Quantity Competition in Networked Markets
Outflow and Inflow Competition

Francesco Nava

Discussion Paper No. TE/2009/542
September 2009
The Suntory-Toyota International Centres for
Economics and Related Disciplines
London School of Economics and Political Science
Houghton Street
London WC2A 2AE
Tel.: 020-7955 6674

Abstract

This paper investigates how quantity competition operates in economies in which a network describes the set of feasible trades. A general equilibrium model is presented in which prices and .ows of goods are endogenously determined. In such economies equilibrium dictates whether an individual buys, sells or does both (which is possible). The first part of the analysis provides sufficient conditions for pure strategy equilibrium existence; characterizes equilibrium prices, flows and markups; and details negative effects on welfare of changes in the network structure. The main contributions show that goods do not cycle, since prices strictly increase along the supply chains; that not all connected players with different marginal rates of substitution trade; and that adding trading relationships may decrease individual and social welfare. The second part of the analysis provides necessary and sufficient conditions for a networked economy to become competitive as the number of players grows large. In this context it shown that no economy in which goods are resold can ever be competitive; and that large well connected economies are competitive.

Contents

1 Introduction

Classical models of competition rely on the anonymity of markets to explain trade and prices in an economy. In such view all exchanges take place in a centralized market and the identity of players has no effect on the prices and on the terms of trade. In recent years different models of competition have been proposed to study economies in which all exchanges take place in decentralized bilateral relationships. Prices and terms of trade in such economies depend on the limitations to the set of feasible exchanges (which are usually expressed as network). Most results have been developed for economies in which the set of buyers and the set of sellers are exogenously determined and in which only trades from sellers to buyers are feasible. The purpose of this study is: (1) to understand how networked markets operate if equilibrium determines who buys, who sells and who does both; and (2) to provide necessary and sufficient conditions on the fundamentals of such decentralized markets for an economy to be perfectly competitive.

The paper presents a general equilibrium trade model for economies in which a network describes the set of feasible trading relationships among individuals. In the model proposed every individual owns an exchange location and selfishly routes scarce resources to his neighbors, knowing how prices are determined at each location. Individuals in the economy simultaneously decide how many goods to sell to their neighbors (i.e. outflow competition). As in a Cournot model, prices at each location guarantee that all good supplied to a location are purchased. Changes in sales distort both the price at which units are purchased and the prices at which units are sold, as trade distorts the marginal rates of substitution of both players involved in a transaction. Traders account for such price distortions and selfishly choose how much to sell to their neighbors on the network in order to maximize their well being. Equilibrium flows of goods endogenously determine whether an individual buys, sells or does both, based on preferences, production possibilities and the position held in the network. Supply chains arise endogenously in equilibrium. Intermediation and significant price dispersion are persistent phenomena in small or poorly connected economies.

The first part of the analysis provides sufficient conditions for the existence of a pure strategy equilibrium; characterizes equilibrium prices, flows and markups; and details negative effects on welfare of changes in the network structure. The main contributions of this part of the analysis show that goods do not cycle, since prices strictly increase along the supply chains; that not all connected players with different marginal rates of substitution trade; and that adding trading relationships may have negative consequences on individual and social welfare.

In the outflow competition model individuals who both buy and sell goods (i.e. resell goods) do so at strictly positive markups, as the ownership of trading locations leads to positive price distortions. Selfish behavior results in price discrimination across individuals at different positions in the trading network. Resale is pervasive and relies both on the scarcity of trading partners and on different prices that reign throughout the economy in equilibrium. Two observations follow directly from the strictly positive retail markups: (i) goods do not cycle and (ii) not all linked players with different marginal rates of substitution trade. No individual would ever purchase units he sold, because a higher price would have to be paid. Moreover individuals with lower willingness to pay for a good may prefer not to sell goods to players with a higher willingness to pay, since trade would increase the price paid for the units purchased. This implication differs from that of the Cournot model in which any two players with different marginal rates of substitution always elect to trade. The final results characterizing flows details the effects on welfare of changes in the network structure. In this context is possible to show that adding trading relationships does not necessarily improve the social welfare. In fact when new links are added, more goods may flow to low value markets as sellers try to elicit higher prices in the markets in which the goods are most desired. More surprisingly the study shows that increasing the set of trading partners of an individual may decrease his own welfare even if he is given the option not to trade on newly created links. Indeed whenever trades on a new link raise the demand of an individual, price discrimination by his suppliers may decrease both the amount of goods sold to him and his individual welfare.

The second part of the analysis provides necessary and sufficient conditions for a networked economy to become competitive as the number of players in the economy grows large. The main contributions of this part show that no economy in which one or more individuals both buy and sell goods can ever be competitive independently of the size and of the structure of the market; and that large well connected economies are always competitive.

In this setup a networked economy becomes competitive only if all trades become direct and individuals either buy or sell goods. The necessity of such condition relies on the strictly positive markups that intermediaries command whenever they are required to distribute goods. Thus in the outflow model competition among those who resell goods can undermine, but not eliminate resale markups if intermediaries are required to supply goods. Sufficient conditions on the network structure are also provided to guarantee that an economy becomes competitive as the number of players grows large. Such conditions require any group of individuals purchasing goods in a competitive equilibrium to face a positive excess supply from the players which are linked to them on the network and which sell goods in the competitive equilibrium. If such condition holds and if all players have sufficiently many suppliers in the economy, all trades become direct and equilibrium markups vanish. Limiting economies can be fully characterized even when they do not become competitive. In such instances retail can persists even with infinitely many producers, intermediaries and consumers. Even though equilibrium markups and gross margins decrease as an economy grows large, they do not disappear for intermediaries of any limiting economy as individuals become price takers as suppliers, but not as buyers. When all local markets become more competitive, social welfare increases and is bounded above by its limiting value. As in most quantity competition models sufficient conditions for equilibrium existence can be relaxed when an economy grows large.

The final part of the analysis presents an alternative quantity competition model in which individuals to decide how much to buy (i.e. inflow competition). Similar results hold, even though the distribution rents differs. More rents flow to buyers and social welfare is generally higher than when players choose how much to sell. Again equilibria are not competitive in economies that are small or with significant intermediation, because of the price distortions inherent to the model.

Literature Review: A recent literature has been devoted to the analysis of trade in buyer-seller networks. In such models the set of buyers and the set seller in an economy are exogenously determined and the network describes which sellers can supply any given buyer in the economy. Several papers discuss such setup, but differ in the way competition proceeds on the network. Kranton and Minehart 2001 [13] model the competition among suppliers, by having them hold simultaneous ascending price auctions. Their paper shows that link patterns which lead to efficient outcomes are equilibria of a network formation game. Corominas-Bosch 2004 [6] models trade on each link as resulting from a non-cooperative bargaining game. Her analysis provides sufficient conditions on the network structure for the equilibrium of the bargaining game to coincide with the Walrasian outcome. Blume, Easley, Kleinberg and Tardos 2007 [3] study a buyer-seller networks in which all trades are mediated by middlemen. In their model since middlemen can make take it or leave it offers to both sides of the market, they capture all the surplus and implement the efficient allocation. Lever [15] discusses Bertrand competition between duopolists on buyer seller networks. Therein he shows that with such model of competition social welfare always increases with the connectedness of the network and that fully connected networks are perfectly competitive. In all the models discussed, however, the position that individuals hold on the network exogenously determines whether they buy, sell or retail. The quantity competition model presented here differs from the basic setup of this literature, because equilibrium dictates which roles individuals have on the supply chains. Other papers have been concerned with the study of trade in more general networked market. Kakade, Kerns and Orthiz [11] study the competitive equilibria of a networked market. As in quantity competition model proposed flows of goods in such economies are endogenous. However, the advantages of more general network structures are tempered by the assumption that all individuals are price takers. Finally a related literature takes sources and sinks of the flows as exogenously determined and studies how the owners of the links should price flows if individuals selfishly rout those flows on a network. Such literature usually takes a Bertrand approach to model competition and was developed to study how internet providers price stream of information if individuals selfishly rout their flows. Two examples of it are: Chawla and Roughgarden 2007 [7] which studies competition with capacity constraints; and Acemouglu and Ozdaglar 2007 [2] which studies competition in parallel-serial networks.

Roadmap: Section 2 is devoted to the outflow competition model. It presents: the basic setup (section 2.1), the outflow competition model (section 2.2), some examples (section 2.3) and sufficient conditions for pure strategy equilibrium existence (section 2.4). The analysis proceeds with the characterization of equilibrium prices and flows (section 2.5) and of the welfare effects of changes in the network structure (section 2.6). Section 3 discusses outflow competition in large economies and provides necessary and sufficient conditions for a competitive equilibrium to emerge. Section 4 presents the inflow competition model. All proofs can be found in appendix.

2 Outflow Competition in Networked Markets

2.1 Basic Setup and Constrained Efficiency

A networked market is one in which not all trades among individuals are feasible. An economy in this setup consists of: a set of individuals, an undirected graph describing what trades are feasible, endowments and payoffs for each individual. If an undirected graph $G = (V, E)$ describes the set of feasible exchanges, the set of vertices of such graph V consists of all the individuals in the economy, while the set of edges of the graph E describes which trades are feasible. Therefore, agent i and agent j can trade if and only if $ij \in E$. Assuming the graph to be undirected requires j to be able to supply to i, whenever i can supply to j. The set of neighbors of player i consists of all those individuals who are linked to i and is denoted by:

$$V_i = \{j \in V | \, ij \in E\}$$

Initially let there be two goods in the economy. For convenience call them consumption q and money m. Each individual in the economy is endowed with a finite amount of the consumption good, call it Q_i for player i, and with a large amount of money. Assume that the payoff of each player is separable in the two goods and linear in money:

$$u_i(q) + m$$

Moreover assume that payoffs on consumption satisfy:

Assumption A1 *For any player $i \in V$ assume that u_i is three times continuously differentiable, strictly increasing and strictly concave on $\mathbb{R}$.*

When the assumptions in A1 hold only on $\mathbb{R}_+$, assumption A1_+ is said to hold. Otherwise, payoffs are defined also on short trading positions, because deviations from suppliers may leave intermediaries on the network with negative consumption holdings. Such payoffs can be interpreted assuming that players have preferences on consumption defined by a map v_i and can produce those goods through cost function γ_i. If so u_i denotes the net payoff on after-trade positions and satisfies:

$$u_i(q) = \arg\max_{y \in \mathbb{R}_+} v_i(q+y) - \gamma_i(y) \ \text{ s.t. } q + y \geq 0$$

An alternative interpretation abstracts from production to focus on pure exchange economies, but requires players to hold positive amounts of consumption after trade. In such view, payoffs on short positions are punishments for default. The differentiability assumption on $\mathbb{R}_-$ is easily relaxed, but different assumptions on the consequences of short positions would have to be invoked.

Denote by q_j^i the flow of consumption good from individual i to individual j. Bold letters are used to denote vectors of flows. Thus $\mathbf{q}^i$ denotes the vector of outflows from i to his neighbors in V_i; $\mathbf{q}$ denotes the Cartesian product of all the $\mathbf{q}^i$'s and $\mathbf{q}^{-i}$ denotes the Cartesian product of $\mathbf{q}^j$ for all $j \neq i$. Since trades can occur only amongst consumers that know each other, $q_j^i = 0$ whenever $ji \notin E$. The price of the flow q_j^i is denoted by p_j^i. In this setup after trade consumption of player i is defined by:

$$q_i = Q_i + \sum_{k \in V_i} [q_i^k - q_k^i]$$

In a model without production, three different assumptions on credit may be invoked to guarantee that trades be feasible. We refer to the first assumption as *full-credit* (F), to the second as *limited-credit* (L) and to the latter as *no-credit* (N). Such assumptions bound the set of feasible outflows from player i, respectively, to:

$$X_i^F(\mathbf{q}^{-i}) = \left\{\mathbf{q}^i \in \mathbb{R}_+^{V_i} \middle| \, q_i \geq 0\right\} \tag{F}$$

$$X_i^L(\mathbf{q}^{-i}) = \left\{\mathbf{q}^i \in \mathbb{R}_+^{V_i} \middle| \, q_i - \max_{k \in V_i} q_i^k \geq 0\right\} \tag{L}$$

$$X_i^N(\mathbf{q}^{-i}) = \left\{\mathbf{q}^i \in \mathbb{R}_+^{V_i} \middle| \, Q_i - \sum_{k \in V_i} q_k^i \geq 0\right\} \tag{N}$$

In the full-credit model individuals can sell all the units they own or purchase. In the limited-credit model, individuals must hold some reserves to make sure that no unilateral deviation leads to default. While in the no-credit model no short sales are allowed in equilibrium. In an economy with production and without capacity constraints, none of this assumptions will need to be invoked. Non-negativity constraints on monetary holdings will be neglected throughout. It is implicitly assumed that money endowments are large enough for such constraints never to bind. Since utility is linear in money, such assumption implies that the monetary endowments will not affect decisions. Omitting such endowments, the welfare of individual i consists of his utility from consumption and of the gains from trade. In particular, given flows $\mathbf{q} \in \mathbb{R}^E_+$ and prices $\mathbf{p} \in \mathbb{R}^E_+$ it is equal to:

$$u_i(q_i) + \sum_{k \in V_i} [p^i_k q^i_k - p^k_i q^k_i]$$

A player resells whenever he buys and sells a positive amount of goods. Individual i's *resale,* r_i, consists of the total amount of goods that are bought to be sold by i:

$$r_i = \min \left\{ \sum_{k \in V_i} q^k_i, \sum_{k \in V_i} q^i_k \right\}$$

A profile of flows $\mathbf{q}^* \in \mathbb{R}^E_+$ said to be *constrained efficient* if it maximizes the sum of individuals' welfare given set of feasible transfers. Because of quasi-linearity the equal Pareto weights assumption will select the feasible allocation that maximizes of the monetary value of the economy. For instance, in a model with no production and with full-credit constrained efficient flows solves:

$$\mathbf{q}^* \in \arg\max_{\mathbf{q} \in \mathbb{R}^E_+} \sum_{i \in V} u_i(q_i) \;\; \text{s.t.} \;\;\; q_i \geq 0 \;\; \text{for } \forall i \in V$$

If the solution to this problem lies in the interior of the domain, the constrain efficient allocation equalizes the marginal rates of substitution of any two consumers i and j that belong to the same component of the networked economy G.[1] If, instead, no feasible profile of flows equalizes marginal rates of substitution across individuals belonging to the same component, feasibility constraints on flows limit the extent of feasible redistribution and pin down the allocation. If the networked economy is connected and credit is full, any constrained efficient allocation is also efficient, since goods can be routed to all locations. But if the economy is not connected redistribution can only equalize marginal rates of substitution within components of the graph, but not across components. Constrained efficiency always pins down an allocation. It does not, however, pin down flows of goods in the economy, unless further assumptions are invoked.

2.2 Outflow Competition

In the outflow competition model the fundamentals of the economy (network, preferences and endowments) are common knowledge among players. Every individual purchasing con-

[1] Any maximal connected subgraph of G is a component of G. See [4].

sumption owns a trading location to which all his neighbors can ship goods to, so long as they satisfy the relevant credit constraints. In particular, competition proceeds as follows: initially individuals simultaneously decide how much consumption to ship each neighbor; then given the chosen flows market prices are determined at each location so that all units supplied are sold. Suppliers expect such prices when choosing their shipments. Buyers pay all of their inflows equally at the marginal value of the last unit purchased, which is the marginal value of the last unit consumed. Thus prices are determined by inverse demand curve with respect to quantity at each node. In particular for $ij \in E$:

$$p_i^j(\mathbf{q}) = p_i(q_i) = u_i'(q_i) = u_i'(Q_i + \textstyle\sum_{k \in V_i}[q_i^k - q_k^i]) \tag{1}$$

Such prices arise because, given the number of units for sale at each node, no higher price would clear the market. The concavity of the utility map requires that $\partial p_i(q_i)/\partial q_i^j < 0$ and $\partial p_i(q_i)/\partial q_j^i > 0$ for any $j \in V_i$. The price paid by consumer i decreases when inflows increase and increases when outflows increase. Thus increasing the amount of goods supplied to a neighbor leads both to an increase in the price paid for all units purchased and to a decrease in the price received for all units sold to that neighbor. Changes in other flows on the network do not affect directly the price paid by i.

