An Approach To Asset-Pricing Under Incomplete and Diverse Perceptions*

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Abstract

We model a competitive market where risk-neutral traders trade one risk-free and one risky asset with limited short-selling of the risky asset. Traders use "incomplete theories" that give statistically correct beliefs about the market price of the risky asset next period conditional upon information contained in their theories this period; they neither condition upon information outside of their theories nor upon current market prices. The more theories are present in the market, the higher is the equilibrium price of the risky asset, which exceeds the most optimistic trader's expectation of its present-discounted value. When the dividend paid by the risky asset is sufficiently persistent, low asset prices are more volatile than high asset prices. Investors with more complete theories do not necessarily earn higher returns than those with less complete theories, who can earn more than the risk-free rate, despite perfect competition.

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1 Introduction

Recent economic events have brought into focus people's limited abilities to understand the *modus operandi* of financial markets and forecast their outcomes. In this paper, we explore the economic interaction of agents who have diverse yet limited degrees of sophistication in their abilities to recognize patterns and connections among economic variables affecting the markets in which they operate.

The classical approach to modelling expectations in markets assumes that the theories of pricing behaviour that traders use to form their expectations are essentially complete and homogeneous; traders have the same understanding as the modeller of price formation, including which variables affect prices. As a consequence, market traders form expectations on the basis of statistically correct models. In this paper, we attempt to dispense with the assumption that traders use complete models, whilst maintaining the assumption that they form expectations that are statistically correct to the extent possible on the basis of the *incomplete* models that describe their possibly limited perception of the environment. Whereas much can be debated about the empirical plausibility of statistical correctness, the inadequacy of the assumption of complete understanding of markets is rather conspicuous. Just as we do not presume that physicists or other scientists have a complete understanding of all the connections and relationships among objects in the physical world, why should we assume that traders or economists do of markets?[1]

Our model of trade is basic. Every period, risk-neutral traders with an infinite horizon choose between holding a risky asset with a known current dividend and an uncertain future dividend, and a risk-free, one-period bond that yields the known current interest rate. Traders have "theories" about which variables affect prices and form expectations on the basis of the theories that they hold and their limited perception of environment. These theories are "statistically correct" within the limits of the traders' perceptions: in every period, every trader's beliefs about the price of

[1]Indeed, Aragones, Gilboa, Postlewaite and Schmeidler (2005) show that determining whether a k-variable subset of a set of explanatory variables can achieve a given level of R^2 in a linear regression is an NP-complete problem, namely computationally difficult.

the risky asset in the subsequent period match the long-run frequencies, conditional upon everything included in that trader's theory. Different traders may hold different theories.

We begin by establishing the existence of an equilibrium pricing function, where short-sales constraints on the risky asset cause its market price to be set by the trader who is most optimistic about next period's average price. Beliefs about market prices after the next period do not affect the equilibrium pricing equation; consequently, it does not matter whether traders' beliefs about such future prices are statistically correct.[2] We show that the more theories are present in the market, the higher is the price of the risky asset. This price exceeds the most optimistic trader's perceived value of holding the asset in perpetuity because market prices incorporate the option to sell the risky asset in the future to a more optimistic trader.

Our framework provides a natural language for ordering traders by sophistication: one trader whose theory is contained inside of a second trader's theory is unambiguously less sophisticated than the second trader. Although greater sophistication allows traders to make better predictions about future market prices, it does not necessarily translate into higher market returns. Instead, we find that traders' sophistication *relative* to others in the marketplace determines their performance. One trader who is less sophisticated than another—but more sophisticated than most of the market—may find himself losing the asset to his more sophisticated rival whenever its next-period price surpasses his expectation and buying whenever its next-period price falls short of his expectation, a form of winner's curse that produces a return below the risk-free rate. Meanwhile, a trader with a coarser theory may never buy the risky asset, earning the risk-free rate. This suggests that the real losers in asset markets may not be those who understand nothing about the evolution of asset prices—they do not invest in equities—but instead those who in fact have predictive theories of price formation that are simply less complete than those of rival traders. In our model, not only can increased sophistication be negatively correlated with portfolio return in the cross section, it can lower a trader's return holding fixed all other traders' theories.

[2]We study the correctness of traders' beliefs over future price paths in Appendix A.

Incomplete-theory traders sometimes suffer a winner's curse yet, surprisingly, may also benefit from a "loser's blessing" when trading against other traders who use theories neither coarser nor finer than their own. In such settings, a trader may unwittingly buy the asset whenever its next-period price exceeds his expectation and not buy whenever its next-period price falls below his expectation. A trader who benefits from such "favourable selection" earns above the risk-free-rate returns even in the face of perfect competition. However, we show that standard monotonicity conditions imply that all selection is adverse and, consequently, that no trader earns above market returns.

In one simple variant of our model, dividends and interest rates depend upon N economic variables, and each trader's theory comprises some subset of these variables. In this setting, we augment in two ways the result that expanding the collection of theories in the market increases prices. First, we establish a form of converse: unless one collection of theories contains a second, we cannot know that the price of the risky asset is uniformly higher in a market containing the first collection than in a market containing the second. For instance, if every trader's theory were refined (or coarsened), in some setting the price of the risky asset would rise in some states of the world but fall in other states of the world.

Second, we explore how expanding the set of theories in the market affects the range of market prices, as measured by the highest price less the lowest one. Incorporating the fact that many economic variables are persistent, we work in a finite-state model where the state transits to itself with probability γ and to the invariant, long-run distribution with probability $1 - \gamma$. With sufficient persistence, expanding the collection of theories present in the market causes the lowest price to increase by more than the highest price. Put differently, if a theory were to exogenously quit the marketplace, then the lowest price would fall by more than the highest price: volatility is higher at the bottom of the market than at the top, a finding consistent with the empirical evidence in Campbell (2003).

DellaVigna and Pollett (2007) establish that predictable changes in future consumer demand for products from certain industries are not fully appreciated. We give an example linking this observation to a gain-loss asymmetry in asset price changes. In our example, dividends are persistent, some investors perceive the current and correctly predict the next-period dividend, whilst others perceive only the current dividend. This heterogeneity creates an asymmetry in the size of gains and losses that does not exist when all traders are sophisticated. The biggest fall in asset prices happens when the current dividend falls from high to low: unsophisticated investors, who are most optimistic and set the price in this case, receive bad news both about the present (low current dividend) and about the future (likely low future prices/dividends). By contrast, the largest rise in asset prices happens when the current dividend is low and future dividends change from low to high: sophisticated investors, who are the most optimistic and set the price in this case, receive good news, yet only about the future. Because the largest price drops accompany a double dose of bad news, whereas the largest price rises accompany only a single dose of good news, asset prices experience bigger one-day drops than one-day gains, an empirical regularity of equity markets (see again Campbell (2003).

Section 2 reviews the related literature. Section 3 presents a simple example in which the risk-free interest rate cycles deterministically and traders differ in the degree to which they comprehend this process. Section 4 introduces the primitives of the model. Section 5 defines equilibrium, proves existence, and establish monotonicity of the pricing function in the collection of theories present in the market. Section 6 presents the N-variable asset model. Section 7 concludes.

2 Related Literature

Our model shares similarities with several recent models of boundedly-rational information processing, all of which maintain the standard equilibrium assumption that agents hold statistically correct beliefs about the distribution of others' actions but depart from standard equilibrium conditions that agents understand the relationship between those actions and other variables of interest.[3] In their "absent-minded driver's paradox", Piccione and Rubinstein (1997) model someone who cannot figure

[3]Spiegler (2011) includes a thoughtful overview of the literature.

out which node in a information set she is at because she cannot remember whether she previously made a decision (to exit a freeway); the driver forms beliefs by calculating the relative frequencies of reaching the different exits. Piccione and Rubinstein (2003) model consumers who observe the entire history of a deterministic price process but fail to understand how it is generated: a consumer's beliefs about next period's price, given some most recent price realisations, are defined as the conditional long-run frequencies. Eyster and Rabin (2005) introduce the concept of cursed equilibrium for Bayesian games, where players only partially appreciate the connection between other players' private information (types) and actions. They illustrate how such "cursedness" produces information-based trade in no-trade settings.

The formal framework closest to our approach is Jehiel's (2005) elegant analogy-based expectations equilibrium (ABEE), originally elaborated in complete-information games and later extended to incomplete-information games by Jehiel and Koessler (2008).[4] In an ABEE, players do not fully appreciate the history-contingent nature of their opponent's play; instead, each player partitions histories into analogy classes and best responds to beliefs that opponents' behaviour strategies are constant across each class. Consequently, players may have only a coarse understanding of how future actions depend upon current actions. This resembles our solution concept in that players imperfectly understand how future prices depend upon current prices. Dekel, Fudenberg and Levine's (2004) self-confirming equilibrium in games of incomplete information shares the feature that players have correct beliefs about the distribution of others' actions yet allows any beliefs about the mapping between opponents' types and actions consistent with that action distribution. For instance, in a game where a low type of Player 1 always plays "low" and a high type always plays "high", opponents may believe that the low type always plays "high" and the high type always "low" (so long as these two types are equally likely). Such certain misattribution of action to type cannot occur in any cursed or analogy-based-expectations equilibrium and falls outside the rubric of error through inattention or neglect that motivates this

[4] Although the approaches in Piccione and Rubinstein (2003), Eyster and Rabin (2005) and Jehiel (2005) share many similarities, as Spiegler (2011) observes, the main ideas were developed independently.

literature.

Our model differs from all these by proposing a market equilibrium approach and not a game-theoretic one. In so doing, it overlaps with a small finance literature that models traders with incorrect beliefs about asset values. In Harrison and Kreps (1978), traders have non-common priors about the dividend process of an asset. Hong and Stein (2003), Scheinkman and Xiong (2003), Xiong and Yan (2010) model heterogeneous beliefs as arising from traders' overestimating the precision of some signals. Whereas these models allow for biased beliefs about fundamentals and hence prices, ours requires that traders' beliefs about next-period prices be unbiased. Heterogeneity in our model is generated not by biases in the traders' beliefs on the interdependence of different variables but by a diverse perception of which variables affect prices: traders may neglect conditioning variables and hence overlook correlations in the data despite having correct marginal beliefs.[5] Because heterogeneity in our model derives from incompleteness, the completeness of traders' theories provides a natural taxonomy of sophistication missing from models of non-common priors.[6] Bianchi and Jehiel (2010) show how ABEE can support bubbles, where traders know that an asset is overpriced but differ in their perceptions of when the bubble will burst. Fuster, Hebert and Laibson (2011, forthcoming) model a macroeconomy where a risky asset pays a dividend whose innovation is "hump-shaped"—positively serially correlated in the short run, followed by mean-reverting—and described by an AR(40) process. All traders use a simpler "natural-expectations" model that the innovation is an AR(p) process, for $p \leq 40$. Fuster, Hebert and Laibson show that traders overreact to dividend shocks and perceive equities as riskier than they are. In addition to making

[5]Morris (1996) models traders with non-common priors about an asset's dividend who learn the process over time. Hong and Stein (1999) model a financial market where some traders learn dividend-relevant information before others, and no one infers from prices. Hong and Stein (2007) provide a lucid overview of the literature on such belief "disagreement" on financial markets.

