Decentralized Bargaining: Efficiency and the Core*

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Abstract

This paper studies market clearing in matching markets. Bargaining in such markets occurs in the context of alternative possible matches that provide endogenous outside options. We ask when will such markets clear efficiently and find inefficiencies, delay and mismatch, to be pervasive. The model is non-cooperative, fully decentralized, and in Markov strategies. Workers and firms bargain with each other to determine who will be matched to whom and on what terms of trade. Once a worker–firm pair reach agreement they exit the market. Delay can extensive and structured with vertically differentiated markets endogenously clearing form the top down. We find mismatch occurs whenever an agent is at risk of loosing a binding endogenous outside option. In both cases inefficiencies are driven by the endogenous evolution of the market.

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1 Introduction

Market clearing is at the center of economics. At one extreme we understand well how thick markets for homogeneous goods clear and at the other we understand bilateral bargaining well. However, many markets lie somewhere inbetween. In this paper we focus on matching markets, particularly labor markets, featuring decentralized negotiations that involve multiple, heterogeneous agents from both sides of the market. Such markets are common. One dimension of heterogeneity we capture is through constraints restricting who can match to whom. Firms might be able to employ only workers they have interviewed, workers might not know about all vacancies, and some people may simply be unqualified for some positions. On top of this, variability in how well suited different workers are to fill different vacancies is the norm, rather than the exception. We take these matching constraints and heterogeneities as given, and ask when decentralized negotiations can clear markets efficiently.

An important feature of the markets we study is that individuals have alternative matches that affect their bargaining positions. However, these outside options are endogenous. As the market clears people exit and the set of alternative matches evolves. This market evolution is a common feature of decentralized matching markets and it generates frictions that prevent markets from clearing efficiently. People delay in the hope that the market will evolve in their favor and accept inefficient matches because of a concern that the market will evolve against them.

We take a standard approach to the problem, extending the canonical Rubinstein (1982) model to a setting with multiple agents on both sides of the market. In each period an agent is selected at random to make a proposal, this proposer chooses someone to make an offer to, and then the offer is accepted or rejected. Once an offer is accepted and a worker–firm / buyer–seller / man–woman pair reach agreement, they exit the market. This creates a dynamic in which the composition of the market is ever changing. While this complicates matters, modeling the endogenous evolution of the market is crucial to understanding bargaining frictions. It is concerns about the market evolving unfavorably that drive inefficiencies, both mismatch and delay. We find that these bargaining frictions are pervasive. They arise whenever the market context and people's alternative matches, other than their efficient partner, matter.

We study the Markov perfect equilibria (MPE) of our bargaining game, where the state is the set of agents who have not yet been matched, and hence who have not exited. Our focus on Markov perfect equilibria is standard in the bargaining literature (for instance, Rubinstein and Wolinsky (1985, 1990), Gale (1987), Polanski and Winter (2010), Abreu and Manea (2012b)), motivated by Maskin and Tirole (2001) and on strategic complexity grounds in the context of bargaining in markets by Sabourian (2004). Nevertheless, it should be noted that with a bargaining protocol similar to ours but without heterogeneous surpluses Abreu and Manea

(2012a) show that clever punishment and reward strategies can be constructed to guarantee the existence an efficient subgame perfect equilibrium. Agranov and Elliott (2015) run a horse race between MPE, efficient subgame perfect equilibria and several other bargaining theories in a laboratory experiment. They find that the MPE organize the data very well and substantially better than the alternative theories, finding empirical support for several predictions we make in this paper.

In the matching markets we study alternative possible matches provide endogenous outside options, with their values depending on the other agreements reached. A crucial insight form our analysis is that these endogenous outside options cannot bound payoffs without generating mismatch. We find that an efficient MPE exists if and only if no such outside options bind the market context does not matter for any of the negotiations. More precisely, as exogenous bargaining frictions are reduced by letting delay costs vanish, an efficient Markov perfect equilibrium exists if and only if a certain condition holds. This condition is simple. Think of the probability of a player being selected to make an offer as the bargaining power of that player, and let the surplus generated by an efficiently matched pair be split in proportion to their bargaining powers. We refer to these payoffs as agents' *Rubinstein payoffs*, as they would obtain if all efficient pairs bargained bilaterally. Our main result establishes that an efficient MPE exists for sufficiently patient players if and only if Rubinstein payoffs are in the core of the market (that is, if no pair of players has a profitable joint deviation and none of the endogenous outside options bind). Moreover, when the condition holds, there is an efficient MPE in which all players receive their Rubinstein payoffs in the limit.

As the market evolves, endogenous outside options are lost. The intuition for our result is that if an outside option is never exercised by a player i, then i's efficient match could just wait for the market to evolve and for i's alternative match to exit. The intuition also identifies a limitation of the result. If i's alternative match never exits before i in equilibrium, then the market will not evolve against i. In this case i's endogenous outside option should in effect be exogenous. Sutton (1986) shows in a two player setting that as agents become perfectly patient exogenous outside options can bind payoffs from below while being exercised with probability 0. So when outside options are exogenous in this way, we might expect there to be an MPE that while not efficient exhibits vanishingly small inefficiencies as players become patient. Investigating this possibility we find there are two and only two ways in which endogenous outside options can behave as though they are exogenous and MPE can exhibit vanishing inefficiencies. Most simply, i's alternative match may have no efficient partner (and so match in an efficient equilibrium with probability 0). In this case we can treat players who are unmatched in the efficient match as exogenous outside options and an adjusted version of our main result continues to hold in the limit.

There is one other possibility. It could be the case that all players except for the most valuable efficient match delay awaiting for an unlikely mismatch of such valuable partners. Remarkably, endogenous delay of this form, resulting in sequential exit from the market, is possible in an MPE. With four players and equal bargaining powers we find necessary and sufficient conditions for such an MPE to exist. Such markets must be highly vertically differentiated and clear form the top. This accords with anecdotal evidence from high-skill labor markets. In sports and in the movie industry, labor markets are often reported to be held up while a star decides what offer to accept.[1]

Delay is possible in our model despite there being complete information. As time progresses and matched pairs exit the market in an endogenous, but random, order, the strength of people's bargaining positions in the network evolves. In equilibrium, people can choose to delay instead of making an offer to their efficient match because they expect the market to evolve in their favor. Even with sequential exit when an agent and their efficient partner both delay, it is because there is a (vanishingly) small probability that an inefficient match will occur that will increase their collective expected payoffs.

1.1 Related Literature

We study *decentralized* bargaining in *thin* markets. Both features differentiate us from the vast literatures we discuss below. The prototypical market we intend to speak to is a labor market for high skill individuals. Such markets are inherently thin, characterized by heterogeneities and matches are reached following decentralized negotiations. These markets are also important. They are high value, and large surplus losses result from a misallocation of resources within them. It is more important that CEOs are allocated to the correct companies than janitors, even though there are many more janitors than CEOs.

1.1.1 Centralized approaches

We find that inefficiencies are pervasive. One way to resolve these inefficiencies would be to arrange matches through a centralized clearinghouse. Although most matching markets operate without recourse to centralized clearing, our analysis provides a motivation for such mechanisms. In many cases centralized matches are motivated by the inability to make transfers.[2] However, one centralized market where transfers are possible is the market for the matching of residents to hospitals. Following a lawsuit filed in 2002, Bulow and Levin (2006) and Niederle (2007) investigate the effect of incorporating wages into a centralized match. Our analysis

[1] See, for example, Telegraph (2012)

[2] In the case of the kidney donor market, for instance, transfers are prohibited by law. Likewise, transfers cannot be used to clear public school matching markets.

suggests an alternative counterfactual of interest – a decentralized market outcome. The bargaining frictions identified in this paper provide an efficiency motivation for maintaining a centralized approach. We find that even with wages freely able to adjust and with frictions going to zero, inefficiencies remain ubiquitous in decentralized markets.

A sizable literature analyzes coalitional bargaining – seminal references include Gul (1989) and Chatterjee et al. (1993). Although such models are typically a better fit for political negotiations and committee decision making, the closest paper to ours in this literature, Okada (2011), links cooperative and non-cooperative approaches as we do.[3] Like us Okada finds conditions under which there does not exist an efficient MPE, and relates these conditions to the core. However, the conditions he identifies are generically violated in assignment economies with at least two players on each side of the market such that there is no efficient equilibrium.[4] In contrast, an efficient MPE exists for a positive measure subset of the parameter space in our decentralized bargaining model.

We also view cooperative approaches as somewhat centralized. The primary cooperative solution concept used to study assignment economies is the core following the pioneering work of Shapley and Shubik (1971). Generalizations of Nash bargaining – Rochford (1984), Kleinberg and Tardos (2008), Kanoria et al (2014) – have been shown to refine the core. We connect the core to non-cooperative bargaining. While our condition for the existence of an efficient MPE requires agents' Rubinstein payoffs to be core payoffs, the connection is deeper. There exist parameter values[5] for which equilibrium payoffs are equal to any core payoffs. In other words, the efficient equilibria span the core, and any core outcome can be justified as an equilibrium of some bargaining game.

A common motivation for refining the core, which is a set-valued solution concept, is to derive more precise predictions about agents' payoffs. Our results show that while the non-cooperative approach can obtain this goal, it requires taking a stance on players' bargaining powers (or, more accurately, offer probabilities). These probabilities are given exogenously by the bargaining protocol. If one does not want to take a stand on what bargaining powers should be, and instead entertains the possibility of all bargaining powers, then the set of equilibrium payoffs is a superset of the core. The core refines the set of non-cooperative bargaining payoffs and not vice versa.

[3] Nguyen (2012, 2014) and Siedlarek (2014) are close to our aims in a different way, as they model coalitional bargaining over networks.

[4] Details available upon request.

[5] Namely, a probability distribution over players which determines how likely each player is to be called on to make an offer.

1.1.2 Large markets

There are many papers that consider decentralized bargaining in large markets, meaning that they either model an infinite number of players or else they assume pairs that exit are replaced by exact replicas.[6] Seminal work includes Rubinstein and Wolinsky (1985), Gale (1987), and Binmore and Herrero (1988). A general conclusion from this literature is that sometimes equilibrium outcomes approximate competitive equilibria when the frictions get small and other times they do not. Lauerman (2013) provides a nice characterization of when these two outcomes can be expected. Most papers study steady state outcomes.[7] The closest papers to ours in this literature are Manea (2011), who studies network bargaining, and Moreno and Wooders (2002) who study non-stationary markets. Unlike us, in their large market setting, Moreno and Wooders (2002) find that outcomes approaching the competitive equilibrium outcomes are obtained as frictions vanish. However, they do find that it is possible to have delay in the limit (outcomes are nevertheless competitive in the limit as the losses due to delay go to zero). Manea (2011) focuses on limit payoffs rather than efficiency.

The search literature also considers large markets. Our approach is complementary to this research agenda. While we focus on bargaining frictions, the search literature typically focuses on other frictions. Indeed, some of the search literature can be viewed as endogenizing the constraint set on possible matches we take as given. Moreover, better understanding of bargaining frictions can help improve our understanding of search incentives – Rogerson, Shimer and Wright (2005).[8]

To place our findings in the context of the search literature we restrict heterogeneities to assortative matching environments in Section 7.2.[9] The efficient (assortative) match can be obtained in our thin market setting, but only under strong conditions that require the market to exhibit a high degree of symmetry. This further emphasizes the potential for incorporating bargaining frictions into the search literature.

1.1.3 Decentralized thin markets

The most closely related work to ours also models non-cooperative bargaining with exit in thin markets. This literature includes Rubinstein and Wolinsky (1990), Corominas-Bosch (2004), Gale and Sabourian (2006), Polanski (2007), Polanski and Winter (2010), Kanoria et

[6]See Manea (2013) for how the replica assumption relates to steady state outcomes in large markets.

[7]Some examples, with a particular focus on network bargaining, include Atakan (2010), Manea (2011) and Polanski and Lazarova (2014).

[8]Some more recent additions to this literature include Albrecht, Gautier, and Vroman (2006), Jacquet and Tan (2007), Galenianos and Kircher (2009), Kircher (2009), Gautier, Teulings, and Van Vuuren (2010), Eeckhout and Kircher (2010), Gautier and Holzner (2013) and Elliott (2014).

[9]This is the standard way heterogeneities are included in the search literature – e.g. Shimer and Smith (2000), Smith (2006), Eeckhout and Kircher (2010).

al. (2011), Abreu and Manea (2012a, 2012b) and Polanski and Vega Redondo (2014). These papers embed different degrees of coordination into their bargaining protocols and some could be classified as centralized. However, like us, others are fully decentralized. Overall the closest papers to ours are Gale and Sabourian (2006) and Abreu and Manea (2012b). They both study the MPE of a decentralized bargaining model with exit. Their main contributions are to provide examples in which no efficient MPE exists.[10] We permit more heterogeneity in surpluses than both and allow a proposer to choose to whom to make an offer to. Both of these changes take the model closer to reality, at least for the applications we have in mind, and while they may appear to complicate the model we show that they eliminate some anomalies permitting a cleaner analysis and allowing us to characterize when an efficient MPE exists.[11]

2 The Assignment Economy

An *assignment economy* consists of a set of players $N = \{1, ..., n\}$ and an n by n matrix S characterizing the surplus that can be generated by any two players in the economy. The ij entry of S, $s_{ij} \geq 0$, denotes the surplus generated when players i and j are matched. The surplus matrix S can be interpreted as a network. The network is assumed to be undirected (so that $s_{ij} = s_{ji}$ for any $i, j \in N$) and bipartite (so that, for some partition (P_1, P_2) of the set of players N, $s_{ij} = 0$ whenever $i, j \in P_k$ for $k \in \{1, 2\}$). The two assumptions imply that the surplus generated in a match is independent of the identity of the player who initiates the match, and that surplus can be generated only by players of different types. By assumption, workers generate surplus only with firms, men generate surplus only with women, and buyers generate surplus only with sellers.

