SPECIFICATION FOR LATTICE PROCESSES

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Abstract. We consider an omnibus test for the correct specification of the dynamics of a sequence $\{x(t)\}_{t\in\mathbb{Z}^d}$ in a lattice. As it happens with causal models and $d = 1$, its asymptotic distribution is not pivotal and depends on the estimator of the unknown parameters of the model under the null hypothesis. One of our main goals of the paper is to provide a transformation to obtain an asymptotic distribution that is free of nuisance parameters. Secondly, we propose a bootstrap analogue of the transformation and show its validity. Third, we discuss the results when $\{x(t)\}_{t\in\mathbb{Z}^d}$ are the errors of a parametric regression model. As a by product, we also discuss the asymptotic normality of the least squares estimators under very mild conditions. Finally, we present a small Monte Carlo experiment to shed some light on the finite sample behaviour of our test.

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1. **INTRODUCTION**

Given a spatial process $\{x(t)\}_{t\in\mathbb{Z}^d}$, $d \geq 1$, it is agreed that one of the main purposes is to obtain a correct description of its covariogram $\{\gamma(s)\}_{s\in\mathbb{Z}^d}$, defined as $\gamma(s) = \mathrm{Cov}(x(t), x(t+s))$. The importance of the covariogram relies on the fact that it plays a key role to obtain good and accurate predictions and/or interpolations. Furthermore, in regression models it enables either correct inferences on the parameters of the model or efficient estimation. Indeed, given a stretch of data $X = \{x(t)\}_{t=1}^{n}$, it is well known that the best predictor (in a linear sense) of $x(t^*)$, where $t^* \neq 1, ..., n$, is given by

$$E(x(t^*) \mid X) = \mathrm{Cov}\left(x(t^*), X'\right) \mathrm{Cov}^{-1}\left(X, X'\right) X,$$

known as the Kriging predictor, see Cressie (1993). Similarly, when X are the errors in a regression model

$$y(t) = \beta_0' z(t) + x(t), \qquad t = 1, ..., n, \tag{1.1}$$

where $Z = \{z(t)\}_{t=1}^{n}$ is a q-dimensional set of fixed regressors, we have that the asymptotic covariance of the least squares estimator of β_0 depends on $\{\gamma(s)\}_{s\in\mathbb{Z}^d}$. In addition, the predictor of say $y(t^*)$ becomes in this case

$$E\left(y(t^*) \mid \{y(t)\}_{t=1}^{n}\right) = \beta_0' z(t^*) + E(x(t^*) \mid X),$$

so that an accurate specification of $\gamma(s)$ is the key to obtain a good predictor of $y(t^*)$.

More specifically, we are interested to check whether the covariogram $\{\gamma(s)\}_{s\in\mathbb{Z}^d}$ follows a particular parametric family, that is $\{\gamma(s)\}_{s\in\mathbb{Z}^d} = \{\gamma(s;\vartheta)\}_{s\in\mathbb{Z}^d}$, where $\vartheta = \left(\theta', \sigma_\varepsilon^2\right)'$ is a $(p+1)$-dimensional vector of unknown parameters. Observing that for any covariance stationary spatial lattice process $\{x(t)\}_{t\in\mathbb{Z}^d}$, the spectral density function $f(\lambda)$ and the covariogram $\{\gamma(s)\}_{s\in\mathbb{Z}^d}$ are related by the expression

$$\gamma(s) = \int_{\Pi^d} e^{-is'\lambda} f(\lambda)\, d\lambda; \quad s \in \mathbb{Z}^d, \tag{1.2}$$

we could have equivalently formulated our interest on whether $f(\lambda) = f(\lambda;\theta)$ for $\lambda \in \Pi^d$. Herewith "$s'\lambda$" means the inner product of two d-dimensional vectors s and λ and $\Pi = [-\pi, \pi]$.

Thus, one of the aims of the paper is to describe a T_p-type omnibus test for the composite hypothesis that the covariogram of the sequence $\{x(t)\}_{t\in\mathbb{Z}^d}$ follows a specified parametric model. One difference with previous work when $d = 1$ is that we allow for models which are also forward looking, i.e. noncausals models, which have gained some consideration recently in economics, see for instance Lanne and Saikonnen (2011) and Davis et al. (2001). In addition, we examine the behaviour and consequences of the test when $\{x(t)\}_{t\in\mathbb{Z}^d}$ are the errors of a parametric regression model. As a by-product, we obtain the asymptotic distribution of the estimator of β_0 under mild conditions. In particular, we show the asymptotic normality when the regressors are deterministic, without the need to assume that the process $\{x(t)\}_{t\in\mathbb{Z}^d}$ is strong mixing as it was assumed in Bolthausen (1982) or more recently in Jenish and Prucha (2009), although our conditions are quite similar to those in Robinson and Thawornkaiwong (2011). Instead, we assume that the process $\{x(t)\}_{t\in\mathbb{Z}^d}$ is a *Generalized linear* process in the sense put forward by Hannan (1970, p.210), see below (1.3). The basic condition that we will need for the asymptotic distribution of the least squares being that the jump of the spectral distribution function of $\{z(t)\}_{t\in\mathbb{Z}^d}$ does not coincide with the discontinuity of the spectral density function of $\{x(t)\}_{t\in\mathbb{Z}^d}$, allowing for strong dependence, see Yajima and Matsuda (2011).

All throughout the paper we shall assume that the spatial linear process $\{x(t)\}_{t\in\mathbb{Z}^d}$ has a representation by the multilateral (noncausal) model

$$x(t)-\mu=\sum_{s\in\mathbb{Z}^d}\psi(s)\,\varepsilon(t-s),\quad \sum_{s\in\mathbb{Z}^d}\psi^2(s)<\infty,\quad \psi(0)=1, \tag{1.3}$$

for some sequence $\{\varepsilon(t)\}_{t\in\mathbb{Z}^d}$ satisfying $\mathbb{E}(\varepsilon(t))=0$ and $\mathbb{E}(\varepsilon(0)\,\varepsilon(t))=\sigma_\varepsilon^2\mathbb{I}(t=0)$, where $\mathbb{I}(\cdot)$ is the indicator function. Under (1.3), we have that

$$f(\lambda)=\frac{\sigma_\varepsilon^2}{(2\pi)^d}\left|\Psi(\lambda)\right|^2$$

where $\Psi(\lambda)=\sum_{s\in\mathbb{Z}^d}\psi(s)\,e^{-is'\lambda}$, which summarizes the covariogram structure of $\{x(t)\}_{t\in\mathbb{Z}^d}$ as seen in (1.2).

Denoting $[0,\pi]\times\Pi^{d-1}$ as $\widetilde{\Pi}^d$, that is $\lambda\in\widetilde{\Pi}^d$ if $\lambda[1]\in[0,\pi]$ and $\lambda[\ell]\in\Pi$ for $\ell=2,...,d$, where $a[\ell]$ denotes the ℓ-th coordinate of the vector a that belongs to $\mathbb{Z}^d$ (or Π^d), we can write the null hypothesis as follows:

$$H_0:\forall\lambda\in\widetilde{\Pi}^d \text{ and for some } \theta_0\in\Theta,\ \ \left|\Psi(\lambda)\right|^2=\left|\Psi_{\theta_0}(\lambda)\right|^2, \tag{1.4}$$

where $\Theta\subset\mathbb{R}^p$ is a compact parameter space and $\Psi_\theta(\lambda)=\sum_{s\in\mathbb{Z}^d}\psi(s;\theta)\,e^{-is'\lambda}$. The alternative hypothesis is the negation of H_0.

A particular parameterization of (1.3) is the $ARMA(-k_1,k_2;-\imath_1,\imath_2)$ field model, see Whittle (1954), defined as

$$\sum_{s=-k_1}^{k_2}\alpha(s)\left(x(t-s)-\mu\right)=\sum_{s=-\imath_1}^{\imath_2}\beta(s)\,\varepsilon(t-s)\qquad \alpha(0)=\beta(0)=1,$$

whose spectral density function is given by

$$f(\lambda)=\frac{\sigma_\varepsilon^2}{(2\pi)^d}\frac{\left|\sum_{s=-\imath_1}^{\imath_2}\beta(s)\,e^{is'\lambda}\right|^2}{\left|\sum_{s=-k_1}^{k_2}\alpha(s)\,e^{is'\lambda}\right|^2}.$$

Notice that the latter model is causal if $\imath_1=k_1=0$. It is worth mentioning that Whittle (1954) showed that any given stationary multilateral scheme on a plane lattice has a unilateral autoregression with the same spectral scheme, although not necessarily of finite order as is the case when $d=1$.

Another parametric model of interest is the extension to the lattice of the classical Bloomfield (1973)'s exponential model. In fact, it was introduced by Whittle (1954, Sec. 6) beforehand and it was also named as the *Cepstrum* model by Solo (1986). These models can be characterized as having a spectral density function defined as

$$f_\vartheta(\lambda)=\sigma_\varepsilon^2\exp\left\{-\sum_{s\prec 0}a(s;\theta)\cos\left(s'\lambda\right)\right\}, \tag{1.5}$$

where "$\prec$" denotes the lexicographical (dictionary) ordering which is defined as

$$s \prec k \Leftrightarrow (\exists \iota > 0)\,(\forall i < \iota)\,(s[i] = k[i] \vee s[\iota] < k[\iota])\,,$$

that is, if one of the terms $s[\iota] < k[\ell]$ and all the previous ones are equal. Observe that if we allowed s in (1.5) to belong to $\mathbb{Z}^d$, the model would not be identified as $\cos(s'\lambda) = \cos(-s'\lambda)$ for all $\lambda \in \Pi^d$ and s. Solo (1986) notes that if $0 < f_\vartheta(\lambda) < M$ the representation of the spectral density in (1.5) exists. Finally, another example is the *neighbourhood structure parameterization* by Zhu, Huang and Reyes (2010).

Due to the complicated notation in this paper, we have decided to gather it at this stage for convenience. The numbers $0, 1$ and π can be either scalars or vectors (of dimension d), which should be clear from the context, whereas $\mathring{\pi} = (0, \pi, ..., \pi)'$ and e_ℓ denotes the unit vector in $\mathbb{Z}^d$ whose ℓ-th element is one and the others are zero. For two vectors a and b, $a \vee b$ and $a \wedge b$ represent the maximum and minimum of the two, respectively, based on the lexicographical ordering, while $a \geq (\leq)\, b$ means that $a[\ell] \geq (\leq)\, b[\ell]$ for all $\ell = 1, ..., d$. For $\tilde{n} = [n/2]$, denote

$$\Pi_n^d = \left\{ \lambda_k[\ell] = \frac{\pi k[\ell]}{\tilde{n}[\ell]}, \quad k[\ell] = 0, \pm 1, ..., \pm \tilde{n}[\ell]\,, \quad \ell = 1, ..., d \right\},$$

where λ_k stands for the Fourier frequencies. Similarly to $\widetilde{\Pi}^d$, we define $\widetilde{\Pi}_n^d = \left\{ \lambda_k \in \Pi_n^d : \lambda_k[1] > 0 \right\}$. We use three different summation operators when they are taken over the Fourier frequencies, namely

$$\sum_{\lambda_s} = \sum_{\lambda_s \in \widetilde{\Pi}_n^d}, \qquad \sum_{\lambda_s}^{\lambda} = \sum_{\substack{\lambda_s \in \widetilde{\Pi}_n^d \\ \lambda_s \leq \lambda}}, \qquad \text{and} \sum_{\lambda_s \prec \lambda} = \sum_{\substack{\lambda_s \in \Pi_n^d; \\ 0 \prec \lambda_s \prec \lambda}},$$

Note that λ_s's in the last summation are taken from Π_n^d.

The remainder of the paper is organized as follows. In the next section, we present the test and examine its asymptotic properties, showing that it is not pivotal as its asymptotic distribution depends on H_0 and on the estimator employed of the unknown parameters ϑ_0. Because the asymptotic critical values are difficult to obtain, Section 3 describes a transformation in the spirit of Brown, Durbin and Evans (1975) such that it converges to a functional of the "standard" Brownian sheet in $[0,1]^d$. The transformation mirrors that of Delgado, Hidalgo and Velasco (2005) to the case when $d > 1$, or when $d = 1$ and the model is not causal. We also describe a bootstrap algorithm to compute the critical values of the transformation. Section 4 describes the local alternatives and it examines the consequences, if any, when $\{x(t)\}_{t \in \mathbb{Z}^d}$ are the errors of a parametric regression model. In addition as by-product, we show the asymptotic normality of the least squares estimator of the parameters under mild conditions. Section 5 presents the results of a Monte Carlo study to shed some light on the finite sample performance of our test and its bootstrap analogue. The proof of our main results are given in Appendix **B** which uses a series of lemmas given in Appendix **A**.

2. THE TEST AND ITS PROPERTIES

Before we introduce and describe the test, we first observe that we can state the null hypothesis (1.4) as

$$H_0 : \forall \lambda \in \widetilde{\Pi}^d \text{ and some } \theta_0 \in \Theta,\ \frac{G_{\theta_0}(\lambda)}{G_{\theta_0}(\pi)} = \frac{\lambda[1]}{\pi} \prod_{\ell=2}^{d} \left(1 + \frac{\lambda[\ell]}{\pi}\right), \tag{2.1}$$

where

$$G_\theta(\lambda) = 2 \int_{-\mathring{\pi}}^{\lambda} |\Psi_\theta(\omega)|^{-2} f(\omega)\, d\omega.$$

Under H_0, $G_{\theta_0}(\lambda) / (2\pi)^d$ is the spectral distribution function of the lattice process $\{\varepsilon(t)\}_{t \in \mathbb{Z}^d}$ and $G_{\theta_0}(\pi) = \sigma_\varepsilon^2$. Notice that by symmetry of $f(\lambda)$, it does not matter which coordinate we choose in the interval $[0, \pi]$, as it will not affect the value of $G_\theta(\lambda)$ and hence the test given below. We shall indicate though that for simplicity of arguments, we focus in the case when $d \leq 3$. Extensions to $d > 3$ can be adapted easily under suitable modifications.

Let $h_n(t) = 2^{-d} \prod_{\ell=1}^{d} h(t[\ell]/n[\ell])$, where $h(\cdot)$ is a function in $[0,1]$, and define the taper *periodogram* of a generic sequence $\{v(t)\}_{t=1}^{n}$ by

$$I_v^T(\lambda) = \frac{1}{\sum_{t=1}^{n} h_n^2(t)} \left| \sum_{t=1}^{n} h_n(t)\, v(t)\, e^{-it'\lambda} \right|^2 .$$

The motivation to employ the taper periodogram instead of the standard periodogram, i.e. when $h(\cdot) = 1$, is due to the adverse properties that $\widehat{\vartheta}$ in (2.3) would have with $h(t) = 1$ as Guyon (1982) observed. Recall that tapering is primarily a technique employed to reduce the bias of the "standard" periodogram, although it increases the variance by a factor $P_4^2 = \left(\int_0^1 h^4\right) / \left(\int_0^1 h^2\right)^2$. Another possibility is the one described by Robinson and Vidal-Sanz (2006), which would be helpful when $d \geq 4$. However as we only consider explicitly the most common scenario when $d \leq 3$, it then suffices to employ $I_v^T(\lambda_s)$.

Given a record $\{x(t)\}_{t=1}^{n}$, and denoting henceforth $N = \Pi_{\ell=1}^{d} n[\ell]$, a natural estimator of $G_{\theta_0}(\lambda)$ is $G_{\widehat{\theta}N}(\lambda)$, where

$$G_{\theta N}(\lambda) = \frac{(2\pi)^d}{N} \sum_{\lambda_s}^{\lambda} \frac{I_x^T(\lambda_s)}{|\Psi_\theta(\lambda_s)|^2}, \tag{2.2}$$

for a given estimate $\hat{\theta}$. The summation in (2.2) is taken over $\widetilde{\Pi}_n^d$ instead of the half space $\{\lambda_s \succ 0\}$ to ease notation and exposition.

For $\hat{\theta}$, we employ the Whittle's (1954) estimator of $\vartheta_0 = \left(\theta_0', \sigma_\varepsilon^2\right)'$ defined as

$$\widehat{\vartheta} = \arg \min_{\vartheta \in \Theta \times \mathbb{R}^+} \mathcal{Q}_N(\vartheta), \tag{2.3}$$

where

$$\mathcal{Q}_N(\vartheta) = \frac{1}{N} \sum_{\lambda_s} \left\{ \log f_\vartheta(\lambda_s) + \frac{I_x^T(\lambda_s)}{(2\pi)^d f_\vartheta(\lambda_s)} \right\}$$

with $f_\vartheta(\lambda_s) = \sigma_\varepsilon^2 |\Psi_\theta(\lambda_s)|^2 / (2\pi)^d$. It is worth pointing out that because our model is multilateral, one consequence is that $\{\varepsilon(t)\}_{t\in\mathbb{Z}^d}$ loses its interpretation as the "prediction" error or as the innovations. The implication of the latter is that the standard least squares estimator of the parameters θ, that is

$$\widehat{\theta}^{LSE} = \arg\min_{\theta\in\Theta} \sum_{\lambda_s} \frac{I_x^T(\lambda_s)}{|\Psi_\theta(\lambda_s)|^2},$$

is an inconsistent estimator of θ_0, see Whittle (1954).

The formulation of H_0 given in (2.1) suggests to use the Bartlett's T_p- *process* $\alpha_{\widehat{\theta}N}(\lambda)$ as the basis to test H_0, where

$$\alpha_{\theta N}(\lambda) = \frac{1}{2^{d/2}P_4} N^{1/2} \left[\frac{G_{\theta N}(\lambda)}{G_{\theta N}(\pi)} - \frac{\lambda[1]}{\pi} \prod_{\ell=2}^{d} \left(1 + \frac{\lambda[\ell]}{\pi}\right) \right], \quad \lambda \in \widetilde{\Pi}^d, \tag{2.4}$$

with $G_{\theta N}(\lambda)$ given in (2.2). From here, we can base the test for H_0 using $\eta\left(\alpha_{\widehat{\theta}N}\right)$ for some continuous functional $\eta : D\left(\widetilde{\Pi}^d\right) \to \mathbb{R}^+$, where $D\left(\widetilde{\Pi}^d\right)$ is the space of càdlàg functions in $\widetilde{\Pi}^d$.