[6]Blume and Easley (2006) show in a model of non-common priors that when one trader's priors are closer to the truth than another's as measured by relative entropy, the second will be driven from the market. A crucial difference between their work and ours is their assumption of complete markets essentially allows traders to bet on differences in beliefs about any event; in our model, traders only "bet" about next-period prices.'

specific assumptions on the information structure, their model differs from ours by not incorporating heterogeneous theories in asset pricing.

Kurz (1994a,b) proposes a theory of expectations of traders who do not know the structural relations of a market. His approach complements ours by focusing on a different dimension of heterogeneity in beliefs. Whereas in our model heterogeneity derives from different traders including different subsets of the relevant variables in their beliefs, Kurz's traders include all of the relevant variables in their beliefs. In the simplest variant of his model, the true data generating process is stationary, but traders may hold beliefs that are non-stationary, as long as those beliefs generate the same asymptotic frequencies as the true data-generating process. In our model, we impose that beliefs are stationary and endogenous uncertainty arises form the agents' incomplete perception of the variables that determine asset prices. In contrast to our uniqueness of equilibrium, Kurz's model admits an infinity of equilibria, some of which involve excess volatility.

One way that our model differs from the finance literature (but resembles the economic literature) is that permits comparisons of the different levels of sophistication among traders. Someone who conditions upon more correlates of dividends and interest rates is more "sophisticated" than someone who conditions upon fewer. Certain results hinge crucially upon the assumption that beliefs about the price in each individual future period are statistically correct.

3 Introductory Examples

Consider the market for an asset that yields a dividend of 1 in every period, traded by risk-neutral agents. There are three possible values of the interest rate, $r_h > r_m > r_l > 0$, which cycles deterministically as follows

$$r_h \longrightarrow r_m \longrightarrow r_l \longrightarrow r_m \longrightarrow r_h ...$$

Equivalently, the interest rate follows a Markov process with transitions

$$(r_h, r_m) \longrightarrow (r_m, r_h) \longrightarrow (r_l, r_m) \longrightarrow (r_m, r_l) \longrightarrow (r_h, r_m) ...$$

where the first component in $(\cdot, \cdot)$ is the interest rate in the current period and the second component is the interest rate in the previous period. Each trader chooses between holding the asset and a short-term bond that lasts for one period and yields the current interest rate. The price of the short-term bond equals one. The asset's dividend and the current interest rate are known to all agents.

The following elementary examples illustrate our approach to modelling incomplete understanding of pricing.

Example 1 *[Complete understanding]: Suppose that all traders understand the evolution of the interest rate and of asset prices. In equilibrium, the asset takes on four possible prices: p_h and p_l at r_h and r_l, respectively; p_{ml} at r_m when followed by r_l; and p_{mh} at r_m when followed by r_h. The equilibrium prices are given by the following equations:*

$$\begin{aligned} \frac{1+p_{ml}}{p_h} &= 1+r_h \\ \frac{1+p_l}{p_{ml}} &= 1+r_m \\ \frac{1+p_{mh}}{p_l} &= 1+r_l \\ \frac{1+p_h}{p_{mh}} &= 1+r_m \end{aligned}$$

When $r_h = 0.09$, $r_m = 0.06$ and $r_l = 0.03$, the equilibrium prices are

$$\begin{aligned} p_{ml} &= 16.96 \\ p_{mh} &= 16.49 \\ p_h &= 16.48 \\ p_l &= 16.98 \end{aligned}$$

Example 2 *[Partial understanding]: Suppose that the agents understand only partially the relationship between the price of the asset and the interest rate or, equivalently, that they do not understand the dynamic behaviour of the interest rate. In particular, these agents understand the dependence of the interest rate at time $t+1$*

on the interest rate at time t but fail to perceive its dependence on the interest rate at time $t-1$. Thus, they know that r_l and r_h are followed by r_m, yet fail to predict when r_m precedes r_l and when it precedes r_h. We assume that all traders' beliefs about the interest rate are "statistically correct" in that they are determined by the long-run average, conditional upon the traders' observations of the current interest rate. An agent's incomplete understanding of the behaviour of the interest rate may be summarised as follows:

$$r_h \longrightarrow r_m$$
$$r_m \longrightarrow r_h \text{ with prob } \tfrac{1}{2},\ r_m \longrightarrow r_l \text{ with prob } \tfrac{1}{2}$$
$$r_l \longrightarrow r_m$$

Their beliefs about the behaviour of the next period price conditional upon the current interest rate may be summarised as follows:

$$r_h \longrightarrow p_m$$
$$r_m \longrightarrow p_l \text{ with prob } \tfrac{1}{2},\ r_m \longrightarrow p_h \text{ with prob } \tfrac{1}{2}$$
$$r_l \longrightarrow p_m$$

The equilibrium prices are given by the following equations:

$$\begin{aligned} \frac{1+p_m}{p_h} &= 1+r_h \\ \frac{1+\frac{1}{2}p_l+\frac{1}{2}p_h}{p_m} &= 1+r_m \\ \frac{1+p_m}{p_l} &= 1+r_l \end{aligned}$$

When $r_h = 0.09$, $r_m = 0.06$, $r_l = 0.03$, the equilibrium prices are

$$\begin{aligned} p_h &= 16.31 \\ p_l &= 17.26 \\ p_m &= 16.78 \end{aligned}$$

Example 3 *[Heterogeneous understanding]: The market contains two types of agents. Some agents fully perceive the relationship between interest rate and price as in Example 1. The remaining agents share the same incomplete understanding of that* *relationship as presented in Example 2. Assume that the equilibrium price equals the reservation price of the most optimistic trader.*[7] *Intuitively, the agents with a partial understanding of prices purchase the asset when the interest rate is* r_m *and moving towards* r_h *since they believe that the next period price is equally likely to be either* p_l *or* p_h*. For symmetric reasons, the agents with complete understanding purchase the asset when the interest rate is* r_m *and moving towards* r_l *since they believe that the next period price is* p_l*. At all other interests rates, both types of agents predict the next-period price correctly. The equilibrium prices are given by the following equations:*

$$\begin{aligned}
\frac{1+p_{ml}}{p_h} &= 1+r_h \\
\frac{1+p_l}{p_{ml}} &= 1+r_m \\
\frac{1+p_{mh}}{p_l} &= 1+r_l \\
\frac{1+\frac{1}{2}p_l+\frac{1}{2}p_h}{p_{mh}} &= 1+r_m
\end{aligned}$$

When $r_h = 0.09$*,* $r_m = 0.06$*,* $r_l = 0.03$*, the equilibrium prices are*

$$\begin{aligned}
p_{ml} &= 18.45 \\
p_{mh} &= 18.11 \\
p_h &= 17.84 \\
p_l &= 18.56
\end{aligned}$$

Note that prices are higher than in Examples 1 and 2. As in Harrison and Kreps (1978), higher diversity of beliefs leads to uniformly higher prices.

Coarse-theory traders earn less than the risk-free rate because they fail to appreciate that they only buy at r_m *when on the verge of a capital loss (incorrectly predicting a* 50% *chance of a capital gain).*[8]

[7] Our formal model includes a set of assumptions sufficient for this pricing rule.

[8] The seeming paradox that every trader does weakly worse (and some traders strictly worse) than the risk-free rate is explainable by the fact that asset prices are "too high" relative to fundamentals;

4 The Model

In this section, we present the formal model and introduce the notion of equilibrium with incomplete understanding of pricing behaviour.

4.1 Dividends and Interest Rates

In each of a countably infinite number of periods, traders on a financial market trade one infinitely-lived risky asset and one riskless asset. The risky asset pays a dividend in every period. The riskless asset is a one-period bond that pays back the principal plus the risk-free rate the following period. The riskless asset is in infinite supply and the risky asset only in finite supply.

Both dividend and interest rate are determined by the "state" of the financial world, which evolves over time. Let S be a Polish space and $\mathcal{S}$ the Borel σ-algebra of subsets of S. For example, S could be finite and $\mathcal{S}$ the collection of all subsets of S. The current state is some $s \in S$. Let $\mathcal{R}$ be the Borel σ-algebra on $\mathbb{R}$. The state of the world evolves according to the Markov process described by the transition function $Q : S \times \mathcal{S} \to [0, 1]$. The function $Q(\cdot, A)$ is $\mathcal{S}$-measurable (with respect to $\mathcal{R}$) for any $A \in \mathcal{S}$, and $Q(s, \cdot)$ is a probability measure on $\mathcal{S}$ for any $s \in S$.[9] Given a non-negative, bounded function $f : S \to \mathbb{R}_+$ that is $\mathcal{S}$-measurable, define the operator Tf such that

$$Tf(s) = \int f(y)Q(s, dy),\ s \in S$$

Obviously, Tf is $\mathcal{S}$-measurable and bounded.[10] The mapping $T^k f(s)$ defines the k^{th} iteration of the above operator. Given a probability measure λ on $\mathcal{S}$, define the

one way to think about this is that the asset's issuer earned an excess return in some un-modeled initial period. Morris (1996) interprets prices above fundamentals in a Harrison-Kreps-style model as IPO overvaluation.

[9] When S is finite all functions are $\mathcal{S}$-measurable.

[10] Measurability is true definition when f is the indicator function and thus when f is a simple function. It then follows for any measurable function f by standard convergence arguments (See Dudley Proposition 4.1.5).

operator $T^*\lambda$ such that

$$T^*\lambda\,(A) = \int Q(s,A)\lambda\,(ds)\,,\; A \in \mathcal{S}$$

We assume that there exists a unique invariant probability measure μ such that $\mu\,(A) = T^*\mu\,(A)$ for any $A \in \mathcal{S}$. Sufficient condition for the existence of such measure are standard (see Theorem 12.12 in Stokey and Lucas (1989)).

A set $A \in \mathcal{S}$ is *μ-invariant* if $Q(s, A) = 1$ for μ almost every $s \in A$. The invariant measure μ is *ergodic* (for Q) if $\mu(A) \in \{0, 1\}$ for each A that is μ-invariant. It is well know that, if the invariant distribution is unique, it is also ergodic.