A *match* is a map $\mu : N \to N$ such that $\mu(\mu(i)) = i$ for any $i \in N$. If $\mu(i) = i$, we say that player i is unmatched. If $\mu(i) = j$, then i and j generate surplus s_{ij}.[12] Let $M(N)$ denote the set of possible matches for a given set of players N. An *efficient match* η for an assignment economy S is a match that satisfies

$$\sum_{i \in N} s_{i\eta(i)} = \max_{\mu \in M(N)} \left\{ \sum_{i \in N} s_{i\mu(i)} \right\}.$$

[10] Abreu and Manea (2012b) also consider markets in which players cannot necessarily be partitioned into buyers and sellers. Indeed, their example in which no efficient MPE exists occurs in such a market. It might then be hoped that inefficiencies do not feature in buyer-seller markets. We show that they do, and identify the conditions under which they do.

[11] We thus take up the challenge posed in Abreu and Manea (2012b) who conclude that "Many open questions remain, including the analysis of network structures which lead to multiplicity or inefficiency of MPEs. It is unclear at this stage whether useful characterizations are attainable."

[12] Note that by the bipartite assumption, $s_{ii} = 0$.

Shapley and Shubik (1971) establish that the efficient match is generically unique for an assignment economy.[13] The analysis therefore restricts attention to economies in which a unique efficient match exists. For the efficient match η, the *core* of the market is characterized by the set of payoff profiles $U \subseteq \mathbb{R}^N_+$ such that for all $u \in U$,

$$[1] \quad u_i + u_j \geq s_{ij} \text{ for any } i, j \in N,$$
$$[2] \quad u_i + u_{\eta(i)} = s_{i\eta(i)} \text{ for any } i \in N.$$

Although these conditions ensure only that there are no profitable pairwise deviations, Shapley and Shubik (1971) establish that this is sufficient for there to be no profitable coalitional deviations. The lowest and the highest payoff that player i can receive in the core will be denoted by $\underline{u}_i$ and $\overline{u}_i$. We occasionally refer to η also as the *core match*.

3 Matching and Bargaining

The analysis considers a non-cooperative, infinite-horizon bargaining protocol in which players choose whom to bargain with. All players discount the future by a common factor $\delta \in (0,1)$. At the beginning of the game, all players are active, but they can become inactive as the game unfolds. In every period, a single player $i \in N$ is selected at random to be the proposer, with probability $p_i > 0$. If proposer i is active, he can make an offer to at most one other active player. We adopt as a convention that a player failing to make an offer chooses to offer to himself. An offer from player i to a player $j \neq i$ consists of a surplus split $x_{ji} \in [0, s_{ij}]$, where x_{ji} denotes the amount of surplus generated by the new match, s_{ij}, that he intends to leave to j. The player receiving the offer can then decide whether to accept (1) or reject (0) the offer. If j rejects the offer, both players remain active, and the game moves to the next stage. Otherwise, players i and j become inactive, and their final payoffs are determined by the discounted value of the shares that they have agreed upon. In particular, if players expect to reach agreement x_{ji} with certainty at stage t, their expected payoffs at the beginning of the game satisfy

$$u_j = \delta^{t-1} x_{ji} \quad \text{and} \quad u_i = \delta^{t-1}(s_{ij} - x_{ji}).$$

In the next stage the proposer is selected according to the same probability distribution. If an inactive player is selected the game moves to the subsequent period. The game ends when the surplus generated by any pair of active players is zero. The structure of the game is common knowledge among players. Information is complete. Thus, all players observe any

[13] For any assignment economy in which the surpluses are perturbed by an independent noise term (drawn from a continuous distribution with no atoms), there is a unique efficient match with probability 1.

offer previously made and the corresponding acceptance decision.

Histories and Strategies: There are two kinds of histories. Denote the set of histories at date t observed by any player after the new proposer has been selected by $P^t = N \times [N^2 \times \mathbb{R}_+ \times \{0,1\}]^{t-1}$. This includes the identity of the current proposer, the identities of past proposers, who they offered to, the offer they made and whether the offer was accepted or rejected. Denote the set of histories of length t observed after an offer has been made by $R^t = N \times \mathbb{R}_+ \times P^t$. Let $R = \cup_t R^t$ and $P = \cup_t P^t$. Finally, let P_i denote the subset of histories in P in which player i is the proposer, and let R_i denote the subset of histories in R in which player i has to decide whether to accept or reject some offer.

We say that player $i \in N$ is *active* at history $h \in P$ if that player has never accepted an offer and has never made an offer that was accepted. For any history $h \in P$, let $A(h) \subseteq N$ denote the set of active players in the game after history h. Throughout, the operator $\Delta(\cdot)$ will map any finite set to its simplex. The strategy of an active player $i \in A(h)$ who was selected to be the proposer consists of a pair of functions, ρ_i and χ_i, such that

$$\rho_i(h) \in \Delta(A(h)) \text{ and } \chi_i(h) \in \mathbb{R}_+^{|A(h)|} \text{ for } h \in P_i.$$

The first map $\rho_i(h)$ describes a probability distribution over players who may receive an offer from i at any given history, while the second map $\chi_i(h)$ identifies the amount of surplus that i would offer to any other active player who might receive such an offer. The strategy of an active player $i \in A(h)$ who receives an offer is instead a single function, α_i, such that

$$\alpha_i(h) \in [0,1] \quad \text{for } h \in R_i.$$

The map $\alpha_i(h)$ describes the probability that an offer is accepted at any possible history. Strategy profiles are usually denoted by omitting the dependence on players, $(\rho, \chi, \alpha) = \{\rho_i, \chi_i, \alpha_i\}_{i \in N}$.

4 Solution Concept and Preliminaries

The analysis restricts attention to Markov perfect equilibria in which strategies depend only on the set of active players in the game.

Definition 1 *A profile of strategies (ρ, χ, α) is a Markov perfect equilibrium (MPE) if strategies are subgame perfect and if strategies coincide whenever active player sets coincide. That is, for any two histories $h, h' \in P$ such that $A(h) = A(h')$:*

[1] $\rho(h) = \rho(h')$ *and* $\chi(h) = \chi(h')$,

[2] $\alpha(i,x|h)=\alpha(i,x|h')$ *for any offer* $(i,x)\in N\times R_+$.

Since MPE strategies depend only on the set of active players and on the offers made, we often write such dependence explicitly, thereby omitting the dependence on histories. Notation $(\rho^\delta,\chi^\delta,\alpha^\delta)$ will occasionally be used to clarify that equilibrium strategies may also depend on the discount factor δ. But, we omit this dependence when redundant.

Results also consider MPE behavior in the limit as the discount factor converges to 1. To simplify the discussion we introduce a notion of limiting equilibrium.

Definition 2 *A limiting Markov perfect equilibrium (LMPE)* $(\bar\rho,\bar\chi,\bar\alpha)$ *is the limit of a selection* $\{\rho^\delta,\chi^\delta,\alpha^\delta\}_{\delta=0}^1$ *of the MPE correspondence as* δ *converges to* 1.

Throughout the text the expression *equilibrium* will refer to an MPE, and the expression *limiting equilibrium* will refer to an LMPE.

Before presenting our characterization of equilibrium behavior, we introduce two efficiency concepts that we apply to both MPE and LMPE and the notion of delay that will be analyzed in the following sections. Let E denote the set of unmatched players in the core of the entire assignment economy, $E=\{i\in N|\eta(i)=i\}$, and let $C(N)$ denote the set of possible active player sets that can arise as core matches drop out of the game,

$$C(N)=\{A|\,A=\cup_{i\in M}\{i,\eta(i)\}\cup E\ \text{ for some }\ M\subseteq N\}.$$

For any MPE (ρ,χ,α) and any set of players $A\subseteq N$, let $\pi_{ij}(A)$ denote the *agreement probability* between players $i\in A$ and $j\in A\backslash i$ when i is selected to make an offer,

$$\pi_{ij}(A)=\underbrace{\rho_i(j|A)}_{\Pr(i\text{ offers to }j)}\cdot\underbrace{\alpha_j(i,\chi_i(j|A)|A)}_{\Pr(j\text{ accepts})},$$

and let $\pi_{ii}(A)$ denote the probability that i does not reach agreement when selected to make an offer,

$$\pi_{ii}(A)=1-\textstyle\sum_{j\in A\backslash i}\pi_{ij}(A).$$

Also, let $V_i(A)$ denote the expected payoff – or equivalently *value* – of an active player i at the beginning of a subgame in which the set of active players is A, and let $v_i(A)$ denote the MPE value of an active player i when he is chosen to be the proposer.

Consider a social planner who is able to choose the actions of players but is constrained by the environment of the game. For a high enough discount factor, this constrained social planner will implement only efficient matches and will do so at the first available opportunity. Such an MPE is said to be strongly efficient. It requires that every player who is matched in the core of the assignment economy agrees on a division of surplus with his core partner at the very first opportunity.

One way in which surplus can be lost is through delay. However, when players interact frequently – that is, when δ is close to 1 – little surplus dissipation will be lost in equilibrium because of delay. We therefore also consider a weaker efficiency criterion that only requires players to eventually match to their respective core partners. An MPE is thus said to be weakly efficient if every player who is matched in the core eventually agrees on a division of surplus with his core partner.[14]

Definition 3 *An MPE* (ρ, χ, α) *is:*

- *strongly efficient if* $\pi_{i\eta(i)}(A) = 1$ *for any* $A \in C(N)$ *and any* $i \in A$*;*
- *weakly efficient if* $\pi_{i\eta(i)}(A) + \pi_{ii}(A) = 1$ *for any* $A \in C(N)$ *and any* $i \in A$*.*

Neither of our efficiency criteria are satisfied when an inefficient match obtains with positive probability. As we assume $\delta < 1$ this includes situations in which an inefficient match occurs with vanishingly small probability as $\delta \to 1$. To address this we apply our two efficiency criteria to LMPE.

Definition 4 *An LMPE* $(\bar{\rho}, \bar{\chi}, \bar{\alpha})$ *is:*

- *strongly efficient if* $\lim_{\delta \to 1} \pi^{\delta}_{i\eta(i)}(A) = 1$ *for any* $A \in C(N)$ *and any* $i \in A$*;*
- *weakly efficient if* $\lim_{\delta \to 1} \pi^{\delta}_{i\eta(i)}(A) + \pi^{\delta}_{ii}(A) = 1$ *for any* $A \in C(N)$ *and any* $i \in A$*.*

While both strongly and weakly efficient LMPE generate the same surplus in the limit, it is instructive to separate them for the purpose of classifying limiting efficient equilibrium play. An important difference between applying our efficiency criterion to MPE and LMPE is that subgames outside $C(N)$ may now be on the equilibrium path for all $\delta < 1$.

Our notion of equilibrium delay disciplines the behavior of active players. It states that an MPE displays delay whenever a player with a positive value forgoes the option to make an acceptable offer with positive probability.

Definition 5 *An MPE* (ρ, χ, α) *displays delay if for some* $A \subseteq N$ *and some player* $i \in A$

$$V_i(A) > 0 \quad \textit{and} \quad \pi_{ii}(A) > 0.$$

[14] In terms of utilitarian welfare strongly efficient MPE maximize the ex-ante sum of MPE values, whereas weakly efficient MPE may not. Even in strongly efficient MPE, however, the sum of values is necessarily below total surplus, as it takes time for the core match to form. Moreover, in a strongly efficient MPE all active player sets in $C(N)$ obtain with positive probability. But, this is not the case for weakly efficient MPE, as the market may clear sequentially.

The definition applies only to players with a positive value, as it is immediate that players with zero continuation value might as well refrain from reaching agreements. In Section 6, we present two examples in which a player (with a positive continuation value) chooses to delay on the equilibrium path.

5 MPE Existence and Characterization

The first result in the analysis provides a proof of equilibrium existence and a preliminary characterization of equilibrium bargaining values. For convenience, let $p_A = \sum_{j \in A} p_j$. The characterization allows for mixed strategy equilibria. Fix an active player set A and consider any Markovian strategy profile (ρ, χ, α) and its associated values and agreement probabilities $(\pi, V) \in [\Delta(A) \times \mathbb{R}]^{|A|}$ where we omit the dependence on A for clarity. As in numerous bargaining models, subgame perfection dictates that a proposer never offers to another player more than that player's present discounted value of staying in the game. As players can choose whom to bargain with, proposers necessarily offer to those players who leave them with the highest surplus, $\text{argmax}_{j \in A \backslash i} \{s_{ij} - \delta V_j\}$, whenever such surplus exceeds the value of remaining unmatched, δV_i. It follows that for any active player set $A \subseteq N$, MPE values $V(A)$ for any player $i \in A$ must be a fixed point of the following system of value equations

$$
\begin{aligned}
v_i &= \max\{\delta V_i, \max_{j \in A \backslash i}\{s_{ij} - \delta V_j\}\}, \\
V_i &= \underbrace{p_i v_i}_{i \text{ proposes}} + \textstyle\sum_{j \in A \backslash i} p_j [\underbrace{(\pi_{ji} + \pi_{jj}) \delta V_i}_{j \text{ proposes to } i \text{ or delays}} + \underbrace{\textstyle\sum_{k \in A \backslash i,j} \pi_{jk} \delta V_i (A \backslash j, k)}_{j \text{ proposes to } k \neq i,j}] + \underbrace{(1 - p_A) \delta V_i}_{\text{no player proposes}},
\end{aligned}
$$

for some profile of agreement probabilities $\pi(A)$ satisfying

$$
\begin{aligned}
\pi_{ij} &= 0 \quad \text{if} \quad v_i > s_{ij} - \delta V_j \text{ and } j \neq i, \\
\pi_{ii} &= 0 \quad \text{if} \quad v_i > \delta V_i.
\end{aligned} \tag{1}
$$

Proposition 1 *An MPE exists. Moreover, $\{\pi(A), V(A)\}_{A \subseteq N}$ is a profile MPE values and agreement probabilities if and only if it solves system (1) at any active player set $A \subseteq N$.*

Existence then follows by Kakutani's fixed point theorem. The result extends Proposition 1 and Lemma 1 in Abreu and Manea (2012b) to environments in which players are allowed to choose whom to offer to and in which the surplus generated in a match depends on the identity of the players. MPE are not unique. But if (π, V) is MPE at an active player set A, $(\pi, \bar{V})$ is not for all $\bar{V} \neq V$.

The result implies that no player would delay when bargaining at an active player set A of an MPE (π, V) if for all $i \in A$ there exists a $j \in A$ such that $\delta V_i + \delta V_j < s_{ij}$. This condition will therefore be violated at some subgame of any MPE displaying delay.

