Price Competition Among Differentiated Products:
A Detailed Study of a Nash Equilibrium

by

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PRICE COMPETITION AMONG DIFFERENTIATED PRODUCTS:
A DETAILED STUDY OF A NASH EQUILIBRIUM

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The present paper explores a model in which each of a number of consumers, who differ in income, chooses to purchase one unit of some one of a set of substitute goods. These are produced at zero cost by various firms, and their different levels of quality are exogenously fixed. Our focus of interest lies in analysing price competition between rival products.¹

We begin by developing conditions which ensure the existence and uniqueness of a non-cooperative price equilibrium (Nash equilibrium).

We then proceed to examine the question of entry of new products. First, we develop conditions which determine whether a single additional product, whose quality is below that of existing products, can enter. We then proceed to show that an upper bound exists to the number of products which can survive with positive

¹. Our model extends and generalizes the analysis of Gabszewicz and Thisse [2,3]. The prior question of the optimal choice of quality by firms has been explored by Shaked and Sutton [4].
market shares at equilibrium: competition between the producers of the 'surviving' high quality products drives their prices down to a level such that no consumer prefers to purchase one of the 'excluded' lower quality products, even at price zero. Then, we turn to the effect of the entry of a single new product whose quality is superior to that of existing products; such a product may always enter, but may induce the exit of a low quality product; we examine the effect of such a process on equilibrium prices.

Lastly, we explore a special case of the model, in which the utility function has a linear form, and where the distribution of consumer incomes is uniform. (A further specialization of this case, in which the qualities of rival products were 'equispaced', was analysed in Jaskold Gabszewicz and Thisse [3]). This linear case allows us to explore the entry of a sequence of high quality products. We show that such a process will lead to the convergence of prices to cost, i.e. zero, in the limit, only if the qualities of the rival products converge, in a suitably defined sense.

The mathematical structure of the present model is of some interest in its own right, as it affords an example of a game with a large number of players - possibly an infinite sequence - in which, at a Nash Equilibrium, only a finite subset 'play', in the following sense: the outcome is independent of the strategies of these 'excluded' players, and there exists some upper bound to the number
of 'included' players, which is independent of the list of players (i.e. the number of firms and the distinct qualities of their products).

1. The Model

A number of firms produce distinct substitute goods. We label their products by an index \( k = 1, \ldots, n \) where firm \( k \) sells product \( k \) at price \( P_k \).

A continuum of consumers, identical in tastes, have differing incomes; the distribution of incomes is described by the density function \( f(\cdot) \) on a support \([a,b]\), \( 0 < a < b \).

Now consumers make indivisible and mutually exclusive purchases from among the \( n \) goods, in the sense that a consumer either makes no purchase, or else buys exactly one unit from one of the \( n \) firms. We denote by \( U(t,k) \) the utility achieved by consuming one unit of product \( k \) and \( t \) units of 'income' (the latter may be thought of as a Hicksian 'composite commodity', measured as a continuous variable); and by \( U(t,0) \) the utility derived from consuming \( t \) units of income only.

We may illustrate this utility function \( U(t,\cdot) \) as shown in Figure 1.
We assume $U(0,k) = 0$; the consumer must consume some 'composite commodity' to achieve positive utility.

The horizontal distance $d$ between $U(t,k)$ and $U(t,\ell)$ at height $u_0$ represents the price increment which a consumer currently consuming product $\ell$ is just willing to pay for $k$ rather than $\ell$. We define a function $r^{k,\ell}(d)$ of the price difference $d$, being the income remaining, after his purchase of good $\ell$, to a consumer who is just willing to pay the price increment $d = p^k - p^\ell$ in order to consume the higher quality good $k$. 

Figure 1: The utility function $U(t,\cdot)$
(Product $k$ is of higher quality than $\ell$)
We introduce the following Assumptions:

A1: \[ f(t) \geq \frac{f}{b} > 0, \quad \forall t \in [a, b]; \]
\[ f(t) = 0, \quad \forall t \notin [a, b], \quad 0 < a < b. \]

A2: \[ \left[ f(t) \right]^2 + f'(t) \cdot \int_t^b f(x) \, dx > 0 \]

A3: \[ U_t(x^k, \ell - d, \ell,k) > U_t(x^k, \ell,k) \]

or equivalently
\[ r_{d}^{k, \ell} > 1 \]

A4: \[ \frac{d}{r_{x^k, \ell}} \bigg|_{d} < 0 \]

A5: \[ \frac{1}{b} \left( r_{d}^{k, \ell} - 1 \right) > \left| \frac{r_{x^k, \ell}}{d} \right| \]

A6: \[ \exists \lambda \in [0, 1] \text{ such that} \]

(1) \[ \lambda \left[ f(t) \right]^2 + f'(t) \int_a^t f(x) \, dx > 0 \]

and (ii) \[ \frac{1 - \lambda}{b} r_{x^k, \ell} \geq \left| \frac{r_{x^k, \ell}}{d} \right| \]

Assumption 1 requires that \( f \) be bounded below by some \( \frac{f}{b} > 0, \)

and that its support be bounded away from zero. This is used in
demonstrating the 'finiteness' property; were it violated it is possible
that an infinite sequence of firms might survive, with positive market
shares, and prices, converging to zero.

Assumption 3 states that richer consumers are willing to pay more for a given improvement in quality, i.e. that \( r^{k,2}(d) \) increases with \( d \).

The latter form of the assumption makes it clear that the property we require is ordinal. In terms of the \( U(t, \cdot) \), it states that the curves become steeper as we move to the left along a horizontal.

Assumption 4 further requires that the increment they are willing to pay rises no more than proportionately with increases in their income. The most natural way of thinking of this restriction, perhaps, is by imagining the present industry as being one among many; and as a consumer's income increases he may begin to spend part of the increment in starting to consume new types of good produced in other industries.

Assumptions 5 and 6, bounding \(|r_{d}\dd|\), require that the utility function does not depart too far from the linear case in which \( r_{d}\dd = 0 \), as when the functions \( U(t,k) \) are rays through the origin. (Section 6 below). In Assumption 6, the degree to which the utility function may depart from linearity is related to the distribution of income. The higher is \( \lambda \), the looser is the restriction on \( f(t) \), and the more stringent is the requirement on the utility function; and conversely.
In the linear case, \( r_{dd} = 0 \), we can set \( \lambda = 1 \) in Assumption 6, whence it becomes similar in form to Assumption 2. For a linear utility function, the uniform, (truncated) normal and lognormal distributions satisfy Assumptions 2, 6 (taking \( \lambda = 1 \) in A.6). (This follows immediately from the observation that \( \frac{f(t)}{\int_t^\infty f(x)dx} \) is increasing, for these distributions). In order to permit an extension to non-linear utility functions, we need to satisfy A.6. with \( \lambda < 1 \). This is feasible so long as \( \left[ \frac{f(t)}{f'(t)} \right]^2 - f'(t) \int_a^t f(x)dx > 0; \) this is satisfied for the uniform and (truncated) normal distributions, and for the (truncated) lognormal on support \([a,b] \), \( a > 0 \).

2. Existence of Equilibrium

For any vector of prices, consumers are partitioned into bands corresponding to the market shares of various firms. The market share of firm \( k \) corresponds to an interval of incomes, which, except for some 'top' and 'bottom' product, is bounded by the shares of two firms which we may label \( k - 1 \) and \( k + 1 \); the marginal consumer who is indifferent between \( k - 1 \) at price \( p^{k-1} \) and \( k \) at price \( p^k \) is of income

\[
t_{k-1,k} = p^{k-1} + r^{k-1,k}(p^k - p^{k-1})
\]
(It follows from Assumption 3 that a consumer of income greater than \( t^{k-1,k} \) strictly prefers good \( k \), and vice versa.)

Trivially, among those products which retain positive market shares, those of higher quality command a higher price, and, by virtue of Assumption 3, the market share of a higher quality product corresponds to a higher income band.

The existence of equilibrium is established in

**THEOREM 1**: Under Assumptions 1, 2, 3, 5, 6 for any \( n > 1 \), there exists a non-cooperative price equilibrium (i.e. a Nash Equilibrium).

**PROOF**: We wish to establish that, for any given vector of prices for competing products, the revenue function of firm \( k \) is a continuous single peaked (quasi-concave) function of its price \( p^k \). The existence of equilibrium then follows immediately [1].

Without loss of generality, assume that those products having a positive market are labelled \( 1...k-1, k+1...n \), and their respective prices denoted as \( (\hat{p}^1, ..., \hat{p}^{k-1}, \hat{p}^{k+1}, ..., \hat{p}^n) \). For \( p^k \) sufficiently high the sales of firm \( k \) are zero; similarly, for \( \hat{p}^k = 0 \), revenue equals zero.
We note that the market share of firm $k$ is sandwiched between that of two neighbouring firms, $k-1$ and $k+1$. As its price falls, it will at some point squeeze out one or both of these neighbouring firms, thus acquiring a new 'neighbour'.

