001
	$\overline{\nu}_F$	.102	.025	.068	.008	.064	.011	.038	.003	.039	.005	.021	.001
	$\overline{\nu}_F^o$	.102	.026	.069	.009	.064	.012	.039	.003	.040	.005	.022	.001
	$\overline{\nu}_U$	.088	.021	.066	.009	.059	.009	.040	.003	.041	.004	.023	.001

TABLE 6

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_I^o, \overline{\nu}_F, \overline{\nu}_F^o, \overline{\nu}_U$ FOR $m = III$

	n	64				128				256			
	γ, δ	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$	$0, .6$	$0, 1.2$	$.4, 1.2$	$.4, 2$
WN	$\overline{\nu}_I$	.103	.026	.071	.009	.061	.010	.037	.003	.038	.004	.020	.001
	$\overline{\nu}_I^o$	.108	.025	.073	.009	.068	.010	.039	.003	.042	.004	.021	.001
	$\overline{\nu}_F$	.105	.029	.075	.009	.062	.011	.041	.003	.039	.005	.021	.001
	$\overline{\nu}_F^o$	.114	.030	.087	.009	.069	.012	.048	.003	.044	.005	.024	.001
	$\overline{\nu}_U$	.086	.025	.078	.010	.052	.010	.046	.003	.033	.004	.025	.001
AR	$\overline{\nu}_I$	.097	.021	.061	.008	.060	.009	.033	.002	.037	.004	.017	.001
	$\overline{\nu}_I^o$	.097	.021	.059	.008	.059	.009	.032	.002	.036	.004	.017	.001
	$\overline{\nu}_F$	.097	.029	.068	.010	.061	.012	.037	.003	.039	.005	.021	.001
	$\overline{\nu}_F^o$	.110	.030	.075	.010	.066	.013	.040	.003	.043	.005	.022	.001
	$\overline{\nu}_U$	.092	.022	.069	.010	.062	.009	.040	.003	.042	.004	.022	.001
MA	$\overline{\nu}_I$	.090	.020	.057	.007	.060	.009	.032	.003	.039	.004	.018	.001
	$\overline{\nu}_I^o$	.091	.020	.057	.007	.059	.009	.032	.003	.039	.004	.018	.001
	$\overline{\nu}_F$	.096	.023	.063	.008	.064	.011	.037	.003	.041	.005	.020	.001
	$\overline{\nu}_F^o$	.105	.024	.068	.008	.071	.011	.039	.003	.045	.005	.021	.001
	$\overline{\nu}_U$	.089	.021	.066	.009	.064	.009	.040	.003	.048	.004	.022	.001

TABLE 7

EMPIRICAL SIZES OF W_I AND W_F FOR $m = I$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.106	.171	.096	.146	.083	.121	.175	.222	.146	.206	.140	.191
WN	0	1.2	.075	.190	.062	.147	.052	.125	.144	.247	.123	.195	.120	.169
	.4	1.2	.094	.179	.078	.139	.061	.122	.157	.244	.132	.198	.116	.165
	.4	2	.062	.172	.072	.154	.066	.118	.125	.222	.131	.198	.122	.176
	0	.6	.256	.202	.222	.206	.203	.173	.360	.274	.319	.286	.290	.259
AR	0	1.2	.190	.103	.142	.078	.128	.053	.267	.149	.225	.101	.204	.084
	.4	1.2	.200	.155	.154	.117	.121	.100	.285	.217	.241	.187	.203	.162
	.4	2	.189	.079	.142	.065	.126	.037	.273	.113	.217	.090	.193	.069
	0	.6	.179	.220	.155	.238	.158	.213	.246	.306	.247	.317	.231	.271
MA	0	1.2	.093	.156	.098	.155	.069	.112	.167	.211	.155	.198	.124	.150
	.4	1.2	.126	.189	.095	.176	.078	.147	.195	.276	.166	.241	.148	.205
	.4	2	.096	.131	.091	.130	.080	.091	.151	.172	.160	.181	.128	.137

