Condorcet was Wrong, Pareto was Right:

Families, Inheritance and Inequality

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Abstract

Using a simple model of family decision making we examine the processes by which the wealth distribution changes over the generations, focusing in particular on the division of fortunes through inheritance and the union of fortunes through marriage. We show that the equilibrium wealth distribution can be characterized in a simple way for a variety of inheritance rules and marriage patterns. The shape of the distribution is principally determined by the size distribution of families. We show how changes in fertility, inheritance rules and inheritance taxation affect long-run inequality.

Keywords: wealth distribution, inheritance, inheritance taxation

JEL codes: D31, D63

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1 Introduction

Why does wealth inequality persist? Would wealth inequality persist even if economic and social policies were reformed? There is no shortage of interesting intellectual approaches and empirical studies that have built on economic and statistical models to provide an insight into both wealth and income distributions, to account for their inequality and their characteristic shape.[1] There is also no shortage of well-meaning "it does not have to be that way" intuition of social reformers and respected philosophers. Contrast the principled optimism of Nicolas de Condorcet

> "It is easy to prove that fortunes naturally tend to equality, and that excessive differences of wealth either cannot exist or must promptly cease, if the civil laws do not establish artificial ways of perpetuating and amassing such fortunes, and if freedom of commerce and industry eliminate the advantage that any prohibitive laws or fiscal privilege gives to acquired wealth." (Condorcet 1798)

with Vilfredo Pareto's pragmatic view expressed about a century later

> "We find ourselves confronted by a natural law, which reveals a tendency for incomes [and wealth] to be arranged in a certain way, at least in the societies and periods considered." (Pareto 1965)

There is abundant evidence of the long-lasting effect on the wealth distribution that can arise from shocks such as stock-market crashes, war and revolution.[2] However, although the impact of dramatic isolated events can be substantial, it is reasonable to expect that the principal mechanisms governing wealth inequality are to be found in the normal, conventional forces underlying the distribution of wealth in modern economies. In this paper we discuss two main types of forces that are at work in all societies: those that divide wealth, principally through gifts and bequests, and those that unite wealth, principally through marriage. These forces may be shaped by social and cultural considerations as well as straightforward economic motives. We show that, under conventional assumptions about the determination of bequests, the forces that divide wealth will lead to the characteristic Pareto distribution: the parameters of the Pareto distribution depend on, among other things, the effective savings rate and the size distribution of families. This conclusion is robust to alternative assumptions about the forces that divide or unite wealth holdings.

We also show that policy measures that focus directly on the mechanisms that perpetuate wealth inequality can have an effect on long-run inequality that

[1] See, for example, Piketty (2014b), Stiglitz (2015a, 2015b) and, for related work on income, Atkinson (2017), Jenkins (2017).

[2] See, for example, Kopczuk and Saez (2004).

is much larger than might be supposed from the short-run impact of such policy measures.

The paper is organized as follows. Section 2 describes the basic analytical framework and places our contribution in context. Section 3 sets out our simplified basic model and introduces a key result. Section 4 shows that this key result is robust under a variety of richer assumptions about the forces that divide and unite wealth and shows how our approach can illuminate several types of inheritance customs and the phenomenon of gender bias in inheritance. Section 5 applies the model to some empirical puzzles and examines the implications of the analysis for inheritance taxation and other aspects of public policy.

2 Background

2.1 Concepts

The analysis presented here focuses on families, family wealth-holdings, and the transfer of wealth from generation to generation, where the family is considered to be an entity that exists over an indefinite number of generations.

In the analysis of wealth distributions it is essential to take account of both market and non-market forces: in general both will shape the development of the wealth distribution. This is one area where the non-market forces cannot be assumed away or disregarded as of peripheral interest: these forces include the formation of families and the nature of decisions taken within families. Non-market forces also include those that are directly attributable to governmental intervention, most notably taxation.

Wealth dynamics in principle cover the changes in the dispersion and shape of the wealth distribution and the mobility of families over time. Within this framework the concept of equilibrium distribution is central to our analysis. Suppose we summarize the (market and non-market) forces that make up the period-to-period wealth dynamics as a single process. The process transforms the wealth distribution in one period to the wealth distribution in the next. Is there a specific wealth distribution that is left unchanged when the process is applied to it? If so, then that distribution is an equilibrium distribution for the given process. Of course, in some cases, there may be more than one equilibrium distribution per process; for some processes there may be no equilibrium distribution. A central part of this paper is to show how one can characterize the class of equilibrium distributions based on reasonable assumptions about the family component of the wealth-generating process.

2.2 Foundations of the approach

Let us place our approach within the context of the extensive theoretical and empirical literature on inheritance, wealth distribution and inequality. We do this by focusing on previous approaches to the processes of wealth division and wealth union.

Wealth division

Parents may bequeath wealth to their children for a variety of motives (Cremer and Pestieau 2006). Here we focus principally on the "joy-of-giving" motive,[3] in which parents receive utility directly from the bequest. Given homothetic preferences – an assumption widely used in papers that adopt a formal model of bequest preference – bequests would be proportional to total wealth, but the distribution of bequests would be indeterminate. However, there is a strong presumption that one ought to assume equal division: in the US of recent years around 75% of bequests are divided equally (Menchik 1980, Wilhelm 1996, Behrman and Rosenzweig 2004);[4] in some countries equal division of bequests is an almost universally applied norm;[5] in some societies equal division may be imposed by law (Kessler and Masson 1988). Given the prevalence of equal division, our basic model, naturally focuses on this case – see section 3. But we do not limit ourselves to the classical inheritance rules analyzed in the literature, like equal division (Laitner 1979, Stiglitz 2015b) or male primogeniture. In section 4.3 we show how the implications of other inheritance practices, applied in the past or in other cultures, naturally follow from that of the basic model.

Wealth union

Marriage unites the wealth of different families and so the question of who marries whom is crucially important for the evolution of the distribution of wealth. In the basic model we assume strict assortative mating, where parents' wealth is perfectly correlated. This has been assumed explicitly in some of the early literature (Blinder 1973, 1976, Atkinson 1980); it is also implicit in the papers that assume asexual reproduction, such as Banerjee and Newman (1991, 1993), Galor and Zeira (1993), Ghatak and Jiang (2002), Stiglitz (2015b). It can be rationalized by the theory of stable matching if there is a positive interaction between spouses' wealth and the production of marital output (Weiss 1997). Empirical evidence of the correlation of parents' wealth is patchy. Charles et al. (2013) estimate that the correlation in parental wealth of husband and wife in the US, after controlling for race and age, is about 0.4. This concerns not the correlation of wealth between husband and wife, but of the wealth levels of their parents. This is an imperfect approximation for what husband and wife will inherit because (i) it is their parents' wealth at a particular point in time (not what they will have when they receive their inheritance, and past gifts inter vivos are not included), and (ii) what husband and wife get depends on the number of siblings they have. Frémeaux (2014) examines the correlation between total inherited wealth (already received and expected), as well as ex-

[3]Within the context of our model we can also analyze the case where parents have an altruistic bequest motive and care about the welfare of their children rather than the amount bequeathed. Here the division of wealth is contingent on children's needs – see section 4.3.

[4]Departures from this rule are found principally in the case of complex families (Francesconi et al. 2015).

[5]Erixson and Ohlsson (2014) found that, in Sweden, bequests are equally distributed in 97% of cases.

pected labor income of husband and wife: he shows that the correlation between inheritances is 0.25 and is significantly higher for inherited wealth than for labor income, especially at the top of the distribution. For our model, in principle the correlation between total wealth of the partners (inheritance and earnings) is relevant. Although the evidence is incomplete, it suggests that reality lies somewhere between perfectly assortative and random mating. Accordingly, in section 4.5, we show how the basic model can be straightforwardly extended to one where parents' wealth is not perfectly correlated.

There are two areas where our approach differs significantly from previous work. First, we allow for a distribution of family sizes in the population; the way wealth is divided among children affects families differently: in all previous papers each family in the population has the same number of children. Second, we characterize the functional form of the equilibrium distribution, rather than discussing the mere existence of one or several equilibrium distributions (Banerjee and Newman 1991, Galor and Zeira 1993, Laitner 1979 and Ghatak and Jiang 2002), a characteristic of the equilibrium distribution like its variance (Atkinson 1980) or a simulation of the distribution of wealth over a limited number of generations (Blinder 1976).

2.3 Analytical framework

We consider a world in which society at any point in time consists of a large number of families that accrue wealth through the growth of their asset holdings and from exogenous forms of income (earnings). Some of this wealth cascades down through the generations through the inheritance process.

Families and time

A family consists of a pair of adults (marriage partners) and a positive integer number of children. Children arrive through a mixture of planning and chance; social convention and government policy influence the planning; the chance element in the number of children is independent of wealth. We simplify matters by assuming that only two generations are alive at any one point in time (no grandparents or grandchildren) and that there are no childless families.

Let time be discrete and indexed by $t = ..., 0, 1, 2, ...$ Although each generation is associated with a pair of periods – those who are children at time t become adults at time $t + 1$ – we can use the time index t to characterize a particular generation within a family line. "Generation t" refers to the generation who are adult at time t. The population is made up of a sequence of generations.