The risky asset yields a dividend $d : S \to [0, \bar{d}]$, where $d\,(s)$ is $\mathcal{S}$-measurable. The one-period, riskless bond pays the interest rate $r : S \to [r_0, r_1]$, where $r\,(s)$ is $\mathcal{S}$-measurable and $r_0 > 0$. The realizations of d and r are known to all agents.

4.2 Expectations

We model an agent's understanding of the next-period price of the risky asset as a subalgebra of $\mathcal{S}$. For example, when $S = S_1 \times S_2$, an agent may fail to perceive that prices depend on S_2 despite recognising a relationship between S_1 and future prices. Or, an agent may have an incomplete, rough description of both S_1 and S_2. We shall assume that the expectations of such agents are correct in that their beliefs are given by the conditional long-run distribution of prices. An agent's subalgebra may or may not include the σ-algebra generated by $d\,(\cdot)$ and $r\,(\cdot)$ as we wish to model the behaviour of agents who observe the dividend and the interest rate without including them in their understanding of the determination of prices.

Given the above invariant measure μ, define the probability space $(S, \mathcal{S}, \mu)$. Let $\mathcal{F}$ be a subalgebra of $\mathcal{S}$. For example, when S is finite, $\mathcal{F}$ can be generated by a partition of S by taking all unions of its elements and a function is $\mathcal{F}$-measurable if it is constant over the elements of such partition. The conditional expectation given $\mathcal{F}$ of a bounded $\mathcal{S}$-measurable function $g : S \to \mathbb{R}_+$ is an $\mathcal{F}$-measurable function

$E(g \mid \mathcal{F})(s)$ such that

$$\int_A E(g \mid \mathcal{F})(s)\, \mu(ds) = \int_A g(s)\, \mu(ds)$$

for any $A \in \mathcal{F}$. Given $B \in \mathcal{S}$, let 1_B denote its indicator function. A regular conditional probability is a function $P_{\mathcal{F}} : \mathcal{S} \times S \to [0,1]$ such that

(1) for each $B \in \mathcal{S}$, $P_{\mathcal{F}}(B, \cdot)$ is $\mathcal{F}$-measurable and

$$P_{\mathcal{F}}(B, s) = E(1_B \mid \mathcal{F})(s)$$

μ-almost surely.

(2) $P_{\mathcal{F}}(\cdot, s)$ is a probability measure on $\mathcal{S}$ for every $s \in S$.

Since S is a Polish space, regular conditional probability exists for any subalgebra of $\mathcal{S}$ by Theorem 10.2.2 in Dudley (2002). When S is finite, regular conditional probability are easily obtained. Let $F(s)$ be the element of the partition generating $\mathcal{F}$ that contains $s \in S$. If $\mu(F(s)) > 0$, set

$$P_{\mathcal{F}}(B, s) = \frac{\mu(F(s) \cap B)}{\mu(F(s))}.$$

Otherwise, $P_{\mathcal{F}}(B, s)$ is arbitrary subject to the restrictions in (1) and (2).

For each subalgebra $\mathcal{F}$ of $\mathcal{S}$, select one regular conditional probability. Given a bounded $\mathcal{S}$-measurable function $g : S \to \mathbb{R}_+$, define the conditional expectation of g as the function $E_{\mathcal{F}}(g) : S \to \mathbb{R}_+$ such that

$$E_{\mathcal{F}}(g)(s) = \int g(y)\, P_{\mathcal{F}}(dy, s).$$

The function $E_{\mathcal{F}}(g)(s)$ is $\mathcal{F}$ measurable and bounded. By Theorem 10.2.5 in Dudley (2002), the function $E_{\mathcal{F}}(g)$ is almost surely identical to the conditional expectation $E(g \mid \mathcal{F})$. Obviously, for any constant c

$$E_{\mathcal{F}}(g + c)(s) = E_{\mathcal{F}}(g)(s) + c$$

and if $g(s) \geq g'(s)$ for any $s \in S$, $E_{\mathcal{F}}(g)(s) \geq E_{\mathcal{F}}(g')(s)$.[11]

[11] Because the derivation of conditional expectations is made via a regular conditional probability, the above monotonicity property holds in every state.

The following simple lemma will be useful later on.

Lemma 1 *Given a bounded $\mathcal{S}$-measurable function $g : S \to \mathbb{R}_+$,*

$$\int E_{\mathcal{F}}(Tg)(s)\mu(ds) = \int g(s)\mu(ds)$$

Proof. The function $E_{\mathcal{F}}(Tg)$ is almost surely identical to the conditional expectation $E(Tg \mid \mathcal{F})$. Thus, by definition,

$$\int E_{\mathcal{F}}(Tg)(s)\mu(ds) = \int Tg(s)\mu(ds)$$

Since the measure μ is invariant,

$$\int Tg(s)\mu(ds) = \int g(s)\mu(ds)$$

by Theorem 8.3 in Stokey and Lucas (1989). ∎

For any **finite** collection Ψ of subalgebras of $\mathcal{S}$, we refer to any element of Ψ as a *theory* and to Ψ as a *collection of theories.* We will sometimes refer to $\mathcal{S}$, the finest possible theory, as the complete theory. It should be noted that S can include lagged variables that are inessential in the Markov process but that, given the agents' partial understanding, can be of use in forming expectations.

5 The Market

The financial market consists of risk-neutral traders who have access to unlimited borrowing at the risk free rate and in every period trade the risk-free and the risky asset. For simplicity, we assume that in each period the agents observe the current dividend and interest rate before trading. The market for the risky asset has no short-selling.[12] Because traders are risk-neutral, the only moment of next period's

[12] Allowing a positive bound on short-selling of the risky asset would not affect equilibrium prices. Shleifer and Vishny (1997) and Stein (2009) describe numerous obstacles to arbitrage that limit short-selling of risky assets.

stochastic price that concerns them is its expectation. In order for the market to clear, each trader must weakly prefer holding the riskless asset to a strategy of buying the risky asset, holding it for one period, and then selling it next period; in addition, some trader must be indifferent between these two.

5.1 Equilibrium

The price of the riskless asset is normalised to equal one. A stationary price function $p : S \to \mathbb{R}_+$ for the risky asset is a bounded measurable function that maps realizations of s to non-negative prices. When the state in period t is s, $Tp(s)$ gives the conditional expected price of the risky asset in period $t+1$. Now let $\mathcal{F}$ be a subalgebra of $\mathcal{S}$. The $\mathcal{F}$-*restricted* expectation of a stationary price function p is $E_{\mathcal{F}}(Tp)$. A trader whose theory is $\mathcal{F}$ forms expectations about the price in period $t+1$ equal to $E_{\mathcal{F}}(Tp)(s)$ in period t when the state is s. Purely for notational convenience, the holder of the risky asset in period t receives its dividend in period $t+1$.

Remark 1 *Given a stationary price function p, the probability measure $\Pi_{\mathcal{F}}$ on $\mathcal{R} \times \mathcal{S}$ such that for any $B \in \mathcal{R}$ and $F \in \mathcal{F}$,*

$$\Pi_{\mathcal{F}}(B \times F) = \int_F Q\left(s, p^{-1}(B)\right) \mu(ds)$$

is the joint distribution of current state and next-period price. The probability measure $\Pi_{\mathcal{F}}$ describes the beliefs of an agent with theory $\mathcal{F}$ about the next-period prices. By Birkhoff's Ergodic Theorem (Theorem 20.7 in Aliprantis and Border (2006)), since μ is ergodic, the frequency of $B \times F$ (obtained via the indicator function defined on the corresponding infinite horizon process) converges almost surely (with respect to the infinite-horizon process) to $\Pi_{\mathcal{F}}(B \times F)$. Hence, the agent's understanding of the relationship between his perception of which variables determine prices and the next-period price is statistically correct. In this sense, we can think of the agent's model as being the limit point of a statistical learning process.

We can round off the "perceived" model of an agent with theory $\mathcal{F}$ as follows. Define the "perceived" transition function $\tilde{Q} : S \times \mathcal{F} \to [0,1]$, where

$$\tilde{Q}(s, A) = E_{\mathcal{F}}(Q(\cdot, A))(s)$$

for any $A \in \mathcal{F}$. The model of an agent with theory $\mathcal{F}$ is then given by S, $\mathcal{F}$, the Markov transition function $\tilde{Q}$, and the probability measure $\Pi_{\mathcal{F}}$. Iterating $\tilde{Q}$ and applying $\Pi_{\mathcal{F}}$ yields beliefs about prices in all future periods. Note that beliefs about prices beyond the next period are not necessarily correct. For instance, in Example 2 a trader at period t facing interest rate r_h would assign probability one-half to p^{t+2} being p_h, despite its true probability being zero. Nevertheless, such errors in perceived correlations do not affect market prices. In Appendix A, we explore the statistical correctness of these beliefs. In particular, we show that μ is the invariant distribution of $\tilde{Q}$, which further distinguishes our model from those of non-common priors like Harrison and Kreps (1978), where the only trader whose model of dividends matches long-run frequencies is one using the correct model.

Definition 1 *A stationary price function p_Ψ is a Ψ-equilibrium price function if*

$$\frac{d(s)}{p_\Psi(s)} + \frac{\max_{\mathcal{F}\in\Psi} E_{\mathcal{F}}(Tp_\Psi)(s)}{p_\Psi(s)} = 1 + r(s) \tag{1}$$

for any $s \in S$.

The above equilibrium condition is analogous to the one in Harrison and Kreps (1978). For the rationale for this condition, consider first one-period buying/selling strategies. At prices below $p_\Psi(s)$, unboundedly large demands from traders with the highest *$\mathcal{F}$-restricted* expectations of the price function exceed the finite supply of the risky asset. At prices above $p_\Psi(s)$, no trader wishes to hold the risky asset. Hence, the market clears at $p_\Psi(s)$. Now consider a trader who at time t wishes to buy the risky asset and hold it for some $\tau > 1$ periods before selling it. Because the equilibrium condition implies that he weakly prefers investing in the riskless asset to the risky asset at time t, he perceives that a strategy of holding the riskless asset for one period before buying the risky asset in period $t+1$ and holding for $\tau-1$ periods does weakly better than his original strategy. This observation leads to the conclusion that equilibrium prices depend only upon one-period tradeoffs.

An agent with an $\mathcal{F}$ theory forms his expectation of the price of the risky asset in period $t+1$ on the basis of his understanding of future prices. The agent's expectations are correct in that his beliefs are consistent with the long-run frequencies of prices. Naturally, if $\Psi=\{\mathcal{S}\}$, then $p_{\Psi}(s)$ is a rational expectation price function.