Let us introduce the following regularity conditions.

***C1*:** (a) $\{\varepsilon(t)\}_{t\in\mathbb{Z}^d}$ in (1.3) is a sequence of zero mean independent identically distributed random variables with $\mathbb{E}\left(\varepsilon^2(t)\right) = \sigma_\varepsilon^2 = 1$ and finite $4th$ moments, denoting its fourth cumulant by κ_ε.

(b) The multilateral *moving average* representation of $\{x(t)\}_{t\in\mathbb{Z}^d}$ in (1.3) can be written as a multilateral *autoregressive* model

$$\sum_{s\in\mathbb{Z}^d} \xi(s)\, x(t-s) = \varepsilon(t) \qquad \xi(0) = 1,$$

where $\xi(s)$ is the coefficient of z^s in the Fourier expansion of $\wp^{-1}(z)$, where

$$\wp(z) = \wp(z[1], ..., z[d]) = \sum_{s\in\mathbb{Z}^d} \psi(s)\, z^s$$

using the notation $z^s = \Pi_{\ell=1}^d z[\ell]^{s[\ell]}$ and the convention $0^0 = 1$,

***C2*:** $n[\ell] \asymp \vec{n} \nearrow \infty$ for $\ell = 1, ..., d$, where "$a \asymp b$" means that $C^{-1} \le a/b \le C$ for some finite positive constant C.

***C3*:** $h(\cdot)$ is the *cosine-bell* taper, that is,

$$h(z) = (1 - \cos(2\pi z)).$$

***C4*:** θ_0 is an interior point of the compact parameter set $\Theta \subset \mathbb{R}^p$.

***C5*:** $|\Psi_\theta(\lambda)| = \left|\sum_{s\in\mathbb{Z}^d} \psi_\theta(s)\, e^{-is'\lambda}\right|$ is a positive and twice continuously differentiable function in θ on $\widetilde{\Pi}^d$ and continuously differentiable in λ for all $\theta \in \Theta$.

***C6*:** If $\theta_1 \neq \theta_2$, then $\Psi_{\theta_1}(\lambda) \neq \Psi_{\theta_2}(\lambda)$ in a set $\Delta \subset \widetilde{\Pi}^d$ with positive Lebesgue measure.

Conditions $C1 - C6$ are similar to those in Hidalgo (2009) and so his comments apply here. Notice that we write explicitly $\mathbb{E}\left(\varepsilon^2(t)\right) = \sigma_\varepsilon^2$ as is a parameter in itself, although for notational simplicity we have assumed that its true value is 1, cf. Condition $C1(a)$.

Let

$$(2.5)\quad \phi_{\vartheta}(\lambda) = \frac{\partial}{\partial\vartheta}\log f_{\vartheta}(\lambda) = \left(\varphi_{\theta}'(\lambda), \sigma_{\varepsilon}^{-2}\right)', \quad \varphi_{\theta}(\lambda) = \frac{\partial}{\partial\theta}\log|\Psi_{\theta}(\lambda)|^2$$

and

$$\Phi_{\vartheta}(\lambda) = (2\pi)^{-d}\int_{-\mathring{\pi}}^{\lambda}\phi_{\vartheta}(\omega)\,d\omega \text{ and } \Lambda_{\vartheta} = (2\pi)^{-d}\int_{-\mathring{\pi}}^{\pi}\phi_{\vartheta}(\omega)\,\phi_{\vartheta}'(\omega)\,d\omega.$$

***C7*:** Λ_{ϑ} is a continuous positive definite matrix at $\vartheta = \vartheta_0$.

Proceeding as in Hidalgo (2009), we have that the Whittle estimator $\widehat{\vartheta}$ in (2.3) satisfies the asymptotic linearization

$$\widehat{\theta} - \theta_0 = -\widetilde{\Lambda}_{\theta_0 N}^{-1}\int_{-\mathring{\pi}}^{\pi}\widetilde{\varphi}_{\theta_0}(\lambda)\,\alpha_{\theta_0 N}(d\lambda) + o_p\left(N^{-1/2}\right),$$

where

$$\widetilde{\varphi}_{\theta}(\lambda) = \varphi_{\theta}(\lambda) - \frac{2}{(2\pi)^d}\int_{-\mathring{\pi}}^{\pi}\varphi_{\theta}(\lambda)\,d\lambda,$$

and defining $\widetilde{\varphi}_{\theta N}(\lambda_s) = \varphi_{\theta}(\lambda_s) - \frac{2}{N}\sum_{\lambda_s}\varphi_{\theta}(\lambda_s)$,

$$\widetilde{\Lambda}_{\theta N} = N^{-1}\sum_{\lambda_k}\widetilde{\varphi}_{\theta N}(\lambda_k)\,\widetilde{\varphi}_{\theta N}'(\lambda_k).$$

Let

$$(2.6)\qquad \dot{\boldsymbol{B}}(\lambda) = \boldsymbol{B}\left(\frac{\lambda}{\pi}\right) - \left\{\frac{\lambda[1]}{\pi 2^{d-1}}\prod_{\ell=2}^{d}\left(1 + \frac{\lambda[\ell]}{\pi}\right)\right\}\boldsymbol{B}(1) \qquad \lambda \in \widetilde{\Pi}^d,$$

where $\left\{\boldsymbol{B}(u) : u \in [0,1]\times[-1,1]^{d-1}\right\}$ denotes a zero mean Gaussian process such that

$$\operatorname{Cov}(\boldsymbol{B}(u), \boldsymbol{B}(v)) = 2^{1-d}\,|u[1]\wedge v[1]|\prod_{\ell=2}^{d}|(u[\ell]\wedge v[\ell]) + 1|,$$

that is, $\boldsymbol{B}$ is a time-changed Brownian sheet. Also let

$$\widetilde{\Lambda}_{\theta} = (2\pi)^{-d}\int_{-\mathring{\pi}}^{\pi}\widetilde{\varphi}_{\theta}(\bar{\lambda})\,\widetilde{\varphi}_{\theta}'(\bar{\lambda})\,d\bar{\lambda}$$

and define

$$(2.7)\qquad \alpha_N^0(\lambda) = \frac{N^{1/2}}{2^{d/2}P_4}\left[\frac{G_N^0(\lambda)}{G_N^0(\pi)} - \frac{\lambda[1]}{\pi}\prod_{\ell=2}^{d}\left(1 + \frac{\lambda[\ell]}{\pi}\right)\right], \quad \lambda \in \widetilde{\Pi}^d,$$

with $G_N^0(\lambda) = \frac{(2\pi)^d}{N}\sum_{\lambda_s}^{\lambda} I_{\varepsilon}^T(\lambda_s)$. Denoting

$$\alpha_{\infty}(\lambda) = \dot{\boldsymbol{B}}(\lambda) - \left(\frac{1}{(2\pi)^d}\int_{-\mathring{\pi}}^{\lambda}\widetilde{\varphi}_{\theta_0}'(\bar{\lambda})\,d\bar{\lambda}\right)\widetilde{\Lambda}_{\theta_0}^{-1}\int_{-\mathring{\pi}}^{\pi}\widetilde{\varphi}_{\theta_0}(\bar{\lambda})\,\dot{\boldsymbol{B}}(d\bar{\lambda}),$$

we then have the following result.

Theorem 1. *Under H_0 and assuming C1 − C7, uniformly in $\lambda \in \widetilde{\Pi}^d$, we have that*

$$\begin{aligned}\alpha_{\widehat{\theta}N}(\lambda) &= \alpha_N^0(\lambda) - \left(\frac{1}{N}\sum_{\lambda_s}^{\lambda}\widetilde{\varphi}'_{\theta_0 N}(\lambda_s)\right)\widetilde{\Lambda}_{\theta_0 N}^{-1}\frac{1}{N}\sum_{\lambda_s}\widetilde{\varphi}'_{\theta_0 N}(\lambda_s) I_\varepsilon^T(\lambda_s) \\ &\quad + o_p(1) \\ &\Rightarrow \alpha_\infty(\lambda).\end{aligned}$$

Proof. See Hidalgo (2009). □

The main conclusion from Theorem 1 is that the asymptotic distribution of the $T_p - process$ $\alpha_{\widehat{\theta}N}(\lambda)$ depends on the model under H_0 and also on the estimator of θ_0. So, the asymptotic critical values of $\eta\left(\alpha_{\widehat{\theta}N}\right)$, for any continuous functional $\eta(\cdot)$, cannot be easily tabulated. To circumvent this type of problem, several approaches have been described. A first approach proposes to use bootstrap algorithms. This is the route employed, among others, by Chen and Romano (2000) or Hainz and Dahlhaus (2000) using the $U_p - process$ and by Hidalgo and Kreiss (2006) who employed the $T_p - process$. Of course, all those works were for $d = 1$, whereas Hidalgo (2009) extends the previous work when $d \geq 1$. A second alternative compares the parametric and nonparametric fits of the spectral density function. This route was followed, among others, by Hong (1996) or Paparoditis (2000) for $d = 1$ and Crujeiras et al. (2008) when $d > 1$. However, the implementation of the test depends on a bandwidth parameter and they are inefficient compared to tests based on $\eta\left(\alpha_{\widehat{\theta}N}\right)$. One additional disadvantage is that there is not a clear procedure as to how to choose the bandwidth parameter that keeping the correct size of the test conveys good power properties. A third approach is to employ a transformation of $\alpha_{\widehat{\theta}N}$ that converges in distribution to the "standard" Brownian sheet. This is the route that we follow in the next section.

3. MARTINGALE TRANSFORMATION: ITS BOOTSTRAP ANALOGUE

In this section we shall present and examine a martingale transformation $\mathcal{L}_{\theta N}(\cdot)$ of $\alpha_{\widehat{\theta}N}(\lambda)$, as well as its bootstrap analogue, when $d \geq 1$. The transformation resembles ideas introduced by Brown, Durbin and Evans (1975) and examined in depth by Khmaladze (1981) and Delgado, Hidalgo and Velasco (2005) when $d = 1$ and the model is causal. Our aim in this section is thus to extend the latter approach to $d > 1$ and/or noncausal models. The approach parallels the existing similarities between Khmaladze's (1981) transformation and the CUSUM of least squares residuals approach followed in Delgado et al. (2005) in that the latter can be considered a discrete version of the former. In our context, as we will see below, we will mirror the transformation given in McKeague et al.'s (1995). More specifically, our aim shall be to show that the transformation $\mathcal{L}_{\widehat{\theta}N}\left(\alpha_{\widehat{\theta}N}\right)$ converges weakly to the time-changed Brownian sheet $\boldsymbol{B}(\cdot)$ defined in (2.6). In addition, we describe a bootstrap analogue of $\mathcal{L}_{\widehat{\theta}N}\left(\alpha_{\widehat{\theta}N}\right)$ showing its validity.

For that purpose, it is worth first noticing that Theorem 1 part (a) indicates that $\alpha_{\widehat{\theta}N}$ has the uniform asymptotic expansion

$$\sup_{\lambda\in\widetilde{\Pi}^d}\left|\alpha_{\widehat{\theta}N}(\lambda)-\frac{2\pi}{G_N^0(\pi)}\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}u_N(\lambda_s)\right|=o_p(1),$$

where

$$u_N(\lambda_s)=\breve{I}_{\varepsilon}^T(\lambda_s)-\widetilde{\varphi}'_{\theta_0N}(\lambda_s)\widetilde{\Lambda}_{\theta_0N}^{-1}N^{-1}\sum_{\lambda_k}\widetilde{\varphi}_{\theta_0N}(\lambda_k)\breve{I}_{\varepsilon}^T(\lambda_k)$$

and $\breve{I}_{\varepsilon}^T(\lambda_s)=I_{\varepsilon}^T(\lambda_s)-\widehat{\sigma}_{\varepsilon}^2$. Here $\widehat{\sigma}_{\varepsilon}^2=G_{\widehat{\theta}N}(\pi)$ which is a $N^{1/2}$-consistent estimator of $\sigma_{\varepsilon}^2=1$, see Hidalgo (2009). Now observing that we can consider $u_N(s)$ as the least squares residuals in the artificial regression model of $I_{\varepsilon}^T(\lambda_s)$ on $\left(1,\varphi'_{\theta_0N}(\lambda_s)\right)'$, it suggests employing the *CUSUM* of recursive least squares residuals to construct asymptotically pivotal tests as originally proposed by Brown, Durbin and Evans (1975). In our case, the recursive estimation is based on the lexicographic ordering in $\widetilde{\Pi}_n^d$, whose the minimum value is $(\pi/\tilde{n}[1],-\pi,...,-\pi)'$.

Let $\widetilde{\Lambda}_{\theta N}(\lambda)=N^{-1}\sum_{\lambda_k\preceq\lambda}\widetilde{\varphi}_{\theta N}(\lambda_k)\widetilde{\varphi}'_{\theta N}(\lambda_k)$ and assume the following condition.

***C8*:** $\widetilde{\Lambda}_{\theta_0N}\left((\pi/\tilde{n}[1],-\pi,...,-\pi)'\right)$ is non-singular a.s. for all n.

Condition $C8$ is very mild and satisfied for all common models used with real data. Recall that p is the dimension of the parameter θ_0. Also, notice that we can directly compute from the model the deterministic matrix $\widetilde{\Lambda}_{\theta_0N}(\cdot)$.

The (scaled) CUSUM of recursive least squares residuals is thus defined as

$$\beta_N^0(\lambda)=\frac{1}{G_N^0(\pi)}\frac{2^{1/2}}{N^{1/2}}\sum_{\lambda_s}^{\lambda}e_N(\lambda_s),\qquad\lambda\in\widetilde{\Pi}^d,$$

where

$$e_N(\lambda_s)=\breve{I}_{\varepsilon}^T(\lambda_s)-\widetilde{\varphi}'_{\theta_0N}(\lambda_s)b_N(\lambda_s)$$

are the least squares residuals with

$$b_N(\lambda_s)=\widetilde{\Lambda}_{\theta_0N}^{-1}(\lambda_s)\frac{1}{N}\sum_{\lambda_k\prec\lambda_s}\widetilde{\varphi}_{\theta_0N}(\lambda_k)\breve{I}_{\varepsilon}^T(\lambda_k).$$

Of course, we could have used the forward least squares residuals, i.e.

$$e_N^f(\lambda_s)=\breve{I}_{\varepsilon}^T(\lambda_s)-\widetilde{\varphi}'_{\theta_0N}(\lambda_s)b_N(\lambda_s),\qquad s\prec\mathring{p},$$

with $\mathring{p}=(\tilde{n}[1],\tilde{n}[2],...,\tilde{n}[d-1],\tilde{n}[d]-p-1)$ and

$$b_N^f(\lambda_s)=\widetilde{\Lambda}_{\theta_0N}^{-1}(\lambda_s)\frac{1}{N}\sum_{\lambda_s\prec\lambda_k}\widetilde{\varphi}_{\theta_0N}(\lambda_k)\breve{I}_{\varepsilon}^T(\lambda_k)$$

being the conclusions the same as with $e_N(\lambda_s)$.

The empirical process β_N^0 is a linear transformation of α_N^0, i.e.

$$\beta_N^0(\lambda)=\mathcal{L}_{\theta_0N}\left(\alpha_N^0(\lambda)\right),\qquad\lambda\in\widetilde{\Pi}^d,$$

where, for any function $g \in D\left(\widetilde{\Pi}^d\right)$,

$$\mathcal{L}_{\theta N}\left(g\left(\lambda\right)\right) = g\left(\lambda\right) - \frac{1}{N}\sum_{\lambda_s}^{\lambda} \widetilde{\varphi}'_{\theta N}\left(\lambda_s\right)\widetilde{\Lambda}^{-1}_{\theta N}\left(\lambda_s\right)\frac{1}{N}\sum_{\lambda_k \prec \lambda_s} \widetilde{\varphi}_{\theta N}\left(\lambda_k\right) g\left(\lambda_k\right).$$

The transformation $\mathcal{L}_{\theta_0 N}$ has the limiting version $\mathcal{L}^0$, defined as

$$\mathcal{L}^0\left(g\left(\lambda\right)\right) = g\left(\lambda\right) - \frac{1}{\left(2\pi\right)^d}\int_{-\mathring{\pi}}^{\lambda} \widetilde{\varphi}_{\theta_0}\left(\bar{\lambda}\right)\widetilde{\Lambda}^{-1}_{\theta_0}\left(\bar{\lambda}\right)\left(\int_{\widetilde{\lambda}\prec\bar{\lambda}} \widetilde{\varphi}_{\theta_0}\left(\widetilde{\lambda}\right) g\left(d\widetilde{\lambda}\right)\right) d\bar{\lambda}.$$

Notice that for $d = 1$, $\mathcal{L}^0\left(\alpha_\infty\right)$ is the martingale innovation of α_∞, see Khmaladze (1981). On the other hand, in our context, $\mathcal{L}^0\left(\alpha_\infty\right)$ becomes the transformation examined by McKeague et al. (1995). That is, let $d = 2$ for simplicity, consider

$$\eta\left(\lambda\right) = B\left(\lambda\right) - K\left(\lambda\right)\xi,$$

where $B\left(\lambda\right)$ is a Brownian sheet in $[0,1]^2$ and $K\left(\lambda\right) = \int_0^{\lambda[1]}\int_0^{\lambda[2]} k\left(s,x\right) ds dx$. Then,

$$W\left(\lambda\right) = \eta\left(\lambda\right) - \int_0^{\lambda[2]}\left[\int_0^{\lambda[1]} k\left(s,x\right)\left\{\frac{\int_x^1 k\left(s,u\right) d\eta\left(s,u\right)}{\int_x^1 k^2\left(s,r\right) dr}\right\} ds\right] dx$$

follows a Brownian sheet. In this sense, $\mathcal{L}_{\theta N}\left(g\left(\lambda\right)\right)$ becomes the discrete version of the latter. In our context $k\left(s,x\right) = \widetilde{\varphi}_\theta\left(s,x\right)$ and ξ is the asymptotic distribution of $N^{1/2}\left(\widehat{\theta} - \theta_0\right)$. Also, it is worth mentioning that the transformation is valid whether any other $N^{1/2}$-consistent estimator of θ_0 were employed.