Consider the function

$$R^k = p^k \int_{t_{k-1,k}}^{t_{k,k+1}} f(t) dt = p^k M_k \quad \text{say}$$

which is formally defined for all $p^k$, and which coincides with the profit of firm $k$ over that range of $p^k$ such that firm $k$ has a positive market share bounded by $(k-1)$ and $(k+1)$. It may be shown, by differentiating and applying Assumptions 1, 2, 3, 5, that $R^k$ thus defined is a single peaked function of $p^k$. This is established in LEMMAS A1, A2 in the Appendix.

Suppose now that $p^k$ falls so far as to drive one of its neighbours, $(k-1)$ say, out of the market. Then its new neighbours are $(k-2), (k+1)$. Again, the profit function $R^k$ for $k$ sandwiched between $(k-2)$ and $(k+1)$ is a single peaked function of $p^k$. The price at which the market share of $k-1$ becomes zero, which we label $p_{k-1}^k$, is implicitly defined by
\[ t_{k-1, k-2} = p^{k-2} + r_{k-2, k-1} (p^{k-1} - p^{k-2}) = p^{k-1} + r_{k-1, k} (p^{k-1} - p^{k-2}) = t_{k-1, k} \]

In order to establish that the revenue function is globally single peaked, it is sufficient to show that at the price \( p^{k}_{k-1} \) thus defined, \( R^{k} \) is steeper than \( R^{k} \), so that, if \( R^{k} \) is increasing at this point, then a fortiori \( R^{k} \) is increasing.

To show this, we note that the derivative of \( R^{k} \) at \( p^{k}_{k-1} \) is (remembering \( t_{k-1, k-2} = t_{k-1, k} \)),

\[ M^{k} + p^{k}_{k-1} f(t^{k, k+1}) (1 - r_{d}^{k, k+1}) - p^{k}_{k-1} f(t^{k-2, k-1}) r_{d}^{k-1, k} \]

We wish to show that, where this is positive, it is less than the derivative of \( R^{k} \), i.e.

\[ M^{k} + p^{k}_{k-1} f(t^{k, k+1}) (1 - r_{d}^{k, k+1}) - p^{k}_{k-1} f(t^{k-2, k-1}) r_{d}^{k-2, k} \]

Comparing these two expressions, it follows that it suffices to show that

\[ r_{d}^{k-1, k} (p^{k}_{k-1} - p^{k-1}) > r_{d}^{k-2, k} (p^{k}_{k-1} - p^{k-2}) \]

Let \( s = p^{k}_{k-1} - p^{k-1} \) and \( v = p^{k}_{k-1} - p^{k-2} \). We then have

\[ r^{k-1, k}(s) + v - s = r^{k-2, k}(v) \]

(see Figure 2) from which we deduce that
\[
\frac{ds}{dv} = \frac{r_d^{k-2,k}(v) - 1}{r_d^{k-1,k}(s) - 1}
\]

But by virtue of Assumption 3 it follows immediately that this derivative is less than unity, whence the desired inequality follows.

Finally, the above proof remains valid for \( k = 1 \), provided \( k = 0 \) is interpreted as a product sold at price \( \hat{p}_0 = 0 \), and for \( k = n \) provided \( t^{k,k+1} \) is set equal to \( b \) in the definition of \( R^k \).

Q.E.D.

Figure 2
A basic property of equilibrium is stated in

**Theorem 2**: Let \((\bar{p}^1, \ldots, \bar{p}^n)\) be an equilibrium price vector. Then

(i) \(M^{k+j}(\bar{p}^1, \ldots, \bar{p}^n) > 0 \) if \(M^k(\bar{p}^{1}, \ldots, \bar{p}^n) > 0\),

\[ k = 1, \ldots, n, \ j = 1, \ldots, n-k; \]

(ii) \(\frac{\bar{p}^{k+j}}{\bar{p}^k} > 1 \) if \(M^k(\bar{p}^{1}, \ldots, \bar{p}^n) > 0\),

\[ k = 1, \ldots, n, \ j = 1, \ldots, n-k \]

**Proof**: (i) Assume, on the contrary, that \(M^{k+j} = 0\) for some \(j\), so that the revenue of seller \(k + j\) is equal to zero. Then, by setting a price equal to \(\bar{p}^k\), seller \(k + j\) could earn positive revenue, contradicting the optimality of \(\bar{p}^{k+j}\).

(ii) This is immediate, for if \(\bar{p}^{k+j} < \bar{p}^k\), then \(M^k = 0\), which is a contradiction.

Q.E.D.

Since this implies that the left and right hand neighbours of firm \(k\) are indeed \(k-1\) and \(k+1\) respectively, at equilibrium, we will ease the burden of notation by writing \(t^{k-1,k}\) as \(t^k\), in what follows. When \(t^k = a\), we suppose that the market share of \(k-1\) is equal to \(\{a\}\).

We may delete the set of products whose market shares are empty. An equilibrium for the original set of products remains an equilibrium for the reduced set of products then remaining.

The case where some product has \(t^k = a\) requires separate comment. Here the availability of product \(k-1\) does affect market equilibrium; if we remove it, the equilibrium prices of the remaining
products change. But if, at equilibrium, $t^k = a$ for some $k$, then product $k-1$ necessarily has $p^{k-1} = 0$. Thus the equilibrium prices of goods $n, \ldots, k$ coincide with their prices in an equilibrium where the quality of the zero good equals $u_{k-1}$.

3. Uniqueness of Equilibrium

We now proceed to characterize $\bar{p}_1, \ldots, \bar{p}_n$ as the solution to a system of $n$ equations, where possibly the first equation is replaced by two inequalities.

For $k > 1$, the function $R^k$ is continuously differentiable at $\bar{p}^k$. Indeed, as $M^{k-1}(\bar{p}^1, \ldots, \bar{p}^n)$ and $M^{k+1}(\bar{p}^1, \ldots, \bar{p}^n)$ are positive, the revenue of firm $k$ takes its maximum at a point where it is defined by $R^k = p^k \int_{t_k}^{k+1} f(t) dt$. For $k = 1$, however, $R^1$ is not differentiable when $p^1$ takes the value $p^1_a$, defined as the solution to $a = \alpha_{0,1}(p^1)$, but the left and right hand derivatives exist. If $\bar{p}^1 \neq p^1_a$, then $\bar{p}^1, \ldots, \bar{p}^n$ is a solution to the system of equations

\[ u^k = M^k + p^k f(t^{k+1}) \left( 1 - r^k \left( p^{k+1} - p^k \right) \right) - p^k f(t^k) r^k \left( p^k - p^{k-1} \right) = 0 \]

\[ k = 1, \ldots, n-1 \]

and

\[ l^k = s^n - p^n f(t^n) r^{n-1, n} \left( p^n - p^{n-1} \right) \]

where $s^n$ denotes the market share of firm $n$, with upper bound $b$. 
If, on the other hand, $\overline{p}^1 = \overline{p}^1_a$ then $\overline{p}^1, \ldots, \overline{p}^n$ is a solution to

$$\lim_{p^1 \to \overline{p}^1_a} J^1 < 0 \quad \text{and} \quad \lim_{p^1 \to \overline{p}^1_a} J^1 > 0$$

$$J^k = 0 \quad \text{for} \quad k = 2, \ldots, n-1$$

$$J^n = 0$$

We begin by using the fact that $t^1 = x^0, 1, 1(t^1)$, from (1), to define the function $p^1 = p^1(t^1)$. Again from (1), we have $t^1 = p^1 + r^1, 2(p^2 - p^1)$; substituting $p^1 = p^1(t^1)$ here, we define the function $p^2 = p^2(t^2, t^1)$. Substituting $p^1(t^1)$ and $p^2(t^2, t^1)$ into $J^1 = 0$, i.e. provided $p^1 \neq p^1_a$, we obtain $J^1(t^1, t^2) = 0$ for $t^1 \neq a$. Continuing in a similar manner, we define $p^k = p^k(t^k, \ldots, t^1)$ and $p^{k+1} = p^{k+1}(t^{k+1}, \ldots, t^1)$ which are substituted into $J^k = 0$ to yield $J^k(t^{k+1}, t^k, \ldots, t^1) = 0$.

The equation $J^1(t^2, t^1) = 0$ is now used to define $t^2$ as a function of $t^1$, labelled $F(t^1)$. (Note that $F(t^1)$ is not defined at $a$.) The system of equations $J^k = 0$, $k = 2, \ldots, n-1$, $J^n = 0$ is used to define $t^2$ as a function of $t^1$, labelled $G(t^1)$. We now characterize the functions $F$ and $G$, respectively, in LEMMAS 1, 2 whose proofs are given in the Appendix.
LEMMA 1: There exists an interval \([0, C]\) and a unique increasing function \(F(t^1)\) defined on \([0, C] - \{a\}\) such that:

(i) \(F(0) = 0\), \(t^1 \in [0, C] - \{a\}\);

(ii) \(F(0) = a\),

(iii) If \(C > a\), \(F\) is two valued at \(t^1 = a\) and \(F(a^-) < F(a^+)\)

(iv) \(J^1(F(a^-), a^-) = J^1(F(a^+), a^+) = 0\)

The form of the function \(F(t^1)\) is illustrated in Figure 3.