6 Examples

Before proceeding to the main analysis, we consider a few examples to illustrate the model, the solution concepts, the efficiency definitions and the main messages. The first example establishes that equilibrium mismatch can occur, the second shows why mismatch inefficiencies can be small in a strongly efficient limiting equilibrium, the third displays equilibrium delay, and the forth shows that mismatch inefficiencies can be small even in a weakly efficient limiting equilibrium when all players endogenously choose to exit the market sequentially one core pair at a time. This section can be skipped.

Example 1: Consider an assignment economy populated by four players who propose with equal probabilities. Surpluses in the market are as depicted in Panel I of Figure 1.

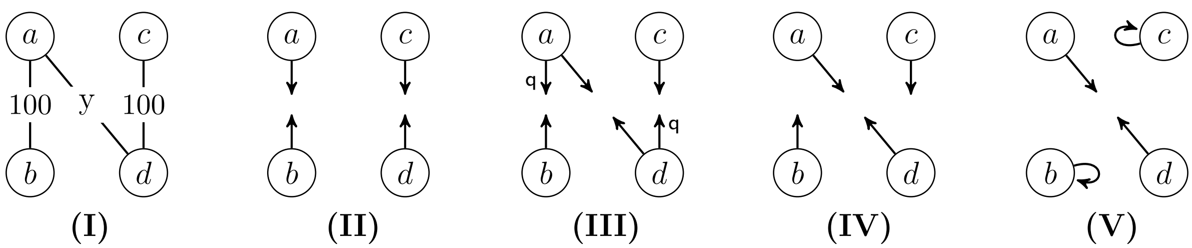


Figure 1: In Panel I the assignment economy; in Panel II MPE agreement probabilities for $y \in [0, 100]$; in Panel III equilibrium probabilities for $y \in (100, 143]$; in Panel IV equilibrium probabilities for $y \in [144, 200]$, in Panel V equilibrium probabilities for $y \in (200, \infty)$. An arrow between two players represents a positive agreement probability. A self-arrow instead represents a positive disagreement probability.

The unique efficient assignment matches player a to b and player c to d whenever $y < 200$, while it matches only player a to d when $y > 200$. Multiple core assignments exist at $y = 200$. Proposition 1 can then be used to derive MPE payoffs and strategies in this game for any discount factor. To make the discussion more transparent, suppose that the discount factor is close to unity. When y is sufficiently small ($y \leq 100$), the characterization establishes that players necessarily bargain only with their core matches. If so, payoffs are pinned down as in a bilateral Rubinstein bargaining game (without outside options), and no player is ever tempted to offer to players other than their core match (Panel II of Figure 1). In this scenario, all players achieve an LMPE payoff of 50. When $y > 100$, however, endogenous outside options become binding. If everyone only offered to their efficient match players a and d would both have a profitable deviation to offer to each other. In equilibrium, when $y \in (100, 1000/7)$, players a and d randomize between offering to their respective core matches and bargaining with each

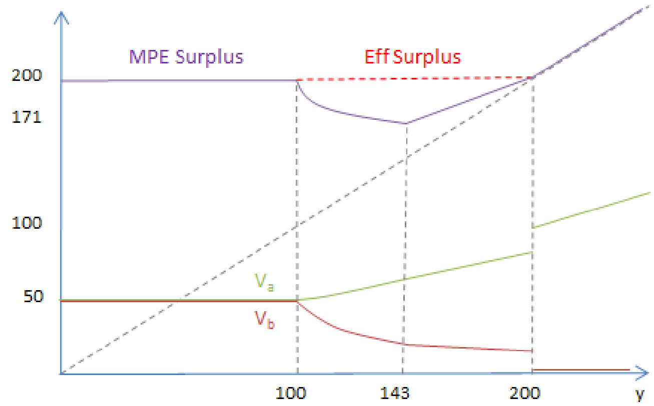


Figure 2: The plot depicts payoffs and surplus as a function of $s_{bc} = y$. The payoff of players a and d is denoted by V_a, whereas V_b denotes the payoff of b and c.

other (Panel III of Figure 1). By offering to each other with positive probability, a and d reduce the continuation values of their efficient partners and randomizing is optimal. The alternative of matching with each other now affects the payoffs a and d agree with their efficient matches, but only because their endogenous outside options are exercised with positive probability. Mismatch therefore occurs with positive probability once endogenous outside options bind. As δ converges to 1, a and d offer to their respective efficient matches with probability $q \in (0, 1)$, and offer to their efficient matches with the complementary probability.

When y grows further, to $y \in [1000/7, 200)$, players a and d stop making offers to their core matches, as they strictly prefer offering to each other (Panel IV of Figure 1). However, they still accept offers made by their respective core matches. There is now mismatch with probability 1/2. The final case is the one in which $y > 200$, and in which the efficient match changes. Players a and d become core partners and continue offering to each other with probability 1. However, b and c stop making offers to players a and d, as any accepted offer would have to be worth more than the entire surplus in the relevant relationship (Panel V of Figure 1). This change affects limiting payoffs discontinuously. When $y < 200$, player b always makes an acceptable offer to a, leaving c to bargaining bilaterally with d with probability 1/4. Thus, c gets a limiting payoff of 50 with probability 1/4. For $y > 200$, however, b stops making acceptable offers to a, and so c receives a payoff of 0 with certainty. Note that this discontinuity occurs precisely at the value of y for which the core match is not unique. Figure 2 depicts LMPE values and surplus as a function of y.

Example 2: The second example is closely related to classical models of bargaining with exogenous outside options. The message from the first example, which holds in general, is that endogenous outside options must be exercised with positive probability to affect payoffs and so mismatch occurs with positive probability whenever endogenous outside options bind.

Although outside options are generally endogenous in our framework, players who never exit the market provide outside options that are effectively exogenous. In any efficient equilibrium they remain available until the game ends. As such, we refer to them as exogenous outside options. These outside options can affect payoffs while being exercised with probability 0 in the limit.

Consider the stylized three-player market depicted in Panel I of Figure 3. The unique core match of the market matches players e and f, while leaving c unmatched. Let the three players propose with equal probability, and assume again that their discount factors are sufficiently close to unity. If so, then players e and f offer to each other with probability 1 in the unique MPE, whereas player c offers to player e with probability $q \in (0, 1)$. Such probability converges to 0 as the discount factor converges to unity. However, the mere presence of player c, significantly affects bargaining outcomes. Players e and f would share the 10 units of surplus evenly were they to bargain in solitude. However, because c never exits the market, he acts as an exogenous outside option for f, which is why player f can extract the same limiting surplus that he would get were he to bargain in solitude with player e while having access to an exogenous outside option with value 8. The limiting payoffs converge to 8 for player f, to 2 for player e, and to 0 for player c. Even though player c does not make an acceptable offer, the equilibrium does not display delay by our definition, because the payoff of player c equals exactly 0 for all δ sufficiently close to 1. While the equilibrium described is not strongly or weakly efficient for $\delta < 1$ because there is positive probability of mismatch, in the limit that probability converges zero, and so the limiting equilibrium is a strongly efficient.

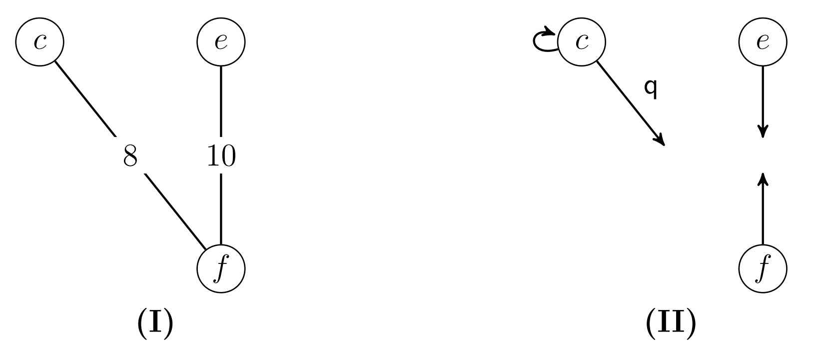


Figure 3: In Panel I the assignment economy; in Panel II agreement probabilities. The limiting equilibrium shown is strongly efficient as $\lim_{\delta \to 1} q^{\delta} = 0$.

Example 3: We saw in example 1 that mismatch occurs because players with endogenous outside options fear the market will evolve against them and they will loose this alternative. In contrast, when the market does not evolve against a player with a binding outside option we saw that the probability of mismatch could become vanishingly small in example 2. We now reinforce the message that the market evolution drives inefficiencies by showing that delay can occur because players expect the market to evolve in their favor.

Consider the six-player assignment economy depicted in Panel I of Figure 4, in which agents are selected to propose with equal probability p. We show an equilibrium exists in which f delays making offers with probability 1 when selected to propose if all other players are still active in the market. Panel II of Figure 4 shows the agreement probabilities in this MPE. To solve this game, we use backward induction. Under the proposed equilibrium, if the protocol selects agent e as the first proposer, agent e makes an offer to agent f that will be accepted. If so, the remaining subgame coincides precisely with the game discussed in Example 1, so we know the MPE payoffs for all the remaining players in the subgame. If agent c is selected as the first proposer and agrees with d, then in the following subgame agents e and f bargain bilaterally, as do agents a and b. If agent a or d is selected as the first proposers instead, then agent b must remain unmatched, while agents c, e, and f are left in precisely the subgame we considered in Example 2. Finally, if agent b is the first proposer, he agrees with a, and players c, d, e, and f are left in a subgame. While we have not solved this subgame yet, in the unique MPE all players offer to their efficient partner and doing so the outside options c and f have of matching with each other is non-binding. Limit payoffs for c, d, e and f and then $50, 50, 5$ and 5. With these subgames in mind, it is easy to write down the value equations for the six agents and solve them. For instance, the value equation for agent c simply amounts to

$$V_c(N) = p[\underbrace{2\delta V_c(\mathcal{E}_2)}_{a \text{ or } d \text{ propose}} + \underbrace{\delta V_c(c,d)}_{b \text{ proposes}} + \underbrace{(100 - \delta V_d)}_{c \text{ proposes}} + \underbrace{\delta V_c(\mathcal{E}_1)}_{e \text{ proposes}} + \underbrace{\delta V_c}_{f \text{ delays}}],$$

where $V_c(\mathcal{E}_i)$ denotes the value of player c in Example $i \in \{1, 2\}$. Solving the value functions establishes that no player has a profitable deviation from the proposed strategies and that player f must delay for all sufficiently high values of δ. Taking limits as $\delta \to 1$ the payoffs of the six players converge to

$$V(N) = (55/3, 230/3, 230/3, 55/3, 13/2, 7/2).$$

Agents a through d achieve the same limiting values as in Example 1. The additional option available to c (of matching with f) does not improve his terms of trade as it never binds. However, in this game f might prefer to delay, as there is a positive probability of c remaining unmatched and the game ending up in Example 2 where player f achieves a payoff of 8. This possibility incentivizes f to delay, and indeed f strictly prefers delaying at $A = N$ for all δ close to 1. Such threats are factored into the limiting payoffs of e, and thus f in the limit will be indifferent between delaying and making an offer to e when selected to propose first. Yet, f will delay with certainty in any LMPE to make full use of his potential future outside option.

Delay in our setting relies only on the endogenous evolution of bargaining positions. This is the case because players can choose whom to bargain with (which implies that no player has to delay to be matched to his equilibrium partner) and because the efficient match is unique (which shuts down possible coordination problems among players). We show in Appendix III that when multiple efficient matches exist, delay can arise just because players fail to coordinate on one of the efficient matches.

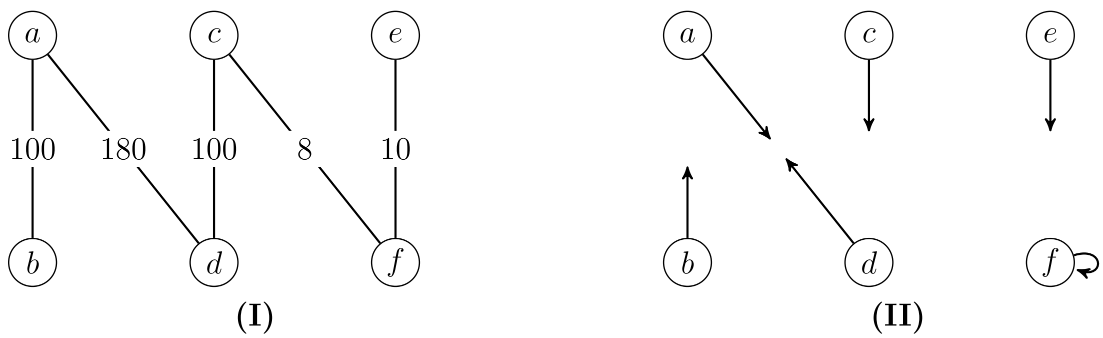


Figure 4: In Panel I the assignment economy; in Panel II agreement probabilities. Player f delays.

Example 4: There is one final way in which outside options can, in principle, be effectively exogenous and bound limiting values without being exercised. There could be sequential exit. Consider a limiting equilibrium in which all players in the market delay with probability 1 except for a single core pair – so that others would agree, only upon seeing agreement of a given core pair. If so, those who delay may act as exogenous outside options for the pair who ends up agreeing. We show that this is the only other way in which the probability of mismatch can be vanishingly small when alternative matches provide binding outside options. Remarkably, sequential exit as just described can happen in limiting equilibria.

Consider the market depicted in Panel I of Figure 5. This market is vertically differentiated. Both b and d generate a higher surplus with a than c while both a and c generate more surplus with b than d. Indeed the market is highly vertically differentiated in so far as the match between a and b generates ten times more surplus than the match between c and d. The efficient match is also assortative. It is efficient for a to match to b and for c to match to d.

There is no strongly or weakly efficient MPE for $\delta < 1$ and no strongly efficient LMPE. There is, however, a weakly efficient LMPE. For high enough $\delta < 1$ there is an MPE in which player c delays with probability 1, player d agrees with a with probability $q^\delta > 0$ and delays with probability $1 - q^\delta$ while a and c always agree with each other. Moreover, $\lim_{\delta \to 1} q^\delta = 0$ so in the limit c and d both delay with probability 1 and wait for a and b to reach agreement before bargaining with each other. The market therefore clears from the top. The limit payoffs of c and d are 5, a receives 80 and b gets 20.