Theorem 2. *Under H_0 and assuming $C1-C8$, $\beta^0_N\left(\lambda\right) \Rightarrow \boldsymbol{B}\left(\lambda/\pi\right)$; $\lambda \in \widetilde{\Pi}^d$.*

Proof. The proof proceeds, if it is not easier, as that of Theorem 4 part (a) and thus it is omitted. □

Because β^0_N cannot be computed in practice, as it depends on θ_0, we employ the finite sample analogue $\beta_{\widehat{\theta}N} = \mathcal{L}_{\widehat{\theta}N}\left(\alpha_{\widehat{\theta}N}\left(\lambda\right)\right)$, where

$$\beta_{\theta N}\left(\lambda\right) = \frac{1}{G_{\theta N}\left(\pi\right) P_4}\frac{2^{d/2}}{N^{1/2}}\sum_{\lambda_s}^{\lambda} e_{\theta N}\left(\lambda_s\right), \quad \lambda \in \widetilde{\Pi}^d$$

with

$$\begin{aligned} e_{\theta N}\left(\lambda_s\right) &= \mathring{I}^T_{x,\theta}\left(\lambda_s\right) - \widetilde{\varphi}'_{\theta N}\left(\lambda_s\right) b_{\theta N}\left(\lambda_s\right), \\ b_{\theta N}\left(\lambda_s\right) &= \widetilde{\Lambda}^{-1}_{\theta N}\left(\lambda_s\right)\frac{1}{N}\sum_{\lambda_k \prec \lambda_s} \widetilde{\varphi}_{\theta N}\left(\lambda_k\right)\mathring{I}^T_{x,\theta}\left(\lambda_k\right), \end{aligned}$$

are the recursive residuals in the linear projection of $\left\{\mathring{I}^T_{x,\theta}\left(\lambda_k\right)\right\}_{0\prec\lambda_k\prec\lambda_s}$ on $\left\{1, \varphi_\theta\left(\lambda_k\right)\right\}_{0\prec\lambda_k\prec\lambda_s}$, where

$$\mathring{I}^T_{x,\theta}\left(\lambda_s\right) = \frac{I^T_x\left(\lambda_s\right)}{\left|\Psi_\theta\left(\lambda_s\right)\right|^2} - \frac{1}{\boldsymbol{J}}\sum_{\lambda_k \prec \lambda_s}\frac{I^T_x\left(\lambda_k\right)}{\left|\Psi_\theta\left(\lambda_k\right)\right|^2}$$

and $\boldsymbol{J} = \#\left\{\lambda_k \in \Pi^d_n : 0 \prec \lambda_k \prec \lambda_s\right\}$.

To establish the asymptotic equivalence between $\beta_N^0(\cdot)$ and $\beta_{\widehat{\theta}N}(\cdot)$, we need an extra smoothness condition on the model under H_0.

***C9*:** For all $\lambda \in \widetilde{\Pi}^d$, $\varphi_\theta(\lambda)$ is twice continuously differentiable in θ.

Theorem 3. *Assuming $C1 - C9$, under H_0,*

$$\sup_{\lambda \in \widetilde{\Pi}^d} \left| \beta_{\widehat{\theta}N}(\lambda) - \beta_N^0(\lambda) \right| = o_p(1).$$

Proof. The proof proceeds, if it is not easier, as that of Theorem 4 part (b) and thus it is omitted. □

From a computational point of view, it is worth observing that

$$\widetilde{\Lambda}_{\theta N}^{-1}(\lambda_{s+1}) = \widetilde{\Lambda}_{\theta N}^{-1}(\lambda_s) - \frac{\widetilde{\Lambda}_{\theta N}^{-1}(\lambda_s)\, \widetilde{\varphi}_{\theta N}(\lambda_{s+1})\, \widetilde{\varphi}_{\theta N}'(\lambda_{s+1})\, \widetilde{\Lambda}_{\theta N}^{-1}(\lambda_s)}{N + \widetilde{\varphi}_{\theta N}'(\lambda_{s+1})\, \widetilde{\Lambda}_{\theta N}^{-1}(\lambda_s)\, \widetilde{\varphi}_{\theta N}(\lambda_{s+1})}$$

and, proceeding as in Brown, Durbin and Evans (1975),

$$b_{\theta N}(\lambda_{s+1}) = b_{\theta N}(\lambda_s) + \widetilde{\Lambda}_{\theta N}^{-1}(\lambda_{s+1})\, \widetilde{\varphi}_{\theta N}(\lambda_{s+1}) \left[\mathring{I}_{x,\theta}^T(\lambda_{s+1}) - \widetilde{\varphi}_{\theta N}'(\lambda_{s+1})\, b_{\theta N}(\lambda_s) \right],$$

where $\lambda_{s+1} = \min \left\{ \lambda_k \in \widetilde{\Pi}_n^d : \lambda_k \succ \lambda_s \right\}$.

Corollary 1. *Let $\eta : D\left(\widetilde{\Pi}^d\right) \to \mathbb{R}^+$ be a continuous functional. Then, under H_0 and the conditions in Theorem 3, we have that*

$$\eta\left(\beta_{\widehat{\theta}N}\right) \xrightarrow{d} \eta(\boldsymbol{B}).$$

Proof. The proof is an immediate consequence of Theorems 2 and 3 and the continuous mapping theorem, so it is omitted. □

Two standard functionals $\eta(\cdot)$ are the Kolmogorov-Smirnov and the Cramer-von Mises defined as

$$\begin{aligned} \widehat{K}_N &= \sup_{\lambda_s \in \widetilde{\Pi}_n^d} \left| \beta_{\widehat{\theta}N}(\lambda_s) \right| \xrightarrow{d} \sup_{\lambda \in \widetilde{\Pi}^d} \left| \boldsymbol{B}\left(\frac{\lambda}{\pi}\right) \right|, \\ \widehat{C}_N &= \frac{2}{N} \sum_{\lambda_s}^{\lambda} \beta_{\widehat{\theta}N}(\lambda_s)^2 \xrightarrow{d} \frac{2}{(2\pi)^d} \int_{-\mathring{\pi}}^{\pi} \boldsymbol{B}^2\left(\frac{\lambda}{\pi}\right) d\lambda. \end{aligned}$$

Note that the limit variables can be represented as the supremum and integral of the d-dimensional standard Brownian sheet by the change-of-variable.

3.1. Bootstrap Approach.

As mentioned at the beginning of the section, we shall present and examine the bootstrap analogue of $\eta\left(\beta_{\widehat{\theta}N}\right)$. To that end, we define for a generic sequence $\{v(t)\}_{t=1}^n$, the *discrete Fourier transform* as

$$w_v(\lambda) = \frac{1}{N^{1/2}} \sum_{t=1}^{n} v(t)\, e^{it'\lambda}.$$

The bootstrap analogue of $\beta_{\theta N}(\lambda)$ is described in the following *3 STEPS*.

STEP 1: We first obtain the residuals $\{\widehat{\varepsilon}(t)\}_{t=1}^{n}$ as

$$\widehat{\varepsilon}(t) = \frac{1}{N^{1/2}} \sum_{s=-\tilde{n}}^{\tilde{n}} e^{-it'\lambda_s} \Psi_{\widehat{\theta}}^{-1}(\lambda_s) w_x(\lambda_s),$$

and we obtain a random sample of size $\mathring{n} = (2n[1], ..., 2n[d])$ with replacement from the empirical distribution function of $\{\widehat{\varepsilon}(t)\}_{t=1}^{n}$. Denote the sample by $\{\varepsilon^*(t)\}_{t=1}^{\mathring{n}}$ and compute $\{\widetilde{x}^*(t)\}_{t=1}^{\mathring{n}}$ by

$$\widetilde{x}^*(t) = \frac{1}{2^{(d+1)/2} N^{1/2}} \sum_{s=-\tilde{n}}^{\tilde{n}} e^{-it'\widetilde{\lambda}_s} \Psi_{\widehat{\theta}}\left(\widetilde{\lambda}_s\right) w_{\varepsilon^*}\left(\widetilde{\lambda}_s\right), \tag{3.1}$$

where $\widetilde{\lambda}_s$ are

$$\widetilde{\lambda}_s[\ell] = \frac{\pi s[\ell]}{n[\ell]}; \quad s[\ell] = 0, \pm 1, ..., \pm n[\ell], \quad \ell = 1, ..., d.$$

Finally, our bootstrap sample is $\{x^*(t)\}_{t=1}^{n} = \{\widetilde{x}^*(t+n)\}_{t=n+1}^{\mathring{n}}$.

Remark 1. (**a**) *Notice that because* $\beta_{\widehat{\theta}N} = \mathcal{L}_{\widehat{\theta}N}\left(\alpha_{\widehat{\theta}N}\right)$ *is independent of the first two moments of* $\{\varepsilon(t)\}_{t\in\mathbb{Z}^d}$*, we do not need to standardize* $\widehat{\varepsilon}(t)$ *to obtain the bootstrap sample.* (**b**) *The motivation to compute the residuals as in* STEP 1 *comes from the observation that, for any generic sequence* $\{v(t)\}_{t=1}^{n}$*, we have the equality*

$$v(t) = \frac{1}{N^{1/2}} \sum_{s=-\tilde{n}}^{\tilde{n}} e^{it'\lambda_s} w_v(\lambda_s),$$

and then that by Lemmas 2 and 3 of Hidalgo (2009)*, we have that* $w_x(\lambda_s) \simeq \Psi_{\theta_0}(\lambda_s) w_{\varepsilon}(\lambda_s)$*, for all* s*.*

STEP 2: The bootstrap analogue of $\widehat{\vartheta} = \left(\widehat{\theta}', \widehat{\sigma}_{\varepsilon}^2\right)'$ is given by

$$\widehat{\vartheta}^* = \widehat{\vartheta} - \left(\sum_{\lambda_s} \phi_{\widehat{\vartheta}}(\lambda_s) \phi'_{\widehat{\vartheta}}(\lambda_s)\right)^{-1} \frac{\partial}{\partial \vartheta} \mathcal{Q}_N^*\left(\widehat{\vartheta}\right), \tag{3.2}$$

where

$$\mathcal{Q}_N^*(\vartheta) = \frac{1}{N} \sum_{\lambda_s} \left\{ \log f_{\vartheta}(\lambda_s) + \frac{I_{x^*}^T(\lambda_s)}{(2\pi)^d f_{\vartheta}(\lambda_s)} \right\}. \tag{3.3}$$

Remark 2. *We can replace the estimator* $\widehat{\vartheta}^*$ *in* (3.2) *by*

$$\widehat{\vartheta}^* = \arg \min_{\vartheta \in \Theta \times \mathbb{R}^+} \mathcal{Q}_N^*(\vartheta).$$

However, for computational simplicity, see Shao and Tu (1995, *pp.*228 *and* 336)*, we keep our definition of* $\widehat{\vartheta}^*$ *in* (3.2)*.*

STEP 3: Compute the bootstrap $T_p - process$ $\alpha_{\widehat{\theta}^* N}^*(\lambda)$, where

$$\alpha_{\theta N}^*(\lambda) = \frac{N^{1/2}}{2^{1/2} P_0} \left[\frac{G_{\theta N}^*(\lambda)}{G_{\theta N}^*(\pi)} - \frac{\lambda[1]}{\pi} \prod_{\ell=2}^{d} \left(1 + \frac{\lambda[\ell]}{\pi}\right) \right], \quad \lambda \in \widetilde{\Pi}^d, \tag{3.4}$$

with $G^*_{\theta N}(\lambda) = (2\pi)^d N^{-1} \sum_{\lambda_s}^{\lambda} |\Psi_\theta(\lambda_s)|^{-2} I^T_{x^*}(\lambda_s)$. Finally we compute the bootstrap analogue of $\beta_{\widehat{\theta}N}$, $\beta^*_{\widehat{\theta}^* N}$, as

$$\begin{aligned} \beta^*_{\theta N}(\lambda) &= \mathcal{L}_{\theta N}\left(\alpha^*_{\theta N}(\lambda)\right) \\ &= \frac{1}{G^*_{\theta N}(\pi)} \frac{2^{1/2}}{N^{1/2}} \sum_{\lambda_s}^{\lambda} e^*_{\theta N}(\lambda_s), \quad \lambda \in \widetilde{\Pi}^d \end{aligned}$$

with

$$\begin{aligned} e^*_{\theta N}(\lambda_s) &= \mathring{I}^T_{x^*,\theta}(\lambda_s) - \widetilde{\varphi}'_\theta(\lambda_s)\, b^*_{\theta N}(\lambda_s), \\ b^*_{\theta N}(\lambda_s) &= \widetilde{\Lambda}^{-1}_{\theta N}(\lambda_s) \frac{1}{\boldsymbol{J}} \sum_{\lambda_k \prec \lambda_s} \widetilde{\varphi}_{\theta N}(\lambda_k)\, \mathring{I}^T_{x^*,\theta}(\lambda_k), \end{aligned}$$

are the recursive residuals in the linear projection of $\left\{\mathring{I}^T_{x^*,\theta}(\lambda_k)\right\}_{0 \prec \lambda_k \prec \lambda_s}$ on $\{1, \varphi_\theta(\lambda_k)\}_{0 \prec \lambda_k \prec \lambda_s}$ with

$$\mathring{I}^T_{x^*,\theta}(\lambda_s) = \frac{I^T_{x^*}(\lambda_s)}{|\Psi_\theta(\lambda_s)|^2} - \frac{1}{\boldsymbol{J}} \sum_{\lambda_k \prec \lambda_s} \frac{I^T_{x^*}(\lambda_k)}{|\Psi_\theta(\lambda_k)|^2}.$$

With $G^{0*}_N(\lambda) = (2\pi)^d N^{-1} \sum_{\lambda_s}^{\lambda} \breve{I}^T_{\varepsilon^*}(\lambda_s)$, $\lambda \in \widetilde{\Pi}^d$, let β^{0*}_N be as

$$\beta^{0*}_N(\lambda) = \frac{1}{G^{0*}_N(\pi)} \frac{2^{1/2}}{N^{1/2}} \sum_{\lambda_s}^{\lambda} e^*_N(\lambda_s), \qquad \lambda \in \widetilde{\Pi}^d,$$

where

$$\begin{aligned} e^*_N(\lambda_s) &= \breve{I}^T_{\varepsilon^*}(\lambda_s) - \widetilde{\varphi}'_{\widehat{\theta}N}(\lambda_s)\, b^*_N(\lambda_s), \\ b^*_N(\lambda_s) &= \widetilde{\Lambda}^{-1}_{\widehat{\theta}N}(\lambda_s) \frac{1}{\boldsymbol{J}} \sum_{\lambda_k \prec \lambda_s} \widetilde{\varphi}_{\widehat{\theta}N}(\lambda_k)\, \breve{I}^T_{\varepsilon^*}(\lambda_k). \end{aligned}$$

Here $\breve{I}^T_{\varepsilon^*}(\lambda_s) = I^T_{\varepsilon^*}(\lambda_s) - \left(\sum_{\lambda_k \prec \lambda_s} I^T_{\varepsilon^*}(\lambda_k)\right)/\boldsymbol{J}$. Let the notation $\overset{*}{\Rightarrow}$ $\left(\text{and } o_{p^*}(1),\ \overset{d^*}{\longrightarrow},\ \text{etc}\right)$ indicate the weak convergence (convergence in probability and distribution, respectively) of a bootstrap statistic conditional on the observed data.

Theorem 4. *Under the maintained hypothesis and $C1 - C9$, we have that*

$$(a) \quad \beta^{0*}_N(\lambda) \overset{*}{\Rightarrow} \boldsymbol{B}(\lambda/\pi), \quad \lambda \in \widetilde{\Pi}^d \quad \textit{in probability.}$$

$$(b) \quad sup_{\lambda \in \widetilde{\Pi}^d} \left|\beta^*_{\widehat{\theta}^* N}(\lambda) - \beta^{0*}_N(\lambda)\right| = o_{p^*}(1).$$

Corollary 2. *Let $\eta : D\left(\widetilde{\Pi}^d\right) \to \mathbb{R}^+$ be as in Corollary 1. Then, under the maintained hypothesis and conditions in Theorem 4, we have that*

$$\eta\left(\beta^*_{\widehat{\theta}^* N}\right) \overset{d^*}{\to} \eta(\boldsymbol{B}) \qquad \textit{in probability.}$$

Proof. The proof is an immediate consequence of Theorem 4 and the continuous mapping theorem, so it is omitted. □

4. **EXTENSION TO REGRESSION MODELS AND LOCAL ALTERNATIVES**

The aim of this section is twofold. On the one hand, we would like to describe the consequences when the sequence $\{x(t)\}_{t=1}^{n}$ is not observable but they are the errors in the parametric regression model. The second aim of this section is to describe the type of local alternatives that $\eta\left(\beta_{\widehat{\theta}N}\right)$ is able to detect.