The pricing equation implicitly assumes that suppliers can pre-commit to deliver flows of consumption to known buyers. Indeed if suppliers had such ability and were to compete locally on prices, equation 1 would still dictate pricing, since no supplier could benefit from a unilateral deviation in prices offered. Price reductions would not affect the quantity sold, while price increases would reduce revenues because of falling sales. This observation was first made by Kreps and Sheinkman while studying Bertrand competition with quantity pre-commitment in [14].[2] The implied pricing favors suppliers in each local market, because the demand curve is used to clear markets. Section 4 explores the consequences of the alternative setup in which suppliers own the trading location and buyers pre-commit to inflows bought.

Suppliers account for the effects that their flows have both on the prices they receive for each unit supplied and on the price they pay for each unit purchased. The problem of supplier $i \in V$ is to choose which quantities to supply to each of his neighbors given price effects on inflows and outflows subject to the relevant credit constraint $c \in \{F, L, N\}$:

$$\max_{\mathbf{q}^i \in X_i^c(\mathbf{q}^{-i})} u_i(q_i) + \textstyle\sum_{k \in V_i} \left[p_k(q_k) q_k^i - p_i(q_i) q_i^k\right]$$

When the credit constraint of player $i \in V$ does not bind, the first order condition for a

[2] A forthcoming note expands their considerations to this setup.

flow q_j^i in the networked economy requires that:

$$p_j(q_j) - u_i'(q_i) + \frac{\partial p_j(q_j)}{\partial q_j^i} q_j^i - \frac{\partial p_i(q_i)}{\partial q_j^i} \sum_{k \in V_i} q_i^k \leq 0$$

Such condition holds with equality whenever a positive quantity is supplied from i to j. It states that the marginal benefit of selling an additional unit, the price, must offset the marginal cost of forgone consumption, the marginal decrease of the price of that outflow and the marginal increase of the prices of all inflows. The first wedge is due to the fact that i is a Cournot supplier of j. While the second wedge is due to the fact i is a monopsonistic buyer at his location.

A necessary condition for trade from i to j to occur is that price player i receives for selling an additional unit to j exceeds the marginal benefit of consuming that unit: $p_j(q_j) > u_i'(q_i)$. In the outflow model, as argued later in detail, such condition is not sufficient for trade to take place. Price distortions curtail the supply of consumption good to each local market. The worst possible use of goods owned is thus consumption and not trade. Hence, when clearing each local market, no buyer would be willing to pay more than such benefit on the last unit purchased. Pricing equation 1 implies that the marginal benefit of the last unit consumed is exactly the price each buyer pays for his inflows.

For clarity's sake let the individual welfare that player i derives from a profile of flows $\mathbf{q}$ be defined by:

$$w_i(\mathbf{q}) = u_i(q_i) + \sum_{k \in V_i} \left[u_k'(q_k) q_k^i - u_i'(q_i) q_i^k \right]$$

In what follows the expression *outflow equilibrium* will be used to refer to a pure strategy Nash equilibria of the outflow competition model. For any of the three credit constraints $c \in \{F, L, N\}$:

Outflow Equilibrium *Flows* $\mathbf{q} \in R_+^E$ *constitute an outflow equilibrium with c-credit if for* $\forall i \in V$*:*

$$\mathbf{q}^i \in \arg\max_{\overline{\mathbf{q}}^i \in X_i^c(\mathbf{q}^{-i})} w_i(\overline{\mathbf{q}}^i, \mathbf{q}^{-i})$$

2.3 A Four Player Example

Consider an economy with four players, labeled $\{a, b, c, d\}$. Let player a be endowed with three units of consumption, b be endowed with one unit, while the remaining two players, c and d, with none. For all players preferences for consumption satisfy $u_i(q) = q^{1/2}$. Constrained efficiency requires all consumers to split consumption good equally when the trade network is connected. Social welfare at such allocation is maximal and equal to 4. Equal sharing, however, will not occur in equilibrium if individuals compete on outflows, even when all trades are feasible.

If the economy is fully connected, in the unique outflow equilibrium player a sells to all his neighbors, player b resells some of the goods bought from a to c and d. Players c and d do not trade among themselves since they are identical and in a symmetric position.[3] Equilibrium flows do not equalize marginal rates of substitutions across individuals. The price paid by consumers c and d for each unit of consumption purchased is 0.63. Such price is higher than the price charged by consumer a to b on the units he purchases from him, 0.47. Even though a has the option to sell directly and at a higher price to the final consumers, c and d, all the units that b resells to them, it is in his best interest to forgo such profits. Indeed he prefers to do so, because he cannot curb the competition from b to supply of the final consumers as deviations on his part would not affect the the sales from b to them.

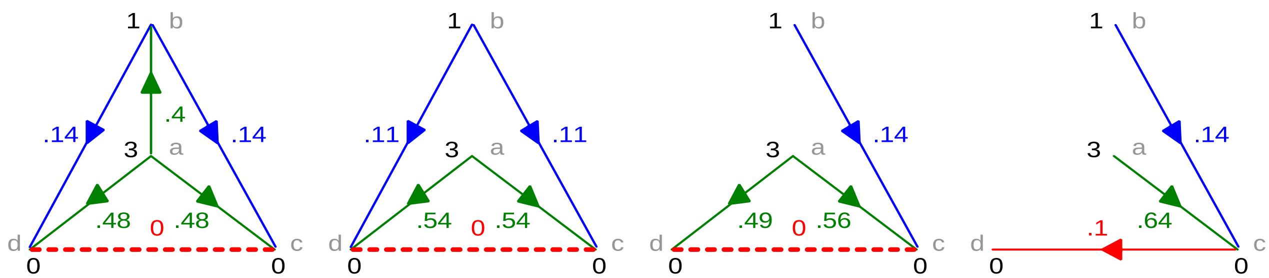

FIGURE 1: The four economies discussed are plotted. In each network: on the vertices are endowments and identities and on the edges are equilibrium flows and their direction.

Equilibrium flows for this economy are reported in first network of figure 1, consumption, purchasing prices and welfare can be found in the first matrix of table 1. In the outflow equilibrium consumers a and b restrict their supply to c and d in order to maximize their gains from trade. The allocation is inefficient and social welfare is equal to 3.91. Player b imposes a 34% markup on all the units he resells. Therefore equilibrium resale at positive profit margins is possible even when resale can be prevented through direct sales.

It could be conjectured that severing the link between consumers a and b, would favor a by giving the opportunity to commit not to sell to b. However, this is not the case. When such link is severed consumption of every player, but for player b, increases. The final consumers, c and d, receive more consumption good at a lower price and are better off. But, consumers a and b are worse off, because direct competition reduces their rents. The equilibrium is still inefficient. Social welfare decreases further to 3.87. A unique price is paid for each unit of consumption purchased by c and d, namely 0.62. This price coincides with the Cournot equilibrium price for the economy without network. Flows for this economy are reported in second network of figure 1, consumption, prices and welfare can be found in the second matrix of table 1.

[3] If the link cd were removed from the trade network, equilibrium flows and prices would not be affected, since no trade takes place between c and d in the fully connected economy.

E1	p	q	w	E2	p	q	w	E3	p	q	w	E4	p	q	w
a	-	1.64	2.08	a	-	1.92	2.05	a	-	1.95	2.08	a	-	2.36	1.92
b	0.47	1.12	1.05	b	-	0.78	1.02	b	-	0.86	1.01	b	-	0.86	1.01
c	0.63	0.62	0.39	c	0.62	0.65	0.40	c	0.60	0.70	0.42	c	0.59	0.71	0.51
d	0.63	0.62	0.39	d	0.62	0.65	0.40	d	0.72	0.49	0.35	d	1.85	0.07	0.13
+	-	4.00	3.91	+	-	4.00	3.87	+	-	4.00	3.86	+	-	4.00	3.57

Table 1: The four matrices report equilibrium prices paid, consumption and welfare for each player and society in each of the four economies.

If link between consumers b and d is further removed from the relationship network, consumer d remains with only a and c as potential suppliers, while player c can still purchase from both a and b. In equilibrium consumers a and b still supply all their neighbors. But, even though consumer c ends up with significantly more consumption good than d, he prefers not to resell anything to d in equilibrium. Indeed, because selling to player d would raise the price paid by c for all units purchased, player c prefers to forgo the revenues he could make from the sale. In these markets it is quite common for connected players with different marginal rates of substitution to prefer not to trade, because a commitment not to resell may significantly reduce the price of the goods purchased. The equilibrium for this economy is characterized in the third network of figure 1 and in the third matrix of table 1. Since player c has two suppliers, while player d has only one that is active, the price c pays for each unit bought is lower than the price paid by d. Competition amongst suppliers reduces prices and increases the quantity supplied. Player a sells more consumption to the competitive market than to the one in which he is a monopolist. Social welfare in this economy is lower than in the previous two examples at 3.86.

Finally consider the economy in which individuals a, b and d only know player c. In such a market players a and b sell to c, who with the units purchased supplies d. The equilibrium for this economy is characterized in the fourth network of figure 1 and in the fourth matrix of table 1. Player c's markup on the units the resells to d is of 210%. Resale occurs despite such markup, because c has access to all suppliers. The amount of resale is constrained by the effects that such a trade bears on the prices paid by player c to his suppliers. Social welfare drops significantly to 3.57. Consumers a and d are negatively affected by the change from the previous environment. Consumer c gains from the previous situation, because he gets monopoly on d and because all the supply in the market is directed towards him.

2.4 Outflow Equilibrium Existence

This section presents sufficient conditions for the existence of an outflow equilibrium under different credit assumptions. For convenience, refer to the expressions:

$$\begin{aligned} C_i(\mathbf{q}^i, \mathbf{q}^{-i}) &= -u_i(q_i) + u_i'(q_i) \textstyle\sum_{k \in V_i} q_i^k \\ R_j^i(q_j^i, \mathbf{q}^{-i}) &= u_j'(q_j) q_j^i \end{aligned}$$

respectively, as player i's *cost* of suppling outflows and as player i's *revenue* from suppling units to market $j \in V_i$. Thus, the welfare of an individual is given by the revenues he makes in each market that he supplies minus the costs of supplying such outflows. Also, define the elasticity of the inverse demand curve with respect to quantity of player i by $\eta_i(q) = -q u_i''(q)/u_i'(q)$.

The proposed proof of existence imposes enough restrictions on the payoffs of each individual to guarantee that best reply maps are singe-valued. First notice that either of the three credit assumptions $c \in \{F, L, N\}$ requires the set of feasible outflows $X_i^c(\mathbf{q}^{-i})$ to be a continuous correspondence with non-empty, compact and convex image sets for every player $i \in V$.

In economies without credit the conditions imposed on the problem require payoffs to be a strictly concave function of the transfers that a player makes, given the vector of transfers made by all others. Since forbidding credit guarantees that no deviation by the suppliers can leave a buyer in a short position, the sufficient conditions for equilibrium existence of this model need to discipline payoffs only on the positive orthant.[4] In particular the following conditions require: costs of supplying outflows to be strictly increasing and convex; revenues in each market to be concave; and therefore equilibrium existence.

Theorem 1 *If assumption $A1_+$ holds and if for any player $i \in V$ and $q \in \mathbb{R}_+$:*

$$-q u_i'''(q)/u_i''(q) \in [-1, 2] \tag{AN}$$

then an outflow equilibrium exists in the no-credit model.

The bounds on the elasticity of the slope of the inverse demand curve with respect to quantity imposed by AN discipline both costs and revenues in any market. In particular, the upperbound requires revenues to be concave in market i. While the lowerbound on the elasticity requires costs to be convex. Since such conditions together with assumption $A1_+$ guarantee that best reply maps are continuous and single-valued, Brouwer's fixed point theorem grants the existence of an outflow equilibrium.

[4] With NC $q_i - \sum_{k \in V_i} q_i^k \geq 0$ for any $i \in V$.

In economies with limited-credit the conditions imposed on the problem require payoffs to be a strictly concave function of the transfers that a player makes, whenever payoffs are increasing in those transfers. Since limiting credit also guarantees that no unilateral deviation by a single supplier can leave a buyer in a short position, sufficient conditions for equilibrium existence will discipline payoffs only on the positive orthant.[5] In particular the following conditions require: revenues in each market to be concave; costs of supplying outflows to be convex whenever revenues of supplying that market are increasing; and therefore equilibrium existence.[6]

Theorem 2 *If assumption A1$_+$ holds and if for any player $i \in V$ and $q \in \mathbb{R}_+$:*

$$-qu_i'''(q)/u_i''(q) \in [-\eta_i(q)/V_i, 2] \tag{AL}$$

then an outflow equilibrium exists both in the limited-credit and no-credit model.

Assumption AL imposes a different lowerbound on the elasticity of the slope of the inverse demand curve with respect to quantity to guarantee that costs are convex whenever the revenues from selling to that market are increasing for any supplier. Since such conditions require payoffs for all players to be strictly concave whenever increasing, it follows that best reply functions are continuous and single-valued and Brouwer's fixed point theorem grants the existence of an outflow equilibrium.

In the full-credit model tighter conditions need to be imposed to make sure that no supplier benefits from unilateral deviations that can leave his buyers in short positions. Moreover payoffs need to be defined and disciplined also on short consumption positions when such transfers are feasible. The assumptions invoked will require one of the following two conditions to hold: (i) costs are convex and revenues are concave whenever increasing; or (ii) revenues are concave and costs are convex whenever revenues increase in that market. In particular the following assumption when coupled with A1, guarantees that either (i) or (ii) holds:

Theorem 3 *If assumption A1 holds and if for any player $i \in V$ and $q \in \mathbb{R}$:*

$$-qu_i'''(q)/u_i''(q) \in [-\eta_i(q)/V_i, 2\eta_i(q)] \tag{AF}$$

then an outflow equilibrium exists in anyone of the three credit models.

Assumption AF requires the elasticity of the slope of the inverse demand curve with respect to quantity to be bounded: below by $-1/V_i$ times the elasticity of the inverse demand

[5] With LC $q_i - q_i^k \geq 0$ for any $k \in V_i$.

[6] That is $\partial R_i^j(q_i^j, \mathbf{q}^{-j})/\partial q_i^j \geq 0$ implies $\partial^2 C_i(\mathbf{q}^i, \mathbf{q}^{-i})/(\partial \mathbf{q}^i)^2 \geq 0$.

curve and above by 2 times the elasticity of the inverse demand curve. Such condition also guarantees that best reply maps are continuous and single-valued and that Brouwer's fixed point theorem applies. Even though assumptions for equilibrium existence in the full-credit model discipline payoffs also on short consumption positions, such conditions can be relaxed on the negative orthant, if one assumes that players with negative after trade positions default and suffer a punishment that does not depend on their trading position. A forthcoming note discusses the issue in detail and provides sufficient conditions for equilibrium existence when such assumptions are invoked.