It is surprising that both c and d can delay with probability 1 until a and b exit, despite ending up matched with each other with probability 1. For $\delta < 1$, c receives a higher limiting payoff when bargaining bilaterally with b than when bargaining bilaterally with d. While these benefits of delaying vanish in the limit because $\lim_{\delta\to 1} q^\delta = 0$, so do the costs of delaying. These costs and benefits vanish at the same rate so whether delay by c and d can be sustained in a limiting equilibrium depends on the values of the surpluses. In this example the benefits of delay outweigh the costs. We solve the general four player equal proposer probability problem in section 7.1. The key features of the market in this example are necessary and sufficient for sequential exit. The market will have to be highly vertically differentiated with an assortative efficient match for sequential, top-down, market clearing to be consistent with players incentives.

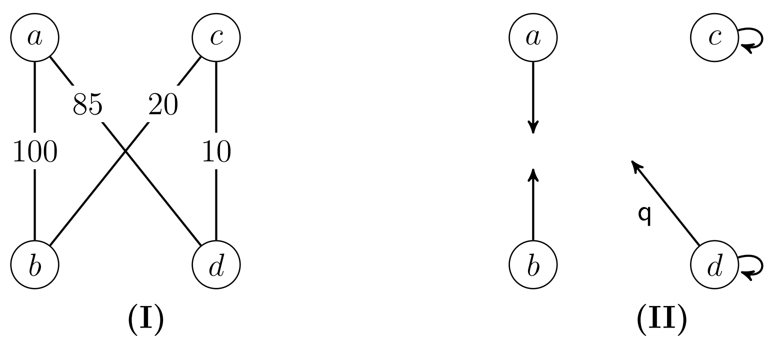


Figure 5: In Panel I the assignment economy; in Panel II MPE agreement probabilities. The MPE features sequential exit. As $\lim_{\delta\to 1} q^\delta = 0$, in the limit c and d wait for a and b to reach an agreement before reaching an agreement themselves.

7 MPE Efficiency and Frictions

We now present the main conclusions on equilibrium welfare. The analysis begins by characterizing payoffs in any efficient MPE and by deriving necessary and sufficient conditions for the existence of such MPE for δ close to 1. The main conclusion here establishes how these conditions relate the primitives of the bargaining model to the core of the assignment economy. Results also provide a characterization of inefficient MPE. The second part of the section derives similar conclusions for limiting equilibria, and shows when players can act as outside options affecting bargaining outcomes without distorting trade. Broadly, the analysis establishes that mismatch and delay arise in decentralized bargaining models whenever someone uses an alternative match to improve their terms of trade and that alternative match might exit the market before them. Whenever the market can evolve, on path, in ways that change players' bargaining positions inefficiencies are a necessary feature of any MPE.

To state results, it is useful to introduce three relevant payoff profiles. The first of these identifies the LMPE values that players would achieve while bargaining bilaterally with their core match. For any player $i \in N$, let σ_i denote the *Rubinstein payoff*[15] of player i,

$$\sigma_i = \frac{p_i}{p_i + p_{\eta(i)}} s_{i\eta(i)}.$$

The second profile identifies the highest payoff that players could achieve while offering to players that are unmatched in the core of the assignment economy. For any player $i \in N$, let ω_i denote the *outside payoff* of player i,

$$\omega_i = \max_{j \in E \cup i} s_{ij}.$$

In the bargaining game, players that are unmatched in the core act as exogenous outside options in efficient equilibria, as they never exit the market. The third and final profile identifies the LMPE payoffs that players would achieve while bargaining bilaterally with their core match when facing exogenous outside options equal to ω (Shaked and Sutton (1984), Sutton (1986), Binmore and Herrero (1988)). For any player $i \in N$, let $\bar{\sigma}_i$ denote the *shifted Rubinstein payoff*,

$$\bar{\sigma}_i = \begin{cases} \omega_i & \text{if} \quad \omega_i \geq \sigma_i \\ s_{i\eta(i)} - \omega_{\eta(i)} & \text{if} \quad \omega_{\eta(i)} \geq \sigma_{\eta(i)} \\ \sigma_i & \quad \text{otherwise} \end{cases}.$$

The first proposition characterizes equilibrium payoffs in any strongly efficient MPE.

Proposition 2 *In any strongly efficient MPE, the payoff of any player $i \in A$ in any subgame $A \in C(N)$ amounts to*

$$V_i(A) = \frac{p_i}{(1-\delta) + \delta(p_i + p_{\eta(i)})} s_{i\eta(i)}.$$

Strongly efficient MPE payoffs are stationary and independent of the set of active players along the equilibrium path, and converge to Rubinstein payoffs. Bargaining is strongly efficient only when outside options, exogenous or endogenous, have no effect on outcomes, and players achieve the same payoff they would get by bargaining with their efficient match in solitude. The result follows by simple manipulation and the observation that behavior in subgames that are off the equilibrium path cannot affect the terms of trade in any equilibrium path subgame, as players would reach such subgames only by exiting the game.

[15] This name was chosen because the seminal reference is Rubinstein (1982).

While we will identify necessary and sufficient conditions for bargaining to be efficient, it will be helpful to highlight two potentially separate sources of distortions, namely, inefficient matching and delay in reaching agreements. Both distortions are driven by the endogenous evolution of bargaining power that results from the random order of play. Whereas mismatch is necessarily a hard friction, as it permanently destroys surplus, delay can be a soft friction, in that its effects on welfare are negligible when discount factors are sufficiently close to unity. Next, we establish that delay cannot be the sole source of frictions in the model, as mismatch is necessary for delay. Pinning down weakly efficient equilibria thus amounts to identifying strongly efficient equilibria.

Theorem 3 *Any weakly efficient MPE is strongly efficient.*

The proof shows that players never delay in any weakly efficient equilibrium, as delay necessarily weakens their bargaining position relative to their core match. As strongly efficient MPE coincide with weakly efficient MPE, henceforth we simply refer to them as *efficient equilibria*.

The next result shows why bargaining outcomes are necessarily inefficient whenever the outside options affect the terms of trade. The result focuses on high frequency of interaction, and provides necessary and sufficient conditions for the existence of efficient MPE.[16] We say that an efficient MPE exists *for all values of* δ *close to* x if it exists for any $\delta \in (0, 1)$ such that $|x - \delta| \leq \varepsilon$ for some $\varepsilon > 0$.

Theorem 4 *An efficient MPE exists for all* δ *close to 1:*

(a) if Rubinstein payoffs are in the interior of the core,

$$\sigma_i + \sigma_j > s_{ij} \quad \text{for all } i, j \in N \text{ such that } j \neq \eta(i). \tag{2}$$

(b) only if Rubinstein payoffs are in the core,

$$\sigma_i + \sigma_j \geq s_{ij} \quad \text{for all } i, j \in N. \tag{3}$$

The proof establishes that players must occasionally agree with players other than their core partner whenever Rubinstein payoffs do not belong to the core. Intuitively, when players consider agreeing with their respective core matches, other partners act as endogenous outside options. However, because outside options vanish as matched players exit the market on the equilibrium path, players must occasionally choose such options for these to affect the outcome.[17] Such behavior necessarily leads to mismatch, surplus dissipation, and possibly delay,

[16]We defer the discussion of low frequency of interaction to Appendix I.

[17]Consider again Example 1, and in particular panel III of Figure 1, so that $y \in (100, 143]$. Suppose an

as players sometimes match with inefficient partners. Consequently, only when Rubinstein payoffs live in the core of the assignment economy is there no profitable deviation away from an efficient MPE.[18]

The sufficient condition for the existence of an efficient MPE is intuitive. Such a condition, however, does not guarantee that every equilibrium is efficient. Indeed, Appendix II presents an example in which condition (2) holds, but in which multiple MPE exist for all δ close to 1. Coordination problems implicit in the partner selection stage are the source of the multiplicity.

The economic content of Theorem 4 is that bargaining inefficiencies are pervasive when negotiations are decentralized. Bargaining often occurs in a market context. For example, workers' possible alternative matches may affect the wages they are able to negotiate. Theorem 4 establishes that whenever this market context matters, bargaining is inefficient. Markets are able to clear efficiently only when all players can optimally bargain bilaterally with their efficient partners, ignoring all alternatives. Moreover, these inefficiencies persist even when the discount factor is high, and so the exogenous frictions imposed by time preferences and the sequential nature of offers become small. In Appendix IV, we explore the consequences of Theorem 4 in classical labor market settings, and show that vertical differentiation and increasing differences are not sufficient for the existence of an efficient MPE.

The efficiency results have several immediate implications, which are summarized in the next result. These imply that (a) any core payoff profile can be implemented as an LMPE by appropriately selecting the vector of proposal probabilities; (b) for any pair $\{i, \eta(i)\}$, proportional changes in their proposal probabilities do not affect limiting bargaining outcomes; (c) efficiency is easier to achieve in economies which have a large core. For convenience, say that surplus S *supports more core payoffs than* S' in the strong set order if any core payoff profile in S' is also a core payoff profile in S.[19]

Proposition 5 *The following are immediate consequences of Theorem 4 as $\delta \to 1$.*

(a) Any interior core payoff $u \in U$ is an LMPE payoff for some probabilities $p \in \Delta(N)$.

(b) If an efficient MPE exists with probabilities $p \in \Delta(N)$, then an efficient MPE (with the same limiting payoffs) exists for any probabilities p' such that $p_i/p_{\eta(i)} = p'_i/p'_{\eta(i)}$ for all $i \in N$.

efficient equilibrium is played, and so, by Theorem 3, $q = 1$. An alternative strategy available to a is to reject all offers from b and to delay when selected to be the proposer until c exits the market. Doing so will result in a bargaining bilaterally with b in the resulting subgame, and in the limit, b will obtain a payoff of 50. Thus, for a to receive a limiting payoff greater than 50, a must exercise his endogenous outside option and match to b with positive probability in equilibrium. Thus there is no efficient equilibrium.

[18]Our theorem does not speak to the non-generic case in which Rubinstein payoffs are on the boundary of the core. In such cases a discount factor equal to 1 may be required to guarantee the existence of an efficient MPE.

[19]A sufficient condition is that, for all $i \in N$, $s_{ij} = s'_{ij}$ whenever $j = \eta(i)$, and $s_{ij} \leq s'_{ij}$ whenever $j \neq \eta(i)$.

(c) Fix probabilities $p \in \Delta(N)$. If surplus S supports more core payoffs than S' and if an efficient MPE exists for S', then an efficient limiting MPE also exists for S.

The first part of the result establishes that the core of the assignment economy can be spanned by varying proposal probabilities. By interpreting players' proposal probability as their bargaining power, the second part shows that at a high frequency of interaction in an efficient MPE a player's bargaining power matters only relative to that of his efficient match. The last part of the result implies that economies with larger cores are more likely to result in efficient bargaining outcomes.

Next, we establish that any MPE without delay must lead to agreement on the core match with positive probability. As we saw that delay can occur in equilibrium in Examples 3 and 4, the no-delay condition is non-trivial.[20]

Proposition 6 *The core match obtains with strictly positive probability in any MPE that does not display delay. However, MPE that display delay exist in some markets.*

The result obtains because in any MPE without delay it is impossible to find a subset of players who prefer to exchange their respective core matches, and thus some players must optimally agree their efficient match.

7.1 Limiting Equilibria and Efficiency

Efficient LMPE can differ considerably from efficient MPE. Theorem 4 considers only $\delta < 1$ and so categorizes equilibria in which mismatch occurs with a vanishingly small probability as δ converges to 1 as inefficient. Moreover, Examples 2 and 4 establish that mismatch can occur in equilibrium with vanishingly small probability. This section studies this possibility asking when inefficiencies can be small in this sense.

The first result of this section extends Proposition 2 showing that strongly efficient LMPE converge to shifted Rubinstein payoffs. Whenever these payoffs differ from Rubinstein payoffs and delay is costless, unmatched players in E act as exogenous outside options without distorting the limiting equilibrium match. In Example 2, for instance, player c had an effect on player f's terms of trade in the limit without ever matching to f. The result also extends the negative efficiency conclusions of Theorem 4 to markets in which delay costs vanish. In the limit, equilibria cannot be efficient if shifted Rubinstein payoffs are outside the core of the assignment economy.

[20] We stress again that Example 2 does not fit our definition of equilibrium delay, as in the unique LMPE the only player who delays has a continuation value equal to zero.

Theorem 7 *In any strongly efficient LMPE, the payoff of any player $i \in A$ in any equilibrium-path subgame $A \in C(N)$ converges to*

$$\lim_{\delta \to 1} V_i(A) = \bar{\sigma}_i.$$

Moreover, a strongly efficient LMPE exists only if shifted Rubinstein payoffs are in the core,

$$\bar{\sigma}_i + \bar{\sigma}_j \geq s_{ij} \quad \text{for all } i, j \in N. \tag{4}$$

Core unmatched players can affect the limiting terms of trade without ever agreeing, because they belong to every equilibrium path subgame. Core matched players, instead, cannot play such a role in a strongly efficient LMPE as they exit the game at the first available instance by agreeing with their core match.

In addition to demonstrating the robustness of the conclusions previously reached, Theorem 7 uncovers a crucial difference between alternative matches that can be lost as the market evolves and alternative matches that cannot be lost as the market evolves. We term the former endogenous outside options and the later exogenous outside options. Furthermore, it clarifies that bargaining frictions arise endogenously and simply as a strategic response to possible changes in market composition. It is because a player is worried that an alternative match currently available to him will exit the market, thereby weakening his market position, that he makes an offer to an inefficient partner even as δ converges to 1. As we have seen in Example 3, similar considerations regarding the evolution of the market can also lead to equilibrium delay.

When shifted Rubinstein payoffs are in the interior of the core they coincide, by construction, with Rubinstein payoffs. If so, by Theorem 4 an efficient equilibrium exists for any sufficiently high value of δ. The same argument establishes that a strongly efficient LMPE exists in this case. Also, strongly efficient LMPE may exist even when shifted Rubinstein payoffs are on the boundary of the core, as was the case in Example 2. But if so, distortions vanish only when the discount factor approaches unity.

Next, we consider weakly efficient LMPE and their properties.[21] The main result establishes that, whereas only core unmatched players can act as outside options in strongly efficient LMPE, all players can potentially act as outside options in a weakly efficient LMPE. However, for this to be the case, the market must clear sequentially, one core match at a time. If so, even players who are ultimately matched can provide exogenous outside options by only matching after some other players have matched.

