

# ROBUST INFERENCE AND TESTING OF CONTINUITY IN THRESHOLD REGRESSION MODELS \*

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## Abstract

This paper is concerned with inference in regression models with either a kink or a jump at an unknown threshold, particularly when we do not know whether the kink or jump is the true specification. One of our main results shows that the statistical properties of the estimator of the threshold parameter are substantially different under the two settings, with a slower rate of convergence under the kink design, and more surprisingly slower than if the correct kink specification were employed in the estimation. We thus propose two testing procedures to distinguish between them. Next, we develop a robust inferential procedure that does not require prior knowledge on whether the regression model is kinky or jumpy. Furthermore, we propose to construct confidence intervals for the unknown threshold by the bootstrap test inversion, also known as grid bootstrap. Finite sample performances of the bootstrap tests and the grid bootstrap confidence intervals are examined and compared against tests and confidence intervals based on the asymptotic distribution through Monte Carlo simulations. Finally, we implement our procedure to an economic empirical application.

JEL Classification: C12, C13, C24.

Key words: Threshold, Change Point, Grid Bootstrap, Kink Design, Continuity Test.

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\*M. Seo gratefully acknowledges the support from Promising-Pioneering Researcher Program through Seoul National University (SNU) in 2016

# 1 INTRODUCTION

Detecting structural breaks or instability in regression models has attracted a huge attention since the work of Quandt (1960), as the surveys by Perron (2006) and Aue and Horváth (2013) demonstrate. On the other hand, the so-called Self-Exciting-Threshold-Autoregressive (SETAR) model of Tong (1990) has become popular in empirical applications due to its parsimony and ability to capture many of the nonlinearities present in macroeconomic data, see Chan and Tsay (1998) or Gonzalo and Wolf (2005) among others for some examples. The SETAR model and its generalizations, better known as threshold models, were brought to econometricians' attention by Hansen (2000), who highlighted their flexibility when modelling economic data in general. Despite the similarities, the literature on structural break models and threshold regression models has mostly followed parallel paths without much reference to each other, although some exceptions exist, see Hansen (2000) and more implicitly in a nonparametric setting by Delgado and Hidalgo (2000).

Broadly speaking, the aforementioned models belong to the class of segmented regression models, which have been examined in the area of approximation theory in mathematics and are better known as splines. The latter methodology is frequently used in nonparametric estimation and routinely employed in functional regression models. However, one key difference between splines and threshold/structural break models is the way in which the points of break (knots) or change between different segments are chosen. In the former approach, the practitioner merely chooses the knots for convenience, whereas in the latter schemes the knots have to be estimated. It is precisely this distinction, regarding the estimation of the thresholds/breaks points or knots, that yields theoretical challenges which otherwise will not be present.

Once one accepts the hypothesis of instability in the regression function via any of the available tests, see e.g. Andrews (1993), Hansen (1996) and Lee *et al.* (2011) among others, one is then interested in deciding whether the “segmented” regression model is a model with a discontinuity (jump) or a model with a kink, since the so-called break/threshold tests are unable to discriminate between the two models. One motivation for such testing is the recent surge in theoretical and empirical literature of regression discontinuity (*RD*) and regression kink (*RK*) designs, see Lee and Lemieux (2010) for a comprehensive review of *RD* design, including an extensive list of empirical works, and Ganong and Jäger (2015) for a list of empirical applications that use the *RK* design. The main identification and estimation results

can be found in Hahn, Todd and Van der Klaauw (2001) for *RD* design and Card, Lee, Pei and Weber (2015) for *RK* design. In most applications, researchers are interested in estimating the size of a discrete change in the level or slope of a conditional expectation function at a known threshold. However in some applications the threshold or change-point should be considered as an unknown parameter and hence needs to be estimated, and there is some ambiguity on whether we shall employ an *RD* or an *RK* design, see e.g. Card, Mas and Rothstein (2008) and Hansen (2015). Furthermore, as argued by Landais (2014), even when the change-point is known e.g. determined by policy, testing if this change-point is indeed the correct one in light of the data offers a useful tool to check the validity of the design being used. As we find in this paper that the unrestricted estimator of threshold has different rates of convergence under the two settings, testing for continuity (kink) of the regression function is required when there is ambiguity on which of *RD* or *RK* designs should be used, or else one needs a unified inference on the location of threshold point which works across both designs.

Another powerful motivation to test for a kink comes from the statistical inferential point of view. As we discuss in Section 2, the kink design can be represented by a set of restrictions on the parameter space of the threshold regression (1) below. Thus, the parameters can be consistently estimated by the unrestricted least squares estimator. Unlike in the linear regression model where one can still make valid inference based on the unconstrained estimation without knowing if the constraint holds true, inferences in our context have very different statistical properties under a kink design when using the unrestricted estimation. More specifically, we discover that the rate of convergence of the estimate of the threshold point becomes  $n^{1/3}$  in contrast to  $n^{1/2}$ , which was obtained by Chan and Tsai (1998) and Feder (1975a) when the (true) constrained of a kink is employed in its estimation. The asymptotic distribution of the threshold estimator is no longer normal but the “*argmax*” of some Gaussian process.<sup>1</sup> On the other hand, we find that the unconstrained estimator of the slope parameters is asymptotically independent of the estimator of the threshold point. This is also the case under the jump design of Hansen (2000) but not under the kink design in Feder (1975a) and others. So, we can conclude that the statistical inference for the threshold regression models hinges too much on the unverified assumption of kink versus jump.

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<sup>1</sup>Gonzalo and Wolf (2005) examined the validity of the subsampling inference for the threshold model. One of their results concerns the unrestricted estimator of threshold point when the model has a kink. Our theoretical findings cast doubts on their result under the continuity restriction.

Thus, one of the main purposes of this paper is to develop testing procedures to distinguish between jump and kink designs. Hansen (2015) had considered inference under the kink design and mentioned “one could imagine testing the assumption of continuity within the threshold model class. This is a difficult problem, one to which we are unaware of a solution, and therefore is not pursued in this paper”. We present two test statistics that are based on the quasi-likelihood ratio. One of the tests is not asymptotically pivotal since it involves multiple restrictions, while the other test is pivotal yet it appears less powerful based on our Monte Carlo experiment. Bootstrap methods are proposed to estimate  $p$ -values of the test. These are in fact closely related to the second goal of the paper and can be viewed as its by-product.

Our second aim is to provide a unified inferential procedure valid for both kink and jump models without resorting to prior knowledge of which framework is true. This is an important issue in the current literature because the literature appears to examine separately these two frameworks assuming they are correctly specified. See Chan (1993), Bai and Perron (1998) or Hansen (2000) who assume that the true segmented regression model is discontinuous with a jump and Chan and Tsay (1998) or Feder (1975*a,b*) and more recently Hansen (2015) who assumes continuity of the regression model. Furthermore, it has an important practical bearing because we often lack such knowledge and the asymptotic distribution of the unconstrained estimator given by e.g. Hansen (2000) is not valid under the kink design. In the meantime, it is worthwhile to mention that these two aims of this paper are not contradictory but complementary in that one is for an efficient inference and the other is for a robust inference. The choice should be left to the practitioners.

To make a robust inference in the threshold regression model, we first show that a quasi likelihood ratio test for the location of the threshold has the same asymptotic distribution up to a scale constant that depends on whether the true regression model has a kink or a jump. Second, we present an estimator for the scale factor based on the ratio of two kernel Nadaraya-Watson estimators. The consistency of this estimator is standard under the jump design but both its numerator and denominator converge to zero in probability under the kink design. This makes to examine its properties not standard. Nonetheless, we show that, similar to L’Hopital rule in real analysis, the ratio of the two degenerating terms still converges in probability to the correct scale factor under an interesting requirement that higher order kernels should not be used. Third, we show that the asymptotic distribution of the unconstrained estimator of the slope parameters when the model has a kink is identical to the one under the jump design, which

results from the asymptotic independence between threshold and slope parameters estimates.

The last, but not least, goal of this paper is to present asymptotically valid bootstrap schemes for the two continuity tests and for the construction of confidence sets for the threshold location. The motivation comes from the observation that the asymptotic critical values sometimes appear to be a poor approximation to the finite sample ones as documented by Hansen (2000) and our Section 6 among others. In addition, the first-order validity of the bootstrap is of theoretical interest and it has not been established even under the (shrinking) jump design of Hansen (2000). The interest stems from two sets of findings in the literature regarding the failure of bootstrap for non-standard estimators: firstly with cube-root estimators such as the maximum score estimator, and secondly with super-consistent estimators such as the estimator of autoregressive coefficients of unit root processes and the threshold estimator under the non-shrinking jump design, see Abrevaya and Huang (2005), Seijo and Sen (2011), and Yu (2014), just to name a few. Note that the unconstrained estimator of the threshold belongs to the cube-root class under the kink design and to the super-consistent class under the shrinking jump design. Unlike failures of bootstrap in the cases listed above, we show that the proposed bootstrap statistics, which build on the wild bootstrap, correctly approximate the sampling distributions of the two scaled quasi likelihood ratio statistics in our settings. This contrast is perhaps due to the fact that the nuisance parameter in the asymptotic distribution under the non-shrinking design is infinite-dimensional while the ones in our continuous or shrinking designs are finite-dimensional scaling terms. Furthermore, we propose bootstrap test inversion confidence interval for the threshold, also known as the grid bootstrap in Hansen (1999), to enhance the finite sample coverage probability.

We then present findings of a small Monte Carlo experiment to evaluate the finite sample performance of our bootstrap procedure. We first compare the Monte Carlo size of tests for correct location of the threshold, based on asymptotic theory of Hansen (2000) and our bootstrap method. We also investigate coverage probabilities of confidence intervals, constructed from asymptotic theory of Hansen (2000) and test inversion based on our bootstrap. We then examine the Monte Carlo size and power performance of the bootstrap-based tests of continuity proposed in Section 4. Some interesting findings emerge. First, inference based on the asymptotic distribution of Hansen (2000) tends to have large size distortions and/or poor coverage probability in the jump designs we consider and does not work in the kink design. Second, the hypothesis testing and test inversion confidence interval based on the bootstrap

proposed in this paper seem to work well in all settings considered, including the kink design. Third, the size and power properties of the continuity tests proposed in this paper perform quite well.

In our empirical application, we employ our tests of continuity on the long span time series data of US real GDP growth and debt-to-GDP ratio data used in Hansen (2015), who had fitted the kink model. Our tests of continuity reject the null of continuity and we present the estimated jump model and grid bootstrap confidence intervals for the threshold. We also consider data from Australia, Sweden and the UK and find substantial variations across countries not only in the values of parameter estimates, but also in the results of tests on the presence of threshold effect and continuity.

In Section 2 we introduce the model, make clear the distinction between the jump and kink designs, state assumptions and explain how to estimate model parameters with and without the continuity restriction. In Section 3, we develop robust inferential methods for model parameters that are valid in both jump and kink settings, despite the slower rate of convergence for the estimate of the threshold under the kink design. In Section 4, testing procedures that distinguish between the jump and kink designs are introduced. We then present in Section 5 bootstrap methods for both inference on parameters and the tests of continuity and establish their validity. Section 6 presents a Monte Carlo experiment to examine the finite sample performance of the methods developed in the paper, followed by Section 7 which contains an empirical application. Section 8 concludes.

## 2 MODEL AND ESTIMATION

We shall consider the following threshold/segmented regression model

$$y_t = \beta'x_t + \delta'x_t\mathbf{1}\{q_t > \gamma\} + \varepsilon_t, \quad (1)$$

where  $\mathbf{1}\{\cdot\}$  denotes the indicator function and  $x_t$  is a  $k$ -dimensional vector of regressors. The parameter  $\gamma$  is referred to as change/break-point or threshold, taking values in a compact parameter space  $\Gamma$ , which is a subset of the interior on the domain of the threshold variable  $q_t$ . It is worth mentioning that all our results hold true also when  $q_t = t$ , which is the case with structural break models. However, we have opted not to include this scenario for the sake

of clarity and notational simplicity. In addition we shall remark that the assumption  $\delta \neq 0$  implies that the model has a break which is either due to a kink or a jump.

As mentioned in the introduction, our main interests are twofold. First to make inferences on the parameters  $\alpha = (\beta', \delta')'$  and  $\gamma$  without knowing whether the threshold is due to a kink or a jump and secondly to test whether the regression model (1) has a kink or a jump. Note that to accept that the model changes at  $\gamma$  implies that  $\delta \neq 0$ . Since  $q_t$  must be one of the elements of the covariate vector  $x_t$  for the model to be continuous, i.e. to have a kink, in what follows we shall denote

$$x_t = (1, x'_{t2}, q_t)'; \quad \delta = (\delta_1, \delta'_2, \delta_3)', \quad (2)$$

where  $\delta$  is partitioned in a way to match that of  $x_t$ . Also we shall abbreviate  $\mathbf{1}_t(\gamma) = \mathbf{1}\{q_t > \gamma\}$  and  $x_t(\gamma) = (x'_t, x'_t \mathbf{1}_t(\gamma))'$ , so that we can write (1) as

$$y_t = \beta' x_t + \delta_1 \mathbf{1}_t(\gamma) + \delta'_2 x_{t2} \mathbf{1}_t(\gamma) + \delta_3 q_t \mathbf{1}_t(\gamma) + \varepsilon_t. \quad (3)$$

Specifically, the following condition characterizes the kink design as a constrained version of (3).

**Assumption C.**  $\delta_{30} \neq 0$  and

$$\delta_{10} + \delta_{30} \gamma_0 = 0; \quad \delta_{20} = 0. \quad (4)$$

As usual a “0” subscript to a parameter indicates its true unknown value. Under (4), we observe that (3) becomes

$$y_t = x'_t \beta_0 + \delta_{30} (q_t - \gamma_0) \mathbf{1}_t(\gamma_0) + \varepsilon_t. \quad (5)$$

Note that  $\delta_{30} \neq 0$  is imposed to ensure the identification of  $\gamma_0$ .

The jump design is the negation of constraint (4) provided that  $E[x_{t2} x'_{t2} | q_t = \gamma_0] > 0$ . In other words,

$$E \left[ (\delta_{10} + \delta_{30} \gamma_0 + \delta'_{20} x_{t2})^2 | q_t = \gamma_0 \right] > 0,$$

which implies that the regression function has a jump (non-zero change) at  $q_t = \gamma_0$  with positive probability. The size of jump is often allowed to decrease as the sample size increases see Yao (1987) and Hansen (2000) among others. This is the case to obtain a more tractable asymptotic distribution for the estimator of  $\gamma$ , although as far as our test is concerned we

allow the jump to be of a fixed size. Before stating some extra regularity assumptions on the model, we need to introduce some extra notation. Let  $f(\cdot)$  denote the density function of  $q_t$  and  $\sigma^2(\gamma) = E(\varepsilon_t^2 | q_t = \gamma)$ , the conditional variance function of error term, while  $\sigma^2 = E(\varepsilon_t^2)$  denotes the unconditional variance. Denote  $k \times k$  matrices  $D(\gamma) = E(x_t x_t' | q_t = \gamma)$ ,  $V(\gamma) = E(x_t x_t' \varepsilon_t^2 | q_t = \gamma)$ , and let  $D = D(\gamma_0)$ ,  $V = V(\gamma_0)$ . Finally, let  $M = E(\mathbf{x}_t \mathbf{x}_t')$  and  $\Omega = E(\mathbf{x}_t \mathbf{x}_t' \varepsilon_t^2)$  with  $\mathbf{x}_t = x_t(\gamma_0)$ . Then, the jump design is characterized by:

**Assumption J.** For some  $0 < \varphi < 1/2$  and  $c \neq 0$ ,  $\delta_0 = \delta_n = cn^{-\varphi}$  and  $c'Vc$  and  $c'Dc$  are positive for all  $n$ .

When  $\varphi$  is greater than or equal to  $1/2$ ,  $\delta_n$  is too small to identify  $\gamma_0$ , so that it is excluded.

**Assumption Z.** Let  $\{x_t, \varepsilon_t\}_{t \in \mathbb{Z}}$  be a strictly stationary, ergodic sequence of random variables such that its  $\rho$ -mixing coefficients satisfy  $\sum_{m=1}^{\infty} \rho_m^{1/2} < \infty$  and  $E(\varepsilon_t | \mathcal{F}_{t-1}) = 0$ , where  $\mathcal{F}_{t-1}$  is the filtration up to time  $t-1$ . Furthermore,  $M, \Omega > 0$ ,  $E\|x_t\|^4 < \infty$ ,  $E\|x_t \varepsilon_t\|^4 < \infty$  and  $E|\varepsilon_t|^{4+\eta} < \infty$  for some  $\eta > 0$ .

**Assumption Q.** The functions  $f(\gamma)$ ,  $V(\gamma)$  and  $D(\gamma)$  are continuous at  $\gamma = \gamma_0$ . For all  $\gamma \in \Gamma$ , the functions  $E(x_t x_t' \mathbf{1}\{q_t \leq \gamma\})$  and  $E(x_{t2} x_{t2}' | q_t = \gamma)$  are positive and continuous, and the functions  $f(\gamma)$ ,  $E(|x_t|^4 | q_t = \gamma)$ , and  $E(|x_t \varepsilon_t|^4 | q_t = \gamma)$  are bounded by some  $C < \infty$ .

Assumptions **Z** and **Q** are commonly imposed on the distribution of  $\{x_t, \varepsilon_t\}$ , see e.g. Hansen (2000), so his comments apply here. The condition for  $E(x_{t2} x_{t2}' | q_t = \gamma)$  is written in term of  $x_{t2}$  as the other elements in  $x_t$  are fixed given  $q_t = \gamma$ .

Before introducing the estimators of the parameters using or not the constraint (4), we shall emphasize that the model (1) encompasses two competing models characterized by Assumption **C** and Assumption **J**. In practice, however, it is common that the parameters  $\alpha$  and  $\gamma$  are estimated without a constraint and statistical inferences are made under Assumption **J**. However as we show in the next section, inferences can be misleading if Assumption **C** holds, although the estimator is consistent whether or not Assumption **C** holds true. On the other hand as expected, when Assumption **C** is taken to hold true and (4) is imposed in the estimation, the resulting estimator will not be even consistent under Assumption **J**. The latter comments suggests one of our tests described below. The following two sections introduces the estimator of the parameters when the constraint (4) holds true but it is not used in their estimation and we presents and discusses their statistical properties. The reason or motivation



to examine this scenario is due to the fact that when Assumption **J** holds true its statistical properties are very well known, see for instance Hansen (2000), as is the case when Assumption **C** holds true and the constraint of kink is employed in the estimation, see Feder (1975a).

## 2.1 Estimators

We first describe the unconstrained estimator of the parameters. We estimate  $\theta_0 = (\alpha'_0, \gamma_0)'$  by the (non-linear) least squares estimator (LSE), that is,

$$\hat{\theta} = (\hat{\alpha}', \hat{\gamma})' := \underset{\theta \in \Theta}{\operatorname{argmin}} \mathbb{S}_n(\theta), \quad (6)$$

where  $\Theta = (\Lambda, \Gamma)$  is a compact set in  $\mathbb{R}^{2k+1}$  and

$$\mathbb{S}_n(\theta) = \frac{1}{n} \sum_{t=1}^n (y_t - \alpha' x_t(\gamma))^2, \quad (7)$$

which is a step function in  $\gamma$  at  $q_t$ 's. Note that we have abbreviate (1) as

$$y_t = \alpha' x_t(\gamma) + \varepsilon_t \quad (8)$$

in an obvious way. For its computation, we shall employ a step-wise algorithm. To that end, one could employ the grid search algorithm on  $\Gamma_n = \Gamma \cap \{q_1, \dots, q_n\}$  to find  $\hat{\gamma}$ . Define the concentrated sum of squared residuals (with some abuse of notation)

$$\mathbb{S}_n(\gamma) = \frac{1}{n} \sum_{t=1}^n (y_t - \hat{\alpha}'(\gamma) x_t(\gamma))^2, \quad (9)$$

where

$$\hat{\alpha}(\gamma) = \underset{\alpha \in \Lambda}{\operatorname{argmin}} \frac{1}{n} \sum_{t=1}^n (y_t - \alpha' x_t(\gamma))^2 \quad (10)$$

is the LSE of  $\alpha$  for a given  $\gamma$ . Then, our estimator of  $\alpha$  is  $\hat{\alpha} = \hat{\alpha}(\hat{\gamma})$ , with

$$\hat{\gamma} = \underset{\gamma \in \Gamma_n}{\operatorname{argmin}} \mathbb{S}_n(\gamma). \quad (11)$$

On the other hand, we can minimize (7) using the constraints in (4) yielding the constrained least squares estimator (CLSE):

$$\tilde{\theta} = (\tilde{\alpha}', \tilde{\gamma})' := \underset{\theta \in \Theta: \delta_1 + \delta_3 \gamma = 0; \delta_2 = 0}{\operatorname{argmin}} \mathbb{S}_n(\theta). \quad (12)$$

### 3 INFERENCE

This section develops inferential methods for  $\theta_0$ , that is, for both the slope and threshold parameters, which do not assume *a priori* whether the model has a jump or a kink, but the model has a kink. The next proposition gives the consistency and rates of convergence for  $\hat{\theta}$  given in (6).

**Proposition 1.** *Under Assumptions **C**, **Z** and **Q**, we have that*

$$\hat{\alpha} - \alpha_0 = O_p(n^{-1/2}) \quad \text{and} \quad \hat{\gamma} - \gamma_0 = O_p(n^{-1/3}).$$

The results of Proposition 1 are very surprising because the convergence rate of  $\hat{\gamma}$  is slower than that of the CLSE  $\tilde{\gamma}$ , which was shown to be  $n^{-1/2}$  in Feder (1975a) and later in Chan and Tsay (1998) or Hansen (2015). That is, using the true restriction on the parameters, it improves the rate of convergence of the estimator of  $\gamma_0$ , not just reducing its asymptotic variance as it is often assumed.

Next we present the asymptotic distribution of  $\hat{\theta}$  under Assumption **C**.