We denote by \(C\) the value of \(t^1\) at which \(F(t^1)\) reaches \(b\); the three cases correspond to \(C < a\), \(C = a\) and \(C > a\) respectively.

Figure 3: The function \(F(t^1)\)
LEMMA 2: Assume that there exists a solution to the system of equations $j^2 = \ldots = j^{N-1} = 1^n = 0$. Then this system implicitly defines a unique decreasing function $t^2 = G(t^1)$ on a domain $[0,d]$.

We now use these results to examine the conditions under which, given an equilibrium in which $n-1$ goods, labelled $2,\ldots,n$, all enjoy positive market shares, a new good, labelled 1, of quality inferior to good 2, can enter the market. In other words, we are concerned with the question of whether each of the $n$ products now present enjoys a positive market share at equilibrium.

Let $p^2, \ldots, p^n$ be an equilibrium for products $2, \ldots, n$ with $t_0^2 < t_3 < \ldots < t^n$, and $\bar{p}^1, \ldots, \bar{p}^n$ an equilibrium for products $1, \ldots, n$ with $t^{-1} < t^{-2} < \ldots < t^{-n}$. Given the quality of product 1, we denote by $p^{2*}$ the solution to $a = r^{1,2}(p^2)$. Thus, if product 1 was available at price zero, then, from equation (1), the poorest consumer would prefer product 1 to product 2 if $p^2$ exceeded $p^{2*}$. We further define $t^*$ as $r^{0,2}(p^{2*})$; this denotes the income of a consumer who would be just willing to purchase good 2 at price $p^{2*}$ rather than consume the 'zero' good; clearly $t^* < a$.

The condition for good 1 to be able to enter the market with positive market share is simply that $t_0^2 > t^*$. 
**REMARK:** It may help, in interpreting Theorem 3, to remember that \( \hat{c}_{0,2} \) is a measure of the extent to which firm 2's price is driven down through competition with good 3, in the equilibrium involving goods 0, 2, 3, ..., n, a lower \( \hat{c}_{0,2} \) corresponding to a lower price. The income \( t^* \), on the other hand, is defined independently of this equilibrium. It is a measure of the price \( p^{2*} \) at which good 2 can drive good 1 out of the market.

**THEOREM 3:**

(i) If \( \hat{c}_{0,2} > t^* \), then \( \hat{E}^2 > a \)

(ii) If \( \hat{c}_{0,2} < t^* \), then \( \hat{E}^2 < a \) and \( \hat{E}^k = \hat{c}^k, k = 3, ..., n \).

**PROOF:**

(i) Let \( G \) be the function associated with the system

\[
\begin{align*}
J^2 &= 0 \\
J^3 &= 0 \\
&\vdots \\
J^n &= 0 \\
I^n &= 0
\end{align*}
\]

if \( t^2 \neq a \), (and with \( J^2 = 0 \) replaced by the appropriate inequalities if \( t^2 = a \)). Establishing that \( \hat{E}^2 > a \) is equivalent to showing that \( G(0) > a \) (see Figure 1).

To find \( G(0) \), we set \( t^1 = 0 \) and so \( t^2 = r_{1,2}(p^2) \) in (2).

We then break the system into two blocks, the first being the equation \( J^2 = 0 \) (or the corresponding pair of inequalities) and the second block being the remaining equations \( J^3 = \ldots = I^n = 0 \). Following Lemmas 1 and 2, we construct two functions \( F^1(t^2) \) and \( G^1(t^2) \) from these blocks.
Now $\hat{t}^0, 2, \hat{t}^3, \ldots, \hat{t}^n$ are a solution to the system

$$
\int_{t^0, 2}^{t^3} f(t) dt + p^2 f(t^3) (1 - r^{2, 3}_d) - p^2 f(t^0, 2) r^{0, 2}_d = 0
$$

$$
J^3 = 0
$$

$$
J^n = 0
$$

(3)

with $t^{0, 2} = r^{0, 2}(p^2)$, if $\hat{t}^0, 2 \neq a$, and to the equivalent system with the first equation replaced by a pair of inequalities if $\hat{t}^0, 2 = a$. Again, we may construct two functions $F^2(t^{0, 2})$ and $G^2(t^{0, 2})$ as in Lemmas 2 and 3.

Clearly, we have $F^2(\hat{t}^{0, 2}) = G^2(\hat{t}^{0, 2})$ if $\hat{t}^{0, 2} \neq a$, and $G^2(\hat{t}^{0, 2}) \in [F^2(a^-), F^2(a^+)]$ otherwise.

First compare the functions $G^1$ and $G^2$. For the same price $p^2$, the corresponding sub-systems of (2) and (3) are identical. Hence, for $t^{0, 2} = r^{0, 2}(p^2)$ and $t^2 = r^{1, 2}(p^2)$, we have $G^1(t^2) = G^2(t^{0, 2})$. Compare now function $F^1$ and $F^2$ when $t^1 < a$. In that case, for any $p^2$ such that $t^2 < a$, we have $t^{0, 2} = r^{0, 2}(p^2) < r^{1, 2}(p^2) = t^2 < a$. Consequently, the first equation is the same in both systems so that $F^1(t^2) = F^2(t^{0, 2})$.

By definition of $F^1$ and $G^1$, we have $F^1(t^2) = G^1(t^2)$ if $t^2 \neq a$, or $G^1(t^2) \in [F^1(a^-), F^1(a^+)]$ if $t^2 = a$.

Take $\underline{t}^2 = a$ (the result is immediate if $\overline{t}^2 = a$).
Denoting as $p^2$ the solution to $\bar{t}^2 = r^{1,2}(p^2)$ and setting $\hat{t} = r^{o,2}(\hat{p}^2)$, we have $G^1(\bar{t}^2) = G^2(\hat{t})$. Assume now that $\bar{t}^2 < a$, contradicting the proposition. Then it follows that $F^1(\bar{t}^2) = F^2(\hat{t})$. Accordingly, $G^2(\hat{t}) = p^2(\hat{t})$; this is possible only if $\hat{t} = \hat{t}^{o,2}$. As $\hat{t}^{o,2} \geq t^*$, we have $p^2 \geq p^{2*}$, so that $\bar{t}^2 = r^{1,2}(\hat{p}^2) \geq r^{1,2}(p^{2*}) = a$, a contradiction.

(ii) Since $\hat{t}^{o,2} < t^*$, $\hat{p}^2 < p^{2*}$ so that $r^{1,2}(\hat{p}^2) < r^{1,2}(p^{2*}) = a$.

(i.e. a consumer of income $a$ prefers product 2 at price $\hat{p}^2$ to product 1 at price zero.) The entrance of product 1 therefore has no impact on the price vector $\hat{p}^2, \ldots, \hat{p}^n$.

Q.E.D.

Remark: It follows that entry with a low quality product is always possible when the market is not covered ($\hat{t}^{o,2} > a$), and that such an entry may be possible even when the market is covered by goods 2 to n ($\hat{t}^{o,2} \in [t^*, a]$). Furthermore, when $\bar{t}^2 = a$, $\hat{p}^1 = 0$, and $\hat{p}^2$ (being the solution to $a = r^{1,2}(p^2)$) can be interpreted as that price, for good 2, which just prevents firm 1 from obtaining a positive market share. Thus $\hat{p}^2$ is a limit price. In this configuration, firm 2 faces a demand schedule which is kinked at the price $\hat{p}^2$. 
We are now in a position to establish the uniqueness of equilibrium.

**Theorem 4:** For \( n > 1 \) there exists a unique price equilibrium under Assumptions 1, 2, 3, 5, 6.

**Proof:** We first note that there can be at most one equilibrium in which some given number \( m \leq n \) products have a non-empty market share. Indeed, assuming that such an equilibrium exists, we can define two functions \( F(t^{n-m+1}) \) and \( G(t^{n-m+1}) \) from the corresponding system of necessary conditions, following Lemmas 1 and 2. These functions are such that \( F(t^{n-m+1}) \) if \( t^{n-m+1} \neq a \), or \( G(t^{n-m+1}) \in [F(a^-), F(a^+)] \) if \( t^{n-m+1} = a \), at the equilibrium. As \( F \) is increasing and \( G \) decreasing, this determines a unique pair of values \( t^{n-m+1} \) and \( t^{n-m+2} \), and so a unique solution to the system of necessary conditions.

It remains to establish that the number of products having a non-empty market share at equilibrium is uniquely defined. The proof is constructive. Starting from the top quality \( n \), it is clear that a unique equilibrium may be found which corresponds to the monopolistic price. Then, considering an equilibrium with \( m \) products, i.e. \( n, \ldots, n-m+1 \), having a non-empty market share, the following two cases may arise. Firstly, we may have \( t^{n-m+1} < t^* \) so that, by Theorem 3,
\[ \hat{P}^{n-m} = 0 \] and the prices of the higher quality products are
unchanged by the entry of further, lower quality, products,
and the latter enjoy zero market shares. Secondly,
\[ \hat{t}^{c,n-m+1} \geq t^* \], and the equilibrium price vector correspond-
ing to the case where \( m \) products are available is no longer
an equilibrium price vector for these products in the presence
of product \( n-m \). Proceeding in this way until \( n \) products
are available, it follows that a unique price equilibrium exists.