[21] We thank Mihai Manea for encouraging us to consider this notion of efficiency, termed asymptotic efficiency in Abreu and Manea 2012a.

To formalize the discussion it is convenient to introduce a notion of sequential agreement.

Definition 6 *A weakly efficient LMPE is a sequential LMPE, if for some $A \in C(N)$ such that $|A \backslash E| \geq 4$ and for some $i \in A \backslash E$*

$$\lim_{\delta \to 1} \pi_{jj}(A) = 1 \quad \text{for any } j \in A \backslash \{i, \eta(i)\}. \tag{5}$$

Sequential LMPE display sequential agreement in that players who certainly end up matched with each other prefer to delay reaching an agreement until a given core match has exited the market. The next result establishes that any weakly efficient LMPE whose limiting payoffs do not converge to shifted Rubinstein payoffs must be sequential.

Theorem 8 *Any weakly efficient LMPE that is not payoff equivalent to a strongly efficient LMPE is sequential.*[22] *Moreover, sequential LMPE exist in some markets.*

When exit is sequential, all players remain in the market until a given core match exits, creating effectively exogenous outside options for this match. Theorem 8 further reinforces our central message that inefficiencies are ubiquitous. Even in a weakly efficient LMPE outside options cannot affect bargained outcomes without being exercised with strictly positive probability if they can be lost on the equilibrium path. Nevertheless, people who are efficiently matched can provide effectively exogenous outside options through sequential exit. It is intriguing that sequential exit can occur in equilibrium. The observation conforms with empirical regularities in some matching markets which can clear from the top down. However, delay is a knife-edge phenomenon in most bargaining models without asymmetric information. It might be thought that sequential LMPE will require very specific parameter values of the bargaining problem. To address this issue systematically, we characterize the set of sequential LMPE in the context of a 4 player market with equal proposer probabilities.

Let $N = \{a, b, c, d\}$ and $p_i = p$ for $i \in N$. To avoid redundancies when stating results we adopt the following labelling conventions:

- ab and cd are the core matches, $s_{ab} + s_{cd} > s_{ad} + s_{bc}$;
- ab is the most valuable core match, $s_{ab} \geq s_{cd}$;
- ad is the most valuable non-core match, $s_{ad} \geq s_{bc}$.

We also omit the dependence on N when obvious. The final result on efficient LMPE characterizes payoffs in a sequential LMPE, and delivers necessary and sufficient conditions for the existence of such an LMPE.

[22] Two LMPE are *payoff equivalent* if ex-ante limiting values in the two equilibria coincide for all players.

Proposition 9 *Given the labelling convention, if a sequential LMPE exists, then for all δ close to 1*

$$\pi_{ab} = \pi_{ba} = \pi_{cc} = \pi_{da} + \pi_{dd} = 1, \quad \pi_{da} > 0, \quad \lim_{\delta \to 1} \pi_{dd} = 1.$$

Moreover, in any such LMPE

$$\begin{array}{ll} \lim_{\delta\to 1} V_a = s_{ad} - \sigma_d & \lim_{\delta\to 1} V_c = \sigma_c \\ \lim_{\delta\to 1} V_b = s_{ab} - s_{ad} + \sigma_d & \lim_{\delta\to 1} V_d = \sigma_d \end{array}$$

Finally, a sequential LMPE exists if and only if

$$s_{ab} > s_{ad} > \frac{s_{ab} + s_{cd}}{2} > s_{bc} > s_{cd} \quad \text{and} \quad \frac{s_{bc} - s_{cd}}{2(s_{ab} - s_{ad})} \geq \frac{s_{bc} + s_{cd}}{s_{ab} + s_{cd}}. \tag{6}$$

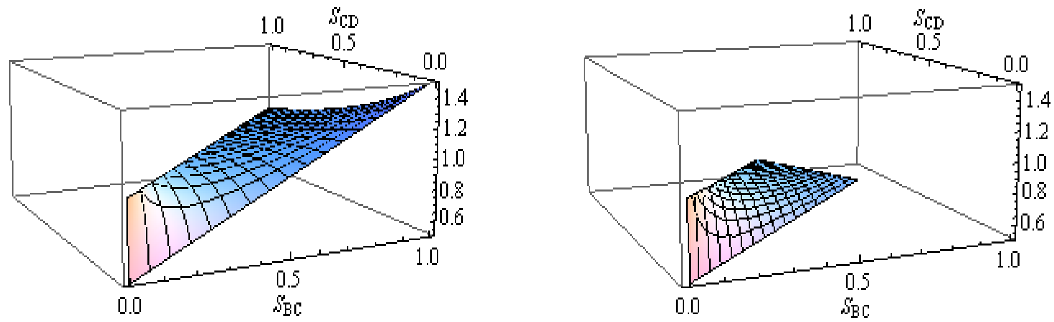
The proposition pins down agreement probabilities at a high frequency of interaction in any sequential LMPE. In such equilibria, exit from the market is sequential. Player a always has the option of matching to d when bargaining with b, as c and d only agree once a and b have agreed. Thus, player d provides an exogenous outside option for a. Limiting payoffs reflect this. In any sequential LMPE payoffs amount to chains of outside options in which disagreeing players receive Rubinstein payoffs.[23] In particular, the dominant player a is able to extract the full residual surplus in his relationship with d, namely $s_{ad} - \sigma_d$, as he exploits the presence of the link ad when bargaining with b. When this improves a's terms of trade, no strongly efficient LMPE exists as shifted Rubinstein payoffs are outside the core. However, sequential LMPE may still exist and in such equilibria b captures only the residual value of his relationship with a, namely $s_{ab} - s_{ad} + \sigma_d$.[24] These limiting equilibria conform to previous intuition. Alternatives within the market can affect the terms of trade only if they remain in the market indefinitely.

One implication of Proposition 9 is that only vertically differentiated markets can clear efficiently in a sequential LMPE.[25] Shifted Rubinstein payoffs must also be outside the core. When the conditions in Proposition 9 are met, the market can clear efficiently with sequential agreement and the first match occurs on the most valuable core match, as $s_{ab} > s_{cd}$. We rationalize top-down sequential exit as an efficient market outcome in a complete information decentralized bargaining game. Delay in bargaining is hard to get, but real world experience suggests that matching markets can occasionally be held up while clearing from the top. The

[23]These chains are similar to the outside option chains identified as characterizing certain core payoff in Elliott (15).

[24]Also, b would have the option to agree with c, and to extract $s_{bc} - \sigma_c$. However, such an option is never preferred.

[25]The market is vertically differentiated here as $s_{ab} > s_{ad} > s_{bc} > s_{cd}$ implies everyone agrees that a and b are the most valuable players on their respective sides of the market.



(I) (II)

Figure 6: Panel I plots the lower bound for ζ for different combinations of s_{cd} and s_{bc}. As $\zeta < 1$, regions of the parameter space where lower bound is greater than 1 are regions in which no sequential LMPE exists. Panel II shows the lower bound only when $\zeta < 1$.

model delivers such behavior as an equilibrium phenomenon in thin markets without any asymmetric information.

To better understand the necessary and sufficient conditions found in Proposition 9 it is instructive to consider the amount of mismatch inefficiency that weakly efficient LMPE are able to eliminate. For convenience normalize $s_{ab} = 1$ and define the fraction of surplus that can be lost because of mismatch $\zeta = (s_{ad} + s_{bc}) / (1 + s_{cd}) \in (0, 1)$. The final and key restriction to the parameter space identified in Proposition 9 can then be restated in terms of this parameter as requiring

$$\zeta \geq \frac{2(1 + s_{bc})(s_{bc} + s_{cd}) - (1 + s_{cd})(s_{bc} - s_{cd})}{2(s_{bc} + s_{cd})(1 + s_{cd})}.$$

We plot this lower bound on the relative efficiency of the wrong matches in Figure 6.

The plot shows that, when s_{cd} is relatively large, there is no weakly efficient LMPE that is not strongly efficient. As we must have $\zeta < 1$ for a sequential LMPE to exist, $s_{cd} < 1 - 2s_{bc}$. Since by Proposition 9 $s_{bc} > s_{cd}$, it must be that $s_{cd}/s_{ab} < 1/3$. So, the less productive core match must be at least three times less productive than the most productive core match. This upper bound on the relative value of s_{cd} becomes much tighter when the welfare loss associated to mismatch ζ is at least 5%. Indeed, for $\zeta \leq 0.95$ a similar calculation establishes that $s_{cd}/s_{ab} < 0.133$; so s_{cd} can be at most 13.3% as productive as s_{ab}.[26] We conclude that sequential LMPE only exist in sufficiently vertically differentiated markets and only in extremely differentiated markets when mismatch generates a considerable amount of inefficiency.

[26]Example 4 in Section 6 provides some specific parameter values for which sequential exit occurs. In this example $s_{cd}/s_{ab} = 0.1$ and $\zeta = 0.955$.

8 Conclusions

We study the Markov perfect equilibria of an extended Rubinstein bargaining game with many buyers and many sellers. Efficient equilibria are closely tied to the core. If all agents bargaining bilaterally with their efficient (core) partner results in an outcome that is in the core, then there exists an efficient Markov perfect equilibrium. If not, then no Markov perfect equilibrium is efficient. There are two types of inefficiency: mismatch and delay. In the limit, inefficiencies due to delay go to 0, but mismatch is a necessary condition for delay, so frictions cannot be driven purely by delay.

The intuition for the inefficiencies we find is the following. The network of potential matches provides endogenous outside options to the agents. However, for an agent to benefit from an outside option that might otherwise disappear (as happens when the alternative match providing the outside option exits the market), the agent must exercise his outside option with positive probability otherwise his eventual match can simply wait for the outside option to be lost. This leads to mismatch. While mismatch is driven by the potential for the market to evolve adversely, delay occurs because agents expect the market to evolve in their favor. Delay can be extensive and highly structured. We find that vertically differentiated markets can be held up while everyone waits for the highest quality worker and firm to reach an agreement.

Endogenous outside options provided by alternative matches are not equivalent to a scenario in which all players have exogenous outside options of the same value. In equilibrium, exogenous outside options can bound payoffs from below without ever being exercised (Sutton 1986). Our analysis shows that this logic relies crucially on the outside options remaining available forever, and fails to hold when agents exit the market.

Our results suggest a conjecture. If people could costlessly rematch endogenous outside options should act as though they are exogenous. They would no longer disappear as the market evolves and pairs match. Agents would not need to exercise an outside option for it to bound their payoff from below, and so mismatch should be eliminated. Inefficiencies should disappear. Moreover, MPE payoffs should be core payoffs. Allowing costless rematch should result in our model microfounding the core. Such a result would further emphasize that bargaining frictions arise as a strategic response to the changing composition of the market, helping to identify when such frictions will be most severe and how they might be ameliorated. While there are substantial technical difficulties that need to be overcome before such a result can be obtained, Elliott and Arganov (2015) test this conjecture in the lab and find support for it. Their results also strongly support the inefficiencies we document when there is exit.

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9 Appendix I: Low Frequency

This short section provides conditions for MPE efficiency at low frequency of interaction. These conditions require that a player's preferred bargaining partner coincide with his core partner. At low frequency of interaction, players necessarily offer a negligible amount to any partner when proposing due high cost of rejecting offers. Thus, players will negotiate with their core match only when the surplus generated with such a player exceeds the surplus that could be generated in any other match. The next result formalizes such observations.

Theorem 10 *A strongly efficient MPE exists:*
(a) for all δ close to 0 if any player strictly prefers his core match,

$$s_{i\eta(i)} > s_{ij} \quad \text{for all } i, j \in N.$$

(b) for all δ close to 0 only if any player weakly prefers his core match,

$$s_{i\eta(i)} \geq s_{ij} \quad \text{for all } i, j \in N.$$

It would be compelling to conclude by arguing that if an efficient MPE exists for arbitrarily high and low values of δ, then it also exists for any intermediate value. However, the conclusion does not hold in general, as the incentive constraints characterizing the existence of efficient MPE are quadratic.

10 Appendix II: Multiplicity

This short section presents an economy in which condition (2) holds, but in which multiple MPE exist for all δ close to 1. Consider the following 4 player economy: $N = \{a, b, c, d\}$;

$$p_a = p_b = 4/10, \;\; p_c = p_d = 1/10;$$
$$s_{ab} = s_{cd} = 36, \;\; s_{ad} = s_{bc} = 35.$$

Clearly it satisfies condition (2) as

$$\sigma_a + \sigma_d = \sigma_b + \sigma_c = 36 > 35.$$

Thus, an efficient MPE always exits for all δ close to 1; and consequently a strongly efficient LMPE exists. However, for all δ close to 1 there also exists an inefficient MPE with the following proposal probabilities

$$\pi_{ad} = \pi_{bc} = \pi_{cd} = \pi_{dc} = 1.$$

In such an equilibrium (setting $V_a = V_b$ and $V_d = V_c$) value equations in (1) reduce to

$$V_a = \frac{4}{10}(35 - \delta V_c) + \frac{2}{10}\delta V_a(ab) + \frac{4}{10}\delta V_a(ad)$$
$$V_d = \frac{1}{10}(36 - \delta V_d) + \frac{1}{2}\delta V_d + \frac{4}{10}\delta V_d(ad).$$

Solving for subgame values, establishes that

$$V_a = \frac{2(350 - 69\delta - 25\delta^2)}{5(5-\delta)(2-\delta)} \text{ and } V_d = \frac{36 - 4\delta}{(5-\delta)(2-\delta)}.$$

Taking limits then implies that $\lim_{\delta\to 1} V_a = 128/5 = 25.6$ and $\lim_{\delta\to 1} V_d = 8$. Limit values then satisfy all the equilibrium incentive constraints as

$$2V_a > 36, \ \ 2V_d < 36,$$
$$V_a + V_d < 35, \ \ 36 - V_d > 35 - V_a.$$

Thus, as incentive constraints are strict and value functions continuous, players strictly prefer to comply with the strategy for all δ close to 1. This establishes that the proposed strategy is an MPE all δ close to 1, and thus an LMPE. Hence, multiple equilibria may exist even when condition (2) holds and the core match is unique. The multiplicity arises here as directed search and partner selection inevitably bring about coordination problems.