We focus on the structure of families within the model. Let the proportion of families with k children be $p_k \geq 0$, $k = 1, 2, ..., K$. The vector $\mathbf{p} := (p_1, p_2, ...p_K)$ is the distribution of families by size. This distribution of families in the population is arbitrary and assumed to be exogenous. By definition $\sum_{k=1}^{K} p_k = 1$

and, for population stationarity, $\mathbf{p}$ must satisfy:

$$\sum_{k=1}^{K} kp_k = 2. \tag{1}$$

Wealth accumulation

We assume that the two adults in the family pool their inherited wealth, even if they enter the family with differing amounts of inherited wealth. For every family in generation t wealth per adult W_t is given by E_t, the lifetime value of earnings (evaluated at the beginning of period t) plus I_t, inheritance (received at the beginning of period t):

$$W_t = E_t + I_t. \tag{2}$$

If there is an exogenous per-period rate of growth of total wealth g and a family chooses to consume an amount C_t then the following budget constraint determines how much is available for bequests B_t:

$$C_t + \frac{B_t}{1+g} \leq E_t + I_t. \tag{3}$$

We assume that each two-adult family has a utility function with three arguments, bequest, current consumption and leisure. Leisure can be written as $1 - E_t/\overline{E}$, where $\overline{E}$ is the maximum possible earnings over the lifetime (if the available time for work during the lifetime is normalized at 1, then $\overline{E}$ is the wage rate). If we further assume that the family's utility is Cobb-Douglas[6] then the family's problem is to choose B_t, C_t, E_t to maximize

$$\gamma \log B_t + [1-\gamma] \log C_t + \nu \log \left(1 - E_t/\overline{E}\right) \tag{4}$$

subject to (3) and

$$B_t, C_t \geq 0, \tag{5}$$
$$E_t \leq \overline{E}, \tag{6}$$
$$E_t \geq 0. \tag{7}$$

In (4) γ is a parameter capturing the taste for bequest relative to one's own consumption and ν is a parameter capturing the taste for leisure. Constraint (3) will be binding at the optimum and constraints (5) and (6) will be automatically satisfied for this type of utility function. So the problem then becomes equivalent to choosing C_t and E_t so as to maximize $\gamma \log(E_t + I_t - C_t) + [1-\gamma] \log(C_t) +$

[6] As in many other standard models of this type of problem such as Becker and Tomes (1979), Banerjee and Newman (1993), Galor and Zeira (1993) and Ghatak and Jiang (2002). The main conclusions of the model would also apply for other forms of the utility function – see section 4.3 – including a more general specification of homothetic preferences.

$\nu \log \left(\overline{E} - E_t\right)$ subject to (7). The solution to this is[7]

$$C_t = [1-\gamma][E_t + I_t], \tag{8}$$

$$E_t = \max\left\{\frac{\overline{E} - \nu I_t}{1+\nu}, 0\right\}. \tag{9}$$

The equation for optimal consumption (8) implies a uniform savings rate γ and that the bequest per adult made to the next generation is

$$B_t = \gamma [1+g] W_t. \tag{10}$$

From (2) and (9) we see that the relationship between earnings and wealth is

$$E_t = \overline{E} - \nu W_t, \tag{11}$$

as long as earnings are positive: the stronger is the taste for leisure, ν, the more rapidly earnings decrease with wealth.

Wealth transfers

At any moment t we have a collection of families characterized by their joint wealth level and the number of children who will eventually inherit some of that wealth. The dynamics of the wealth distribution amongst families depends not only on the specific assumptions made about the way wealth grows in each period and people's savings behavior but also the processes by which wealth is transferred through family mechanisms.

There are two such family mechanisms of particular interest here. First we need to consider the way in which new families are formed in each generation. Clearly if the rich marry the rich and the poor marry the poor there will be a very different dynamic from the case where there is a zero correlation between amounts of wealth that the two adults bring to the family (Blinder 1973). The second mechanism is the way which parents distribute their wealth. Again we would expect a very different dynamic if the population followed a policy of primogeniture compared with the case where bequests are divided equally.

There is, in addition, a third important non-market mechanism. Taxation on wealth, or on a component of wealth, or on the transfer of wealth, could change the dynamic of change of the wealth distribution.

3 Basic model

Our basic model is used to provide an insight on a fundamental result concerning the equilibrium distribution in this type of economy. We make two key

[7]One of the implications of the widely-used Cobb-Douglas preference model is that B_t is independent of the number of children. The empirical evidence on the relationship between bequests and the number of children is unclear (Altonji and Villanueva 2007, Menchik and David 1983, Pestieau 2003). There is some suggestion that there is slight reduction in total bequest as the number of children is increased: if our model were to be modified to accommodate this phenomenon then the conclusions reached in sections 3 and 4 would be reinforced.

assumptions about the family: (i) that there is strict "assortative mating" – in each family the parents' inherited wealth is perfectly correlated and, (ii) that all parents follow a policy of equal division of bequests amongst their children. In this model each family is characterized by the joint wealth level of the marriage partners and by the number of children.

3.1 Wealth dynamics

With two parents of equal wealth and k children sharing in the inheritance, the relationship between inheritance and bequest is

$$I_t = \frac{2}{k} B_{t-1}. \tag{12}$$

Combining equations (2), (10) and (12) we find the following difference equation for a family with k children:

$$W_t = E_t + \frac{2\beta}{k} W_{t-1}, \tag{13}$$

where E_t depends on W_t (see equation 11) and β is an exogenous one-generation growth factor given by

$$\beta := \gamma[1 + g]. \tag{14}$$

In summary, if a person in the current generation has wealth W_t and comes from a family where there were k children, the parents must each have had wealth $\frac{k}{2\beta}[W_t - E_t]$. This simple fact enables us to derive the evolution of the wealth distribution from generation $t-1$ to t. We use equation (13) along with the assumptions of the basic model to derive F_t as a function of F_{t-1}, where F_t is the distribution function of individual wealth at time t.

We assume that the support of F_t is an interval $\mathbb{W}$. There are two important sub-cases of (13) to consider, corresponding to two separate parts of $\mathbb{W}$. From equation (11) it is clear that $E_t = 0$ if $W_t \geq \overline{W} := \overline{E}/v$. This condition effectively partitions society into two self-selecting groups: those with sufficiently high wealth who choose not to work (the "idle rich"); and those who find it necessary to supplement their small inheritances with earnings in the labor market.

The case $E_t > 0$

In this case $W_t < \overline{W}$, characterizing a subset of $\mathbb{W}$ where everyone works: (11) and (13) then imply the following wealth dynamics

$$W_t = \frac{\overline{E} + \frac{2\beta}{k} W_{t-1}}{1 + \nu}. \tag{15}$$

Recalling that there is a proportion p_k of families with k children, then, for an arbitrary W in a subset of $\mathbb{W}$:

$$F_t(W) = \sum_{k=1}^{K} \frac{1}{2} k p_k F_{t-1}\left(k \frac{[1+\nu]\,W - \overline{E}}{2\beta}\right). \tag{16}$$

For a given value of W, the proportion of the population that has wealth W or less in period t, $F_t(W)$, is found as a weighted sum of the values of $F_{t-1}(\cdot)$ at K distinct wealth levels in the previous generation. Equilibrium requires that the distribution function remains unchanged through the generations, so that $F_{t-1} = F_t$, for all t. Call this equilibrium distribution F_*. Defining the constants $a_k := \frac{1}{2} k p_k$, $b_k := \frac{k}{2\beta}[1+\nu]$, $c_k := -\frac{k}{2\beta}\overline{E}$, the above argument implies that the equilibrium distribution must satisfy

$$F_*(W) = \sum_{k=1}^{K} a_k \; F_*(b_k W + c_k), \tag{17}$$

for any value of W such that $W < \overline{W}$ and $b_k W + c_k < \overline{W}$, for all k.

The case $E_t = 0$

In this case $W_t \geq \overline{W}$; we are focusing on a part of the population consisting solely of the "idle rich" or rentiers. It is of particular interest, not just because it is close to other inheritance models in the wealth-distribution literature, [8] but also because the zero-earnings simplification enables us to establish a striking result.

If people are so rich that they choose not to work, then

$$W_t = \frac{2\beta}{k} W_{t-1} \tag{18}$$

The characterization of the equilibrium condition follows the same lines as in the previous case. However, given $E_t = 0$, we now have $b_k := \frac{k}{2\beta}$ and $c_k = 0$ in equation (17) and so the equilibrium condition becomes:

$$F_*(W) = \sum_{k=1}^{K} a_k \; F_*(b_k W), \tag{19}$$

where $W \geq \overline{W}$ and $b_k W$ for all k. Observe that, given the assumptions of the basic model, $\sum_{k=1}^{K} a_k = 1$; so, with $F_*(\cdot)$ increasing, we need to have $\frac{W}{2\beta} < W < \frac{KW}{2\beta}$, which implies $\frac{1}{2} < \beta < \frac{K}{2}$.

From Theorem 1 (in the Appendix) we find that if (19) is true on an appropriate domain $\overline{\mathbb{W}}$, this implies that the equilibrium wealth distribution must

[8] See, for example, the classic papers by Blinder (1973) and Atkinson (1980).

belong to the extended Pareto Type I family, throughout the region where the equal-division inheritance rule applies. In other words F_* must satisfy

$$F_*(W) = a + b\frac{W^{-\alpha} - 1}{\alpha}, \tag{20}$$

where a and b are constants of normalisation and the parameter α could in principle take any real value. Clearly the applicability of this result depends on the specification of the domain $\overline{\mathbb{W}}$: we will address this after we have considered the rest of the basic model.

3.2 Closure of the model

Although the basic model with $E_t = 0$ tells an interesting story about what goes on in the sub-population of the rentiers in the upper tail of the wealth distribution, it clearly does not adequately describe the whole distribution. In order to make progress we need to discuss an important component that is missing from the model and some suggestions for fitting such a component into the model.

To identify the missing component, let us continue with the structure and assumptions of the basic model, whereby each family sees its bequeathable wealth increase by an exogenous factor β over a generation and divides bequests equally among their offspring. Consider a family with k children and wealth per adult sufficiently large that they choose not to work. The children of this family will fall out of the interval $\left[\overline{W}, \infty\right)$ if parental wealth falls into the interval $\left[\overline{W}, \frac{k}{2\beta}\overline{W}\right)$, where $k > 2\beta$. As a result of children "dropping out", the probability mass in $\left[\overline{W}, \infty\right)$ would decrease for parental wealth satisfying $W \in \mathbb{W}_0$ where

$$\mathbb{W}_0 := \left[\overline{W}, \frac{K\overline{W}}{2\beta}\right). \tag{21}$$

Given a wealth distribution F the overall decrease in mass would be given by

$$\sum_{k=\underline{k}}^{K} p_k \left[F\left(\frac{k\overline{W},}{2\beta}\right) - F\left(\overline{W}\right)\right]. \tag{22}$$

where $\underline{k}$ is the smallest integer greater than or equal to 2β. Clearly one can only have an equilibrium distribution of wealth if those families dropping out of $\left[\overline{W}, \infty\right)$ are replaced by previously low-wealth families moving in. We need to consider appropriate sources of new wealth that will ensure that this replacement takes place.