It can be readily verified that common families of preferences meet the proposed conditions for outflow equilibrium existence:

Corollary 4 *If $u_i(q) = \beta_i q^{\alpha_i} + \varepsilon_i$ for $\alpha_i \in (0,1)$, $\beta_i \in \mathbb{R}_{++}$, $\varepsilon_i \in \mathbb{R}$ and any $i \in V$, then an outflow equilibrium exists in economies with none or limited credit.*

Corollary 5 *If $u_i(q) = -\beta_i e^{-\alpha_i q} + \varepsilon_i$ for $\alpha_i \in \mathbb{R}_{++}$, $\beta_i \in \mathbb{R}_{++}$, $\varepsilon_i \in \mathbb{R}$ and any $i \in V$, then an outflow equilibrium exists in anyone of the three credit models.*

Sufficient conditions for uniqueness in economies with four or fewer players have been found. It is conjectured that such conditions are valid even in economies with more than four player.

2.5 Basic Properties of Outflow Equilibria

This section presents results about equilibrium flow patterns and markups in economies with a finite number of players. Recall that, as argued in section 2.3, the following condition is necessary for consumption to be sold from individual i to j:

$$u_j'(q_j) = p_j(q_j) > u_i'(q_i) \tag{2}$$

An immediate implication of condition 2 is that consumption flows only in one direction on every link, since condition 2 and the reverse inequality cannot hold at once. Thus at most $|E|/2$ flows are positive in equilibrium. In contrast to the Cournot model condition 2 is not sufficient for a trade from i to j to take place. In the outflow model individuals purchasing goods sell to their neighbors only if the gains from trade can compensate them for the monopsony price distortion on their inflows. Equilibrium retail markups are positive because player i purchasing consumption and $q_j^i > 0$ together imply:

$$p_j(q_j) - p_i(q_i) = -u_j''(q_j)q_j^i - u_i''(q_i)\sum\nolimits_{k\in V_i} q_i^k > -u_i''(q_i)\sum\nolimits_{k\in V_i} q_i^k > 0$$

Thus buyers and intermediaries with similar marginal rates of substitution usually refrain from trading even if linked (as in the third example of section 2.3). Resale is pervasive in these economies and has two motives. The more immediate motive is that trading networks have a limited number of links that can be used to transfer goods. The second motive is that sellers have incentives to price discriminate buyers by selling to all individuals to which they are linked at different prices, provided that they make gains on each trade. Which explains why even fully connected economies display equilibrium resale.

For convenience define an individual to be a *source* (*sink*) if he does not buy (sell) consumption. The following proposition summarizes some useful properties of the outflow equilibrium flows. These properties follow directly from the optimality individual decisions:

Proposition 6 *If assumption A1 holds in any outflow equilibrium* $\mathbf{q} \in \mathbb{R}_+^E$*:*
(a) Goods do not cycle and prices strictly increase along the supply chains
(b) Players with marginal utility higher/lower than their neighbors are sources/sinks
(c) Sources sell to all their neighbors with strictly higher marginal utility
(d) If $i, j \in V_k$ *and* $u_i'(q_i) < u_j'(q_j)$*, then* i *buys from* k *only if* j *buys from* k

Part (a) is a straightforward consequence of goods being resold at strictly positive markups. In fact, because the marginal utility of consumption strictly increases along the supply chain, it can never be that an individual buys some of the units he previously sold. Since goods do not cycle flows of goods move from sources to sinks. A flow pattern, however, can have more than one source and/or sink in equilibrium. Part (b) shows that individuals with lower marginal utility than their neighbors are sources and that individuals with higher marginal utility than their neighbors are sinks. Individuals with lower marginal utility than all their neighbors would never buy, because only players with lower marginal utility could supply them. Similarly individuals with higher marginal utility than their neighbors could never sell. Part (c) shows that sources sell to every neighbor with higher marginal utility. For sources condition 2 is not only necessary, but also sufficient for a trade to take place, because sources have no inflows and because outflow price distortions vanish with outflows. Part (d) shows that if two players have a neighbor in common, that neighbor sells to the low marginal utility player only if he sells to the high marginal utility player. Intuitively because the lower marginal utility neighbor pays a lower price, he is supplied only if the high marginal utility neighbor has been supplied first.

If all individuals in the economy are linked and can trade with each other, the positions in the market are symmetric. In such economies differences in flows and welfare driven by endowments and preferences. In particular differences in marginal utilities dictate which of two players sells more to all his neighbors:

Proposition 7 *If assumption A1 holds and if* $u_i'(q_i) < u_j'(q_j)$*, then in any outflow equilibrium of the complete networked economy: (1)* $q_k^i \geq q_k^j$ *and (2)* $q_k^i > 0$ *implies* $q_k^i > q_k^j$*.*

Even though such claim is intuitive it requires some discipline on inflows. In particular, the claim requires the ranking of marginal utilities to coincide with the ranking of supply costs. The complete network structure guarantees that such discipline is maintained. Therefore low marginal utility players sell more. However, without further assumptions it is impossible to guarantee that players with low marginal utilities also buy less from their neighbors. Section 7.2 of the web-appendix provides sufficient conditions on preferences to make sure that the latter holds in a complete network economy.

Whenever endowments are inefficient, equilibrium consumption in any economy populated by a finite number of players is inefficient. In general poorly endowed consumers tend to consume less than what would be efficient and well endowed players tend overconsume. Exceptions to this rule of thumb are possible for specific network and endowment configurations, but never lead to equilibrium efficiency. Occasionally well connected but poorly endowed players reap large gains both in consumption and money through retrade. If endowments are efficient, no trade takes place and a unique price reigns throughout the economy, namely the competitive equilibrium price.

The last result presented in this section considers what happens to this economy if players have several instances to trade on the network. In such environment endowments at each trading round are the final allocation of the previous round. The current version of the result makes use of two strong assumptions: agents do not discount and act myopically. If such assumptions hold, it can be shown that as the number of instances in which to trade on the network grows large outflow equilibrium outcomes become constrained efficient. Thus if players have arbitrarily many instances to trade the limiting allocation of consumption will be efficient and the limiting prices will be competitive in each component of the network. But, since prices paid for each flow differ along the sequence of trades the distribution of individual welfare will not coincide with that of a Walrasian economy.

Proposition 8 *If individuals do not discount and do not account for future actions, as the number of instances in which to trade grows large the sequence of outflow equilibrium trades converges to the constrained efficient outcome whenever such outcome is interior.*

The result hinges on two observations. The first is that all individuals with the lowest equilibrium marginal utility at round t always have an incentive to sell to all their neighbors at round $t + 1$. In turn this implies that the sequence of outflow equilibrium allocations converges. The second observation is that such sequence cannot converge to an allocation that is not constrained efficient, since at the limiting allocation individuals which benefit from a deviation would exist. Requiring the efficient allocation to be interior guarantees that goods do not remain stuck with suppliers which value them little and that can only sell them to individuals which value them less. Since individuals are myopic they are willing to pay different prices for the same flow at different trading instances. Limiting prices are competitive. A generalization of proposition 8 to forward looking players should account for the fact that individuals do not want to pay different prices along the sequence of equilibria. Such result, however, is still under investigation.

Comments on Prices, Welfare and Market Power: In the outflow competition model individuals on the network can be interpreted as a separate local markets. Competitors use their access to different locations to price discriminate their customers. Since the outflow model prevents price discrimination within each market, discriminating across markets is welfare maximizing for suppliers. Neighbors' endowments and access to markets determines buying prices. High equilibrium marginal utility neighbors pay more than low marginal utility neighbors. Goods are often exchanged below the competitive equilibrium price (as in the first example of section 2.3). Because of price discrimination suppliers prefer having access to more local markets. Suppliers occasionally prefer to sell goods to competitors at a discount, even though such goods are used to compete against them to supply consumers at higher prices. This occurs when the sales to competitors bring enough revenues to overcome the losses due to increased competition in the high value markets. Resale is pervasive in such economies, even fully connected economies. It is driven by the profit opportunities that the different prices in the economy present to players. The monopsony wedge is the main force limiting resale, because it increases the cost of supplying goods at each step.

Whenever the economy's endowment is efficient, no trade occurs and the competitive equilibrium price reigns on all links. Thus, transfers of consumption good that bring economy to the competitive outcome always exist. But, no economy of finite size and with an inefficient endowment profile, can ever become efficient as a result of trades in such markets. In fact, even though goods flow to under-endowed individuals the distortions caused by the price effects cannot vanish in any finite size economy. In a networked market not only the endowments, but also the positions held in the market determine individual welfare. Since rents are higher in markets with fewer competitors, having access to more of such markets benefits a seller. Buyers benefit from having access to more sellers since competition decreases the markup that supplier are able to charge to them. Poorly endowed, but well connected individuals can thrive in these economies by selling at a markup most of the units bought to low competition markets. Gains from trades are distributed along the supply chain.

2.6 Adding Links and Welfare

This section details the effects on individual and social welfare of changes in the network. Two prototypical examples are introduced. The first shows that adding a connection can reduce social welfare, an instance of Braess's Paradox. The second, instead, shows that adding a link may reduce the welfare of one of the two players on the newly created relationship. Such examples motivate the study of networks that are not complete.

Braess's Paradox: Consider a market with three consumers $\{a, b, c\}$ and in which individual a is endowed with two units of consumption, b with one unit and the c with none. Preferences of all players satisfy $u(q) = q^{1/2}$. In this economy increasing the set of trading relationships can reduce social welfare. In particular, if the trade network is increased from $\{ac\}$ to $\{ac, ab\}$, social welfare drops. If only players a and c are allowed to trade, player a sells 0.4 units to b at a price of 0.8 and social welfare stands at 2.9. But if consumer a is allowed to trade with b as well as c, he elects to supply both neighbors: b with 0.2 units and c with 0.36 units at different prices. Individual a's price discrimination of b and c leads to a reduction in the amount sold to c and to the drop in social welfare to 2.89. Player a prefers to curtail his supply to c to extract higher marginal rents, since he can recoup that loss in revenue by selling to b. Equilibrium flows, prices and quantities for the two economies are reported in figure 2 (left and center) and table 2 (left and center).

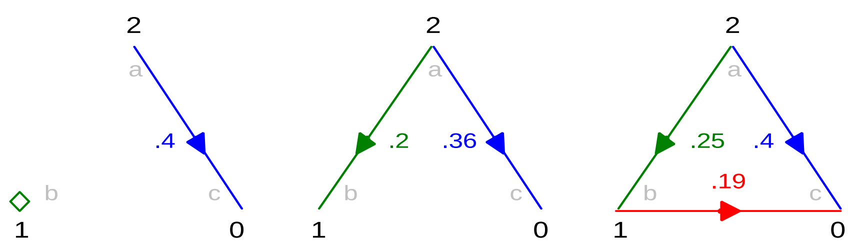

FIGURE 2: On the vertices of a network endowments and identities and on the edges flows.

Despite the negative result about increasing trading relationships, no example has been found in which all of the missing connections reduce social welfare. In general a trading relationship can be found that if added to the economy increases social welfare. In the example presented adding link bc improves social welfare and the complete network is the welfare maximizing market structure. The flows and prices for such network are reported in figure 2 (right) and in table 2 (right).

E1	p	q	w
a	-	1.60	1.58
b	-	1.00	1.00
c	0.79	0.40	0.32
+	-	3.00	2.90

E2	p	q	w
a	-	1.44	1.59
b	0.46	1.20	1.00
c	0.83	0.36	0.30
+	-	3.00	2.89

E3	p	q	w
1	-	1.35	1.54
2	0.49	1.06	1.03
3	0.65	0.59	0.39
+	-	3.00	2.96

TABLE 2: Prices paid, consumption and welfare. Left $\{ac\}$, center $\{ac, ab\}$, right $\{ac, ab, bc\}$.

Individuals Prefer Not to be Linked: It may appear that adding a link to a networked market always improves the individual welfare of the two players being linked. When individuals compete on quantities, however, general equilibrium effects can decrease the welfare of either member of the new trading relationship. The next example presents an economy in which adding a link reduces the welfare of the supplier on the new link.

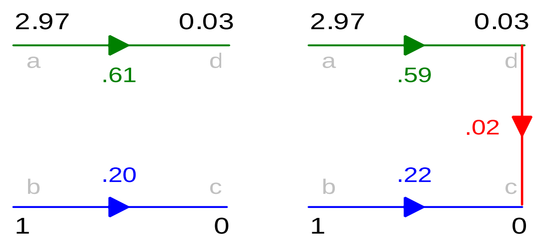

FIGURE 3: On the vertices of a network endowments and identities and on the edges flows.

Consider an economy with four players $\{a, b, c, d\}$ and with endowments $\{2.97, 1, 0, 0.03\}$. In this example when the set of feasible trades increases from $\{ad, bc\}$ to $\{ad, bc, dc\}$, player d's welfare decreases. If only trades in $\{ad, bc\}$ are feasible in the unique equilibrium of this economy players a and b supply their respective customers as monopolies. But when the link cd is added to the network, player d competes with b to supply c. In the unique equilibrium consumer d is worse off than when he could not sell to c, since his payoff decreases from 0.42 to 0.41. Even though player d chooses to supply c, having the option to sell affects the quantity sold to him from a and thus reduces his welfare. All gains from trade on the newly created link are either kept by c or transferred to a. Player a being the monopoly supplier of d is able to extract more rents, because of the steeper demand schedule he faces when d resells. Flows, allocations and prices for the two economies are reported in table 3 and in figure 3.

E1	p	q	w
a	-	2.36	1.92
b	-	0.80	1.12
c	1.12	0.20	0.22
d	0.63	0.64	0.42
+	-	4.00	3.68

E2	p	q	w
a	-	2.38	1.93
b	-	0.78	1.11
c	1.02	0.24	0.24
d	0.65	0.60	0.41
+	-	4.00	3.69

TABLE 3: Prices paid, consumption and welfare. Left $\{ad, bc\}$, right $\{ad, dc, bc\}$.

The two examples presented displayed the negative consequences of adding trading relationships. It would be interesting to argue that despite such negative scenarios a link always exists that, if added, does not decrease either social or individual welfare. If such conjecture were to hold, the complete network would be welfare maximizing and pairwise stable. But even though the claim holds true for all simulations carried out, the proof of such result remains an open question. Two positive results about adding connection are proven in what follows. The first shows that appropriately connecting copies of an economy always increases social welfare (section 3). The second provides sufficient conditions on the network structure to guarantee that equilibrium social welfare becomes efficient as an economy grows large (section 3).

3 Large Economies and the Competitive Equilibrium

This section presents necessary and sufficient conditions on the network structure to guarantee that the outflow equilibria of a networked economy become competitive as number of players grows large. It is shown that resale must vanish for an economy to become competitive and that social welfare increases as an economy grows large. A complete characterization of flows in the limiting economy is also presented. Sufficient conditions for equilibrium existence weaken as an economy grows large. For sake of clarity all results are presented in the context of replica economies. Section 7.5 in the web-appendix shows how results extend to arbitrary sequences of networked markets.