Q.E.D.

4. The 'Finiteness' Property

We now proceed to establish a basic property of the model, which
concerns the fact that the number of products enjoying a positive
market share at equilibrium is bounded by a number which is independ-
ent of the qualities of the various products, and depends only on
the distribution of consumer incomes. The idea is that price
competition between the 'surviving' high quality products will drive
their prices down to a level such that even the poorest consumer
will prefer to purchase the lowest 'surviving' product at its
equilibrium (positive) price rather than purchase one of the lower
quality products, even at price zero.\(^2\)

---

2. In general, this property depends on the relation \( c(u) \) between (unit
variable) cost and quality. Shaked and Sutton (1981b) fully character-
ize the class of cost functions for which the finiteness property holds,
in the context of the 'linear' case of the model (Section 5 below).
Loosely, what is required is that \( c(u) \) does 'not increase too quickly',
as will be true of industries where the burden of quality improvements
fall primarily on \( R & D \) or capital costs rather than on increases in
labour and material inputs. The present 'zero cost' framework is a
special case of this.
THEOREM 5: Under Assumptions 1-4, the number of firms which enjoy a positive price and positive market share at equilibrium is bounded above by \((1 + N/af)\), independently of the qualities of the products available.

REMARK: Note that \(f(t)\) is the density function of incomes, and \(N = \int_a^b f(t)dt\). The upper bound thus defined depends on the shape of the distribution of income, in the sense that multiplying \(f(t)\) by a constant, for all \(t\), leaves this bound unchanged.

PROOF: For \(1 < k \leq n\), define
\[
\tilde{t}^k = p_{r_d}^{k,k-1,k} + p_1^{k-1}(1 - r_d^{k-1,k})
\]
(Note that \(\tilde{t}^k = \tilde{t}^k\) when \(r_d^{k-1,k}\) is linear). Rewriting \(J^k, \tilde{r}^n\) in terms of \(\tilde{t}^k\), we get
\[
J^k = M^k - \tilde{t}^k f(t^k) + [p f(t^{k+1})(1 - r_d^{k,k+1}) + p f(t^k)(1 - r_d^{k-1,k})]
\]
for \(1 < k < n\), and
\[
\tilde{r}_n = S^n - \tilde{t}^n f(t^n) + p^{n-1} f(t^n)(1 - r_d^{n-1,n})
\]
In equilibrium we have that \(J^k = \tilde{r}_n = 0\); hence
\[
M^k > \tilde{t}^k f(t^k), \text{ for } 1 < k < n
\]
and
\[
M^n > \tilde{t}^n f(t^n)
\]
(since the last expression in \(J^k, \tilde{r}_n\) is negative).
Using Assumption 4, we have
\[ r^{k+1} - d_r^k \leq 0 \]
and for \( d = p^k - p^{k-1} \), we have \( E^k \leq t^k \). Hence, for \( 1 < k \leq n \), we obtain
\[ M^k \geq t^k f(t^k) \]

Since \( E^k \geq a \) and \( f(t^k) \geq f \), we have
\[ M^k \geq a \bar{f} \]

Summing over \( 1 < k \leq n \) yields
\[ N \geq \sum M^k > (n - 1) a \bar{f} \]
or
\[ 1 + N/a \bar{f} > n. \]  
Q.E.D.

We now turn to the question of the entry of a new product of quality superior to product \( n \); entry is of course always possible; here, moreover, it leads to a fall in the prices of all existing products.

(It is important to note that this does not imply that the vector of prices converges to zero, merely that the price of any particular good will fall, until the good disappears from the market, as a sequence of higher quality products enter. We analyse the behaviour of the vector of market prices in the next section.)
Theorem 6: Let $\bar{p}^k(n)$ be the equilibrium price of product $k$ when products $1, \ldots, n$ are available. Then

$\bar{p}^k(n + 1) < \bar{p}^k(n)$ for $k = 2, \ldots, n$ and

$\bar{p}^1(n + 1) \leq \bar{p}^1(n)$.

The proof of this result is given in the Appendix.
5. The Structure of Market Shares

Clearly, it is generally the case that a firm selling a higher quality product enjoys greater revenue; we here note that a stronger property holds for a wide class of income distributions, viz. that market shares increase with product quality.

For any single peaked \( f(t), \ a \leq t \leq b, \) let \( d \) be the (unique) point for which

\[
\int_{a}^{d} f(t) \, dt = d \cdot f(d)
\]

(If no such \( d \) exists, then \( \int_{a}^{d} f(t) \, dt < d \cdot f(d), \ a \leq t \leq b; \) here we set \( d = \infty \)). Let \( m \) denote the median of \( f(t) \).

**Lemma 3** ("Regularity")

(i) If \( m < d \) then for any \( n \) qualities, at equilibrium

\[
M^n > M^{n-1} > \ldots > M^1
\]

(ii) Furthermore, if \( m < d \) there exists \( \epsilon > 0 \) such that

\[
M^k > (1 + \epsilon)M^{k-1}, \quad k = 2, 3, \ldots, n
\]

**Remark:** The point \( d \) necessarily lies to the right of the peak. Our condition implies simply that the distribution shall not be too heavily skewed to the right; it is satisfied for a uniform, normal or lognormal distribution.
**PROOF:** (i) First observe that Assumption 2 ensures that \( f(t)/\int_{t}^{b} f(x) \, dx \)
and hence \( tf(t)/\int_{t}^{b} f(x) \, dx \), increases in \( t \). Note also that since \( m \leq d \),

\[
\frac{df(d)}{\int_{t}^{b} f(t) \, dt} > 1
\]

From the proof of Theorem 5, we have, in equilibrium

\[
M^n = \int_{t^n}^{b} f(x) \, dx > t^n f(t^n)
\]

Hence the market share of firm \( n \) extends below \( d \), i.e. \( t^n < d \).

Since to the left of \( d \), we have \( tf(t) > \int_{a}^{t} f(x) \, dx \),

\[
M^n > t^n f(t^n) > \int_{a}^{t^n} f(x) \, dx \geq M^1 + M^2 + \ldots + M^{n-1}
\]

Thus

\[
M^n > M^{n-1}
\]

For all \( k \), we have, from the Proof of Theorem 5,

\[
M^k > t^k f(t^k)
\]

But \( t^k < t^n < d \), whence

\[
t^k f(t^k) > \int_{a}^{t^k} f(x) \, dx \geq M^1 + \ldots + M^{k-1}
\]

Thus

\[
M^k > M^{k-1}
\]
(ii) By Assumption 2, \( f(t) / \int_t^b f(x) \, dx \) is increasing. Therefore, when \( m < d \), the (unique) point \( v \) satisfying

\[
\int_v^b f(x) \, dx = v f(v)
\]

lies to the left of \( d \). (If no such \( v \) exists, define \( v = a \)). Moreover, the market share of product \( n \) extends below \( v \).

Note that

\[
\min \left( tf(t) - \int_t^a f(x) \, dx \right), \quad a \leq t \leq v
\]

is positive; and denote it

\[
\varepsilon \cdot \int_a^b f(x) \, dx.
\]

Then, following the first part of the proof

\[
M^n > t^k f(t^k) > \varepsilon \cdot \int_a^b f(x) \, dx + M^{k-1} + \ldots + M^1
\]

\[
> (1 + \varepsilon) M^{k-1}.
\]

Q.E.D.

**REMARK:** The condition \( m < d \) is in fact also necessary for regularity.

* A rather lengthy argument shows that for every distribution for which \( d < m \) it is possible, where the utility function

\*

See Appendix
is of the 'linear' form, to find qualities $u_1, u_2, u_3$, where $u_3$ is sufficiently large and $u_2$ sufficiently close to $u_1$ such that

$$M^3 < M^2 > M^1.$$ 

(No counter-example exists for 2 products; for 2 good regularity always holds, independently of the condition $m \leq d$).
6. The Linear Case

We now turn to a special case of the model, in which the distribution of incomes is uniform (so that, without loss of
generality, we may write \( f(t) = 1 \) on \([a,b]\)) and the utility function
takes the special 'linear' form,

\[
U(t,k) = u_k t
\]

whence

\[
r_{k,k+1}^d(d) = \frac{u_{k+1}}{u_{k+1} - u_k} \cdot d
\]

and

\[
r_{d}^{k,k+1} = \frac{u_{k+1}}{u_{k+1} - u_k}
\]

Now this implies that \( r_{d}^{k,k+1} \) is constant for all \( d \).