TABLE 8

EMPIRICAL SIZES OF W_I^o AND W_F^o FOR $m = I$

		α	.05						.10					
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o
	0	.6	.103	.177	.090	.143	.078	.117	.175	.224	.151	.203	.133	.180
WN	0	1.2	.073	.192	.065	.149	.050	.126	.137	.247	.122	.194	.122	.172
	.4	1.2	.091	.177	.077	.134	.053	.120	.163	.239	.126	.192	.111	.165
	.4	2	.064	.169	.075	.156	.065	.119	.123	.224	.128	.199	.117	.174
	0	.6	.248	.203	.212	.202	.177	.161	.345	.280	.303	.274	.269	.239
AR	0	1.2	.187	.106	.144	.078	.127	.055	.268	.149	.221	.108	.203	.088
	.4	1.2	.194	.159	.153	.132	.117	.111	.281	.224	.235	.199	.202	.172
	.4	2	.185	.078	.145	.067	.125	.037	.277	.117	.217	.094	.195	.071
	0	.6	.168	.215	.145	.225	.143	.197	.236	.301	.234	.302	.209	.266
MA	0	1.2	.091	.154	.097	.158	.068	.108	.170	.208	.157	.204	.121	.156
	.4	1.2	.122	.196	.093	.177	.076	.157	.193	.275	.167	.261	.144	.215
	.4	2	.091	.132	.092	.128	.082	.091	.150	.171	.157	.180	.129	.137

TABLE 9

EMPIRICAL SIZES OF W_I AND W_F FOR $m = III$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.6	.127	.185	.100	.146	.090	.123	.198	.233	.159	.204	.155	.192
WN	0	1.2	.074	.192	.065	.142	.052	.126	.141	.246	.122	.199	.123	.170
	.4	1.2	.104	.187	.086	.142	.064	.121	.171	.251	.139	.200	.121	.168
	.4	2	.067	.174	.071	.152	.062	.117	.121	.219	.126	.202	.122	.172
	0	.6	.316	.259	.274	.266	.255	.240	.391	.353	.353	.350	.346	.332
AR	0	1.2	.191	.102	.147	.076	.127	.052	.278	.142	.216	.103	.204	.084
	.4	1.2	.216	.159	.166	.113	.132	.097	.297	.227	.252	.183	.207	.154
	.4	2	.195	.077	.139	.066	.126	.039	.268	.116	.223	.096	.197	.071
	0	.6	.237	.308	.242	.318	.238	.281	.326	.410	.328	.395	.323	.373
MA	0	1.2	.099	.160	.099	.152	.066	.104	.172	.204	.160	.199	.121	.147
	.4	1.2	.134	.201	.108	.177	.087	.149	.216	.276	.184	.241	.156	.203
	.4	2	.090	.139	.089	.136	.078	.090	.151	.179	.165	.183	.129	.139

TABLE 10

EMPIRICAL SIZES OF W_I^o AND W_F^o FOR $m = III$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o	W_I^o	W_F^o
	0	.6	.149	.219	.131	.186	.118	.163	.217	.275	.205	.252	.188	.243
WN	0	1.2	.066	.192	.066	.142	.051	.127	.143	.250	.128	.196	.123	.174
	.4	1.2	.117	.228	.103	.164	.079	.143	.192	.286	.153	.225	.139	.201
	.4	2	.066	.172	.074	.155	.063	.118	.125	.223	.124	.196	.118	.176
	0	.6	.279	.306	.231	.264	.208	.227	.367	.368	.328	.351	.294	.320
AR	0	1.2	.189	.101	.145	.077	.127	.055	.274	.147	.218	.108	.204	.089
	.4	1.2	.203	.173	.157	.131	.115	.111	.282	.255	.243	.198	.201	.165
	.4	2	.189	.079	.142	.065	.125	.038	.269	.115	.221	.093	.194	.073
	0	.6	.211	.307	.202	.316	.201	.283	.283	.401	.271	.393	.268	.355
MA	0	1.2	.097	.157	.095	.149	.065	.105	.174	.210	.155	.206	.120	.156
	.4	1.2	.128	.216	.103	.189	.083	.157	.210	.300	.173	.251	.151	.222
	.4	2	.089	.134	.090	.131	.078	.090	.150	.170	.156	.179	.126	.135