Remark 2 *In our model, agents are oblivious to the incompleteness of their theories, not understanding that prices are determined by variables not included in their theories. In particular, agents do not condition on the current price to form expectations about unobserved variables, as in standard rational-expectation models. This is a common feature of models of asset pricing with heterogeneous beliefs, as pointed out in Hong and Stein (2007). It should also be noted that if the agents have complete theories but lack information about some variables, allowing them to condition on the equilibrium price function in the equilibrium condition (1) would not necessarily yield a rational-expectation equilibrium. The reason is that, in general, partial observations on Markov processes do not necessarily have a Markov structure. For the same reason, our model is not equivalent to one in which traders understand the environment but fail to extract information on unobserved variables from the equilibrium price. Such model is not necessarily stationary and would not yield a stationary equilibrium.*

The fact that traders do not condition on the current price to extract information does not imply that prices do not have an informational role. In Walrasian Equilibria, for instance, prices convey information about mutual gains from trade that is fully exploited by traders. Information is extracted and aggregated unintentionally via the unmodelled equilibrating process that leads to equilibrium prices.

5.2 Existence

The next result shows that an equilibrium price function exists and is unique for any specification of the agents' understanding of this market. Because traders' theories are exogenous and do not depend upon the equilibrium price function, our model does not suffer the non-existence problems of rational-expectations equilibrium.

Theorem 2 *For any collection Ψ there exists a unique Ψ-equilibrium price function.*

Proof. Define the mapping

$$\mathcal{T}(p)(s) = \frac{1}{1+r(s)}\left(d(s) + \max_{\mathcal{F}\in\Psi} E_{\mathcal{F}}(Tp)(s)\right)$$

from the set of bounded measurable functions over S to itself.[13] Note that $\mathcal{T}$ is monotone and that, given a constant $c \geq 0$

$$\mathcal{T}(z+c) \leq \mathcal{T}(z) + \frac{c}{1+r_0}.$$

Since the set of bounded measurable functions over S is a closed subspace of the set of bounded functions in the uniform metric, Blackwell's sufficient conditions (see Theorem 3.53 in Aliprantis and Border (2006)) are satisfied. ■

5.3 Properties of the Equilibrium Price Function

5.3.1 Monotonicity

The following result shows that enlarging the number of theories in the market leads to uniformly higher prices. The equilibrium price distribution generated by the larger collection of theories first-order stochastically dominates that generated by a smaller one.

Theorem 3 *Let Ψ and Ψ' be two collections of theories, where $\Psi \subset \Psi'$. For each $s \in S$, $p_{\Psi}(s) \leq p_{\Psi'}(s)$.*

[13] See for example Theorem 4.27 in Aliprantis and Border (2006).

Proof. For any bounded measurable function $p : S \to \mathbb{R}_+$, the contraction mapping in the proof of Theorem 2 has the property that $\mathcal{T}_\Psi (p) \leq \mathcal{T}_{\Psi'} (p)$. The claim then follows from the fact that the mapping is monotonic. ■

Intuitively, the more theories present in the market, the more optimistic is the most optimistic trader about next period's price. The introduction of a new theory $\mathcal{A}$ into a market with preexisting theories Ψ affects prices only if in some state s, $\max_{\mathcal{F}\in\Psi} E_{\mathcal{F}} (Tp_\Psi) (s) < E_{\mathcal{A}} (Tp_\Psi) (s)$. In fact, when the collection of theories present in the market expands, either almost all prices remain unchanged or almost all prices increase; it cannot happen that some prices rise whilst others remain unchanged.

Proposition 4 *Let Ψ and Ψ' be two collections of theories where $\Psi \subset \Psi'$. Then, either $p_\Psi (s) = p_{\Psi'} (s)$ μ-almost surely or $p_\Psi (s) < p_{\Psi'} (s)$ μ-almost surely.*

Proof. Suppose that there exists some $A \in \mathcal{S}$ for which $0 < \mu(A) < 1$ and for any $s \in A$, $p_\Psi (s) < p_{\Psi'} (s)$, whilst for any $s \in A^c$, $p_\Psi (s) = p_{\Psi'} (s)$, where $A^c = S \backslash A$. Because μ is ergodic, there exists some $\mathcal{S}$-measurable $B \subset A^c$ such that $Q (s, A^c) < 1$ for $s \in B$ and $\mu (B) > 0$, as A^c is not invariant. Thus, $Q (s, A) > 0$ for any $s \in B$, which implies that

$$Tp_{\Psi'} (s) > Tp_\Psi (s) \text{ for any } s \in B.$$

Consider any $\mathcal{F} \in \Psi$. Since $E_{\mathcal{F}} (Tp_\Psi) (s)$ and $E_{\mathcal{F}} (Tp_{\Psi'}) (s)$ are $\mathcal{F}$-measurable, the set

$$X_{\mathcal{F}} = \{s \in S : E_{\mathcal{F}} (Tp_\Psi) (s) = E_{\mathcal{F}} (Tp_{\Psi'}) (s)\}$$

is in $\mathcal{F}$. Obviously, $B \subseteq \bigcup\limits_{\mathcal{F}\in\Psi} X_{\mathcal{F}}$. Then, by the definition of conditional expectation,

$$\begin{aligned} \int_{X_{\mathcal{F}}} Tp_\Psi (s) \mu (ds) &= \int_{X_{\mathcal{F}}} E_{\mathcal{F}} (Tp_\Psi) (s) \mu (ds) \\ &= \int_{X_{\mathcal{F}}} E_{\mathcal{F}} (Tp_{\Psi'}) (s) = \int_{X_{\mathcal{F}}} Tp_{\Psi'} (s) \mu (ds) \end{aligned}$$

and thus

$$\int_{\bigcup\limits_{\mathcal{F}\in\Psi} X_{\mathcal{F}}} Tp_\Psi (s) \mu (ds) = \int_{\bigcup\limits_{\mathcal{F}\in\Psi} X_{\mathcal{F}}} Tp_{\Psi'} (s) \mu (ds)$$

which is a contradiction since $\mu(B) > 0$. ■

The intuition for the above result is simple. When the price rises in state s, it also rises in state $s' \neq s$ because (loosely speaking) there is positive probability that s' transits to s or to some s'' that transits to s, etc.; ergodicity ensures that a rise in price in one state propagates to (almost) all others.

5.3.2 Prices and Fundamental Values

One important question is how prices relate to real fundamentals and perceived fundamentals. We define the fundamental value of an asset as the expected present-discounted value of holding it in perpetuity, where expectations are taken with respect to $\mathcal{S}$, the finest possible theory. Recall that $T^k f(s)$ is the k^{th} iteration of the operator T on the function f. For notational simplicity, in this subsection we assume that the interest rate is deterministic and equal to r.

Definition 2 *The* fundamental value *of the risky asset in state s is*

$$V(s) := \frac{1}{1+r}\left(d(s) + \sum_{k=1}^{\infty} \frac{T^k d(s)}{(1+r)^k}\right).$$

Fundamentals in state s equal the correct expected present-discounted value of the infinite stream of dividend payments that ownership confers upon the asset holder. Obviously,

$$V(s) = \frac{1}{1+r}\left(d(s) + TV(s)\right)$$

Agents with coarser theories perceive fundamentals to differ from $V(s)$. We define the *$\mathcal{F}$-perceived fundamental value* of the risky asset recursively where expectations are taken with respect to $\mathcal{F}$.

Definition 3 *The* $\mathcal{F}$-perceived fundamental value *of the risky asset is the function $V_{\mathcal{F}} : S \to \mathbb{R}_+$ such that*

$$V_{\mathcal{F}}(s) := \frac{1}{1+r}\left(d(s) + E_{\mathcal{F}}\left(TV_{\mathcal{F}}\right)(s)\right) \tag{2}$$

The following lemma shows that the perceived long-run value of holding the asset is equal to ex-ante value of receiving the expected dividend in perpetuity

Lemma 5 *For any theory $\mathcal{F}$,*

$$\int V_{\mathcal{F}}(s)\mu\,(ds) = \frac{1}{r}\int d(s)\mu\,(ds)$$

Proof. It follows from equation (2) and Lemma 1. ■

With a single theory present in the market, the price equals perceived fundamentals under that theory.

Corollary 6 *If $\Psi = \{\mathcal{F}\}$, then in each s, $p_{\Psi}(s) = V_{\mathcal{F}}(s)$.*

Proof. It follows by the definition of $V_{\mathcal{F}}$ since the equilibrium price function is unique by Theorem 1 ■

When all traders in the market use the complete theory, the price coincides with the rational-expectations equilibrium price. When the market includes more than one theory, Theorem 3 tells us that prices must be at least as high as the maximum perceived fundamentals in every state. The following results shows that, unless all $\mathcal{F}$-perceived fundamental values are almost surely identical, prices must exceed the maximum fundamental value almost surely.

Proposition 7 *For any collection of theories Ψ and $s \in S$,*

$$p_{\Psi}(s) \geq \max_{\mathcal{F}\in\Psi} V_{\mathcal{F}}(s).$$

Moreover

(i) if for some $\mathcal{G}$ and $\mathcal{G}'$ in Ψ, the set

$$\{s \in S : V_{\mathcal{G}}(s) \neq V_{\mathcal{G}'}(s)\}$$

has positive μ measure, then

$$p_{\Psi}\,(s) > \max_{\mathcal{F}\in\Psi} V_{\mathcal{F}}(s) \ \mu\text{-almost surely.}$$

(ii) if for any $\mathcal{G}$ and $\mathcal{G}'$ in Ψ, the set

$$\{s \in S : V_{\mathcal{G}}(s) \neq V_{\mathcal{G}'}(s)\}$$

has zero μ measure, then

$$p_{\Psi}(s) = V_{\mathcal{F}}(s) \ \mu\text{-almost surely for any } \mathcal{F} \in \Psi.$$

Proof. The first statement follows trivially from Corollary 6 and Theorem 3.
To prove (i), first note that, by Theorem 3 and Corollary 6, if $\Psi'' = \Psi' \cup \{\mathcal{H}\}$ and

$$p_{\Psi'}(s) > \max_{\mathcal{F}\in\Psi'} V_{\mathcal{F}}(s) \ \mu\text{-almost surely}$$

then

$$p_{\Psi''}(s) > \max_{\mathcal{F}\in\Psi''} V_{\mathcal{F}}(s) \ \mu\text{-almost surely.}$$

since, by Lemma 5, it is impossible that $V_{\mathcal{H}}(s) > \max_{\mathcal{F}\in\Psi'} V_{\mathcal{F}}(s)$ μ-almost surely. Thus, it is sufficient to show that if $\Psi = \{\mathcal{F}, \mathcal{G}\}$ and $V_{\mathcal{F}}(s) \neq V_{\mathcal{G}}(s)$ for a set of positive measure, then

$$p_{\Psi}(s) > \max\{V_{\mathcal{F}}(s), V_{\mathcal{G}}(s)\}, \ \mu\text{-almost surely.}$$