11 Appendix III: Coordination Problems and Delay

Example 5: Consider the six-player assignment economy depicted in Panel I of Figure 5, in which agents a and f propose with probability 1/4, whereas all other players propose with probability 1/8. In such an example, the efficient match is fully pinned down by the value of parameter y. We consider values of $y \in [2, 3]$. Panel II of Figure 5 shows the equilibrium offer probabilities of the players.

Agents b and e have binding outside options. We consider whether there can be an equilibrium in which a and f delay making an offer. Agents c and d will make offers to each other, so if agent a delays, and agent c or d is selected before either agent b or agent e, then agent a will end up bargaining bilaterally with agent b. As a will end up in a strong position vis-a-vis b in this scenario, it could in principle be possible that a optimally delays. To explore this possibility, we assume that agents a and f delay with probability $1-q$, and we look for conditions on q and y under which there is an equilibrium with the offer pattern shown in Panel II of Figure 5. Finding agents' expected payoffs in the relevant subgames and taking the limit, we find that as $\delta \to 1$,

$$\begin{aligned} V_a(N) = V_f(N) &\to \frac{16+q(17-3y)}{24+12q}, \\ V_b(N) = V_e(N) &\to \frac{7+3y}{12}, \\ V_c(N) = V_d(N) &\to 1. \end{aligned}$$

Note that these expected payoffs are all strictly positive for all $y \in [2,3]$ and all $q \in [0,1]$. It can be verified that $\partial V_a/\partial q > 0$ for $y < 3$, but $\partial V_a/\partial q = 0$ for $y = 3$. In this example, there can be no equilibrium delay for $y < 3$, as we must have $q = 1$ for agents a and f to be playing best responses. However, there might be equilibrium delay for $y = 3$. In fact, it can be shown that there is an equilibrium with $q = 0$ and $y = 3$. The reason for this discontinuity is that there are multiple efficient matches when $y = 3$. Although agent a delays, there is an efficient match in which he is left alone. With heterogeneities, instances of multiple efficient matches are non-generic. When the core match is unique, delay occurs only because of fundamental strategic reasons, as we saw in the previous example.

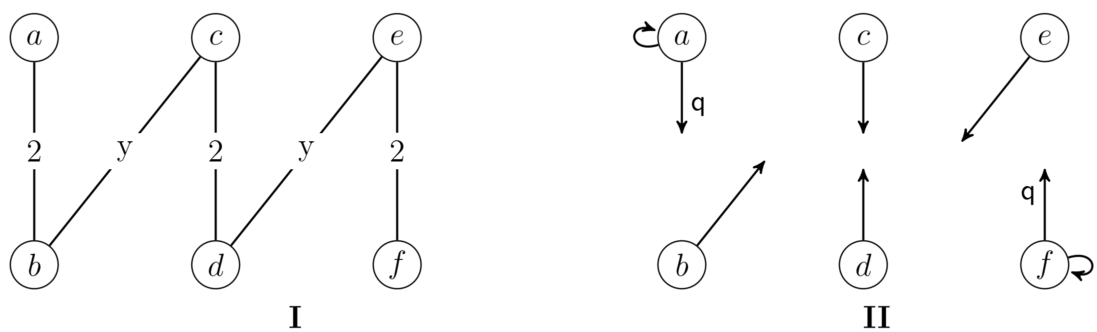

Figure 7: In Panel I the assignment economy; in Panel II agreement probabilities.

12 Appendix IV: Assortative Matching

For this section it will be helpful to partition the set of agents into workers and firms. While only a limited exploration of heterogeneities has been reported in the labor market search literature, one form of heterogeneity that has been extensively analyzed is vertically differentiated markets with assortative matching (see, for example, Shimer and Smith (2000), Smith (2006), Eeckhout and Kircher (2010)). More specifically, let W and F denote the sets of worker types and firm types, respectively, and suppose that $W = \{1, ..., w\}$ and $F = \{1, ..., f\}$, and that the surplus generated by a worker i and firm j is given by a function $S : W \times F \to \mathbb{R}_+$ satisfying the following conditions:

[C1] $S(i, j) > S(i', j)$ if and only if $i < i'$;

[C2] $S(i, j) > S(i, j')$ if and only if $j < j'$;

[C3] $S(i, j) - S(i, j') > S(i', j) - S(i', j')$ if and only if $i < i'$ and $j < j'$.

Condition C1 requires workers to be vertically differentiated, C2 requires firms to be vertically differentiated, and C3 requires increasing differences in the surpluses that worker–firm pairs can generate. As before maintain that surplus is generated only in matches between workers and firms. In contrast to our previous notation, however, there can now be a worker-type i and a firm-type i. Thus typically $S(i, i) \neq 0$ and $S(i, j) \neq S(j, i)$ unless the productivity of the i^{th} ranked worker and j^{th} ranked firm is the same as the productivity of the j^{th} ranked worker and i^{th} ranked firm. Let the set of functions satisfying these conditions be denoted by $\bar{\mathcal{S}}$. It is well known that in such markets the unique surplus-maximizing – or core – match is the assortative match in which worker k is matched to firm k if $k \leq \min\{w, f\}$, while all the remaining agents are unmatched.

We can use our previous results to find conditions under which decentralized bargaining would result in an efficient and thus assortative match. We say an MPE is *assortative* – or efficient – if worker k is always matched to firm k for all $k \leq \min\{w, f\}$. For convenience, let p_k denote the proposal probability of firm k, and let q_k denote the proposal probability of worker k.

Proposition 11 *If $w = f$, $p_k = q_k = p$ for all $k \leq \max\{w, f\}$, and $S(i, j) = S(j, i)$ for all $i, j \leq \min\{w, f\}$, then for all δ sufficiently close to 1 there is an assortative MPE. However, if any of these conditions fails, there exists a map $S \in \bar{\mathcal{S}}$ such that for all δ sufficiently close to 1 there is no assortative MPE .*

Proposition 11 shows that, although there are conditions under which the assortative match obtains in decentralized markets, these conditions are fairly restrictive and require the market to be highly symmetric. There must be the same number of workers as firms, the k^{th} ranked worker and firm must have the same proposal probabilities, and the productivity of the i^{th} ranked worker and j^{th} ranked firm must be the same as the j^{th} ranked worker and i^{th} ranked firm. In fact, conditions C1, C2, and C3 alone do not guarantee that decentralized bargaining is efficient, as uncertainty about future market evolution can still distort trade. Even under conditions that guarantee that the assortative matching is efficient, focusing on efficient bargaining outcomes can thus be highly restrictive.

13 Appendix V: Proofs

Proof of Proposition 1. We first establish the characterization for MPE values, and then proceed to establish existence. Fix a discount factor $\delta \in (0,1)$. Consider an MPE strategy profile (ρ, χ, α) and its corresponding MPE payoffs $V(A) \in \mathbb{R}^{|A|}$ for any active player set $A \subseteq N$. Fix any subset $A \subseteq N$. By subgame perfection, we know that the acceptance decision by a player $j \in A$ faced with an offer x must be such that he accepts an offer if $x > \delta V_j(A)$, and rejects it if $x < \delta V_j(A)$. Clearly, this implies that it cannot be optimal to offer $x > \delta V_j(A)$ to player j, as the proposer could profitably deviate to an offer in $(\delta V_j(A), x)$. Thus, in any MPE every player would offer at most $\delta V_j(A)$ to player j, and the only offers player j may accept with positive probability are offers of $\delta V_j(A)$ with positive probability. Therefore, a proposer $i \in A$ would make offers with positive probability only to a player j that maximizes his residual payoff $s_{ij} - \delta V_j(A)$. Recall that $\pi_{ij}(A)$ is the joint probability that player i offers $\delta V_j(A)$ to player j and that the offer is accepted, and that $\pi_{ii}(A)$ is the joint probability that i does not agree when proposing. We frequently abuse notation by dropping the dependence of π_{ij} on A where it should not cause confusion. The payoff of any player $k \in A \backslash i, j$ at the beginning of the following period is given by $V_k(A \backslash i, j)$ if an agreement was reached, and by $V_k(A)$ otherwise. Therefore, at a history in which the set of active players is A and in which i is the proposer, the expected payoff of a player $k \in A \backslash i$ must be given by

$$\textstyle\sum_{j \in A \backslash i,k} \pi_{ij} \delta V_k(A \backslash i, j) + (1 - \sum_{j \in A \backslash i,k} \pi_{ij}) \delta V_k(A).$$

When i is chosen to propose, if $\delta[V_i(A) + V_j(A)] < s_{ij}$ for some $j \in A \backslash i$, then i offers with certainty to players j who maximize $s_{ij} - \delta V_j(A)$, and agreement obtains with certainty. The latter observation obtains from the following argument. If $\pi_{ii} > 0$, then the expected payoff conditional on offering $\delta V_j(A)$ to players j who maximize $s_{ij} - \delta V_j(A)$ is

$$\textstyle\sum_{j \in A \backslash i} \pi_{ij} (s_{ij} - \delta V_j(A)) + \left(1 - \sum_{j \in A \backslash i} \pi_{ij}\right) \delta V_i(A) < s_{ij} - \delta V_j(A).$$

The payoff conditional on offering $\delta V_j(A) + \varepsilon$, for $\varepsilon > 0$ is $s_{ij} - \delta V_j(A) - \varepsilon$, as players accept with probability 1 any offer exceeding $\delta V_j(A)$. Hence, it cannot be optimal to offer more than $\delta V_j(A)$ (or, for that matter, it cannot be optimal to offer less than $\delta V_j(A)$) since all such offers are rejected and since $\delta V_i(A) \leq s_{ij} - \delta V_j(A) - \varepsilon$. Thus, if $\pi_{ii} > 0$ and $\delta[V_i(A) + V_j(A)] < s_{ij}$, a profitable deviation always exists. Therefore, $\delta[V_i(A) + V_j(A)] < s_{ij}$ for some $j \in A \backslash i$ implies $\pi_{ii} = 0$. Similarly, $\delta[V_i(A) + V_j(A)] > s_{ij}$ for any $j \in A \backslash i$ implies $\pi_{ii} = 1$. If $\max_{j \in A \backslash i} \{s_{ij} - \delta[V_i(A) + V_j(A)]\} = 0$, then $\pi_{ii} \in [0,1]$. Next consider the correspondence

$f_i(V|A) : \mathbb{R}^{|A|} \rightrightarrows \mathbb{R}^{|A|}$, where

$$f_i(V|A) = \left\{ \begin{array}{l} \left[\underbrace{\textstyle\sum_{j\in A\backslash i,k} \pi_{ij}\delta V_k(A\backslash i, j) + \pi_{ik}\delta V_k}_{k^{\text{th}} \text{ Entry for } k\in A\backslash i}, \underbrace{(1-\pi_{ii}) \max{}_{j\in A\backslash i}\{s_{ij} - \delta V_j\}}_{i^{\text{th}} \text{ Entry}} \right] + \pi_{ii}\delta V \Bigg| \\ \quad \pi_{ii} = 0 \quad \text{if} \qquad\qquad \max{}_{j\in A\backslash i}\{s_{ij} - \delta V_j(A)\} > \delta V_i(A) \\ \quad \pi_{ik} = 0 \quad \text{if} \quad s_{ik} - \delta V_k(A) < \max\{\delta V_i(A), \max_{j\in A\backslash i}\{s_{ij} - \delta V_j(A)\}\} \end{array} \right\};$$

where the expressions in the square brackets give the components of a $|A| \times 1$ vector. Let $f_{ik}(V|A)$ denote the k$^{\text{th}}$ entry of $f_i(V|A)$. The correspondence $f_{ik}(\cdot|A)$ identifies the set of expected payoffs compatible with our partial equilibrium analysis for a player $k \in A$ and for any history in which A is the set of active players and i is the proposer. Next, define the correspondence

$$F(V|A) = \textstyle\sum_{i\in A} p_i f_i(V|A) + \left(1 - \sum_{i\in A} p_i\right) \delta V. \tag{7}$$

The k$^{\text{th}}$ entry of such a correspondence, $F_k(\cdot|A)$, identifies the set of possible expected payoffs for a player $k \in A$ for any history in which A is the set of active players. Thus, the argument establishes that V is an MPE payoff only if it is a fixed point of the correspondence in (7), $V \in F(V|A)$.

Next, we establish that the converse must hold too. In particular, we argue that if $V(A) \in F(V(A)|A)$ for any subset $A \subseteq N$, then $V(A)$ is an MPE payoff profile for any subgame in which A is the set of active players. At any subgame in which A are the active players, consider a strategy in which any player $i \in A$ chooses $\rho_i(A) = \pi_i$, $\chi_i(j, A) = \delta V_j(A)$, and

$$\alpha_i(j, x, A) = \left\{ \begin{array}{lll} 1 & \text{if} & x \geq \delta V_i(A) \\ 0 & \text{if} & x < \delta V_i(A) \end{array} \right. .$$

For any finite set of players N, the proposed strategy clearly constitutes an MPE in any subgame in which no more than one player is active, as any such subgame is eventless. By induction, suppose that the proposed strategy is an MPE for any subset of active players of size $k \leq n-1$, in order to show that it is an MPE for any subgame in which the set of active players has size $k+1$. Consider a subgame in which the set of active players A has cardinality $k+1$. Fix an MPE payoff profile $V(A')$ for all subgames in which the cardinality of the set of active players A' does not exceed k. Furthermore, given such values, suppose that we can find a payoff profile $V(A)$ such that $V(A) \in F(V(A)|A)$ (we establish below that such a fixed point exists). If so, no player receiving an offer can profitably deviate from strategy α, as no change in the acceptance rule can strictly increase his payoff. Similarly, given the acceptance rule, the proposer's strategy (ρ, χ) is optimal given that offers are made only to those players who leave the highest residual surplus to the proposer (provided that such surplus exceeds the value of being unmatched). Thus, $V(A)$ is an MPE payoff in any subgame with a set of active players A. Consequently, if $V(A) \in f(V(A)|A)$ for any subset $A \subseteq N$, then $V(A)$ is an MPE payoff profile.