(This property holds for a wider class of utility functions than
the special case just cited, and in fact is enough to allow the
equilibrium equations to be written in the following simplified form.)
The equation for \( t^k \) now becomes

\[
\begin{align*}
t^k &= p^{k-1} + x_d^{k-1,k} (p^k - p^{k-1}) = p^{k-1} (1 - x_d^{k-1,k}) + p x_d^{k-1,k}
\end{align*}
\]  

(5)

Define

\[
\begin{align*}
a^k &= \frac{x_d^{k-1,k} - 1}{x_d^{k-2,k-1}} = \frac{u_{k-1} - u_{k-2}}{u_k - u_{k-1}}
\end{align*}
\]

being a measure of the relative qualities of products \( k-2, k-1 \) and \( k \). Solving for \( p^k \) (from the equations for \( t^1, t^2, ..., t^k \)) we find:

\[
\begin{align*}
p x_d^{k-1,k} &= t^k + t^{k-1}a^k + t^{k-2}a^{k-1}a^k + ... + t^1a^2...a^k
\end{align*}
\]

whence our system of equilibrium relationships becomes

\[
\begin{align*}
\left\{ \begin{array}{l}
j^1 = t^2 - t^1 - t^1 (1 + a^2) = 0 \quad t^1 > a \\
j^1 = t^2 - a - t^1 a^2 = 0 \quad t^1 < a \\
j^k = t^{k+1} - t^k - (1 + a^{k+1}) (t^k + t^{k-1}a^k + ... + t^1a^2...a^k) = 0 \quad 1 < k < n \\
j^n = b - t^n - (t^n + t^{n-1}a^n + ... + t^1a^2...a^n) = 0
\end{array} \right.
\]

We now proceed to investigate the effect of introducing a sequence of high quality products; this will be associated with the possible exit of low quality products. What concerns us is
whether, as such products enter, prices converge to zero (i.e. marginal cost) or not. We first show that if the qualities of successive goods converge, in the sense that $u_n - u_{n-1} \to 0$, whence $r_{d,l,n} = \frac{u_n}{u_n - u_{n-1}} \to \infty$, then prices will indeed converge to zero.

To show this, we simply note that the equilibrium condition for the top (nth) product is

$$\int_{t}^{b} f(t)dt = \frac{n}{n-1, n} \frac{p}{r_d}$$

Now as the left hand side is finite, it follows that as $r_d \to \infty$, then $p^n$, and so all other prices, converge to zero.

We now turn to the case where qualities do not converge. Here, we show that prices will not in general converge to zero. We do this by describing a particular example, constructed as follows. Notice that the $a^2, ..., a^n$ defined above represent, as it were, the distribution of product qualities: for, given $a^2, ..., a^n$, and a first quality $u_1$, it is possible to find a set of qualities $2, ..., n$ such that the qualities $1, 2, ..., n$ are distributed according to $a^2, ..., a^n$. Now, from the form of the equilibrium relationships just cited, it is immediately clear that $t^1, ..., t^n$ are determined uniquely by $a^2, ..., a^n$, i.e. two sets of products whose qualities are similarly distributed will partition
the market in the same way - though their respective prices will not in fact be the same. In fact, we will now show that for two sets of products \( (1, 2, \ldots, n) \) and \( (1', 2', \ldots, n') \), such that \( u_1' > u_1 \) and \( a^{2'} = a^2, \ldots, a^{n'} = a^n \), the set of equilibrium prices \( p_1', \ldots, p_n' \) are strictly dominated by \( p_1', \ldots, p_n' \).

To show this, write

\[
\begin{align*}
  u_k &= u_1 + (u_1 - u_0) \left[ \frac{1}{a^2} + \frac{1}{a^2 \alpha} + \ldots + \frac{1}{a^2 \ldots a^k} \right] \\
  u_k' &= u_1' + (u_1' - u_0) \left[ \frac{1}{a^2} + \frac{1}{a^2 \alpha} + \ldots + \frac{1}{a^2 \ldots a^k} \right]
\end{align*}
\]

Hence (since \( u_1' > u_1 \)) we have \( u_k' > u_k \).

Moreover, it follows that

\[
\frac{u_k'}{u_k' - u_k - 1} < \frac{u_k}{u_k - u_k - 1}
\]

for, since

\[
\frac{u_k'}{u_k' - u_k - 1} = \frac{u_1 + (u_1 - u_0) \left[ \frac{1}{a^2} + \ldots + \frac{1}{a^2 \ldots a^k} \right]}{(u_1' - u_0) \frac{1}{a^2 \ldots a^k}}
\]
\[
\begin{align*}
= & \frac{u_1}{u_1-u_0} \cdot a^2 \ldots a^k + a^2 \ldots a^k \left[ \frac{1}{a^2} + \ldots + \frac{1}{a^2} \right] \\
= & \frac{1}{1-\frac{u_0}{u_1}} \cdot a^2 \ldots a^k + a^2 \ldots a^k \left[ \frac{1}{a^2} + \ldots + \frac{1}{a^2} \right] 
\end{align*}
\]

Thus, \( r_{d,k}^{k-1,k'} < r_{d}^{k-1,k} \). Now since \( t^{k'} = t^k \), for all \( k \), it follows immediately on inspection of (5), (6) above that \( p_{k}^{'} > p_{k} \) for all \( p_{k} > 0 \).

The relationship between the vector of product qualities and the vector of prices for both the converging and non-converging cases is illustrated by Figure 4.

\[\text{Figure 4: The introduction of higher quality products} \]
\[\text{(i) converging case} \quad \text{(ii) non-converging case.} \]
\[\text{Old Product Set (o) New Product Set (e).} \]
Thus the entrance of a sequence of new products of successively higher qualities does not necessarily lead to the convergence of prices to zero.\(^3\)

We turn, finally, to a second application of this special, linear, case of the model. While the problem of choice of product quality lies outside the scope of the present paper, one basic property which is prior to any consideration of that problem is the following: will an increase in the quality of the 'top quality' (nth) product lead to an increase in the profit of the firm producing it? This is a non-trivial problem in the present model,\(^4\)\(^5\) but, in the linear case, an unambiguous answer is available.

Let \( u_n \) increase to \(+\infty\), whence \( r_{d,n}^{n-1} \to 1 \) and

\[
a^n = \frac{r_{d,n}^{n-1} - 1}{r_{d,n}^{n-2} - 1} \to 0
\]

The revenue of the top product is

\[
R = \int_a^b p(t) f(t) dt = \frac{t + d}{t^{n-1,n}} \frac{r_{d,n}^{n-1,n} - 1}{r_{d,n}^{n-1,n}} \int_a^b f(t) dt
\]

(The proof does not require that \( f(t) \) be uniform.)

---

3. Thus, in particular, the fact that prices converge to zero in the model of [1] reflects the use of a particular assumption on the spacing of successive qualities.

4. It was left as an open question in [1]

5. The problem of quality choice for firms producing lower quality products is more delicate: raising quality in this case will make the product more similar to a higher quality neighbour so that price competition is more 'intense'. See [2].
Differentiating with respect to \( a^n \), and noting

\[
\left. \frac{r_{n-1,n}}{a} \right| = r_{n-2,n-1}
\]

we obtain

\[
R'_n = \frac{(t^n + p = n-1 a r_d n-2,n-1 + p n-1 r_d n-2,n-1)}{r_{n-1,n}} \int_t^b f(t) dt
\]

\[- \frac{r_{n-2,n-1}}{r_d} \int_t^b p^n f(t) dt - p^n f(t^n) t^n'
\]

\[
= t^n \left( \frac{1}{r_{n-1,n}} \int_t^b f(t) dt - p^n f(t^n) \right) + p n-1 \frac{a r_d n-2,n-1}{r_{n-1,n}} \int_t^b f(t) dt
\]

\[+ (p^n - p^n) \frac{r_{n-2,n-1}}{r_d} \int_t^b f(t) dt
\]

Now the first term here is zero, by virtue of the equilibrium condition for \( p^n \). The last term is negative, as \( p^{n-1} < p^n \).

We shall show that \( p^{n-1} < 0 \). In our equilibrium conditions, \( a^n \) does not appear in the equations for \( J^1 \) to \( J^{n-2} \). Hence we can eliminate \( t^{n-1}, \ldots, t^2 \) as functions of \( t^1 \) from \( J^1, \ldots, J^{n-2} \). Using the equation for \( J_{n-1} \) we can write \( t^n \) as a function of \( t^1 \) and \( a^n t^n (t^1, a^n) \), increasing in both arguments. Hence, for the original \( t^1 \) (i.e. for the original value of \( a^n \)) and the new value of \( a^n \), we have

\[
R^n (t^1) < 0
\]
(Note that \( I^n \) is a decreasing function of \( t^n, \ldots, t^1, a^n \) and \( t^{n-1}, \ldots, t^1 \) remain unchanged while \( a^n, t^n \) increase.)

So the value of \( t^1 \) which solves \( I^n = 0 \) at the new, higher \( a^n \), is now less. Thus \( t^{n-1}, \ldots, t^n \) are now less, so that \( p^{n-1} \) (being an increasing function of \( t^{n-1}, \ldots, t^n \)) is less, viz. \( p^{n-1} < 0 \).

This implies that \( R'_n < 0 \); as the quality \( u_n \to \infty \), i.e. \( a^n \to 0 \), then the revenue of the nth firm increases.

(Note: This proof is valid even when increasing the quality of the nth good causes some products to disappear from the market, as the functions \( t^k(t^1) \) are still defined.)
The Appendix is devoted to a number of proofs omitted from the text.