4.1. **Regression models.**

With regard to our first aim. Let's consider the model in (1.1), that is

$$y(t) = \beta_0' z(t) + x(t), \qquad t = 1, ..., n, \tag{4.1}$$

where $z(t)$ is the q-dimensional regressor. Recall that as we have excluded the frequency $\lambda = 0$ in the computation of $\eta\left(\beta_{\widehat{\theta}N}\right)$, we have effectively covered in the previous section the scenario when $z(t) = 1$. In our present context and denoting by $\widehat{\beta}$ the least squares estimator, the test becomes $\eta\left(\beta_{\widetilde{\theta}N}\right)$, where

$$\widetilde{\vartheta} = \arg \min_{\vartheta \in \Theta \times \mathbb{R}^+} \mathcal{Q}_N(\vartheta)$$

with

$$\mathcal{Q}_N(\vartheta) = \frac{1}{N} \sum_{\lambda_s} \left\{ \log f_\vartheta(\lambda_s) + \frac{I_{\widehat{x}}^T(\lambda_s)}{(2\pi)^d f_\vartheta(\lambda_s)} \right\}$$

and $\{\widehat{x}(t)\}_{t=1}^{n} = \left\{ y(t) - \widehat{\beta}' z(t) \right\}_{t=1}^{n}$ is the set of the least squares residuals.

Before we state the asymptotic properties of the least squares estimator

$$\widehat{\beta} = \left(\sum_{t=1}^{n} z(t) z'(t) \right)^{-1} \sum_{t=1}^{n} z(t) y(t),$$

let's introduce the following condition denoted as Grenander condition on the deterministic regressors Z, which denotes the $n \times q$ matrix stacking $z(t)$'s. The case with stochastic regressors will be examined later.

***Grenander Condition*:** Let $z_s(t)$ denote the s-th element of the vector $z(t)$ and $A_n = diag\left(\sqrt{\sum_{t=1}^{n} z_s^2(t)}\right)_{s=1}^{q}$. Then, for all $s = 1, ..., q$, as $n \to \infty$,

(*i*) $\sum_{t=1}^{n} z_s^2(t) \to \infty$,
(*ii*) $\max_{1 \le u \le n} \frac{z_s^2(u)}{\sum_{t=1}^{n} z_s^2(t)} \to 0$,
(*iii*) $A_n^{-1} \sum_{t=s+1}^{n} z(t-s) z'(t) A_n^{-1} \to \mathcal{R}(s) = \int_{-\pi}^{\pi} e^{is'\lambda} \mathcal{M}(d\lambda)$,

where $\mathcal{M}(\lambda_2) - \mathcal{M}(\lambda_1)$ is a Hermitian nonnegative matrix and $\mathcal{R} = \mathcal{R}(0) > 0$ and $t - s = (t[\ell] - s[\ell])_{\ell=1}^{d}$.

Examples of deterministic sequences $\{z(t)\}_{t \in \mathbb{Z}^d}$ satisfying the Grenander's conditions are spatial-trend polynomials, see e.g. §3.4 of Cressie (1993). That is, in case of $d = 2$,

$$z(t) = \left[t[1]^s \, t[2]^k \right]_{0 \le s, k \le r}.$$

If $r = 2$,

$$z(t) = \left(1, t[1], t[2], t[1]^2, t[2]^2, t[1]t[2]\right)' \tag{4.2}$$

and hence $q = 6$. In this case, using that

$$\frac{1}{m^{\kappa+1}} \sum_{k=1}^{m} k^{\kappa} \underset{m \nearrow \infty}{\rightarrow} \frac{1}{\kappa+1}, \quad \kappa > -1$$

we obtain

$$\mathcal{R}(s) = \begin{pmatrix} 1 & 3^{1/2}/2 & 3^{1/2}/2 & 5^{1/2}/3 & 5^{1/2}/3 & 3/4 \\ & 1 & 3/4 & 15^{1/2}/4 & 15^{1/2}/6 & 27^{1/2}/6 \\ & & 1 & 15^{1/2}/6 & 15^{1/2}/4 & 27^{1/2}/6 \\ & & & 1 & 15^{1/2}/9 & 45^{1/2}/8 \\ & & & & 1 & 45^{1/2}/8 \\ & & & & & 1 \end{pmatrix}.$$

One consequence of $\mathcal{R}(s)$ being independent of s is that $\mathcal{M}(\lambda)$ has a jump at the origin, and the size of the jump is $\mathcal{R}$. That is,

$$\mathcal{M}(\lambda) = \begin{cases} 0 \text{ if } \lambda[1] < 0 \text{ or } \lambda[2] < 0 \\ \mathcal{R}(s) = \mathcal{R} \text{ if } \lambda[1] \geq 0 \text{ and } \lambda[2] \geq 0. \end{cases}$$

Let's now introduce a slightly milder condition on the spectral density function of the sequence $\{x(t)\}_{t\in\mathbb{Z}^2}$.

C1': (a) The *Generalized Linear process* $\{x(t)\}_{t\in\mathbb{Z}^2}$ in (1.3) has a spectral density function $f(\lambda)$, which is positive and piecewise continuous.
(b) The jumps of $\mathcal{M}(\lambda)$ do not coincide with the discontinuities of $f(\lambda)$.

We have then the following proposition.

Proposition 1. *Under $C1', C2$ and the Grenander conditions, we have that*

$$A_n\left(\widehat{\beta} - \beta_0\right) \rightarrow \mathcal{N}\left(0, \mathcal{R}^{-1} \int_{-\pi}^{\pi} f(\lambda)\, \mathcal{M}(d\lambda)\, \mathcal{R}^{-1}\right).$$

We now comment on the condition $C1'$ and the results on Proposition 1. First, we observe that $C1'$ indicates that the *Generalized linear process* $\{x(t)\}_{t\in\mathbb{Z}^2}$ does not need to satisfy the standard strong mixing conditions for central limit theorem of the least squares to hold true. Moreover, the condition that $\sum_{s\in\mathbb{Z}^d} \psi^2(s) < \infty$ implies that it is possible to allow for long memory and still the results of the latter proposition hold. Of course, the conditions in Jenish and Prucha (2009) ruled out long memory or jumps in the spectral density function, however they allowed for nonlinear processes, say the errors $x(t) = g(\breve{x}(t))$, where $\breve{x}(t)$ is a Generalized linear process. Recall that as we allow for the spectral density function to have jumps, due to results of Ibragimov and Rozanov (1978), it implies that $\{x(t)\}_{t\in\mathbb{Z}^2}$ cannot be strong-mixing. Moreover, our results improve the results in Mardia and Marshall (1984). Finally, the results of Proposition 1 indicates that the fast Fourier transform at λ_0 of $\{x(t)\}_{t=1}^{n}$ satisfies the Central limit theorem if $\lambda_0 \neq \breve{\lambda}$ where $\breve{\lambda}$ is a jump/discontinuity point of $f(\lambda)$.

From Proposition 1, we have the following corollary.

Corollary 3. *Under $C1'$ and $C1-C7$ and the Grenander conditions, we have that*

$$\widetilde{\vartheta}-\widehat{\vartheta}=O_p\left(N^{-1}\right).$$

So, the first conclusion we have is that the asymptotic distribution of the Whittle estimator of ϑ_0 is unaffected by using the residuals instead of the (un)observable $X=\{x(t)\}_{t=1}^{n}$ and that the asymptotic distribution of $A_n\left(\widehat{\beta}-\beta_0\right)$ and $N^{1/2}\left(\widetilde{\vartheta}-\vartheta_0\right)$ are independent.

Denote

$$\widehat{\alpha}_{\theta N}(\lambda)=\frac{1}{2^{d/2}P_4}N^{1/2}\left[\frac{\widehat{G}_{\theta N}(\lambda)}{\widehat{G}_{\theta N}(\pi)}-\frac{\lambda[1]}{\pi}\prod_{\ell=2}^{d}\left(1+\frac{\lambda[\ell]}{\pi}\right)\right],\quad \lambda\in\widetilde{\Pi}^d, \tag{4.3}$$

where

$$\widehat{G}_{\theta N}(\lambda)=\frac{(2\pi)^d}{N}\sum_{\lambda_s}^{\lambda}\frac{I_{\widehat{x}}^{T}(\lambda_s)}{|\Psi_\theta(\lambda_s)|^2} \tag{4.4}$$

are (2.4) and (2.2) but with the residuals $\widehat{x}(t)$ instead of the errors $x(t)$. Similarly, the martingale transformation

$$\widehat{\beta}_{\theta N}(\lambda)=\frac{1}{\widehat{G}_{\theta N}(\pi)P_4}\frac{2^{d/2}}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\widehat{e}_{\theta N}(\lambda_s),\quad \lambda\in\widetilde{\Pi}^d$$

with

$$\begin{aligned}\widehat{e}_{\theta N}(\lambda_s) &= \mathring{I}_{\widehat{x},\theta}^{T}(\lambda_s)-\widetilde{\varphi}_{\theta N}'(\lambda_s)\widehat{b}_{\theta N}(\lambda_s),\\ \widehat{b}_{\theta N}(\lambda_s) &= \widetilde{\Lambda}_{\theta N}^{-1}(\lambda_s)\frac{1}{N}\sum_{\lambda_k\prec\lambda_s}\widetilde{\varphi}_{\theta N}(\lambda_k)\mathring{I}_{\widehat{x},\theta}^{T}(\lambda_k),\end{aligned}$$

are the recursive residuals in the linear projection of $\left\{\mathring{I}_{\widehat{x},\theta}^{T}(\lambda_k)\right\}_{0\prec\lambda_k\prec\lambda_s}$ on $\{1,\varphi_\theta(\lambda_k)\}_{0\prec\lambda_k\prec\lambda_s}$, where

$$\mathring{I}_{\widehat{x},\theta}^{T}(\lambda_s)=\frac{I_{\widehat{x}}^{T}(\lambda_s)}{|\Psi_\theta(\lambda_s)|^2}-\frac{1}{\boldsymbol{J}}\sum_{\lambda_k\prec\lambda_s}\frac{I_{\widehat{x}}^{T}(\lambda_k)}{|\Psi_\theta(\lambda_k)|^2}.$$

With the help of Corollary 3, we have the following theorem:

Theorem 5. *Under $C1-C9$ and the Grenander conditions, we have that*

$$\begin{aligned}(a)\quad \sup_{\lambda\in\widetilde{\Pi}^d}\left|\widehat{\alpha}_{\widetilde{\theta}N}(\lambda)-\alpha_{\widehat{\theta}N}(\lambda)\right| &= o_p(1)\\ (b)\quad \sup_{\lambda\in\widetilde{\Pi}^d}\left|\widehat{\beta}_{\widetilde{\theta}N}(\lambda)-\beta_{\widehat{\theta}N}(\lambda)\right| &= o_p(1).\end{aligned}$$

So, the conclusion from Theorem 5 is that, up to first order asymptotics, the behaviour of the test based on functionals of $\alpha_{\theta N}(\lambda)$ or $\beta_{\theta N}(\lambda)$ is unaltered. Furthermore, the bootstrap can be performed by applying the same algorithm as described in Section 3.1 to the regression residuals $\{\widehat{x}(t)\}_{t=1}^{n}$ due to the asymptotic independence implied by Corollary 3. Alternatively, we can add one more step between Step 1 and 2. That is, do Step 1 with the regression residuals $\{\widehat{x}(t)\}_{t=1}^{n}$and obtain $\{x^*(t)\}_{t=1}^{n}$. Next, generate

$\left\{y^*(t) = z(t)'\widehat{\beta} + x^*(t)\right\}_{t=1}^n$ and compute the OLS residuals $\{\widehat{x}^*(t)\}_{t=1}^n$ by the OLS of $\{y^*(t)\}_{t=1}^n$ on $\{z(t)\}_{t=1}^n$. Finally, run Step 2 with $\{\widehat{x}^*(t)\}_{t=1}^n$.

4.2. **Local alternatives.**

Regarding our second aim of this section, we will see that $\eta\left(\beta_{\widehat{\theta}N}\right)$ is able to detect local alternatives of the type

$$H_{1N}: |\Psi(\lambda)|^2 = |\Psi_{\theta_0}(\lambda)|^2 \left(1 + \tau \frac{l(\lambda)}{N^{1/2}} + \frac{s_N(\lambda)}{N}\right),\ \lambda \in \widetilde{\Pi}^d \text{ for some } \theta_0 \in \Theta,$$

where $\int_{-\pi}^{\pi} l(\lambda)\, d\lambda = 0$, $l(\lambda)$ satisfies the same properties as φ_{θ_0} in $C9$, τ is a constant, possibly unknown, and for some finite N_0, $\sup_{N>N_0} |s_N(\cdot)|$ is an integrable function. Let us consider a couple of examples for $d = 2$.

Example 1. *We wish to study departures of total independence (the white noise) hypothesis in the direction of the first-order isotropic conditional autoregressive (CAR) scheme*

$$\mathbb{E}\{x(t)|...\} = \frac{\theta_0}{N^{1/2}}(x(t-e_1) + x(t+e_1) + x(t-e_2) + x(t+e_2)).$$

In this case, we have that

$$\frac{|\Psi(\lambda)|^2}{|\Psi_{\theta_0}(\lambda)|^2} = 1 - 2\frac{\theta_0}{N^{1/2}}\{\cos(\lambda[1]) + \cos(\lambda[2])\},$$

so that $l(\lambda) = -2\{\cos(\lambda[1]) + \cos(\lambda[2])\}$ *and* $\tau = \theta_0$, *and the remainder function* $s_N(\lambda)$ *being equal to zero.*

(Recall that the general CAR formulation, see Besag (1974), *is given by*

$$\mathbb{E}\{x(t)|x(r):\ r \neq t\} = \sum_{s\in\mathbb{Z}^d\setminus\{0\}} \delta(s)\, x(t-s). \tag{4.5}$$

Example 2. *Suppose now that we wish to study departures of total independence (white noise) hypothesis in the direction of a first-order (isotropic) simultaneous autoregressive (SAR) model, see Whittle* (1954),

$$\begin{aligned} x(t) &= \frac{\theta_0}{N^{1/2}}(x(t-e_1) + x(t+e_1) + x(t-e_2) + x(t+e_2)) \\ &\quad + \varepsilon(t). \end{aligned}$$

Then, we obtain that

$$\frac{|\Psi(\lambda)|^2}{|\Psi_{\theta_0}(\lambda)|^2} = 1 - 2\frac{\theta_0}{N^{1/2}}\{\cos(\lambda[1]) + \cos(\lambda[2])\} + \frac{\theta_0}{N} s_N(\lambda),$$

so that, we have that

$$l(\lambda) = -2\{\cos(\lambda[1]) + \cos(\lambda[2])\} \qquad \text{and } \tau = \theta_0,$$

and $s_N(\lambda)$ *is a function of* $\cos(\lambda[1]), \cos(2\lambda[1])$, $\cos(\lambda[2])$ *and* $\cos(2\lambda[2])$, *which satisfies that* $|s_N(\lambda)| < C$.

Remark 3. *It is worth mentioning that the class of* CAR *models is more general than the* SAR *models. In fact, as Cressie* (1993, *Ch.6*) *observed, any* SAR *model has a* CAR *representation but not vice versa.*

Now, for $\lambda \in \widetilde{\Pi}^d$, let us define

(4.6)

$$\boldsymbol{L}(\lambda) = \frac{1}{(2\pi)^d} \int_{-\mathring{\pi}}^{\lambda} \left\{ l\left(\overline{\lambda}\right) - \widetilde{\varphi}'_{\theta_0}\left(\overline{\lambda}\right) \widetilde{\Lambda}_{\theta_0}^{-1}\left(\overline{\lambda}\right) \frac{1}{(2\pi)^d} \int_{\widetilde{\lambda} \prec \overline{\lambda}} \widetilde{\varphi}_{\theta_0}\left(\widetilde{\lambda}\right) l\left(\widetilde{\lambda}\right) d\widetilde{\lambda} \right\} d\overline{\lambda}$$

and

$$\boldsymbol{M}(\lambda) = \boldsymbol{B}(\lambda/\pi) + \tau \cdot \boldsymbol{L}(\lambda), \quad \lambda \in \widetilde{\Pi}^d.$$

Then, we have the following theorem.

Theorem 6. *Assuming the same conditions of Theorem 3, under* H_{1N}, $\beta_{\widehat{\theta}N} \Rightarrow \boldsymbol{M}$.

Proof. The proof follows by Theorem 3 and standard arguments, so it is omitted. □

Our approach is in contrast with classical Portmanteau tests which are based on the statistic

$$\widetilde{Q}_{q_N N} = N \sum_{s=1}^{q_N} \widetilde{\rho}_N^2(s), \tag{4.7}$$

where $\widetilde{\rho}_N(s)$ is some estimate of the "$s-th$" autocorrelation of $\{\varepsilon(t)\}_{t\in\mathbb{Z}^d}$ after the model has been fitted and q_N is a bandwidth parameter. It can be shown (as in the case $d=1$) that $\widetilde{Q}_{q_N N}$ is approximately distributed as a $\chi^2_{q_N - p}$ under H_0 and assuming that q_N diverges with $\vec{n}$. In addition the test based on $\widetilde{Q}_{q_N N}$ is unable to detect alternatives converging to the null at the rate $q_N^{1/4} N^{-1/2}$, which is slower than the rate $N^{-1/2}$ of our tests. Moreover, the performance of the test can be quite sensitive to the choice of q_N as a particular choice of q_N for which the level of the test is close to the nominal one, it turns out that particular choice delivers a test with low power.

5. MONTE CARLO EXPERIMENT

We examine the finite sample performance of our tests. In particular, we compare Cramer-von Mises tests based on the T_p-process $\alpha_{\widehat{\theta}N}$ and the martingale transformed process $\beta_{\widehat{\theta}N}$. Because the test based on $\alpha_{\widehat{\theta}N}$ is not pivotal, its critical value is computed via bootstrap algorithms. On the other hand, for the test based on the martingale transformation $\beta_{\widehat{\theta}N}(\lambda)$ we employ both the asymptotic critical values as well as those from its bootstrap approach. For all the specifications and sample sizes considered in the experiment, the number of Monte Carlo simulations is 1000. However, to simplify and speed up the computations, we have the bootstrap distribution G_n^* be approximated by the WARP algorithm (Giacomini, Politis and White, 2007). The WARP algorithm permits to approximate the Monte Carlo distribution of the bootstrap test generating only one additional bootstrap replication for each Monte Carlo sample, $\mathcal{X}_{n,m}^{*(1)}$, $m = 1, \ldots, 1000$. Then the empirical distribution of all 1000 bootstrap resamples of our statistic of interest from every independent replication are used jointly to approximate the distribution of the bootstrap test. The results are denoted by T_p, $\widehat{C}_N^*$ and $\widehat{C}_N$, respectively in the Tables 5.1 through 5.4 below.