TABLE 11

MONTE CARLO BIAS OF $\overline{\nu}_I, \overline{\nu}_{2I}, \overline{\nu}_F, \overline{\nu}_{2F}, \overline{\nu}_U$ FOR $\rho = .5$, $\phi = \psi = 0$

	n	64				128				256			
	γ, δ	$0, .4$	$.2, .4$	$.4, .8$	$.7, 1$	$0, .4$	$.2, .4$	$.4, .8$	$.7, 1$	$0, .4$	$.2, .4$	$.4, .8$	$.7, 1$
I	$\overline{\nu}_I$	.060	.182	.055	.100	.043	.164	.040	.080	.028	.131	.025	.055
	$\overline{\nu}_{2I}$	.025	.120	.024	.053	.019	.110	.018	.044	.010	.079	.009	.025
	$\overline{\nu}_F$	.072	.204	.069	.122	.054	.194	.052	.107	.031	.140	.031	.066
	$\overline{\nu}_{2F}$	.031	.164	.031	.077	.026	.162	.026	.075	.005	.089	.009	.033
	$\overline{\nu}_U$	.149	.276	.140	.194	.119	.254	.109	.163	.096	.226	.085	.132
II	$\overline{\nu}_I$	.119	.263	.092	.142	.093	.240	.068	.116	.063	.203	.044	.082
	$\overline{\nu}_{2I}$	.069	.216	.054	.101	.050	.191	.037	.078	.029	.153	.021	.050
	$\overline{\nu}_F$	.141	.282	.116	.173	.108	.256	.088	.144	.070	.212	.054	.098
	$\overline{\nu}_{2F}$	.105	.259	.091	.150	.075	.225	.065	.121	.038	.172	.032	.072
	$\overline{\nu}_U$	.211	.321	.161	.204	.181	.301	.131	.176	.146	.273	.099	.138
III	$\overline{\nu}_I$	.177	.318	.120	.169	.127	.279	.084	.133	.085	.235	.052	.092
	$\overline{\nu}_{2I}$	.123	.288	.085	.138	.078	.242	.053	.101	.044	.189	.028	.062
	$\overline{\nu}_F$	.197	.329	.143	.195	.143	.289	.103	.157	.093	.242	.064	.109
	$\overline{\nu}_{2F}$	.164	.315	.123	.182	.106	.266	.081	.140	.057	.207	.042	.086
	$\overline{\nu}_U$	.259	.351	.174	.208	.212	.323	.137	.178	.170	.292	.103	.140

TABLE 12

MONTE CARLO S.D. OF $\overline{\nu}_I, \overline{\nu}_{2I}, \overline{\nu}_F, \overline{\nu}_{2F}, \overline{\nu}_U$ FOR $\rho = .5$, $\phi = \psi = 0$

	n	64				128				256			
	γ, δ	$0, .4$	$.2, .4$	$.4, .8$	$.7, 1$	$0, .4$	$.2, .4$	$.4, .8$	$.7, 1$	$0, .4$	$.2, .4$	$.4, .8$	$.7, 1$
I	$\overline{\nu}_I$	.262	.521	.261	.382	.185	.429	.188	.297	.115	.300	.116	.197
	$\overline{\nu}_{2I}$	.288	.648	.286	.427	.199	.520	.199	.319	.121	.357	.121	.206
	$\overline{\nu}_F$	.291	.530	.286	.413	.207	.455	.207	.336	.129	.325	.128	.221
	$\overline{\nu}_{2F}$	.380	.708	.363	.514	.268	.637	.248	.408	.162	.440	.157	.266
	$\overline{\nu}_U$	.201	.342	.221	.354	.146	.289	.174	.308	.094	.207	.113	.219
II	$\overline{\nu}_I$	.172	.268	.185	.283	.117	.206	.129	.222	.084	.171	.089	.159
	$\overline{\nu}_{2I}$	.196	.332	.202	.292	.130	.252	.135	.214	.091	.209	.092	.153
	$\overline{\nu}_F$	.179	.267	.191	.285	.121	.205	.136	.233	.088	.170	.094	.171
	$\overline{\nu}_{2F}$	.214	.328	.221	.311	.142	.246	.153	.242	.103	.212	.107	.177
	$\overline{\nu}_U$	.137	.201	.174	.298	.097	.158	.133	.256	.070	.127	.095	.197
III	$\overline{\nu}_I$	.144	.198	.169	.273	.105	.167	.123	.223	.076	.140	.084	.159
	$\overline{\nu}_{2I}$	.164	.233	.178	.269	.116	.196	.125	.209	.082	.167	.085	.146
	$\overline{\nu}_F$	.153	.204	.171	.269	.110	.170	.128	.228	.078	.139	.087	.167
	$\overline{\nu}_{2F}$	.179	.241	.187	.275	.125	.198	.137	.228	.088	.164	.095	.163
	$\overline{\nu}_U$	.123	.167	.166	.289	.091	.140	.130	.252	.066	.112	.093	.195