**Lemma A.1:** Any stationary point of \( R_k \) is a local maximum.

**Proof:** Let \( p^k \) be a solution of

\[
\frac{\partial R^k}{\partial p} = M^k + p^k f(t^k, \ell) (1 - x_d^k) - p^k f(h, k), \quad x_d^k = 0.
\]

Consider the second derivative of \( R_k \) w.r.t. \( p^k \) after some rearrangements:

\[
(1 - x_d^k) [ f(t^k, \ell) + p^k f'(t^k, \ell) (1 - x_d^k) ] + f(t^k, \ell) [1 - x_d^k + p^k r_d^k] - x_d^k [ f(h, k) + p^k f'(h, k) x_d^k ]
\]

\[
= f(t^h, k) [ r_d^k + p^k r_d^k ].
\]

Take successively the expressions in brackets

1) \( f(t^k, \ell) + p^k f'(t^k, \ell) (1 - x_d^k) \geq f(t^k, \ell) + p^k f'(t^k, \ell) (1 - x_d^k) \)

by Assumption 3 and the first order condition,

\[
= f(t^k, \ell) - M^k | f'(t^k, \ell) | \cdot \frac{1}{f(t^k, \ell)}
\]

\[
+ p^k f(t^h, k) x_d^k | f'(t^h, \ell) | \cdot \frac{1}{f(t^h, \ell)} > f(t^k, \ell) - M^k | f'(t^k, \ell) | \cdot \frac{1}{f(t^k, \ell)} > 0 \quad \text{by Assumption 2.}
\]
2) \(1 - r_d^{k,\ell} + \hat{p}_d^{k,\ell} < 1 - r_d^{k,\ell} + b|r_d^{k,\ell}| < 0\) by Assumption 5.

3) \(f(t^h, k) + \hat{p}_d f'(t^h, k) r_d^{h,k} > 0\) when \(f'\) is positive
\[> f(t^h, k) + f'(t^h, k) \cdot \frac{\hat{p}_d^{k} - \hat{p}_d^{h}}{f(t^h, k)}\]
(by the first order condition when \(f'\) is negative
\[> 0\) by Assumption 2.

4) \(r_d^{h,k} + \hat{p}_d^{k} r_d^{k} > r_d^{h,k} - b|r_d^{k}| > 0\) by Assumption 6.

Consequently, the second derivative of \(R^k\) at \(\hat{p}_d^{k}\) is negative.

A similar proof holds for \(k = n\).

Q.E.D.

Let \(h < k < \ell\) be three goods with \(\hat{p}_d^{h}\) and \(\hat{p}_d^{\ell}\) given such that \(\hat{c}^{h,\ell} = \hat{p}_d^{h} + c^{h,\ell}\) \((\hat{p}_d^{\ell} - \hat{p}_d^{h})\) belongs to \([a, b]\). Denote by \(\hat{p}_{\text{max}}\) the unique solution to \(t^{h,\ell} = \hat{p}_d^{h} + c^{h,k}(\hat{p}_d^{k} - \hat{p}_d^{h})\).

**Lemma A.2:** The function \(\phi_{h,\ell}^{k} = \int_{t^{h,\ell}}^{t^{k,\ell}} f(t) dt\) is quasi-concave on \([\hat{p}_d^{h}, \hat{p}_{\text{max}}]\).

**Proof:** We aim to show that \(\phi_{h,\ell}^{k}\) is either single-peaked or monotonic. Two cases may arise. In the first case, \(\hat{p}_d^{h} \leq a\). Then \(\hat{p}_d^{h} < p_a^{k}\) where \(p_a^{k}\) is the unique solution to \(a = \hat{p}_d^{h} + c^{h,k}(p_a^{k} - \hat{p}_d^{h})\). The function \(\phi_{h,\ell}^{k}\) is therefore continuously differentiable on
\[ \hat{p}^h, \hat{p}_{\text{max}} [- (p^k_a)] \text{ and continuous at } p^k_a. \] By Lemma A.1, we know that any stationary point of \( \phi^k_{h, \ell} \), if it exists, is a local maximum of \( \phi^k_{h, \ell} \). To complete the proof, it then suffices to show that \( p^k_a \) is not a local minimum of \( \phi^k_{h, \ell} \). Assume that \( \phi^k_{h, \ell} \) increases in the neighbourhood of \( p^k_a \) for \( p^k > p^k_a \). Accordingly, the right-derivative of \( \phi^k_{h, \ell} \) at \( p^k_a \) given by \( M^k + \frac{p^k_a f(t^k, \ell)}{p^k_a f(t^k, \ell) + (1 - r^k_d)} - p^k_a f(a) r^k_h, \) is positive. This implies that the left-derivative of \( \phi^k_{h, \ell} \), given by \( M^k + \frac{p^k_a f(t^k, \ell)}{p^k_a f(t^k, \ell) + (1 - r^k_d)} \), is also positive so that \( \phi^k_{h, \ell} \) is increasing in the neighbourhood of \( p^k_a \) for \( p^k < p^k_a \).

Consider now the case when \( \hat{p}^h > a \). Then, \( \phi^k_{h, \ell} \) is everywhere continuously differentiable and the property follows immediately from Lemma A.1. An argument similar to that developed above holds for \( k = n \) provided \( t^{k, \ell} \) is set equal to \( b \) in the definition of \( \phi^k_{h, \ell} \).

Q.E.D.

**Lemma A.3:** For any \( k = 1, \ldots, n \), \( p^k(t^k, \ldots, t^l) \) is an increasing function of \( t^k \), with \( h \leq k \).

**Proof:** The proof is by induction. Let \( k = 1 \). As \( t^1 = r^{0,1}(p^1) \) and \( r^{0,1}_d > 0 \), \( p^1(t^1) \) is increasing in \( t^1 \). Let \( k > 1 \).
Then \( p^k \) is defined by \( t^k = p^{k-1} + r^{k-1,k}(p^k - p^{k-1}) \), so that the derivative of \( p^k \) w.r.t. \( t^j \), \( j < k \), satisfies

\[
0 = \frac{\partial p^{k-1}}{\partial t^j} + r^{k-1,k} \cdot \left( \frac{\partial p^k}{\partial t^j} - \frac{\partial p^{k-1}}{\partial t^j} \right).
\]

From that, we deduce

\[
\frac{\partial p^k}{\partial t^j} = \frac{(r_d^{k-1,k} - 1)}{r_d^{k-1,k}} \cdot \frac{\partial p^{k-1}}{\partial t^j} > 0, \quad \text{since} \quad \frac{\partial p^k}{\partial t^j} > 0 \quad \text{by assumption.}
\]

Finally, the derivative of \( p^k \) w.r.t. \( t^k \) satisfies

\[
1 = r_d^{k-1,k} \cdot \frac{\partial p^k}{\partial t^k},
\]

from which it follows that \( \frac{\partial p^k}{\partial t^k} > 0 \).

Q.E.D.

**LEMMA A.4:**

(i) \( J^k \) increases with \( t^{k+1} \) in a neighbourhood of a point \((t^{k+1}, \ldots, t^1)\) which satisfies \( a < t^{k+1} < b \) and \( J^k(t^{k+1}, \ldots, t^1) = 0 \).

(ii) \( J^k \) decreases with \( t^k \) in a neighbourhood of a point \((t^{k+1}, \ldots, t^1)\) which satisfies \( J^k(t^{k+1}, \ldots, t^1) = 0 \).

(iii) \( J^k \) decreases with \( t^{k-1}, \ldots, t^1 \).

(iv) \( I^n \) decreases with \( t^n, \ldots, t^1 \) in a neighbourhood of a point \((t^n, \ldots, t^1)\) which satisfies \( I^n(t^n, \ldots, t^1) = 0 \).
PROOF: (i) \[
\frac{\partial J}{\partial t^{k+1}} = f(t^{k+1}) + p f'(t^{k+1})(1 - r_{d}^{k,k+1}) - p f(t^{k+1}) \frac{r_{d}^{k,k+1}}{k,k+1} \\
= \left[ \lambda f(t^{k+1}) + p f'(t^{k+1})(1 - r_{d}^{k,k+1}) \right] \\
+ f(t^{k+1}) \left[ (1-\lambda)p \frac{r_{d}^{k,k+1}}{k,k+1} \right]
\]

The first bracketed term is positive, by an argument analogous to that of part (1) of Lemma A.1. The second term is positive by Assumption 6.