Earnings

An important source of this new wealth consists of savings by families with positive earnings. The children of some of these families will receive a large inheritance and so will, in effect, replace the children of rentiers with wealth given by (21) who drop out of $\left[\overline{W},\infty\right)$. The equal-division rule and the one-generation growth factor β also apply in the region $W<\overline{W}$, so a child from a family with k children moves up into the interval $\mathbb{W}_0$ only if the following condition holds on each of the parents' wealth[9]:

$$W\in\left(\frac{k\overline{W}}{2\beta},\frac{K\overline{W}}{4\beta^2}\right). \tag{23}$$

It is clear from (23) that families cannot move up into the drop-out zone given in (21) if they have more than $\overline{k}$ children, where $\overline{k}$ is the largest integer less than or equal to $K/2\beta$. As a result of children moving up into $\mathbb{W}_0$ the mass of the wealth distribution in $\left[\overline{W},\infty\right)$ changes as follows:

$$\sum_{k=1}^{\overline{k}}\frac{k}{2}p_k\left[F\left(\frac{K\overline{W}}{4\beta^2}\right)-F\left(\frac{k\overline{W}}{2\beta}\right)\right]. \tag{24}$$

Wealth subsidies

There is also an argument based on the use of the resources raised through a progressive tax-subsidy scheme based on wealth or inheritance (for more on this see section 5.2). Suppose, for convenience, that earnings are zero everywhere and that the tax raised is used to subsidize relatively low-wealth families – those where adult wealth is below $\overline{W}$.

As an example of this suppose that, for $W<\overline{W}$, the total wealth to be passed on to the next generation is $\beta\left[W+\sigma\left(W\right)\right]$ where σ is a subsidy function that has the property that it is non-increasing in W and that $\sigma\left(\overline{W}\right)=0$. The process of dropping out of the support above $\overline{W}$ is similar to the one described above, but the behavior of the wealth distribution below $\overline{W}$ will be different. It is possible, given the subsidy scheme, to determine the part of the wealth distribution over an interval $\left[\hat{W},\overline{W}\right]$ compatible with equilibrium above $\overline{W}$ in the same way as for the model with earnings. The only difference is that there is also the budgetary requirement that

$$\int_0^{\overline{W}}\sigma\left(W\right)dF\left(W\right)\leq\left[1+g\right]\int_{\overline{W}}^{\infty}T\left(W\right)dF\left(W\right), \tag{25}$$

where $T(W)$ is the total tax paid by someone with wealth W at or above $\overline{W}$. However, as the wealth distribution below $\hat{W}$ is not determined, the budgetary requirement can be ignored to derive the wealth distribution over $\left[\hat{W},\overline{W}\right]$ that is compatible with equilibrium above $\overline{W}$.

[9]Parental wealth per adult cannot be lower than $k\overline{W}/2\beta$, because then the child never gets more than $\overline{W}$, and cannot be higher than $K\overline{W}/4\beta^2$, because then, if parents have one child, the child ends up above that part of the support of the wealth distribution where descendants drop out.

3.3 Equilibrium distribution

First let us consider the appropriate domain over which the functional equation (19) is to apply. In the light of the discussion of section 3.2 this domain must exclude the set $\mathbb{W}_0$ (defined in equation 21) because, for some families, if $W_{t-1} \in \mathbb{W}_0$ (where there are zero earnings) then $W_t < \overline{W}$ (where there are positive earnings); by similar reasoning it must exclude

$$\mathbb{W}_1 := \left[\overline{W}, 2\beta\overline{W}\right) \tag{26}$$

since, if $W_t \in \mathbb{W}_1$, then $W_{t-1} < \overline{W}$ for some families. The union of $\mathbb{W}_0$ and $\mathbb{W}_1$ is a "transition zone" in which we find non-working adults who either have children who will choose to work when they become adults in the next period or whose parents had chosen to work in the preceding period. Clearly the relevant domain $\overline{\mathbb{W}}$ for F_* in (19) consists of the points in $\left[\overline{W}, \infty\right)$ that do not belong to the transition zone, namely

$$\overline{\mathbb{W}} = \left[\theta\overline{W}, \infty\right), \text{ where } \theta := \max\left(\frac{K}{2\beta}, 2\beta\right). \tag{27}$$

Now consider the shape of the equilibrium distribution over $\overline{\mathbb{W}}$. Substituting (20) in (17) we see that, for $W \in \overline{\mathbb{W}}$, this is given by the Pareto distribution

$$F_*(W) = 1 - AW^{-\alpha}, \tag{28}$$

where A is a constant and α is the root of the following implicit equation in α:

$$\sum_{k=1}^{K} p_k \left[\frac{k}{2}\right]^{1-\alpha} = \beta^{-\alpha}. \tag{29}$$

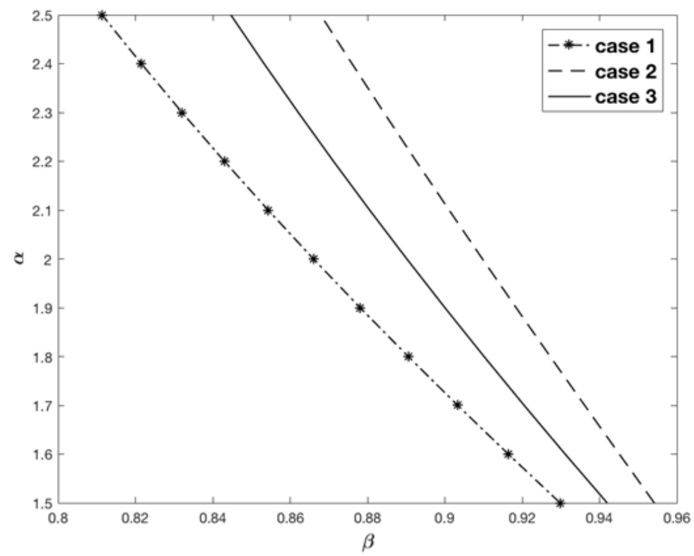
The parameter α will depend on the growth factor β and on the distribution of families by size $\mathbf{p}$. To see the implications of this for wealth inequality take the three cases of the family-size distributions in Table 1: the distribution in case 1 does not second-order dominate the distribution in case 2 but the latter dominates case 3. However, the ordering by inequality of the equilibrium wealth distributions is not determined by the inequality ordering of the family-size distribution. To see this refer to Figure 1 depicting the equilibrium relationship between α and β for each of the three cases in Table 1. It is clear that for every value of the growth β the value of α in case 2 is higher than that for case 3. Equilibrium inequality amongst the rentiers is unambiguously higher for the case with the more widely dispersed family-size distributions. However, of the three cases, the one with the lowest α (highest inequality) is case 1.

The proportion of single-child families in each case is the key factor here: ultimately it is the wealth of these little emperors/empresses that drives equilibrium inequality at the top. The reason why this is so is that the growth in family wealth depends on the number of children: the more children there are in a family, the lower the growth rate of wealth. It is clear from (29) that in

	p_1	p_2	p_3	p_4	p_5	p_6
case 1	0.50	0	0.50	0	0	0
case 2	0.30	0.45	0.20	0.05	0	0
case 3	0.35	0.45	0.10	0.06	0.03	0.01

Table 1: Three family structures

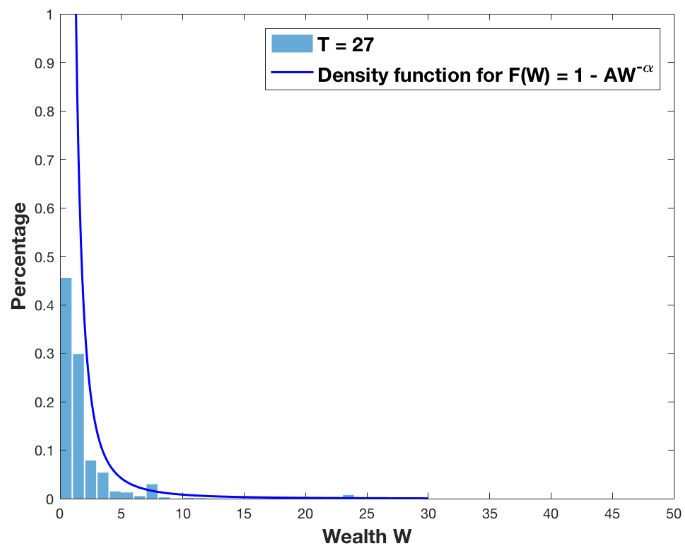
Figure 1: Pareto tail parameter α and the growth factor β for three distributions of families by size



equilibrium we must have $\beta < 1$.[10] So, in the upper tail, where $E_t = 0$, in any one family wealth must fall if $k \geq 2$. In equilibrium all these decreasing-wealth families must be balanced by increasing-wealth families for whom $k < 2$; but, given that the number of children is an integer, this means that the decreasing-wealth families are balanced by families where $k = 1$. One implication of these facts is that, higher up in the equilibrium distribution, we find dynasties with a history of having few children; and, the higher up in the distribution the more families with histories of fewer and fewer children we encounter.

[10] This follows from two observations: (i)$\alpha$is strictly decreasing in β; (ii) if $\alpha = 1$ then $\beta = 1$. So if we had $\beta \geq 1$ in (29) we would have $\alpha \leq 1$ which would imply a distribution with an infinite mean ($\alpha = 1$) or the non-existence of an equilibrium distribution ($\alpha < 1$).