By Lemma 5, if $V_{\mathcal{F}}(s) \neq V_{\mathcal{G}}(s)$ for a set of positive μ measure, the sets

$$\{s \in S : V_{\mathcal{F}}(s) > V_{\mathcal{G}}(s)\}$$

$$\{s \in S : V_{\mathcal{F}}(s) < V_{\mathcal{G}}(s)\}$$

each have positive measure. When the market includes only the theory $\mathcal{F}$, Corollary 6 gives that $p_{\{\mathcal{F}\}}(s) = V_{\mathcal{F}}(s)$ in every state s. The claim then follows by Proposition 4.
To prove (ii), consider a set B for which $\mu(B) = 1$ and $V_{\mathcal{F}}(s) = V_{\mathcal{G}}(s)$ for any $\mathcal{F}, \mathcal{G} \in \Psi$ and $s \in B$. Since $\int Q(s, B)\,\mu(ds) = \mu(B)$, there exist a set $C_1 \in \mathcal{S}$ such that $Q(s, B) = 1$ for any $s \in C_1$ and $\mu(C_1) = 1$. Obviously, we can assume that $C_1 \subseteq B$. By definition, since

$$\int P_{\mathcal{F}}(B, s)\,\mu(ds) = 1$$

for any $\mathcal{F} \in \Psi$, there exists a set $D_1 \in \mathcal{S}$ such that $P_{\mathcal{F}}(s, B) = 1$ for any $s \in D_1$ and any $\mathcal{F} \in \Psi$, and $\mu(D_1) = 1$. Define $A_1 = C_1 \cap D_1$. Obviously, $\mu(A_1) = 1$ and $A_1 \subseteq B$. Thus, we can construct a decreasing sequence of sets $\{A_n\}_{n=1}^{\infty}$ such that, for any n, $A_{n+1} \subseteq A_n$, $\mu(A_n) = 1$, and
(a) $Q(s, A_n) = 1$ for any $s \in A_{n+1}$, and
(b) $P_{\mathcal{F}}(s, A_n) = 1$ for any $s \in A_{n+1}$ and any $\mathcal{F} \in \Psi$.
Define $A = \cap_{n=1}^{\infty} A_n$. By standard continuity properties of measures,
(c) $\mu(A) = 1$ and
(d) $Q(s, A) = P_{\mathcal{F}}(s, A) = 1$ for any $s \in A$ and any $\mathcal{F} \in \Psi$.
By construction and Corollary 6, for any $s \in A$ and any $\mathcal{F} \in \Psi$

$$V_{\mathcal{F}}(s) = \frac{1}{1+r}\left(d(s) + \max_{\mathcal{G} \in \Psi} E_{\mathcal{G}}(TV_{\mathcal{F}})(s)\right)$$

Hence, the mapping

$$\mathcal{T}(p)(s) = \frac{1}{1+r}\left(d(s) + \max_{\mathcal{F} \in \Psi} E_{\mathcal{F}}(Tp)(s)\right)$$

is such that if $p(s) = V_{\mathcal{F}}(s)$ for $s \in A$, then $\mathcal{T}(p)(s) = V_{\mathcal{F}}(s)$ for $s \in A$. Since $\mu(A) = 1$ the claim follows. ■

Thus, the above proposition shows that either all theories generate the same perceived fundamentals in almost every state or the price exceeds the perceived fundamentals of every theory in almost every state. Prices can strictly exceed the highest perceived fundamentals due to the speculative motive in trade. The owner of the asset in the current period at state s may sell the asset next period in state $s' \neq s$ at a price higher than her own perceived fundamentals in s'; recognising this possibility, she is willing to pay more in the current period than her own perceived fundamentals in state s. Once more, ergodicity ensures that prices higher than the highest perceived fundamentals in some state propagate to (almost) all others.

5.3.3 Sophisticated traders

One possibly surprising feature of the model is that the most sophisticated traders hold the most (and the least) optimistic price expectation (across all states), for the expectations of less sophisticated traders are an average of the expectations more sophisticated ones. In particular, consider a collection of theories Ψ and let $\mathcal{A}, \mathcal{B} \in \Psi$ be such that $\mathcal{A} \subset \mathcal{B}$. Then, μ-almost surely,

$$E_{\mathcal{A}}(Tp_{\Psi})(s) = \int E_{\mathcal{B}}(Tp_{\Psi})(y) P_{\mathcal{A}}(dy, s)$$

When the number of states is finite, if one trader has finer information that all others, then there exists *some* state in which she holds more optimistic beliefs about the next-period price than any other trader does in *any* state. Such a trader must therefore buy the asset in some state. One might conjecture that whenever $\mathcal{A} \subset \mathcal{B}$ and the trader holding theory $\mathcal{A}$ buys the asset in some state s, then the trader with the finer theory $\mathcal{B}$ buys the asset in some state s'. Although this is true in a market containing only theories $\mathcal{A}$ and $\mathcal{B}$, it is not generally valid.

Example 4 $S = \{a, b, c, d, e\}$, *with all states transiting back to themselves with probability* α *close to one, and otherwise to the uniform distribution. Trader* i *holds the theory generated by the partition* $\mathcal{P}_i$*, where*

$$\begin{aligned}
\mathcal{P}_1 &= \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\},\ \mathcal{P}_2 = \{\{a, b, c\}, \{d\}, \{e\}\} \\
\mathcal{P}_3 &= \{\{a, b, d\}, \{c\}, \{e\}\},\ \mathcal{P}_4 = \{\{a, b, c, d\}, \{e\}\} \\
\mathcal{P}_5 &= \{\{a, b, c, d, e\}\}.
\end{aligned}$$

The dividend equals 1 *in states* a *and* b *and equals* 0 *in states* c, d *and* e*; the interest rate* r *is constant. In equilibrium, Trader 1 buys in states* a, b*; Trader 2 buys in state* c*; Trader 3 buys in state* d*; and Trader 5 buys in state* e*. Trader 4 never buys the asset, unlike the coarser-theory Trader 5.*

In the example, removing from the market any theory other than that held by Trader 4 would cause all prices to fall strictly; removing only Trader-4's theory does not affect prices. This illustrates the conclusion of Proposition 4 that all prices move in tandem as the collection of theories in the market expands or contracts.

6 The N-Variable Asset Model

In this section, we use a special version of the general model in which the state space is described by N (potentially) relevant "economic" variables. In particular, we assume that $S = \times_{n=1}^{N} X_n$, where $X_n \subset \mathbb{R}$ for each $n = 1, ..., N$, and that traders' subalgebras are sections of S; that is, each trader perceives some variables but ignores others. In this case, we can index the set of possible theories by the subsets of $\{1, 2, ..., N\}$. This model will be called an N-variable asset model.

6.1 Non-Monotonicity

We begin our analysis of the N-variable asset model by establishing a result that yields a converse of the statement in Theorem 3.

Theorem 8 *Consider two collections Γ and Θ of subsets of $\{1, ..., N\}$. Suppose that $\Gamma \not\subseteq \Theta$ and $\Theta \not\subseteq \Gamma$. Then, there exists an N-variable asset model with finite S such that $p_\Gamma(s) > p_\Theta(s)$ and $p_\Gamma(s') < p_\Theta(s')$ for some s and s' in the support of μ.*

Proof. In Appendix. ■

6.2 Rates of Return

The relationship between rates of return on purchasing the risky asset and the properties of the theories of the buyer is complex. For instance, Example 4 shows that having a better theory does not necessarily imply higher returns. Trader 2 receives a rate of return below the interest rate as his expectations, conditional upon buying the asset, are too optimistic: he does not factor in that, in states a and b, he never acquires the asset. In state c, Trader 4 attaches a positive probability that the state is d and has lower expectations than Trader 2; never acquiring the asset, Trader 4 earns the risk-free rate in every period. To earn below market returns in this example, a trader must neither be too well nor too poorly informed.[14] Note that none of the agents achieves a return above the interest rate as profits are competed away even for Trader 1 whose theory is complete.

Not only can more sophisticated investors earn lower equilibrium rates of return than less sophisticated investors, but greater sophistication can lessen a trader's return *holding all other traders' theories fixed.* In one sense this follows trivially from Example 4: fixing the theories of the five traders described in the market, an entrant would earn higher returns by sharing the theory of Trader 4 than by sharing that of Trader 3; this follows immediately from Example 4 because no entrant who shares an existing theory affects prices. What about when traders are unique in holding their theories? The next example illustrates the possibility that sophistication can lower returns in this case as well.

Example 5 *$S = \{a, b, c, d, e, f\}$, with all states transiting back to themselves with probability α close to one, and otherwise to the uniform distribution. Trader i holds the theory generated by the partition $\mathcal{P}_i$, where*

$$\begin{aligned}
\mathcal{P}_1 &= \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}\},\ \mathcal{P}_2 = \{\{a, b, c\}, \{d\}, \{e\}, \{f\}\} \\
\mathcal{P}_3 &= \{\{a, b, d\}, \{c\}, \{e\}, \{f\}\},\ \mathcal{P}_4 = \{\{a, e\}, \{b, c\}, \{d\}, \{f\}\} \\
\mathcal{P}_5 &= \{\{a, b, d, f\}, \{c\}, \{e\}\}
\end{aligned}$$

Consider two possibilities for Trader 6's information,

$$\mathcal{P}_6 = \{\{a, b, c, e, f\}, \{d\}\} \quad or \quad \hat{\mathcal{P}}_6 = \{\{a, b, f\}, \{c\}, \{d\}, \{e\}\}$$

The dividend equals 1 in states a and b and equals 0 in states c, d, e and f; the interest rate r is constant. Regardless of Trader 6's theory, in equilibrium, Trader 1 buys in states a, b; Trader 2 buys in state c; Trader 3 buys in state d; and Trader 4 buys in state e. When Trader 6's theory is $\mathcal{P}_6$, Trader 5 buys in state f; Trader 6 does not buy and earns the risk-free rate. When Trader 6's theory is $\hat{\mathcal{P}}_6$, Trader 6 buys in state f and earns below the risk-free rate.

[14] Ettinger and Jehiel (2010) make a related observation about deception in strategic settings. To be deceived, a receiver must neither be sophisticated enough to see through the manipulation nor unsophisticated enough not to respond to it.