(ii) \[
\frac{\partial J}{\partial t^{k}} = - f(t^{k}) - \frac{\partial p^{k}}{\partial t^{k}} \left[ f(t^{k+1}) (r_{d}^{k,k+1} - 1) + f(t^{k}) r_{d}^{k-1,k} \right] \\
- p f(t^{k+1}) r_{d}^{k,k+1} \left( \frac{\partial p^{k+1}}{\partial t^{k}} - \frac{\partial p^{k}}{\partial t^{k}} \right) - p f'(t^{k}) r_{d}^{k-1,k} \\
- p f(t^{k}) r_{d}^{k-1,k} \frac{\partial p^{k}}{\partial t^{k}}.
\]

Using the expressions for \(\frac{\partial p^{k+1}}{\partial t^{k}}\) and \(\frac{\partial p^{k}}{\partial t^{k}}\) derived in the proof of Lemma A.2, we obtain:

\[
\frac{\partial J}{\partial t^{k}} = - f(t^{k+1}) \left[ r_{d}^{k,k+1} - 1 - p r_{d}^{k,k+1} \right] - f(t^{k}) \left[ 1 + p r_{d}^{k-1,k} \right] \\
\]

By Assumptions 3 and 4, the first term is negative. The second term is negative by an argument similar to that
used in part (3) of Lemma A.1. Finally, the third term is also negative by Assumption 5.

\[
\frac{\partial \psi^k}{\partial t_j} = - \frac{\partial P^k}{\partial t_j} \left[ f(t^{k+1}) \left( r^k \frac{\partial}{\partial t_j} - 1 \right) + f(t^k) r_{d}^{k-1, k} \right] \\
- p^k f(t^k) r_{d}^{k-1, k} \left( \frac{\partial P^k}{\partial t_j} - \frac{\partial P^{k-1}}{\partial t_j} \right) \\
- p^k f(t^k) r_{d}^{k-1, k} \left( \frac{\partial P^k}{\partial t_j} - \frac{\partial P^{k-1}}{\partial t_j} \right)
\]

From the definition of \( t^k \), we deduce that

\[
\frac{\partial P^{k-1}}{\partial t_j} = \frac{r_{d}^{k-1, k}}{r_{d}^{k-1, k-1}} \frac{\partial P^k}{\partial t_j}.
\]

Then

\[
\frac{\partial \psi^k}{\partial t_j} = \frac{\partial P^k}{\partial t_j} \left[ - f(t^{k+1}) \left( r^k \frac{\partial}{\partial t_j} - 1 + \frac{r_{d}^{k, k+1}}{r_{d}^k} \right) \\
- f(t^k) r_{d}^{k-1, k} \frac{r_{d}^{k-1, k}}{r_{d}^{k-1, k-1}} \cdot p^k \right]
\]

Using Assumption 5 we may show that the bracketed term is negative. Hence, the statement follows by Lemma A.3.

(iv) This follows the above argument as \( s^k \) is a special case of \( \psi^k \) for \( t^{n+1} = b \) and \( f(t^{n+1}) = 0 \).

Q.E.D.
Note that $J^k(t^{k+1}, a, t^{k-1}, \ldots, t^1)$ is not uniquely determined. Indeed,

$$J^k(t^{k+1}, a, t^{k-1}, \ldots, t^1) = \lim_{t \to a} J^k(t^{k+1}, t, t^{k-1}, \ldots, t^1)$$

is different from

$$J^k(t^{k+1}, a^+, t^{k-1}, \ldots, t^1) = \lim_{t \to a} J^k(t^{k+1}, t, t^{k-1}, \ldots, t^1).$$

**LEMMA A.5:** For every vector $(t^k, \ldots, t^1)$ with $t^k \neq a$, there is at most one value $\tilde{t}^{k+1} \in [a, b]$ such that $J^k(\tilde{t}^{k+1}, t^k, \ldots, t^1) = 0$.

Furthermore, the equations $J^k(t^{k+1}, a^-, t^{k-1}, \ldots, t^1) = 0$ and $J^k(t^{k+1}, a^+, t^{k-1}, \ldots, t^1) = 0$ have at most one solution $t^{k+1}$ in $[a, b]$.

For every vector $(t^{k+1}, t^{k-1}, \ldots, t^1)$, there is at most one value $\hat{t}^k \in [a, b]$ such that $J^k(t^{k+1}, \hat{t}^k, \ldots, t^1) = 0$.

**PROOF:**

First, let $t^k \neq a$. Assume that there exist two values $\bar{s}_1$ and $\bar{s}_2$ such that $a < \bar{s}_1 < \bar{s}_2 < b$ and

$$J^k(\bar{s}_1, t^k, \ldots, t^1) = J^k(\bar{s}_2, t^k, \ldots, t^1) = 0.$$ 

Hence, by Lemma A.4 there exists some $\delta > 0$ sufficiently small such that $J^k(\bar{s}_2 - \delta, t^k, \ldots, t^1) < 0 < J^k(\bar{s}_1 + \delta, t^k, \ldots, t^1)$.

Given that $t^k \neq a$, $J^k$ is a continuous function of $t^{k+1}$.

Consequently, by the mean value theorem, there exists $\bar{s}_3$

such that $\bar{s}_1 < \bar{s}_3 < \bar{s}_2$ and $J^k(\bar{s}_3, t^k, \ldots, t^1) = 0$. In a similar fashion, it can be shown that $J^k(\bar{s}, t^k, \ldots, t^1) = 0$ for all $\bar{s} \in [\bar{s}_1, \bar{s}_2]$. But this contradicts Lemma A.4.

Indeed, we know that $J^k(t^{k+1}, \ldots, t^1)$ is increasing in $t^{k+1}$.
in the neighbourhood of \( j^k(t^{k+1}, \ldots, t^1) = 0 \).

The second part of the Lemma can be proved in a similar manner.

Q.E.D.

We are now in a position to establish Lemmas 1 and 2, and Theorem 6, of the main text.

**Lemma 1:** There exists an interval \([0, C]\) and a unique increasing function \( F(t^1) \) defined on \([0, C] - \{a\}\) such that:

1. \( j^1(F(t^1), t^1) = 0, \forall t' \in [0, C] - \{a\}; \)
2. If \( C > a, F \) is two valued at \( t^1 = a \) and \( F(a^-) < F(a^+); \)
3. \( j^1(F(a^-), a^-) = j^1(F(a^+), a^+), a^+ = 0. \)

**Proof:** To start with, we note that \( j^1(a, 0) = 0 \), so that \( F(0) = a \).

By Lemma A.4 \( j^1 \) is increasing in \( t^2 \) in the neighbourhood of \((a, 0)\). Then, by the implicit function theorem, we can determine \( t^2 \) as a function of \( t^1 \) in this neighbourhood, and the resulting value of \( t^2 \) is uniquely determined for a given \( t^1 \), by Lemma A.5. The function thus obtained is increasing since \( j^1 \) increases in \( t^2 \) and decreases in \( t^1 \) in a neighbourhood of \( j^1(t^2, t^1) = 0 \) by Lemma A.4.

We can proceed in this manner to extend the function \( t^2 = F(t^1) \) as the solution to \( j^1(t^2, t^1) = 0 \), as long as the conditions...
$t^2 \leq b$ and $t^1 \leq a$ are satisfied. If $F(t^1)$ attains the value $b$ for $t^1 < a$, then we have $C = t^1$ (this case is illustrated in Figure 3(i) of the main text).

Let us now assume that $F(a^-) < b$. Two cases arise. In the first, $J^1(t^2, a^+) = 0$ has no solution, and we set $C = a$ (see Figure 3 (ii)). In the second, $\hat{t}^2$ exists such that $\hat{t}^2 < b$ and $J^1(\hat{t}^2, a^+) = 0$. Then we have $\hat{t}^2 = F(a^+) > F(a^-)$. Indeed, from $J^1(\hat{t}^2, a^+) = 0$ we deduce that

$$J^1(t^2, a^-) = \int_a^{\hat{t}^2} f(t) \, dt + \int_{\hat{t}^2}^{t^2} f(t) (1 - r_d) \, dt$$

is positive. Hence, $F(a^-)$ cannot be equal to $F(a^+)$. Assume now that $F(a^-) > F(a^+)$. In this case, there must be another solution to $J^1(t^2, a^-) = 0$ between $\hat{t}^2$ and $F(a^-)$, violating Lemma A.5. Consequently, $F(a^+)$ lies above $F(a^-)$. We can then proceed as before and define uniquely $t^2 = F(t^1)$ as the solution to $J^1(t^2, t^1) = 0$. This may be done as long as $F(t^1) \leq b$; when $F(t^1) = b$, we set $C = t^1$ (see Figure 3 (iii)).

Q.E.D.
For certain \( p^2 \), the best reply of seller 1 is \( p^1_a \), i.e. \( t^1 = a \), in which case seller 1 adopts a corner solution corresponding to a kink in his demand function.

**Lemma 2:** Assume that there exists a solution to the system of equations
\[
j^2 = \ldots = j^{n-1} = t^n = 0.
\]
Then the system implicitly defines a unique decreasing function \( t^2 = g(t^1) \) on a domain \([0,d]\).

**Proof:** Let \( t^1, \ldots, t^n \) be a solution to the system. Substituting \( t^2 \) and \( t^1 \) into \( j^2 \), there is a unique value \( \overline{t}^3 \) of \( t^3 \) such that \( j^2(t^3, \overline{t}^2, t^1) = 0 \) by Lemma A.5. By Lemma A.4, \( J^2 \) is increasing in \( t^3 \) and decreasing in \( t^2 \) and \( t^1 \) in a neighbourhood of \((\overline{t}^3, \overline{t}^2, t^1)\). Then, by the implicit function theorem and Lemma A.5, we can uniquely determine \( t^3 \) as an increasing function \( H^3 \) of \( t^2 \) and \( t^1 \) in this neighbourhood.