Competitive Equilibrium CE: *The competitive equilibrium of the economy* $\{V, \mathbf{Q}, \mathbf{u}\}$ *consists of a price* $p^* \in R_+$ *for consumption and of an allocation* $\mathbf{q}^* \in R_+^V$ *such that: (1) each player's consumption is optimal given the price; (2) the market of consumption clears.*

By assumption A1 a competitive equilibrium exists and is efficient. For convenience, define the competitive equilibrium net-trade of player i by $e_i = Q_i - q_i^*$. Let D denote the set of individuals buying goods in the CE and let S denote the set of players selling goods in the CE:

$$\begin{aligned} D &= \{i \in V | e_i < 0\} \\ S &= \{i \in V | e_i > 0\} \end{aligned}$$

Players in D are called *competitive buyers* and players in S are *competitive sellers*. For any group of competitive buyers T, let S_T denote the set of competitive sellers that are neighbors of at least one of them. Similarly for any group of competitive sellers T, let D_T denote the set of competitive buyers that are neighbors of at least one of them:

$$\begin{aligned} S_T &= \textstyle\bigcup_{i \in T} V_i \cap S \quad \text{for} \quad T \subseteq D \\ D_T &= \textstyle\bigcup_{i \in T} V_i \cap D \quad \text{for} \quad T \subseteq S \end{aligned}$$

Finally define the CE excess supply faced by a group of buyers and the CE excess demand faced by a group sellers respectively by:

$$\begin{array}{rll} \sigma(T,\mathbf{e}) = \sum_{i\in S_T} e_i + \sum_{j\in T} e_j & \text{for} & T\subseteq D \\ \delta(T,\mathbf{e}) = -\sum_{j\in D_T} e_j - \sum_{i\in T} e_i & \text{for} & T\subseteq S \end{array}$$

The following condition on the economy is necessary and sufficient the existence of direct flows from competitive sellers to competitive buyers that support the CE allocation.

Marriage Condition MC: *Economy* $\{G,\mathbf{Q},\mathbf{u}\}$ *satisfies MC if* $\sigma(T,\mathbf{e})\geq 0$ *for any* $T\subseteq D$.

The marriage condition requires any group of competitive buyers to face a non-negative excess supply. A simple economy in which MC holds is one in which every competitive buyer is linked to every competitive seller: $V_i \supseteq S$ for any $i\in D$. If so, the marriage condition holds trivially because CE requires the market for consumption to clear, $\sigma(D,\mathbf{e})=0$. It can be shown that market clearing also requires MC to be equivalent to the assumption that any group of competitive suppliers faces a non-negative excess demand. An extension of Hall's marriage theorem to this setup shows that MC is equivalent to the existence of direct flows from competitive sellers to competitive buyers that support the CE allocation.

Lemma 9 *An economy* $\{G,\mathbf{Q},\mathbf{u}\}$ *satisfies MC if and only if* $\delta(T,\mathbf{e})\geq 0$ *for any* $T\subseteq S$ *if and only if there exists* $\mathbf{q}\in\mathbb{R}^E_+$ *such that:*

$$e_j + \sum_{i\in S_j} q^i_j = 0 \quad \text{for } \forall j\in D \qquad (i)$$

$$e_i - \sum_{j\in D_i} q^i_j = 0 \quad \text{for } \forall i\in S \qquad (ii)$$

As in Hall's theorem the more surprising part of the lemma is that condition MC is not only necessary, but also sufficient for the existence of competitive flows. Because condition MC imposes discipline both on the excess supply and on the excess demand functions, flows support a competitive equilibrium on both sides of the market.

To impose some discipline on the economy as it grows large this section makes use of a convenient definition of replica economy.

Replica Economy: $\{G(z),\mathbf{Q}(z),\mathbf{u}(z)\}$ *is a* z*-replica of* $\{G,\mathbf{Q},\mathbf{u}\}$ *for some* $z\in N_+$ *if:*

$$V(z) = \{i.s | i\in V \ \ \& \ \ s\in\{1,...,z\}\} \qquad \text{(R1)}$$

$$E(z) = \{(i.s)(j.t) | ij\in E \ \ \& \ \ s,t\in\{1,...,z\}\} \qquad \text{(R2)}$$

$$Q_{i.s}(z) = Q_i \ \ \& \ \ u_{i.s}(z) = u_i \qquad \text{(R3)}$$

The first condition states that the players belonging the z-replica are z copies of the players in the original economy. The second condition requires any copies of two players which were linked in the original economy to be linked. While the third condition states that all copies of a player have the same endowment and utility that he has. The competitive equilibrium does not change as the economy gets replicated. Efficient per-capita social welfare remains constant along any sequence of replica economies. Therefore any copy of a competitive buyer of the original economy remains a competitive buyer in any replica and similarly for sellers.

Consider a sequence of replica economies $\{G(z), \mathbf{Q}(z), \mathbf{u}(z)\}_{z=1}^{\infty}$. The first result of the section shows that the fraction of goods sold directly from competitive seller to competitive buyers converges to one in any economy that becomes competitive if replicated arbitrarily many times. Indeed resale has to vanish in competitive markets, because intermediaries command a rent whenever they are necessary to distribute goods.

Proposition 10 *If a sequence of outflow equilibria of replicas converges to the competitive equilibrium, then equilibrium resale of any individual vanishes as the economy grows large.*

The proof of the claim hinges on two observations: all trades occur at one price in a competitive economy; and no individual ever resells any positive quantity of goods at a zero markup. In the outflow model increasing the size of the economy causes individuals to become price takers as sellers, since each local market becomes more competitive. But individuals never become price takers as buyers since they maintain their monopsony power when purchasing goods at their local market. Thus the wedge on inflow prices cannot disappear if resale persists.

The next theorem is central to the analysis of large economies and shows that condition MC is both necessary and sufficient to guarantee that a sequence of outflow equilibrium allocations and prices to converges to competitive equilibrium as the replicas grow large.

Theorem 11 *Condition MC holds in an economy if and only if a sequence of outflow equilibria of its replicas converges to the competitive equilibrium.*

The proof of the theorem shows that competition among the infinitely many suppliers in each local market eliminates rents on all trades if and only if condition MC holds. If so, outflow equilibrium trades converge to the efficient allocation and a unique price reigns throughout the limiting economy. The claim views the anonymous Walrasian market place as an approximation of a non-anonymous market in which a large number of buyers and sellers can trade. The theorem implies that equilibrium outcomes become competitive in any networked economy in which all competitive buyers can trade with all competitive sellers. Any complete network economy trivially satisfies condition MC and therefore when replicated converges to the competitive equilibrium. Such market structure maximizes social welfare in the economy as the number of players grows large. Condition MC hinges on the definitions of D and S and in turn on that of competitive equilibrium. Thus the knowledge of endowments and preferences is required to test condition MC whenever the network is not complete.

Theorem 11 also implies that no economy in which condition MC fails can be competitive, since intermediation would be necessary for the CE allocation to be attained. In the outflow model middlemen serve no purpose in large and well connected economies. Thus they play no role in competitive economies, as they charge more for goods that can be provided directly by sellers. Different assumptions can be envisioned for middlemen to play positive roles in a networked market. However, if players compete on quantities and if their marginal benefits are decreasing, any complication of the model will still have to account for a force trying to eradicate the presence of middlemen in limiting competitive economies. The proposed definition of replica imposes more discipline on the sequence of economies than required. Section 7.5 shows that claims 11 and 10 hold true with slight modifications even for arbitrary sequences of growing economies.

As in most quantity competition models existence is easier to proof as an economy grows large. If one restricts attention to symmetric equilibria along the sequence of replicas, it is possible to define by $\overline{q}_j^i = \lim_{z\to\infty} z q_{j.s}^{i.t}(z)$ the amount of goods sold in the limiting economy from an individual of type i to all individuals of type j. In any symmetric equilibrium of the limiting economy optimality of flows requires:

$$u_j'(\overline{q}_j) - u_i'(\overline{q}_i) + u_i''(\overline{q}_i)\sum_{k\in V_i}\overline{q}_i^k - \mu_i \leq 0$$

Outflow price distortions vanish in a symmetric equilibrium, because in the limit economy infinitely many individuals compete to supply each neighbor. The price effects on inflows, instead, always remains positive for those individuals who both buy and sell goods in the limit economy. However, since the outflow wedges were the complicating factor in the proof of existence stronger results can be stated for the limit economy.

Proposition 12 *If assumption A1 holds and if:*
(1) $u_i''' \geq 0$*, then a symmetric outflow equilibrium exists in the limit economy*
(2) MC holds, then a competitive outflow equilibrium exist in the limit economy
(3) $V_i \supseteq S$ *for any* $i \in D$*, then a unique outflow equilibrium exist in the limit economy*

In either case revenues in each local market are concave in the limiting economy. Proposition 12 shows that costs of supplying outflows are convex: in (1) by the assumption on the third derivative; in (2) and (3) since condition MC holds and requires resale to vanish in any competitive limiting equilibrium. The stronger conditions on the market structure imposed in (3) guarantee that all equilibria become competitive, since all competitive sellers and buyers can directly trade.

The last result of this section provides sufficient conditions for per-capita social welfare to increase, when an economy is replicated. Since the definition of replica requires every local market to become more competitive as the economy grows large, the result holds with some generality.

Proposition 13 *If conditions for existence are met and if in any z-replica there exists a unique symmetric equilibrium, then symmetric equilibrium per-capita social welfare increases every time the economy is replicated.*

If a unique symmetric equilibrium exists, per-capita social welfare monotonically increases as economy grows large. Thus even an economy in which the retail sector does not vanish becomes more competitive, though not perfectly competitive, as it grows large. This proposition relies on the definition of replica to establish a link between social welfare and network structure by exploiting the nature of the Jacobian matrix of the complementarity problem defining the equilibrium.

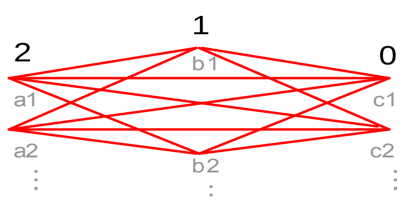

FIGURE 4A: A 2-replica of $\{ab, bc, ac\}$

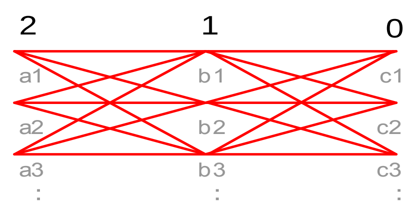

FIGURE 4B: A 3-replica of $\{ab, bc\}$

To conclude the discussion on large markets consider two examples of replica economies. The two initial economies differ in the market structure, but have common preferences and endowments. In both economies there are three players: player a is endowed with two units, player b with one unit and player c with none. Call them respectively: the producer, the intermediary and the consumer. All three players have a common utility function for consumption, namely $u(q) = q^{1/2}$. In the first example, depicted in figure 4A, all players are connected. Such economy satisfies condition MC and therefore if replicated arbitrarily many times converges to the competitive equilibrium. Equilibrium consumption of all three types of players converges to one. Consumption of producers decreases monotonically, consumption of consumers increases monotonically. Intermediaries' consumption first increases and then drops. The price paid by those players first declines and then converges from below to 1/2, the CE price. The price paid by consumers, instead, monotonically decreases to it's competitive equilibrium value. Equilibrium resale vanishes in the limiting economy and intermediaries do not trade in the limit. Per capita social welfare increases monotonically as the equilibrium converges to the competitive outcome. Figures 5A and 5B depict the sequences of consumption and prices of the replicas.

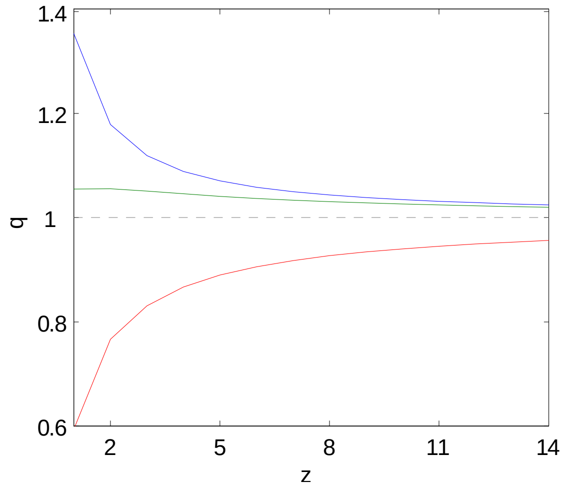

Figure 5A: On the vertical axis consumption on the horizontal the replica

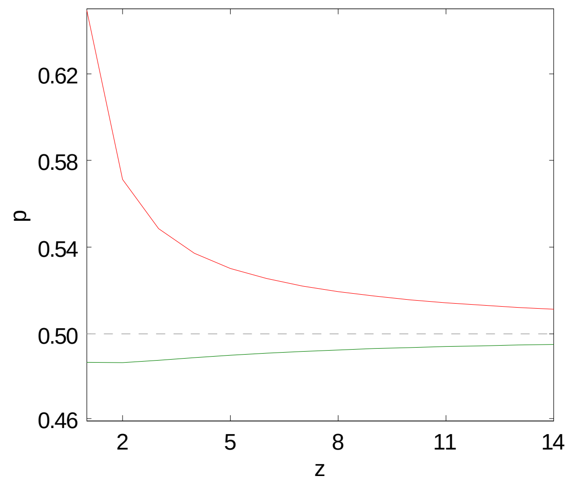

Figure 5B: On the vertical axis prices on the horizontal axis the replica

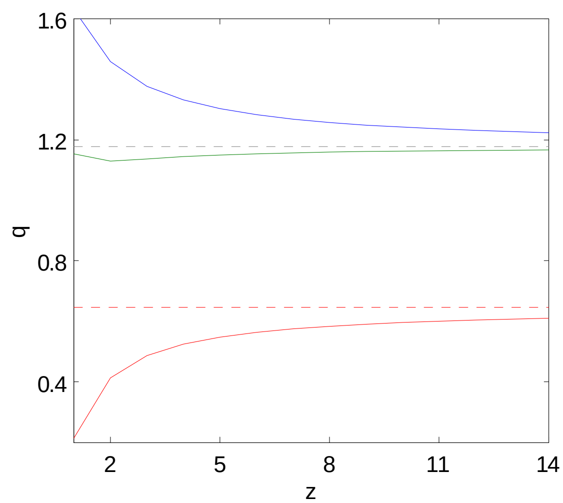

Figure 6A: On the vertical axis consumption on the horizontal the replica

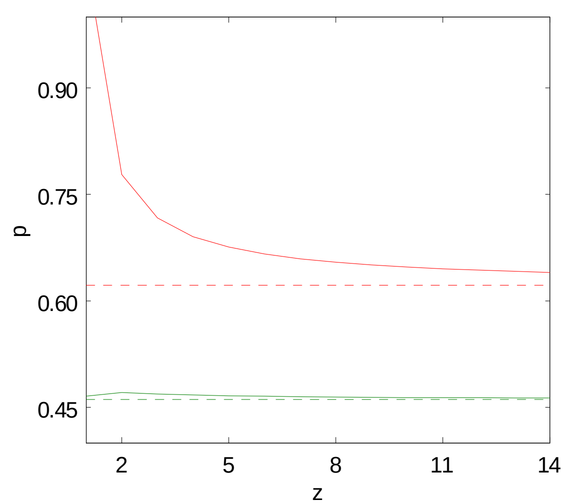

Figure 6B: On the vertical axis prices on the horizontal axis the replica

Now consider the economy, depicted in figure 4B, in which the links between producers and consumers have been removed. In such market intermediaries act as middlemen buying from producers in order to sell to consumers. Such economy does not satisfy MC, since no direct sale from producers to consumers is feasible. Thus no sequence of outflow equilibria of it's replicas will ever converge to the competitive equilibrium. Equilibrium consumption of the three types of players does not converge. In the limiting economy consumption of producers converges to that of the intermediaries and exceeds that of the consumers. Consumption of producers decreases monotonically, while that of consumers increases monotonically. The price paid by intermediaries first grows and then declines converging to a value below the CE price. The price paid by consumers instead monotonically decreases, but remains significantly above the CE price. The limiting markup made by middlemen converges to approximately 30%. Per-capita social welfare increases monotonically as the economy grows large, but remains inefficient. Figures 6A and 6B depict the sequences of consumption and prices of the replicas of such economy.