A more sophisticated Trader 6 earns lower returns in this example, holding all other traders' theories fixed. Becoming more sophisticated than Trader 5 leads him to swap places with Trader 5 in losing out to the more sophisticated Trader 1.

The next example shows that despite perfect competition some agents can earn a return above the interest rate.

Example 6 *Suppose that $N = 2$ and $X_i = \{0,1\}$, $i = 1,2$. The transitions are such that a state s transits to itself with probability α and to each other state with probability $\frac{1-\alpha}{3}$. Obviously, the invariant distribution is uniform. The interest rate is constant and equal to $r = 0.05$. The dividend function is*

$$d(x_1, x_2) = \frac{3}{4}x_1 + 0.95 \cdot x_2 - x_1 x_2$$

The market has three theories, namely $\{1\}$, $\{2\}$, and $\varnothing$. The equilibrium, as α approaches 1, is characterised by the following equations:

$$\begin{aligned}
\tfrac{3}{4} - 0.05 + \tfrac{1}{2}(p_{01} + p_{11}) &= (1+r)\, p_{11} \\
0.95 + \tfrac{1}{2}(p_{01} + p_{11}) &= (1+r)\, p_{01} \\
\tfrac{3}{4} + \tfrac{1}{2}(p_{10} + p_{11}) &= (1+r)\, p_{10} \\
\tfrac{1}{4}(p_{01} + p_{00} + p_{10} + p_{11}) &= (1+r)\, p_{00}
\end{aligned}$$

In particular, a trader with theory $\{2\}$ buys the asset in states $(1,1)$ and $(0,1)$, a trader with theory $\{1\}$ buys the asset in state $(1,0)$, and a trader with theory $\varnothing$ buys the asset in state $(0,0)$. The equilibrium prices are, as α approaches 1,

$$p_{00} = 13.57,\ p_{01} = 14.62,\ p_{10} = 14.44,\ p_{11} = 14.38$$

When α approaches 1, a trader with theory $\{1\}$ earns a return $\frac{\frac{3}{4}}{14.44} = 0.052 > r$ in state $(1,0)$ and earns r in all other states. The reason why this trader's return is above the risk free rate is "favourable selection". The dividend at $(1,1)$ is lower than the dividend at $(1,0)$. A trader with theory $\{1\}$ believes that the state is equally likely to be $(1,0)$ or $(1,1)$. However, his expectations are too pessimistic because he does not factor in that at state $(1,1)$ the asset is acquired by the agent with theory $\{2\}$.

Although some traders may earn returns in excess of the risk-free rate, it is impossible for all traders to earn above-market returns in that it is impossible that the return of holding one unit of the risky asset in every state

$$\int_S (d(s) + Tp_\Psi(s) - p_\Psi(s))\,\mu(ds)$$

exceeds

$$\int_S p_\Psi(s)\,r(s)\mu(ds).$$

To see this, note that the equilibrium equation implies that for any theory $\mathcal{F}$,

$$d(s) + E_{\mathcal{F}}(Tp_\Psi)(s) - p_\Psi(s) \le p_\Psi(s)\,r(s)$$

and that, by definition,

$$\int_S Tp_\Psi(s)\,\mu(ds) = \int_S E_{\mathcal{F}}(Tp_\Psi)\,\mu(ds)$$

Under certain monotonicity conditions, no theory can earn returns above the risk-free interest as all selection is adverse. For each theory $\mathcal{F}$ belonging to a collection of theories Ψ present in the market, let $R(p_\Psi;\mathcal{F})$ be the expected equilibrium return of a trader with theory $\mathcal{F}$ from allocating an amount equal to the price of one unit of the risky asset to purchasing either the risky or the risk-free asset. Define

$$B_{\mathcal{F}} = \{s \in S : E_{\mathcal{F}}(Tp_\Psi)(s) \ge E_{\mathcal{G}}(Tp_\Psi)(s), \forall \mathcal{G} \in \Psi\}$$

to be the set of states in which a trader with theory $\mathcal{F}$ perceives the risky asset as paying a return at least as large as the interest rate. Then

$$R(p_\Psi;\mathcal{F}) = \int_{B^c_{\mathcal{F}}} p(s)\,r(s)\mu(ds) + \int_{B_{\mathcal{F}}} (Tp_\Psi(s) + d(s) - p(s))\,\mu(ds)$$

Suppose that there exists a continuous density function $g : S \times S \to \mathbb{R}_+$ such that $g(s,z) > 0$ for any $s, z \in S$ and

(i) $\int\limits_S g(z,s)\,dz = \int\limits_S g(s,z)\,dz$ for any $s \in S$;

(ii) $Q(s,A) = \dfrac{\int\limits_A g(x,s)\,dx}{\int\limits_S g(x,s)\,dx}$

Note that these properties entail no loss of generality whenever a transition function is obtained via a conditional density $f(z \mid s)$ and an invariant distribution with a density function $m(s)$ exists for which

$$m(z) = \int f(z \mid s)\, m(s)\, ds.^{15}$$

In particular, the marginal density $g_M(s) = \int\limits_S g(z,s)\,dz$ yields an invariant distribution density. Given a theory $\mathcal{F} \subset \{1, 2, ..., N\}$, write s as $(s_{-\mathcal{F}}, s_{\mathcal{F}})$, where $s_{-\mathcal{F}}$ includes the components of s not in $\mathcal{F}$ and $s_{\mathcal{F}}$ the components in $\mathcal{F}$, and let $g_M(s_{-\mathcal{F}} \mid s_{\mathcal{F}})$ denote the conditional density. Let $S(s_{\mathcal{F}})$ be the section of S in which the components in $\mathcal{F}$ are equal to $s_{\mathcal{F}}$. Given a price function $p : S \to \mathbb{R}_+$, we set, for any $(z_{-\mathcal{F}}, s_{\mathcal{F}}) \in S(s_{\mathcal{F}})$,

$$E_{\mathcal{F}}(Tp)(z_{-\mathcal{F}}, s_{\mathcal{F}}) = \int Tp(s)\, g_M(s_{-\mathcal{F}} \mid s_{\mathcal{F}})\, ds_{-\mathcal{F}}$$

Proposition 9 *Suppose that g is affiliated. If d is increasing and r is decreasing in s,*

$$R(p_\Psi; \mathcal{F}) \leq \int_S p_\Psi(s)\, r(s) g_M(ds)$$

for any $\mathcal{F} \in \Psi$.

Proof. By standard properties of affiliated variables, the contraction mapping in Theorem 2 maps increasing functions to increasing functions. Hence, there exists a unique and increasing Ψ-equilibrium price function $p_\Psi(\cdot)$. Now note that $E_{\mathcal{F}}(Tp_\Psi)(s)$ is increasing in s and so is $\max_{\mathcal{F}\in\Psi} E_{\mathcal{F}}(Tp_\Psi)(s)$. Consider $s = (s_{-\mathcal{F}}, s_{\mathcal{F}})$ and $s' =$

[15] Define $g(z,s) = f(z \mid s)\, m(s)$. Then,

$$m(s) = \int f(z \mid s)\, m(s)\, dy$$

$\left(s'_{-\mathcal{F}}, s_{\mathcal{F}}\right)$, $s' \geq s$. Obviously, $E_{\mathcal{F}}\left(Tp_{\Psi}\right)(s) = E_{\mathcal{F}}\left(Tp_{\Psi}\right)(s')$. Hence, if $s \notin B_{\mathcal{F}}$, then $s' \notin B_{\mathcal{F}}$. That is, the set $B^c_{\mathcal{F}} \cap S\left(s_{\mathcal{F}}\right)$ is an increasing set in $S\left(s_{\mathcal{F}}\right)$. Recall that, given $B \in \mathcal{S}$, 1_B denotes its indicator function. By Milgrom and Weber (Theorem 22 (iii), 1982), since $g_M\left(s_{-\mathcal{F}} \mid s_{\mathcal{F}}\right)$ is affiliated in $s_{-\mathcal{F}}$, we have that, for any $\left(z_{-\mathcal{F}}, s_{\mathcal{F}}\right) \in S\left(s_{\mathcal{F}}\right)$,

$$E_{\mathcal{F}}\left(Tp_{\Psi}\right)\left(z_{-\mathcal{F}}, s_{\mathcal{F}}\right) \geq \frac{\int 1_{B_{\mathcal{F}}}(s)\, Tp_{\Psi}(s)\, g_M(s)\, ds_{-\mathcal{F}}}{\int 1_{B_{\mathcal{F}}}(s)\, g_M(s)\, ds_{-\mathcal{F}}}$$

that is, the expected price conditional upon $s_{\mathcal{F}}$ and purchasing the asset cannot exceed the expected price conditional upon $s_{\mathcal{F}}$ alone. By definition and the equilibrium equation,

$$\int 1_{B_{\mathcal{F}}}(s)\left(E_{\mathcal{F}}\left(Tp_{\Psi}\right)(s) + d(s) - p_{\Psi}(s)\right) g_M(s)ds = \int 1_{B_{\mathcal{F}}}(s)\, p_{\Psi}(s) r(s)\, g_M(s)ds$$

Since

$$\int 1_{B_{\mathcal{F}}}(s) \frac{\int 1_{B_{\mathcal{F}}}(s)\, Tp_{\Psi}(s)\, g_M(s)\, ds_{-\mathcal{F}}}{\int 1_{B_{\mathcal{F}}}(s)\, g_M(s)\, ds_{-\mathcal{F}}} g_M(s)ds =$$

$$\int \frac{\int 1_{B_{\mathcal{F}}}(s)\, Tp_{\Psi}(s)\, g_M(s)\, ds_{-\mathcal{F}}}{\int 1_{B_{\mathcal{F}}}(s)\, g_M(s)\, ds_{-\mathcal{F}}} \left(\int 1_{B_{\mathcal{F}}}(s)\, g_M(s)ds_{-\mathcal{F}}\right) ds_{\mathcal{F}} =$$

$$\int 1_{B_{\mathcal{F}}}(s)\, Tp_{\Psi}(s)\, g_M(s)ds$$

By simple substitutions, it follows that

$$R\left(p_{\Psi}; \mathcal{F}\right) \leq \int p_{\Psi}(s)\, r(s) g_M(ds)$$

for any $\mathcal{F} \in \Psi$. ■

Intuitively, when traders' conditioning variables are affiliated, dividends increasing and interest rate decreasing in these variables, then all selection is adverse: Trader i does not buy the asset because some Trader j's theory provides positive news about its value, unbeknownst to i.