Figure 2: Simulated equilibrium distribution starting from perfect equality



3.4 Behavior out of equilibrium

The focus on equilibrium distributions is convenient for the succinct discussion of comparative statics, but it would be of little practical interest if the underlying model of wealth dynamics did not converge reasonably quickly. Here we set out three simulations using the basic model. The wealth dynamics is governed by (15) if $W_t < \overline{W}$ and (18) otherwise. It is clear from equation (18) that if β were greater than 1 then the process would not converge to an equilibrium: the expected value of the number of children is 2 and, given $\beta > 1$ we would have, for a family with $k = 2$, $W_t > W_{t-1}$ for all t – a more general version of this point is treated in section 4.2 . It is also clear that if $\overline{E}$ were zero then there is an accumulation of ever-smaller inheritances at the bottom of the distribution: again there would be no convergence to equilibrium.

We carried out several simulations to find out how fast the model converges to an equilibrium. Parameter values are $\overline{E} = 2$, $\nu = 2$, and $\beta = 0.95$, the number of families is 10,000 and the distribution of family size is given by case 1 in Table 1. First we assume that there is no inequality initially: every family starts with per capita wealth equal to 10. After 27 periods the equilibrium distribution is reached. Figure 2 gives a histogram of the equilibrium distribution, and the density of the theoretical distribution. Figure 3 gives the Pareto diagram: the relationship between the $\log(W)$ and $\log(1 - F(W))$.

In the Pareto diagram, every dot corresponds to one of the observations of the simulated equilibrium distribution. For Pareto distributions, the Pareto

Figure 3: Simulated equilibrium distribution starting from perfect equality

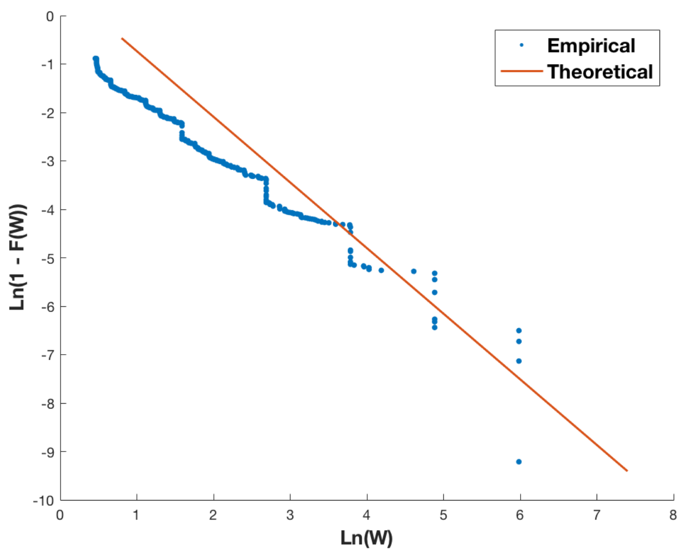


diagram is a straight line with slope equal to minus α. The line in the graph has a slope equal to the value of $-\alpha$ of the equilibrium distribution predicted by equation (29). It is clear that the empirical CDF of the simulated distribution has the same slope. We also simulated starting with a random drawing from a uniform distribution defined over $[5, 15]$. The simulated distribution converges to the same equilibrium distribution as above. The only difference is that convergence is faster: it only takes 17 rounds to obtain the equilibrium.

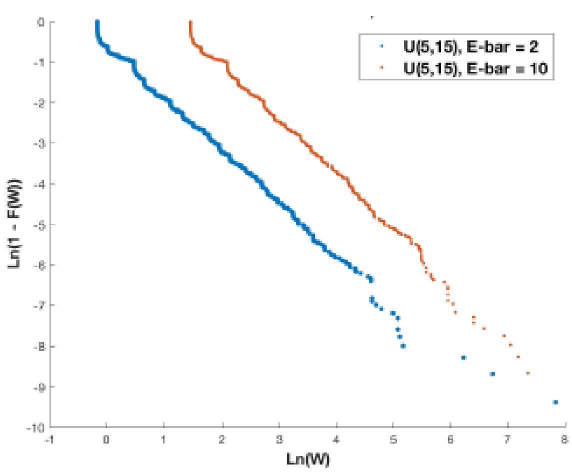
4 Development of the approach

The basic model analyzed in section 3 provides a useful starting point but needs to be re-examined in order to take into account important real-world aspects of wealth dynamics. We begin with two features of the model that, in different ways, affect the possibility of convergence to equilibrium: the labor market and population growth – see sections 4.1 and 4.2. We then turn to generalizing the forces of wealth division and wealth union – see sections 4.3 to 4.5.

4.1 The labor market

The labor market plays an interesting role in our inheritance model. It acts as an optional resource for those who find that their inheritance is insufficient for their own lifetime needs and bequest plans. If there is an equilibrium distribution then $\overline{E}$, the potential earnings attainable, has an important role in the distributional dynamics. Consider what would happen if $\overline{E}$ were reduced. Figure 4 shows the equilibrium distribution for two specific values of the labor parameter $\overline{E}$: decreasing $\overline{E}$ shifts the Pareto diagram of the equilibrium wealth distribution to the left; but the slope of the line characterizing the equilibrium distribution remains unchanged: the equilibrium distribution shifts to the left without changing the value of α. Why this happens is clear from equation (15): the "bounce-back" effect experienced by members of a dynasty who drop out of the rentier class is reduced. However, if $\overline{E}$ were to go to zero, the equilibrium would collapse: one has the situation where some members of a dynasty with large numbers of children may find that, generation by generation, their inheritances dwindle to zero. This may seem fanciful if one is thinking only in terms of an advanced economy with a well-developed labor market. But, if wealth consists of agricultural property that provides the entire basis for maintaining one's livelihood, it is clear that this process could be catastrophic.[11]

Figure 4: Simulated equilibrium wealth distribution with differing values of $\overline{E}$



Heterogeneous earnings $\overline{E}$ and heterogeneous tastes for leisure can easily be incorporated into the model with only small modifications. The effect is to alter the width and location of the transition zone $\mathbb{W}_1$ in equation (26): it is now defined by the maximum value of $\overline{W}$ taken across all values of $\overline{E}$ and

[11]It has been argued that this sort of process contributed to the dreadful breakdown of society and civil war in Rwanda of the early 1990s – see André and Platteau (1998).

ν. However, this heterogeneity has no impact on the structure of equilibrium inequality in the upper tail. This is illustrated in Figure 4: different values of $\overline{E}$ produce Pareto distributions that have the same slope but different implicit intercepts. So a mixture of the distributions corresponding to these values of $\overline{E}$ will produce a distribution with the same slope: the implied equilibrium wealth inequality in the upper tail is unchanged by allowing a variation in $\overline{E}$.

4.2 Population growth

The basic model assumes a stationary population. But it is easy to extend it to deal with population growth, where we continue to assume equal division among children. Define the *population growth factor* as $\pi := \sum_{k=1}^{K} \frac{1}{2} k p_k$ (the population growth rate is $\pi - 1$). One simply replaces each instance of p_k in the discussion of section 3 with the normalised proportion p_k/π. So, the equilibrium condition for those who do not work is again (19), but now with $a_k = \frac{kp_k}{2\pi}$. Following the same argument as for the basic model, the equilibrium distribution, if it exists, is given by the Pareto distribution (28), and α is the solution to the following equation:

$$\sum_{k=1}^{K} p_k \left[\frac{k}{2\pi}\right]^{1-\alpha} = \left[\frac{\beta}{\pi}\right]^{-\alpha}, \tag{30}$$

which is a straightforward generalization of (29).

We can now see that the condition for convergence to equilibrium given in section 3.4 needs to be slightly modified. Now we will get convergence to an equilibrium only if

$$\beta < \pi. \tag{31}$$

The generation-to-generation growth factor on wealth must not exceed the growth factor on population.

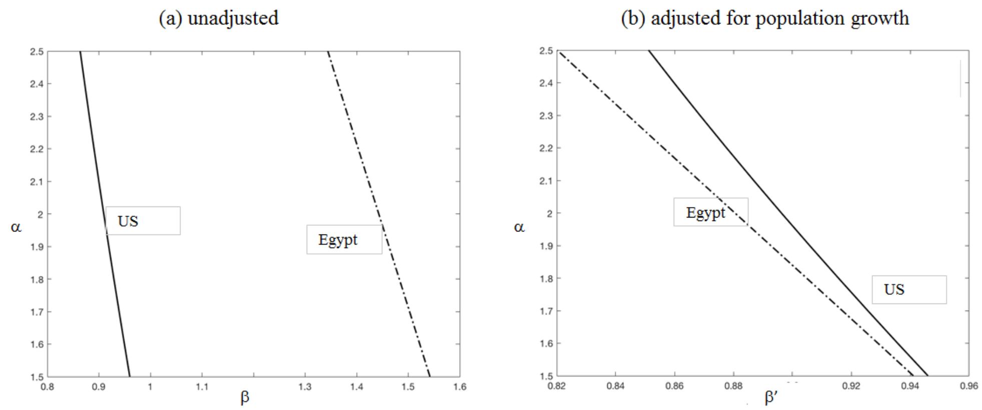
Comparing distributions for non-stationary populations requires some care. Figure 5 shows an example where there is a significant difference in the population growth factor between two distributions: the case of the US and Egypt.[12] In part (a) of the figure we have the simple (α, β)-plot which, at first glance, suggests that the US curve is below and to the left of the Egypt curve and that Egypt will not achieve an equilibrium wealth distribution. However, the population growth factor is almost sixty percent higher in Egypt than in the US.

[12] The distributions used for Figure 5 are derived from the University of Minnesota's Integrated Public Use Microdata Series:

- US: Version 6 (2015) [IPUMS Sample: 1970-2016] http://doi.org/10.18128/D010.V6.0;
- Egypt: International Version 6.5 (2017) [IPUMS Sample: 1986-2006] http://doi.org/10.18128/D020.V6.5.