The outflow model recognizes that the second economy cannot mimic an anonymous Walrasian market because intermediaries are required to exchange goods. Such example pins down the relevant prices for the two anonymous market squares that arise in the limit. Intermediaries in such interpretation are the only players allowed to enter both market squares and collect a rent by transferring goods between the two markets.

4 Inflow Competition in Networked Markets

This section presents the inflow competition model and compares it to the outflow model. A more detailed discussion and examples of it can be found in the web-appendix (sections 7.3 and 7.4).

Inflow Competition: In the inflow competition model every individual selling consumption owns a trading location from which all his neighbors can buy goods from. Competition proceeds as follows: initially individuals simultaneously decide how much consumption to buy from each neighbor; then given the chosen flows market prices are determined at each location so that demand and supply for consumption clear at that node. Buyers expect such prices when choosing their purchases. At each location all goods are sold at the marginal cost of the last unit sold, which is the marginal value of the last unit consumed at that location. The inverse supply curve at the node $j \in V$ satisfies for any $i \in V_j$:

$$p_i^j(\mathbf{q}) = p^j(q_j) = u_j'(q_j)$$

Again an argument à la Kreps and Scheinkman shows that if individuals can commit to their inflows, price competition amongst buyers leads to such prices in each local market.[7] By assumption A1, the price that consumer j receives for the units he sells decreases when inflows increase and increases when outflows increase: $\partial p^j(q_j)/\partial q_i^j > 0$ and $\partial p^j(q_j)/\partial q_j^i < 0$ for any $i \in V_j$. Changes in other flows in the network do not affect the price paid by j. Buyers take into account the effects that their choices have both on the prices received

[7] Individuals offering a lower price are worse off since part of their demand is not met. Individual offering a higher price are worse off since their demand can be met at the lower price.

for each unit sold and on the prices paid for each unit purchased. In a full-credit inflow model buyers' purchases are only limited by the availability of outflows at neighboring locations. Such assumption bounds the set of feasible inflows of player i to $X_i^F(\mathbf{q}_{-i}) = \left\{ \mathbf{q}_i \in \mathbb{R}_+^{V_i} \middle| q_j \geq 0 \text{ if } j \in V_i \right\}$. Thus in the full-credit inflow model individual $i \in V$ solves the following problem:

$$\max_{\mathbf{q}_i \in X_i^F(\mathbf{q}_{-i})} u_i(q_i) + \sum_{k \in V_i} [p^i(q_i) q_k^i - p^k(q_k) q_i^k]$$

In the interior of the domain optimality for the flow q_i^j requires that:

$$u_i'(q_i) - p^j(q_j) - \frac{\partial p^j(q_j)}{\partial q_i^j} q_i^j + \frac{\partial p^i(q_i)}{\partial q_i^j} \sum_{k \in V_i} q_k^i \leq 0$$

Such condition holds with equality whenever a positive quantity is purchased, $q_i^j > 0$. It states that the marginal benefit of buying consumption must offset the price paid, monopoly price distortion on all units sold and the Cournot distortion on the units purchased by seller j. This equation differs from the outflow equation because a different set of price distortions is considered. In the outflow model the suppliers can commit outflows, while in the inflow model the buyers have the option to commit to their inflows. The group having such power benefits in equilibrium by appropriating most gains from trade. Since rents in the inflow model go to buyers, more goods flow to them. Thus an inflow economy is in general more efficient than an outflow economy, as more goods flow to individuals with higher marginal values. In what follows the expression *inflow equilibrium* will be used to refer to a pure strategy Nash equilibria of the inflow competition model.

Result Survey: Sufficient conditions for inflow equilibrium existence require slightly different restrictions on preferences than in the outflow model. In particular:

Theorem 14 *If assumption A1 holds and if for any player $i \in V$ and $q \in \mathbb{R}$:*

$$-q u_i'''(q)/u_i''(q) \in [-2\eta_i(q), \eta_i(q)] \tag{AFI}$$

then an inflow equilibrium exists in full credit model.

A detailed discussion of existence under different credit assumptions is reported in the web-appendix. In the inflow model sellers supply all their customers at a single price. Buyers, however, purchase the same good from different suppliers at different prices. Again it is in their best interest to do so given the choices of others, because price distortions would increase their expenses if they were to concentrate their demand on a single neighboring market. As in the outflow model resale is pervasive, but connected individuals with different marginal rates of substitution do not necessarily trade. In this setup a sufficient condition for trade to place between a seller and a buyer is that the gains from trade exceed the outflow price distortion of the buyer. Examples reported in the web-appendix show that adding links may still reduce social welfare or the welfare of one of the two individuals being connected. If markets are two-sided, markups in the inflow model are smaller than in the outflow model, because they depend on the concavity of the most endowed player. But for arbitrary network structures general equilibrium effects prevent such claim to hold in full generality. Thus the inflow model tends to allocate goods more efficiently since rents flow to the weaker side of the market.

The results for large economies hold unchanged. Again any economy in which the retail sector does not vanish cannot become competitive. Necessary and sufficient conditions on the network structure for a competitive equilibrium to emerge as the economy grows large coincide in the two models. As in the outflow model, sufficient conditions for equilibrium existence simplify when the limiting economy is competitive.

5 Conclusions

Two questions were at the heart of this study: (1) how do networked oligopolistic markets operate? (2) what are necessary and sufficient conditions on the network structure for such economies to become competitive as the number of players grows large?

To tackle the first question two general equilibrium models of quantity competition in networked markets were introduced. In such model individuals had ownership of a trading location and had to choose how many goods to sell to (purchase from) their neighboring locations. Both models had similar implications on trade patterns and pricing, but had different implications on the distribution of rents. In either model flows and prices were endogenously determined, goods did not cycle and prices were strictly increasing along the supply chains. Trade patterns and the distribution of rents were determined by the network structure and by price discrimination across neighboring local markets. Even well connected economies displayed significant price dispersion and non-trivial trade patterns across local markets. Since ownership of the trading location by individuals required goods to be resold at positive markups, not all neighboring individuals with different marginal rates of substitution would necessarily trade. Moreover, price setting behavior implied that adding trading relationships could negatively affect both social welfare and the welfare of anyone of the two individuals on the newly created trading relationship. Such negative scenarios, however, would disappear as an economy grew large and well connected.

To tackle the second question necessary and sufficient conditions on the network structure were presented for a networked economy to become competitive as the number of individuals grows large. It was shown: that no economy that required individuals to resell goods could ever become competitive, since retail would only occur at positive markups; and that economies would become competitive if and only if any group of players purchasing goods were to face a positive excess supply from the competitive sellers linked to them. As in most quantity competition models sufficient conditions for pure strategy equilibrium existence simplified in large and well-connected economies.

The analysis presented suggested Walrasian markets as good approximations of large networked markets in which sufficiently many sellers could compete to supply any buyer and in which no specific individual or group of individuals was necessary to intermediate goods in the economy.

References

[1] Ahuja R., Magnanti T., Orlin J., "Network Flows", Prentice Hall, 1993.

[2] Acemouglu D., Ozdaglar A., "Competition in Parallel Serial-Networks", IEEE Journal on Selected Areas in Communication, 25/6, 2007.

[3] Blume L., Easley D., Kleinberg J., Tardos E., "Trading Networks with Price-Setting Agents", Electronic Commerce, Proceedings of the 8th ACM conference, 2007.

[4] Bollobas B., "Modern Graph Theory", Springer, 1998.

[5] Cournot A., "Recherches sur les Principes Mathematiques de la Théorie des Richesses", 1838, Translation Macmillan, 1897.

[6] Corominas-Bosh M., "Two Sided Networked Markets", Journal of Economic Theory 115/1, 2004.

[7] Chawla S., Roughgarden T., "Bertrand Competition in Networks", Working Paper, 2007.

[8] Goyal S., Joshi S., "Networks of Collaboration in Oligopoly", Games and Economic Behavior 43, 2003.

[9] Jackson M., "Social and Economic Networks", Princeton University Press, 2008.

[10] Karamardian D., "The Complementarity Problem", Mathematical Programming 2, 1972.

[11] Kakade S., Kearns M., Ortiz L., "Graphical Economics", Lecture Notes in Computer Science 3120, 2004.

[12] Kolstad C., Mathiesen L., "Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium", Review of Economic Studies 54/4, 1987.

[13] Kranton R., Minehart D., "Competition in Buyer-Seller Networks", American Economic Review 91/3, 2001.

[14] Kreps D., Sheinkman J., "Quantity Pre-commitment and Bertrand Competition Yield Cournot Outcomes", Bell Journal of Economics 14/2, 1983.

[15] Lever C., "Price Competition on a Buyer-Seller Network", Working Paper, 2009.

[16] Manea M., "Bargaining on Stationary Networks", Working Paper, 2009.

6 Appendix A: Proofs

Outflow Equilibrium Existence

Theorem 1 *If assumption $A1_+$ holds and if for any player $i \in V$ and $q \in R_+$:*

$$-qu_i'''(q)/u_i''(q) \in [-1, 2] \tag{AN}$$

then an outflow equilibrium exists in the no-credit model.

Proof. The no-credit assumption guarantees that the set of feasible outflows $X_i^N(\mathbf{q}^{-i})$ is a continuous correspondence with non-empty, closed, bounded, convex, image sets for every player $i \in V$. Sufficient conditions for the best reply maps to be single-valued are that player i's revenues from the sales to market $j \in V_i$ be concave in q_j^i, that his costs of supplying units be convex in the vector of outflows $\mathbf{q}^i$ and that one of the two conditions be strict. Revenues are concave in each market if for any $ij \in E$:

$$\partial^2 R_j^i(q_j^i, \mathbf{q}^{-i})/(\partial q_j^i)^2 = 2u_j''(q_j) + q_j^i u_j'''(q_j) \leq 0 \tag{E1}$$

Moreover since q_i is a linear function of every outflow q_j^i and since outflows affect costs only though consumption q_i, costs $C_i(\mathbf{q}^i, \mathbf{q}^{-i})$ are a convex function of the vector $\mathbf{q}^i$, if $C_i(\mathbf{q}^i, \mathbf{q}^{-i})$ is convex in q_i. Therefore costs are convex in the vector of outflows $\mathbf{q}^i$ if:

$$\partial^2 C_i(\mathbf{q}^i, \mathbf{q}^{-i})/(\partial q_i)^2 = -u_i''(q_i) + u_i'''(q_i) \textstyle\sum_{k \in V_i} q_i^k \geq 0 \tag{E2}$$

Assumptions $A1_+$ and AN imply that E1 and E2 hold, with at least one of the two holding strictly. In particular since in the no-credit model $q_i - q_i^j \geq 0$, $A1_+$ and the upperbound in

AN imply that revenues are concave since:

$$2u_j''(q_j) + q_j^i u_j'''(q_j) \leq (2u_j''(q_j) + q_j u_j'''(q_j))\mathbb{I}(u_j'''(q_j) > 0) \leq 0$$

where $\mathbb{I}(\cdot)$ denotes the indicator function.[8] Since in the no-credit model $q_i - \sum_{k \in V_i} q_i^k \geq 0$, $A1_+$ and the lowerbound in AN imply that costs are convex since:

$$-u_i''(q_i) + u_i'''(q_i) \textstyle\sum_{k \in V_i} q_i^k \geq (-u_i''(q_i) + u_i'''(q_i)q_i)\mathbb{I}(u_i'''(q_i) < 0) \geq 0$$

Moreover since both indicator maps cannot hold at once either revenues are strictly concave or costs are strictly convex. Thus assumptions $A1_+$ and AN imply that the payoffs for each i are strictly concave and continuous. The strict concavity of individual payoff functions and the fact that for all $\mathbf{q}^{-i}$, $X_i^N(\mathbf{q}^{-i})$ is a non-empty, closed, bounded, convex set imply that the best-response correspondences are single-valued. Continuity of the correspondences $X_i^N(\mathbf{q}^{-i})$ and continuity of the payoff functions imply (by Berge's theorem of the maximum) that the best response functions are continuous. Therefore the existence of outflow equilibrium is guaranteed by Brouwer's fixed point theorem. ■

Theorem 2 *If assumption $A1_+$ holds and if for any player $i \in V$ and $q \in R_+$:*

$$-qu_i'''(q)/u_i''(q) \in [-\eta_i(q)/V_i, 2] \tag{AL}$$

then an outflow equilibrium exists both in the limited-credit and no-credit model.

Proof. Either credit assumption $c \in \{N, L\}$ guarantees that the set of feasible outflows $X_i^c(\mathbf{q}^{-i})$ is a continuous correspondence with non-empty, closed, bounded, convex, image sets for every player $i \in V$. Sufficient conditions for the best reply maps to be single-valued are that player i's revenues from the sales to market $j \in V_i$ be concave in q_j^i, that his costs of supplying units be convex in the vector of outflows $\mathbf{q}^i$ whenever revenues from selling units to i are increasing. Indeed it is not necessary to discipline costs if revenues are decreasing since such outflows would never be a best reply for the suppliers of that individual.

To grant existence it suffices to show $A1_+$ and AL imply that E1 holds and that E2 holds whenever revenues are increasing. The concavity of revenues can be shown as in the proof of 1, since either credit assumption still requires $q_i - q_i^j \geq 0$, since $A1_+$ still holds and because the upperbound in AN coincides with that in AL. Asking revenues be increasing in market i requires:

$$u_i'(q_i) + u_i''(q_i)q_i^j \geq 0 \quad \Rightarrow \quad u_i'(q_i)V_i + u_i''(q_i) \textstyle\sum_{j \in V_i} q_i^j \geq 0$$

[8] In particular $\mathbb{I}(A) = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$.

Moreover if such condition holds, A1$_+$ and the lowerbound in AL imply that costs are convex since:

$$-u_i''(q_i) + u_i'''(q_i) \sum_{k\in V_i} q_i^k \geq u_i'(q_i)V_i \left(-\frac{u_i'''(q_i)}{u_i''(q_i)} - \frac{u_i''(q_i)}{u_i'(q_i)V_i}\right) \mathbb{I}(u_i'''(q_i) < 0) \geq 0$$

Again one of the two conditions on revenues and costs holds strictly. Thus assumptions A1$_+$ and AL imply that the payoffs for each player are strictly concave and continuous whenever increasing. The strict concavity of individual payoff functions implied by AL and the fact that for all $\mathbf{q}^{-i}$, $X_i^c(\mathbf{q}^{-i})$ is a non-empty, closed, bounded, convex set imply that the best-response correspondences are single-valued. Continuity of the correspondence $X_i^c(\mathbf{q}^{-i})$ and continuity of the payoff function imply (by Berge's theorem of the maximum) that the best response functions are continuous. Therefore the existence of outflow equilibrium is guaranteed by Brouwer's fixed point theorem. ■

Theorem 3 *If assumption A1 holds and if for any player $i \in V$ and $q \in R$:*

$$-qu_i'''(q)/u_i''(q) \in [-\eta_i(q)/V_i, 2\eta_i(q)] \tag{AF}$$

then an outflow equilibrium exists in anyone of the three credit models.

Proof. Either of the three credit assumptions $c \in \{N, L, F\}$ guarantees that the set of feasible outflows $X_i^c(\mathbf{q}^{-i})$ is a continuous correspondence with non-empty, closed, bounded, convex, image sets for every player $i \in V$. Sufficient conditions for the best reply maps to be single-valued are that player i's revenues from the sales to market $j \in V_i$ be concave when increasing in q_j^i, that his costs of supplying units be convex in the vector of outflows $\mathbf{q}^i$ whenever revenues from selling units to i are increasing. Indeed it is not necessary to discipline payoffs if revenues are decreasing since such outflows could never be a best reply for players selling to i.