If only theory $\mathcal{F}$ is present in the market, then

$$R\left(p_{\Psi}; \mathcal{F}\right) = \int_S p_{\Psi}(s)\, r(s) g_M(ds)$$

However, the relationship between rates of return and price changes can be counterintituive as in the failure of the classic expectations hypothesis according to which, when long-term bond yield spread with the short rate is positive, the long-term bond price is expected to fall. Consider the following example.

Example 7 *Suppose that* $S = \{0,1\}$, $d(0) = d(1) = 1$, $r(0) < r(1)$, $Q(1,\{1\}) = Q(0,\{0\}) = \gamma > 0.5$, $\Psi = \{\varnothing\}$. *Then,*

$$1 + \frac{p_\Psi(0) + p_\Psi(1)}{2} = (1 + r(0))\, p_\Psi(0)$$
$$1 + \frac{p_\Psi(0) + p_\Psi(1)}{2} = (1 + r(1))\, p_\Psi(1)$$

Obviously, $p_\Psi(0) > p_\Psi(1)$. *Then, when the long-term bond yield,* $\frac{1}{p_\Psi(s)}$, *is larger (smaller) than* $r(s)$, *the probability that* $p(s)$ *remains low (high) is* γ*: yields higher than the short-term rate are not offset by the price falls predicted by the classic expectations hypothesis. This agrees with evidence found by Campbell and Shiller (1991).*

6.3 Price Volatility

In Section 5.3.1, we showed that entry of new theories increases the price of the risky asset. In this section, we will investigate how the entry and exit of theories affects the variability of prices. We will restrict attention to a special class of transition functions for the state space.

Definition 4 *An* N*-variable asset model is* state persistent *if (i)* S *is finite and* $\mu(s) > 0$ *for any* $s \in S$*; (ii) there exists* $\gamma > 0$ *such that for each* $s, s' \in S$,

$$Q(s, \{s'\}) = \begin{cases} (1-\gamma)\,\mu(s') & \text{if } s' \neq s \\ (1-\gamma)\,\mu(s') + \gamma. & \text{if } s' = s \end{cases}$$

Note that one can specify an arbitrary invariant distribution to obtain a transition function with state persistence. The next proposition shows how the support of the price distribution changes as theories entry and exit the market. In particular, it shows that when persistence is sufficiently high, increasing the number of theories, in addition to raising all prices, decreases their range. Let $s^{max} := \arg\max_{s \in S} d(s)$ and $s^{min} := \arg\min_{s \in S} d(s)$.

Proposition 10 *Consider a state-persistent N-variable asset model and two collections of theories Γ and Θ. Suppose that (i) $\mu(\cdot)$ is affiliated; (ii) d is increasing and r is decreasing in s. If $\varnothing, \{1, \ldots, N\} \in \Gamma$ and $\Gamma \subset \Theta$, then*

$$\max_{s \in S} p_\Theta(s) - \min_{s \in S} p_\Theta(s) \leq \max_{s \in S} p_\Gamma(s) - \min_{s \in S} p_\Gamma(s)$$

if and only if

$$(1-\gamma)(1 + r(s^{min})) \leq 1 + r(s^{max}) - \gamma.$$

Proof. For any $\Psi \subset 2^{\{1,\ldots,N\}}$, $p_\Psi(s)$ is increasing in s. This follows from noting that

$$Tp_\Psi(s) = \gamma p_\Psi(s) + (1-\gamma) \int_S p_\Psi(s)\,\mu(ds)$$

Thus, since $\mu(\cdot)$ is affiliated, the contraction mapping in Theorem 2 maps increasing functions to increasing functions. When the state is s^{max}, the agents with theory $\{1, \ldots, N\}$ have the highest expectation and thus, for any $\Psi \subset 2^{\{1,\ldots,N\}}$,

$$\frac{d(s^{max}) + \gamma p_\Psi(s^{max}) + (1-\gamma)E[p_\Psi]}{p_\Psi(s^{max})} = 1 + r\left(s^{max}\right) \tag{3}$$

where $E[p_\Psi]$ denotes the unconditional expectation of $p_\Psi(\cdot)$ under $\mu(\cdot)$. Given a theory $\mathcal{F} \subset \{1, 2, ..., N\}$, write s as $(s_{-\mathcal{F}}, s_\mathcal{F})$, where $s_{-\mathcal{F}}$ includes the components of s not in $\mathcal{F}$ and $s_\mathcal{F}$ the components in $\mathcal{F}$. Since

$$E_\mathcal{F}\left(Tp_\Psi\right)\left(s^{min}\right) = \gamma \int_S p_\Psi\left(z_{-\mathcal{F}}, s^{min}_\mathcal{F}\right) \mu\left(dz_{-\mathcal{F}} \mid s^{min}_\mathcal{F}\right) + (1-\gamma)E[p_\Psi]$$

it follows from the affiliation of μ that

$$E_\mathcal{F}\left(Tp_\Psi\right)\left(s^{min}\right) \leq E[p_\Psi]$$

Hence, the lowest price is set by the empty theory; that is,

$$\frac{d(s^{min}) + E[p_\Psi]}{p_\Psi\left(s^{min}\right)} = 1 + r\left(s^{min}\right) \tag{4}$$

Substituting $E[p_\Psi]$ from Equation (4) into Equation (3) gives

$$d(s^{max}) - (1-\gamma)d(s^{min}) + (1-\gamma)(1 + r(s^{min}))p_\Psi(s^{min}) = p_\Psi(s^{max})(1 + r(s^{max}) - \gamma),$$

By Theorem 3, $p_\Gamma(s) \leq p_\Theta(s)$ for any $s \in S$. Thus, if

$$(1-\gamma)(1+r(s^{min})) \leq 1+r(s^{max})-\gamma,$$

the highest price cannot increase by more than the lowest price. The opposite holds if the inequality is reversed. ■

This result implies that, if interest rates are relatively constant or the state sufficiently persistent, prices are more volatile at bottom than at top in the sense that a theory's exogenous entry or exit from the market changes low prices more than high prices. Complete-theory traders set the price at the top of the market and recognise that high persistence implies that next-period's price coincides with this period's price with high probability; hence, the increase in expected price that accompanies an expansion of the set of theories present in the market by Theorem 3 has little effect on the top price. Empty-theory traders set the price at the bottom of the market and expect next period's price to equal the unconditional expected price; when it rises, so too does this lowest price. Campbell (2003) presents evidence that stock volatility is indeed countercyclical.

6.4 Variability in Prices and Fundamentals

We conclude our analysis with observations that underline the complexity of the relationship between prices and fundamentals.

First, in a model of heterogeneous beliefs in which agents have complete theories as in Harrison and Kreps (1978), the stationary equilibrium price function depends only on the fundamentals. However, if we allow for incomplete theories, stationary prices may have to depend on other variables. Thus, endogenous uncertainty is added to the randomness generated by fundamentals.

Second, we link Della Vigna and Pollett's (2007) observation that market participants underappreciate predictable patterns in future dividends to the observation that the largest one-day equity-price losses exceed in magnitude the largest one-day equity-price gains (Campbell, 2003).

Example 8 *Suppose that an asset can yield two dividends,* $d_h = 1$ *and* $d_l = 0$*, and that* $d^{t+1} = d^t$ *with probability* α*. Assume that* $x_1^t = d^t$*,* $x_2^t = d^{t+1}$*, and that the collection of theories is* $\Psi = \{\{1,2\},\{2\}\}$*. Thus, one set of agents foresees the next-period dividend while the other does not. When* $\alpha = 0.9$ *and* $r = 0.05$*, the equilibrium prices are*

$$\begin{aligned} p_\Psi(1,1) &= 19.24 \\ p_\Psi(0,1) &= 18.29 \\ p_\Psi(1,0) &= 18.86 \\ p_\Psi(0,0) &= 14.86 \end{aligned}$$

The largest one-period decline in the price of the risky asset is $p_\Psi(1,0) - p_\Psi(0,0) = 4.00$*, whilst the largest one-period rise in price is only* $p_\Psi(0,1) - p_\Psi(0,0) = 3.43$*.*

If the collection of theories is $\Psi' = \{\{1,2\}\}$*, that is, all the agents have complete understanding of the pricing process, the equilibrium prices are*

$$\begin{aligned} p_{\Psi'}(1,1) &= 12.38 \\ p_{\Psi'}(0,1) &= 11.43 \\ p_{\Psi'}(1,0) &= 8.57 \\ p_{\Psi'}(0,0) &= 7.62 \end{aligned}$$

Here, the largest one-period decline in the price of the risky asset is $p_{\Psi'}(1,1) - p_{\Psi'}(1,0) = 3.81$*, and the largest one-period rise in price is only* $p_{\Psi'}(0,1) - p_{\Psi'}(0,0) = 3.81$*.*[16]

Heterogeneous theories makes big losses larger than big gains because losses happen when incomplete-theory traders receive two types of bad news: first, the current dividend is low; second, future prices (and dividends) are likely to be low. By contrast, the biggest gains occur when complete-theory investors receive only the single piece of good news that future dividends and prices are likely to be high; current dividends have not yet changed. Gains and losses on equities markets are indeed asymmetric, with large losses more frequent than same-sized gains (Campbell, 2003).

Third, because they are determined by the most optimistic price expectations,

[16] At the ninth decimal place, the rise exceeds the fall.

prices can also move in spurts even when the fundamentals move gradually, as in the following example.

Example 9 *Consider a state-persistent model with $N = 2$, $X_i = \{0, 1\}$, and uniform μ. Suppose that $r(x_1, x_2) = r$ for any (x_1, x_2), $d(1,1) = 6$, $d(1,0) = 4$, $d(0,1) = 2$, $d(0,0) = 0$. Consider the collection of theories $\Psi = \{\{1\}, \{1,2\}\}$. When $\gamma = \frac{13}{15}$ and $r = 0.05$, the equilibrium prices are*

$$
\begin{aligned}
p_\Psi(1,1) &= 86.59 \\
p_\Psi(1,0) &= 83.34 \\
p_\Psi(0,1) &= 64.77 \\
p_\Psi(0,0) &= 61.52
\end{aligned}
$$

Only once the bad news about dividends has percolated through to all traders does the price of the risky asset fall significantly. By contrast, prices adjust gradually by tracking dividends if the collection of theories is $\Psi' = \{\{1, 2\}\}$. In particular, the equilibrium prices are

$$
\begin{aligned}
p_{\Psi'}(1,1) &= 76.36 \\
p_{\Psi'}(1,0) &= 65.46 \\
p_{\Psi'}(0,1) &= 54.55 \\
p_{\Psi'}(0,0) &= 43.64
\end{aligned}
$$

7 Conclusion

In this paper, we develop a framework where traders use incomplete models of the myriad connections among economic variables relevant for dividends and interest rates: each trader's model uses a subset of the predictive variables and is statistically correct about the next-period price. Heterogeneity increases asset prices but reduces their spread in a sufficiently persistent world. Less sophisticated traders can earn higher returns than their more sophisticated counterparts, and some traders may beat the market despite perfect competition. In some settings, heterogeneity creates an asymmetry between price rises and falls, with large falls more likely than large gains. Prices can be overly rigid because bad news about fundamentals must infiltrate all traders' theories before significantly bringing down prices.