US data on children are from the Current Population Survey and cover all children, including step-children and adopted children, residing with each individual. Egypt data on children are from the census survey and provide a count of the person's own (both biological and adopted) children living in the the same household. In each case the data are truncated to include only females aged between 45 and 50. The resulting table is as follows, where the final column shows the one-generational population growth factor implied by $\boldsymbol{p}^{\mathrm{US}}$ and $\boldsymbol{p}^{\mathrm{E}}$, respectively.

Figure 5: Equilibrium inequality: two populations



If we adjust both series for this difference and plot (α, β'), where $\beta' := \beta/\pi$, a very different picture emerges: both countries have an equilibrium wealth distribution and, for given β', the value of α will be lower in Egypt than in the US.

4.3 Wealth division: unequal shares

The basic model of section 3 assumes equal division amongst inheritors. It is an assumption that has much to commend it: a perception of fairness, widespread custom and neatness in economic modelling. However, there are other inheritance practices and we should consider the viability of the analysis of the basic model in the light of these alternatives. We need to address both the issue of how to modify the underlying wealth dynamics in the light of alternative inheritance practices and the question of whether we would still expect to find an equilibrium distribution with a Pareto tail.

General division rules

Some of the alternative inheritance rules involve dividing the inheritance unequally, without consideration of the gender of the child. In the Basque country, traditionally one child inherited everything, regardless of its gender.[13] In unilateral inheritance practices only children from one gender can inherit wealth.

k	1	2	3	4	5	6	7	8	9	π
$\boldsymbol{p}^{\mathrm{US}}$	0.325	0.423	0.183	0.048	0.016	0.004	0.001	0.001	0	1.015
$\boldsymbol{p}^{\mathrm{E}}$	0.105	0.216	0.281	0.205	0.122	0.04	0.019	0.008	0.004	1.639

[13] This rule was written down and legalized in the 15th and 16th centuries and abolished by the French revolution - see Arrizabalaga (2005) for more detail.

The classical example is male unigeniture, in which a son (typically, but not necessarily, the eldest) inherits all wealth. This rule was practiced in much of Europe; it was first abolished in France by the French revolution and later other countries followed suit.[14] In Africa, wealth is typically divided among all sons. Historically shares were not necessarily equal, with a larger share going to the son who has special duties.[15] These privileged shares are increasingly questioned by other brothers, such that shares become more and more equal. Bilateral inheritance rules divide the father's property among the sons, and the mother's property among the daughters. These rules occur less frequently, but Edmonson (1979) suggests that bilateral primogeniture was practiced by the Classic Maya. Crisologo and Van de gaer (2001) observed bilateral equal division inheritance practices in Cordillera village in the Philippines.

In seventeenth and early eighteenth century Massachusetts, Connecticut, New Hampshire, Rhode Island, Pennsylvania, and Delaware, the eldest son received double the amount his brothers and sisters received (Shammas 1987). This particular provision originated from Massachusetts's colonial laws and was taken over by the other colonies. It was inspired by Mosaic law, as described in Deuteronomy XXI: 17 (Haskins and Ewing 1958).[16] It is intriguing that exactly the same rule was observed in ancient Egypt (Lippert 2013).

Unequal division: model

Consider an extension of the basic model that allows for an unequal division of wealth. A family with k children bequeaths a share $\omega_k^j > 0$ to its jth child where, for all k, $\sum_{j=1}^{k} \omega_k^j = 1$. As a result, in the region where there are zero earnings, the evolution of the wealth distribution is now governed by the following equation:

$$F_t(W) = \sum_{k=1}^{K} \sum_{j=1}^{k} {}^1\!/_2 p_k F_{t-1}\left(\frac{W}{2\beta\omega_k^j}\right). \tag{32}$$

So the equilibrium distribution must satisfy

$$F_*(W) = \sum_{k=1}^{K} \sum_{j=1}^{k} a_k F_*\left(b_k^j W\right), \tag{33}$$

[14] See Piketty (2014a). The elite in medieval England practiced male primogeniture, for the bulk of the population customs varied from borough to borough: male primogeniture, male ultimogeniture and equal division among sons were applied (Shammas 1987).

[15] Examples of special duties are custodianship of family land to be distributed later to younger brothers, the power to settle intra-family conflict or caring for parents during old age (Platteau and Balend 2001).

[16] Deuteronomy XXI:15 : "If a man has two wives, and he loves one but not the other, and both bear him sons but the firstborn is the son of the wife he does not love," 16: "when he wills his property to his sons, he must not give the rights of the firstborn to the son of the wife he loves in preference to his actual firstborn, the son of the wife he does not love", 17: "He must acknowledge the son of his unloved wife as the firstborn by giving him a double share of all he has. That son is the first sign of his father's strength. The right of the firstborn belongs to him."

where $a_k = \frac{p_k}{2}$ and $b_k^j = \frac{1}{2\beta\omega_k^j}$. Observe that $\sum_{k=1}^{K}\sum_{j=1}^{k} a_k = 1$, such that (33) can only hold true if $\frac{1}{2\beta}W < W < \frac{1}{2\beta\underline{\omega}}W$, where $\underline{\omega}$ is the smallest of all the shares. The first inequality requires that $\beta > 1/2$, the second that $\underline{\omega} < [2\beta]^{-1}$. Following the same argument as in section 3 the equilibrium distribution, if it exists, is characterized in the upper tail by the Pareto distribution (28) where now α is the solution to the following equation:

$$\beta^{-\alpha} = 2^{\alpha-1}\sum_{k=1}^{K} p_k \sum_{j=1}^{k}\left(\omega_k^j\right)^{\alpha}. \tag{34}$$

To see what is going on, take a special case that corresponds to most of the practical examples mentioned earlier. Suppose there is just one "favored" child in the family and all the other children (if any) get equal shares in what is left after the favored child has received his/her inheritance: $\omega_k^1 = \frac{1+\xi}{k+\xi}$ and for $j \neq 1$, $\omega_k^j = \frac{1}{k+\xi}$, where $\xi \geq 0$ is the favoritism parameter or "firstborn premium". In this case equation (32) becomes

$$F_t(W) = \sum_{k=1}^{K} {}^1\!/\!_2 p_k \left[F_{t-1}\left(\frac{W[k+\xi]}{2\beta[1+\xi]}\right) + [k-1]\,F_{t-1}\left(\frac{W[k+\xi]}{2\beta}\right)\right]. \tag{35}$$

If $\xi = 0$ (no favoritism) then $\overline{\mathbb{W}}$, the domain over which (35) holds, is (27), as before, and the solution in equation (34) reduces to (29). Otherwise, if $\xi \geq 4\beta^2 - K$, $\overline{\mathbb{W}} = \left[\frac{K+\xi}{2\beta}\overline{W}, \infty\right)$. So, the greater is the favoritism parameter, the smaller is $\overline{\mathbb{W}}$ and the Pareto tail is to be found further up in the wealth distribution. One consequence of this concerns the way the model works under primogeniture: here $\xi \to \infty$ and it is clear that the lower bound of $\overline{\mathbb{W}}$ also becomes infinite. What happens in this extreme case is that in every family with more than one child, everyone other than the favored child inherits zero wealth and so chooses to work: the transition zone becomes the entire interval $\left[\overline{W}, \infty\right)$.

Special inheritance rules

The analysis of favoritism in inheritance can be usefully extended beyond the scope of this version of the model, in ways that take account of gender roles in the family. *Unilateral rules* in their pure form assume that all wealth is owned –and therefore inherited by only one sex (usually men): this case is discussed further in section 4.4. *Bilateral rules* divide the father's property among the sons, and the mother's property among the daughters. These rules can be analyzed in a way similar to unilateral inheritance rules: male and female wealth will follow separate Pareto distributions. The broader aspects of gender differences in inheritance are treated in section 4.4.

With unilateral or bilateral inheritance rules, it does not matter who marries whom for the evolution of the wealth distribution. When inheritance is passed on to children regardless of their sex, as in the rules analyzed in section 3 and in this section, marriage unites the wealth of both spouses.

The bequest motive

In the basic model there is no need to distinguish between the "joy of giving" approach to bequests and the "altruistic" version of the bequest motive. However, if we make the model richer by assuming the offspring may differ in needs, then two families with the same number of children will divide their wealth differently. Let $\psi_k^j(\cdot) : \left[\underline{\omega}_k^j, \overline{\omega}_k^j\right] \to [0,1]$ denote the cumulative distribution of the share going to child j in a family of k children for every value of $k = 1, \ldots, K$ and every value of $j = 1, \ldots, k$. Equation (32), describing the evolution of the distribution of wealth, must now be replaced by

$$F_t(W) = \sum_{k=1}^{K} \sum_{j=1}^{k} {}^1\!/\!{}_2 p_k \int_{\underline{\omega}_k^j}^{\overline{\omega}_k^j} F_{t-1}\left(\frac{W}{2\beta\omega}\right) d\psi_k^j(\omega),$$

such that the equilibrium distribution must satisfy

$$F_*(W) = \sum_{k=1}^{K} \sum_{j=1}^{k} {}^1\!/\!{}_2 p_k \int_{\underline{\omega}_k^j}^{\overline{\omega}_k^j} F_*\left(\frac{W}{2\beta\omega}\right) d\psi_k^j(\omega).$$

Replacing the weights a_k in (19) by ${}^1\!/\!{}_2 p_k d\psi_k^j(\omega)$ this equation leads to a weighted quasilinear mean, provided that $\beta > \frac{1}{2}$ and the smallest of all $\underline{\omega}_k^j$ is smaller than $[2\beta]^{-1}$. Assuming that the distribution functions $\psi_k^j(\cdot)$ are wealth-independent, Theorem 1 applies and the equilibrium distribution of wealth in the interval must belong to the extended Pareto type I family. The equilibrium distribution, if it exists, is given by the Pareto distribution (28), and α will be the solution to the following equation:

$$\beta^{-\alpha} = 2^{\alpha-1} \sum_{k=1}^{K} p_k \sum_{j=1}^{k} \int_{\underline{\omega}_k^j}^{\overline{\omega}_k^j} \omega^{\alpha} d\psi_k^j(\omega),$$

which is a straightforward generalization of (34).