**Theorem 1.** *Let Assumptions **Z** and **Q** hold true and  $B_1(\cdot)$  and  $B_2(\cdot)$  two independent standard Brownian motions. Define  $W(g) := B_1(-g)1\{g < 0\} + B_2(g)1\{g > 0\}$ . Then, under Assumption **C**, we have that*

$$\begin{aligned} n^{1/2}(\hat{\alpha} - \alpha_0) &\xrightarrow{d} \mathcal{N}(0, M^{-1}\Omega M^{-1}) \\ n^{1/3}(\hat{\gamma} - \gamma_0) &\xrightarrow{d} \operatorname{argmax}_{g \in \mathbb{R}} \left( 2\delta_{30} \sqrt{\frac{\sigma^2(\gamma_0) f(\gamma_0)}{3}} W(g^3) + \frac{\delta_{30}^2}{3} f(\gamma_0) |g|^3 \right), \end{aligned}$$

where the two limit distributions are independent one of each other.

The asymptotic independence is a consequence of the different convergence rates between two sets of estimators  $\hat{\alpha}$  and  $\hat{\gamma}$  by similar reasoning as in Chan (1993), albeit the rate for  $\hat{\gamma}$  is slower than that for  $\hat{\alpha}$  in our case. The asymptotic independence is not the case for the CLSE  $\tilde{\gamma}$  and  $\tilde{\alpha}$ , which converge at the same rate as mentioned above and are jointly asymptotically normal with a non-diagonal variance covariance matrix.

The conclusions of Theorem 1 suggests that Gonzalo and Wolf's (2005) subsampling procedure might not be correct as they use the normalization  $n^{1/2}$  instead of the correct one  $n^{1/3}$ . On the other hand, it is worth mentioning that Seo and Linton (2007) considered the

smoothed least squares estimator for the same setup. The convergence rate for their smoothed least squares estimator for  $\gamma$  was slower than our cube root rate under their assumptions for the smoothing parameter.

**Remark 1.** *The findings in Theorem 1 can be easily extended to more general settings such as those implicit in Feder (1975a, b), where  $x_t' \delta \mathbf{1}_t(\gamma)$  is replaced by  $g(x_t; \delta, \gamma)$  under the condition that the first  $(m-1)$  derivatives of the function  $g(x_t; \delta, \gamma)$  with respect to  $q$  are zero at  $\gamma_0$ . In this case we would have that  $\hat{\gamma} - \gamma_0 = O_p(n^{-1/(2m+1)})$  whereas Feder obtained that  $\hat{\gamma} - \gamma_0 = O_p(n^{-1/2m})$  imposing the continuity constraint.*

It seems appropriate to present here a heuristic discussion to illustrate why the constrained and unconstrained estimators of  $\gamma_0$  have different rates of convergence. We shall abbreviate  $\mathbf{1}_t(\gamma_0)$  by  $\mathbf{1}_t$  and  $\mathbf{1}\{\gamma_1 < q_t < \gamma_2\}$  by  $\mathbf{1}_t(\gamma_1; \gamma_2)$ , so that  $\mathbf{1}_t(\gamma; \infty) = \mathbf{1}_t(\gamma)$ . For simplicity of illustration, we begin with the model with a kink design as in (5) with  $\beta_0 = 0$ ,  $x_t = (1, q_t)'$  and  $\delta = (\delta_1, \delta_3)'$ . In addition we shall assume  $\gamma_0 = 0$  without loss of generality since we can always rename the variable  $q_t - \gamma_0$  as  $q_t$ . Then, we have that

$$(\hat{\delta}, \hat{\gamma}) = \underset{\delta, \gamma}{\operatorname{argmin}} \mathbb{S}_n(\delta; \gamma) := \frac{1}{n} \sum_{t=1}^n (y_t - (\delta_1 + \delta_3 q_t) \mathbf{1}_t(\gamma))^2.$$

On the other hand, the *CLSE* is given by

$$(\tilde{\delta}_3, \tilde{\gamma})' = \underset{\delta_3, \gamma}{\operatorname{argmin}} \tilde{\mathbb{S}}_n(\delta_3; \gamma) := \frac{1}{n} \sum_{t=1}^n (y_t - \delta_3 (q_t - \gamma) \mathbf{1}_t(\gamma))^2.$$

It is well known that the rates of convergence of an M-estimator is governed by the local behavior of its criterion function around the true value. In our framework, we have that Standard algebra yields that

$$\begin{aligned} & \mathbb{S}_n(\delta; \gamma) - \mathbb{S}_n(\delta_0; 0) \\ &= \frac{1}{n} \sum_{t=1}^n (\delta_{30} q_t \mathbf{1}_t - (\delta_1 + \delta_3 q_t) \mathbf{1}_t(\gamma))^2 + \frac{2}{n} \sum_{t=1}^n \varepsilon_t (\delta_{30} q_t \mathbf{1}_t - (\delta_1 + \delta_3 q_t) \mathbf{1}_t(\gamma)), \end{aligned} \tag{13}$$

whereas

$$\begin{aligned} & \tilde{\mathbb{S}}_n(\delta_3; \gamma) - \tilde{\mathbb{S}}_n(\delta_{30}; 0) \\ &= \frac{1}{n} \sum_{t=1}^n (\delta_{30} q_t \mathbf{1}_t - \delta_3 (q_t - \gamma) \mathbf{1}_t(\gamma))^2 + \frac{2}{n} \sum_{t=1}^n \varepsilon_t (\delta_{30} q_t \mathbf{1}_t - \delta_3 (q_t - \gamma) \mathbf{1}_t(\gamma)). \end{aligned} \tag{14}$$

We first examine the behaviour of (13). Assuming the consistency of the estimators, we have that the expectation of the right side of (13) is

$$\begin{aligned} & E(\delta_{30}q_t\mathbf{1}_t - (\delta_1 + \delta_3q_t)\mathbf{1}_t(\gamma))^2 \\ &= E(\delta_1 + (\delta_3 - \delta_{30})q_t)^2\mathbf{1}_t(\gamma) + E(\delta_1 + \delta_3q_t)^2\mathbf{1}_t(0; \gamma) \\ &\sim \|\delta - \delta_0\|^2 + \gamma^3 \end{aligned}$$

since  $E\{q_t^2\mathbf{1}_t(0; \gamma)\} = |\gamma|^3(f(0) + o(1))$ , whereas

$$\text{Var}\left(\frac{1}{n}\sum_{t=1}^n \varepsilon_t(\delta_{30}q_t\mathbf{1}_t - (\delta_1 + \delta_3q_t)\mathbf{1}_t(\gamma))\right) \sim \frac{\|\delta - \delta_0\|^2 + |\gamma|^3}{n}.$$

Thus, the last two displayed expressions suggest that

$$\hat{\delta} - \delta_0 = O_p(n^{-1/2}) \quad \text{and} \quad \hat{\gamma} = O_p(n^{-1/3}),$$

as these rates of convergence balance the speeds at which the bias and standard deviation of  $\mathbb{S}_n(\delta; \gamma) - \mathbb{S}_n(\delta_0; \gamma_0 = 0)$  converge to zero.

Similarly for (14), we have that

$$\begin{aligned} & E(\delta_{30}q_t\mathbf{1}_t - \delta_3(q_t - \gamma)\mathbf{1}_t(\gamma))^2 \sim |\delta_3 - \delta_{30}|^2 + \gamma^2 \\ \text{Var}\left(\frac{2}{n}\sum_{t=1}^n \varepsilon_t(\delta_{30}q_t\mathbf{1}_t - \delta_3(q_t - \gamma)\mathbf{1}_t(\gamma))\right) &\sim \frac{|\delta_3 - \delta_{30}|^2 + |\gamma|^2}{n}, \end{aligned}$$

so that we expect

$$\tilde{\delta}_3 - \delta_{30} = O_p(n^{-1/2}) \quad \text{and} \quad \tilde{\gamma} = O_p(n^{-1/2}),$$

which coincides with the rates of convergence that both Feder (1975*a, b*) and Chan and Tsai (1998) obtained.

An intuitive explanation of this phenomenon is to appeal to “misspecification”. When we estimate the unconstrained model (1), although it encompasses both continuous and discontinuous models, the resulting regression function is almost surely discontinuous, since the region of parameter space that fulfils Assumption **C** has Lebesgue measure zero.

Next, we present robust and unified inference methods for the slope parameter  $\alpha$  and the threshold  $\gamma$  that are valid under both Assumption **C** and under Assumption **J**. For clarity, we examine inference for  $\alpha$  and  $\gamma$  in two separate sections.

### 3.1 Inference on Slope Parameter $\alpha$

We state our first result in this section. That is,

**Corollary 1.** *Let Assumptions **Z** and **Q** hold true. Then, under either Assumption **C** or Assumption **J**,*

$$n^{1/2}(\hat{\alpha} - \alpha_0) \xrightarrow{d} \mathcal{N}(0, M^{-1}\Omega M^{-1}).$$

Comparing the result we obtained with those in Hansen (2000), we obtain the interesting result that the asymptotic distribution of  $\hat{\alpha}$  is identical under both Assumptions **C** and **J**, so that the constraints do not help to obtain more efficient estimators for the slope parameters. The results of Corollary 1 indicates that provided consistent estimators for  $M$  and  $\Omega$ , say the respective sample moments

$$\widehat{M} = \frac{1}{n} \sum_{t=1}^n x_t(\hat{\gamma}) x_t(\hat{\gamma})'; \quad \widehat{\Omega} = \frac{1}{n} \sum_{t=1}^n x_t(\hat{\gamma}) x_t(\hat{\gamma})' \widehat{\varepsilon}_t^2,$$

where  $\widehat{\varepsilon}_t = y_t - x_t(\hat{\gamma})' \hat{\alpha}$ , we can obtain confidence regions or perform standard hypothesis testing on  $\alpha_0$  regardless whether the true  $\gamma_0$  were known *a priori*, and more importantly whether the model has a kink or a jump. This “oracle” property of  $\hat{\alpha}$  is an interesting feature that does not hold true for the CLSE  $\tilde{\alpha}$ , whose asymptotic distribution is affected by that of  $\tilde{\gamma}$ , as was first noticed and shown by Feder (1975a) and later extended to time series data by Chan and Tsay (1998).

### 3.2 Inference on the Threshold Location $\gamma$

The main purpose of this section is to develop a method to construct confidence regions that are valid regardless of whether the change is due to a kink or a jump. Conventionally, inference on  $\gamma$  has been done under the assumption that the model has a kink or has a jump, that is the practitioner chooses between a jump or a kink designs before estimating the threshold point. More specifically, if we decide that the model has a jump, then one follows e.g. Hansen (2000), whereas if one has chosen the kink design then one needs to employ the asymptotic normal inference as in Feder (1975a) and others. One of our findings is that Hansen (2000) results

are not valid if the model had a kink and likewise Feder's results are not valid if the model followed a jump design.

Thus, one of our aims is to develop robust confidence regions that are valid under both designs or better without using if the mode has or has not a jump. To that end, we need to find a statistic whose asymptotic distribution is invariant to the true parameter values, that is, a statistic whose asymptotic distribution does not change suddenly under the continuity restriction (4), or just that its asymptotic results are valid whether or not (4) holds true. For that purpose, we begin introducing a Gaussian quasi likelihood ratio statistic based on the unconstrained model (1). Specifically, let

$$QLR_n(\gamma) = n \frac{\mathbb{S}_n(\gamma) - \mathbb{S}_n(\hat{\gamma})}{\mathbb{S}_n(\hat{\gamma})},$$

where  $\mathbb{S}_n(\gamma)$  is defined in (9). Under Assumptions **Q**, **Z** and **J**, Hansen (2000) showed that

$$QLR_n \xrightarrow{d} \xi \max_{g \in \mathbb{R}} (2W(g) - |g|), \quad (15)$$

where

$$\xi = \frac{E((x'_t c \varepsilon_t)^2 | q_t = \gamma_0)}{\sigma^2 E((x'_t c)^2 | q_t = \gamma_0)}$$

with the distribution function of  $\max_{g \in \mathbb{R}} (2W(g) - |g|)$  being  $F(z) = (1 - e^{-z/2})^2$ .

We now derive the following asymptotic distribution under Assumption **C**.

**Proposition 2.** *Suppose that Assumptions **C**, **Z** and **Q** hold true. Then, as  $n \rightarrow \infty$ ,*

$$QLR_n \xrightarrow{d} \zeta \max_{g \in \mathbb{R}} (2W(g) - |g|),$$

where

$$\zeta = \frac{\sigma^2(\gamma_0)}{\sigma^2}.$$

The results of the last proposition and that in (15) indicate that the only difference between the two limit distributions is the scaling factor. This is the case despite the fact the estimator  $\hat{\gamma}$  exhibits a different rate of convergence under Assumption **J** or **C**.

Next, we propose an estimator of the unknown scaling of  $QLR_n$  that is consistent for  $\xi$  under Assumption **J** and consistent for  $\zeta$  under Assumption **C**, thus adapting to the unknown

true scaling in each situation. We begin with a natural estimator of  $\xi$ , which is a ratio of two Nadaraya-Watson estimators of the conditional expectations. That is,

$$\widehat{\xi} = \frac{\frac{1}{n} \sum_{t=1}^n (\widehat{\delta}' x_t)^2 \widehat{\varepsilon}_t^2 K\left(\frac{q_t - \widehat{\gamma}}{a}\right)}{\mathbb{S}_n(\widehat{\theta}) \frac{1}{n} \sum_{t=1}^n (\widehat{\delta}' x_t)^2 K\left(\frac{q_t - \widehat{\gamma}}{a}\right)}, \quad (16)$$

where  $K(\cdot)$  and  $a$  are, respectively, the kernel function and bandwidth parameter and  $\widehat{\varepsilon}_t$ 's are the least squares residuals. The consistency of  $\widehat{\xi}$  to  $\xi$  is standard, as argued in Hansen (2000).

However, it is not trivial to establish that  $\widehat{\xi} \xrightarrow{P} \zeta$  under Assumption **C** because both numerator and denominator degenerates asymptotically under Assumption **C**. It turns out that we need to impose some unconventional restrictions on the kernel function  $K$  and the bandwidth  $a$ . Specifically, we assume

**Assumption K.** *Assume the following for  $K(\cdot)$  and  $a$ .*

**K1**  $K(\cdot)$  is symmetric and  $\kappa_\ell = \int_{-\infty}^{\infty} u^\ell K(u) du < C$  for  $\ell \leq 4$  and  $\kappa_2 \neq 0$ .

**K2**  $K(\cdot)$  is twice continuously differentiable with the first derivative  $K'(\cdot)$  and for all  $u$  such that  $|w/u| \leq C$  as  $w \rightarrow 0$   $K'(u+w)/K'(u) \rightarrow 1$ .

**K3**  $K(u) = \int \phi(v) e^{ivu} dv$ , where the characteristic function  $\phi(v)$  satisfies that  $v\phi(v)$  is integrable.

**K4**  $a^{-3}n^{-1} + a \rightarrow 0$  as  $n \rightarrow \infty$ .

It is clear that the Epanechnikov and the Gaussian kernel functions satisfy **K1**, **K2** and **K3**. One important observation is that **K1** rules out higher order kernels by assuming  $\kappa_2 \neq 0$ . The consequence of dropping the assumption that  $\kappa_2 \neq 0$  is discussed in detail in the remark 2 that follows the next proposition.

**Proposition 3.** *Suppose Assumptions **Z**, **Q** and **K** hold true. Then, under Assumption **C**,*

$$\widehat{\xi} \xrightarrow{P} \zeta,$$

while  $\widehat{\xi} \xrightarrow{P} \xi$  under Assumption **J**.

**Remark 2.** We now comment on the consequence of dropping the assumption that  $\kappa_2 \neq 0$ . If we allowed for higher order kernels, that is  $\kappa_2 = 0$  and  $\kappa_3 = 0$  but  $\kappa_4 \neq 0$ ,  $\widehat{\xi}$  would not be consistent. Indeed, Proposition 3 and Lemma 2 in the Appendix indicate that, without loss of generality for  $\gamma_0 = 0$  and  $\sigma^2 = 1$ ,  $\widehat{\xi}$  converges in probability to

$$\frac{\frac{\partial^2}{\partial q^2} f(q) g_0(q) |_{q=0}}{\frac{\partial^2}{\partial q^2} f(q) g_0^*(q) |_{q=0}},$$

where  $g_r(q) = E(x_{t2}^r \varepsilon_t^2 | q_t = q)$  and  $g_r^*(q) = E(x_{t2}^r | q_t = q)$ . This is the case because dropping in **K1** the assumption of  $\kappa_2 \neq 0$  and letting  $\kappa_2 = \kappa_3 = 0$ , the numerator in (16) will be

$$\kappa_4 \delta_3^2 a^4 \frac{\partial^2}{\partial q^2} (f(0) g_0(0)) (1 + o_p(1)),$$

whereas the denominator in (16) becomes

$$\kappa_4 \delta_3^2 a^4 \frac{\partial^2}{\partial q^2} (f(0) g_0^*(0)) (1 + o_p(1)).$$

So that, unless  $E(\varepsilon_t^2 | q_t = \gamma_0) = E(\varepsilon_t^2)$ , we obtain that (similar to the L'Hopital rule):

$$\widehat{\xi} \xrightarrow{P} \frac{\frac{\partial^2}{\partial q^2} f(q) g_0(q) |_{q=0}}{\frac{\partial^2}{\partial q^2} f(q) g_0^*(q) |_{q=0}} = \frac{\frac{\partial^2}{\partial q^2} (f(q) E[\varepsilon_t^2 | q_t = q]) |_{q=0}}{\frac{\partial^2}{\partial q^2} f(q) |_{q=0}} \neq \zeta,$$

and hence  $\widehat{\xi}$  would not be a consistent estimator of the scale factor  $\zeta$ .

We can construct the 100s percent confidence set of  $\gamma_0$  by

$$\widehat{\Gamma}_s = \left\{ \gamma \in \Gamma : \widehat{\xi}^{-1} QLR_n(\gamma) \leq F^{-1}(s) \right\}. \quad (17)$$

As we argued already, this confidence set is valid under both scenarios, as the next theorem shows.

**Theorem 2.** Suppose that Assumption **K**, **Z** and **Q** hold and let either Assumption **J** or **C** hold. Then, for any  $s \in (0, 1)$ ,

$$P\{\gamma_0 \in \widehat{\Gamma}_s\} \rightarrow s.$$



## 4 TESTING CONTINUITY

This section presents tests for the continuity restriction (4) given in Assumption **C**, that is

$$H_0 : \delta_{10} + \delta_{30}\gamma_0 = 0 \quad \text{and} \quad \delta_{20} = 0, \quad (18)$$

against its negation<sup>2</sup>

$$H_1 : \delta_{10} + \delta_{30}\gamma_0 \neq 0 \quad \text{and/or} \quad \delta_{20} \neq 0.$$

This testing problem is non-standard. First, the score type test is not straightforward due to non-differentiability of the criterion function  $\mathbb{S}_n(\gamma)$  with respect to  $\gamma$ . Second, as the unconstrained estimators  $\hat{\gamma}$  and  $\hat{\delta}$  converge at different rates, the construction of a Wald type test is not obvious. Thus, we first consider a quasi likelihood ratio statistic, which compares the constrained sum of squared residuals with the unconstrained one, i.e.

$$Q_n = n \frac{\tilde{\mathbb{S}}_n - \hat{\mathbb{S}}_n}{\hat{\mathbb{S}}_n} \quad (19)$$

where  $\hat{\mathbb{S}}_n = \mathbb{S}_n(\hat{\theta})$  and  $\tilde{\mathbb{S}}_n = \mathbb{S}_n(\tilde{\theta})$ , with

$$\mathbb{S}_n(\hat{\theta}) = \min_{\alpha, \gamma} \frac{1}{n} \sum_{t=1}^n (y_t - \alpha(\gamma)' x_t(\gamma))^2$$

and

$$\begin{aligned} \mathbb{S}_n(\tilde{\theta}) &= \min_{\alpha, \gamma: \delta_1 + \delta_3 \gamma_0 = 0; \delta_2 = 0} \frac{1}{n} \sum_{t=1}^n (y_t - \alpha(\gamma)' x_t(\gamma))^2 \\ &= \min_{\beta, \gamma, \delta_3} \frac{1}{n} \sum_{t=1}^n (y_t - x_t' \beta - \delta_3 (q_t - \gamma) \mathbf{1}_t(\gamma))^2 \end{aligned} \quad (20)$$

using our abbreviation of the model given in (8).

The asymptotic distribution of  $Q_n$  is in fact a by-product of derivation of the asymptotic distribution of  $\hat{\theta}$  and that of  $\tilde{\theta}$ .

---

<sup>2</sup>The condition of  $\delta_3 \neq 0$  in Assumption **C** is an auxiliary assumption to ensure the identification of the change-point  $\gamma_0$  under the kink design. This is not needed under the jump design as either  $\delta_{20} \neq 0$  or  $\delta_{10} + \delta_{30}\gamma_0 \neq 0$  is sufficient to identify  $\gamma_0$ .

**Theorem 3.** Let  $I_k$  and  $0_{a \times b}$  denote the identity matrix of dimension  $k$  and the matrix of zeros of dimension  $a \times b$ , respectively, and

$$R = \begin{pmatrix} I_k & 0_{k \times k} \\ 0_{1 \times k} & -\gamma_0 : 1 : 0_{1 \times (k-2)} \\ 0_{1 \times k} & -\beta_{30} - \delta_{30} : 0_{1 \times (k-1)} \end{pmatrix}.$$

Define a Gaussian process

$$\begin{aligned} \mathbb{K}(h, g, \ell) &= \ell' E R \mathbf{x}_t \mathbf{x}_t' R' \ell + h' E \mathbf{x}_t \mathbf{x}_t' h - 2(\ell' R + h') B \\ &\quad + (2\delta_{30} \sqrt{\frac{\sigma^2(\gamma_0) f(\gamma_0)}{3}} W(g^3) + \frac{\delta_{30}^2}{3} f(\gamma_0) |g|^3), \end{aligned}$$

where  $B$  is  $\mathcal{N}(0, \Omega)$  and independent of the gaussian process  $W$  that was introduced in Theorem 1. Then, under Assumptions **Q**, **Z** and **C**,

$$Q_n \xrightarrow{d} \min_{h:g=0, \ell=0} \mathbb{K}(h, g, \ell) + \min_{g:h=0, \ell=0} \mathbb{K}(h, g, \ell) - \min_{\ell:g=0, h=0} \mathbb{K}(h, g, \ell) / \sigma^2.$$

As the limiting distribution of  $Q_n$  is not pivotal, it seems of not much use the derive an explicit expression of its limit distribution, and hence we do not pursue it here. Instead, we turn to the bootstrap to estimate  $p$ -value of the test statistic in the subsequent section.