Although in our model "theories" are fairly general, we have been unable to include the case in which traders believe that future prices are affected by past prices, as in trend spotting. The main difficulty is the existence of a stationary price equilibrium function since theories are in this case endogenous. We hope to address this issue in future research.

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8 Appendix A: Expectations and Limit Frequencies

In the main model, the inclusion of lagged variables in the agents' theories was implicit. In this subsection, we will make this inclusion explicit with some abuse of notation. Define S^L to be the L-fold Cartesian product of S and $\mathcal{S}^L$ to be the product σ-algebra. The corresponding Markov process derived from Q has the transition function $Q^L : S^L \times \mathcal{S}^L \rightarrow [0,1]$, where the function $Q^L(\cdot, A)$ is $\mathcal{S}^L$-measurable for any $A \in \mathcal{S}^L$, and $Q^L(s, \cdot)$ is a probability measure on $\mathcal{S}^L$ for any $s \in S^L$. Let μ^L denote the unique invariant measure.

Given the probability space $\left(S^L, \mathcal{S}^L, \mu^L\right)$ and subalgebra $\mathcal{F}$ of $\mathcal{S}$, let $P^L_{\mathcal{F}} : \mathcal{S}^L \times S^L \rightarrow [0,1]$ be regular conditional probability such that for each $B \in \mathcal{S}^L$, $P^L_{\mathcal{F}}(B, \cdot)$ is $\mathcal{F}^L$-measurable. Given a bounded $\mathcal{S}^L$-measurable function $g : S^L \rightarrow \mathbb{R}_+$, define the conditional expectation of g as the function $E^L_{\mathcal{F}}(g) : S^L \rightarrow \mathbb{R}_+$ such that

$$E^L_{\mathcal{F}}(g)(s) = \int g(y)\, P^L_{\mathcal{F}}(dy, s)\,.$$

Now define the "perceived" transition function $\tilde{Q}^L : S^L \times \mathcal{F}^L \rightarrow [0,1]$ where

$$\tilde{Q}^L(s, A) = E^L_{\mathcal{F}}\left(Q^L(\cdot, A)\right)(s)$$

for any $A \in \mathcal{F}^L$. We assume that $\tilde{Q}^L$ has a unique invariant distribution. Monotonicity conditions that yield this can be easily stated in standard specifications such as the model of Section 6.2. Then the restriction of μ^L to $\mathcal{F}^L$, $\mu^L_{\mathcal{F}}$, is the unique invariant distribution as by the definition of conditional expectation,

$$\int \tilde{Q}^L(s, A)\, \mu^L(ds) = \int Q^L(s, A)\, \mu^L(ds) = \mu^L(A)$$

for any $A \in \mathcal{F}^L$. Hence, if the agent describes the dynamic environment as a stationary Markov process with transition function $\tilde{Q}^L$, the theoretical limiting frequencies will be given by $\mu^L_{\mathcal{F}}$ and will coincide almost surely with the empirical frequencies by Birkhoff's Ergodic Theorem. Hence, the joint probabilities of events spanning at most L periods will correspond to the empirical frequencies almost surely. Obviously, the agent can be mistaken about the stationarity of the stochastic process governing the evolution of the system. In this case, if he detects non-stationarity, he might add other variables to his theory. However, our focus is not learning but instead on the economic consequences of coarse theories—perhaps themselves the outcome of a stalled learning process.

9 Appendix B

Proof of Theorem 6. Take $\mathcal{B} \in \Gamma$ that is not an element of Θ and $C \in \Theta$ that is not an element of Γ. Suppose without loss of generality that $\mathcal{B} \nsubseteq \mathcal{C}$. Consider an N variable asset model for which $d, x_i \in \{0,1\}$ for each $i = 1, \ldots, N$. For $D \subseteq \{1, ..., N\}$ define x^D to be the state such that $x_i^D = 1$ if and only if $i \in D$. Assume that
(1) $d(x) = 1$ if and only if $x_i = 1$ for all $i = 1, ..., N$.
(2) the transition function is such that:
(i) the state $(1, 1, ..., 1)$ transits to itself with probability α and to the state $(0, 0, ..., 0)$ with probability $1 - \alpha$;
(ii) for any proper subset D of $\mathcal{B}$ that is not a proper subset of $\mathcal{C}$, the state x^D transits to itself with probability β and to the state $(0, 0, ..., 0)$ with probability $1 - \beta$;
(iii) for any proper subset D of $\mathcal{C}$, the state x^D transits to itself with probability α and to the state $(0, 0, ..., 0)$ with probability $1 - \alpha$;
(iv) the state $(0, 0, ..., 0)$ transits to itself with probability α, to the states in (i) and (iii) with probability γ_1 each, to the states in (ii) with probability γ_2 each, and to any other state with probability γ_3 each.
(v) any other state transits to $(0, 0, ..., 0)$ with probability 1.
Thus, the transitions are represented by the following matrix

$$\begin{array}{c} \text{(i)\&(iii)} \\ \text{(ii)} \\ \text{(iv)} \\ \text{(v)} \end{array} \begin{bmatrix} \alpha & 0 & 1-\alpha & 0 \\ 0 & \beta & 1-\beta & 0 \\ \gamma_1 & \gamma_2 & \alpha & \gamma_3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The unique invariant distribution is obtained solving

$$\alpha y_1 + \gamma_1 y_3 = y_1$$
$$\beta y_2 + \gamma_2 y_3 = y_2$$
$$\gamma_3 y_3 = y_4$$
$$y_1 + y_2 + y_3 + y_4 = 1$$

Thus, the invariant probability of a state in (i) or (iii) is

$$\mu_i = -\frac{\gamma_1 - \beta\gamma_1}{\gamma_1(\beta - 1) + \gamma_2(\alpha - 1) - (\gamma_3 + 1)(\beta - 1)(\alpha - 1)} \frac{1}{1 + N_{iii}}$$

where N_{iii} is the number of states in (iii); the invariant probability of a state in (ii) is

$$\mu_{ii} = -\frac{\gamma_2 - \alpha\gamma_2}{\gamma_1 (\beta - 1) + \gamma_2 (\alpha - 1) - (\gamma_3 + 1) (\beta - 1) (\alpha - 1)} \frac{1}{N_{ii}}$$

where N_{ii} is the number of states in (ii); the invariant probability of the state $(0, 0, ..., 0)$ is

$$\mu_0 = \frac{\alpha + \beta - \alpha\beta - 1}{\gamma_1 (\beta - 1) + \gamma_2 (\alpha - 1) - (\gamma_3 + 1) (\beta - 1) (\alpha - 1)}$$

and the the invariant probability of a state in (v) is

$$\mu_v = \gamma_3 \frac{\alpha + \beta - \alpha\beta - 1}{\gamma_1 (\beta - 1) + \gamma_2 (\alpha - 1) - (\gamma_3 + 1) (\beta - 1) (\alpha - 1)} \frac{1}{N_v}$$

where N_v is the number of states in (v).
Now choose $\beta = \alpha^k$, $\gamma_1 = \gamma_2 = \gamma_3 = \dfrac{1 - \alpha}{3}$. Then, as α converges to 1,

$$\begin{aligned}
\mu_i &\to \frac{k}{4k+1} \\
\mu_{ii} &\to \frac{1}{4k+1} \\
\mu_0 &\to 3\frac{k}{4k+1} \\
\mu_v &\to 0
\end{aligned}$$

For any collection of theories Ψ, define $p_{\Psi}^{\max} = \max_{x \in S} p_{\Psi}(x)$. Since for any theory the expected price is no larger than $p_{\Psi}^{\max}$, from the pricing equation we have,

$$(1 + r) p_{\Psi}^{\max} \leq 1 + p_{\Psi}^{\max},$$

which implies that $p_{\Psi}^{\max} \leq \frac{1}{r}$.

We now establish the following properties of the pricing function for $\Psi = \Gamma, \Theta$: as α converges to 1, $p_{\Psi}(1, 1, ..., 1)$ converges to $\frac{1}{r}$; when x differs from $(1, 1, ..., 1)$, $p_{\Psi}(x)$ is bounded from above by $\frac{1}{r(1+r)}$. The pricing equation implies that

$$1 + \alpha p_{\Psi}(1, 1, ..., 1) \leq (1 + r) p_{\Psi}(1, 1, ..., 1)$$

Hence, $p_\Psi(1,1,...,1) \geq \dfrac{1}{1-\alpha+r}$. Also, for $x \neq (1,1,...,1)$,

$$p_\Psi^{\max} \geq (1+r)\, p_\Psi(x),$$

and, thus, $p_\Psi\,(x) \leq \dfrac{1}{r(1+r)}$.

Now suppose that the state is $x^{\mathcal{B}}$. As $\alpha \to 1$, an agent with theory $\mathcal{B}$ believes that the next period price is $p_\Psi(1,1,...,1)$ with probability close to one. Any agent who observes an $x_i = 0$ believes that the next price is at most $\dfrac{1}{r(1+r)}$ with probability close to one. Any agent with a theory that is a proper subset of $\mathcal{B}$ believes that the price is equal to at most $\dfrac{1}{r(1+r)}$ with non-vanishing probability. Hence, $p_\Gamma\left(x^{\mathcal{B}}\right) > p_\Theta\left(x^{\mathcal{B}}\right)$. Suppose now that the state is $x^{\mathcal{C}}$. Choose k so that, as $\alpha \to 1$, an agent with theory $\mathcal{C}$ believes that the state next period is $(1,1,...,1)$ with probability greater than ρ, where ρ is such that

$$\rho > \frac{2+r}{2+2r}.$$

Any agent who observes an $x_i = 0$, believes that the next price is at most $\dfrac{1}{r(1+r)}$ with probability close to one. Any agent holding a theory that is a proper subset of $\mathcal{C}$ believes that the price is equal to $\dfrac{1}{r(1+r)}$ with probability of at least $\dfrac{1}{2}$. Hence, from the above inequality, $p_\Gamma\left(x^{\mathcal{C}}\right) < p_\Theta\left(x^{\mathcal{C}}\right)$ ■