4.4 Wealth division: Gender bias

Although gender bias in inheritance is a familiar theme in popular fiction from the time of Jane Austen onwards, the economic implications for wealth distribution of this phenomenon remain relatively unexplored.[17] We can extend the idea of favoritism in section 4.3 to deal with the problem of developing the model to cover gender bias.

[17] Division of inheritances between families having one son and one daughter and where one gender is systematically favored in the division of inheritance is analyzed in Blinder (1973, 1976) and Atkinson (1980). We extend the analysis to deal with families of different sizes.

Suppose boys are favored over girls (obviously one can reverse the role of boys and girls in the following argument): in any family a boy inherits $1+\zeta$ times what a girl would inherit, where $\zeta \geq 0$ is the "boy premium". Then, in a family with k children, the share of a boy in the inheritance is given by $\frac{1+\zeta}{k+b\zeta}$ and that of a girl is given by $\frac{1}{k+b\zeta}$ where b is the number of boys, $0 \leq b \leq k$. If boys systematically get a premium in the inheritance process then male and female wealth will no longer have the same distribution, and the notion of strict assortative mating would have to be adjusted. Assuming a large population, for every boy that has inherited an amount $[1+\epsilon]\, I$ there will be exactly one girl that has inherited $[1-\epsilon]\, I$ where $\epsilon := \frac{\zeta}{2+\zeta}$. Assortative mating on the basis of rank order in the distribution of gender-specific wealth implies that a family inherited wealth $2I$ if and only if the husband inherited $[1+\epsilon]\, I$ and the wife $[1-\epsilon]\, I$. We now find that, amongst the rentiers, male and female distributions develop separately, as follows

$$F_t^{\mathrm{m}}(W) = \sum_{k=1}^{K}\sum_{b=1}^{k} {}^1\!/\!_2\, p_{kb}\, b F_{t-1}^{\mathrm{m}}\left(\frac{W\,[k+b\zeta]}{\beta\,[2+\zeta]}\right) \tag{36}$$

$$F_t^{\mathrm{f}}(W) = \sum_{k=1}^{K}\sum_{b=0}^{k-1} {}^1\!/\!_2\, [p_k - p_{kb}]\,[k-b]\, F_{t-1}^{\mathrm{f}}\left(\frac{W\,[k+b\zeta]}{\beta\,[2+\zeta]}\right) \tag{37}$$

where p_{kb} is the proportion of families with k children, of which b are boys. Notice that, if the boy premium becomes extreme ($\zeta \to \infty$), the female equation (37) is irrelevant and, as long as wealth is divided equally among boys, the male equation collapses to $F_t^{\mathrm{m}}(W) = \sum_{k=1}^{K}\sum_{b=1}^{k} {}^1\!/\!_2\, p_{kb}\, b F_{t-1}\left(\frac{bW}{2\beta}\right)$, a minor modification of the basic model in section 3: there will again be a Pareto tail and equilibrium inequality will depend on the distribution of boys in the population.

But of course favoritism may come in more than one flavor. Suppose there is also a premium on being the firstborn in the family, in addition to the boy premium ζ. If this privilege is experienced by both boys and girls, then a firstborn son gets an inheritance that is $1+\zeta+\xi$ times the inheritance of a younger sister, where, as in section 4.3 , ξ is the "firstborn premium"; if the firstborn were a daughter she would inherit $1+\xi$ times the inheritance of a younger sister. Then, instead of equation (36) the evolution of male wealth is governed by

$$F_t^{\mathrm{m}}(W) = \sum_{k=1}^{K}\sum_{b=1}^{k} {}^1\!/\!_2\, p_{kb}$$

$$\left[F_{t-1}\left(\frac{W\,[k+b\zeta+\xi]}{\beta\,[1+\zeta+\xi]}\cdot\frac{1+\zeta}{2+\zeta}\right) + [b-1]\, F_{t-1}\left(\frac{W\,[k+b\zeta+\xi]}{\beta\,[2+\zeta]}\right)\right] \tag{38}$$

Comparing this with equation (35) we may immediately conclude that the qualitative effect of the firstborn premium works exactly as in the case with no gender bias. If ξ increases then the Pareto tail starts further up the distribution and, if $\xi \to \infty$, the Pareto tail disappears.

4.5 Wealth union: marriage patterns

The assumption that people only marry those who have inherited wealth equal to their own is one of the most restrictive features of the basic model. However, the analysis can be adapted in a way that permits a simple modification of the approach in section 3. Here we focus on the upper tail where all wealth is inherited.

In this variant of the model each family is characterized by the triple (W, δ, k), where W is the wealth of the poorer of the two marriage partners, and $\delta \geq 0$ is a parameter giving the disparity between the wealth of the two partners. A person with wealth W marries someone with wealth $W[1+\delta]$, for W, $W[1+\delta] \in \mathbb{W}$; so the wealth received by a child in a (W, δ, k)-family is $\frac{[2+\delta]}{k}W$.

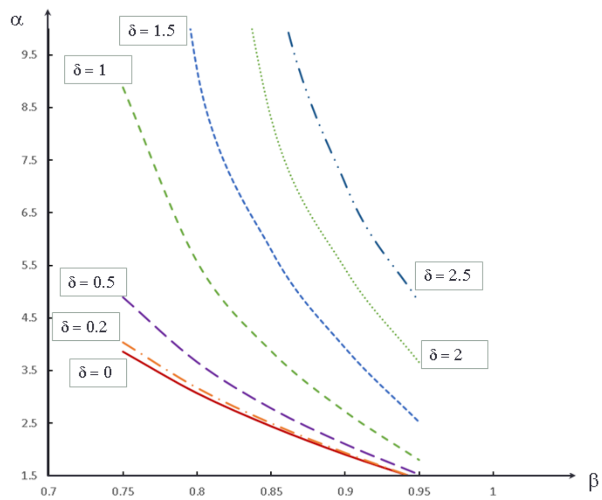
Making this version of the model operational requires an assumption about the distribution of marriage partners - the distribution of δ. An additional equilibrium condition is also required since those who "marry up" must be matched by those who marry beneath them. This induces a constraint on the admissible class of distributions of δ. The key assumption is that the distribution of δ is independent of W: the proportionate disparity of wealth between marriage partners is the same for the wealthy and for those with little wealth. If Φ is the distribution function of δ then the equilibrium condition becomes:

$$F_*(W) = \sum_{k=1}^{K} {}^1\!/\!_2 k p_k \int F_* \left(\frac{kW}{[2+\delta]\,\beta} \right) \mathrm{d}\Phi(\delta). \tag{39}$$

On replacing the weights a_k in (19) by $\frac{1}{2}kp_k\mathrm{d}\Phi(\delta)$ it is clear that (39) will again yield a weighted quasilinear mean. Therefore, we may draw the following conclusion: Given the wealth-independence of the distribution of the spousal wealth-disparity parameter δ, the equilibrium distribution of wealth must have an upper tail that belongs to the extended Pareto class, as in equation (20).

Take a specific example in which the pattern of families is the same again as it was in Case 2 of Table 1 above, and in which the distribution of the wealth-disparity parameter is very simple - only one value of δ is possible. Now, instead of the two partners having equal wealth before marriage, suppose that one partner is just twice as wealthy as the other ($\delta = 1$), and consider the effect that this would have on the degree of inequality of the equilibrium wealth distribution in the model. Although this important modification to the model will not change the resulting functional form of the wealth distribution, it will change the particular member of the class of equilibrium wealth distributions that corresponds to a specific instance of the family-size distribution $\mathbf{p}$. For example, take the bequests made to children of families with a combined wealth-level of 2 units. In the basic model of section 3, 15 percent of their descendants leave 2β, 45 percent leave β, 30 percent leave $\frac{2}{3}\beta$ and 10 percent leave $\frac{1}{2}\beta$. But now, in each of these four groups of descendants a proportion λ will marry partners twice as wealthy, and so the average wealth will be 1.5 times the climber's premarital wealth: the remaining $1-\lambda$ will marry partners half as wealthy, and for them the average wealth after marriage will be only three-quarters of their

Figure 6: The (α, β)-relationship and marriage



own pre-marital wealth.

The value of λ that is consistent with the assumed marriage rule in equilibrium will depend on the distribution of wealth. Since this modified model implies that wealth has a Pareto distribution with parameter α, and since there must be $[1+\delta]^\alpha$ times as many marriages of spouses in the range $[1 \pm \epsilon]$ with spouses in the range $\frac{[1\pm\epsilon]}{[1+\delta]}$ as with spouses in the range $[1 \pm \epsilon][1+\delta]$, this is only possible if $\lambda = 1/\left[1 + [1+\delta]^\alpha\right]$.

If the family size-distribution parameters $\mathbf{p}$ satisfy the stationarity condition (1) the general implicit equation giving α as a function of β and δ may be written:

$$\sum_{k=1}^{K} p_k k^{1-\alpha} = \frac{1 + [1+\delta]^\alpha}{\beta^\alpha [2+\delta]^\alpha}. \tag{40}$$

Clearly the assumptions of this modified model will change the relationship between the parameter β and the equilibrium inequality of wealth distribution (which is inversely related to the Pareto coefficient α). Take, for example, case 2 of the family-size distribution $\mathbf{p}$ in Table 1. By solving for α from (40) for alternative values of β and δ, we obtain Figure 6. Quite modest increases in spousal wealth-disparity have a considerable impact on α and hence on equilibrium inequality. For example an increase in δ from 0.2 to 0.5 increases the value of α by 5.6% to 23.1%, depending on the value of β. Figure 6 also gives us a way of calibrating the inequality-increasing effect of a move toward assortative mating. If the status quo were $\delta = 1$ (everyone marries someone with twice [half] as much wealth), then the switch to $\delta = 0$ (strict assortative mating) has roughly the same impact on α as a rise in the growth factor on wealth β from 0.9 to 0.95.