Three different models are considered for $\{x(t)\}_{t\in\mathbb{Z}}$ as competing models. These models are the first- and second-order simultaneous autoregressive model and the first-order simultaneous moving average model, denoted by $SAR(1)$, $SAR(2)$ and $SMA(1)$, respectively. Specifically, for $d=2$, they are

$$\begin{aligned} SAR(1) &: \\ x(t) &= \theta\left(x(t-e_1)+x(t+e_1)+x(t-e_2)+x(t+e_2)\right)+\varepsilon(t), \\ SAR(2) &: \\ x(t) &= \theta\left(x(t-2e_1)+x(t+2e_1)+x(t-2e_2)+x(t+2e_2)\right)+\varepsilon(t), \\ SMA(1) &: \\ x(t) &= \theta\left(\varepsilon(t-e_1)+\varepsilon(t+e_1)+\varepsilon(t-e_2)+\varepsilon(t+e_2)\right)+\varepsilon(t), \end{aligned}$$

where $\varepsilon(t)$ is an independent and identically distributed mean zero sequence in $\mathbb{Z}^2$. For all the three specifications, we have considered $\theta=0,0.1$ and 0.2 with sample sizes $n=(20,20),(20,40)$ and $(40,40)$. Note that the white noise model is included in our specification by choosing $\theta=0$. We consider two cases. First, we observe $\{x(t)\}_{t=1}^{n}$ directly and second, we observe $\{y(t),z(t)\}$ as specified in section 4.1.

The type I error is examined using three null models, namely the white noise model, $SAR(1)$ and $SMA(1)$ with $\theta=0.1$ and 0.2. The white noise model is estimated under both $SAR(1)$ and $SMA(1)$ specifications.

TABLE 5.1 ABOUT HERE

Table 5.1 reports the rejection frequencies of the three tests for three different significance levels, $0.1, 0.05$ and 0.01. The true data generating processes are indicated in each panel and the white noise cases are indicated by $SMA(1)$ and $SAR(1)$, respectively, depending on which model is used in the estimation. The outcome of the Monte-Carlo experiment seems to indicate that our procedure performs reasonable well. All the tests exhibit rejection rates similar to corresponding levels for all the scenarios. The bootstrap test, $\widehat{C}_N^*$, appears to be more conservative than its corresponding asymptotic one $\widehat{C}_N$, while there is some variation in the performance of the T_p test across different scenarios. All the results seem to be within Monte Carlo error band.

Table 5.2 reports empirical powers of the tests. We considered three scenarios. In the first one, we generated the sample from a $SMA(1)$ process but we wrongly estimated a $SAR(1)$ model. The second scenario we generated a $SAR(1)$ process but we estimated a $SMA(1)$ model; and finally in the third scenario we generated a $SAR(2)$ model but we estimated a $SAR(1)$.

TABLE 5.2 ABOUT HERE

We can signal out some features of the tests. First, the power of each test increases as the sample size increases excluding some exception in the T_p test when $\theta=0.1$; second, the power also increases as the alternative model deviates more from the null model; and third, it appears that neither of the tests dominates the others. The tests based on the transformed process has more power than the T_p test when the true data generating process is

$SMA(1)$ or $SAR(1)$. On the other hand, the latter has more power than the former when it is $SAR(2)$. While the $\widehat{C}_N$ test shows more rejection than the $\widehat{C}_N^*$ test, it seems to be a reflection of the under-rejection tendency of the bootstrap test over the asymptotic test as noted in Table 5.1.

TABLE 5.3 and 5.4 ABOUT HERE

Finally, Table 5.3 and 5.4 report the empirical sizes and powers of the test when $\{x(t)\}$ is the regression residuals as in section 4.1. In particular, $z(t)$ is specified as in (4.2) and the true regression coefficients are set as $\beta_0 = (1, ..., 1)'$. As predicted by our theory, the error in estimating β_0 does not seem to affect the performance of our test much at least in our simulation design and the discussion on the previous tables applies to here as well.

6. Conclusion

We have developed a distribution free test for the specification of a spartial process observed in a lattice and its bootstrap analogue. It allows for the process to be observed as residuals from a regression. Both the asymptotic and bootstrap tests seem to work well in small samples as demonstrated by a set of Monte Carlo simulations. In particular, it is encouraging to practitioners that the asymptotic test has reasonable finite sample size property as it saves the computation time.

7. **APPENDIX A**

We first introduce some more notation. For a generic function $g(\lambda)$, we abbreviate $g(\lambda_s)$ by g_s and C will denote a generic positive and finite constant. Then,

$$\sum_{v \le \lambda_s \le u} g_s = \sum_{\lambda_s \in \widetilde{\Pi}_n^d; v[\ell] \le \lambda_s[\ell] \le u[\ell], \forall \ell} g(\lambda_s),$$

for example. We also drop for simplicity any reference to "T" in w_ξ^T or I_ξ^T, and we shall denote $\zeta(\lambda;\theta) : \widetilde{\Pi}^d \times \Theta \to \mathbb{R}^p$ a function twice continuously differentiable in λ and θ, abbreviating $\zeta(\lambda;\theta_0)$ and $\zeta\left(\lambda;\widehat{\theta}\right)$ respectively by $\zeta(\lambda)$ and $\widehat{\zeta}(\lambda)$.

Lemma 1. *Assume $C1-C8$. Then,*

$$\begin{aligned} (a) \quad \widehat{\vartheta}^* - \widehat{\vartheta} &= o_{p^*}(1) \\ (b) \quad N^{1/2}\left(\widehat{\theta}^* - \widehat{\theta}\right) &= \left(\frac{1}{N}\sum_{\lambda_s} \widetilde{\varphi}_{\widehat{\theta}s}\widetilde{\varphi}'_{\widehat{\theta}s}\right)^{-1} \frac{1}{N^{1/2}}\sum_{\lambda_s} \widetilde{\varphi}_{\widehat{\theta}s} I_{\varepsilon^* s} + o_{p^*}(1). \end{aligned}$$

Proof. Part (a). The proof is quite immediate. Indeed, (3.3) is

$$\frac{1}{N}\sum_{\lambda_s} \frac{f_{\widehat{\vartheta}s}}{f_{\vartheta s}}\left(\frac{I_{x^* s}}{(2\pi)^d f_{\widehat{\vartheta}s}} - 1\right) + \frac{1}{N}\left\{\sum_{\lambda_s} \frac{f_{\widehat{\vartheta}s}}{f_{\vartheta s}} - \log\frac{f_{\widehat{\vartheta}s}}{f_{\vartheta s}} + \log f_{\widehat{\vartheta}s}\right\}. \tag{7.1}$$

Now, the difference between the second term of (7.1) and

$$\int_{-\mathring{\pi}}^{\pi}\left\{\frac{f_{\widehat{\vartheta}}(\lambda)}{f_{\vartheta}(\lambda)}-\log\left(\frac{f_{\widehat{\vartheta}}(\lambda)}{f_{\vartheta}(\lambda)}\right)\right\}d\lambda+\int_{-\mathring{\pi}}^{\pi}\log f_{\widehat{\vartheta}}(\lambda)\,d\lambda$$

converges to zero in probability using Brillinger (1981, p.15) and that uniformly in λ, $\left|f_{\widehat{\vartheta}}(\lambda)-f_{\vartheta_0}(\lambda)\right|=o_p(1)$ by the mean value theorem and $C5$. Moreover, the last displayed expression is greater than or equal to $\frac{(2\pi)^d}{2}+\int_{-\mathring{\pi}}^{\pi}\log f_{\widehat{\vartheta}}(\lambda)\,d\lambda$ with equality when $f_{\widehat{\vartheta}}(\lambda)=f_{\vartheta}(\lambda)$ for all $\lambda\in\widetilde{\Pi}^d$, which is the case only if $\vartheta=\widehat{\vartheta}$ by $C6$. On the other hand, the first term of (7.1) converges to zero uniformly in ϑ by Lemma 15 of Hidalgo (2009) because $f_{\vartheta}^{-1}(\lambda)\,f_{\widehat{\vartheta}}(\lambda)$ is a twice continuos differentiable function by $C5$. From here the conclusion of the lemma proceeds as in Theorem 1 of Hannan (1973), so we omit its details.

Part (b). It follows by an obvious extension of Lemma 14 of Hidalgo (2009), and thus it is omitted. □

Lemma 2. *Assume $C1-C8$. Under H_0, uniform in $\lambda\in\widetilde{\Pi}^d$,*

$$\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\zeta_s\left(\frac{I_{x^*s}}{\left|\Psi_{\widehat{\theta}^*s}\right|^2}-I_{\varepsilon^*s}\right)=-\left(\frac{1}{N}\sum_{\lambda_s}^{\lambda}\zeta_s\varphi'_{\widehat{\theta}s}\right)N^{1/2}\left(\widehat{\theta}^*-\widehat{\theta}\right)+o_{p^*}(1)\,. \tag{7.2}$$

Proof. See Lemma 17 of Hidalgo (2009). □

We now introduce the following notation. For $v_1\prec v_2\in\widetilde{\Pi}^d$, with $\dot{\varepsilon}^*(t)=h(t)\,\varepsilon^*(t)$,

$$\mathcal{E}_{1,N}^*(v_1,v_2)=\left(\frac{1}{N}\sum_{v_1\le\lambda_s\le v_2}\zeta_s\right)\left(\frac{N^{1/2}}{\sum_{t=1}^n h^2(t)}\sum_{t=1}^n\left(\dot{\varepsilon}^*(t)^2-h^2(t)\,\widehat{\sigma}_{\varepsilon}^2\right)\right) \tag{7.3}$$

$$\mathcal{E}_{2,N}^*(v_1,v_2)=\frac{1}{N}\sum_{v_1\le\lambda_s\le v_2}\zeta_s\frac{N^{1/2}}{\sum_{t=1}^n h^2(t)}\sum_{t_1\ne t_2=1}^n\dot{\varepsilon}^*(t_1)\,\dot{\varepsilon}^*(t_2)\,e^{i(t_1-t_2)'\lambda_s}. \tag{7.4}$$

Observe that $\mathcal{E}_{1,N}^*(v_1,v_2)+\mathcal{E}_{2,N}^*(v_1,v_2)=N^{-1/2}\sum_{v_1\le\lambda_s\le v_2}\zeta_s\left(I_{\varepsilon^*s}-\widehat{\sigma}_{\varepsilon}^2\right)$. Also for $\ell=1,...,d$, we define

$$\mathcal{E}_{1,N}^{*(\ell)}(v_1[\ell],v_2[\ell])=\left(\frac{1}{n[\ell]}\sum_{s[\ell]=[\tilde{n}v_1[\ell]/\pi]}^{[\tilde{n}v_2[\ell]/\pi]}\zeta_{s[\ell]}\right)\left(\frac{N^{1/2}}{\sum_{t=1}^n h^2(t)}\sum_{t=1}^n\left(\dot{\varepsilon}^*(t)^2-h^2(t)\,\widehat{\sigma}_{\varepsilon}^2\right)\right)$$

$$\begin{aligned}&\mathcal{E}_{2,N}^{*(\ell)}(v_1[\ell],v_2[\ell])\\ =\;&\frac{1}{N}\sum_{\lambda_s;[\tilde{n}v_1[\ell]/\pi]<s[\ell]<[\tilde{n}v_2[\ell]/\pi]}\zeta_s\frac{N^{1/2}}{\sum_{t=1}^n h^2(t)}\sum_{t_1\ne t_2=1}^n\dot{\varepsilon}^*(t_1)\,\dot{\varepsilon}^*(t_2)\,e^{i(t_1-t_2)'\lambda_s}.\end{aligned}$$

We define $H_N(\cdot,\cdot)$ as a $O_p(1)$ sequence of random variables.

Next we prove that the processes $\left(\lambda[1]\prod_{\ell=2}^{d}(\pi+\lambda[\ell])\right)^{-\upsilon}\mathcal{E}^{*}_{c,N}(-\mathring{\pi}[\ell],\lambda)$, $c=1,2$, are tight for some value of $\upsilon>0$. From Bickel and Wichura (1971), it suffices to show the following lemma.

Lemma 3. *Assuming $C1$, for any $0\le\nu<1/4$ and $\ell=1,...,d$,*

$$
\begin{aligned}
&(a)\ \ \mathbb{E}^{*}\left(\frac{\mathcal{E}^{*(\ell)}_{1,N}(-\mathring{\pi}[\ell],\lambda_1[\ell])}{\lambda_1^{\nu}[\ell]}-\frac{\mathcal{E}^{*(\ell)}_{1,N}(-\mathring{\pi}[\ell],\lambda_2[\ell])}{\lambda_2^{\nu}[\ell]}\right)^{2}\\
=\ & H_N(\lambda_1[\ell],\lambda_2[\ell])(\lambda_2[\ell]-\lambda_1[\ell])^{2-2\nu}\\
&(b)\ \ \mathbb{E}^{*}\left(\frac{\mathcal{E}^{*(\ell)}_{2,N}(-\mathring{\pi}[\ell],\lambda_1[\ell])}{\lambda_1^{\nu}[\ell]}-\frac{\mathcal{E}^{*(\ell)}_{2,N}(-\mathring{\pi}[\ell],\lambda_2[\ell])}{\lambda_2^{\nu}[\ell]}\right)^{4}\\
=\ & H_N(\lambda_1[\ell],\lambda_2[\ell])(\lambda_2[\ell]-\lambda_1[\ell])^{2-4\nu}
\end{aligned}
$$

for all $0<\lambda_1[1]<\lambda_2[1]<\pi$ and $-\pi<\lambda_1[\ell]<\lambda_2[\ell]<\pi$ for $\ell=2,...,d$.

Proof. The proof proceeds, with standard modifications, as that of Lemma 9 of Hidalgo (2009) and thus it is omitted. □

In what follows we shall abbreviate $\widetilde{\varphi}'_{\theta q}\widetilde{\Lambda}^{-1}_{\theta N}(\lambda_q)$ by $\Im_{\theta N}(q)$ and we write

$$
\varkappa^{*}_{s}=\frac{I_{x^{*}s}}{\left|\Psi_{\widehat{\theta}^{*}s}\right|^{2}}-I_{\varepsilon^{*}s};\quad \jmath^{*}_{s}=I_{\varepsilon^{*}s}-\widehat{\sigma}^{2}_{\varepsilon}. \tag{7.5}
$$

Lemma 4. *Assuming $C1-C9$, for all $\varepsilon>0$, in probability*

(7.6)

$$
\lim_{\lambda_0\to-\mathring{\pi}}\ \overline{\lim_{\overrightarrow{n}\to\infty}}\ \Pr{}^{*}\left\{\sup_{-\mathring{\pi}\prec\lambda\preceq\lambda_0}\left|\frac{1}{N}\sum_{\lambda\le\lambda_k\le\lambda_0}\frac{\Im_{\widehat{\theta}N}(k)}{N^{1/2}}\sum_{\lambda_s\prec\lambda_k}\widetilde{\varphi}_{\widehat{\theta}N}(\lambda_s)(\varkappa^{*}_{s}+\jmath^{*}_{s})\right|>\epsilon\right\}=0.
$$

Proof. Take $\lambda_0\prec-\mathring{\pi}/2$ without loss of generality. The triangle inequality implies that

$$
\sup_{-\mathring{\pi}\preceq\lambda\preceq\lambda_0}\left|\frac{1}{N}\sum_{\lambda\le\lambda_k\le\lambda_0}\frac{\Im_{\widehat{\theta}N}(k)}{N^{1/2}}\sum_{\lambda_s\prec\lambda_k}\widetilde{\varphi}_{\widehat{\theta}N}(\lambda_s)(\varkappa^{*}_{s}+\jmath^{*}_{s})\right| \tag{7.7}
$$

$$
\begin{aligned}
\le\ & \frac{C}{N}\sum_{\lambda\le\lambda_k\le\lambda_0}\left\|\Im_{\widehat{\theta}N}(k)\right\|g_N(k)^{\frac{\delta}{2}}\left\{\sup_{-\tilde{n}\le k\le[\lambda_0]_{\tilde{n}}}\left\|\frac{g_N(k)^{-\frac{\delta}{2}}}{N^{1/2}}\sum_{\lambda_s\prec\lambda_k}\widetilde{\varphi}_{\widehat{\theta}N}(\lambda_s)\varkappa^{*}_{s}\right\|\right.\\
&\left.+\sup_{-\tilde{n}<k\le[\lambda_0]_{\tilde{n}}}\left\|\frac{g_N(k)^{-\frac{\delta}{2}}}{N^{1/2}}\sum_{\lambda_s\prec\lambda_k}\widetilde{\varphi}_{\widehat{\theta}N}(\lambda_s)\jmath^{*}_{s}\right\|\right\},
\end{aligned}
$$

for any $0<\delta<1$, where $g_N(k)=N^{-1}(k[1]/n[1])\prod_{\ell=2}^{d}|1+k[\ell]/n[\ell]|$.