An alternative to the statistic  $Q_n$  is to use  $QLR_n(\gamma)$  described in Section 3.2 evaluated at  $\gamma = \tilde{\gamma}$ . Its rational is that under the null hypothesis of continuity both estimators  $\hat{\gamma}$  and  $\tilde{\gamma}$  converge to the true value  $\gamma_0$  with the latter having a faster rate of convergence. Hence the continuity test using  $QLR_n(\tilde{\gamma})$  has the same limit distribution as  $QLR_n(\gamma_0)$  due to Propositions 2 and 3 of Section 3.2. On the other hand, under the alternative hypothesis of the jump design, one expects that the estimator of the location of the jump  $\tilde{\gamma}$  is inconsistent and thus  $\hat{\gamma}$  and  $\tilde{\gamma}$  would converge to different values, leading to consistency of the test. Furthermore, the bootstrap procedure given in Section 5.2.1 below can be applied too.

## 5 BOOTSTRAP

This section provides a bootstrap procedure for the test of continuity based on the  $Q_n$  statistic and develops a bootstrap-based test inversion confidence interval for the unknown threshold parameter  $\gamma_0$ , which is valid under both Assumptions **J** and **C**. The latter bootstrap can also be used for the continuity test based on  $QLR_n(\tilde{\gamma})$ .

We do not discuss the bootstrap for  $\alpha_0$  in detail but note that the bootstrap for the linear regression can be employed, see e.g. Shao and Tu (1995), since we can treat  $\hat{\gamma}$  as  $\gamma_0$  for the inference on  $\alpha_0$  due to the arguments leading to the asymptotic independence between  $\hat{\alpha}$  and  $\hat{\gamma}$ .

## 5.1 Bootstrapping Continuity Test

We proceed as follows:

**STEP 1** Obtain both LSE  $\hat{\theta} = (\hat{\alpha}', \hat{\gamma})'$  and CLSE  $\tilde{\theta} = (\tilde{\alpha}', \tilde{\gamma})'$  of  $\theta_0 = (\alpha_0', \gamma_0)'$  as described in Section 2.1, and compute the least squares residuals

$$\hat{\varepsilon}_t = y_t - \hat{\alpha}' x_t(\hat{\gamma}), \quad t = 1, \dots, n.$$

**STEP 2** Generate  $\{\eta_t\}_{t=1}^n$  as independent and identically distributed zero mean random variables with unit variance and finite fourth moments, and compute

$$y_t^* = \tilde{\alpha}' x_t(\tilde{\gamma}) + \hat{\varepsilon}_t \eta_t, \quad t = 1, \dots, n.$$

**STEP 3** Using  $\{y_t^*\}_{t=1}^n$  and  $\{x_t\}_{t=1}^n$ , construct the bootstrap statistic  $Q_n^*$  as in (19) of Section 4. Specifically,

$$Q_n^* = n \frac{\tilde{S}_n^* - \hat{S}_n^*}{\hat{S}_n^*},$$

where

$$\begin{aligned} \hat{S}_n^* &= \min_{\theta} \frac{1}{n} \sum_{t=1}^n (y_t^* - \alpha(\gamma)' x_t(\gamma))^2, \\ \tilde{S}_n^* &= \min_{\theta: \delta_1 + \delta_3 \gamma_0 = 0; \delta_2 = 0} \frac{1}{n} \sum_{t=1}^n (y_t^* - \alpha(\gamma)' x_t(\gamma))^2. \end{aligned}$$

**STEP 4** Compute the bootstrap  $p$ -value,  $p^*$  by repeating *STEPS 2-3*  $B$  times and obtain the proportion of times that  $Q_n^*$  exceeds the sample statistic  $Q_n$  given in (19).

## 5.2 Bootstrap Test Inversion Confidence Interval for $\gamma_0$

We propose using the bootstrap test inversion method, also known as the grid bootstrap, of Dömbgen (1991) to build confidence intervals for the parameter  $\gamma$ , see also Carpenter (1999) and Hansen (1999). Such test inversion bootstrap confidence interval (BCI) is known to have certain optimality property as in e.g. Brown, Casella and Hwang (1995) from the Bayesian perspective. Mikusheva (2007) showed that test inversion BCI attains correct coverage probability uniformly over the parameter space for the sum of coefficients in autoregressive models, despite the behaviour of the estimator not being uniform over the parameter space.

For a given confidence level  $s$ , one can exploit the duality between hypothesis testing and confidence interval by inverting tests to obtain a confidence region

$$\hat{\Gamma}_s^* = \left\{ \gamma \in \Gamma : \hat{\xi}(\gamma)^{-1} QLR_n(\gamma) \leq q_n^*(s|\gamma) \right\} ,$$

where  $q_n^*(s|\gamma)$  is the bootstrap estimate of the  $s$ th quantile of the statistic  $\hat{\xi}(\gamma)^{-1} QLR_n(\gamma)$  when  $\gamma_0 = \gamma$ . In other words, it denotes the bootstrap critical value of level  $(1 - s)$  testing for  $\mathcal{H}_0 : \gamma_0 = \gamma$ . In practice, one would estimate  $q_n^*(s|\gamma)$  over a grid of  $\gamma$ 's and use some smoothing method such as linear interpolation or kernel averaging to obtain a smoothed bootstrap quantile function over a range of  $\gamma$ . The region  $\hat{\Gamma}_s^*$  is known as  $s$ -level grid bootstrap confidence interval (BCI) of  $\gamma$  in the terminology of Hansen (1999).

Figure 1 illustrates how this confidence interval can be obtained in practice. The  $QLR_n(\gamma)$  line is the linear interpolation of the rescaled  $QLR_n(\gamma)$  statistic over the grid of  $\gamma$  at 50 points. The ACV line is the asymptotic critical value of Hansen (2000). The true value of  $\gamma_0$  was 2. We estimated bootstrap quantile function (described in the sequel) at 17 grid points and present the interpolated line as Grid quantile plot. The vertical arrow at intersections between  $QLR_n(\gamma)$  and ACV yield the asymptotic confidence interval (ACI), while the vertical broken arrows indicate grid BCI based on the bootstrap.

Now, we describe the bootstrap procedure for the grid bootstrap. We repeat the following procedure for each values of  $\gamma_j \in \{\gamma_1, \dots, \gamma_g\}$ .

### 5.2.1 Bootstrap Algorithm for each $\gamma_j$

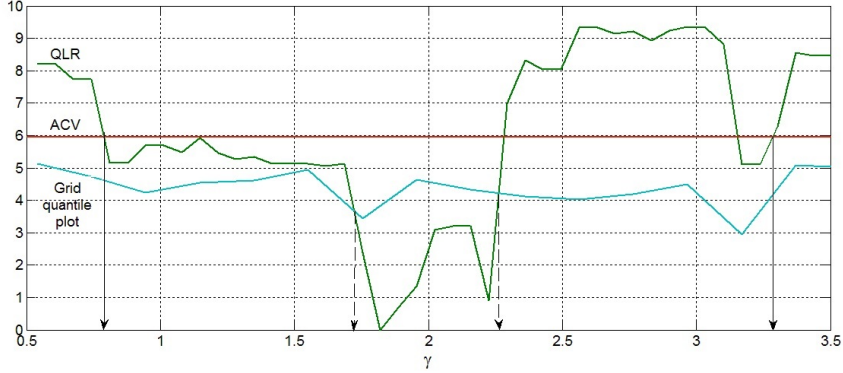


Figure 1: 90% asymptotic and grid bootstrap confidence intervals,  $n = 100$ ,  $\gamma_0 = 2$

**STEP 1** Obtain LSE  $(\hat{\alpha}', \hat{\gamma})'$  by minimizing (7) and compute the LSE residuals

$$\hat{\varepsilon}_t = y_t - \hat{\alpha}'x_t(\hat{\gamma}) \quad t = 1, \dots, n.$$

**STEP 2** Generate  $\{\eta_t\}_{t=1}^n$  as independent and identically distributed zero mean random variables with unit variance and finite fourth moments, and compute

$$y_t^* = \hat{\alpha}'x_t(\gamma_j) + \hat{\varepsilon}_t\eta_t, \quad t = 1, \dots, n.$$

**STEP 3** Obtain the least squares estimate using  $\{y_t^*\}_{t=1}^n$  and  $\{x_t\}_{t=1}^n$ ,

$$\hat{\theta}^* = \underset{\theta}{\operatorname{argmin}} \mathbb{S}_n^*(\theta) := \frac{1}{n} \sum_{t=1}^n (y_t^* - x_t(\gamma)' \alpha)^2. \quad (21)$$

**STEP 4** Compute the bootstrap analogues of  $QLR_n$  and  $\hat{\xi}$  as

$$QLR_n^* = n \frac{\mathbb{S}_n^*(\gamma_j) - \mathbb{S}_n^*(\hat{\gamma}^*)}{\mathbb{S}_n^*(\hat{\gamma}^*)},$$

and

$$\hat{\xi}^* = \frac{\sum_{t=1}^n (\hat{\delta}^{*'}x_t)^2 \hat{\varepsilon}_t^{*2} K\left(\frac{q_t - \hat{\gamma}^*}{a}\right)}{\mathbb{S}_n(\hat{\theta}^*) \sum_{t=1}^n (\hat{\delta}^{*'}x_t)^2 K\left(\frac{q_t - \hat{\gamma}^*}{a}\right)}, \quad (22)$$

where  $\mathbb{S}_n^*(\gamma)$  is defined analogously defined as  $\mathbb{S}_n(\gamma)$  in (9) by replacing  $y_t$  with  $y_t^*$ .

**STEP 5** Compute the bootstrap sth quantile  $q_n^*(s|\gamma_j)$  from the empirical distribution of  $\widehat{\xi}^{*-1}QLR_n^*$  by repeating STEPs 2-4.

One can use the above algorithm setting  $\gamma_j = \tilde{\gamma}$  for the test of continuity based on  $QLR_n(\tilde{\gamma})$ .

### 5.3 Validity of Bootstrap

This section derives the convergences of the bootstrap LSE  $\widehat{\alpha}^*$  and  $\widehat{\gamma}^*$  for both continuous and discontinuous setups and shows the consistency of the bootstrap statistic  $\widehat{\xi}^*$ . These results then yield the validity of the bootstrap continuity test and that of the bootstrap test inversion confidence set following the same arguments in the proof of Theorem 3 and that of Theorem 2, respectively.

As usual, the superscript “\*” indicates the bootstrap quantities and convergences of bootstrap statistics conditional on the original data.

**Theorem 4.** *Suppose that Assumptions **Z** and **Q** hold true.*

(a) *Under Assumption **C**,  $\widehat{\alpha}^*$  and  $\widehat{\gamma}^*$  are asymptotically independent and*

$$n^{1/2}(\widehat{\alpha}^* - \tilde{\alpha}) \xrightarrow{d^*} \mathcal{N}(0, M^{-1}\Omega M^{-1}), \quad \text{in Probability,}$$

$$n^{1/3}(\widehat{\gamma}^* - \gamma_0) \xrightarrow{d^*} \arg \max_{g \in \mathbb{R}} \left( 2\delta_{30} \sqrt{\frac{\sigma^2(\gamma_0)}{3}} f(\gamma_0) W(g^3) + \frac{\delta_{30}^2}{3} f(\gamma_0) |g|^3 \right), \quad \text{in Probability.}$$

(b) *Under Assumption **J**,  $\widehat{\alpha}^*$  and  $\widehat{\gamma}^*$  are asymptotically independent and*

$$n^{1/2}(\widehat{\alpha}^* - \tilde{\alpha}) \xrightarrow{d^*} \mathcal{N}(0, M^{-1}\Omega M^{-1}), \quad \text{in } P,$$

$$n^{1-2\varphi}(\widehat{\gamma}^* - \gamma_0) \xrightarrow{d^*} \frac{2c'Vc}{(c'Dc)^2 f(\gamma_0)} \arg \max_{g \in \mathbb{R}} (2W(g) - |g|), \quad \text{in Probability.}$$

Our results can be compared with those already obtained in literature regarding the validity of bootstrap for non-standard estimators. First, our consistency result seems to contradict Seijo and Sen’s (2011) result on the inconsistency of a residual-based bootstrap and the non-parametric bootstrap (with independent and identically distributed data) for the case where  $\varphi = 0$ , see also Yu (2014). The reason behind such contradictory conclusions lies in the observation that our setup differs from theirs in an important and vital way: they consider the case

of a fixed size of the break whereas we consider the situation that  $\delta = \delta_n$  decreases with the sample size. Thus, their limiting distribution depends on the whole conditional distribution of  $\varepsilon_t \eta_t c' x_t$  given  $q_t = \gamma_0$  in a complicated manner, whereas ours contains only an unknown scaling factor.

Next, the consistency of  $\widehat{\xi}^*$  is established in the following proposition.

**Proposition 4.** *Suppose Assumptions **Z**, **Q** and **K** hold and either of Assumption **J** or Assumption **C** hold true. Then,*

$$\widehat{\xi}^* - \widehat{\xi} = o_{p^*}(1).$$

We are ready to state the validity of bootstrap procedure.

**Theorem 5.** *Suppose Assumptions **Z**, **Q** and **K** hold. Then, under Assumption **C**,*

$$Q_n^* \xrightarrow{d^*} Q, \text{ in Probability,}$$

where  $Q$  denotes the limit variable in Theorem 3. Now, suppose either Assumption **J** or Assumption **C** hold true in addition to Assumptions **Z**, **Q** and **K**. Then,

$$\widehat{\xi}^{*-1} Q L R_n^* \xrightarrow{d^*} \max_{g \in \mathbb{R}} (2W(g) - |g|), \text{ in Probability.}$$

## 6 Monte Carlo Experiment

We generate data based on the following 4 designs, with setting C representing the kink case and Setting D the structural break model.

$$A : y_t = 2 + 3x_t + \delta x_t 1\{q_t > \gamma_0\} + \varepsilon_t, \quad E(q_t) = 2, \gamma_0 = 2, 2.674, \delta = \sqrt{10}/4n^{1/4},$$

$$B : y_t = 2 + 3q_t + \delta q_t 1\{q_t > \gamma_0\} + \varepsilon_t, \quad E(q_t) = 2, \gamma_0 = 2, 2.674, \delta = \sqrt{10}/4n^{1/4},$$

$$C : y_t = 2 + 3q_t + \delta q_t 1\{q_t > \gamma_0\} + \varepsilon_t, \quad E(q_t) = 0, -0.674, \gamma_0 = 0, \delta = 2,$$

$$D : y_t = 2 + 3q_t + \delta q_t 1\{t > \gamma_0\} + \varepsilon_t, \quad \gamma_0 = \lfloor n/2 \rfloor, \lfloor 3n/4 \rfloor, \delta = \sqrt{10}/4n^{1/4}.$$

We set  $\varepsilon_t = |q_t| e_t$  where  $\{e_t\}_{t \geq 1}$  and  $\{q_t\}_{t \geq 1}$  were generated as mutually independent and independent and identically distributed normal random variables with unit variance, leading to conditional heteroscedasticity of form  $E(\varepsilon_t^2 | q_t) = q_t^2$ . We use sample sizes  $n = 100, 250, 500$ , which yield  $\delta = 0.25, 0.1988, 0.1672$  respectively, for settings A, B and D. For the grid  $\Gamma_n$  we

used in estimation of  $\gamma_0$ , we discard 10% of extreme values of realized  $q_t$  and use  $n/2$  number of equidistant points. In  $\hat{\xi}$ , Epanechnikov kernel and minimum-MSE bandwidth choice, given in Härdle and Linton (1994), were used.

Columns 4-6 of Table 1 present Monte Carlo size of test of  $H_0 : \gamma = \gamma_0$  when  $\gamma_0 = E(q_t)$  for nominal sizes  $s = 0.9, 0.95, 0.99$ . We carried out 10,000 iterations with one bootstrap per iteration, using the warp-speed method of Giacomini, Dimitris and White (2013). Using asymptotic critical values delivers poor Monte Carlo sizes in settings A and B with substantial oversizing, in contrast to the bootstrap sizes which are satisfactory for  $n = 250, 500$ . For the kink design C, asymptotic test does not work across all  $n$  as expected, with results getting worse for  $n = 500$ , while bootstrap sizes are good for larger sample sizes  $n = 250, 500$ . For D, asymptotic test reports satisfactory size results but the bootstrap method performs even better.

Columns 8-10 of Table 1 report the coverage probabilities of confidence intervals for  $\gamma_0$ , when  $\gamma_0$  is median of  $q_t$ , and columns 11-13 present the case when  $\gamma_0$  is the third quartile of  $q_t$ , for confidence levels  $\zeta = 0.9, 0.95, 0.99$ . Results are based on 1,000 iterations and in each iteration, we generated bootstrap quantile plots by interpolating bootstrap quantiles obtained at 10 equidistant points of the realized support of  $q_t$  from 399 bootstraps, and found intersections with the sample  $QLR_n$  plot formed by interpolating between  $n/2$  number of equidistant points after discarding 10% of extreme values of realized  $q_t$ . The coverage probability results are better when  $\gamma_0$  is the third quartile of  $q_t$  in settings A and B for both methods and similar across the two  $\gamma_0$ 's in Setting D. For Setting C, bootstrap coverage probability is better when  $\gamma_0$  is the third quartile of  $q_t$  compared to when it is the median, and the asymptotic confidence interval does not work for both  $\gamma_0$ 's. In setting A, the asymptotic and bootstrap methods perform similarly, reporting lower than nominal coverage probabilities which improve with larger  $n$ . In setting B, the bootstrap method delivers substantially better coverage probabilities than the asymptotic confidence intervals and in setting D asymptotic method yields results somewhat better than that of bootstrap method. In setting C, the asymptotic coverage probabilities becomes close to 1 for all values of  $\zeta$  for  $n = 250, 500$ , while bootstrap coverage probabilities are satisfactory for  $n = 250, 500$ .

We also present results for when  $\varphi = 1/8$  in Table 2 for settings A, B and D, with  $\delta = 0.4446, 0.3965, 0.3636$  for  $n = 100, 250, 500$  respectively, to see if slower rate of shrinkage in  $\delta$  affects the results. Both the size and coverage probability results are better when  $\varphi = 1/8$



Table 1: Monte Carlo size of test  $H_0 : \gamma = \gamma_0$  and coverage probability of confidence intervals of  $\gamma_0$ , models A-D

		Size			Coverage Probability							
		$\gamma_0$	median of $q_t$			$\gamma_0$	median of $q_t$			third quart. of $q_t$		
		$s \setminus n$	100	250	500	$\zeta \setminus n$	100	250	500	100	250	500
A	Asym	0.01	0.095	0.059	0.044	0.9	0.733	0.770	0.774	0.811	0.834	0.844
		0.05	0.195	0.153	0.130	0.95	0.818	0.832	0.857	0.870	0.895	0.914
		0.1	0.290	0.242	0.200	0.99	0.916	0.938	0.950	0.953	0.971	0.980
	B/rap	0.01	0.003	0.015	0.009	0.9	0.756	0.810	0.840	0.783	0.826	0.852
		0.05	0.052	0.055	0.037	0.95	0.833	0.880	0.910	0.859	0.892	0.915
		0.1	0.106	0.095	0.083	0.99	0.928	0.959	0.969	0.935	0.965	0.980
B	Asym	0.01	0.185	0.145	0.155	0.9	0.608	0.612	0.658	0.740	0.730	0.725
		0.05	0.344	0.293	0.268	0.95	0.687	0.707	0.742	0.813	0.817	0.827
		0.1	0.437	0.379	0.365	0.99	0.831	0.851	0.859	0.905	0.924	0.926
	B/rap	0.01	0.022	0.013	0.021	0.9	0.770	0.836	0.866	0.868	0.882	0.878
		0.05	0.101	0.066	0.071	0.95	0.853	0.894	0.924	0.932	0.943	0.943
		0.1	0.203	0.126	0.133	0.99	0.946	0.972	0.982	0.975	0.984	0.980
C	Asym	0.01	0.123	0.028	0.005	0.9	0.802	0.946	0.975	0.749	0.925	0.972
		0.05	0.168	0.043	0.015	0.95	0.84	0.965	0.983	0.784	0.945	0.98
		0.1	0.200	0.056	0.024	0.99	0.892	0.982	0.992	0.852	0.966	0.99
	B/rap	0.01	0.027	0.014	0.012	0.9	0.768	0.854	0.805	0.828	0.894	0.877
		0.05	0.091	0.054	0.052	0.95	0.817	0.918	0.889	0.88	0.949	0.943
		0.1	0.153	0.108	0.104	0.99	0.905	0.979	0.975	0.954	0.981	0.984
D	Asym	0.01	0.006	0.003	0.004	0.9	0.846	0.878	0.883	0.849	0.880	0.892
		0.05	0.036	0.028	0.031	0.95	0.918	0.929	0.931	0.904	0.935	0.944
		0.1	0.074	0.067	0.070	0.99	0.968	0.990	0.985	0.974	0.982	0.992
	B/rap	0.01	0.006	0.004	0.011	0.9	0.794	0.840	0.869	0.786	0.840	0.865
		0.05	0.046	0.040	0.062	0.95	0.865	0.898	0.919	0.865	0.910	0.930
		0.1	0.085	0.092	0.120	0.99	0.944	0.973	0.978	0.952	0.972	0.983

Size results for test of  $H_0 : \gamma = \gamma_0$  with nominal size  $s$  based on Hansen (2000)'s asymptotic distribution(Asym), and bootstrap(B/rap).