We can proceed in this way and extend \( H^3 \) as long \( t^3 < b \) and \( t^2 \neq a, b \). Substituting \( H^3 \) in \( J^3 \) for \( t^3 \), we get
\[
j^3(t^4, H^3(t^2, t^1), t^2, t^1) = 0 \quad \text{and} \quad J^3 \text{ is increasing in } t^2 \text{ and } t^1 \text{ in a neighbourhood of } (t^2, t^1) \text{ such that } J^3 = 0.
\]
Then, as above, we can uniquely define \( t^4 \) as an increasing function \( H^4(t^2, t^1) \) provided \( t^4 < b \) and \( t^2 \neq a, b \). In a similar manner, we construct \( t^{n-1} = H^{n-1}(t^2, t^1) \) and \( t^n = H^n(t^2, t^1) \) from \( J^{n-2} = 0 \) and \( J^{n-1} = 0 \) respectively, where \( H^{n-1} \) and \( H^n \) are increasing in a neighbourhood of \((t^2, t^1)\). Let us now introduce those functions \( t^n \); we get
\[
J^n(H^n(t^2, t^1), H^{n-1}(t^2, t^1), \ldots, t^2, t^1) = 0.
\]
Then, by the implicit function theorem and Lemmas A.4 and A.5, we can obtain \( t^2 \) as a decreasing function of \( t^1 \) as long as \( G(t^1) > a \). Defining \( t^1 \) as the solution to \( G(t^1) = a \), we set \( d = \min(t^1, b) \).

Now assume that the same process starts from another solution \( \hat{t}^1, \ldots, \hat{t}^n \) to the system. Then, as above, it is possible to construct a function \( \hat{H}^3(t^2, t^1) \) such that \( J^2(\hat{H}^3(t^2, t^1), t^2, t^1) = 0 \). But then, by Lemma A.5, we have \( \hat{H}^3(t^2, t^1) = H^3(t^2, t^1) \). This implies that function \( G \) is independent of the solution from which it is constructed.

Q.E.D.

**Theorem 6:** Let \( \bar{p}^k(n) \) be the equilibrium price of product \( k \) when products \( 1, \ldots, n \) are entered. Then, \( \bar{p}^k(n + 1) < \bar{p}^k(n) \) for \( k = 2, \ldots, n \) and \( \bar{p}^1(n + 1) \leq \bar{p}^1(n) \).

**Proof:** Let \( \bar{t}^1(n), \ldots, \bar{t}^n(n) \) and \( \bar{t}^1(n + 1), \ldots, \bar{t}^n(n + 1) \) be the equilibrium values of the \( t^k \), \( k = 1, \ldots, n \), when the sets of available products are \( (1, \ldots, n) \) and \( (1, \ldots, n + 1) \) respectively. Without loss of generality, we may assume that the market share of seller 1 at equilibrium, when products \( (1 \ldots n) \) are available, is positive, i.e. \( \bar{t}^2(n) > a \). At the equilibrium for products \( (1, \ldots, n+1) \), the following two situations may arise:
(i) the market share of seller 1 is non-empty, i.e. \( \bar{t} \geq 2(n+1) \geq a \), and

(ii) the market share of seller 1 is empty, i.e. \( \bar{t} \leq 2(n+1) < a \).

(i) Assuming that \( \bar{t} \geq 2(n+1) \geq a \), two cases arise. In the
first, \( \bar{t} = \bar{t} \geq 2(n+1) = a \). As in Lemma 2, we define
\( t^k = H^k(t^2, a) \) as an increasing function of \( t^2 \), \( k = 3, \ldots, n+1 \).
Introducing these functions for \( t^k \), \( k = 3, \ldots, n \), into \( I^n \),
we get \( I^n(H^n(\bar{t} \geq 2(n), a), \ldots, \bar{t} \geq 2(n), a) = 0 \), \( I^n \) being
decreasing in a neighbourhood of \( \bar{t} \geq 2(n) \).

Now, \( I^{n+1}(t^{n+1}, \ldots, a) + J^n(t^n, \ldots, a) = \frac{b}{t^{n+1}} \int_{t^n}^{t^{n+1}} f(t) \, dt + \int_{t^n}^{t^{n+1}} f(t) \, dt \)
\[ = p^n f(t^n) x_d^{-n-1} t^n - f(t^{n+1})[p^{n+1} x_d^n + p^n (x_d^n - 1)] \]
\[ = I^n(t^n, \ldots, a) - f(t^{n+1})[p^{n+1} x_d^n + p^n (x_d^n - 1)] \]

Introducing \( H^k(\bar{t} \geq 2(n+1), a) \) for \( t^k \), \( k = 3, \ldots, n+1 \), in this
expression, we see that the LHS is equal to zero, so that
\( \bar{I} = n(H^n(\bar{t} \geq 2(n+1), a), \ldots, \bar{t} \geq 2(n+1), a) \) is positive. Hence, we
must have \( \bar{t} \geq 2(n+1) < \bar{t} \geq 2(n) \), otherwise \( \bar{I} \) would be equal
to zero on \([\bar{t} \geq 2(n), \bar{t} \geq 2(n+1)] \) by an argument similar to that
developed in the proof of Lemma A.5, contradicting the fact
that \( \bar{I} \) is decreasing in the neighbourhood of \( \bar{t} \geq 2(n) \).
Consequently, \( \bar{t}^k(n+1) = H^k(\bar{t} \geq 2(n+1), a) < H^k(\bar{t} \geq 2(n), a) = \bar{t}^k(n) \),
so that \( p^k(n+1) < p^k(n) \) by Lemma A.3. Note that
\( \bar{p}^{-1}(n+1) = \bar{p}^{-1}(n) \) since \( \bar{t}^{-1}(n+1) = \bar{t}^{-1}(n) \).
The Regularity Condition: A Counterexample

Assume $m > d$, and that utility functions are of the 'linear' form. We show that there exist $u_3$, $u_2$, $u_1$ such that $M^3 < M^2 > M^1$.

For 3 products, the equilibrium conditions take the form (where $t^1 < a$),

$$M^3 - f(t^3)(t^3 + t^2a^3 + t^1a^2a^3) = 0$$  \hspace{1cm} (I)

$$M^2 - (a^3f(t^3) + f(t^2))(t^2 + t^1a^2) = 0$$  \hspace{1cm} (II)

$$M^1 - t^1a^2f(t^2) = 0$$  \hspace{1cm} (III)

Since $m > d$, $\int_a^m f(x)dx > mf(m)$. (To see this, note that for all $x > d : \int_a^x f(x)dx > tf(t)$ and $\int_a^m f(x)dx = \int_a^m f(x)dx$).

Hence by choosing $a^3$, $a^2$ sufficiently small ($u_3 \to \infty$) we can make $I^3$ positive for $t^3 = m$, irrespective of $t^1$ and $t^2$. Now $I^3$ is certainly negative for $t^3 = b$, hence the equilibrium value of $t^3$ lies to the right of $m$ and so

$$M^3 < \frac{1}{2} \int_a^b f(x)dx$$

Let $K = t^1a^2$. By choosing $K, a^3$ sufficiently small, we can make the value of $t^2$ satisfying II greater than $a$. To show this, we establish that $J^2$ is positive for $t^2 = a$; now this is certainly the case if

$$\int_a^b f(x)dx - af(a) > 0$$
But
\[ \int_{a}^{d} f(x) \, dx > \int_{a}^{d} \frac{d}{a} f(x) \, dx = df(d). \]

Thus we aim to establish \( df(d) > af(a) \).

We show that \( xf(x) \) is increasing between \( a \) and \( m \). For, since \( m > d \),

\[ \frac{mf(m)}{\int_{m}^{b} f(x) \, dx} < 1 \]

But \( tf(t) / \int_{t}^{b} f(x) \, dx \) is increasing; thus to the left of \( m \)

\[ \frac{tf(t)}{\int_{t}^{b} f(x) \, dx} < 1 \]

But if at any point \( tf(t) \) is decreasing we have from A2 that

\[ tf(t) > \int_{t}^{b} f(x) \, dx \]

whence we obtain a contradiction.

Hence \( df(d) > af(a) \) as required.

It follows that \( J^2 \) is positive at \( t^2 = a \). Moreover, \( J^2 \) is
clearly negative at \( t^2 = t^3 \). Thus, at equilibrium, \( t^2 > a \).

However, we can make \( t^2 \) as small as we wish, so that

\[ M^2 = \int_{t^2}^{t^3} f(x) \, dx \]

will exceed \( \frac{1}{2} \int_{a}^{b} f(x) \, dx \), from which our result follows immediately.
To do this, we observe that by choosing $K$ sufficiently small, the solution of $J^1 = 0$ will be arbitrarily close to $a$. We need, moreover, to ensure (for consistency) that $\int_a^t f(t^2) < 0$ so that at equilibrium, $t^1 < a$.

This is done by choosing $t^1 < a$ sufficiently small and $\alpha^2$ sufficiently large so that $t^1 < a$ and $t^1 \alpha^2 = K$, (where $K$ is as determined above). This is done by choosing $u_2$ sufficiently close to $u_1$. 
REFERENCES