5 Applications and policy

5.1 Changing structures: The puzzle of the French Revolution

As we have seen in the introduction, the Marquis de Condorcet was an optimist on equality. Indeed during the French revolution there was a widespread belief that constitutional and legal reform would provide the key eliminating inherited inequalities (Piketty 2014b, pages 363-364). In particular the abolition of the form of primogeniture that had existed in pre-revolutionary France was seen as a powerful force for change. Perhaps the Napoleonic Code would generate a more equitable society through a change in inheritance rules. However, the results in terms of wealth inequality appear to have been rather modest.

We can throw some light on this by posing two counterfactuals within the context of our inheritance model. (A) What if the Napoleonic Code had not abolished primogeniture? (B) What if the post-revolutionary population structure had been different? We use estimates on the distribution by family size available from Lévy and Henry (1960)[18] and the version of our standard model that allows for favoring the firstborn.[19]

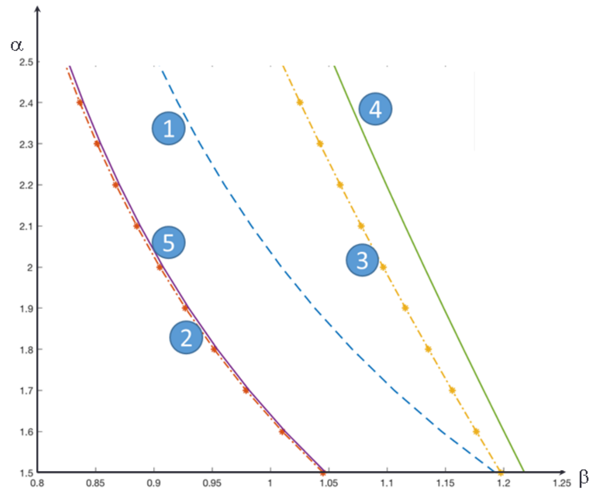
Figure 7 is the counterpart to Figure 1, showing the situation before and after the French revolution and the two counterfactuals. Curve 1 represents the (α, β)-curve for pre-revolutionary France (1771-1780), based on the Lévy-Henry data and taking our approximation to primogeniture. The move from curve 1 to curve 2 shows counterfactual A: what would have happened to the equilibrium wealth distribution if the inheritance rules in France had remained unchanged, but the population structure had changed to that observed in the early 19th century (1801-1805). Under counterfactual A it is clear that for any β there is a much lower α, so that equilibrium wealth inequality would have undoubtedly increased substantially. The actual change that occurred is depicted by the shift from curve 1 to curve 3, which shows post-revolutionary France, allowing for the changes in the population structure and the change in the inheritance rules. Since curve 3 lies above curve 1 almost everywhere it is clear that equilibrium

[18] The three distributions used for Figure 7 are shown in the accompanying table. $\mathbf{p}^{\text{pre}}$ comes directly from Lévy and Henry (1960). The data concern fertility of women aged above 45 and the population proportions were adjusted to eliminate households without children: this results in 3.77 children per woman. The fertility of women aged 15-45 between 1771-80 was 15% higher than between 1801-05. Distribution $\mathbf{p}^{\text{post}}$ for 1801-05 is obtained from $\mathbf{p}^{\text{pre}}$ where the households with 7 children have been re-assigned to have only one child: this leads to a reduction in the number of children per woman to 2.62 – a reduction of 15%, a number that is plausible – see Bourgeois-Pichat (1951). Distribution $\mathbf{p}^{\text{adj}}$ is formed from $\mathbf{p}^{\text{post}}$ by adjusting the distribution to maintain the same proportion of little emperors ($k = 1$) as in $\mathbf{p}^{\text{pre}}$, while maintaining exactly the same number of children per woman as $\mathbf{p}^{\text{post}}$.

k	1	2	3	4	5	6	7
$\mathbf{p}^{\text{pre}}$	0.231	0.231	0.154	0.154	0.154	0	0.077
$\mathbf{p}^{\text{post}}$	0.308	0.231	0.154	0.154	0.154	0	0
$\mathbf{p}^{\text{adj}}$	0.231	0.308	0.192	0.154	0.115	0	0

[19] See section 4.3; we assumed $\xi = 10$ for the favoritism parameter in pre-revolutionary society and $\xi = 0$ for the situation where the Napoleonic code applies.

Figure 7: The French revolution and two counterfactuals



wealth inequality decreased for all values of β, but perhaps not by much.

Why, then was the reduction in equilibrium wealth inequality not larger? Part of the answer lies in the change in the population structure that occurred just after the French revolution: there appears to have been a substantial increase in the proportion of little emperors, which would have boosted equilibrium wealth inequality. So, for Counterfactual B we construct a post-revolutionary distribution of families by size that adjusts downwards the proportion of little emperors to the pre-revolutionary levels, but that allows for the change in inheritance rules under the Napoleonic code. This would take the (α, β)-curve from curve 1 to curve 4: in this case it is clear that equilibrium inequality would have increased α – reduced equilibrium inequality – for every β, even more than the actual change (curve 1 to curve 3). Finally we can check what would have happened if the conditions in both counterfactual A and counterfactual B were applied (no change in the inheritance rule, and no change in the proportion of emperors): this results in curve 5, very close to curve 2.

5.2 Taxation and redistribution

The reduction of inequality is often seen as a policy objective and taxation is often used as a means to achieving that objective. There is a conventional argument that taxes on wealth or on the transfer of wealth through bequests can only have a very limited impact on inequality because of the small sums that that such taxes can raise. The conventional argument is as follows. Consider a tax at a constant marginal rate τ_W on wealth in excess of a given break-even level; wealth below this level is subsidized at the same marginal rate. This results in the following relation between post-tax wealth W' and pre-tax wealth

W :

$$W' = [1 - \tau_W] W + \tau_W \theta \mu, \tag{41}$$

where $\theta\mu$ is the break-even wealth level defined as a proportion θ of mean pretax wealth μ. The conventional (relative) Gini coefficient is the absolute Gini divided by the mean. Under the tax/subsidy scheme in (41) the proportionate reduction in inequality is given by[20]

$$\frac{\tau_W \theta}{1 - \tau_W + \tau_W \theta}.$$

So, if the break-even level is 10 percent of the mean, a one percent tax on wealth induces a nominal proportionate reduction in the Gini coefficient of 0.1 percent; if the break-even level is 25 percent of the mean, a one percent tax on wealth induces a nominal proportionate reduction in the Gini coefficient of 0.25 percent.

However, if an equilibrium distribution exists, a tax on wealth or on wealth transfers will also affect the equilibrium and hence the inequality of wealth in the upper tail. If a tax τ is imposed on bequests we find optimal pre-tax bequests (given by equation 10) remain unchanged and that equation (12) now becomes

$$I_t = 2B_{t-1} [1 - \tau] / k. \tag{42}$$

The wealth-dynamics equation (13) remains the same but the exogenous growth factor β is replaced by

$$\tilde{\beta} := s[1 + g][1 - \tau] = \beta[1 - \tau], \tag{43}$$

and equilibrium α is determined as the root of

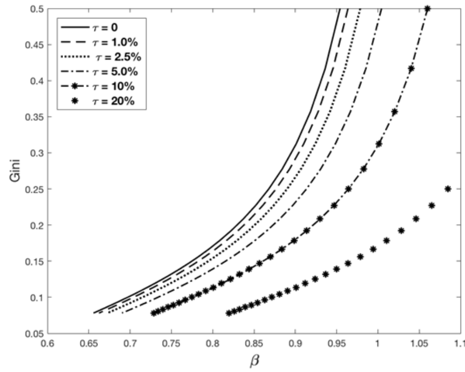
$$\sum_{k=1}^{K} p_k \left[\frac{k}{2}\right]^{1-\alpha} = \tilde{\beta}^{-\alpha}. \tag{44}$$

The implications for long-run equilibrium are immediate: if an equilibrium exists then introducing the bequest tax will reduce $\tilde{\beta}$ and so will increase the equilibrium α; this means that the bequest tax must reduce (pre-tax) bequest- and wealth-inequality in the upper tail.[21] It may also mean that, in a situation where the absence of tax would allow an explosive growth of wealth in the upper tail, introducing the tax will ensure the existence of an equilibrium distribution. Using the standard result that the Gini coefficient for a Pareto distribution with parameter α is given by $\frac{1}{2\alpha-1}$ we can use equations (43) and (44) to plot the relationship between equilibrium inequality and the underlying β. Figure 8 shows this for different rates of inheritance tax. So, for example if $\beta = 0.9$ equilibrium

[20]This follows because the after-tax Gini coefficient of wealth G' is related to the before-tax Gini G as follows: $G' = [1 - \tau_W] G\mu / [[1 - \tau_W] \mu + \tau_W \theta \mu]$.

[21]In our model we do not find the phenomenon suggested by Stiglitz in a multi-generational model with general equilibrium effects, whereby bequest taxation has an effect on savings that actually increases inequality (Stiglitz 1976, 1978)

Figure 8: Equilibrium inequality and inheritance tax



Gini in the absence of tax is 0.311; with a tax rate of 1%, then equilibrium Gini is reduced by 6.7 percent to 0.291. If the tax rate goes up to 2.5% the Gini falls to about 0.266, 15 percent of no-tax equilibrium inequality, and if the tax rate goes up to 20%, equilibrium Gini falls to 0.113.

5.3 Family policy and long-run inequality

Sometimes state actors have used control of family size as a policy instrument for a variety of reasons. A side-effect that is sometimes overlooked is the implication on long-run wealth inequality through the effect of changing the size distribution of potential inheritors. A simple example of this can be drawn from Figure 1: if $\beta = 0.9$ a family policy that changed the distribution of families by size from case 3 to case 2 in Table 1 (no families with more than four children, maintaining a stationary population) would have the effect of increasing equilibrium α from 1.90 to 2.11: this means a drop in the upper-tail Gini coefficient of wealth from 0.357 to 0.311.