Coverage probability results for  $\gamma_0$  with asymptotic confidence interval based on Hansen (2000) and grid bootstrap confidence interval,

with nominal confidence level  $\zeta$ . A:  $q_t \neq x_t$ , B:  $q_t = x_t$  with a jump, C:  $q_t = x_t$  with a kink, D:  $q_t = t$ . In A, B and D,

$\delta = \delta_n = \sqrt{10}/4n^{1/4}$ , in C  $\delta = 2$ .

compared to  $\varphi = 1/4$  in settings A and B while for D they were similar between the two  $\varphi$ 's. For both size and coverage probability results, patterns similar to Table 1 discussed in the previous paragraphs emerge, with the exception that in setting A, bootstrap coverage probabilities are similar between the two values of  $\gamma_0$ .

Now we turn to finite sample performance of the two tests of continuity proposed in Section 4. Table 3 presents Monte Carlo size results of the two tests of continuity when 10,000 iterations are taken with warp-speed method for nominal size of test  $s = 0.1, 0.05, 0.01$ . In columns 3-5, the size results show that  $QLR_n$  test is undersized for larger  $n$  while  $Q_n$  test reports more satisfactory size performance. Table 4 reports Monte-Carlo power results for the two tests of continuity for nominal size of test  $s$ . Power results naturally are affected by the size of  $\delta$  and three sets of  $\delta_n$  have been tried. As expected, power improves as  $\delta$  gets larger and as  $n$  increases. It can be seen that  $Q_n$  test yields better power performance compared to the  $QLR_n$  test.

## 7 Empirical Application: Growth and debt

The so-called Reinhart-Rogoff hypothesis postulates that above some threshold (90% being their estimate of this threshold), higher debt-to-GDP ratio is associated with lower GDP growth rate. There have been many studies that utilize the threshold regression models to assess this hypothesis, including Hansen (2015) who fitted a kink model to a time series of US annual data, see Hansen (2015) for references on earlier studies which fitted jump models to various data sets. Hansen (2015, p.3) mentions that “one could imagine testing the assumption of continuity within the threshold model class. This is a difficult problem, one to which we are unaware of a solution, and therefore is not pursued in this paper”. As we have developed testing procedures for continuity in this paper, we follow up on Hansen (2015)'s investigation.

Hansen (2015) used US annual data on real GDP growth rate in year  $t$  ( $y_t$ ) and debt-to-GDP ratio from the *previous* year ( $q_t$ ) for the period spanning 1792-2009 ( $n = 218$ ) and reported the following estimated equation with standard errors in parenthesis:

$$\hat{y}_t = 3.78 + 0.28y_{t-1} + \begin{cases} \frac{0.033(q_t - 43.8)}{(0.026)}, & \text{if } q_t < 43.8 \\ -\frac{0.067(q_t - 43.8)}{(0.048)}, & \text{if } q_t \geq 43.8 \end{cases} \begin{matrix} (12.1) \\ (12.1) \end{matrix}$$

We carried out our tests of continuity given in Section 4 with 10,000 bootstraps and obtained

Table 2: Monte Carlo size of test  $H_0 : \gamma = \gamma_0$  and coverage probability of confidence intervals of  $\gamma_0$ ,  $\varphi = 1/8$ , models A, B, D

		Size				Coverage Probability						
		$\gamma_0$ $s \setminus n$	median of $q_t$			$\gamma_0$ $\zeta \setminus n$	median of $q_t$			third quart. of $q_t$		
			100	250	500		100	250	500	100	250	500
A	Asym	0.01	0.068	0.037	0.029	0.9	0.79	0.837	0.897	0.817	0.835	0.872
		0.05	0.164	0.092	0.077	0.95	0.856	0.898	0.923	0.873	0.91	0.914
		0.1	0.214	0.15	0.129	0.99	0.933	0.961	0.975	0.949	0.964	0.972
	B/rap	0.01	0.006	0.009	0.008	0.9	0.791	0.846	0.881	0.792	0.827	0.871
		0.05	0.046	0.052	0.049	0.95	0.858	0.907	0.93	0.859	0.9	0.917
		0.1	0.099	0.095	0.105	0.99	0.936	0.968	0.98	0.938	0.963	0.972
B	Asym	0.01	0.155	0.098	0.079	0.9	0.661	0.72	0.786	0.771	0.779	0.791
		0.05	0.285	0.207	0.158	0.95	0.745	0.802	0.852	0.852	0.844	0.855
		0.1	0.368	0.275	0.224	0.99	0.86	0.886	0.921	0.925	0.941	0.938
	B/rap	0.01	0.029	0.009	0.017	0.9	0.797	0.871	0.904	0.886	0.891	0.888
		0.05	0.093	0.073	0.065	0.95	0.878	0.917	0.945	0.936	0.946	0.943
		0.1	0.171	0.113	0.109	0.99	0.95	0.981	0.99	0.984	0.984	0.98
D	Asym	0.01	0.01	0.008	0.006	0.9	0.88	0.889	0.898	0.876	0.894	0.91
		0.05	0.043	0.04	0.041	0.95	0.93	0.942	0.946	0.922	0.945	0.956
		0.1	0.088	0.093	0.082	0.99	0.972	0.988	0.987	0.975	0.984	0.991
	B/rap	0.01	0.007	0.013	0.007	0.9	0.816	0.853	0.864	0.813	0.853	0.881
		0.05	0.053	0.059	0.05	0.95	0.887	0.912	0.929	0.879	0.924	0.938
		0.1	0.099	0.129	0.116	0.99	0.961	0.978	0.983	0.951	0.974	0.986

Size results for test of  $H_0 : \gamma = \gamma_0$  with nominal size  $s$  based on Hansen (2000)'s asymptotic distribution(Asym), and bootstrap(B/rap).

Coverage probability results for  $\gamma_0$  with asymptotic confidence interval based on Hansen (2000) and grid bootstrap confidence interval,

with nominal confidence level  $\zeta$ . A:  $q_t \neq x_t$ , B:  $q_t = x_t$  with a jump, D:  $q_t = t$ .  $\delta = \delta_n = \sqrt{10}/4n^{1/8}$ .

Table 3: Monte Carlo size of two tests of continuity, Setting C

$n$	$s$	0.1	0.05	0.01
100	$QLR_n$	0.0947	0.055	0.0097
	$Q_n$	0.1195	0.0737	0.0204
250	$QLR_n$	0.0642	0.0387	0.0077
	$Q_n$	0.0832	0.0477	0.0122
500	$QLR_n$	0.0674	0.0341	0.0067
	$Q_n$	0.0897	0.0408	0.0076
1000	$QLR_n$	0.0851	0.0434	0.0092
	$Q_n$	0.105	0.0491	0.0109

Table 4: Monte Carlo power of two tests of continuity, Setting B

$n$	$\delta$	$\sqrt{10}/4n^{1/4}$			$\sqrt{10}/2n^{1/4}$			$\sqrt{10}/n^{1/4}$		
	$s$	0.1	0.05	0.01	0.1	0.05	0.01	0.1	0.05	0.01
100	$QLR_n$	0.088	0.0421	0.0064	0.1036	0.0479	0.0058	0.2392	0.1196	0.0185
	$Q_n$	0.1313	0.0661	0.0134	0.1525	0.0726	0.013	0.3282	0.1918	0.0365
250	$QLR_n$	0.0715	0.0284	0.0032	0.0993	0.0374	0.0034	0.3674	0.1971	0.0345
	$Q_n$	0.1205	0.0564	0.0097	0.1502	0.068	0.0098	0.4684	0.3028	0.0623
500	$QLR_n$	0.0765	0.0276	0.0034	0.1286	0.0512	0.0052	0.5804	0.4128	0.127
	$Q_n$	0.1151	0.0574	0.0089	0.1656	0.0787	0.0117	0.637	0.4797	0.1685

$Q_n$  test's  $p$ -value of 0.029 and  $QLR_n$  test's  $p$ -value of 0.039, and hence reject the null of continuity at 5% nominal level. Before fitting the jump model to the data, we first tested for the presence of threshold effect using the testing procedure of Hansen (1996) with 1,000 bootstrap replications, and obtained  $p$ -value of 0.047, rejecting the null hypothesis of no threshold effect. This is in contrast to the the  $p$ -value of 0.15 obtained by Hansen's (2015) test for presence of threshold effect when imposing the kink model. Hansen (2015) had remained inconclusive on the presence of kink threshold effect, since the bootstrap method used there did not account for the time series nature of data and the high  $p$ -value could have been due to modest power of the test.

The fitted jump model is given by:

$$\hat{y}_t = \begin{cases} 4.82 - 0.052y_{t-1} - 0.114q_t, & \text{if } q_t < 17.2 \\ (0.87) & (0.16) & (0.049) \\ 2.78 + 0.49 y_{t-1} - 0.017q_t, & \text{if } q_t \geq 17.2 \\ (0.74) & (0.082) & (0.012) \end{cases}$$

The sizes of the two regimes were 99 (below 17.2% ) and 109 (above 17.2%). We obtained grid bootstrap confidence intervals for  $\gamma_0$  to be (10.5, 39) for 95% confidence level and (10.8, 38.6) for 90%, based on 399 bootstrap iterations. Bootstrap quantiles were obtained at 38 grid points, which included  $\hat{\gamma}$ ,  $\tilde{\gamma}$  and equidistant points on the realised support of  $q_t$  after discarding 7.5% of the largest and smallest values of  $q_t$  in the sample. We find the points of intersection between the linearly interpolated bootstrap quantile line and the linear interpolation of sample  $QLR_n(\gamma)$  test statistics for  $H_0 : \gamma_0 = \gamma_j$  at grid points  $\gamma_j$  consisting of 73 equidistant points and  $\hat{\gamma}$ ,  $\tilde{\gamma}$ , as shown in Figure 2 for 90% confidence level. These confidence intervals do not contain the CLSE  $\tilde{\gamma} = 43.8$ , which is not surprising in the light of rejection of continuity.

As the estimated threshold under the jump model is noticeably small at 17.2%, our estimated jump model which suggests insignificance of effect of  $q_t$  on  $y_t$  above the threshold does not necessarily contradict the Reinhart-Rogoff hypothesis. To see if this could be an indication of presence of further threshold points, we applied Hansen (1996)'s testing procedure for presence of threshold effect on the lower and upper subsamples with 1000 bootstraps and obtained  $p$ -values of 0.025 and 0.016, respectively. Hence, we conclude that the US time series data should be fitted to a threshold regression model with multiple threshold points, and that the continuity tests suggest that imposing the kink restriction is not warranted.

To see if such conclusion holds across different countries, we proceeded by first applying Hansen (1996)'s test for the presence of threshold effect on Reinhart and Rogoff's (2010) data

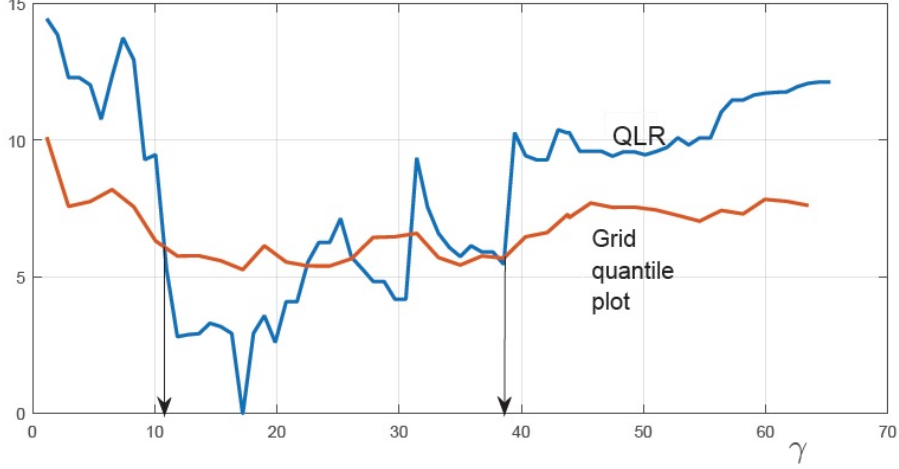


Figure 2: 90% grid bootstrap confidence interval for the US

for countries with relatively long time spans without missing observations. For Australia ( $n = 107$ ) and the UK ( $n = 178$ ), the  $p$ -values with 1000 bootstraps were 0.795 and 0.98 so we conclude that there is no threshold effect for these countries in the relationship between the GDP growth and the debt-to-GDP ratio.

For data from Sweden for the period 1881-2009 ( $n = 129$ ), the  $p$ -value for Hansen (1996)'s test of presence of threshold effect with 1000 bootstraps for the whole sample is 0.048, while for the lower and upper regimes, divided by  $\hat{\gamma}$ , they were 0.979 and 0.131, respectively. Applying our continuity tests based on 10,000 bootstraps yield  $p$ -value of 0.091 for  $Q_n$  test and 0.097 for  $QLR_n$  test. The estimated jump model is:

$$\hat{y}_t = \begin{cases} \begin{matrix} 1.12 - 0.2 y_{t-1} + 0.13 q_t, & \text{if } q_t < 21.3 \\ (2.17) & (0.24) & (0.11) \end{matrix} \\ \begin{matrix} 1.86 + 0.48 y_{t-1} - 0.004 q_t, & \text{if } q_t \geq 21.3 \\ (0.58) & (0.11) & (0.0082) \end{matrix} \end{cases}$$

with the lower regime having 61 observations and upper regime containing 68. Here again, the coefficient of debt-to-GDP ratio is not statistically significant.

The grid bootstrap confidence intervals for  $\gamma_0$  were  $(15.3, \infty)$  and  $(16.4, \infty)$  for 95% and 90% confidence levels. Shown in Figures 3 are linear interpolation of 90% bootstrap quantiles

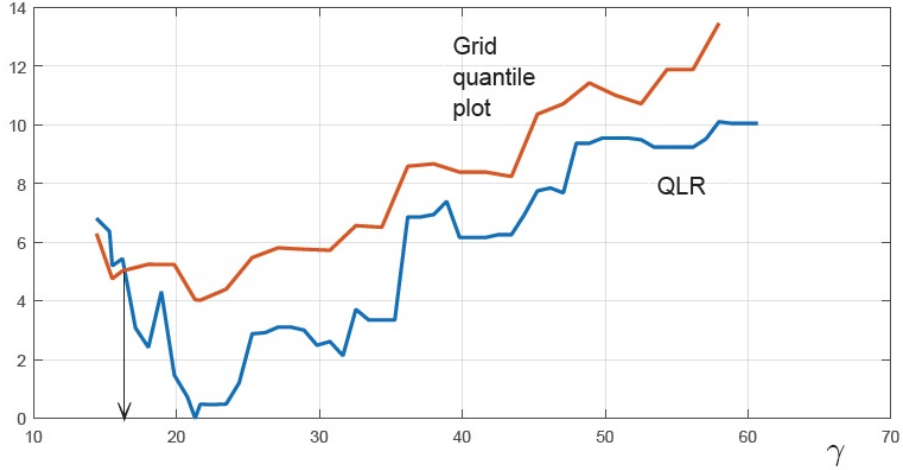


Figure 3: 90% grid bootstrap confidence interval for Sweden

at 27 grid points with 399 bootstraps and linear interpolation of QLR test statistic at each of 54 grid points. This is in line with our finding that the confidence interval for  $\gamma_0$  tends to become much wider as the model becomes a kink model as reflected by the cube root convergence rate.

The coefficients of debt-to-GDP ratio were also not significant in the estimated kink model, which need to be read with caution in the light of the continuity test:

$$\hat{y}_t = \frac{2.89}{(0.58)} + \frac{0.048}{(0.13)} y_{t-1} + \begin{cases} +0.24(q_t - 15.5), & \text{if } q_t < 15.5 \\ (0.3) & (5.75) \\ -0.0008(q_t - 15.5), & \text{if } q_t \geq 15.5 \\ (0.014) & (5.75) \end{cases}$$

whereby the lower regime had 15 observations and the upper regime contained 114 observations.

We conclude that there is substantial heterogeneity across countries in the relationship between the GDP growth and the debt-to-GDP ratio, not only in the values of model parameters, but also in the kinds of models that are suitable. Along with the existing tests for presence of threshold effect, our tests of continuity need to be carried out before fitting and interpreting threshold regression models, especially the kink model.

## 8 CONCLUSION

This paper has developed a unified framework for inference within a class of threshold regression models that encompass jump and kink designs, uncovering a hitherto-unknown convergence rate result for the unrestricted estimate under the kink design. We have also offered two testing procedures that distinguish between the jump and kink designs. Moreover, we provided assumptions under which the bootstrap is valid for the threshold regression model, regardless of whether the model is continuous or not. Our Monte Carlo simulation further demonstrates that its finite sample performance is better than that using the asymptotic critical values.

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## A PROOFS OF MAIN THEOREMS

Let us introduce some notation first. In what follows  $C, C_1, \dots$  denote generic positive finite constants, which may vary from line to line or expression to expression. Recall that  $x_t = (1, x'_{t2}, q_t)'$  and  $x_{t1} = (1, x'_{t2})'$  and that  $\mathbf{1}_t(a; b) = \mathbf{1}\{a < q_t < b\}$  and  $\mathbf{1}_t(b) = \mathbf{1}\{b < q_t\}$ . Finally, we abbreviate  $\psi - \psi_0$  by  $\bar{\psi}$  for any parameter  $\psi$ .

### A.1 Proof of Proposition 1

Without loss of generality we assume that  $\hat{\gamma} \geq \gamma_0$  and  $\gamma_0 = 0$ , so that  $\delta_{10} = 0$  and  $\delta_{20} = 0$  under Assumption C. By definition, we have that

$$\begin{aligned} \mathbb{S}_n(\theta) - \mathbb{S}_n(\theta_0) &= \frac{1}{n} \sum_{t=1}^n \left\{ (y_t - \alpha' x_t(\gamma))^2 - \varepsilon_t^2 \right\} \\ &= \frac{1}{n} \sum_{t=1}^n \left\{ \left( \bar{\beta}' x_t + \bar{\delta}' x_t \mathbf{1}_t(\gamma) + \delta_0' x_t \mathbf{1}_t(0; \gamma) + \varepsilon_t \right)^2 - \varepsilon_t^2 \right\}. \end{aligned}$$

By standard algebra and denoting  $v = \beta + \delta$ ,

$$\begin{aligned} &\bar{\beta}' x_t + \bar{\delta}' x_t \mathbf{1}_t(\gamma) + \delta_0' x_t \mathbf{1}_t(0; \gamma) \\ &= \bar{v}' x_t \mathbf{1}_t(\gamma) + (\bar{\beta} + \delta_0)' x_t \mathbf{1}_t(0; \gamma) + \bar{\beta}' x_t \mathbf{1}_t(-\infty; 0), \end{aligned}$$

which implies, because of the orthogonality of the terms on the right of the last displayed expression, that

$$\mathbb{S}_n(\theta) - \mathbb{S}_n(\theta_0) = \mathbb{A}_{n1}(\theta) + \mathbb{A}_{n2}(\theta) + \mathbb{A}_{n3}(\theta) + \mathbb{B}_{n1}(\theta) + \mathbb{B}_{n2}(\theta) + \mathbb{B}_{n3}(\theta),$$

where

$$\begin{aligned} \mathbb{A}_{n1}(\theta) &= \bar{v}' \frac{1}{n} \sum_{t=1}^n x_t x_t' \mathbf{1}_t(\gamma) \bar{v}; & \mathbb{A}_{n2}(\theta) &= \bar{\beta}' \frac{1}{n} \sum_{t=1}^n x_t x_t' \mathbf{1}_t(-\infty; 0) \bar{\beta} \\ \mathbb{A}_{n3}(\theta) &= (\bar{\beta} + \delta_0)' \frac{1}{n} \sum_{t=1}^n x_t x_t' \mathbf{1}_t(0; \gamma) (\bar{\beta} + \delta_0) \\ \mathbb{B}_{n1}(\theta) &= \bar{v}' \frac{2}{n} \sum_{t=1}^n x_t \varepsilon_t \mathbf{1}_t(\gamma); & \mathbb{B}_{n2}(\theta) &= \bar{\beta}' \frac{2}{n} \sum_{t=1}^n x_t \varepsilon_t \mathbf{1}_t(-\infty; 0) \\ \mathbb{B}_{n3}(\theta) &= (\bar{\beta} + \delta_0)' \frac{2}{n} \sum_{t=1}^n x_t \varepsilon_t \mathbf{1}_t(0; \gamma). \end{aligned}$$

**Consistency.** It suffices to show that for any  $\epsilon > 0$ ,  $\eta > 0$ , there is  $n_0$  such that for all  $n > n_0$ ,  $\Pr \left\{ \left\| \hat{\theta} - \theta_0 \right\| > \eta \right\} < \epsilon$ , which is implied by

$$\Pr \left\{ \inf_{\|\hat{\theta}\| > \eta} \sum_{\ell=1}^3 E(\mathbb{A}_{n\ell}(\theta)) + \mathbb{D}_{n\ell}(\theta) \leq 0 \right\} < \epsilon, \quad (23)$$

where  $\mathbb{D}_{n\ell}(\theta) = \mathbb{B}_{n\ell}(\theta) + (\mathbb{A}_{n\ell}(\theta) - E(\mathbb{A}_{n\ell}(\theta)))$  for  $\ell = 1, 2, 3$ .