First $C7$ implies that $\left\|\widetilde{\Lambda}_{\theta_0}(\lambda)\right\|\ge C^{-1}|\lambda_0[\ell]+\mathring{\pi}[\ell]|$ and hence because

$\widehat{\theta}-\theta_0 = O_p\left(N^{-1/2}\right)$ we have that $\left\|\widetilde{\Lambda}_{\widehat{\theta}N}\left(\lambda_k\right)-\widetilde{\Lambda}_{\theta_0}\left(\lambda_k\right)\right\| = o_p\left(1\right)$. So,

$$\left\|\widetilde{\Lambda}_{\widehat{\theta}N}^{-1}\left(\lambda_k\right)\right\| \leq Cg_N\left(k\right)^{-1} \tag{7.8}$$

which implies that the first factor on the right of (7.7) is bounded by

$$C\left|\frac{1}{N}\sum_{\lambda_k}^{\lambda_0}\left\|\widetilde{\varphi}_{\widehat{\theta}k}\right\| g_N\left(k\right)^{\frac{\delta}{2}-1}\right| = O_p\left(\prod_{\ell=1}^{d}\left|\lambda_0\left[\ell\right]+\mathring{\pi}\left[\ell\right]\right|^{\frac{\delta}{2}}\right).$$

Next, by Lemma 3, the second term inside the braces on the right of (7.7) is $O_p\left(1\right)$ for $\delta>0$ small enough, whereas Lemmas 3 and 1 imply that the first term on the right of (7.7) is bounded by

$$\begin{aligned}&\sup_{-\tilde{n}<k\leq[\lambda_0]_{\tilde{n}}}\left\|\frac{g_N\left(k\right)^{-\frac{\delta}{2}}}{N}\sum_{\lambda_s}^{\lambda_k}\widetilde{\varphi}_{\widehat{\theta}s}\widetilde{\varphi}_{\widehat{\theta}s}'\right\| O_{p^*}\left(1\right)+o_{p^*}\left(\sup_{-\tilde{n}<k\leq[\lambda_0]_{\tilde{n}}}\frac{g_N\left(k\right)^{-\frac{\delta}{2}}}{N^{\delta}}\right)\\ &= O_{p^*}\left(\prod_{\ell=1}^{d}\left|\lambda_0\left[\ell\right]+\mathring{\pi}\left[\ell\right]\right|^{\frac{\delta}{2}}\right)\end{aligned}$$

because $n^{-1}\left[\ell\right]\leq\widetilde{n}^{-1}\left[\ell\right]\leq\inf_{-\tilde{n}<k\leq[\lambda_0]_{\tilde{n}}}\left(k\left[\ell\right]/\widetilde{n}\left[\ell\right]\right)$, $0<\delta<1$ and an obvious extension of Brillinger (1981, p.15). So we conclude that (7.7) $=$ $O_{p^*}\left(\prod_{\ell=1}^{d}\left|\lambda_0\left[\ell\right]+\mathring{\pi}\left[\ell\right]\right|^{\frac{\delta}{2}}\right)$ and hence (7.6) holds true because $\delta>0$. □

Lemma 5. *Assuming $C1-C8$,*

$$\sup_{\lambda\in\widetilde{\Pi}^d}\left\|\sum_{\lambda_s}^{\lambda}\left(\varphi_{\widehat{\theta}^*N}\left(\lambda_s\right)-\varphi_{\widehat{\theta}N}\left(\lambda_s\right)\right)\left(\varkappa_s^*+\jmath_s^*\right)\right\| = O_{p^*}\left(1\right). \tag{7.9}$$

Proof. The expression inside the norm on the left of (7.9) is

$$\begin{aligned}&\sum_{\lambda_s}^{\lambda}\frac{\partial}{\partial\theta}\varphi_{\widehat{\theta}N}\left(\lambda_s\right)\varkappa_s^*\left(\widehat{\theta}^*-\widehat{\theta}\right)+\sum_{\lambda_s}^{\lambda}\frac{\partial}{\partial\theta}\varphi_{\widehat{\theta}N}\left(\lambda_s\right)\jmath_s^*\left(\widehat{\theta}^*-\widehat{\theta}\right)\\ &+\sum_{\lambda_s}^{\lambda}\left(\varphi_{\widehat{\theta}^*N}\left(\lambda_s\right)-\varphi_{\widehat{\theta}N}\left(\lambda_s\right)-\frac{\partial}{\partial\theta}\varphi_{\widehat{\theta}N}\left(\lambda_s\right)\left(\widehat{\theta}^*-\widehat{\theta}\right)\right)\left(\varkappa_s^*+\jmath_s^*\right).\end{aligned} \tag{7.10}$$

By $C9$ and then noting that $|a-b|\leq(a-b)+2b$ for $a>0$ and $b>0$, the norm of the third term of (7.10) is bounded by

$$\begin{aligned}&C\left\|\widehat{\theta}^*-\widehat{\theta}\right\|^2\sum_{\lambda_s}\left|\varkappa_s^*+\jmath_s^*\right| = O_p\left(\left\|\widehat{\theta}^*-\widehat{\theta}\right\|^2\right)\left\{\sum_{\lambda_s}\left(\varkappa_s^*+\jmath_s^*\right)+\frac{\widehat{\sigma}_{\varepsilon}^2}{\pi}\sum_{\lambda_s}1\right\}\\ &= O_{p^*}\left(1\right)\end{aligned}$$

by Lemma 1 and then using Lemmas 3 and 18 of Hidalgo (2009). So, uniformly in λ, the third term of (7.10) is $o_{p^*}\left(1\right)$. Likewise, the first term of (7.10) is $O_{p^*}\left(1\right)$ uniformly in λ using Lemma 4 with $\widehat{\zeta}\left(\lambda\right)=\frac{\partial}{\partial\theta}\varphi_{\widehat{\theta}}\left(\lambda\right)$ and Lemma 1. Finally, the second term of (7.10) is $O_{p^*}\left(1\right)$ by Lemma 18 of Hidalgo (2009) with $\widehat{\zeta}\left(\lambda\right)=\frac{\partial}{\partial\theta}\varphi_{\widehat{\theta}}\left(\lambda\right)$. □

Lemma 6. *Assuming $C1-C9$, for all $\varepsilon>0$, in probability*

(7.11)
$$\lim_{\lambda_0\to-\mathring{\pi}}\overline{\lim_{\vec{n}\to\infty}}\Pr{}^*\left\{\sup_{-\mathring{\pi}\prec\lambda\preceq\lambda_0}\left|\frac{1}{N}\sum_{\lambda\leq\lambda_k\leq\lambda_0}\frac{\Im_{\widehat{\theta}^*N}(k)}{N^{1/2}}\sum_{s\prec k}\widetilde{\varphi}_{\widehat{\theta}^*N}(\lambda_s)(\varkappa_s^*+\jmath_s^*)\right|>\varepsilon\right\}=0.$$

Proof. Notice that Lemma 1 implies that it suffices to show (7.11) in the set $\left\{\left\|\widehat{\theta}^*-\widehat{\theta}\right\|<CN^{-1/2}m_N^{-1}\right\}$, where $m_N+m_N^{-1}N^{-1/2}\to0$. On the other hand, Lemmas 2 and 3 imply that, uniformly in k,

$$\frac{1}{N^{1/2}}\sum_{s\prec k}\widetilde{\varphi}_{\widehat{\theta}^*N}(\lambda_s)\varkappa_s^*=-\left(\frac{\widehat{\sigma}_\varepsilon^2}{N}\sum_{s\prec k}\widetilde{\varphi}_{\widehat{\theta}^*N}(\lambda_s)\widetilde{\varphi}'_{\widehat{\theta}^*N}(\lambda_s)\right)N^{1/2}\left(\widehat{\theta}-\widehat{\theta}^*\right)+o_{p^*}(1)$$

$$\frac{1}{N^{1/2}}\sum_{s\prec k}\widetilde{\varphi}_{\widehat{\theta}^*N}(\lambda_s)\jmath_s^*=\frac{1}{N^{1/2}}\sum_{s\prec k}\widetilde{\varphi}_{\widehat{\theta}^*N}(\lambda_s)\jmath_s^*+O_{p^*}\left(n^{-1/2}\right)\tag{7.12}$$

proceeding as in the proof of (7.9) but with $\varkappa_s^*+\jmath_s^*$ replaced by $\jmath_s^*$ there. Observe that we can take $\lambda_0\prec-\mathring{\pi}/2$. Next, $C8$ implies that

$$\sup_{-\tilde{n}<k\leq[\lambda_0]_{\tilde{n}}}\left\|\widetilde{\Lambda}_{\widehat{\theta}^*N}(k)-\widetilde{\Lambda}_{\widehat{\theta}N}(k)\right\|=O_{p^*}\left(\left\|\widehat{\theta}^*-\widehat{\theta}\right\|\right)\prod_{\ell=1}^d\left|\lambda_0[\ell]+\mathring{\pi}[\ell]\right|$$

which, together with (7.8), implies that $\left\|\widetilde{\Lambda}_{\widehat{\theta}^*N}^{-1}(k)\right\|=O_p\left(g_N(k)^{-1}\right)$.

So, we have that for $0<\delta<1/2$,

$$\sup_{-\mathring{\pi}\prec\lambda\preceq\lambda_0}\left\|\frac{1}{\widetilde{n}}\sum_{\lambda\leq\lambda_k\leq\lambda_0}\frac{\Im_{\widehat{\theta}^*n}(k)}{N^{1/2}}\sum_{s\prec k}\widetilde{\varphi}_{\widehat{\theta}^*N}(\lambda_s)(\varkappa_s^*+\jmath_s^*)\right\|\tag{7.13}$$

$$\begin{aligned}=\;&O_{p^*}(1)\sup_{-\mathring{\pi}\prec\lambda\preceq\lambda_0}\left|\frac{1}{N}\sum_{\lambda\leq\lambda_k\leq\lambda_0}\left\|\widetilde{\varphi}_{\widehat{\theta}N}(k)\right\|g_N(k)^{(-1+\delta/2)}\right|\\&\times\left\{\sup_{\lambda_k\leq\lambda_0}\left\|g_N(k)^{-\delta/2}\frac{1}{N^{1/2}}\sum_{s\prec k}\widetilde{\varphi}_{\widehat{\theta}N}(\lambda_s)\jmath_s^*\right\|+O_{p^*}\left(\prod_{\ell=1}^d\left|\lambda_0[\ell]+\mathring{\pi}[\ell]\right|^{\frac{\delta}{2}}\right)\right\},\end{aligned}$$

by (7.12) and because $C2$ implies that $\overrightarrow{n}\leq\inf_{\lambda_k\leq\lambda_0}(k[\ell]/\widetilde{n}[\ell])$. But Lemma 4 implies that $\sup_{\lambda_k\leq\lambda_0}\left\|g_N(k)^{-\delta/2}N^{-1/2}\sum_{s\prec k}\widetilde{\varphi}_{\widehat{\theta}N}(\lambda_s)\jmath_s^*\right\|=O_{p^*}(1)$ and $C3$ implies that

$$\sup_{-\mathring{\pi}\prec\lambda\preceq\lambda_0}\frac{1}{N}\sum_{\lambda\leq\lambda_k\leq\lambda_0}\left\|\widetilde{\varphi}_{\widehat{\theta}N}(k)\right\|g_N(k)^{(-1+\delta/2)}=O_p\left(\prod_{\ell=1}^d\left|\lambda[\ell]+\mathring{\pi}[\ell]\right|^{\frac{\delta}{2}}\right),$$

so it is the left side of (7.13). From here, we conclude because $\delta>0$. □

8. APPENDIX B

8.1. PROOF OF THEOREM 4.

We begin with part (a). Using $G_N^*(\pi) = \widehat{\sigma}_\varepsilon^2 + o_{p^*}(1)$ and recalling that $\Im_{\theta N}(\lambda_s) = \widetilde{\varphi}'_{\theta N}(\lambda_s)\widetilde{\Lambda}^{-1}_{\theta N}(\lambda_s)$, we obtain that, uniform in $\lambda \in \widetilde{\Pi}^d$,

$$\widehat{\alpha}^*_{\widehat{\theta}^* N}(\lambda) = \frac{(2\pi)^{2d}}{\widehat{\sigma}_\varepsilon^2}\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda} \jmath_s^* - \frac{(2\pi)^{2d}}{\widehat{\sigma}_\varepsilon^2}\widehat{\Upsilon}_N(\lambda) + o_{p^*}(1), \tag{8.1}$$

where $\widehat{\Upsilon}_N(\lambda) = N^{-3/2}\widehat{\sigma}_\varepsilon^{-2}\sum_{\lambda_s}^{\lambda}\left(\Im_{\widehat{\theta}N}(\lambda_s)\sum_{p^*\prec s}^{[\lambda]}\widetilde{\varphi}_{\widehat{\theta}N}(k)\jmath_k^*\right)$ and $\jmath_k^*$ as defined in (7.5).

Suppose, to be shown later, that the convergence in $\lambda \preceq \lambda_0$ holds true for any $\lambda_0 \in \widetilde{\Pi}^d$. Then, because the Brownian sheet $\boldsymbol{B}(\lambda/\pi)$ and the limit of $N^{-1/2}\sum_{\lambda_s}^{\lambda}\jmath_s^*$ are continuous in $\widetilde{\Pi}^d$, Billingsley's (1968) Theorem 4.2 implies that it suffices to show that for all $\varepsilon > 0$,

$$\lim_{\lambda_0 \to -\mathring{\pi}}\ \overline{\lim_{\vec{n}\to\infty}}\ \Pr{}^*\left\{\sup_{-\mathring{\pi}\prec\lambda\preceq\lambda_0}\left|\widehat{\Upsilon}_N(\lambda_0) - \widehat{\Upsilon}_N(\lambda)\right| > \varepsilon\right\} = 0,$$

in probability. But this holds true by Lemma 5, cf. the second term on the right of (7.7).

So, to complete the proof we need to show that, for any $\lambda_0 \in \widetilde{\Pi}^d$, the first two terms on the right of (8.1) converge in bootstrap to $\pi^{-d/2}\boldsymbol{B}(\lambda/\pi)$ in $-\mathring{\pi} \prec \lambda \preceq \lambda_0$ in probability. Fidi's convergence follows by Lemma 18 Hidalgo (2009) part (b) after we write $\widehat{\Upsilon}_N(\lambda)$ as

$$\frac{(2\pi)^{2d}}{\widehat{\sigma}_\varepsilon^2}\frac{1}{N^{1/2}}\sum_{\lambda_k}\left(\frac{1}{N}\sum_{\lambda_s}^{\lambda_k\wedge\lambda}\Im_{\widehat{\theta}N}(s)\right)\widetilde{\varphi}_{\widehat{\theta}N}(k)\jmath_k^*$$

and $\left(N^{-1}\sum_{\lambda_s}^{\lambda_k\wedge\lambda}\Im_{\widehat{\theta}N}(s)\right)\widetilde{\varphi}_{\widehat{\theta}N}(k)$ satisfies the same conditions of Lemma 18 Hidalgo (2009) for $\widehat{\zeta}(\lambda)$. Then, it suffices to prove tightness. Since $N^{-1/2}\sum_{\lambda_s}^{\lambda}\jmath_s^*$ is tight by Lemma 2, we only need to show the tightness condition of $\widehat{\Upsilon}_N(\lambda)$. By Billingsley's (1968) Theorem 15.6, it suffices to show that

$$\mathbb{E}^*\left(\left|\widehat{\Upsilon}_N(\vartheta) - \widehat{\Upsilon}_N(\mu)\right|\left|\widehat{\Upsilon}_N(\lambda) - \widehat{\Upsilon}_N(\vartheta)\right|\right) = O_p(1)\prod_{\ell=1}^{d}|\lambda[\ell] - \mu[\ell]|^{2\delta}$$

for all $-\mathring{\pi}[\ell] \le \mu[\ell] < \vartheta[\ell] < \lambda[\ell] \le \pi$ and some $\delta > 1/2$. Observe that we can take $\widetilde{n}^{-1} < |\lambda[\ell] - \mu[\ell]|$ since otherwise the last inequality is trivial. Because $(\lambda - \vartheta)(\vartheta - \mu) < (\lambda - \mu)^2$ by the Cauchy-Schwarz's inequality, it suffices to show the last displayed equality holds for $\mathbb{E}^*\left|\widehat{\Upsilon}_N(\lambda) - \widehat{\Upsilon}_N(\mu)\right|^2$ which is

$$\begin{aligned}
&\frac{1}{\widehat{\sigma}_\varepsilon^4}\frac{1}{N^3}\sum_{\mu\le\lambda_s,\lambda_k\le\lambda}\Im_{\widehat{\theta}N}(s)\sum_{\ell_1\prec s}\sum_{\ell_2\prec k}\widetilde{\varphi}_{\widehat{\theta}N}(\ell_1)\widetilde{\varphi}'_{\widehat{\theta}N}(\ell_2)\mathbb{E}^*\left(\jmath_{\ell_1}^*\jmath_{\ell_2}^*\right)\Im'_{\widehat{\theta}N}(k)\\
= {}& H_N(\lambda,\mu)\frac{1}{N^2}\sum_{\mu\le\lambda_s,\lambda_k\le\lambda}\left\|\Im_{\widehat{\theta}N}(s)\right\|\left\|\Im_{\widehat{\theta}N}(k)\right\|\\
= {}& H_N(\lambda,\mu)\left(\left|\widetilde{\Im}_{\widehat{\theta}}(\mu,\lambda)\right|^2 + N^{-2}\right),
\end{aligned}$$

because $\left\| N^{-1} \sum_{\lambda_s}^{\lambda} \left\| \Im_{\widehat{\theta}N}(s) \right\| - \widetilde{\Im}_{\widehat{\theta}}(\lambda) \right\| = O_p\left(N^{-1}\right)$ by Lemma 12 of Hidalgo (2009) with $\widetilde{\Im}_{\widehat{\theta}}(\mu,\lambda) = \pi^{-1} \int_{-\mu}^{\lambda} \Im_{\widehat{\theta}}(w)\, dw$. From here we conclude the proof of part (a) by Billingsley's (1968) Theorem 15.6, because $\widetilde{\Im}_{\theta}(\lambda)$ is a monotonic, continuous and nondecreasing function such that $\left| \widetilde{\Im}_{\widehat{\theta}}(\lambda) - \widetilde{\Im}_{\widehat{\theta}}(\mu) \right| = O_p(1) \prod_{\ell=1}^{d} |\lambda[\ell] - \mu[\ell]|^{\delta}$, $\delta > 1/2$ and $\widetilde{n}^{-1}[\ell] \le |\lambda[\ell] - \mu[\ell]|$. To show part (b), by definition of $\beta^*_{\theta N}$ and β^*_N, it suffices to show that

$$\left| \frac{1}{N^{1/2}} \sum_{\lambda_k}^{\lambda} \varkappa^*_k - \Im_{\widehat{\theta}N}(k) \frac{1}{N} \sum_{\lambda_s}^{\lambda_k} \widetilde{\varphi}_{\widehat{\theta}N}(s)\, \varkappa^*_s \right| \tag{8.2}$$