Assumptions A1 and AF imply that E1 and E2 hold, whenever revenues are increasing. Because the lowerbound in AF coincides with that in AL, the convexity of costs can be shown as in the proof of 2. Moreover if revenues increase $u_i'(q_i) + u_i''(q_i)q_i^j \geq 0$, A1 and the upperbound in AF imply that revenues are concave since:

$$2u_j''(q_j) + q_j^i u_j'''(q_j) \leq -u_i'(q_i)\left(\frac{u_i'''(q_i)}{u_i''(q_i)} - \frac{2u_i''(q_i)}{u_i'(q_i)}\right) \mathbb{I}(u_j'''(q_j) > 0) \leq 0$$

Again one of the two conditions on revenues and costs holds strictly. Thus assumptions *A1* and AF imply that the payoffs for each player are strictly concave and continuous whenever increasing. As in the previous theorem such observations suffice to show that best reply maps are continuous and single-valued. Therefore the existence of outflow equilibrium is again guaranteed by Brouwer's fixed point theorem. ■

Basic Properties of the Outflow Model

Proposition 6 *If assumption A1 holds in any outflow equilibrium $q \in R^E_+$:*
(a) Goods do not cycle and prices strictly increase along the supply chains
(b) Individuals with higher (lower) marginal utility than all their neighbors are sources (sinks)
(c) Sources sell to all their neighbors with strictly higher marginal utility
(d) If $i, j \in V_k$ and $u_i'(q_i) < u_j'(q_j)$, then i buys from k only if j buys from k

Proof. (a) Let $T(q) = \{ij \in E | q^i_j > 0\}$ be the set of active trading links. If $ij \in T(\mathbf{q})$ by first order optimality it must be that $u_i'(q_i) < u_j'(q_j)$. If by way of contradiction a cycle $c = \{ij, jk, ..., li\} \in T(\mathbf{q})$, one gets a contradiction since:

$$u_i'(q_i) < u_j'(q_j) < u_k'(q_k) < ... < u_l'(q_l) < u_i'(q_i)$$

(b) If for $i \in V$ and for any $j \in V_i$ equilibrium dictates that $u_i'(q_i) \leq u_j'(q_j)$, then i cannot buy from any neighbor, since $u_i'(q_i) > u_j'(q_j)$ is necessary for $q^j_i > 0$. Therefore his neighbors prefer not to sell to him and $q^j_i = 0$. Similarly when $u_i'(q_k) \geq u_j'(q_j)$ for any $j \in V_i$, player i cannot be selling to any neighbor, since $u_i'(q_i) < u_j'(q_j)$ is necessary for $q^i_j > 0$.
(c) By part (b) if i is a source $u_i''(q_i) \sum_{k \in V_i} q^k_i = 0$. Which in turn implies that, if A1 holds, player i sells to any neighbor $j \in V_i$ with $u_i'(q_i) < u_j'(q_j)$, since there always exists $q^i_j > 0$ for which:

$$-u_i'(q_i) + u_j'(q_j) + q^i_j u_j''(q_j) = 0$$

(d) If A1 holds, optimality of the trade from k to i requires:

$$u_k'(q_k) - u_k''(q_k) \textstyle\sum_{l \in V_k} q^l_k = u_i'(q_i) + q^k_i u_i''(q_i) < u_i'(q_i)$$

Thus if the stated assumptions hold, it must be that:

$$u_k'(q_k) - u_k''(q_k) \textstyle\sum_{l \in V_k} q^l_k < u_j'(q_j)$$

Which is both necessary and sufficient for a trade from k to j to occur. ■

Proposition 7 *If assumption A1 holds and if $u_i'(q_i) < u_j'(q_j)$, then in any outflow equilibrium of the complete networked economy: (1) $q^i_k \geq q^j_k$ and (2) $q^i_k > 0$ implies $q^i_k > q^j_k$.*

Proof. The first part of the proof shows that the marginal costs of supplying units of player j exceed those of player i. Such claim is trivial if $i \in \arg\min_{k \in V} u_k'(q_k)$, since by assumption and by part (c) of proposition 6 marginal costs satisfy:

$$u_i'(q_i) < u_j'(q_j) \leq u_j'(q_j) - u_j''(q_j) \textstyle\sum_{l \in V_j} q^l_j$$

If instead $i \notin \arg\min_{k \in V} u_k'(q_k)$, let $Y_i = \left\{k \in V | q_i^k > 0\right\}$. Then notice that by parts (c) and (d) of proposition 6 it must be that $\emptyset \neq Y_i \subseteq Y_j$. By the optimality of trades from any $h \in Y_i$ to both i and j it must be that:

$$u_i'(q_i) + u_i''(q_i)q_i^h = u_j'(q_j) + u_j''(q_j)q_j^h$$

By adding such conditions for all $h \in Y_i$ and some manipulation, one gets the desired condition on marginal costs since:

$$u_j'(q_j) - u_j''(q_j) \sum_{l \in V_j} q_j^l - u_i'(q_i) + u_i''(q_i) \sum_{l \in V_i} q_i^l = (u_j'(q_j) - u_i'(q_i))(|Y_i|+1) - u_j''(q_j) \sum_{l \in Y_j \setminus Y_i} q_j^l > 0$$

Since the marginal costs of supplying units of player j exceed those of player i, if both players sell a strictly positive amount of consumption to individual k, the marginal revenue of player j must exceed that of player i:

$$u_k'(q_k) + u_k''(q_k)q_k^j > u_k'(q_k) + u_k''(q_k)q_k^i$$

Thus $q_k^i > q_k^j$. If instead player i does not supply k, neither does j, since marginal costs always exceed marginal revenues:

$$u_j'(q_j) - u_j''(q_j) \sum_{l \in V_j} q_j^l > u_i'(q_i) + u_i''(q_i) \sum_{l \in V_i} q_i^l > u_k'(q_k)$$

■

Proposition 8 *If individuals do not discount and do not account for future actions, as the number of instances in which to trade grows large the sequence of outflow equilibrium trades converges to the constrained efficient outcome whenever such outcome is interior.*

Proof. The proof proceeds as follows: first it shows that the lowest marginal utility individuals at any iteration, always sell at the next iteration; then it shows that this requires the sequence of equilibrium allocations to converge; finally it shows that such sequence has to converge to the constrained efficient outcome. It is without loss of generality to restrict attention to connected networks, since separate components of the network will have no influence on each other. Let $q_j^i(t)$ denote the flow of goods from i to j at the t^{th} round of trading, for $t \in \{1, 2...\}$. Consider an inefficient round $t-1$ equilibrium outcome $\mathbf{q}(t-1)$ and an individual $i \in \arg\min_{i \in V} u_i'(q_i(t-1))$. For any such player i it must be that $q_i(t-1) > 0$ because the efficient outcome being interior implies that:

$$q_i(t-1) > q_i^* > 0$$

Since i is a source at round $t-1$, he sells to all his neighbors with strictly higher marginal utility by part (c) of proposition 6. Since the graph is connected there exists $i \in \arg\min_{k\in V} u_k'(q_k(t-1))$ and $j \in V_i$ such that:

$$u_i'(q_i(t-1)) < u_j'(q_j(t-1))$$

or else the outflow equilibrium outcome would be efficient. Therefore $q_j^i(t) > 0$, since i cannot have any inflows at round t by proposition 6. Since the minimal equilibrium marginal utility strictly increases at each iteration and is bounded above by $u_i'(q_i^*)$, it must converge. If minimal marginal utility converges to such upper-bound, the allocation is efficient in the limit since the marginal utility of all players in the economy converges. However, if the sequence of minimal marginal utilities did not converge to such upper-bound, a contradiction arises. In particular, suppose that $\lim_{t\to\infty} \min_{k\in V} u_k'(q_k(t-1)) < u_i'(q_i^*)$ and define $\overline{q}_i = \lim_{t\to\infty} q_i(t-1)$. If so, $\overline{\mathbf{q}}$ cannot be a limiting outcome. Indeed if individuals had an additional round to trade at $\overline{\mathbf{q}}$, they would have a incentives to do so, because again $i \in \arg\min_{k\in V} u_k'(\overline{q}_k)$ and $j \in V_i$ such that $u_i'(\overline{q}_i) < u_j'(\overline{q}_j)$ would exist. A contradiction. Hence, it must be that:

$$\lim_{t\to\infty} \min_{k\in V} u_k'(q_k(t-1)) = u_i'(q_i^*)$$

Marginal utilities of all players converge and limiting outcomes are constrained efficient. ■

Large Economies and the Competitive Equilibrium

Lemma 9 An economy $\{G, \mathbf{Q}, \mathbf{u}\}$ satisfies MC if and only if $\delta(T, \mathbf{e}) \geq 0$ for any $T \subseteq S$ if and only if there exists $\mathbf{q} \in \mathbb{R}_+^E$ such that:

$$\begin{aligned} e_j + \textstyle\sum_{i\in S_j} q_j^i &= 0 \quad \text{for } \forall j \in D && \text{(i)} \\ e_i - \textstyle\sum_{j\in D_i} q_j^i &= 0 \quad \text{for } \forall i \in S && \text{(ii)} \end{aligned}$$

Proof. The first part of the proof shows that $\sigma(H, \mathbf{e}) \geq 0$ for any $H \subseteq D$ if and only if $\delta(H, \mathbf{e}) \geq 0$ for any $H \subseteq S$. Suppose that $\sigma(H, \mathbf{e}) \geq 0$ for $\forall H \subseteq D$. Notice that for any $T \subseteq S$ it must be that $S \backslash T \supseteq S_{D\backslash D_T}$. Therefore:

$$\begin{aligned} -\delta(T, \mathbf{e}) &= -\delta(S, \mathbf{e}) - \sum_{i\in S\backslash T} e_i - \sum_{j\in D\backslash D_T} e_j = -\left[\sum_{j\in D\backslash D_T} e_j + \sum_{i\in S_{D\backslash D_T}} e_i\right] - \sum_{i\in (S\backslash T)\backslash S_{D\backslash D_T}} e_i = \\ &= -\sigma(D\backslash D_T, \mathbf{e}) - \textstyle\sum_{i\in S\backslash (T\cap S_{D\backslash D_T})} e_i \leq -\sigma(D\backslash D_T, \mathbf{e}) \leq 0 \end{aligned}$$

Where the second equality holds since $\delta(S, \mathbf{e}) = 0$. The condition holds with equality if $S\backslash(T \cap S_{D\backslash D_T}) = \emptyset$ and $\sigma(D\backslash D_T, \mathbf{e}) = 0$. Similarly if $\delta(H, \mathbf{e}) \geq 0$ for $\forall H \subseteq S$, then

$D \backslash T \supseteq D_{S \backslash S_T}$ for any $T \subseteq D$ and:

$$\sigma(T, \mathbf{e}) = \delta(S \backslash S_T, \mathbf{e}) - \sum_{j \in D \backslash (T \cap D_{S \backslash S_T})} e_j \geq \delta(S \backslash S_T, \mathbf{e}) \geq 0$$

The next part of the proof shows that condition MC is necessary for i and ii. Indeed suppose that condition MC were violated at some $T \subset D$. So that $\sigma(T, Q) < 0$. Additionally assume that condition ii holds. If so:

$$\sum_{i \in S_T} e_i = \sum_{i \in S_T} \sum_{j \in D_i} q_j^i \geq \sum_{i \in S_T} \sum_{j \in T \cap D_i} q_j^i = \sum_{j \in T} \sum_{i \in S_j} q_j^i$$

But if such were the case, the following would hold:

$$\begin{aligned} 0 \quad &> \quad \sigma(T, \mathbf{e}) = \sum_{i \in S_T} e_i + \sum_{j \in T} e_j \geq \sum_{j \in T} \sum_{i \in S_j} q_j^i + \sum_{j \in T} e_j \\ &\Rightarrow \quad -\sum_{j \in T} e_j > \sum_{j \in T} \sum_{i \in S_j} q_j^i \end{aligned}$$

Which would imply the existence of a player $i \in D$ for which condition i fails.

To prove that MC is sufficient proceed by induction on D. Notice that if $|D| = 1$ the theorem holds trivially because condition MC and feasibility imply that $S_j = S$ for $j \in D$. Flows $q_j^i = e_i$ for any $\forall i \in S$ satisfy both conditions i and ii since $\sum_{i \in V} e_i = 0$. Then suppose that MC is sufficient whenever $|D| \leq m - 1$ to prove that MC is sufficient for $|D| = m$. Initially assume that there exists $H \subset D$ such that $\sigma(H, \mathbf{e}) = 0$. Consider the subgraph with $D' = H$, $S' = S_H$ and the edges restricted to those subsets $E' = E \cap \{ij | i \in S' \cap j \in D'\}$. This subgraph satisfies condition MC trivially, since no condition was altered, but for the fewer players. Thus by the induction assumption on H since $|H| < m$ it is possible to find flows $\mathbf{q} \in \mathbb{R}_+^{E'}$ such that conditions i and ii hold in the subgraph:

$$\begin{aligned} e_j + \sum_{i \in S_j} q_j^i &= 0 \quad \text{for} \quad \forall j \in H \\ e_i - \sum_{j \in D_i \cap H} q_j^i &= 0 \quad \text{for} \quad \forall i \in S_H \end{aligned}$$

It remains to be shown that given such flows the remaining players of the original graph still satisfy the conditions MC. Define $\mathbf{e}'$ the quantities adjusted for such flows and consider the subgraph with $D'' = D \backslash H$, $S'' = S \backslash S_H$ and $E'' = E \cap \{ij | i \in S'' \cap j \in D''\}$. Notice that for any $K \subset D \backslash H$ it must be that:

$$\begin{aligned} \sigma''(K, \mathbf{e}') &= \sigma''(K, \mathbf{e}') + \sigma''(H, \mathbf{e}') - \sum_{i \in S_H \cap S_K} e_i' = \\ &= \sigma(K, \mathbf{e}) - \sum_{i \in S_K} \sum_{j \in D_i \cap H} q_j^i + \sigma(H, \mathbf{e}) - \sum_{i \in S_H \cap S_K} (e_i - \sum_{j \in D_i \cap H} q_j^i) = \\ &= \sigma(K, \mathbf{e}) + \sigma(H, \mathbf{e}) - \sum_{i \in S_H \cap S_K} e_i = \sigma(K \cup H, \mathbf{e}) \geq 0 \end{aligned}$$

Since $\sigma''(H, \mathbf{e}') = \sigma(H, \mathbf{e}) = 0$ and $\sum_{i \in S_H \cap S_K} e'_i = 0$. Which in turn implies by induction that flows $\mathbf{q}' \in \mathbb{R}_+^{E''}$ exist that satisfy condition i and ii, since $|B \backslash H| < m$.