To establish existence, also proceed by induction. Existence follows, in any subgame in which no more than one player is active, as such subgames are eventless. Assume by induction that an MPE exists for any subset of active players of size $k \leq n-1$, in order to show that it exists for any subgame in which the set of active players A has size $k+1$. If so, consider MPE strategies for all subgames of size k and derive MPE payoffs for all such subgames. Given such values, construct the correspondence $F(\cdot|A)$ as in (7). Observe that the correspondence $f_i(\cdot|A)$ is upper-hemicontinuous with non-empty convex images. Similarly, the correspondence $F(\cdot|A)$ is upper-hemicontinuous with non-empty convex images, as it is a convex combination of the correspondences $f_i(\cdot|A)$ for $i \in A$. By Kakutani's fixed point Theorem $F(\cdot|A)$ has a fixed point. Moreover, such a fixed point is an MPE payoff of this subgame, and can be used to construct consistent MPE strategies in every subgame, as argued above. ■

Proof of Proposition 2. Consider any MPE strategy prescribing that any player $i \in N$ offers to his core match $\eta(i)$ with probability 1 at any active player set $A \in C(N)$. If players follow the prescribed strategies, only core matches are ever consummated. Thus, if all players comply with the proposed strategy, only subgames $A \in C(N)$ occur on the equilibrium path. As the core match maximizes the total surplus in an assignment economy, the core match of a player cannot change when other core pairs exit the market. Thus, the core match of a player coincides with that at any subgame $A \in C(N)$. By Proposition 1, we know that any proposer $i \in A$ necessarily offers an amount equal to $\delta V_{\eta(i)}(A)$ and that any receiver $i \in A$ accepts any offer exceeding $\delta V_i(A)$. As players negotiate with only core partners on the equilibrium path, at any $A \in C(N)$ we guess that

$$V_i(A) = V_i(A \backslash j, \eta(j)) \quad \text{whenever } i \notin \{j, \eta(j)\}. \tag{8}$$

Thus, at any $A \in C(N)$, equilibrium payoffs for every player $i \in A$ satisfy

$$V_i(A) = p_i(s_{i\eta(i)} - \delta V_{\eta(i)}(A)) + (1-p_i)\delta V_i(A).$$

Solving the latter equation for player i with the one for player $\eta(i)$ implies that

$$V_i(A) = \frac{p_i}{1-\delta+\delta p_i}\left(s_{i\eta(i)} - \delta \frac{p_{\eta(i)}}{1-\delta+\delta p_{\eta(i)}}(s_{i\eta(i)} - \delta V_i(A))\right),$$

which after some manipulation yields

$$V_i(A) = \frac{p_i}{1-\delta+\delta p_i+\delta p_{\eta(i)}} s_{i\eta(i)}. \tag{9}$$

It can be (trivially) verified that $V_i(A)$ satisfies (8). Taking limits as the discount factor approaches unity, we obtain

$$\lim_{\delta\to 1} V_i(A) = \frac{p_i}{p_i+p_{\eta(i)}} s_{i\eta(i)} = \sigma_i. \tag{10}$$

■

Proof Theorem 3. Suppose, by contradiction, that there exists a weakly efficient MPE that is not strongly efficient. If so, along any equilibrium path, players either match to their core partner or defer reaching an agreement, which implies that any equilibrium-path subgame has a set of active players A which belongs to $C(N)$. Formally, such a requirement amounts to finding a fixed point of the MPE characterization in Proposition 1 which satisfies $\pi_{ii}(A)+\pi_{i\eta(i)}(A)=1$ for any $i\in A$ and any $A\in C(N)$. If such an equilibrium were to exist, an argument equivalent to the proof of Proposition 2 would imply that for any $i\in A$ and any $A\in C(N)$,

$$V_i(A) = \frac{p_i\pi_{i\eta(i)}(A)}{1-\delta+\delta p_i\pi_{i\eta(i)}(A)+\delta p_{\eta(i)}\pi_{\eta(i)i}(A)} s_{i\eta(i)}.$$

This would lead to a contradiction, as $V_i(A)$ strictly increases in $\pi_{i\eta(i)}(A)$, which implies that any player i would always strictly prefer offering immediately to his core match rather than deferring to future rounds. ■

Proof of Theorem 4. First, we establish part (a). Payoffs in any subgame $A\in C(N)$ of a strongly efficient MPE are pinned down by Proposition 2 for any $\delta\in(0,1)$. We show that complying with efficient strategies yields an equilibrium for any sufficiently high value of δ. Recall that any player $j\in A$ accepts any offer that is worth at least $\delta V_j(A)$. Suppose, by contradiction, that some player $i\in A$ at some subgame $A\in C(N)$ has a profitable deviation which entails offering to $j\neq\eta(i)$ when all players comply with strongly efficient strategies. For such an offer to be profitable for i, at any sufficiently high δ it must be that

$$s_{ij}-\delta V_j(A) > s_{i\eta(i)}-\delta V_{\eta(i)}(A). \tag{11}$$

However, by taking limits, as δ converges to 1, on both sides of this inequality, we obtain

$$s_{ij}-\sigma_j > s_{i\eta(i)}-\sigma_{\eta(i)} = \sigma_i.$$

This obviously contradicts the assumption that Rubinstein payoffs are in the interior of the core: $\sigma_i + \sigma_j > s_{ij}$ for all $i, j \in A$ such that $j \neq \eta(i)$. Thus, any player $i \in A$ at any subgame $A \in C(N)$ cannot have a profitable deviation when making offers if the discount factor is sufficiently high, which implies the existence of a strongly efficient MPE for any δ close to 1. Next, we establish part (b). By contradiction, assume that a strongly efficient MPE exists for any δ close to 1, but that $\sigma_i + \sigma_j < s_{ij}$ for some pair $i, j \in N$. Recall that player i has a strictly profitable deviation from a strongly efficient equilibrium if condition (11) holds. Then as δ converges to 1, that inequality converges to $s_{ij} - \sigma_j > \sigma_i$, which holds by assumption. Thus, player i must have a profitable deviation for any sufficiently high value of δ. ■

Proof of Proposition 5. To establish the first implication, let u be a vector of core payoffs associated to the core match η. Consider two players $i, j \in N$ such that $\eta(i) = j$, and set

$$\frac{p_i}{p_j} = \frac{u_i}{s_{ij} - u_i}.$$

This condition ensures that i and j receive their core payoffs, u_i and u_j, if everyone plays the strategies characterized in the proof of Proposition 2. This removes at most $N/2$ degrees of freedom from the vector p. Thus, it is straightforward to find a probability vector p that satisfies the above condition for all $i \in N$.

The second part of the result is a trivial consequence of the Rubinstein payoffs not being affected by proportional changes in probabilities. The third part is also straightforward. Let $U(S)$ denote the core when the surplus matrix is S. Observe that if the surplus changes from S to S', it must be that $s_{i\eta(i)} = s'_{i\eta(i)}$ for any $i \in N$. This is because the core match cannot change when S changes to S', and because $s_{i\eta(i)} \neq s'_{i\eta(i)}$ implies that any core payoff in S would not belong to S' (since $u_i + u_{\eta(i)} = s_{i\eta(i)}$ for any $u \in U(S)$). Thus, Rubinstein payoffs in the two markets must coincide,

$$\sigma = (\sigma_1, ..., \sigma_n) = (\sigma'_1, ..., \sigma'_n) = \sigma'.$$

The conclusion then follows immediately from these observations, since $\sigma \in U(S) \subseteq U(S')$. ■

Proof of Proposition 6. To prove this result, it is useful to introduce the notions of an offer graph and a cyclical offer graph. For any subgame with active player set $A \subseteq N$ and any MPE, the *offer graph* (A, G) consists of a directed graph with vertices in A and with edges satisfying

$$ij \in G \quad \Leftrightarrow \quad i \in A \ \text{ and } \ j \in \{k \mid \pi_{ik}(A) > 0\} \cup \eta(i).$$

We say that an offer graph is cyclical whenever there exists a subset of active players choosing to make offers so as to exchange their respective core partners with one another. Formally, an offer graph is *cyclical* if there exists a map $\varphi : N \to N \backslash i$ and a set of players $F \subseteq P_k \cap A$ for $k \in \{1, 2\}$ such that

$$\begin{aligned}
&(1)\ \varphi(i) = j \quad \Rightarrow \quad ji \in G, \\
&(2)\ \varphi(i) \neq \eta(i) \ \text{ for some } i \in F, \\
&(3)\ \{\varphi(i)|i \in F\} = \{\eta(i)|i \in F\}.
\end{aligned}$$

Next, we establishes that MPE offer graphs are never cyclical. If offers were cyclical, a subset of players who prefer offering to one another's core matches instead of their own core match would exist. These players would have to achieve a higher aggregate surplus by matching with non-core partners, thereby violating the efficiency properties of the core. Formally, suppose the offer graph is cyclical. By revealed preferences for any player $i \in F$ and $\varphi(i)$ such that $\pi_{i\varphi(i)}(A) > 0$, subgame perfection requires that

$$s_{i\varphi(i)} - \delta V_{\varphi(i)} \geq s_{i\eta(i)} - \delta V_{\eta(i)}.$$

Furthermore, because of cyclicality, by summing over all players in F we would have that

$$\textstyle\sum_{i \in F}(s_{i\varphi(i)} - \delta V_{\varphi(i)}) \geq \sum_{i \in F}(s_{i\eta(i)} - \delta V_{\eta(i)}) \quad \Leftrightarrow \quad \sum_{i \in F} s_{i\varphi(i)} \geq \sum_{i \in F} s_{i\eta(i)}.$$

However, this would contradict the efficiency of the core match, as a unique core match was assumed to exist.

Next, we establish that the core match always obtains with positive probability in an MPE without delay. The uniqueness of the core match and the non-negativity of surpluses imply that all players on one side of the market are matched at the unique core allocation.[27] Fix an MPE without delay. No delay implies that at any subgame any player with a positive value who is selected to be the proposer makes offers that result in agreement with probability 1. Without loss of generality, suppose that $P_1 \cap A \geq P_2 \cap A$. If for any A there exists $i \in P_1 \cap A$ such that $\pi_{i\eta(i)}(A) > 0$, the conclusion obviously holds. Thus assume that this is not the case. Then $\pi_{i\eta(i)}(A) = 0$ for some A and any $i \in P_1 \cap A$. Next, we show that this leads to a contradiction, as the offer graph would necessarily be cyclical. Pick any match φ satisfying $\varphi(i) = j$ for $\pi_{ij}(A) > 0$, and $\varphi(i) \neq \eta(i)$ for any $i \in P_1 \cap A$. Such a match exists because players in $P_1 \cap A$ do not delay, and because $\pi_{i\eta(i)}(A) = 0$. Observe that, since the core match is unique, $P_2 = \{\eta(i)|i \in P_1 \cap A\} \cap P_2$. Furthermore, by construction it must be that $P_2 \supseteq \{\varphi(i)|i \in P_1 \cap A\} \cap P_2$. Since $\eta(i) \neq \eta(k)$ for any $i, k \in P_1 \cap A$, there must exist a

[27] This is the only result in which the assumption on non-negativity of the surplus is substantive.

set $F \subseteq P_2$ such that

$$\{\varphi(i)|i \in F\} = \{\eta(i)|i \in F\},$$

as otherwise a player $i \in P_1 \cap A$ would exist such that $\varphi(i) = \eta(i)$. This in turn implies the desired contradiction to the first part of the proposition, as the offer graph would necessarily be cyclical.

The existence of MPE and of LMPE with delay is follows by examples 3 and 4. ■

Proof of Theorem 7. To pin down LMPE values, for any player $i \in N$, define the outside option partner for player i as follows

$$\lambda(i) = \begin{cases} \arg\max_{j \in E} s_{ij} & \text{if} \quad \omega_i > 0 \\ i & \text{if} \quad \omega_i = 0 \end{cases}$$

Therefore, $\omega_i = s_{i\lambda(i)}$. An LMPE is strongly efficient if at any active player set $A \in C(N)$, all players $i \notin E$ make acceptable offers to their core matches $\eta(i)$ with a probability that converges to 1 (that is, $\lim_{\delta \to 1} \pi_{i\eta(i)}(A) = 1$), and all players $i \in E$ delay with a probability that converges to 1 (that is, $\lim_{\delta \to 1} \pi_{ii}(A) = 1$). In the limit, if all players follow a strongly efficient strategy, only core matches are consummated with probability 1, and only subgames $A \in C(N)$ occur on the equilibrium path with positive probability. As the core match maximizes the total surplus in an assignment economy, the core match of a player coincides in any subgame $A \in C(N)$. Further, if players that are unmatched in the core never exit the market, outside options coincide in any subgame $A \in C(N)$ (that is, $E \subseteq A$). Therefore, as the discount factor converges to unity, we have that

$$\lim_{\delta \to 1} V_j(A) = \bar{V}_j \text{ for any } j \in A \text{ and } A \in C(N).$$

As before, we occasionally omit the dependence on δ and on the active player set A, whenever it is superfluous.