First  $\|\hat{\theta}\| > \eta$  implies that either (i)  $\|\bar{\gamma}\| > \eta/3$  and  $\|\bar{\beta}\| \leq \eta/3$ , or (ii)  $\|\bar{\beta}\| > \eta/3$  or  $\|\bar{v}\| > \eta/3$ . When (i) holds true, it is clear that

$$\inf_{\|\bar{v}\| > \eta/3} E(\mathbb{A}_{n1}(\theta)) > C\eta^2 \quad \text{or} \quad \inf_{\|\bar{\beta}\| > \eta/3} E(\mathbb{A}_{n2}(\theta)) > C\eta^2 \quad (24)$$

whereas when (ii) holds true, we have that

$$\inf_{\|\bar{\gamma}\| > \eta/3, \|\bar{\beta}\| \leq \eta/3} E \left( \frac{1}{n} \sum_{t=1}^n (x_t' (\bar{\beta} + \delta_0))^2 \mathbf{1}_t(0; \gamma) \right) > C\eta^3, \quad (25)$$

because Assumption Q implies that  $E(x_t x_t' \mathbf{1}_t(\gamma))$ ,  $E(x_t x_t' \mathbf{1}_t(-\infty; 0))$  and  $E(x_t x_t' \mathbf{1}_t(0; \gamma))$  are positive definite matrices uniformly in  $\gamma > \eta$  and  $\|\bar{\beta} + \delta_0\| > \eta/3$  if  $\|\bar{\beta}\| \leq \eta/3$  because we can always choose  $\eta$  such that  $|\delta_0| \geq 2\eta/3$ . we have that

$$C_1 \leq \frac{E\mathbb{A}_{n3}(\theta)}{(\bar{\tau}_1, \bar{\tau}_2)' E(x_{t1} x_{t1}' \mathbf{1}_t(0; \gamma)) (\bar{\tau}_1, \bar{\tau}_2)' + \bar{\tau}_3^2 E(q_t^2 \mathbf{1}_t(0; \gamma))} \leq C_2, \quad (26)$$

where  $\bar{\tau} = (\beta_0 - \beta) + \delta_0$ . The motivation for the last displayed inequality comes from the fact that, say, implies that  $E\{x_t x_t' \mathbf{1}_t(\gamma_1; \gamma_2)\}$  is a strictly positive and finite definite matrix which implies that for any vector  $a' = (a'_1, a_2)$ ,

$$C^{-1} \leq \frac{a' E\{x_t x_t' \mathbf{1}_t(\gamma_1; \gamma_2)\} a}{a'_1 E\{x_{t1} x'_{t1} \mathbf{1}_t(\gamma_1; \gamma_2)\} a_1 + a_2^2 E\{q_t^2 \mathbf{1}_t(\gamma_1; \gamma_2)\}} \leq C$$

for some finite constant  $C > 0$ . So, (24) and (25) implies that

$$\inf_{\|\bar{\theta}\| > \eta} \sum_{\ell=1}^3 E(\mathbb{A}_{n\ell}(\theta)) > C\eta^3. \quad (27)$$

On the other hand, Lemma 1 and the uniform law of large numbers, respectively, imply that

$$\sup_{\|\bar{\theta}\| > \eta} \|\mathbb{B}_{n\ell}(\theta)\| = O_p(n^{-1/2}) \quad \ell = 1, 2, 3; \quad \sup_{\gamma_1, \gamma_2} \|\mathbb{F}_n(\gamma_1; \gamma_2)\| = o_p(1),$$

where  $\mathbb{F}_n(\gamma_1; \gamma_2) = \frac{1}{n} \sum_{t=1}^n (x_t x_t' \mathbf{1}_t(\gamma_1; \gamma_2) - E(x_t x_t' \mathbf{1}_t(\gamma_1; \gamma_2)))$ , and hence

$$\sup_{\|\bar{\theta}\| > \eta/3} \left\| \sum_{\ell=1}^3 \mathbb{D}_{n\ell}(\theta) \right\| = o_p(1). \quad (28)$$

Thus  $\hat{\theta} - \theta_0 = o_p(1)$  because the left side of (23) is bounded by

$$\Pr \left\{ \inf_{\|\bar{\theta}\| > \eta} \sum_{\ell=1}^3 E(\mathbb{A}_{n\ell}(\theta)) \leq \sup_{\|\bar{\theta}\| > \eta/3} \left\| \sum_{\ell=1}^3 \mathbb{D}_{n\ell}(\theta) \right\| \right\} \rightarrow 0,$$

using (27) and (28).

**Convergence Rate.** We shall show next that for any  $\epsilon > 0$  there exist  $C > 0$ ,  $\eta > 0$ ,  $n_0$  such that for  $n > n_0$  we have that

$$\Pr \left\{ \frac{C}{n^{1/2}} < \|\bar{v}\|, \|\bar{\beta}\| < \eta; \frac{C}{n^{1/3}} < \|\bar{\gamma}\| < \eta \sum_{\ell=1}^3 E(\mathbb{A}_{n\ell}(\theta)) + \mathbb{D}_{n\ell}(\theta) \leq 0 \right\} < \epsilon. \quad (29)$$

Since  $\Pr\{X_n + Y_n < 0\} \leq \Pr\{X_n < 0\} + \Pr\{Y_n < 0\}$  for any sequence  $X_n$  and  $Y_n$  and  $\inf_x \{f(x) + g(x)\} \geq \inf_x f(x) + \inf_x g(x)$  for any functions  $f$  and  $g$ , it suffices to show that for each  $\ell = 1, 2, 3$

$$\Pr \left\{ \frac{C}{n^{1/2}} < \|\bar{v}\|, \|\bar{\beta}\| < \eta; \frac{C}{n^{1/3}} < \|\bar{\gamma}\| < \eta \quad E(\mathbb{A}_{n\ell}(\theta)) / 2 + (\mathbb{A}_{n\ell}(\theta) - E(\mathbb{A}_{n\ell}(\theta))) \leq 0 \right\} < \epsilon \quad (30)$$

$$\Pr \left\{ \inf_{\substack{\frac{C}{n^{1/2}} < \|\bar{v}\|, \|\bar{\beta}\| < \eta; \\ \frac{C}{n^{1/3}} < \|\bar{\gamma}\| < \eta}} E(\mathbb{A}_{n\ell}(\theta)) / 2 + \mathbb{B}_{n\ell}(\theta) \leq 0 \right\} < \epsilon. \quad (31)$$

To that end, we shall first examine

$$\Pr \left\{ \inf_{\Xi_j(\psi); \Xi_j(\beta); \Xi_k(\gamma)} E(\mathbb{A}_{n\ell}(\theta)) / 2 + \mathbb{B}_{n\ell}(\theta) \leq 0 \right\}, \quad \ell = 1, 2, 3,$$

where

$$\begin{aligned} \Xi_j(\psi) &= \left\{ \psi : \frac{C}{n^{1/2}} 2^{j-1} < \|\bar{\psi}\| < \frac{C}{n^{1/2}} 2^j \right\}; \quad j = 1, \dots, \log_2 \frac{\eta}{C} n^{1/2} \\ \Xi_k(\gamma) &= \left\{ \gamma : \frac{C}{n^{1/3}} 2^{k-1} < \bar{\gamma} < \frac{C}{n^{1/3}} 2^k \right\}; \quad k = 1, \dots, \log_2 \frac{\eta}{C} n^{1/3}. \end{aligned} \quad (32)$$

Recall that we have assumed that  $\gamma \geq 0$ , as the case  $\gamma \leq 0$  follows similarly.

First by standard arguments,

$$\begin{aligned} & \Pr \left\{ \inf_{\Xi_j(\psi); \Xi_k(\gamma)} E(\mathbb{A}_{n1}(\theta)) / 2 + \mathbb{B}_{n1}(\theta) \leq 0 \right\} \\ & \leq \Pr \left\{ \inf_{\Xi_j(\psi)} \|\bar{v}\| \lambda_{\min}(E x_t x_t' \mathbf{1}_t(0)) \leq \sup_{\Xi_k(\gamma)} \left\| \frac{4}{n^{1/2}} \sum_{t=1}^n x_t \varepsilon_t \mathbf{1}_t(\gamma) \right\| \right\} \\ & \leq \Pr \left\{ C 2^{j-2} \leq \sup_{\{\gamma: \|\bar{\gamma}\| < \eta\}} \left\| \frac{1}{n^{1/2}} \sum_{t=1}^n x_t \varepsilon_t \mathbf{1}_t(\gamma) \right\| \right\} \\ & \leq C^{-1} 2^{-j+2} \eta^{1/2} \end{aligned} \quad (33)$$

by Lemma 1 and the Markov's inequality. Observe that the latter inequality is independent of  $\Xi_k(\gamma)$ . Since  $\sum_{j=1}^{\infty} 2^{-j} < \infty$ , the probability in (31) can be made arbitrary small for large  $C$  or small  $\eta$ , thus satisfying the condition (31). (30) follows similarly as is the case for  $\ell = 2$  and thus it is omitted.

We next examine (30) and (31) for  $\ell = 3$ . Observing (26) and the arguments that follow, defining

$$\tilde{\mathbb{A}}_{n3}(\theta) = \bar{\tau}^2 E(q_t^2 \mathbf{1}_t(0; \gamma)); \quad \tilde{\mathbb{B}}_{n3}(\theta) = \bar{\tau} \frac{2}{n} \sum_{t=1}^n q_t \varepsilon_t \mathbf{1}_t(0; \gamma),$$

it suffices to show (30) and (31) for  $\tilde{\mathbb{A}}_{n3}(\theta)$  and  $\tilde{\mathbb{B}}_{n3}(\theta)$ . To that end, because  $\bar{\tau} > C_1$  as

$|\delta_{30}| > C_1 > 0$ , we obtain, since  $E q_t^2 \mathbf{1}_t(0; \eta) \geq C_1 \eta^3$

$$\begin{aligned}
& \Pr \left\{ \inf_{\Xi_j(v); \Xi_k(\gamma)} E \left( \tilde{\mathbb{A}}_{n3}(\theta) / 2 \right) + \tilde{\mathbb{B}}_{n3}(\theta) \leq 0 \right\} \\
& \leq \Pr \left\{ \inf_{\Xi_k(\gamma)} \|\tau_0\| E \left( q_t^2 \mathbf{1}_t(0; \gamma) \right) \leq \sup_{\Xi_k(\gamma)} \left\| \frac{4}{n} \sum_{t=1}^n q_t \varepsilon_t \mathbf{1}_t(0; \gamma) \right\| \right\} \\
& \leq \Pr \left\{ \frac{C}{n} 2^{3(k-2)} \leq \sup_{\Xi_k(\gamma)} \left\| \frac{1}{n} \sum_{t=1}^n q_t \varepsilon_t \mathbf{1}_t(0; \gamma) \right\| \right\} \\
& \leq C^{-1} 2^{-3k/2},
\end{aligned} \tag{34}$$

by Lemma 1 and Markov's inequality. Notice that this bound is independent of  $\Xi_j(v)$ . But by summability of  $2^{-3k/2}$ , we conclude that (31) holds true for  $\ell = 3$  by choosing  $C$  large enough.

We now conclude the proof after we note that the left side of (29) is bounded by

$$\begin{aligned}
& \Pr \left\{ \max_{j,k} \inf_{\Xi_j(v); \Xi_j(\beta); \Xi_k(\gamma)} \sum_{\ell=1}^3 \{ E \mathbb{A}_{n\ell}(\theta) + \mathbb{B}_{n\ell}(\theta) \} \leq 0 \right\} \\
& \leq C^{-1} \left( \sum_{j=1}^{\log_2 \frac{n}{C} n^{1/2}} 2^{-2j} + \sum_{k=1}^{\log_2 \frac{n}{C} n^{1/3}} 2^{-3k/2} \right) < \epsilon
\end{aligned}$$

using (33) – (34). ■

## A.2 Proof of Theorem 1

Because the “arg min” is a continuous mapping, see Kim and Pollard (1990), and the convergence rates of  $\hat{\alpha}$  and  $\hat{\gamma}$  are obtained, it suffices to examine the weak limit of

$$\begin{aligned}
\mathbb{G}_n(h, g) &= n \left( \mathbb{S}_n \left( \alpha_0 + \frac{h}{n^{1/2}}, \gamma_0 + \frac{g}{n^{1/3}} \right) - \mathbb{S}_n(\alpha_0, \gamma_0) \right) \\
&= \sum_{t=1}^n \left\{ \left( \varepsilon_t - \frac{h'}{n^{1/2}} x_t \left( \frac{g}{n^{1/3}} \right) - \delta_{30} q_t \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) \right)^2 - \varepsilon_t^2 \right\},
\end{aligned}$$

over any compact set, where we assume  $\gamma_0 = 0$  as before for notational convenience. Let  $\|h\|, |g| \leq C$ . First, due to the uniform law of large numbers it follows that

$$\sup_{|g| \leq C} \left| \frac{1}{n} \sum_{t=1}^n \left\{ x_t \left( \frac{g}{n^{1/3}} \right) x_t' \left( \frac{g}{n^{1/3}} \right) - \mathbf{x}_t \mathbf{x}_t' \right\} \right| = o_p(1)$$

whereas Lemma 1 and the expansion of  $E \left\{ x_t \left( \frac{g}{n^{1/3}} \right) q_t \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) \right\}$  as in (36) imply that

$$\begin{aligned} \sup_{|g| \leq C} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^n \left\{ x_t \left( \frac{g}{n^{1/3}} \right) q_t \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) \right\} \right| &= O_p \left( n^{-1/6} \right) \\ \sup_{|g| \leq C} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^n \left( x_t \left( \frac{g}{n^{1/3}} \right) - \mathbf{x}_t \right) \varepsilon_t \right| &= O_p \left( n^{-1/6} \right). \end{aligned}$$

Therefore,

$$\sup_{\|h\|, |g| \leq C} \left| \mathbb{G}_n(h, g) - \tilde{\mathbb{G}}_n(h, g) \right| = o_p(1), \quad (35)$$

where

$$\begin{aligned} \tilde{\mathbb{G}}_n(h, g) &= \left\{ h' \frac{1}{n} \sum_{t=1}^n \mathbf{x}_t \mathbf{x}_t' h - h' \frac{2}{n^{1/2}} \sum_{t=1}^n \mathbf{x}_t \varepsilon_t \right\} \\ &\quad + \delta_{30} \left\{ \delta_{30} \sum_{t=1}^n q_t^2 \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) - 2 \sum_{t=1}^n q_t \varepsilon_t \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) \right\} \\ &=: \tilde{\mathbb{G}}_n^1(h) + \tilde{\mathbb{G}}_n^2(g). \end{aligned}$$

The consequence of (35) is then that the minimizer of  $\mathbb{G}_n(h, g)$  is asymptotically equivalent to that of  $\tilde{\mathbb{G}}_n(h, g)$ . Thus, it suffices to show the weak convergence of  $\tilde{\mathbb{G}}_n^1(h)$  and  $\tilde{\mathbb{G}}_n^2(g)$  and that

$$\tilde{h} := \arg \min_{h \in \mathbb{R}} \tilde{\mathbb{G}}_n^1(h); \quad \tilde{g} := \operatorname{argmin}_{g \in \mathbb{R}} \tilde{\mathbb{G}}_n^2(g)$$

are  $O_p(1)$ . The convergence of  $\tilde{\mathbb{G}}_n^1(h)$  and its minimization is straightforward since it is a quadratic function of  $h$ .

Next, the first term of  $\tilde{\mathbb{G}}_n^2(g)$  converges to  $3^{-1} \delta_{30}^2 f(0) |g|^3$  uniformly in probability because Lemma 1, i.e. (53), implies the uniform law of large numbers and the Taylor series expansion up to the third order yields

$$n E q_t^2 \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) = n \int_0^{\frac{g}{n^{1/3}}} q^2 f(q) dq = n \frac{2f \left( \frac{\tilde{g}}{n^{1/3}} \right)}{3!} \left( \frac{g}{n^{1/3}} \right)^3 \rightarrow 3^{-1} f(0) g^3, \quad (36)$$

where  $\tilde{g} \in (0, g)$ . When  $g < 0$ , it follows similarly as in this case the derivative should be multiplied by  $-1$ , so that the limit becomes  $3^{-1} f(0) |g|^3$ .

The second term in the definition of  $\tilde{\mathbb{G}}_n^2(g)$ , that is,  $-2 \sum_{t=1}^n q_t \varepsilon_t \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right)$  converges weakly to  $2\delta_{30} \sqrt{3^{-1} f(0) \sigma_\varepsilon^2(0)} W(g^3)$ . To see this note that Lemma 1, i.e. (52), yields the



tightness of the process as explained in Remark 3. For the finite dimensional convergence, we can verify the conditions for martingale difference sequence CLT (e.g. Hall and Heyde's (1980) Theorem 3.2). In particular, we need to show that for  $u_{nt} = \sqrt{n}q_t\varepsilon_t\mathbf{1}_t\left(0; \frac{g}{n^{1/3}}\right)$ ,

$$(i) \quad n^{-1/2} \max_{1 \leq t \leq n} |u_{nt}| \xrightarrow{p} 0$$

$$(ii) \quad \frac{1}{n} \sum_{t=1}^n u_{nt}^2 \xrightarrow{p} \frac{1}{3} E(\varepsilon_t^2 | q_t = 0) f(0) g^3$$

For (i), note that  $En^{-2} \max_t |u_{nt}|^4 \leq n^{-1} E |u_{nt}|^4 = nE q_t^4 \varepsilon_t^4 \mathbf{1}_t\left(0; \frac{g}{n^{1/3}}\right) \rightarrow 0$  as  $n \rightarrow \infty$ . For (ii), apply the same argument for the first term in  $\tilde{\mathbb{G}}_n^2(g)$  and an expansion similar to that in (36). We now characterize the covariance kernel, note that if  $g_1$  and  $g_2$  have different signs then the cross product becomes zero and for  $g_2 > g_1 > 0$ , similarly as with (36), we have that

$$nE \left( \varepsilon_t^2 (q_t - \gamma_0)^2 \mathbf{1} \left\{ \frac{g_1}{n^{1/3}} < q_t < \frac{g_2}{n^{1/3}} \right\} \right) = \frac{f(\gamma_0)}{3} \sigma_\varepsilon^2(\gamma_0) (g_2^3 - g_1^3) + o(1).$$

The cases for  $g_1 > g_2 > 0$  or  $g_2 < g_1 < 0$  are similar and thus omitted.

Finally, the covariance between  $n^{-1/2} \sum_{t=1}^n \mathbf{x}_t \varepsilon_t$  and  $\sum_{t=1}^n q_t \varepsilon_t \mathbf{1}_t\left(0; g/n^{1/3}\right)$  vanishes for the same reasoning, yielding the independence between  $\tilde{h}$  and  $\tilde{g}$  and thus the asymptotic independence between  $\hat{\alpha}$  and the threshold estimator  $\hat{\gamma}$ . ■

### A.3 Proof of Corollary 1

This is a corollary of Theorem 1, Lemma A.12 of Hansen (2000), Theorem 2 of Chan (1993). ■

### A.4 Proof of Proposition 2

Due to the asymptotic independence between  $\hat{\alpha}$  and  $\hat{\gamma}$  in Theorem 1, specifically (35) in the proof, we have that

$$n(\mathbb{S}_n(\hat{\alpha}(\gamma_0); \gamma_0) - \mathbb{S}_n(\hat{\alpha}; \hat{\gamma})) = n(\mathbb{S}_n(\alpha_0; \gamma_0) - \mathbb{S}_n(\alpha_0; \hat{\gamma})) + o_p(1),$$

which corresponds to  $\min_g \tilde{\mathbb{G}}_n^2(g)$  in the proof of Theorem 1. It also shows that

$$\min_g \tilde{\mathbb{G}}_n^2(g) \xrightarrow{d} f(\gamma_0) \min_{g \in \mathbb{R}} \left( 2\delta_{30} \sqrt{3^{-1} f(\gamma_0) \sigma_\varepsilon^2(\gamma_0)} W(g^3) + 3^{-1} \delta_{30}^2 f(\gamma_0) |g|^3 \right).$$

Finally, the desired result follows from applying the change-of-variable  $g^3 = 3\phi\sigma_\varepsilon^2(\gamma_0)/\delta_{30}^2 f(\gamma_0)$  because of the distributional equivalence  $W(a^2g) \stackrel{d}{=} aW(g)$  (and  $W(s) \stackrel{d}{=} -W(s)$ ) and the fact that  $\min_x g(x) = -\max_x -g(x)$  for any function  $g$ .  $\blacksquare$

## A.5 Proof of Proposition 3

Recalling our notation in (2) and that  $\delta_1 + \delta_3\gamma_0 = 0$  and  $\delta_2 = 0$  under Assumption C, we then have that

$$\widehat{\delta}'x_t = \left(\widehat{\delta}_1 - \delta_1\right) + \widehat{\delta}'_2 x_{2t} + \left(\widehat{\delta}_3 - \delta_3\right) q_t + \delta_3 (q_t - \gamma_0). \quad (37)$$

Because we can rename  $q_t - \gamma_0$  as  $q_t$ , we shall assume without loss of generality that  $\gamma_0 = 0$  so that  $\delta_1 = 0$ .

Consider the case where  $\widehat{\gamma} > 0$ . The proof when  $\widehat{\gamma} < 0$  is analogous and thus it is omitted. By construction, we have that

$$\widehat{\varepsilon}_t = \varepsilon_t + \left(\widehat{\beta} - \beta\right)' x_t + \left(\widehat{\delta} - \delta\right)' x_t \mathbf{1}_t(\widehat{\gamma}) + \delta_3 q_t \mathbf{1}_t(0; \widehat{\gamma}).$$

Because  $(\delta_1, \delta'_2) = 0$  and  $\widehat{\beta} - \beta = O_p(n^{-1/2})$ ,  $\widehat{\delta} - \delta = O_p(n^{-1/2})$  and  $\widehat{\gamma} = O_p(n^{-1/3})$ , we obtain that

$$\begin{aligned} \widehat{\varepsilon}_t^2 &= \varepsilon_t^2 + O_p(n^{-1}) + (\delta_3 q_t)^2 \mathbf{1}_t(0; \widehat{\gamma}) + 2\delta_3 \varepsilon_t q_t \mathbf{1}_t(0; \widehat{\gamma}) \\ &\quad + O_p(n^{-1/2}) \varepsilon_t x_t (1 + \mathbf{1}_t(\widehat{\gamma})) + 2\delta_3 \|x_t\| q_t \mathbf{1}_t(0; \widehat{\gamma}) O_p(n^{-1/2}) \\ &= \varepsilon_t^2 + O_p(n^{-1/2}) \|x_t\| \varepsilon_t + 2\delta_3 \varepsilon_t q_t \mathbf{1}_t(0; \widehat{\gamma}) + \|x_t\| O_p(n^{-2/3}). \end{aligned} \quad (38)$$

Now (37) implies that  $\left(\widehat{\delta}'x_t\right)^2 = \delta_3^2 q_t^2 + O_p(n^{-1/2}) \delta_3 \|x_t\| q_t + O_p(n^{-1})$ . So, by Lemma 2 and 3 and by the standard arguments using  $na^3 \rightarrow \infty$ , we conclude that the behaviour of numerator of (16) is that of

$$\frac{1}{na^3} \sum_{t=1}^n \delta_3^2 q_t^2 \varepsilon_t^2 K\left(\frac{q_t - \widehat{\gamma}}{a}\right) = \kappa_2 \delta_3^2 a^2 \sigma^2(0) f(0) (1 + o_p(1))$$

when  $\kappa_2 \neq 0$ , that is we do not assume higher order kernels. Observe that  $g_0(q)$  in Lemma 2 corresponds to  $\sigma^2(q)$ . More specifically, the contribution due to other terms in (38) are indeed negligible by Lemma 3.