Finally if for any $T \subset D$ we have that $\sigma(T, \mathbf{e}) > 0$, consider $H \in \arg\min_{T \subset D} \sigma(T, \mathbf{e})$. Choose any profile of flows from S_H to $D \backslash H$ such that:

$$\sum_{j \in D \backslash H} \sum_{i \in S_H \cap S_j} q_j^i = \sigma(H, \mathbf{e})$$

Notice that after such transfers condition MC holds in the economy:

$$\sigma(T, \mathbf{e}') \geq \sigma(T, \mathbf{e}) - \sigma(H, \mathbf{e}) \geq 0$$

Moreover after such transfers the economy satisfies all the conditions required in the previous proof of existence of flows satisfying i and ii, since $\sigma(H, \mathbf{e}') = 0$. Thus MC is sufficient. ■

Proposition 10 *If a sequence of outflow equilibria of replicas converges to the CE, then equilibrium resale of any individual vanishes as the economy grows large.*

Proof. Whenever the equilibrium of the replicas converges to the CE, it must be that $\lim_{z \to \infty}(p_j(z) - p_i(z)) = 0$ for any two players $i, j \in V(z)$ for which $\lim_{z \to \infty} q_i(z) > 0$ and $\lim_{z \to \infty} q_j(z) > 0$. Suppose by contradiction that for some player $i \in V(z)$:

$$\lim_{z \to \infty} r_i(z) = \lim_{z \to \infty} \min \left\{ \sum_{k \in V_i(z)} q_i^k(z), \sum_{k \in V_i(z)} q_k^i(z) \right\} > 0$$

Let $\mu_i(z) \geq 0$ denote the multiplier on the non-negativity constraint of player i. If so, first order optimality for flows from i to his neighbors $j \in V_i(z)$ require:

$$\lim_{z \to \infty}(p_j(z) - p_i(z)) = \lim_{z \to \infty} \left(\mu_i(z) - u''_i(q_i(z)) \sum_{k \in V_i(z)} q_i^k(z) - u''_j(q_j(z)) q_j^i \right) > 0$$

Which contradicts the assumption that the economy becomes competitive. ■

Theorem 11 *Condition MC holds in an economy if and only if a sequence of outflow equilibria of its replicas converges to the CE.*

Proof. First the necessity of MC is proven. Notice that MC holds in an economy if and only if MC holds in any of its replicas. If MC were not hold, by lemma 9 no direct flows would exist from competitive sellers to competitive buyers that support the CE allocation in the original economy. Thus resale is necessary for the competitive equilibrium allocation to be an outcome of such economy. Define the minimal competitive resale in the z-replica economy as:

$$r(z) = \min_{\mathbf{q} \in \mathbb{R}_+^{E(z)}} \max_{i \in V(z)} r_i(z) \;\; \text{s.t. (i) and (ii)}$$

Notice that MC fails if and only if $r(1) > 0$. Moreover the definition of replica requires $r(1) = r(z)$, because minimizing the maximum requires players of the same type to resell the identical amounts and because taking averages across players of each type competitive flows of the original economy obtain with $r(1) = r(z)$. Thus any profile of flows leading to the CE allocation would require at least one player to resell a positive amount of goods even in the limit economy. But, by proposition 10 no such outcome could be a limiting outflow equilibrium since no resale can take place in competitive limit economies.

The next part of the proof shows that MC is sufficient for the existence of a competitive limit economy In particular consider flows in the original economy $\mathbf{q} \in \mathbb{R}_+^E$ satisfying (i) and (ii). Such flow exist because MC holds. Then define the sequence of flows $\mathbf{q}(z) \in \mathbb{R}_+^{E(z)}$ as follows: $q_{j.t}^{i.s}(z) = q_j^i/z$ for any $(i.s)(j.t) \in E(z)$. Such flows are direct and satisfy conditions (i) and (ii) for the replica z. Moreover $\lim_{z\to\infty} q_{j.t}^{i.s}(z) = 0$. Thus $\lim_{z\to\infty} \mathbf{q}(z)$ satisfies all the conditions for an outflow equilibrium in the limit economy. In particular, if such are flows chosen by others, no player will be able to sell goods at a price above the CE value, since deviations on his part can only reduce prices in the limit as $\lim_{z\to\infty} q_{j.t}^{i.s}(z) = 0$. Since gains from deviating from $\mathbf{q}(z)$ decrease along the sequence of replicas and vanish in the limit, the limit of $\mathbf{q}(z)$ is competitive and belongs to the limit of the outflow equilibrium correspondence. ■

Proposition 12 *If assumption A1 holds and if:*

(1) $u_i''' \geq 0$, then a symmetric outflow equilibrium exists in the limit economy

(2) MC holds, then a competitive outflow equilibrium exist in the limit economy

(3) $V_i \supseteq S$ for any $i \in D$, then a unique outflow equilibrium exist in the limit economy

Proof. (1) Since in any symmetric equilibrium of the limiting economy the outflow wedges vanish. Revenues in each market are concave. Since the third derivative is positive costs of supplying units are convex. Therefore existence of a symmetric equilibrium in the limit economy follows just as in theorem 3.

(2) This is a consequence of the price distortion vanishing any limiting competitive economy (which requires concave revenues and convex costs) and of theorem 11 (which shows that MC implies that a limiting outcome is competitive).

(3) First notice that for any sequence of outflow equilibrium flow $\mathbf{q}(z) \in \mathbb{R}_+^{E(z)}$ it must be that $\min_{i.s\in V(z)} u_{i.s}'(q_{i.s}(z)) \in S$. Because such a player does not purchase consumption by part (c) of proposition 6 and because by definition of competitive equilibrium $q_{i.s}(z) \geq q_i^*$, player i cannot belong to D as some player in S would have sold units to him. Since $V_j \supseteq S$ *for any* $j \in D$ implies $V_j \supseteq D$ *for any* $j \in S$, player $i \in S$ would sell a positive amount of consumption to all his neighbors $V_i \supseteq D$ with strictly higher marginal utility than him. By contradiction suppose that there exists a sequence of outflow equilibria that does not converge to the competitive equilibrium. If so there exists a player whose marginal utility does not converge to the competitive value. Let $V_+(z) = \{k \in V(z) | q_k(z) > 0\}$ and let $i.s$

be the player for which:

$$\left(u'_{i.s}(q_{i.s}(z)) - u'_{j.t}(q_{j.t}(z))\right) \leq 0 \text{ for } \forall j.t \in V_+(z)$$

Since $V_k \supseteq S$ *for any* $k \in D$ implies $V_k \supseteq D$ *for any* $k \in S$, player $i.s \in S(z)$ would sell a positive amount of consumption to all his neighbors $V_i(z) \supseteq D(z)$ with strictly higher marginal utility than him. Because $\lim_{z\to\infty} |D(z)| = \lim_{z\to\infty} z\,|D| = \infty$ and because $D(z) \subseteq V_+(z) \cap V_{i.s}(z)$, the number of players connected to $i.s$ with equilibrium marginal utility strictly higher than $i.s$ diverges to infinity, $\lim_{z\to\infty} |V_+(z) \cap V_{i.s}(z)| = \infty$. If player $i.s$ were a source, he would be selling a positive amount of consumption to all linked players with strictly higher marginal utility by part (c) of proposition 6. In fact for a limiting outflow equilibrium not to be competitive, requires that for any $j.t \in \lim_{z\to\infty} D(z)$:

$$\lim_{z\to\infty} \left(u'_{i.s}(q_{i.s}(z)) - u'_{j.t}(q_{j.t}(z))\right) < 0$$

Which requires $i.s$ to sells a positive amount of goods to all those players in equilibrium as z grows large and thus $\lim_{z\to\infty} \left(q^{i.s}_{j.t}(z)\right) > 0$. But this gives rise to a contradiction, since $\lim_{z\to\infty} (q_{i.s}(z)) > 0$ and $Q_i < \infty$ imply that by assumption no player can sell a positive amount of goods to infinitely many players. It remains to be shown that the amount of goods resold by player $i.s$ has to vanish to apply proposition 6 in the limit and complete the proof. If for no player $\lim_{z\to\infty} (q_{j.t}(z)) = 0$ the result holds immediately, since only such players can sell to $i.s$. Thus suppose that $V(z)\backslash V_+(z)$ is non-empty in the limit. Let $k.l$ be the player for which:

$$\left(u'_{k.l}(q_{k.l}(z)) - u'_{j.t}(q_{j.t}(z))\right) \leq 0 \text{ for } \forall j.t \in V(z)$$

Players in $V(z)\backslash V_+(z)$ sell to $i.s$ only if $k.l$ sells to $i.s$, since their opportunity cost is higher. But if $k.l$ sells to $i.s$, then by part (c) of proposition 6 $k.l$ sells also to all players with lower marginal utility. Because $\lim_{z\to\infty} |V_+(z) \cap V_{k.l}(z)| = \infty$ and because $k.l$ benefits from selling to all players in $V_+(z) \cap V_{k.l}(z)$, it must that the amount he sells to all, but finitely many players must vanish. In particular since infinitely may players have higher marginal utility than $i.s$ it must be that $\lim_{z\to\infty} q^{k.l}_{i.s}(z) = 0^+$. Necessary conditions for optimality along the sequence would require that, since for any $j.t \in V_{k.l}(z)$ with marginal utility bigger than $i.s$:

$$u'_{k.l}(q_{k.l}(r)) + \mu_{k.l} < u'_{i.s}(q_{i.s}(z)) < u'_{j.t}(q_{j.t}(z))$$

which implies that the limit economy satisfies for all, but finitely many $j.t \in V_{k.l}(z)$, including $i.s$:

$$\lim_{z\to\infty} u'_{k.l}(q_{k.l}(z)) + \mu_{k.l} = \lim_{z\to\infty} u'_{i.s}(q_{i.s}(z)) = \lim_{z\to\infty} u'_{j.t}(q_{j.t}(z))$$

But since resale occurs only at positive markups this implies that in the limit $i.s$ would sell only to finitely many players unless $\lim_{z\to\infty} R_{i.s}(z) = 0$. Moreover because infinitely players have similar incentives to sell to the finitely many players buying goods at positive markups in the limit economy, such players do not exist. Thus, since $\lim_{z\to\infty} R_{i.s}(z) = 0$ no equilibrium that is not competitive can ever exist in the limit. ■

Proposition 13 *If conditions for existence are met and if in any z-replica there exists a unique symmetric equilibrium, then symmetric equilibrium per-capita social welfare increases every time the economy is replicated.*

Proof. Define the total quantity sold from an individual of type i to all individuals of type j in the symmetric equilibrium of an z-replica economy by $\overline{q}_j^i(z) = zq_j^i(z)$. The inequalities defining the symmetric equilibrium of a z-replica (a complementarity problem) can be written in terms of such quantities (omitting the dependence on z):

$$\begin{aligned} f_j^i(\mathbf{q}, \boldsymbol{\mu}|z) &= -u_j'(\overline{q}_j) - u_j''(\overline{q}_j)(\overline{q}_j^i/z) + u_i'(\overline{q}_i) - u_i''(\overline{q}_i)\textstyle\sum_{k\in V_i}^k \overline{q}_i^k + \mu_i \geq 0 \\ f_i(\mathbf{q}, \boldsymbol{\mu}|z) &= \overline{q}_i \geq 0 \end{aligned}$$

Notice that in such system of equations the replica counter z appears only once. Moreover for $z = 1$ such conditions require optimality in the original economy. By assumption any replica economy possesses a unique equilibrium and conditions for existence are met. The complementarity problem has a unique solution only if the Jacobian of that problem is positive definite at the unique symmetric equilibrium (see [12]):

$$J_T(\overline{\mathbf{q}}, \boldsymbol{\mu}|z) = \nabla_{T(z)} f(\overline{\mathbf{q}}, \boldsymbol{\mu}|z) > 0$$

where only the principal minor of Jacobian associated the active indices is considered. The indices active in the z-replica are defined by:

$$T(\overline{\mathbf{q}}, \boldsymbol{\mu}|z) = \left\{ij\in E|\overline{q}_j^i(z) > 0\right\} \cup \{i \in V|\mu_i(z) > 0\}$$

By the implicit function theorem it must be that at the unique equilibrium of the z-replica:

$$\frac{\partial f}{\partial \overline{\mathbf{q}}}\frac{\partial \overline{\mathbf{q}}}{\partial z} + \frac{\partial f}{\partial \boldsymbol{\mu}}\frac{\partial \boldsymbol{\mu}}{\partial z} + \frac{\partial f}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial(\overline{\mathbf{q}}, \boldsymbol{\mu})}{\partial z} = -J_T(\overline{\mathbf{q}}, \boldsymbol{\mu}|z)^{-1}\frac{\partial f}{\partial z}$$

Moreover by the definition of the complementarity problem it must be that:

$$\frac{\partial f_j^i(\overline{\mathbf{q}}, \boldsymbol{\mu}|z)}{\partial z} = \frac{u_j''(\overline{q}_j)\overline{q}_j^i}{z^2} \quad \text{and} \quad \frac{\partial f_i(\overline{\mathbf{q}}, \boldsymbol{\mu}|z)}{\partial z} = 0$$

For notational convenience label players so that in the unique outflow equilibrium of the original economy $q_1 \geq q_2 \geq ... \geq q_V$. Also define $x = \left\{u_j''(\overline{q}_j)\overline{q}_j^i\right\}_{ij\in E}$ and $Z = \left\{z_{kl}^{ij}\right\}_{ij,kl\in E}$ for:

$$z_{kl}^{ij} = \begin{cases} 1/z & \text{if} \quad ij = kl \\ 1 & \text{if} \quad j = k \cap \overline{q}_j^i > 0 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

For such notation and letting $E(\overline{\mathbf{q}}|z) = \left\{ij{\in}E|\overline{q}_j^i(z) > 0\right\}$, one gets that:

$$\begin{aligned} Zx &= \left\{u_j''(\overline{q}_j)(\overline{q}_j^i/z) + u_i''(\overline{q}_i)\textstyle\sum_{k\in V_i}\overline{q}_i^k\right\}_{ij\in E} \\ \frac{\partial\overline{\mathbf{q}}}{\partial z} &= -(1/z^2)J_E(\overline{\mathbf{q}},\boldsymbol{\mu}|z)^{-1}x \end{aligned}$$

Where $J_E(\overline{\mathbf{q}},\boldsymbol{\mu}|z)^{-1}$ is the leading minor of $J_T(\overline{\mathbf{q}},\boldsymbol{\mu}|z)^{-1}$ associated with indexes in $E(\overline{\mathbf{q}}|z)$. The matrix Z is positive definite, since for an appropriate ordering of links it is lower triangular and because all elements on the main diagonal are positive. The matrix can be arranged in a triangular fashion for any profile of equilibrium flows, because goods do not cycle in the economy. Differentiating per-capita social welfare with respect to z one gets that:

$$\begin{aligned} \frac{\partial W(\overline{\mathbf{q}})}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{1}{V}\textstyle\sum_{i\in V}u_i(\overline{q}_i)\right) = \frac{1}{V}\textstyle\sum_{ij\in E}(\partial\overline{q}_j^i/\partial z)(u_j'(\overline{q}_j) - u_i'(\overline{q}_i)) = \\ &= -(1/V)\textstyle\sum_{ij\in E}(\partial\overline{q}_j^i/\partial z)\left(u_j''(\overline{q}_j)\overline{q}_j^i + u_i''(\overline{q}_i)\textstyle\sum_{k\in V_i}\overline{q}_i^k\right) = \\ &= -\frac{1}{V}x'Z'\frac{\partial\overline{\mathbf{q}}}{\partial z} = \frac{1}{Vz^2}x'Z'J_E(\overline{\mathbf{q}},\boldsymbol{\mu}|z)^{-1}x \geq 0 \end{aligned}$$

The last expression is positive since it is a bilinear form and because both Z' and $J_E(\overline{\mathbf{q}},\boldsymbol{\mu}|z)$ are positive definite. In fact because both are positive definite, consider the positive definite square root H of $J_E(\overline{\mathbf{q}},\boldsymbol{\mu}|z)^{-1}$ (i.e. $J_E(\overline{\mathbf{q}},\boldsymbol{\mu}|z)HH = I$) then $Z'J_E(\overline{\mathbf{q}},\boldsymbol{\mu}|z)^{-1} = H^{-1}(HZ'H)H$. Therefore $Z'J_E(\overline{\mathbf{q}},\boldsymbol{\mu}|z)^{-1}$ and $HZ'H$ have the same eigenvalues. Since $HZ'H = H'Z'H$, such matrix is positive definite and thus has only non-negative eigenvalues. The third equality uses the observation that $\partial\overline{q}_j^i/\partial z \neq 0$ implies that the first order condition holds with equality at the original allocation. If in fact $\partial\overline{q}_j^i/\partial z < 0$ then $\overline{q}_i > 0$, but $(z-1)q_j^i(z-1) > 0$ also implies $\overline{q}_j^i = zq_j^i(z) > 0$. If $\partial\overline{q}_j^i/\partial z > 0$ then $\overline{q}_j^i > 0$ and first order conditions hold at z as well since $(z-1)q_i(z-1) < 0$. ■

7 Appendix B: Further Results

7.1 More on Existence & Uniqueness

Non-differentiable negative orthant...