(8.3)

$$\frac{1}{G_{\widehat{\theta}^*N}(\pi)} \left(\frac{1}{N} \sum_{\lambda_k}^{\lambda} \Im_{\widehat{\theta}N}(k) \frac{1}{N^{1/2}} \sum_{s \prec k} \widetilde{\varphi}_{\widehat{\theta}N}(s) \left(\frac{I_{x^*s}}{\left|\Psi_{\widehat{\theta}^*s}\right|^2} - \frac{G_{\widehat{\theta}^*N}(\pi)}{2\pi} \right) \right)$$
$$- \frac{1}{G_{\widehat{\theta}^*N}(\pi)} \left(\frac{1}{N} \sum_{\lambda_k}^{\lambda} \Im_{\widehat{\theta}^*N}(k) \frac{1}{N^{1/2}} \sum_{s \prec k} \widetilde{\varphi}_{\widehat{\theta}^*N}(s) \left(\frac{I_{x^*s}}{\left|\Psi_{\widehat{\theta}^*s}\right|^2} - \frac{G_{\widehat{\theta}^*N}(\pi)}{2\pi} \right) \right)$$

converge to zero uniformly in $\lambda \in \widetilde{\Pi}^d$. Expression (8.2) is $o_{p^*}(1)$, uniformly in $\lambda \in \widetilde{\Pi}^d$, because as we argued with (57) in Delgado et al. (2011)

$$- \frac{\widetilde{\varphi}'_{\widehat{\theta}^*N}(s)}{G_{\widehat{\theta}^*N}(\pi)} \Im^{-1}_{\widehat{\theta}^*N} \frac{1}{N^{1/2}} \sum_{s \prec k} \widetilde{\varphi}_{\widehat{\theta}^*N}(k)\, I_{\varepsilon^*k} = 0.$$

Next, because

$$\frac{1}{N} \sum_{\lambda_k}^{\lambda} \left\| \widetilde{\varphi}_{\widehat{\theta}N}(k) \right\| \left\| \widetilde{\Lambda}^{-1}_{\widehat{\theta}N}(k) \right\| \frac{1}{N} \sum_{s \prec k} \left\| \widetilde{\varphi}_{\widehat{\theta}N}(s) \right\|$$
$$\le \ C \frac{1}{N} \sum_{\lambda_k}^{\lambda} \left\| \widetilde{\varphi}_{\widehat{\theta}N}(k) \right\| \left\| \widetilde{\Lambda}^{-1}_{\widehat{\theta}N}(k)\, g_N(k) \right\| \le C \frac{1}{N} \sum_{\lambda_k}^{\lambda} \left\| \widetilde{\varphi}_{\widehat{\theta}N}(k) \right\| = O_p(1)$$

by integrability of $\varphi_{\widehat{\theta}^*}(\lambda)$ and (7.8), it implies that the contribution into (8.2) due to the term $o_{p^*}(1)$ on part (a) of Theorem 1 is negligible.

Next we examine (8.3). Because $G_{\widehat{\theta}^*N}(\pi) - G_{\widehat{\theta}N}(\pi) = o_{p^*}\left(N^{-1/2}\right)$ by Lemma 3 and $G_{\widehat{\theta}N}(\pi) - G_{\theta_0 N}(\pi) = o_p\left(N^{-1/2}\right)$ by Lemma 15 of Hidalgo (2009), it suffices to show that

(8.4)

$$\frac{1}{N} \sum_{\lambda_k}^{\lambda} \left\{ \frac{\Im_{\widehat{\theta}N}(k)}{N^{1/2}} \sum_{\lambda_k \prec \lambda_s} \widetilde{\varphi}_{\widehat{\theta}N}(s) \left(\varkappa^*_s + \jmath^*_s \right) - \frac{\Im_{\widehat{\theta}^*N}(k)}{N^{1/2}} \sum_{s \prec k} \widetilde{\varphi}_{\widehat{\theta}^*N}(s) \left(\varkappa^*_s + \jmath^*_s \right) \right\}$$

converges to zero uniformly in $\lambda \in \widetilde{\Pi}^d$ after observing that

$$\sup_{\lambda \in \widetilde{\Pi}^d} \left| \sum_{\lambda_k}^{\lambda} \Im_{\widehat{\theta}^*N}(k) \sum_{s \prec k} \widetilde{\varphi}_{\widehat{\theta}^*N}(s) - \sum_{\lambda_k}^{\lambda} \Im_{\widehat{\theta}N}(k) \sum_{s \prec k} \gamma_{\widehat{\theta}s} \right| = 0.$$

First, we observe that Lemmas 3 and 5 imply that it suffices to show the uniform convergence in $-\mathring{\pi} \prec \lambda \preceq \lambda_0$ for any $\lambda_0 \prec 0$. But (8.4) is

$$\frac{1}{N}\sum_{\lambda_k}^{\lambda} \Im_{\widehat{\theta}^* N}(k) \frac{1}{N^{1/2}} \sum_{s \prec k} \left(\widetilde{\varphi}_{\widehat{\theta}N}(s) - \widetilde{\varphi}_{\widehat{\theta}^* N}(s)\right)\left(\varkappa_s^* + \jmath_s^*\right) \tag{8.5}$$

$$+\frac{1}{N}\sum_{\lambda_k}^{\lambda} \left(\Im_{\widehat{\theta}N}(k) - \Im_{\widehat{\theta}^* N}(k)\right) \frac{1}{N^{1/2}} \sum_{s \prec k} \widetilde{\varphi}_{\widehat{\theta}N}(s)\left(\varkappa_s^* + \jmath_s^*\right). \tag{8.6}$$

So, the theorem follows if (8.5) and (8.6) are both $o_{p^*}(1)$ uniformly in $-\mathring{\pi} \prec \lambda \preceq \lambda_0$. To that end, we first show that

$$\sup_{\lambda \in \widetilde{\Pi}^d} \frac{1}{N}\sum_{\lambda_s}^{\lambda} \left\|\widetilde{\varphi}_{\widehat{\theta}N}(s) - \widetilde{\varphi}_{\widehat{\theta}^* N}(s)\right\| = o_{p^*}(1), \tag{8.7}$$

$$\sup_{-\mathring{\pi} \prec \lambda \preceq \lambda_0} \left\|\widetilde{\Lambda}^{-1}_{\widehat{\theta}N}(\lambda) - \widetilde{\Lambda}^{-1}_{\widehat{\theta}}(\lambda)\right\| = o_p(1), \tag{8.8}$$

$$\sup_{-\mathring{\pi} \prec \lambda \preceq \lambda_0} \left\|\widetilde{\Lambda}^{-1}_{\widehat{\theta}^* N}(\lambda) - \widetilde{\Lambda}^{-1}_{\widehat{\theta}N}(\lambda)\right\| = o_{p^*}(1). \tag{8.9}$$

First, (8.7) follows proceeding as with the proof of (7.9) in Lemma 5 but without the factor $\varkappa_s^* + \jmath_s^*$, (8.8) follows because $C8$ implies that $\widetilde{\Lambda}_{\theta_0}(\lambda_0) > 0$ and because by $C3$ $\left\|\widetilde{\varphi}_{\widehat{\theta}}(\lambda)\widetilde{\varphi}'_{\widehat{\theta}}(\lambda)\right\|$ satisfies the same conditions of $\zeta(\lambda)$ in Lemma 12 of Hidalgo (2009), so that

$$\sup_{-\mathring{\pi} \prec \lambda \preceq \lambda_0} \left\|\widetilde{\Lambda}_{\widehat{\theta}}(\lambda) - \widetilde{\Lambda}_{\widehat{\theta}N}(\lambda)\right\| = O\left(n^{-1}\right),$$

whereas (8.9) follows proceeding as with the proof of (8.7) and (8.8).

Now we show that (8.5) is $o_p(1)$ uniformly in $-\mathring{\pi} \prec \lambda \preceq \lambda_0$, which follows by Lemma 5 and (8.7) – (8.9) noting that $\left(\widetilde{\varphi}'_{\widehat{\theta}N}(s) - \widetilde{\varphi}'_{\widehat{\theta}^* N}(s)\right) = \left(\phi'_{\widehat{\theta}N}(s) - \phi'_{\widehat{\theta}^* N}(s), 0\right)$, so does (8.6) by (8.7) and (8.9) and that

$$\sup_{-\mathring{\pi} \prec \lambda \preceq \lambda_0} \left|\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda} \widetilde{\varphi}_{\widehat{\theta}s}\left(\varkappa_s^* + \jmath_s^*\right)\right| = O_{p^*}(1)$$

by Lemmas 1 and 2 with $\widehat{\zeta}(\lambda) = \widetilde{\varphi}_{\widehat{\theta}}(\lambda)$ there and observing Lemma 1 and that Lemma 12 of Hidalgo (2009) implies that $N^{-1}\sum_{\lambda_s}^{\lambda} \widetilde{\varphi}_{\widehat{\theta}N}(s)\widetilde{\varphi}'_{\widehat{\theta}N}(s) \to_P \int_{-\pi}^{\lambda} \widetilde{\varphi}_{\theta_0}(\omega)\widetilde{\varphi}'_{\theta_0}(\omega)\,d\omega$. □

8.2. PROOF OF PROPOSITION 1 AND COROLLARY 3.

8.2.1. *Proof of Proposition 1.*

First we notice that it suffices to show that

$$A_n^{-1}\sum_{t=1}^{n} z(t)\,x(t) \xrightarrow{d} \mathcal{N}\left(0, \int_{-\pi}^{\pi} f(\lambda)\,\mathcal{M}(d\lambda)\right). \tag{8.10}$$

To that end, we shall show first that

$$A_n^{-1}E\left(\sum_{t=1}^{n} z(t)\,x(t)\sum_{t=1}^{n} z'(t)\,x(t)\right)A_n^{-1}\to\int_{-\pi}^{\pi} f(\lambda)\,\mathcal{M}(d\lambda).$$

For that purpose, we first notice that by Weierstrass approximation Theorem, we have that we can find two trigonometric polynomials $f_x^{(1)}(\lambda)$ and $f_x^{(2)}(\lambda)$ such that $f_x^{(2)}(\lambda)-f_x^{(1)}(\lambda)\le\epsilon$ and $f_x^{(1)}(\lambda)\le f(\lambda)\le f_x^{(2)}(\lambda)$. When the spectral density function is not continuous, we can employ the construction given in Hannan (1970). Observe that the latter implies that

$$f_x^{(1)}(\lambda)\,\mathcal{Z}(\lambda)\,\mathcal{Z}^*(\lambda)\le f(\lambda)\,\mathcal{Z}(\lambda)\,\mathcal{Z}^*(\lambda)\le f_x^{(2)}(\lambda)\,\mathcal{Z}(\lambda)\,\mathcal{Z}^*(\lambda),$$

where $\mathcal{Z}(\lambda)=A_n^{-1}\sum_{t=1}^{n} z(t)\,e^{it'\lambda}$ and

$$\begin{aligned}\int_{-\pi}^{\pi}\left(f_x^{(2)}(\lambda)-f_x^{(1)}(\lambda)\right)\mathcal{Z}(\lambda)\,\mathcal{Z}^*(\lambda)\,d\lambda &\le \epsilon\int_{-\pi}^{\pi}\mathcal{Z}(\lambda)\,\mathcal{Z}^*(\lambda)\,d\lambda\\ &= \epsilon A_n^{-1}\sum_{t=1}^{n} z(t)\,z'(t)\,A_n^{-1}. \qquad (8.11)\\ &\to \epsilon\mathcal{R}.\end{aligned}$$

So, it suffices to show (8.10) with $x(t)$ being replaced by $\ddot{x}(t)$, where

$$\ddot{x}(t)=\sum_{s\in\mathbb{M}^d}\psi(s)\,\varepsilon(t-s),$$

and $\mathbb{M}^d=\{s:|s[\ell]|<J,\quad \ell=1,...,d\}$. This is a moving average of finite order. Now, by standard algebra,

$$\alpha' A_n^{-1}\sum_{t=1}^{n} z(t)\,\ddot{x}(t)=\sum_{s\in\mathbb{M}^d}\psi(s)\sum_{t=1}^{n}\left(\sum_{r=1}^{q}\alpha[r]\,\frac{z_r(t)}{A_n[r]}\right)\varepsilon(t-s),$$

where α is a q-dimensional vector with norm 1. Now, for each $s\in\mathbb{M}^d$, the term on right side of , $\sum_{t=1}^{n}\left(\sum_{r=1}^{q}\alpha[r]\,\frac{z_r(t)}{A_n[r]}\right)\varepsilon(t-s)$, converges in distribution to a normal random variable if the Lindeberg's condition is satisfied. However, this is the case as $\varepsilon(t)$ is iid and thus $\varepsilon^2(t)$ is uniformly integrable, and for any $\delta>0$

$$\begin{aligned}&\sum_{t=1}^{n} E\left(\frac{z_s(t)^2}{A_n[s]^2}\right)\varepsilon^2(t-s)\,\mathbb{I}\left\{\left(\frac{z_s(t)^2}{A_n[r]^2}\right)\varepsilon^2(t-s)>\delta\right\}\\ \le\ &\mathrm{E}\varepsilon^2(t)\,\mathbb{I}\left\{\varepsilon^2(t)>\min_u\frac{A_n[r]^2}{z_{u,r}^2}\delta\right\}\to 0\end{aligned}$$

since $\left(\sum_{t=1}^{n} z_r(t)^2 A_n[r]^{-2}\right)=1$ and $\max_u z_r^2(u)\left(\sum_{t=1}^{n} z_r(t)^2\right)^{-1}\to 0$ for all $r=1,...,q$.

8.2.2. ***Proof of Corollary 3.***

We now show that

$$\widehat{\vartheta}-\widetilde{\vartheta}=O_p\left(N^{-1}\right).$$

To that end, it suffices to check Robinson (1988), that is

$$\frac{1}{N}\sum_{\lambda_s}\phi_{\widehat{\vartheta},s}\left\{\frac{I_{\widehat{x},s}}{(2\pi)^d f_{\widehat{\vartheta},s}}-1\right\}=O_p\left(N^{-1}\right). \tag{8.12}$$

The left side of (8.12), except the multiplicative constant $(2\pi)^{-d}$, is

$$\begin{aligned}\frac{1}{N}\sum_{\lambda_s}\phi_{\widehat{\vartheta},s}\frac{I_{\widehat{x},s}-I_{x,s}}{f_{\widehat{\vartheta},s}} &= \left(\widehat{\beta}-\beta\right)'\frac{1}{N}\sum_{\lambda_s}\phi_{\widehat{\vartheta},s}\frac{I_{z,s}}{f_{\widehat{\vartheta},s}}\left(\widehat{\beta}-\beta\right)\\ &\quad -2\left(\widehat{\beta}-\beta\right)'\frac{1}{N}\sum_{\lambda_s}\phi_{\widehat{\vartheta},s}\frac{\mathrm{Re}\left(w_{z,s}\overline{w_{x,s}}\right)}{f_{\widehat{\vartheta},s}}.\end{aligned} \tag{8.13}$$

First by standard linearization and that $\widehat{\vartheta}-\vartheta_0=O_p\left(N^{-1/2}\right)$, we have that

$$\frac{1}{N}\sum_{\lambda_s}\phi_{\widehat{\vartheta},s}\frac{I_{z,s}}{f_{\widehat{\vartheta},s}}=\frac{1}{N}\sum_{\lambda_s}\phi_{\vartheta_0,s}\frac{I_{z,s}}{f_{\vartheta_0,s}}\left(1+O_p\left(N^{-1/2}\right)\right).$$

Next Proposition 1 implies that the behaviour of

$$\left(\widehat{\beta}-\beta\right)'\frac{1}{N}\sum_{\lambda_s}\phi_{\vartheta_0,s}\frac{I_{z,s}}{f_{\vartheta_0,s}}\left(\widehat{\beta}-\beta\right),$$

where $\zeta_\vartheta(\lambda)=f_\vartheta^{-1}(\lambda)\,\phi_\vartheta(\lambda)$, is that of

$$A_n^{-1}\frac{1}{N}\sum_{\lambda_s}\zeta_{\vartheta_0,s}I_{z,s}A_n^{-1}=\frac{1}{N^2}\sum_{\lambda_s}\sum_{r=-n+1}^{n-1}e^{ir'\lambda_s}\zeta_{\vartheta_0,s}A_n^{-1}\sum_{t=1}^{n-r}z(t)\,z'(t+r)\,A_n^{-1}$$

by standard algebra. But by Grenander conditions, the right side of the last displayed expression is

$$\frac{1}{N}\int_{-\mathring{\pi}}^{\pi}\zeta_{\vartheta_0}(\lambda)\,\mathcal{M}(d\lambda).$$

So, the first term of the right of (8.13) is $O_p\left(N^{-1}\right)$. Next as we have done with the first term on the right of (8.13), the second term is governed by the behaviour of

$$\begin{aligned}A_n^{-1}\frac{1}{N}\sum_{\lambda_s}\zeta_{\widehat{\vartheta},s}w_{z,s}\overline{w_{x,s}} &= A_n^{-1}\frac{1}{N}\sum_{\lambda_s}\zeta_{\vartheta_0,s}w_{z,s}\overline{w_{x,s}}\\ &\quad +O_p\left(N^{-1/2}\right)A_n^{-1}\frac{1}{N}\sum_{\lambda_s}\left(\frac{\partial}{\partial\vartheta}\zeta_{\vartheta_0,s}\right)w_{z,s}\overline{w_{x,s}}\\ &\quad +O_p\left(N^{-1}\right)A_n^{-1}\frac{1}{N}\sum_{\lambda_s}\left\|\left(\frac{\partial^2}{\partial\vartheta^2}\zeta_{\check{\vartheta},s}\right)w_{z,s}\overline{w_{x,s}}\right\|,\end{aligned}$$

where $\check{\vartheta}$ is an intermediate point between $\widehat{\vartheta}$ and ϑ_0. From here it is standard to conclude that

$$A_n^{-1}\frac{1}{N}\sum_{\lambda_s}\zeta_{\widehat{\vartheta},s}w_{z,s}\overline{w_{x,s}}=O_p\left(N^{-1}\right)$$

because

$$\mathbb{E}\left|A_n^{-1}\frac{1}{N}\sum_{\lambda_s}\left\{\zeta_{\vartheta_0,s}+\frac{\partial}{\partial\vartheta}\zeta_{\vartheta_0,s}\right\}w_{z,s}\overline{w_{x,s}}\right|^2=O_p\left(N^{-2}\right) \tag{8.14}$$

as we now prove. First, by Lemma 3 of Hidalgo (2009),

$$\mathbb{E}\left|A_n^{-1}\frac{1}{N}\sum_{\lambda_s}\zeta_{\vartheta_0,s}w_{z,s}\overline{w_{x,s}}\right|^2=\mathbb{E}\left|A_n^{-1}\frac{1}{N}\sum_{\lambda_s}\frac{\phi_{\vartheta_0,s}}{f^{1/2}_{\vartheta_0,s}}w_{z,s}\overline{w_{\varepsilon,s}}\right|^2(1+o(1)).$$

Now, form here it is obvious that (8.14) holds true as $\mathbb{E}\left(w_{\varepsilon,s}w_{\varepsilon,k}\right)=\mathbb{I}\left(s=k\right)$. So, we have that second term of the right of (8.13) is also $O_p\left(N^{-1}\right)$ and hence (8.12) is shown.