Similarly, the leading term in the denominator in (16) is

$$\frac{1}{na^3} \sum_{t=1}^n (\widehat{\delta}' x_t)^2 K\left(\frac{q_t - \widehat{\gamma}}{a}\right) = \kappa_2 \delta_3^2 a^2 f(0) (1 + o_p(1)).$$

So, the convergence in (16) follows from the last two displayed expressions. Finally, it is standard to show that  $\mathbb{S}_n(\widehat{\theta}) - \sigma^2 = o_p(1)$ . This completes the proof of the proposition. ■

## A.6 Proof of Theorem 2

It is known that the distribution function of  $\max_{g \in \mathbb{R}} (2W(g) - |g|)$  is  $F$ , as in Hansen (2000). Thus, under Assumption C, Proposition 2 and 3 in this paper yields the conclusion, while under Assumption J, Theorem 2 of Hansen (2000) verified the conclusion. ■

## A.7 Proof of Theorem 3

Recall the following facts observed in the proof Theorem 1: for the unconstrained estimator  $\widehat{\alpha}$  and  $\widehat{\gamma}$ , we could write

$$n(\mathbb{S}_n(\widehat{\alpha}, \widehat{\gamma}) - \mathbb{S}_n(\alpha_0, \gamma_0)) = \min_h \widetilde{\mathbb{G}}_n^1(h) + \min_g \widetilde{\mathbb{G}}_n^2(g) + o_p(1),$$

where

$$\begin{aligned} \widetilde{\mathbb{G}}_n^1(h) &= \left\{ h' \frac{1}{n} \sum_{t=1}^n \mathbf{x}_t \mathbf{x}_t' h + h' \frac{1}{n^{1/2}} \sum_{t=1}^n \mathbf{x}_t \varepsilon_t \right\} \\ \widetilde{\mathbb{G}}_n^2(g) &= \delta_{30} \left\{ \delta_{30} \sum_{t=1}^n q_t^2 \mathbf{1}_t \left(0; \frac{g}{n^{1/3}}\right) + \sum_{t=1}^n q_t \varepsilon_t \mathbf{1}_t \left(0; \frac{g}{n^{1/3}}\right) \right\} \end{aligned}$$

and  $\widetilde{\mathbb{G}}_n^j(\cdot)$ ,  $j = 1, 2$ , are mutually independent. Similarly, for the constrained estimator  $\widetilde{\alpha}$  and  $\widetilde{\gamma}$  we can write, see e.g. Chan and Tsay (1998) or Hansen (2015), that

$$n(\mathbb{S}_n(\widetilde{\alpha}, \widetilde{\gamma}) - \mathbb{S}_n(\alpha_0, \gamma_0)) = \min_{\ell} \mathbb{H}_n(\ell) + o_p(1),$$

where

$$\mathbb{H}_n(\ell) = \ell' \frac{1}{n} \sum_{t=1}^n \bar{\mathbf{x}}_t \bar{\mathbf{x}}_t' \ell - 2\ell' \frac{1}{\sqrt{n}} \sum_{t=1}^n \bar{\mathbf{x}}_t \varepsilon_t$$

and  $\bar{\mathbf{x}}_t = (x_t', (q_t - \gamma_0) 1\{q_t > \gamma_0\}, -(\beta_{30} + \delta_{30}) 1\{q_t > \gamma_0\})' = R\mathbf{x}_t$ . Note that  $\mathbb{H}_n(\ell) + \tilde{\mathbb{G}}_n^1(h)$  converges weakly as a function of  $h$  and  $\ell$  since both  $\ell' \frac{1}{n} \sum_{t=1}^n \bar{\mathbf{x}}_t \bar{\mathbf{x}}_t' \ell$  and  $h' \frac{1}{n} \sum_{t=1}^n \mathbf{x}_t \mathbf{x}_t' h$  converges uniformly in probability by ULLN and  $\ell' \frac{1}{\sqrt{n}} \sum_{t=1}^n \bar{\mathbf{x}}_t \varepsilon_t + h' \frac{1}{\sqrt{n}} \sum_{t=1}^n \mathbf{x}_t \varepsilon_t$  converges weakly by the linearity, the CLT and Cramer-Rao device. The weak convergence of  $\tilde{\mathbb{G}}_n^2(g)$  and its asymptotic independence from  $\tilde{\mathbb{G}}_n^1(h)$  is given in Theorem 1 and by the same reasoning it is independent of  $\mathbb{H}_n(\ell)$  asymptotically. To sum up, let

$$\begin{aligned} \mathbb{K}(h, g, \ell) &= \ell' E \bar{\mathbf{x}}_t \bar{\mathbf{x}}_t' \ell + h' E \mathbf{x}_t \mathbf{x}_t' h - 2(\ell' R, h') B \\ &\quad + \left( 2\delta_{30} \sqrt{\frac{\sigma^2(\gamma_0) f(\gamma_0)}{3}} W(g^3) + \frac{\delta_{30}^2}{3} f(\gamma_0) |g|^3 \right), \end{aligned}$$

where  $B$  is a  $N(0, \Omega)$  and independent of the gaussian process  $W$ , which is defined in Theorem 1 and let

$$\mathbb{K}_n(h, g, \ell) = \tilde{\mathbb{G}}_n^1(h) + \tilde{\mathbb{G}}_n^2(g) + \mathbb{H}_n(\ell).$$

Then, it follows from the preceding discussion that

$$\mathbb{K}_n(h, g, \ell) \Rightarrow \mathbb{K}(h, g, \ell).$$

Furthermore,

$$\begin{aligned} &n(\mathbb{S}_n(\tilde{\alpha}, \tilde{\gamma}) - \mathbb{S}_n(\hat{\alpha}, \hat{\gamma})) \\ &= \min_{h:g=0, \ell=0} \mathbb{K}_n(h, g, \ell) + \min_{g:h=0, \ell=0} \mathbb{K}_n(h, g, \ell) - \min_{\ell:g=0, h=0} \mathbb{K}_n(h, g, \ell) + o_p(1) \\ &\xrightarrow{d} \min_{h:g=0, \ell=0} \mathbb{K}(h, g, \ell) + \min_{g:h=0, \ell=0} \mathbb{K}(h, g, \ell) - \min_{\ell:g=0, h=0} \mathbb{K}(h, g, \ell), \end{aligned}$$

due to the continuous mapping theorem as the (constrained) minimum is a continuous operator and the fact that  $\tilde{\mathbb{G}}_n^1(h)$ ,  $\tilde{\mathbb{G}}_n^2(g)$  and  $\mathbb{H}_n(\ell)$  are zero at the origin. Certainly this limit is  $O_p(1)$  and does not degenerate since  $\tilde{\mathbb{G}}_n^2(g)$  is asymptotically independent of the other terms. The convergence of  $\hat{\mathbb{S}}_n$  is straightforward by the standard algebra and the ULLN and CLT and thus details are omitted. ■

## A.8 Proof of Theorem 4

Recalling our definition of  $\widehat{\alpha}^*$  and  $\widehat{\gamma}^*$  in (21), we begin by showing their consistency and rate of convergence.

**Proposition 5.** *Suppose that Assumptions Z and Q hold. Then,*

(a) *Under Assumption C,*

$$\widehat{\alpha}^* - \widehat{\alpha} = O_{p^*} \left( n^{-1/2} \right) \quad \text{and} \quad \widehat{\gamma}^* - \gamma_0 = O_{p^*} \left( n^{-1/3} \right).$$

(b) *Under Assumption J,*

$$\widehat{\alpha}^* - \widehat{\alpha} = O_{p^*} \left( n^{-1/2} \right) \quad \text{and} \quad \widehat{\gamma}^* - \gamma_0 = O_{p^*} \left( n^{2\varphi-1} \right).$$

**Proof of Proposition 5** Assuming without loss of generality that  $\gamma \geq \widehat{\gamma} = \gamma_0$  and abbreviating  $\widehat{\psi} - \psi$  by  $\bar{\psi}$  for any parameter  $\psi$ , proceeding as in Proposition 1, we obtain that

$$\begin{aligned} \mathbb{S}_n^*(\theta) - \mathbb{S}_n^*(\widehat{\theta}) &= \frac{1}{n} \sum_{t=1}^n \left\{ \left( \bar{\beta}' x_t + \bar{\delta}' x_t \mathbf{1}_t(\gamma) + \widehat{\delta}' x_t \mathbf{1}_t(\widehat{\gamma}; \gamma) + \varepsilon_t^* \right)^2 - \varepsilon_t^{*2} \right\} \\ &= \widehat{\mathbb{A}}_{n1}(\theta) + \widehat{\mathbb{A}}_{n2}(\theta) + \widehat{\mathbb{A}}_{n3}(\theta) + \mathbb{B}_{n1}^*(\theta) + \mathbb{B}_{n2}^*(\theta) + \mathbb{B}_{n3}^*(\theta), \end{aligned}$$

where

$$\begin{aligned} \widehat{\mathbb{A}}_{n1}(\theta) &= \bar{v}' M_n^x(\gamma) \bar{v}; \quad \widehat{\mathbb{A}}_{n2}(\theta) = \bar{\beta}' M_n^x(-\infty; \widehat{\gamma}) \bar{\beta} \\ \widehat{\mathbb{A}}_{n3}(\theta) &= \left( \bar{\beta} + \widehat{\delta} \right)' M_n^x(\widehat{\gamma}; \gamma) \left( \bar{\beta} + \widehat{\delta} \right) \\ \mathbb{B}_{n1}^*(\theta) &= \bar{v}' \frac{2}{n} \sum_{t=1}^n x_t \varepsilon_t^* \mathbf{1}_t(\gamma); \quad \mathbb{B}_{n2}^*(\theta) = \bar{\beta}' \frac{2}{n} \sum_{t=1}^n x_t \varepsilon_t^* \mathbf{1}_t(-\infty; \widehat{\gamma}) \\ \mathbb{B}_{n3}^*(\theta) &= \left( \bar{\beta} + \widehat{\delta} \right)' \frac{2}{n} \sum_{t=1}^n x_t \varepsilon_t^* \mathbf{1}_t(\widehat{\gamma}; \gamma), \end{aligned}$$

where, in what follows, for a generic sequence  $\{z_t\}_{t \in \mathbb{Z}}$  we employ the notation  $M_n^z(\gamma) = \frac{1}{n} \sum_{t=1}^n z_t z_t' \mathbf{1}_t(\gamma)$  and  $M_n^z(\gamma_1; \gamma_2) = \frac{1}{n} \sum_{t=1}^n z_t z_t' \mathbf{1}_t(\gamma_1; \gamma_2)$ . It is also worth recalling that for  $n$  large enough  $0 < \sup_{\gamma \in \Gamma} \|M_n^x(\gamma)\| = H_n$  and  $0 < \sup_{\gamma_1 < \gamma_2} \|M_n^x(\gamma_1; \gamma_2)\| = H_n$ , where in what follows  $H_n$  denotes a sequence of strictly positive  $O_p(1)$  random variables. Finally as we have in the proof of Proposition 1, because  $E(x_t x_t' \mathbf{1}_t(\gamma))$  and  $E(x_t x_t' \mathbf{1}_t(0; \gamma))$  are strictly finite positive

definite matrices,  $M_n^x(-\infty; \gamma) - E(x_t x_t' \mathbf{1}_t(-\infty; \gamma)) = O_p(n^{-1/2})$  and  $M_n^x(\gamma) - E(x_t x_t' \mathbf{1}_t(\gamma)) = O_p(n^{-1/2})$  uniformly in  $\gamma \in \Gamma$ , we have that

$$\begin{aligned} C_1 H_n &\leq \frac{\widehat{\mathbb{A}}_{n2}(\theta)}{\left(\overline{\beta}_1, \overline{\beta}_2'\right) M_n^{x_1}(-\infty; 0) \left(\overline{\beta}_1, \overline{\beta}_2'\right)' + \overline{\beta}_3^2 M_n^q \mathbf{1}_t(-\infty; 0)} \leq C_2 H_n \\ C_1 H_n &\leq \frac{\widehat{\mathbb{A}}_{n3}(\theta)}{\left(\overline{\tau}_1, \overline{\tau}_2'\right) M_n^{x_1}(0; \gamma) \left(\overline{\tau}_1, \overline{\tau}_2'\right)' + \overline{\tau}_3^2 M_n^q(0; \gamma)} \leq C_2 H_n, \end{aligned} \quad (39)$$

where  $\overline{\tau} = \left(\widehat{\beta} - \beta\right) + \widehat{\delta}$ . The motivation is that we employ in the proof of Proposition 1, after observing that Proposition 1 implies that  $\widehat{\gamma} - \gamma_0 = O_p(n^{-1/3})$  and Lemma 1 that uniformly in  $\gamma_1 < \gamma_2 \in \Gamma$ ,

$$M_n^x(\gamma_1; \gamma_2) - E x_t x_t' \mathbf{1}_t(\gamma_1; \gamma_2) = O_p(n^{-1/2})$$

together with the fact that  $M_n^x(-\infty; \widehat{\gamma}) = M_n^x(-\infty; \gamma_0) + M_n^x(\gamma_0; \widehat{\gamma})$ .

**Consistency.** We begin with part (a). Arguing as in the proof of Proposition 1, it suffices to show that

$$\Pr^* \left\{ \inf_{\|\overline{\theta}\| > \eta} \sum_{\ell=1}^3 \widehat{\mathbb{A}}_{n\ell}(\theta) + \mathbb{B}_{n\ell}^*(\theta) \leq 0 \right\} \leq \epsilon H_n. \quad (40)$$

First, when  $\|\overline{\theta}\| > \eta$ , it implies that either (i)  $\|\overline{\gamma}\| > \eta/2$  or (ii)  $\|\overline{\beta}\|, \|\overline{v}\| > \eta/2$ . When (ii) holds true, it is clear that

$$\inf_{\|\overline{v}\| > \eta/2} \widehat{\mathbb{A}}_{n\ell}(\theta) > \eta^2 H_n \quad \ell = 1, 2 \quad (41)$$

whereas when (i) holds true, we obtain that

$$\inf_{\|\overline{\gamma}\| > \eta/2} M_n^x(\widehat{\gamma}; \gamma) > \eta H_n, \quad (42)$$

because  $E(x_t x_t' \mathbf{1}_t(\gamma))$  and  $E(x_t x_t' \mathbf{1}_t(0; \gamma))$  are strictly positive definite matrices, since say  $E(x_t x_t' \mathbf{1}_t(0; \gamma)) - E(x_t x_t' \mathbf{1}_t(0; \eta/4))$  is a positive definite matrix when  $\|\overline{\gamma}\| > \eta/2$ ,  $M_n^x(\widehat{\gamma}; \gamma) = E(x_t x_t' \mathbf{1}_t(0; \gamma)) (1 + o_p(1))$  and  $\widehat{\mathbb{A}}_{n\ell}(\theta) - E(\mathbb{A}_{n\ell}(\theta)) = o_p(1)$ . Recall that  $E(a' x_t \mathbf{1}_t(0; \eta)) > \eta \min_{q \in (0, \eta)} f(q) E(a' x_t)$ . So, (41) and (42) implies that

$$\inf_{\|\overline{\theta}\| > \eta} \sum_{\ell=1}^3 \widehat{\mathbb{A}}_{n\ell}(\theta) > \eta^2 H_n. \quad (43)$$

On the other hand, Lemma 4 implies that

$$E^* \left( \sup_{\gamma} \left\| \frac{1}{n^{1/2}} \sum_{t=1}^n x_t \varepsilon_t^* \mathbf{1}_t(\gamma) \right\| \right)^2 + E^* \left( \sup_{\gamma} \left\| \frac{1}{n^{1/2}} \sum_{t=1}^n x_t \varepsilon_t^* \mathbf{1}_t(-\infty; \gamma) \right\| \right)^2 = H_n, \quad (44)$$

so that

$$E^* \sup_{\|\bar{\theta}\| > \eta/2} \|\mathbb{B}_{n\ell}^*(\theta)\| = n^{-1/2} H_n \quad \ell = 1, 2, 3. \quad (45)$$

Thus (43) and (45) yields that  $\widehat{\theta}^* - \widehat{\theta} = o_{p^*}(1)$  because the left side of (40) is bounded by

$$\Pr^* \left\{ \inf_{\|\bar{\theta}\| > \eta} \sum_{\ell=1}^3 \widehat{\mathbb{A}}_{n\ell}(\theta) \leq \sup_{\|\bar{\theta}\| > \eta} \left\| \sum_{\ell=1}^3 \mathbb{B}_{n\ell}^*(\theta) \right\| \right\}$$

and then Markov's inequality. This concludes the consistency proof.

**Convergence rate.** To that end, we shall show that for some  $C > 0$  large enough and  $\epsilon > 0$ ,

$$\Pr^* \left\{ \inf_{\frac{C}{n^{1/2}} < \|\bar{v}\|; \|\bar{\beta}\| < \eta; \frac{C}{n^{1/3}} < \|\gamma\| < \eta} \sum_{\ell=1}^3 \widehat{\mathbb{A}}_{n\ell}(\theta) + \mathbb{B}_{n\ell}^*(\theta) \leq 0 \right\} < \epsilon H_n. \quad (46)$$

To that end, we shall first examine

$$\Pr^* \left\{ \inf_{\Xi_j(v); \Xi_j(\beta); \Xi_k(\gamma)} \sum_{\ell=1}^3 \widehat{\mathbb{A}}_{n\ell}(\theta) + \mathbb{B}_{n\ell}^*(\theta) \leq 0 \right\}$$

where for some  $j = 1, \dots, \log_2 \frac{\eta}{C} n^{1/2}$  and  $k = 1, \dots, \log_2 \frac{\eta}{C} n^{1/3}$ , and  $\Xi_j(v)$  and  $\Xi_k(\gamma)$  are defined similarly to (32). Recall that we have assumed that  $\gamma \geq 0$  since when  $\gamma \leq 0$  the proof follows similarly.

Now Lemma 4 implies that

$$\begin{aligned} & \Pr^* \left\{ \inf_{\Xi_j(v); \Xi_j(\beta); \Xi_k(\gamma)} \widehat{\mathbb{A}}_{n1}(\theta) + \mathbb{B}_{n1}^*(\theta) \leq 0 \right\} \\ & \leq \Pr^* \left\{ \inf_{\Xi_j(v); \Xi_j(\beta)} \|\bar{v}\| \|M_n^x(\gamma)\| \leq \sup_{\Xi_k(\gamma)} \left\| \frac{2}{n^{1/2}} \sum_{t=1}^n x_t \varepsilon_t^* \mathbf{1}_t(\gamma) \right\| \right\} \\ & \leq \Pr^* \left\{ \|M_n^x(\gamma)\| C 2^{j-1} \leq \sup_{\{\gamma: \|\gamma\| < \eta\}} \left\| \frac{1}{n^{1/2}} \sum_{t=1}^n x_t \varepsilon_t^* \mathbf{1}_t(\gamma) \right\| \right\} \\ & \leq C^{-1} 2^{-2j} H_n. \end{aligned} \quad (47)$$

Observe that the bound in (47) is independent of  $k$ , i.e. the set  $\Xi_k(\gamma)$ . Defining

$$\begin{aligned}\tilde{\mathbb{A}}_{n2}(\theta) &= \left(\bar{\beta}_1, \bar{\beta}'_2\right) M_n^x(-\infty; 0) \left(\bar{\beta}_1, \bar{\beta}'_2\right)' \\ \tilde{\mathbb{B}}_{n2}^*(\theta) &= \left(\bar{\beta}_1, \bar{\beta}'_2\right) \frac{2}{n} \sum_{t=1}^n x_{t1} \varepsilon_t^* \mathbf{1}_t(-\infty; \gamma),\end{aligned}$$

(39) yields that

$$\begin{aligned}& \Pr^* \left\{ \inf_{\Xi_j(v); \Xi_k(\gamma)} \tilde{\mathbb{A}}_{n2}^*(\theta) + \tilde{\mathbb{B}}_{n2}^*(\theta) \leq 0 \right\} \\ & \leq \Pr^* \left\{ \inf_{\Xi_j(v)} \left\| \left(\bar{\beta}_1, \bar{\beta}'_2\right) \right\| M_n^{x1}(-\infty; 0) \leq \sup_{\Xi_k(\gamma)} \left\| \frac{2}{n} \sum_{t=1}^n x_{t1} \varepsilon_t^* \mathbf{1}_t(-\infty; \gamma) \right\| \right\} \\ & \leq \Pr^* \left\{ \|M_n^{x1}(-\infty; 0)\| C 2^{j-1} \leq \sup_{\{\gamma: \|\gamma\| < \eta\}} \left\| \frac{1}{n^{1/2}} \sum_{t=1}^n x_{t1} \varepsilon_t^* \mathbf{1}_t(-\infty; \gamma) \right\| \right\} \quad (48) \\ & \leq C^{-1} 2^{-2j} H_n,\end{aligned}$$

by Lemma 4, which once again the bound is independent of  $k$ .