TABLE 13

EMPIRICAL SIZES OF W_I AND W_F FOR $\rho = .5$, $\phi = \psi = 0$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F	W_I	W_F
	0	.4	.172	.263	.151	.240	.111	.209	.247	.335	.228	.298	.192	.284
I	.2	.4	.275	.336	.283	.327	.263	.311	.362	.416	.354	.398	.357	.381
	.4	.8	.170	.242	.161	.240	.108	.194	.244	.314	.232	.297	.199	.264
	.7	1	.222	.275	.221	.263	.193	.228	.308	.345	.292	.341	.277	.299
	0	.4	.273	.343	.263	.361	.232	.320	.361	.425	.346	.443	.326	.400
II	.2	.4	.434	.473	.490	.549	.518	.536	.511	.550	.578	.622	.588	.605
	.4	.8	.252	.325	.236	.343	.206	.300	.329	.399	.317	.422	.294	.383
	.7	1	.339	.398	.380	.448	.344	.406	.430	.476	.470	.527	.423	.485
	0	.4	.418	.488	.407	.481	.365	.419	.521	.571	.502	.569	.462	.515
III	.2	.4	.618	.646	.665	.701	.682	.706	.694	.708	.741	.768	.747	.764
	.4	.8	.322	.392	.301	.401	.261	.349	.412	.473	.388	.476	.348	.432
	.7	1	.444	.488	.451	.513	.417	.480	.514	.561	.529	.607	.506	.548

TABLE 14

EMPIRICAL SIZES OF W_{2I} AND W_{2F} FOR $\rho = .5$, $\phi = \psi = 0$

		α			.05						.10			
		n	64	64	128	128	256	256	64	64	128	128	256	256
m	γ	δ	W_{2I}	W_{2F}	W_{2I}	W_{2F}	W_{2I}	W_{2F}	W_{2I}	W_{2F}	W_{2I}	W_{2F}	W_{2I}	W_{2F}
	0	.4	.206	.312	.158	.274	.121	.242	.287	.391	.246	.336	.210	.309
I	.2	.4	.352	.430	.322	.394	.300	.353	.455	.507	.412	.475	.384	.431
	.4	.8	.197	.312	.162	.267	.121	.228	.286	.380	.247	.333	.211	.295
	.7	1	.275	.333	.231	.285	.207	.259	.362	.414	.298	.364	.275	.336
	0	.4	.262	.347	.221	.356	.196	.321	.338	.433	.308	.438	.281	.402
II	.2	.4	.439	.495	.447	.529	.446	.504	.513	.573	.522	.606	.522	.574
	.4	.8	.262	.354	.227	.371	.192	.310	.346	.446	.304	.444	.281	.398
	.7	1	.344	.415	.332	.441	.306	.394	.433	.499	.421	.523	.381	.460
	0	.4	.329	.448	.276	.412	.220	.339	.412	.526	.358	.488	.310	.424
III	.2	.4	.563	.623	.582	.647	.562	.613	.636	.691	.649	.703	.631	.672
	.4	.8	.292	.400	.246	.395	.204	.336	.379	.484	.339	.472	.286	.411
	.7	1	.415	.492	.404	.497	.341	.440	.502	.566	.488	.571	.441	.528