7.2 More on Complete Networks

This section provides completes the characterization of the outflow equilibrium trades for economies in which all individuals are connected. Proposition 7 did show that in such economies low marginal utility players to sell more. However, if all players have a common utility function u and if such utility function satisfies assumptions that are sufficient for equilibrium existence, stronger results can be derived. Specifically assume that:

Assumption A2 *For any player $i \in V$ assume that $u_i = u$ and that:*

$$-qu'''(q)/u''(q) \in [0, \eta(q)]$$

Assumption A2 is satisfied if u is by any CARA utility function. When A2 holds the completeness of the network guarantees that inflows can be ranked across players.

Proposition 15 *If assumptions A1 and A2 hold in any outflow equilibrium of the complete networked economy $Q_i > Q_j$ if and only if $q_i > q_j$ if and only if:*

(1) $q_k^i \geq q_k^j$ and $q_k^i > 0$ implies $q_k^i > q_k^j$
(2) $q_i^k \leq q_j^k$ and $q_j^k > 0$ implies $q_i^k < q_j^k$
(3) $q_j^i \geq q_i^j$ and $q_j^i > 0$ implies $q_i^j = 0$

Whenever the assumptions A1 and A2 hold, individuals consuming more goods buy less from their neighbors and sell more to their neighbors when compared to individuals consuming less. The additional assumptions on the utility function were required to motivate any individual to sell more goods to those local markets in which the demand is steeper. Since those consuming more buy less and sell more, it must be that they started with more endowment. Thus if assumption A2 holds in any outflow equilibrium of the complete network economy an individual consumes more goods if and only if he starts with more goods. When A1 and A2 hold and the network is complete, the flow pattern is completely pinned down by the endowments alone. By proposition 6 it must be that the individuals with biggest endowment never buy, while individuals with the smallest endowment never sell.

7.3 More on Inflow Competition

The first order condition for the inflow model written only in terms of utility functions states that:

$$u_i'(q_i) - u_j'(q_j) + u_j''(q_j)q_i^j + u_i''(q_i)\sum_{k \in V_i} q_k^i \leq 0$$

The following credit constraints respectivey defined as full, limited and no-credit can be used to guarantee existence of inflow equilibria.

$$X_i^F(\mathbf{q}_{-i}) = \left\{\mathbf{q}_i \in \mathbb{R}_+^{V_i} \middle| \, q_j \geq 0 \;\text{ if }\; j \in V_i\right\} \tag{FI}$$
$$X_i^L(\mathbf{q}_{-i}) = \left\{\mathbf{q}_i \in \mathbb{R}_+^{V_i} \middle| \, q_j \geq q_i^j \;\text{ if }\; j \in V_i\right\} \tag{LI}$$
$$X_i^N(\mathbf{q}_{-i}) = \left\{\mathbf{q}_i \in \mathbb{R}_+^{V_i} \middle| \, q_j \geq \textstyle\sum_{k\in V_j} q_k^j \;\text{ if }\; j \in V_i\right\} \tag{NI}$$

In particular, in addititon to theorem 14, the following conditions are sufficient for the existence of a pure strategy equilibrium of the inflow model.

Theorem 16 *If assumption $A1_+$ holds and if for any player $i \in V$ and $q \in \mathbb{R}_+$:*

$$-qu_i'''(q)/u_i''(q) \in [-2, 1] \tag{ANI}$$

then an outflow equilibrium exists both in the limited-credit and no-credit model.

Theorem 17 *If assumption $A1_+$ holds and if for any player $i \in V$ and $q \in \mathbb{R}_+$:*

$$-qu_i'''(q)/u_i''(q) \in [-2\eta_i(q), 1] \tag{ALI}$$

then an outflow equilibrium exists both in the limited-credit and no-credit model.

The remaining results for the inflow model coincide with those results proven in the outflow model. Price distortions in the two models differ, but their implications on competition, flows and markups will be similar. The major differences between the two model are due to the fact that sellers, rather that buyers, own the trading locations in the inflow setup. Such property right will in general lead more rents to flow to buyers, because of their ability to commit to their purchases.

7.4 Inflow Competition Examples

Again consider an economy with three players $\{a, b, c\}$ and with endowments $\{2, 1, 0\}$. All players have a common utility map $u(q) = q^{1/2}$. If only a and c were allowed to trade, 0.75 units would be exchanged between them at a price of 0.45. More goods are traded at a lower price in the inflow equilibrium of this economy than in the outflow equilibrium discussed in section 2.6. Social welfare in the inflow model exceeds that of the outflow outcome.

If the link between players a and b were added, the inflow equilibrium social welfare would drop. As was the case for the outflow model, such change curtails the flow of consumption to the least endowed consumer. In this economy the player a sells 0.71 and 0.1 units respectively

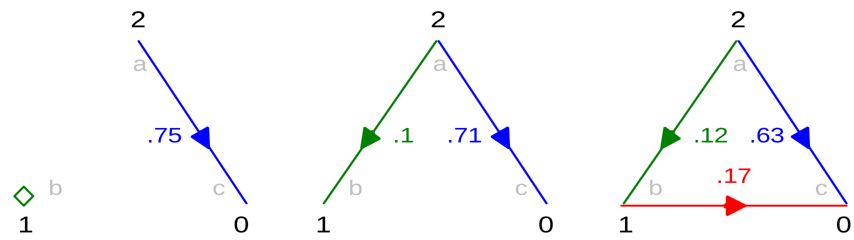

Figure 1: FIGURE 7: On the vertices of a network endowments and identities and on the edges flows.

to players c and b at a price per unit of 0.46. Such change in flows negatively affects the player with the highest demand for consumption and therefore reduces equilibrium social welfare. Again the inflow economy outperforms the outflow outcome, since rents are appropriated by buyers rather than suppliers.

E1	p	q	w
1	0.45	1.25	1.45
2	-	1.00	1.00
3	-	0.75	0.54
+	-	3.00	2.99

E2	p	q	w
1	0.46	1.19	1.46
2	-	1.10	1.00
3	-	0.71	0.52
+	-	3.00	2.98

E3	p	q	w
1	0.45	1.25	1.45
2	0.51	0.95	1.01
3	-	0.80	0.53
+	-	3.00	2.99

TABLE 4: Prices received, consumption and welfare. Left $\{ac\}$, center $\{ac, ab\}$, right $\{ac, ab, bc\}$.

Finally consider the fully connected network. In the inflow equilibrium of this economy two prices reign. The player a sells to both players at a price of 0.45. Player b sells to c at a price of 0.51 all the units purchased from a plus some of his own. All trades take place in equilibrium and it is in the best interest of player c to buy from both suppliers at different prices. Though equilibrium flows are significant, social welfare in the economy is still inefficient. The inflow economy still outperforms the outflow economy in terms of efficiency. Table 4 and figure 7 report inflow equilibrium prices, flows and allocations for the three examples.

7.5 Large Markets Without Replica

A sequence of networked economies $\{G(z), \mathbf{Q}(z), \mathbf{u}(z)\}_{z\in\mathbb{N}}$ is said to increase if for any $z \in \mathbb{N}$ and $i \in V(z)$:

$$(1) \quad V(z) \subset V(z+1) \;\; \& \;\; E(z) \subset E(z+1)$$
$$(2) \quad Q_i(z) = Q_i \;\; \& \;\; u_i(z) = u_i$$

The first conditions states that the number of players and connections grows. The second states that player's tastes and endowments do not depend on the market structure. Let $D(z)$

denote the set of individuals buying goods in the competitive equilibrium $\mathbf{q}^*(z) \in \mathbb{R}_+^{V(z)}$ of the z-th economy and let $S(z)$ denote the set of players selling goods:

$$D(z) = \{i \in V | Q_i < q_i^*(z)\}$$
$$S(z) = \{i \in V | Q_i > q_i^*(z)\}$$

Definition 1 *An networked economy becomes competitive, if a selection of the outflow equilibrium correspondence converges to the CE.*

Given such definitions it is possible to state the following two results about competition in large markets:

Proposition 18 *Consider an increasing sequence of economies if the economy becomes competitive, then outflow equilibrium resale by any individual vanishes.*

Theorem 19 *Consider an increasing sequence of economies if $\exists Z \in \mathbb{N}$ such that condition MC holds in any economy $z > Z$ and if $\lim_{z\to\infty} |D_i(z)| = \lim_{z\to\infty} |S_j(z)| = \infty$ for any $i \in D(z)$ and $j \in S(z)$, then the networked economy becomes competitive.*

Proposition 20 *Consider an increasing sequence of economies if $\exists Z \in \mathbb{N}$ such that $V_i(z) \supseteq S(z)$ for any $i \in D(z)$ and $z > Z$ and if $\lim_{z\to\infty} |D(z)| = \infty$, then any equilibrium of the networked economy becomes competitive.*

As for replica economies no goods can be resold in a competitive economy, since resale occurs only at positive markups. As the number of competitive buyers and sellers grows large and if all buyers meet all sellers the networked economy becomes competitive. In fact if all direct trades are available, retailers get squeezed out of the market, since the rents from selling become arbitrarily small.

7.6 Web-Appendix Proofs

The Complete Network Economy

Proposition 15 *If A1 and A2 hold in any pure strategy equilibrium of the complete networked economy $Q_i > Q_j$* if and only if *$q_i > q_j$ if and only if*
(1) $q_k^i \geq q_k^j$ and $q_k^i > 0$ implies $q_k^i > q_k^j$
(2) $q_i^k \leq q_j^k$ and $q_j^k > 0$ implies $q_i^k < q_j^k$
(3) $q_j^i \leq q_i^j$ and $q_j^i > 0$ implies $q_i^j = 0$

Proof. First we show that $q_i > q_j$ implies (1), (2) and (3). Conditions (1) and (3) follow directly from proposition 6 and 7 given that assumption A2 implies that $q_i > q_j$ if and only if $u'(q_i) < u'(q_j)$. The proof of condition (2) instead relies heavily on the symmetry implicit in A2. If $q_j^k > 0$ and $q_i^k = 0$ the claim is trivial. Also recall that by proposition 6 if $q_j^k = 0$ then $q_i^k = 0$. So suppose that both are positive, $q_j^k > 0$ and $q_i^k > 0$. Then by optimality for the two trades it must be that:

$$\begin{aligned} & u'(q_k) - u''(q_k)\textstyle\sum_{l\in V_k} q_k^l = u'(q_i) + q_i^k u''(q_i) = u'(q_j) + q_j^k u''(q_j) \geq 0 \\ & \Rightarrow q_i^k - q_j^k = \left((u_i'' u_j' - u_i' u_j'') + (u_k' - u_k'' \textstyle\sum_{l\in V_k} q_k^l)(u_j'' - u_i'')\right)/u_i'' u_j'' \end{aligned} \tag{3}$$

Thus rewriting and the two equalities one gets that:Notice that the denominator is always positive. The second term in the numerator is negative since $q_j < q_i$ implies $u''(q_j) < u''(q_i)$ by $u''' > 0$. The first term is also negative since assumption A2 and $q_j < q_i$ imply:

$$-u''(q_j)/u'(q_j) \leq -u''(q_i)/u'(q_i) \quad \Leftrightarrow \quad u''(q_i)u'(q_j) - u'(q_i)u''(q_j) \leq 0$$

Now we prove that (1), (2) and (3) imply $q_i > q_j$. First suppose that $q_i, q_j < q_k = \max_l q_l$. Then since both i and j buy from k condition 3 needs to hold. Since assumption it must be that $q_i^k - q_j^k < 0$ it must be that:

$$\left((u_i'' u_j' - u_i' u_j'') + (u_k' - u_k'' \textstyle\sum_{l\in V_k} q_k^l)(u_j'' - u_i'')\right)/u_i'' u_j'' < 0 \tag{4}$$

But notice that assumptions A1 and A2 require that:

$$u''(q_j) > u''(q_i) \quad \Leftrightarrow \quad q_j < q_i \quad \Leftrightarrow \quad u'(q_i)u''(q_j) - u''(q_i)u'(q_j) < 0$$

Therefore since the denominator of 4 is positive and since both terms in the numerator have the same sign, they must both be negative which implies $q_j < q_i$.

Thus consider the case in which $\max\{q_i, q_j\} = \max_l q_l > \min\{q_i, q_j\}$. If $q_j > q_i$, then proposition 6 would require $q_i^j > 0$ which would contradict condition (3). Finally if $q_i = q_j = \max_l q_l$, then $q_k^i > 0$ implies $q_k^i = q_k^j$, since by proposition 6 both trade with k and:

$$\begin{aligned} u'(q_i) &= u'(q_k) + q_k^i u''(q_k) = u'(q_j) = u'(q_k) + q_k^j u''(q_k) \\ &\Rightarrow q_k^i - q_k^j = (u'(q_i) - u'(q_j))/u''(q_k) = 0 \end{aligned}$$

Which contradicts condition (1). Thus A2 and conditions (1), (2), (3) imply $q_i > q_j$.

Now we show that $q_i > q_j$ implies $Q_i > Q_j$. If $q_i > q_j$ in equilibrium then conditions (1)-(3) holds. Condition (1) and the network being complete imply that $\sum_{k\in V_i} q_k^i > \sum_{k\in V_j} q_k^j$. While condition (2) requires that $\sum_{k\in V_j} q_j^k > \sum_{k\in V_i} q_i^k$. Thus one gets that:

$$Q_i - Q_j = \left(\textstyle\sum_{k\in V_i} q_k^i - \sum_{k\in V_j} q_k^j\right) + \left(\sum_{k\in V_j} q_j^k - \sum_{k\in V_i} q_i^k\right) > 0$$

To prove the converse notice that if $Q_i > Q_j$ and $q_i \leq q_j$ then condition (1) implies $\sum_{k\in V_i} q_k^i \leq \sum_{k\in V_j} q_k^j$ and condition (2) implies $\sum_{k\in V_j} q_j^k \leq \sum_{k\in V_i} q_i^k$. Thus:

$$Q_i - Q_j = \left(\sum_{k\in V_i} q_k^i - \sum_{k\in V_j} q_k^j\right) + \left(\sum_{k\in V_j} q_j^k - \sum_{k\in V_i} q_i^k\right) < 0$$

A contradiction. ■

Large Markets Without Replica

Proposition 18 *Consider an increasing sequence of economies if the economy becomes competitive, then outflow equilibrium resale by any individual vanishes.*

Proof. The proof of proposition 10 made no reference to replicas and thus applies to arbitrary sequences of economies. ■

Theorem 19 *Consider an increasing sequence of economies if* $\exists Z \in \mathbb{N}$ *such that condition MC holds in any economy* $z > Z$*, then an outflow equilibrium becomes competitive.*

Proposition 20 Consider an increasing sequence *of economies* if $\exists Z \in \mathbb{N}$ such that $V_i(z) \supseteq S(z)$ for any $i \in D(z)$ and $z > Z$ and if $\lim_{z\to\infty} |D(z)| = \infty$, then any outflow equilibrium becomes competitive.

Proof. Existence of a competitive outflow equilibrium is implied because MC holds. The proof of uniqueness in part (3) of proposition 6 does not rely on the definition of replica and thus the same argument applies. ■