Next, define

$$\tilde{\mathbb{A}}_{n3}(\theta) = \hat{\tau}^2 q_t^2 \mathbf{1}_t(0; \gamma); \quad \tilde{\mathbb{B}}_{n3}^*(\theta) = \hat{\tau} \frac{2}{n} \sum_{t=1}^n q_t \varepsilon_t^* \mathbf{1}_t(0; \gamma),$$

then, because  $\hat{\tau} = H_n + C_1$ ,

$$\begin{aligned}& \Pr^* \left\{ \inf_{\Xi_j(v); \Xi_k(\gamma)} \tilde{\mathbb{A}}_{n3}(\theta) + \hat{\tau} \tilde{\mathbb{B}}_{n3}^*(\theta) \leq 0 \right\} \\ & \leq \Pr^* \left\{ \inf_{\Xi_j(v)} \|\hat{\tau}\| \frac{1}{n} \sum_{t=1}^n q_t^2 \mathbf{1}_t(0; \gamma) \leq \sup_{\Xi_k(\gamma)} \left\| \tilde{\mathbb{B}}_{n3}^*(\theta) / \tau_0 \right\| \right\} \\ & \leq \Pr^* \left\{ \frac{C}{n} 2^{3(k-1)} \leq \sup_{\Xi_k(\gamma)} \left\| \tilde{\mathbb{B}}_{n3}^*(\theta) / \hat{\tau} \right\| \right\} \quad (49) \\ & \leq C^{-1} 2^{-3k/2} H_n,\end{aligned}$$

by Lemma 4 and Markov's inequality. Observe that the latter displayed bound is independent of  $j$ , i.e. the set  $\Xi_j(v)$ .



So, the left side of (46) is bounded by

$$\begin{aligned} & \Pr^* \left\{ \max_{j,k} \inf_{\Xi_j(v); \Xi_k(\gamma)} \sum_{\ell=1}^3 \widehat{\mathbb{A}}_{n\ell}(\theta) + \mathbb{B}_{n\ell}^*(\theta) \leq 0 \right\} \\ & \leq C^{-1} \left( \sum_{j=1}^{\log_2 \frac{\eta}{C} n^{1/2}} 2^{-2j} + \sum_{k=1}^{\log_2 \frac{\eta}{C} n^{1/3}} 2^{-3k/2} \right) < \epsilon H_n. \end{aligned}$$

using (47) – (49). This concludes the proof of part (a).

The proof of part (b) is similarly handled after obvious changes, so it is omitted.  $\blacksquare$

We now discuss the asymptotic distribution of the bootstrap estimators. We begin with part (a). We assume  $\gamma_0 = 0$  to simplify notation. Because the “arg max” is continuous as mentioned in Theorem 2, it suffices to examine the weak limit of

$$\begin{aligned} \mathbb{G}_n^*(h, g) &= n \left( \mathbb{S}_n^* \left( \tilde{\alpha} + \frac{h}{n^{1/2}}, \frac{g}{n^{1/3}} \right) - \mathbb{S}_n^*(\tilde{\alpha}, 0) \right) \\ &= \sum_{t=1}^n \left\{ \left( \frac{h'}{n^{1/2}} x_t \left( \frac{g}{n^{1/3}} \right) + \tilde{\delta}'_t q_t \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) + \varepsilon_t^* \right)^2 - \varepsilon_t^{*2} \right\}, \end{aligned}$$

where  $\|h\|, |g| \leq C$ .

First, recall that  $\tilde{\delta}_1 = O_p(n^{-1/2})$  and  $\tilde{\delta}_2 = O_p(n^{-1/2})$  under Assumption C and note that Lemma 1 and Lemma 4 imply that, uniformly in  $\|h\|, |g| < C$ ,

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n \left\{ x_t \left( \frac{g}{n^{1/3}} \right) x'_t \left( \frac{g}{n^{1/3}} \right) - \mathbf{x}_t \mathbf{x}'_t \right\} &= O_p(n^{-1/3}) \\ \frac{1}{n^{1/2}} \sum_{t=1}^n \left\{ x_t \left( \frac{g}{n^{1/3}} \right) q_t \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) \right\} &= O_p(n^{-1/6}) \\ E^* \left\| \frac{1}{n^{1/2}} \sum_{t=1}^n \left( x_t \left( \frac{g}{n^{1/3}} \right) - \mathbf{x}_t \right) \varepsilon_t^* \right\|^2 &= O_p(n^{-1/3}). \end{aligned}$$

Thus, the latter implies that

$$E^* \sup_{h, g \in \mathbb{R}} \left| \mathbb{G}_n^*(h, g) - \widetilde{\mathbb{G}}_n^*(h, g) \right| = O_p(n^{-1/6}), \quad (50)$$

where

$$\begin{aligned}\tilde{\mathbb{G}}_n^*(h, g) &= \left\{ h' \frac{1}{n} \sum_{t=1}^n \mathbf{x}_t \mathbf{x}_t' h + h' \frac{1}{n^{1/2}} \sum_{t=1}^n \mathbf{x}_t \varepsilon_t^* \right\} \\ &\quad + \tilde{\delta}_3 \left\{ \tilde{\delta}_3 \sum_{t=1}^n q_t^2 \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) + \sum_{t=1}^n q_t \varepsilon_t^* \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) \right\} \\ &=: \tilde{\mathbb{G}}_{1n}^*(h) + \tilde{\mathbb{G}}_{2n}^*(g).\end{aligned}$$

The consequence of (50) is then that the minimizer of  $\mathbb{G}_n^*(h, g)$  is asymptotically equivalent to that of  $\tilde{\mathbb{G}}_n^*(h, g)$ . Thus, it suffices to show the weak convergence of  $\tilde{\mathbb{G}}_{1n}^*(h)$  and  $\tilde{\mathbb{G}}_{2n}^*(g)$  and that

$$\tilde{h} =: \arg \max_{h \in \mathbb{R}} \tilde{\mathbb{G}}_{1n}^*(h); \quad \tilde{g} =: \arg \max_{g \in \mathbb{R}} \tilde{\mathbb{G}}_{2n}^*(g)$$

are  $O_{p^*}(1)$ . The convergence of  $\tilde{\mathbb{G}}_{1n}^*(h)$  and its minimization follows by standard arguments as it is a quadratic function of  $h$  so that it suffices to examine  $\tilde{\mathbb{G}}_{2n}^*(g)$  and its minimum.

Turning to the second term in the definition of  $\tilde{\mathbb{G}}_{2n}^*(g)$ , we show that it converges to  $2\delta_{30} \sqrt{3^{-1} f(0) \sigma_\varepsilon^2(0)} W(g^3)$  weakly (in probability). To this end, note that Lemma 4's, and the Remark 4 that follows, yields the tightness of the process as explained in Remark 3. For the finite dimensional convergence, it follows by standard arguments as

$$E^* \left( \sum_{t=1}^n q_t \varepsilon_t^* \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right) \right)^2 = \sum_{t=1}^n q_t^2 \hat{\varepsilon}_t^2 \mathbf{1}_t \left( 0; \frac{g}{n^{1/3}} \right)$$

which converges in probability to  $3^{-1} f(0) \sigma_\varepsilon^2(0) g^3$  and the Lindeberg's condition follows easily.

Part (b) is also proved similarly and thus omitted for the sake of space.  $\blacksquare$

## A.9 Proof of Proposition 4

As before we assume  $\gamma_0 = 0$ . We show this proposition under Assumption C and the case with Assumption J is similar and thus omitted. Let  $\hat{\gamma}^* > 0$ . The case when  $\hat{\gamma}^* < 0$  is analogous and thus omitted. We shall examine the behaviour of the numerator of (22), that of its denominator being similarly handled. By construction,

$$\hat{\varepsilon}_t^* = \varepsilon_t^* + \left( \hat{\beta}^* - \tilde{\beta} \right)' x_t + \left( \hat{\delta}^* - \tilde{\delta} \right)' x_t \mathbf{1}_t(\hat{\gamma}^*) + \left( \tilde{\delta}_1 + \tilde{\delta}_3 q_t \right) \mathbf{1}_t(0; \hat{\gamma}^*).$$

Recall that when the constraint given in (4) holds true  $\tilde{\delta}_2$  and  $\tilde{\delta}_1$  are both  $O_p(n^{-1/2})$ . On the other hand Proposition 5 yields that  $\hat{\beta}^* - \tilde{\beta} = O_{p^*}(n^{-1/2})$ ,  $\hat{\delta}^* - \tilde{\delta} = O_{p^*}(n^{-1/2})$  and  $\hat{\gamma}^* = O_{p^*}(n^{-1/3})$ . Then,  $(\hat{\delta}'x_t)^2 = \tilde{\delta}'^2x_t^2 + O_{p^*}(n^{-1/2})\tilde{\delta}'x_tq_t + O_{p^*}(n^{-1})$ . And, proceeding as we did in the proof of Proposition 3, we easily deduce that

$$\hat{\varepsilon}_t^{*2} = \varepsilon_t^{*2} + O_{p^*}(n^{-1/2})x_t\varepsilon_t^* + 2\tilde{\delta}_3\varepsilon_t^*q_t\mathbf{1}_t(0; \hat{\gamma}^*) + x_tO_{p^*}(n^{-2/3}). \quad (51)$$

By obvious arguments and those in (74), it suffices to examine the behaviour of

$$\frac{1}{na} \sum_{t=1}^n (\tilde{\delta}'x_t)^2 \varepsilon_t^{*2} K\left(\frac{q_t - \hat{\gamma}^*}{a}\right).$$

Now, because  $\tilde{\delta}_2$  and  $\tilde{\delta}_1$  are both  $O_p(n^{-1/2})$  when (4) holds true the behaviour of the last displayed expression is governed by

$$\frac{1}{na} \sum_{t=1}^n \tilde{\delta}_3^2 q_t^2 \varepsilon_t^{*2} K\left(\frac{q_t - \hat{\gamma}^*}{a}\right)$$

which is  $\kappa_2\delta_{30}^2a^2E^*[\varepsilon_t^{*2} | q_t = \gamma_0]f(0)(1 + o_{p^*}(1))$  by Lemma 5 when  $\kappa_2 \neq 0$ , that is we do not assume higher order kernels. Notice that, by standard results, the contribution due to other terms in (51) are indeed negligible by Lemma 6.

Likewise the denominator in (22), is

$$\frac{1}{na} \sum_{t=1}^n (\tilde{\delta}'x_t)^2 K\left(\frac{q_t - \hat{\gamma}^*}{a}\right) = \kappa_2\delta_{30}^2a^2f(0)(1 + o_{p^*}(1)).$$

So, the convergence in (22) follows from the last two displayed expressions. Finally, it is standard that  $\mathbb{S}_n(\hat{\theta}^*) - \sigma^2 = o_{p^*}(1)$ . This completes the proof of the proposition.  $\blacksquare$

## A.10 Proof of Theorem 5

This is a direct consequence of Theorem 4 and Proposition 4 and the same arguments as in the proof of Theorem 3 and that of Theorem 2, respectively.  $\blacksquare$

## B AUXILIARY LEMMAS

We begin with a set of maximal inequalities, which play a central role in deriving convergence rates and tightness of various empirical processes. For  $j = 1$  or  $2$ , let

$$\begin{aligned} J_n(\gamma, \gamma') &= \frac{1}{n^{1/2}} \sum_{t=1}^n \varepsilon_t x_t \mathbf{1}_t(\gamma; \gamma') \\ J_{1n}(\gamma, \gamma') &= \frac{1}{n^{1/2}} \sum_{t=1}^n \varepsilon_t |q_t - \gamma|^j \mathbf{1}_t(\gamma; \gamma') \\ J_{2n}(\gamma) &= \frac{1}{n^{1/2}} \sum_{t=1}^n \left\{ |q_t - \gamma_0|^j \mathbf{1}_t(\gamma_0; \gamma) - E |q_t - \gamma_0|^j \mathbf{1}_t(\gamma_0; \gamma) \right\} \end{aligned}$$

and for some sequence  $\{z_t\}_{t=1}^n$ ,

$$J_{3n}(\gamma) = \frac{1}{n^{1/2}} \sum_{t=1}^n (z_t \mathbf{1}_t(\gamma_0; \gamma) - E z_t \mathbf{1}_t(\gamma_0; \gamma)).$$

**Lemma 1.** *Suppose Assumptions Z and Q hold for the sequence  $\{x_t, \varepsilon_t\}_{t=1}^n$ . In addition, for  $J_{3n}(\gamma)$ , assume that  $\{z_t, q_t\}_{t=1}^n$  be a sequence of strictly stationary, ergodic, and  $\rho$ -mixing with  $\sum_{m=1}^{\infty} \rho_m^{1/2} < \infty$ ,  $E|z_t|^4 < \infty$  and, for all  $\gamma \in \Gamma$ ,  $E(|z_t|^4 |q_t = \gamma) < C < \infty$ . Then, there exists  $n_0 < \infty$  such that for all  $\gamma'$  in a neighbourhood of  $\gamma_0$  and for all  $n > n_0$  and  $\epsilon \geq n_0^{-1}$ ,*

$$\begin{aligned} \text{(a)} \quad E \sup_{\gamma' < \gamma < \gamma' + \epsilon} |J_n(\gamma', \gamma)| &\leq C\epsilon^{1/2} \\ \text{(b)} \quad E \sup_{\gamma' < \gamma < \gamma' + \epsilon} |J_{1n}(\gamma', \gamma)| &\leq C\epsilon^{1/2} (\epsilon + |\gamma_0 - \gamma'|)^j \end{aligned} \tag{52}$$

$$\text{(c)} \quad E \sup_{\gamma_0 < \gamma < \gamma_0 + \epsilon} |J_{2n}(\gamma)| \leq C\epsilon^{j+1/2} \tag{53}$$

$$\text{(d)} \quad E \sup_{\gamma_0 < \gamma < \gamma_0 + \epsilon} |J_{3n}(\gamma)| \leq C\epsilon^{1/2}, \tag{54}$$

where  $j = 1$  or  $2$ .

*Proof.* Part (a) proceeds as in Hansen's (2000) Lemma A.3, so it is omitted.

Next part (b). This is almost identical to that of Hansen's (2000) Lemma A.3 once observing that if  $|\gamma_1 - \gamma'| \leq \epsilon$  and  $|\gamma_2 - \gamma'| \leq \epsilon$  and  $h_t(\gamma_1, \gamma_2) = |\varepsilon_t(q_t - \gamma_0)|^j \mathbf{1}_t(\gamma_1, \gamma_2)$ , then the bound in his Lemma A.1 (12) should be updated to

$$E h_i^r(\gamma_1, \gamma_2) \leq C \int_{\gamma_1}^{\gamma_2} |q - \gamma_0|^{jr} dq \leq C |\gamma_1 - \gamma_2| \epsilon_1^{jr},$$

where  $C < \infty$  and  $\epsilon_1 = (\epsilon + |\gamma_0 - \gamma'|)$ , since  $E(|\varepsilon_t^r| |q_t)$  and the density  $f(q)$  of  $q_t$  are bounded around  $q_t = \gamma_0$ . Hansen's bound in (13) should be changed to  $|\gamma_1 - \gamma_2| \epsilon_1^{j^r}$  for the same reason. Then, these new bounds imply that the bounds (15) and (16) in his Lemma A.3 and the bounds (18) and (20) in the proof of his Lemma A.2 should change to  $|\gamma_1 - \gamma_2|^2 \epsilon_1^{4j}$  and  $n^{-1} |\gamma_1 - \gamma_2| \epsilon_1^{4j} + |\gamma_1 - \gamma_2|^2 \epsilon_1^{4j}$ , respectively, to yield the desired bound in (52).

Part (c). For notational simplicity we assume that  $\gamma_0 = 0$ . Let  $\gamma_k = k/n$ , for  $k = 1, \dots, m$ , where  $m = [\epsilon n] + 1$ . By triangle inequality,

$$\sup_{\gamma_0 < \gamma < \gamma_0 + \epsilon} |J_{2n}(\gamma)| \leq \max_{k=1, \dots, m-1} |J_{2n}(\gamma_k)| + \max_{k=1, \dots, m} \sup_{\gamma_{k-1} \leq \gamma \leq \gamma_k} |J_{2n}(\gamma) - J_{2n}(\gamma_{k-1})|. \quad (55)$$

Now because  $f(\cdot)$  is continuous differentiable at  $\gamma_0$ , standard algebra yields that

$$E |q_t|^j \mathbf{1}_t(\gamma_{k-1}; \gamma_k) \leq C \gamma_k^j / n. \quad (56)$$

Next, using (56)

$$\begin{aligned} & \sup_{\gamma_{k-1} \leq \gamma \leq \gamma_k} \left| \frac{1}{n^{1/2}} \sum_{t=1}^n |q_t|^j \mathbf{1}_t(\gamma_{k-1}; \gamma) \right| \\ & \leq (J_{2n}(\gamma_k) - J_{2n}(\gamma_{k-1})) + n^{1/2} E |q_t|^j \mathbf{1}_t(\gamma_{k-1}; \gamma_k) \\ & = (J_{2n}(\gamma_k) - J_{2n}(\gamma_{k-1})) + C \gamma_k^j / n^{1/2}. \end{aligned}$$

Thus, using the inequality  $(\sup_{j=1, \dots, \ell} |c_j|)^4 \leq \sum_{j=1}^{\ell} |c_j|^4$ , we conclude that second term on the right of (55) has absolute moment bounded by

$$\left( \sum_{k=1}^m E |J_{2n}(\gamma_k) - J_{2n}(\gamma_{k-1})|^4 \right)^{1/4} + C \gamma_m^j / n^{1/2}. \quad (57)$$

However, from Lemma 3.6 of Peligrad (1982), for any  $k > i$ ,

$$E |J_{2n}(\gamma_k) - J_{2n}(\gamma_i)|^4 \leq C \left( n^{-1} E |q_t|^{4j} \mathbf{1}_t(\gamma_i; \gamma_k) + \left( E |q_t|^{2j} \mathbf{1}_t(\gamma_i; \gamma_k) \right)^2 \right).$$

So, using again (56) and that  $m = [\epsilon n] + 1$  and  $n^{-1} < \epsilon$ , we conclude that the first moment of the second term on the right of (55) is  $C \epsilon^{j+1/2}$ .

Next the first moment of the first term on the right of (55) is also bounded by  $C \epsilon^{j+1/2}$  by Billingsley's (1968) Theorem 12.2 using the last displayed inequality.

Finally part **(d)**. This is similar to that of (53). It is sufficient to note that, with  $J_{3n}(\gamma)$ , the bounds in (56) and (57) change to  $C/n^{1/2}$  and  $C\epsilon^2$ , respectively. This yields the results as  $n^{-1} < \epsilon$ . ■

**Remark 3.** *One of the consequences of the previous lemma **(a)** and **(b)**, which allows the maximal inequality to hold for any  $\gamma'$  in a neighbourhood of  $\gamma_0$ , is that*

$$nE \sup_{g_1 < g < g_1 + \epsilon} |J_n(\gamma_0 + g/r_n) - J_n(\gamma_0 + g_1/r_n)| \leq C(\epsilon + g_1)\epsilon^{1/2},$$

which can be made small by choosing small  $\epsilon$  and  $r_n \rightarrow \infty$ . This is used to verify the stochastic equicontinuity of the rescaled and reparameterized empirical processes in the proof of Theorem 1.

The following two lemmas are used in the proof of Proposition 3. Before we state our next lemma, we need to introduce some notation. In what follows

$$\begin{aligned} g_r(q) &= E(x_{t2}^r \varepsilon_t^2 | q_t = q); \quad g_r^*(q) = E(x_{t2}^r | q_t = q) \\ h_{r,k}(q) &= \sum_{j=0}^{4-k} a^j \kappa_{j+k} \frac{\partial^j}{\partial q^j} (f(q) g_r(q)), \quad k \leq 4 \\ h_{r,k}^*(q) &= \sum_{j=0}^{4-k} a^j \kappa_{j+k} \frac{\partial^j}{\partial q^j} (f(q) g_r^*(q)), \quad k \leq 4. \end{aligned} \tag{58}$$

Note that we have implicitly assumed that  $g_r(q)$  and  $f(q)$  have four continuous derivatives. Also, without loss of generality, we assume  $\gamma_0 = 0$  and  $x_{t2}$  is a scalar to ease notation.

**Lemma 2.** *Under **K1**, **K2** and **K4**, we have that for integers  $0 \leq \ell, r \leq 4$ ,*

$$\begin{aligned} \frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^2 x_{t2}^r q_t^\ell K\left(\frac{q_t - \widehat{\gamma}}{a}\right) - h_{r,\ell}(0) &= o_p(1) \\ \frac{1}{na^{1+\ell}} \sum_{t=1}^n x_{t2}^r q_t^\ell K\left(\frac{q_t - \widehat{\gamma}}{a}\right) - h_{r,\ell}^*(0) &= o_p(1). \end{aligned} \tag{59}$$

*Proof.* First, observe that we are using the normalization  $(na^{1+\ell})^{-1}$  instead of the standard  $(na)^{-1}$ . This is due to the factor  $q_t^\ell$ . We shall consider only the first equality in (59), the second

one being similarly handled. Now abbreviating  $K_t(\gamma) = K\left(\frac{q_t - \gamma}{a}\right)$ , we have that standard kernel arguments imply

$$\frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^2 x_{t2}^r q_t^\ell K_t(0) - h_{r,\ell}(0) = O_p\left((na)^{-1/2}\right) + o\left(a^{4-\ell}\right).$$

So, to complete the proof of the lemma, it suffices to show that

$$\frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^2 x_{t2}^r q_t^\ell \{K_t(\hat{\gamma}) - K_t(0)\} = o_p(1). \quad (60)$$

Proposition 1 implies that there exists  $C$  such that  $\Pr\{|\hat{\gamma}| > Cn^{-1/3}\} \leq \eta$ , for any  $\eta > 0$ . So, we only need to show that (60) holds true when  $|\hat{\gamma}| \leq Cn^{-1/3}$ . In that case, we have that the left side of (60) is bounded by

$$\begin{aligned} & \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^2 x_{t2}^r q_t^\ell \{K_t(\gamma) - K_t(0)\} \right| \\ \leq & \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^2 x_{t2}^r q_t^\ell \{K_t(\gamma) - K_t(0)\} \mathbf{1}\left(|q_t| < a^{1/2}\right) \right| \\ & + \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^2 x_{t2}^r q_t^\ell \{K_t(\gamma) - K_t(0)\} \mathbf{1}\left(|q_t| \geq a^{1/2}\right) \right|. \end{aligned} \quad (61)$$

The expectation of second term on the right of (61) is bounded by

$$\begin{aligned} & \frac{C_1}{na} \sum_{t=1}^n E\left(\varepsilon_t^2 |x_{t2}|^r \left|\frac{q_t}{a}\right|^\ell K\left(\frac{q_t}{a}\right) \mathbf{1}\left(|q_t| \geq a^{1/2}\right)\right) \\ \leq & \frac{C_1}{a} \int_q \left|\frac{q}{a}\right|^\ell g_r(q) f(q) K\left(\frac{q}{a}\right) \mathbf{1}\left(|q_t| \geq a^{1/2}\right) dq \\ = & C_1 \int_{|q| \geq a^{-1/2}} |q|^\ell g_r(aq) f(aq) K(q) dq \\ = & o\left(a^{2-\ell/4}\right), \end{aligned}$$

because by **K1**,  $\kappa_\ell < C_1$ , for  $\ell \leq 4$ .