This example is oversimplified because, by construction, we are comparing stationary populations. As we saw in section 4.2, it is necessary to adjust β in order to make coherent comparisons between populations that have substantially different growth rates. So, in assessing the impact of family policy upon equilibrium wealth inequality, one can break this down into (i) into a part that is attributable to the change in the growth rate of population $(\pi - 1)$ and (ii) a part that is attributable to a change in the size distribution of families.

6 Conclusions

Condorcet was wrong: even in the absence of institutions and customs that favor one sex or favor the firstborn, the inheritance process will generate and perpetuate inequality in the long run.

Pareto was right: in a simple inheritance model the equilibrium distribution of wealth must have a Paretian tail. But where the tail begins depends on the exact form of the inheritance rules, on the labor market and on the support provided to low-wealth households through the tax system. Equilibrium wealth inequality depends on the savings rate, the growth rate of wealth, the inheritance tax rate and the distribution of families by size. But if the growth rate of wealth is too high – if condition (31) is violated – then no equilibrium distribution exists.

These conclusions are remarkably robust. We show that, in almost all cases, the equilibrium distribution has a Pareto tail. The result holds when parents divide their inheritance differently, to take account of needs amongst their children, or to favor one or more children on the basis of their birth order or their sex. It is also true under a variety of marriage patterns. The model helps explain how wealth inequality can emerge even when each generation follows apparently egalitarian practices. It shows how inheritance taxation at very modest rates can restrain the growth of wealth inequality or reduce the equilibrium level of inequality. It also helps to clarify the long-run impact of family policy on inequality.

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Appendix

Here we characterize the equilibrium distribution in the basic model for the sub-case where $E = 0$. We assume that wealth satisfies $W \in \overline{\mathbb{W}}$, where $\overline{\mathbb{W}}$ is a proper interval that does not contain zero. So, in principle, $\overline{\mathbb{W}}$ may be open or closed, wholly to the right of the origin and bounded or unbounded above, or wholly to the left of the origin and bounded or unbounded below. We also assume (a) there is a bounded interval $\mathbb{W} \subseteq \overline{\mathbb{W}}$ over which the equilibrium distribution function F_* is continuous and (b) that $1 \in \mathbb{W}$. In practice this second assumption involves no loss of generality since one can rescale all wealth values by multiplying through by a constant.

Theorem 1 *The function F_* in (19) must satisfy either*

$$F_*(W) = A \log(W) + B,\ W \in \mathbb{W}, \tag{45}$$

or

$$F_*(W) = AW^{\alpha} + B,\ W \in \mathbb{W}, \tag{46}$$

where A, α are nonzero real numbers.

Proof. By definition of a distribution function $W' \geq W \iff F_*(W') \geq F_*(W)$. There are two cases: (1) F is constant over $\mathbb{W}$; (2) there exist some $W, W' \in \mathbb{W}$ such that $F_*(W') > F_*(W)$. Case (1) is trivial since it means that there is no-one with wealth in $\mathbb{W}$. In case (2), because of the assumed continuity of F_* over $\mathbb{W}$, there must be an interval $\mathbb{W}' \subseteq \mathbb{W}$ for which F_* is increasing. Equation (17) implies

$$W = F_*^{-1}\left(\sum_{k=1}^{K} a_k\ F_*(W_k)\right), \tag{47}$$

where $W_k := b_k W$. In other words the equilibrium condition for the wealth interval is equivalent to requiring that the younger generation's wealth be a quasilinear weighted mean of K values of wealth in the older generation, where the quasilinear mean is constructed using the (transformed) distribution function. However we can further restrict the distribution function F_*.

For some positive scalar $\lambda \neq 1$ we must have:

$$\lambda W = F_*^{-1}\left(\sum_{k=1}^{K} a_k\ F_*(\lambda W_k)\right), \tag{48}$$

or

$$W = H^{-1}\left(\sum_{k=1}^{K} a_k\ H(W_k)\right), \tag{49}$$

where $H(W) := F_*(\lambda W)$. Using (48) and (49) we have

$$F^{-1}\left(\sum_{k=1}^{K} a_k\ F_*(W_k)\right) = H^{-1}\left(\sum_{k=1}^{K} a_k\ H(W_k)\right). \tag{50}$$

Without loss of generality we may take $a_1, a_2 > 0$, and set $W_3, ..., W_K$ equal to some arbitrary constants $\bar{W}_3, ..., \bar{W}_K$. Then introduce:

$$z_i := a_i F(W_i),\ i = 1, 2. \tag{51}$$

We now have:

$$W_i = F_*^{-1}\left(\frac{z_i}{a_i}\right),\ i = 1, 2. \tag{52}$$

Since

$$H\left(F_*^{-1}\left(\sum_{k=1}^{K} a_k F_*\left(W_k\right)\right)\right) = \sum_{k=1}^{K} a_k\ H\left(W_k\right),$$

we have

$$H\left(F_*^{-1}\left(z_1 + z_2 + \sum_{k=3}^{K} a_k\ F_*\left(\bar{W}_k\right)\right)\right) = a_1 H\left(W_1\right) + a_2 H\left(W_2\right) + \sum_{k=3}^{K} a_k\ H\left(\bar{W}_k\right).$$

and so:

$$H\left(F_*^{-1}\left(z_1 + z_2 + \sum_{k=3}^{K} a_k\ F_*\left(\bar{W}_k\right)\right)\right) - \sum_{k=3}^{K} a_k\ H\left(\bar{W}_k\right)$$
$$= a_1 H\left(F_*^{-1}\left(\frac{z_1}{a_1}\right)\right) + a_2 H\left(F_*^{-1}\left(\frac{z_2}{a_2}\right)\right). \tag{53}$$

Also define functions Ξ, Ψ, Φ:

$$\Xi(z) := H\left(F_*^{-1}\left(z + \sum_{k=3}^{K} a_k\ F_*\left(\bar{W}_k\right)\right)\right) - \sum_{k=3}^{K} a_k\ H\left(\bar{W}_k\right), \tag{54}$$

$$\Psi(z_1) := a_1 H\left(F_*^{-1}\left(\frac{z_1}{a_1}\right)\right), \tag{55}$$

$$\Phi(z_2) := a_2 H\left(F_*^{-1}\left(\frac{z_2}{a_2}\right)\right). \tag{56}$$

Using these variables and functions we may rewrite (53) as:

$$\Xi(z_1 + z_2) = \Psi(z_1) + \Phi(z_2),\ z_1, z_2 \in Z, \tag{57}$$

which is a restricted Pexider equation. The standard solution to this is

$$\Psi(z) = hz + C_1, \Phi(z) = hz + C_2, \Xi(z) = hz + C_1 + C_2, \tag{58}$$

where h, C_1, C_2 are constants, with $h > 0$. So, since F and H are strictly increasing functions, (55) and (58) yield:

$$a_1 H\left(F_*^{-1}\left(\frac{z_1}{a_1}\right)\right) = hz + C_1$$

$$H\left(F_*^{-1}\left(\frac{z_1}{a_1}\right)\right) = h\frac{z_1}{a_1} + \frac{C_1}{a_1}.$$

Using the definition of $z_1$we see

$$H(W) = hF_*(W) + g,$$

where $g := \frac{C_1}{a_1}$. In the present case it is clear from (48) and (49) that the values g and h may depend on the value of λ that had been chosen, so that we may write:

$$F_*(\lambda W) = h(\lambda)F_*(W) + g(\lambda), \forall\lambda \text{ and } W, \lambda W \in \mathbb{W}. \tag{59}$$

There are two cases to consider in solving (59). First, if h is independent of λ (let us say $h(\lambda) = 1$ for all λ), then (59) implies

$$\bar{F}(\lambda W) = \bar{F}(W) + \bar{F}(\lambda), \tag{60}$$

where

$$\bar{F}(W) := F_*(W) - F_*(1). \tag{61}$$

In view of the assumed continuity of F_*, equation (60) implies:

$$\bar{F}(W) = A\log(W),\ A \neq 0. \tag{62}$$

Alternatively, if h is not independent of λ we must have:[22]

$$h(\lambda W) = h(\lambda)h(W), \forall\lambda \text{ and } W, \lambda W \in \mathbb{W}, \tag{63}$$

which implies $h(W) > 0$ for all $W \in \mathbb{W}$.[23] Take the following logarithmic transformations of the variables and of the function h

$$\left.\begin{array}{c} \phi(y) := h(\log(W)), \\ y := \log(W)\ z = \log(\lambda), \end{array}\right\} \tag{64}$$

and define $\mathbb{Z} := \{\log W :\ W \in \mathbb{W}\}$. We then find that (63) becomes

$$\phi(y + z) = \phi(y) + \phi(z),\ \forall y, z, y + z \in \mathbb{Z}, \tag{65}$$

which is the standard Cauchy equation on a hexagon. Since $1 \in \mathbb{W}$, we have $0 \in \mathbb{Z}$, and under these conditions (65) has a unique extension from Z to $\mathbb{R}$.[24] The solution to (65) is then:[25]

$$\phi(y) = \alpha y,\ \alpha \in \mathbb{R}\backslash\{0\},\ \forall y \in \mathbb{Z}, \tag{66}$$

which implies

$$h(W) = W^\alpha,\ W \in \mathbb{W}. \tag{67}$$

■

[22]See Aczél (1987) page 26.

[23]The reason for this is that $1 \in \mathbb{W}$, so that if $W \in \mathbb{W}$, then (63) implies $\sqrt{W} \in \mathbb{W}$ and, using (63) again, $h(W) = h(\sqrt{W})^2 \geq 0$. However, $h(W)$ cannot be zero for any $W \in \mathbb{W}$ without violating the strict monotonicity of h over $\mathbb{W}$.

[24]See, for example, Aczél and Dhombres (1989) Chapter 7, Theorem 5.

[25]See Aczél (1987), page 20.