8.3. **PROOF OF THEOREM 5.**

We will only show part (a) as part (b) is handled similarly. The proof proceeds very similarly to Corollary 3. Indeed, except multiplicative constants,

$$\sup_{\lambda\in\widetilde{\Pi}^d}\left|\widehat{\alpha}_{\widetilde{\theta}N}(\lambda)-\alpha_{\widehat{\theta}N}(\lambda)\right|\widehat{G}_{\theta N}(\lambda)=\frac{(2\pi)^d}{N}\sum_{\lambda_s}^{\lambda}\frac{I^T_{\widehat{x}}(\lambda_s)}{\left|\Psi_\theta(\lambda_s)\right|^2},$$

where

$$\widehat{\alpha}_{\widetilde{\theta}N}(\lambda)-\alpha_{\widehat{\theta}N}(\lambda)=N^{1/2}\left(\frac{\widehat{G}_{\widetilde{\theta}N}(\lambda)}{\widehat{G}_{\widetilde{\theta}N}(\pi)}-\frac{G_{\widehat{\theta}N}(\lambda)}{G_{\widehat{\theta}N}(\pi)}\right).$$

Now, by standard delta methods, it suffices to show that

$$\sup_{\lambda\in\widetilde{\Pi}^d}\left|N^{1/2}\left(\widehat{G}_{\widetilde{\theta}N}(\lambda)-G_{\widehat{\theta}N}(\lambda)\right)\right|=o_p(1).$$

But,

$$\begin{aligned}N^{1/2}\left(\widehat{G}_{\widetilde{\theta}N}(\lambda)-G_{\widehat{\theta}N}(\lambda)\right) &= \frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\left\{\frac{I_{\widehat{x},s}}{\left|\Psi_{\widetilde{\vartheta},s}\right|^2}-\frac{I_{x,s}}{\left|\Psi_{\widehat{\vartheta},s}\right|^2}\right\}\\ &= \frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\frac{I_{\widehat{x},s}-I_{x,s}}{\left|\Psi_{\widehat{\vartheta},s}\right|^2} \qquad (8.15)\\ &\quad+\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}I_{\widehat{x},s}\left\{\frac{1}{\left|\Psi_{\widetilde{\vartheta},s}\right|^2}-\frac{1}{\left|\Psi_{\widehat{\vartheta},s}\right|^2}\right\}.\end{aligned}$$

First, it is straightforward to show that

$$\sup_{\lambda\in\widetilde{\Pi}^d}\left|\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}I_{\widehat{x},s}\left\{\frac{1}{\left|\Psi_{\widetilde{\vartheta},s}\right|^2}-\frac{1}{\left|\Psi_{\widehat{\vartheta},s}\right|^2}\right\}\right|=o_p(1)$$

because

$$\sup_{\lambda\in\widetilde{\Pi}^d}\left|\left|\Psi_{\widetilde{\vartheta},s}\right|^2-\left|\Psi_{\widetilde{\vartheta},s}\right|^2\right| = \left(\widetilde{\vartheta}-\widetilde{\vartheta}\right)\sup_{\lambda\in\widetilde{\Pi}^d}\left|\frac{\partial}{\partial\vartheta}\left|\Psi_{\check{\vartheta}}(\lambda)\right|^2\right|$$
$$= O_p\left(\left|\widetilde{\vartheta}-\widetilde{\vartheta}\right|\right).$$

In addition

$$\sup_{\lambda\in\widetilde{\Pi}^d}\left|\frac{1}{N}\sum_{\lambda_s}^{\lambda} I_{\widehat{x},s}\right| = \frac{1}{N}\sum_{\lambda_s} I_{\widehat{x},s}$$
$$= \frac{1}{N}\sum_{\lambda_s}\left(I_{\widehat{x},s}-I_{x,s}\right)+\frac{1}{N}\sum_{\lambda_s} I_{x,s}$$
$$= O_p(1)$$

as a consequence of Corollary 3.

To conclude the proof it remains to show that the first in the far right of (8.15) satisfies that

$$\sup_{\lambda\in\widetilde{\Pi}^d}\left|\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\frac{I_{\widehat{x},s}-I_{x,s}}{\left|\Psi_{\widehat{\vartheta},s}\right|^2}\right| = o_p(1). \tag{8.16}$$

Now as in (8.13),

$$\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\frac{I_{\widehat{x},s}-I_{x,s}}{\left|\Psi_{\widehat{\vartheta},s}\right|^2} = \left(\widehat{\beta}-\beta\right)'\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\frac{I_{z,s}}{\left|\Psi_{\widehat{\vartheta},s}\right|^2}\left(\widehat{\beta}-\beta\right) - 2\left(\widehat{\beta}-\beta\right)'\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\frac{\operatorname{Re}\left(w_{z,s}\overline{w_{x,s}}\right)}{\left|\Psi_{\widehat{\vartheta},s}\right|^2}. \tag{8.17}$$

The contribution of the first term on the right of (8.17) into the left of (8.16) is bounded by

$$A_n^{-1}\frac{1}{N^{1/2}}\sum_{\lambda_s}\frac{I_{z,s}}{\left|\Psi_{\vartheta_0,s}\right|^2}A_n^{-1} = o_p(1)$$

as we showed in Corollary 3. Finally, the contribution of the second term on the right of (8.17) into the left of (8.16) is given by that of

$$\sup_{\lambda\in\widetilde{\Pi}^d}\left|A_n^{-1}\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\frac{w_{z,s}\overline{w_{\varepsilon,s}}}{\left|\Psi_{\vartheta_0,s}\right|}\right|$$

proceeding as in Lemma 3 of Hidalgo (2009). Now,

$$\mathbb{E}\left|A_n^{-1}\frac{1}{N^{1/2}}\sum_{\lambda_s}^{\lambda}\frac{w_{z,s}\overline{w_{\varepsilon,s}}}{\left|\Psi_{\vartheta_0,s}\right|}\right|^2 = o(1)$$

because $\mathbb{E}\left(w_{\varepsilon,s}w_{\varepsilon,k}\right)=\mathbb{I}\left(s=k\right)$. On the other hand,

$$\mathbb{E}\left|A_n^{-1}\frac{1}{N^{1/2}}\sum_{\lambda_1\leq\lambda_s\leq\lambda_2}\frac{w_{z,s}\overline{w_{\varepsilon,s}}}{\left|\Psi_{\vartheta_0,s}\right|}\right|^2\leq C\left|\lambda_2-\lambda_1\right|^{1+\delta}$$

proceeding as in Lemma 9 of Hidalgo (2009). This concludes the proof of part (a) and the theorem.

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Table 5.1
SIZE OF THE TESTS

		White Noise					
		$SMA(1)$			$SAR(1)$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
$(20, 20)$	0.1	0.119	0.098	0.105	0.087	0.097	0.106
	0.05	0.046	0.043	0.065	0.043	0.047	0.065
	0.01	0.016	0.012	0.02	0.015	0.01	0.022
$(20, 40)$	0.1	0.102	0.103	0.121	0.097	0.108	0.121
	0.05	0.048	0.052	0.074	0.045	0.057	0.082
	0.01	0.015	0.008	0.021	0.006	0.012	0.021
$(40, 40)$	0.1	0.089	0.111	0.123	0.097	0.107	0.123
	0.05	0.055	0.078	0.076	0.057	0.068	0.078
	0.01	0.005	0.024	0.020	0.012	0.006	0.020

		$SMA(1)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
$(20, 20)$	0.1	0.110	0.095	0.107	0.082	0.079	0.088
	0.05	0.054	0.050	0.064	0.044	0.044	0.052
	0.01	0.019	0.015	0.021	0.005	0.008	0.009
$(20, 40)$	0.1	0.093	0.104	0.126	0.102	0.109	0.112
	0.05	0.045	0.037	0.066	0.047	0.048	0.061
	0.01	0.008	0.008	0.015	0.005	0.013	0.018
$(40, 40)$	0.1	0.088	0.082	0.101	0.111	0.103	0.116
	0.05	0.042	0.044	0.052	0.051	0.052	0.066
	0.01	0.011	0.017	0.021	0.011	0.013	0.021

		$SAR(1)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
$(20, 20)$	0.1	0.101	0.099	0.121	0.079	0.104	0.092
	0.05	0.047	0.046	0.066	0.038	0.048	0.048
	0.01	0.009	0.010	0.021	0.008	0.012	0.012
$(20, 40)$	0.1	0.106	0.092	0.104	0.095	0.092	0.098
	0.05	0.067	0.053	0.055	0.050	0.044	0.059
	0.01	0.015	0.018	0.025	0.010	0.004	0.025
$(40, 40)$	0.1	0.105	0.112	0.111	0.110	0.103	0.109
	0.05	0.049	0.05	0.055	0.059	0.055	0.064
	0.01	0.01	0.004	0.011	0.011	0.007	0.015

Table 5.2
POWER OF THE TESTS

		$SMA(1)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
$(20, 20)$	0.1	0.119	0.125	0.143	0.270	0.320	0.334
	0.05	0.063	0.054	0.088	0.158	0.211	0.225
	0.01	0.021	0.014	0.033	0.044	0.090	0.100
$(20, 40)$	0.1	0.165	0.130	0.151	0.394	0.405	0.460
	0.05	0.096	0.079	0.102	0.233	0.294	0.342
	0.01	0.025	0.017	0.046	0.089	0.173	0.205
$(40, 40)$	0.1	0.154	0.149	0.160	0.493	0.685	0.705
	0.05	0.104	0.083	0.094	0.328	0.569	0.556
	0.01	0.031	0.023	0.033	0.145	0.323	0.345

		$SAR(1)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
$(20, 20)$	0.1	0.109	0.118	0.128	0.431	0.597	0.601
	0.05	0.047	0.066	0.074	0.320	0.443	0.496
	0.01	0.007	0.014	0.022	0.086	0.243	0.334
$(20, 40)$	0.1	0.106	0.118	0.127	0.704	0.793	0.821
	0.05	0.050	0.071	0.074	0.547	0.695	0.733
	0.01	0.004	0.014	0.022	0.267	0.476	0.592
$(40, 40)$	0.1	0.088	0.136	0.149	0.917	0.977	0.980
	0.05	0.042	0.077	0.087	0.831	0.955	0.961
	0.01	0.004	0.027	0.027	0.618	0.866	0.891

		$SAR(2)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
$(20, 20)$	0.1	0.323	0.139	0.153	0.994	0.874	0.872
	0.05	0.204	0.070	0.086	0.926	0.662	0.682
	0.01	0.101	0.014	0.025	0.601	0.124	0.308
$(20, 40)$	0.1	0.509	0.216	0.262	1.000	0.998	0.998
	0.05	0.361	0.090	0.152	1.000	0.980	0.988
	0.01	0.124	0.010	0.036	0.996	0.628	0.858
$(40, 40)$	0.1	0.811	0.458	0.541	1.000	1.000	1.000
	0.05	0.543	0.231	0.306	1.000	1.000	1.000
	0.01	0.205	0.067	0.095	1.000	1.000	1.000

Table 5.3

SIZE OF THE TESTS (FROM THE RESIDUALS)

		White Noise					
		$SMA(1)$			$SAR(1)$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
(20,20)	0.1	0.101	0.102	0.106	0.091	0.093	0.107
	0.05	0.049	0.053	0.062	0.050	0.042	0.065
	0.01	0.018	0.011	0.021	0.014	0.011	0.023
(20,40)	0.1	0.100	0.098	0.118	0.094	0.110	0.116
	0.05	0.055	0.041	0.064	0.048	0.054	0.062
	0.01	0.009	0.009	0.022	0.007	0.023	0.023
(40,40)	0.1	0.093	0.090	0.105	0.104	0.084	0.106
	0.05	0.039	0.051	0.055	0.044	0.047	0.056
	0.01	0.013	0.020	0.024	0.015	0.019	0.023

		$SMA(1)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
(20,20)	0.1	0.086	0.086	0.086	0.084	0.060	0.090
	0.05	0.043	0.032	0.046	0.033	0.025	0.039
	0.01	0.005	0.009	0.011	0.003	0.006	0.011
(20,40)	0.1	0.109	0.093	0.107	0.106	0.094	0.106
	0.05	0.047	0.048	0.064	0.049	0.045	0.064
	0.01	0.009	0.008	0.014	0.007	0.009	0.023
(40,40)	0.1	0.088	0.108	0.119	0.128	0.093	0.109
	0.05	0.047	0.062	0.077	0.057	0.047	0.056
	0.01	0.014	0.012	0.018	0.012	0.011	0.020

		$SAR(1)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
(20,20)	0.1	0.105	0.097	0.119	0.082	0.093	0.093
	0.05	0.048	0.051	0.067	0.032	0.043	0.054
	0.01	0.009	0.010	0.019	0.008	0.011	0.012
(20,40)	0.1	0.101	0.097	0.105	0.116	0.098	0.111
	0.05	0.047	0.043	0.063	0.056	0.057	0.068
	0.01	0.007	0.016	0.020	0.014	0.015	0.024
(40,40)	0.1	0.107	0.083	0.094	0.096	0.099	0.117
	0.05	0.054	0.037	0.047	0.048	0.035	0.072
	0.01	0.010	0.008	0.014	0.011	0.008	0.018

Table 5.4
POWER OF THE TESTS (FROM THE RESIDUALS)

		$SMA(1)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
(20,20)	0.1	0.113	0.106	0.116	0.252	0.320	0.323
	0.05	0.061	0.062	0.071	0.167	0.203	0.234
	0.01	0.013	0.014	0.022	0.053	0.056	0.110
(20,40)	0.1	0.136	0.148	0.151	0.343	0.440	0.466
	0.05	0.060	0.078	0.081	0.246	0.320	0.356
	0.01	0.014	0.010	0.033	0.096	0.126	0.203
(40,40)	0.1	0.147	0.162	0.175	0.510	0.684	0.721
	0.05	0.073	0.073	0.092	0.387	0.543	0.575
	0.01	0.027	0.022	0.035	0.217	0.286	0.379

		$SAR(1)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
(20,20)	0.1	0.102	0.122	0.124	0.378	0.581	0.581
	0.05	0.043	0.071	0.070	0.249	0.425	0.470
	0.01	0.009	0.014	0.019	0.084	0.194	0.314
(20,40)	0.1	0.068	0.111	0.124	0.681	0.762	0.811
	0.05	0.029	0.059	0.074	0.542	0.682	0.714
	0.01	0.003	0.008	0.018	0.275	0.436	0.565
(40,40)	0.1	0.068	0.142	0.142	0.923	0.969	0.975
	0.05	0.027	0.072	0.075	0.857	0.937	0.951
	0.01	0.003	0.009	0.018	0.611	0.882	0.905

		$SAR(2)$					
		$\theta = 0.1$			$\theta = 0.2$		
n	level	T_p	$\hat{C}_N^*$	$\hat{C}_N$	T_p	$\hat{C}_N^*$	$\hat{C}_N$
(20,20)	0.1	0.328	0.143	0.146	0.987	0.837	0.857
	0.05	0.180	0.079	0.088	0.918	0.580	0.682
	0.01	0.053	0.013	0.027	0.580	0.199	0.327
(20,40)	0.1	0.554	0.232	0.278	1.000	0.998	0.999
	0.05	0.365	0.121	0.159	1.000	0.968	0.987
	0.01	0.145	0.021	0.048	0.987	0.673	0.859
(40,40)	0.1	0.851	0.482	0.530	1.000	1.000	1.000
	0.05	0.687	0.232	0.295	1.000	1.000	1.000
	0.01	0.327	0.030	0.079	1.000	1.000	1.000

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