For some  $0 < \psi < 1$ , the first term on the right of (61) is bounded by

$$\begin{aligned}
& \frac{C}{n^{1/3}} \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^2} \sum_{t=1}^n \varepsilon_t^2 |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K' \left( \frac{q_t - \psi\gamma}{a} \right) \mathbf{1} \left( |q_t| < a^{1/2} \right) \right| \\
\leq & \frac{C}{n^{1/3}} \left| \frac{1}{na^2} \sum_{t=1}^n \varepsilon_t^2 |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K' \left( \frac{q_t}{a} \right) \mathbf{1} \left( a^{3/2} < |q_t| < a^{1/2} \right) \right| \\
& + \frac{C}{n^{1/3}} \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^2} \sum_{t=1}^n \varepsilon_t^2 |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K' \left( \frac{q_t - \phi\gamma}{a} \right) \mathbf{1} \left( |q_t| < a^{3/2} \right) \right|
\end{aligned} \tag{62}$$

because **K4** implies that  $\gamma = o(a)$  when  $|\gamma| \leq Cn^{-1/3}$ , and hence if  $a^{3/2} < |q_t| < a^{1/2}$  we have  $\left| K' \left( \frac{q_t - \phi\gamma}{a} \right) / K' \left( \frac{q_t}{a} \right) \right| \leq C_1$  by **K2**. But, it is well known that the first moment of the first term on the right of (62) is bounded, whereas that of the second term on the right is also bounded because  $E \left| \frac{q_t}{a} \right|^\ell \mathbf{1} \left( |q_t| < a^{3/2} \right) < a^{(\ell+3)/2}$  and

$$\left| K' \left( \frac{q_t - \phi\gamma}{a} \right) - K'_t(0) \right| \mathbf{1} \left( |q_t| < a^{3/2} \right) \leq Ca^{1/2}. \tag{63}$$

So, the expectation of the first term on the right of (61) is  $O(n^{-1/3})$ . This concludes the proof of the lemma. ■

**Lemma 3.** *Under **K1** – **K4**, we have that for integers  $0 \leq r, \ell \leq 4$ ,*

$$\frac{1}{na} \sum_{t=1}^n x_{t2}^r q_t^\ell K_t(\hat{\gamma}) \varepsilon_t = o_p \left( a^\ell n^{1/2} \right). \tag{64}$$

*Proof.* To simplify the notation, we assume that  $r = 0$ . The left side of (64) is

$$\frac{1}{na} \sum_{t=1}^n q_t^\ell \{K_t(\hat{\gamma}) - K_t(0)\} \varepsilon_t + \frac{1}{na} \sum_{t=1}^n q_t^\ell K_t(0) \varepsilon_t.$$

The second term is easily shown to be  $O_p(n^{-1/2} a^{\ell-1/2})$ . Next the first term of the last displayed expression is

$$\begin{aligned}
& \frac{1}{na} \sum_{t=1}^n q_t^\ell \{K_t(\hat{\gamma}) - K_t(0)\} \varepsilon_t \mathbf{1} \left( |q_t| < a^\zeta \right) \\
& + \frac{1}{na} \sum_{t=1}^n q_t^\ell \{K_t(\hat{\gamma}) - K_t(0)\} \varepsilon_t \mathbf{1} \left( |q_t| \geq a^\zeta \right),
\end{aligned} \tag{65}$$



where  $\zeta = 1 - 2/\ell$ , if  $\ell > 2$ , and  $\zeta < 1$  if  $\ell \leq 2$ . The second term of (65) is

$$a^\ell \frac{1}{na} \sum_{t=1}^n \left(\frac{q_t}{a}\right)^\ell \{K_t(\widehat{\gamma}) - K_t(0)\} \varepsilon_t \mathbf{1}(|q_t| \geq a^\zeta),$$

whose first absolute moment is bounded by

$$a^{\ell-1} \int_{|q| \geq a^\zeta} \left(\frac{q}{a}\right)^\ell K\left(\frac{q}{a}\right) f_q(q) dq \leq C_1 a^\ell \int_{|q| \geq a^{\zeta-1}} q^\ell K(q) f_q(aq) dq = o(a^\ell)$$

because by **K1**,  $\kappa_4 < \infty$ . So to complete the proof we need to examine the first term of (65), which using the characteristic function of the kernel function is

$$\int \phi(av) \left(e^{iv\widehat{\gamma}} - 1\right) \left\{ \frac{1}{n} \sum_{t=1}^n q_t^\ell \varepsilon_t e^{ivq_t} \mathbf{1}(|q_t| < a^\zeta) \right\} dv.$$

But its clear that the last displayed expression is bounded by

$$\begin{aligned} & \widehat{\gamma} \int v |\phi(av)| \left| \frac{1}{n} \sum_{t=1}^n q_t^\ell \varepsilon_t e^{ivq_t} \mathbf{1}(|q_t| < a^\zeta) \right| dv = O_p\left(a^{\ell\zeta} n^{-1/2} \widehat{\gamma}\right) \int v |\phi(av)| dv \\ & = O_p\left(a^\ell (na^3)^{-4/3} n^{1/2}\right) \end{aligned}$$

using that  $\zeta = 1 - 2/\ell$ , if  $\ell \geq 2$  and  $\zeta < 1$  when  $0 \leq \ell < 2$ ,  $\widehat{\gamma} = O_p(n^{-1/3})$  and **K4**. This concludes the proof of the lemma. ■

We now extend the maximal inequalities in Lemma 1 to its bootstrap analogues. Define  $J_n^*(\gamma, \gamma')$  and  $J_{1n}^*(\gamma, \gamma')$  by replacing  $\varepsilon_t$  in  $J_n$  and  $J_{1n}$  with  $\widehat{\varepsilon}_t \eta_t$ , that is

$$\begin{aligned} J_n^*(\gamma, \gamma') &= \frac{1}{n^{1/2}} \sum_{t=1}^n x_t \mathbf{1}_t(\gamma, \gamma') \widehat{\varepsilon}_t \eta_t \\ J_{1n}^*(\gamma, \gamma') &= \frac{1}{n^{1/2}} \sum_{t=1}^n |q_t - \gamma|^j \mathbf{1}_t(\gamma; \gamma') \widehat{\varepsilon}_t \eta_t, \end{aligned}$$

and recall that  $H_n$  denotes a sequence of positive  $O_p(1)$  random variables.

**Lemma 4.** *Under Assumption Z, we have that for all  $\epsilon, \varsigma > 0$ , there exists  $\zeta > 0$  such that*

$$\Pr^* \left\{ \sup_{\gamma' < \gamma < \gamma' + \epsilon} |J_n^*(\gamma', \gamma)| > \epsilon \right\} \leq \zeta \varsigma H_n, \quad (66)$$

$$\Pr^* \left\{ \sup_{\gamma' < \gamma < \gamma' + \epsilon} |J_{1n}^*(\gamma', \gamma)| > C\epsilon^{1/2} (\epsilon + |\gamma_0 - \gamma'|)^j \right\} \leq \zeta \varsigma H_n. \quad (67)$$

*Proof.* We shall assume for notational simplicity that  $\gamma_0 < \widehat{\gamma}$ , and that  $\gamma_j = \gamma_1 + \frac{\zeta}{m}j$  and  $n\zeta/2 < m < n\zeta$ , as  $n$  can be chosen such that  $n\zeta > 1$ . By definition,

$$\begin{aligned} J_n^*(\gamma_k, \gamma_j) &= \frac{1}{n^{1/2}} \sum_{t=1}^n x_t \varepsilon_t \mathbf{1}_t(\gamma_j; \gamma_k) \eta_t \\ &\quad + \frac{1}{n^{1/2}} \sum_{t=1}^n x_t x'_t \mathbf{1}_t(\gamma_j; \gamma_k) \eta_t (\widehat{\beta} - \beta) \\ &\quad + \frac{1}{n^{1/2}} \sum_{t=1}^n x_t x'_t \mathbf{1}_t(\gamma_0) \mathbf{1}_t(\gamma_j; \gamma_k) \eta_t (\widehat{\delta} - \delta) \\ &\quad + \frac{1}{n^{1/2}} \sum_{t=1}^n x_t x'_t \mathbf{1}_t(\gamma_0; \widehat{\gamma}) \mathbf{1}_t(\gamma_j; \gamma_k) \eta_t \widehat{\delta}. \end{aligned}$$

Now by standard inequalities and that  $\eta_t \sim iid(0, 1)$  with a finite fourth moments, the fourth (bootstrap) moment of the right side of last displayed equation is bounded by

$$\begin{aligned} &\left| \frac{1}{n} \sum_{t=1}^n \|x_t\|^2 \varepsilon_t^2 \mathbf{1}_t(\gamma_j; \gamma_k) \right|^2 + \|\widehat{\beta} - \beta\|^4 \left| \frac{1}{n} \sum_{t=1}^n \|x_t\|^4 \mathbf{1}_t(\gamma_j; \gamma_k) \right|^2 \\ &+ \|\widehat{\delta} - \delta\|^4 \left| \frac{1}{n} \sum_{t=1}^n \|x_t\|^4 \mathbf{1}_t(\gamma_j; \gamma_k) \mathbf{1}_t(\gamma_0) \right|^2 \\ &+ \|\widehat{\delta}\|^4 \left| \frac{1}{n} \sum_{t=1}^n \|x_t\|^4 \mathbf{1}_t(\gamma_j; \gamma_k) \mathbf{1}_t(\gamma_0; \widehat{\gamma}) \right|^2. \end{aligned} \tag{68}$$

Because for fixed  $\zeta > 0$ , there exists  $n_0$  such that for  $n > n_0$ ,  $Cn^{-1} < \zeta$ , the expectation of the first term of (68) is bounded by

$$C \left[ (k-j) \zeta_m + \left( \frac{(k-j) \zeta_m}{n} \right)^{1/2} \right]^2 \leq C (k-j)^2 \zeta_m^2,$$

arguing similarly as in Hansen's (2000) Lemma A.3 and  $\zeta_m = \zeta/m$ .

Next, recalling that  $\widehat{\gamma} = \gamma_0 + D/n^{1/3}$ , because  $\mathbf{1}(\gamma_j < q_t < \gamma_k) \mathbf{1}(\gamma_0 < q_t < \widehat{\gamma}) \leq \mathbf{1}(\gamma_j < q_t < \gamma_k)$ , the expectation of the fourth term of (68) is bounded by

$$\begin{aligned} &\left| E \left\{ \|x_t\|^4 \mathbf{1}_t(\gamma_j; \gamma_k) \right\} \right|^2 + \left| \frac{1}{n} \sum_{t=1}^n \left\{ \|x_t\|^4 \mathbf{1}_t(\gamma_j; \gamma_k) - E \left\{ \|x_t\|^4 \mathbf{1}_t(\gamma_j; \gamma_k) \right\} \right\} \right|^2 \\ &\leq C (k-j)^2 \zeta_m^2. \end{aligned}$$

Finally, the second and third terms of (68) are

$$H_n \frac{1}{n^3} \sum_{t=1}^n E \left( \|x_t\|^8 \mathbf{1}_t(\gamma_j; \gamma_k) \right) = H_n (k-j)^2 \zeta_n^2.$$

From here we now conclude that (66) holds true, so is the lemma proceeding as in Hansen's (2000) Lemma A.3 and in particular his expressions (20) – (22) because if a sequence of random variables has finite first moments, it implies that it is  $O_p(1)$ . The proof of (67) proceeds similarly and thus omitted. ■

**Remark 4.** *One of the consequences of the previous lemma is that*

$$nE^* \sup_{g_1 < g < g_1 + \epsilon} |J_n^*(\gamma_0 + g/r_n) - J_n^*(\gamma_0 + g_1/r_n)| = (\epsilon + g_1) \epsilon^{1/2} H_n,$$

which can be made small by choosing small  $\epsilon$  and  $r_n \rightarrow \infty$ .

**Lemma 5.** *Under **K1**, **K2** and **K4** , we have that for integers  $0 \leq \ell, r \leq 4$ ,*

$$\begin{aligned} \frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^{*2} x_{t2}^r q_t^\ell K \left( \frac{q_t - \hat{\gamma}^*}{a} \right) - h_{r,\ell}(0) &= o_{p^*}(1) \\ \frac{1}{na^{1+\ell}} \sum_{t=1}^n x_{t2}^r q_t^\ell K \left( \frac{q_t - \hat{\gamma}^*}{a} \right) - h_{r,\ell}^*(0) &= o_{p^*}(1). \end{aligned} \quad (69)$$

*Proof.* We shall consider only the first equality in (69), the second one being similarly handled. Now standard kernel arguments imply

$$\frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^{*2} x_{t2}^r q_t^\ell K_t(0) - h_{r,\ell}(0) = O_{p^*} \left( (na)^{-1/2} \right) + o_p \left( a^{4-\ell} \right).$$

So, to complete the proof of the lemma, it suffices to show that

$$\frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^{*2} x_{t2}^r q_t^\ell \{K_t(\hat{\gamma}^*) - K_t(0)\} = o_{p^*}(1). \quad (70)$$

Proposition 5 implies that there exists  $C > 0$  such that  $\Pr^* \{|\hat{\gamma}^*| > Cn^{-1/3}\} \leq H_n$ . So, we only need to show that (60) holds true when  $|\hat{\gamma}^*| \leq Cn^{-1/3}$ , so that we have that the left side

of (70) is bounded by

$$\begin{aligned}
& \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^{*2} x_{t2}^r q_t^\ell \{K_t(\gamma) - K_t(0)\} \right| \\
\leq & \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^{*2} x_{t2}^r q_t^\ell \{K_t(\gamma) - K_t(0)\} \mathbf{1}(|q_t| < a^{1/2}) \right| \\
& + \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^{1+\ell}} \sum_{t=1}^n \varepsilon_t^{*2} x_{t2}^r q_t^\ell \{K_t(\gamma) - K_t(0)\} \mathbf{1}(|q_t| \geq a^{1/2}) \right|.
\end{aligned} \tag{71}$$

The expectation of second term on the right of (71) is bounded by

$$\begin{aligned}
& \frac{C_1}{na} \sum_{t=1}^n E^* \left( \varepsilon_t^{*2} |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K \left( \frac{q_t}{a} \right) \mathbf{1}(|q_t| \geq a^{1/2}) \right) \\
= & \frac{C_1}{na} \sum_{t=1}^n |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K \left( \frac{q_t}{a} \right) \mathbf{1}(|q_t| \geq a^{1/2}) \frac{1}{n} \sum_{s=1}^n \widehat{\varepsilon}_t^2 \\
= & \frac{C_1}{na} \sum_{t=1}^n |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K \left( \frac{q_t}{a} \right) \mathbf{1}(|q_t| \geq a^{1/2}) H_n,
\end{aligned}$$

where  $C_1$  denotes a generic positive finite constant. Now,

$$E \frac{1}{na} \sum_{t=1}^n |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K \left( \frac{q_t}{a} \right) \mathbf{1}(|q_t| \geq a^{1/2}) = o(a^{2-\ell/4})$$

proceeding as we did in Lemma 2. So, we conclude that right of (71) is  $o(a^{2-\ell/4}) H_n$ .

For some  $0 < \psi < 1$ , the first term on the right of (71) is bounded by

$$\begin{aligned}
& \frac{C_1}{n^{1/3}} \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^2} \sum_{t=1}^n \varepsilon_t^{*2} |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K' \left( \frac{q_t - \psi\gamma}{a} \right) \mathbf{1}(|q_t| < a^{1/2}) \right| \\
\leq & \frac{C_1}{n^{1/3}} \left| \frac{1}{na^2} \sum_{t=1}^n \varepsilon_t^{*2} |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K' \left( \frac{q_t}{a} \right) \mathbf{1}(a^{3/2} < |q_t| < a^{1/2}) \right| \\
& + \frac{C_1}{n^{1/3}} \sup_{|\gamma| \leq Cn^{-1/3}} \left| \frac{1}{na^2} \sum_{t=1}^n \varepsilon_t^{*2} |x_{t2}|^r \left| \frac{q_t}{a} \right|^\ell K' \left( \frac{q_t - \phi\gamma}{a} \right) \mathbf{1}(|q_t| < a^{3/2}) \right|
\end{aligned} \tag{72}$$

because **K4** implies that  $\gamma = o(a)$  when  $|\gamma| \leq Cn^{-1/3}$ , and hence  $\left| K' \left( \frac{q_t - \phi\gamma}{a} \right) / K' \left( \frac{q_t}{a} \right) \right| \leq C_1$  by **K2** if  $a^{3/2} < |q_t| < a^{1/2}$ . But, it is well known that the first moment of the first term on the

right of (72) is bounded, whereas that of the second term on the right is also bounded because  $E \left| \frac{q_t}{a} \right|^\ell \mathbf{1}(|q_t| < a^{3/2}) < a^{(\ell+3)/2}$  and (63). So, the expectation of the first term on the right of (71) is  $O_p(n^{-1/3})$ . This concludes the proof of the lemma. ■

**Lemma 6.** *Under **K1** – **K4**, we have that for integers  $0 \leq r, \ell \leq 4$ ,*

$$\frac{1}{na} \sum_{t=1}^n x_{t2}^r q_t^\ell K_t(\hat{\gamma}^*) \varepsilon_t^* = o_{p^*} \left( a^\ell n^{1/2} \right). \quad (73)$$

*Proof.* To simplify the notation, we assume that  $r = 0$ . The left side of (73) is

$$\frac{1}{na} \sum_{t=1}^n q_t^\ell \{K_t(\hat{\gamma}^*) - K_t(0)\} \varepsilon_t^* + \frac{1}{na} \sum_{t=1}^n q_t^\ell K_t(0) \varepsilon_t^*.$$

The second term is easily shown to be  $O_{p^*}(n^{-1/2}a^{\ell-1/2})$ , whereas the first term is

$$\begin{aligned} & \frac{1}{na} \sum_{t=1}^n q_t^\ell \{K_t(\hat{\gamma}^*) - K_t(0)\} \varepsilon_t^* \mathbf{1}(|q_t| < a^\zeta) \\ & + \frac{1}{na} \sum_{t=1}^n q_t^\ell \{K_t(\hat{\gamma}^*) - K_t(0)\} \varepsilon_t^* \mathbf{1}(|q_t| \geq a^\zeta), \end{aligned} \quad (74)$$

where  $\zeta = 1 - 2/\ell$  if  $\ell > 2$  and  $\zeta < 1$  if  $\ell \leq 2$ . The second term of (74) is

$$a^\ell \frac{1}{na} \sum_{t=1}^n \left( \frac{q_t}{a} \right)^\ell \{K_t(\hat{\gamma}^*) - K_t(0)\} \varepsilon_t^* \mathbf{1}(|q_t| \geq a^\zeta),$$

whose first absolute bootstrap moment is

$$\begin{aligned} & a^\ell \frac{1}{na} \sum_{t=1}^n \left| \frac{q_t}{a} \right|^\ell |K_t(\hat{\gamma}^*) - K_t(0)| \mathbf{1}(|q_t| \geq a^\zeta) \frac{1}{n} \sum_{s=1}^n |\hat{\varepsilon}_s| \\ & a^\ell \frac{1}{na} \sum_{t=1}^n \left| \frac{q_t}{a} \right|^\ell |K_t(\hat{\gamma}^*) - K_t(0)| \mathbf{1}(|q_t| \geq a^\zeta) H_n. \end{aligned}$$

Now, proceed as in Lemma 5 to conclude that second term of (74) is  $O_{p^*}(a^\ell)$ . So, to complete the proof we need to examine the first term of (74) which, as we did with the first term of (65), is

$$\int \phi(av) \left( e^{iv\hat{\gamma}^*} - 1 \right) \left\{ \frac{1}{n} \sum_{t=1}^n q_t^\ell \varepsilon_t^* e^{ivq_t} \mathbf{1}(|q_t| < a^\zeta) \right\} dv.$$

But it is clear that the last displayed expression is bounded by

$$\begin{aligned} & \widehat{\gamma}^* \int v |\phi(av)| \left| \frac{1}{n} \sum_{t=1}^n q_t^\ell \varepsilon_t^* e^{ivq_t} \mathbf{1}(|q_t| < a^\zeta) \right| dv = O_{p^*} \left( a^{\ell\zeta} n^{-1/2} \widehat{\gamma}^* \right) \int v |\phi(av)| dv \\ & = O_{p^*} \left( a^\ell (na^3)^{-4/3} n^{1/2} \right) \end{aligned}$$

using **K4** and that  $\zeta = 1 - 2/\ell$  if  $\ell \geq 2$  and  $\zeta < 1$  when  $0 \leq \ell < 2$ ,  $\widehat{\gamma}^* = O_{p^*}(n^{-1/3})$  and that by standard arguments, it yields

$$E^* \left| \frac{1}{n} \sum_{t=1}^n q_t^\ell \varepsilon_t^* e^{ivq_t} \mathbf{1}(|q_t| < a^\zeta) \right|^2 = O_p \left( a^{2\ell\zeta} n^{-1} \right).$$

This concludes the proof of the lemma. ■