To establish that any strongly efficient strategy compatible with equilibrium necessarily yields shifted Rubinstein payoffs as limiting payoffs, we proceed by induction on the size of the active player set within $C(N)$, and show that for any $A \in C(N)$, any strongly efficient LMPE necessarily satisfies

$$\lim_{\delta \to 1} V_j(A) = \bar{\sigma}_j \quad \text{for any} \quad j \in A. \tag{12}$$

First, consider the smallest active player set in $C(N)$, namely, $A = E$, when such set is not empty. If so, $s_{ij} = 0$ for any $i, j \in E$. Obviously, $V_j(E) = \bar{\sigma}_j = 0$ for any $j \in E$. Next, consider any active player set $A = E \cup \{i, \eta(i)\}$ for some $i \in N \backslash E$. Clearly, not both players in $\{i, \eta(i)\}$ can have binding outside options. If they did, then

$$s_{i\lambda(i)} + s_{\eta(i)\lambda(\eta(i))} \geq s_{i\eta(i)},$$

and an alternative match that generates a weakly higher surplus would be feasible (since both $\lambda(i)$ and $\lambda(\eta(i))$ would be unmatched in the core), thereby contradicting the optimality of the core match or its uniqueness. Without loss of generality, if a player has a binding outside option, let that player be i, so that $\bar{\sigma}_i = \max\{\omega_i, \sigma_i\}$ and $\bar{\sigma}_{\eta(i)} = s_{i\eta(i)} - \bar{\sigma}_i$. Observe that if a player $j \in E$ plays a strategy converging to efficiency, then for sufficiently high δ he must weakly prefer delaying to offering to a player $\{i, \eta(i)\}$, as $\lim_{\delta \to 1} \pi_{jj}(A) = 1$. If so, then $v_j(A) = \delta V_j(A)$ and the valuation of such a player necessarily satisfies

$$\lim_{\delta \to 1} V_j(A) = (1 - p_i - p_{\eta(i)}) \lim_{\delta \to 1} V_j(A) = 0$$

by the characterization in Proposition 1, the definition of strongly efficient LMPE, and the linearity of the limit operator. Therefore, condition (12) holds for any player $j \in E$. Next, consider player $j \in \{i, \eta(i)\}$. If complying with a strongly efficient strategy is a limiting equilibrium, then for sufficiently high δ it must be that $v_j(A) = s_{j\eta(j)} - \delta V_{\eta(j)}(A)$, as $\lim_{\delta \to 1} \pi_{j\eta(j)}(A) = 1$. If so, then for any player $k \in E$,

$$s_{j\eta(j)} - \delta V_{\eta(j)}(A) \geq s_{jk} - \delta V_k(A),$$

which in turn implies that

$$\lim_{\delta \to 1} V_j(A) = p_j \left(s_{j\eta(j)} - \lim_{\delta \to 1} V_{\eta(j)}(A) \right) + (1 - p_j) \lim_{\delta \to 1} V_j(A) \geq s_{jk},$$

which establishes that $\lim_{\delta \to 1} V_j(A) \geq \omega_j$. First, suppose that $\lim_{\delta \to 1} V_j(A) > \omega_j$ for any player $j \in \{i, \eta(i)\}$. If so, then for any player $k \in E$ and any δ close to 1, it must be that $\delta V_k(A) + \delta V_j(A) > s_{jk}$, and thus no player k can ever agree with j. If so, the strategy must be strictly efficient and the result follows by the first part of the proof, as

$$\lim_{\delta \to 1} V_j(A) = \lim_{\delta \to 1} \frac{p_j}{1 - \delta + \delta p_j + \delta p_{\eta(j)}} s_{j\eta(j)} = \sigma_j > \omega_j.$$

Otherwise, suppose that $\lim_{\delta \to 1} V_i(A) = \omega_i$. If so, the characterization in Proposition 1 implies that

$$\lim_{\delta \to 1} V_{\eta(i)}(A) = p_{\eta(i)} \left(s_{i\eta(i)} - \lim_{\delta \to 1} V_{\eta(i)}(A) \right) + (1 - p_{\eta(i)}) \lim_{\delta \to 1} V_{\eta(i)}(A) = s_{i\eta(i)} - \omega_i.$$

The previous observations together imply that $\lim_{\delta\to 1} V_k(A) = \bar{\sigma}_k$ for any $k \in A$, as $\lim_{\delta\to 1} V_i(A) = \bar{\sigma}_i = \max\{\omega_i, \sigma_i\}$ and $\lim_{\delta\to 1} V_{\eta(i)}(A) = s_{i\eta(i)} - \bar{\sigma}_i$.

Next, assume by induction that the same conclusion holds for any active player set $A \in C(N)$ with cardinality $|A| = |E| + 2k$, to show that it holds for any set $A \in C(N)$ with cardinality $|A| = |E| + 2(k+1)$. Consider such a set A. If a player $j \in E$ complies with a strongly efficient strategy, then $v_j(A) = \delta V_j(A)$ for δ close to 1, and the valuation necessarily satisfies

$$\begin{aligned}\lim_{\delta\to 1} V_j(A) &= (1 - p_{A\setminus E}) \lim_{\delta\to 1} V_j(A) + \textstyle\sum_{k\in A\setminus E} p_k \lim_{\delta\to 1} V_j(A\setminus k, \eta(k)) \\ &= (1 - p_{A\setminus E}) \lim_{\delta\to 1} V_j(A) = 0,\end{aligned}$$

where the first equality follows from the characterization in Proposition 1 and the definition of strongly efficient strategy, while the second equality follows from the induction hypothesis. If a player $j \in A\setminus E$ complies with a strongly efficient strategy, then $v_j(A) = s_{j\eta(j)} - \delta V_{\eta(j)}(A)$ for δ close to 1. Thus, for $B = A\setminus[E \cup \{j, \eta(j)\}]$, the valuation necessarily satisfies

$$\begin{aligned}\lim_{\delta\to 1} V_j(A) &= (1 - p_B - p_j) \lim_{\delta\to 1} V_j(A) + p_j(s_{j\eta(j)} - \lim_{\delta\to 1} V_{\eta(j)}(A)) + \textstyle\sum_{k\in B} p_k \lim_{\delta\to 1} V_j(A\setminus k, \eta(k)) \\ &= (1 - p_B - p_j) \lim_{\delta\to 1} V_j(A) + p_j(s_{j\eta(j)} - \lim_{\delta\to 1} V_{\eta(j)}(A)) + p_B \bar{\sigma}_j,\end{aligned}$$

where equalities hold for the reasons stated above. In this case, the limiting value equations for players j and $\eta(j)$ admit a unique solution at

$$\lim_{\delta\to 1} V_j(A) = \bar{\sigma}_j \quad \text{and} \quad \lim_{\delta\to 1} V_{\eta(j)}(A) = \bar{\sigma}_{\eta(j)}.$$

To prove the second part of the result observe that limiting efficiency mandates play according to the strategies characterized above and payoffs converging to shifted Rubinstein payoffs,

$$\lim_{\delta\to 1} V_i(A) = \bar{\sigma}_i \text{ for any } i \in A \text{ and any } A \in C(N).$$

Towards a contradiction, suppose that agents comply with these strategies and that $\bar{\sigma}_i + \bar{\sigma}_j < s_{ij}$ for some pair $i, j \in N$. If so, the definition of shifted Rubinstein payoffs implies that $j \notin \{\eta(i), \lambda(i)\}$. It is then straightforward to see that i has a profitable deviation when selected to make the first offer in the game. Subgame perfection ensures that j would accept any offer greater than $\delta V_j(A)$. Now if the player complies with the prescribed strategy by offering to his core partner, his limiting payoff amounts to

$$\lim_{\delta\to 1} v_i(A) = \bar{\sigma}_i.$$

However, by deviating and offering to j exactly $\delta V_j(A)$, his payoff increases to

$$\lim_{\delta\to 1} s_{ij} - \delta V_j(\delta) = s_{ij} - \bar{\sigma}_j > \bar{\sigma}_i.$$

Thus, for any value of δ sufficiently close to 1, player i has a strict incentive to deviate and make an acceptable offer to j. ■

Proof Theorem 8. In a weakly efficient LMPE, $\lim_{\delta\to 1}\left[\pi_{i\eta(i)}(A) + \pi_{ii}(A)\right] = 1$ for any player $i \in A$ for every $A \in C(N)$. Thus, all players $i \in E$ delay with a probability converging to 1 (that is, $\lim_{\delta\to 1} \pi_{ii}(A) = 1$). In the limit, if all players comply with such strategies, only core matches are consummated with probability 1, and only subgames $A \in C(N)$ occur on the equilibrium path with positive probability. As before, we omit the dependence on δ throughout as every endogenous variable depends on it.

For convenience, define the limiting agreement probability for any player $j \in A\backslash E$ as $\beta_j(A) = \lim_{\delta\to 1} p_j \pi_{j\eta(j)}(A)$, and let $\beta_B(A) = \sum_{k\in B}\beta_k(A)$ for any $B \subseteq A$. Observe that for any active player set $A \in C(N)$ such that $A\backslash E \neq \emptyset$, there exists a player $i \in A\backslash E$ such that $\beta_i(A) > 0$. This is the case since weak efficiency and $\beta_i(A) = 0$ for all player $i \in A\backslash E$, imply $\lim_{\delta\to 1} \pi_{ii}(A) = 1$ for all players $i \in A\backslash E$. But, if so, for δ close to 1, any player i must prefer delaying to offering to $\eta(i)$ which implies that

$$\delta V_i(A) + \delta V_{\eta(i)}(A) \geq s_{i\eta(i)}. \tag{13}$$

As before, this is cannot be the case since it would imply that

$$\textstyle\sum_{i\in A} V_i(A) \geq \sum_{i\in A\backslash E} V_i(A) \geq (1/\delta)\sum_{i\in A\cap P_1} s_{i\eta(i)} > TS(A),$$

where the first and third inequalities are trivial while the second holds by adding the inequalities in (13) and observing that $s_{i\eta(i)} = 0$ if $i \in E$. The observation also implies $\beta_{\eta(i)}(A) > 0$. Thus, in any weakly efficient LMPE subgame there exists a core match with positive agreement probabilities in the limit.

To establish that any weakly efficient LMPE that is not strongly efficient must be sequential, we proceed by induction on the size of the active player set within $C(N)$, and show that for there exists $A \in C(N)$ such that only one core match agrees. That is for some $i \in A\backslash E$ such that (5) holds. First, consider the smallest active player set in $C(N)$, namely, $A = E$, when such set is not empty. If so, any weakly efficient LMPE is strongly efficient as the two definitions coincide. Next, consider any active player set $A = E \cup \{i, \eta(i)\}$ for some $i \in N\backslash E$. Clearly, there must be agreement on the core match, that is $\pi_{i\eta(i)}(A) = \pi_{\eta(i)i}(A) = 1$, as $\delta V_i(A) + \delta V_{\eta(i)}(A) < s_{i\eta(i)}$ by feasibility. This implies that again any weakly efficient LMPE is strongly efficient.

Next, assume by induction that, in the market considered, any weakly efficient LMPE is strongly efficient for any active player set $A \in C(N)$ with cardinality $|A| = |E| + 2k$. Consider any set $A \in C(N)$ with cardinality $|A| = |E| + 2(k+1)$. If a player $j \in E$ complies with a weakly efficient strategy, then $v_j(A) = \delta V_j(A)$ for δ close to 1. If so, the valuation necessarily of j satisfies

$$\begin{aligned}\lim_{\delta\to 1} V_j(A) &= (1-\beta_{A\backslash E}(A)) \lim_{\delta\to 1} V_j(A) + \sum_{k\in A\backslash E} \beta_k(A) \lim_{\delta\to 1} V_j(A\backslash k, \eta(k)) \\ &= (1-\beta_{A\backslash E}(A)) \lim_{\delta\to 1} V_j(A) = 0,\end{aligned}$$

where the first equality follows by taking limits of value equations and the definition of weakly efficient strategy, where the second equality follows by the induction hypothesis, and where the third equality trivially obtains as $\beta_{A\backslash E}(A) > 0$ given that at least 1 core match agrees with positive probability in the limit.

If a player $j \in A\backslash E$ complies with a weakly efficient strategy, then for δ close to 1 it must be that $v_j(A) = \max\{\delta V_j(A), s_{j\eta(j)} - \delta V_{\eta(j)}(A)\}$ by weak efficiency. Taking limits of value equations for any $j \in A\backslash E$ while defining $A(j) = A\backslash[E \cup \{j, \eta(j)\}]$ establishes that

$$\begin{aligned}\lim_{\delta\to 1} V_j(A) &= (1-\beta_{A(j)} - p_j) \lim_{\delta\to 1} V_j(A) + p_j \lim_{\delta\to 1} v_j(A) + \sum_{k\in A(j)} \beta_k \lim_{\delta\to 1} V_j(A\backslash k, \eta(k)) \\ &= (1-\beta_{A(j)} - \beta_j) \lim_{\delta\to 1} V_j(A) + \beta_j \left[s_{j\eta(j)} - \lim_{\delta\to 1} V_{\eta(j)}(A)\right] + \beta_{A(j)}\bar{\sigma}_j, \qquad (14)\end{aligned}$$

where the second equality follows by induction. First suppose that $\beta_j(A) = 0$ for all players $j \in A\backslash[E \cup \{i, \eta(i)\}]$. If so, the equilibrium must be sequential by definition.[28] Next consider a weakly efficient LMPE in which least 2 core matches in A. If so, $\beta_i(A) > 0$ and $\beta_j(A) > 0$ for $i \neq \eta(j)$, and thus $\beta_{A(j)} > 0$ for any $j \in A\backslash E$. But, if so, the limiting value equations (14) for players j and $\eta(j)$ admit a unique solution at

$$\lim_{\delta\to 1} V_j(A) = \bar{\sigma}_j \quad \text{and} \quad \lim_{\delta\to 1} V_{\eta(j)}(A) = \bar{\sigma}_{\eta(j)},$$

for any $\beta_j(A) \in [0, p_j]$. The weakly efficient LMPE must be payoff equivalent to a strongly efficient LMPE at A thereby fulfilling the induction hypothesis. This establishes that any weakly efficient LMPE that is not strongly efficient must be sequential.

The existence of sequential LMPE immediately follows by Example 4. ■

Proof Proposition 9. For convenience, when $A = N$, value functions and proposal proba-

[28] If so, by induction $\lim_{\delta\to 1} V_j(A) = \bar{\sigma}_j$ for all $j \in A\backslash\{i, \eta(i)\}$ as $A(j) \neq \emptyset$ and $\lim_{\delta\to 1} V_j(A\backslash k, \eta(k)) = \bar{\sigma}_j$.

bilities omit the dependence on the active player set A. First observe that players on one of the two core matches never delay in any weakly efficient LMPE for all δ close to 1. Delay on both core matches would require

$$\delta V_a + \delta V_b \geq s_{ab} \quad \text{and} \quad \delta V_c + \delta V_d \geq s_{cd}, \tag{15}$$

which violates feasibility as $\sum_{i \in N} V_i > TS$. Thus, in any weakly efficient LMPE there exists a core match in which no player delays. Call such match $i\eta(i)$ so that $\pi_{ii} + \pi_{\eta(i)\eta(i)} = 0$. Next observe that players agree on at most one of the two non-core with positive probability in any weakly efficient LMPE for all δ close to 1. Agreement on both non-core matches would require

$$\delta V_a + \delta V_d = s_{ad} \quad \text{and} \quad \delta V_b + \delta V_c = s_{bc}.$$

But this would violate the weak efficiency of the limiting equilibrium as

$$\lim_{\delta \to 1} \sum_{k \in N} V_k = s_{ad} + s_{bc} < TS.$$

Thus, in any weakly efficient LMPE there exists a non-core match with disagreement. Call such match ij so that $\pi_{ij} = 0$. This establishes that $\pi_{i\eta(i)} = 1$ and that $\pi_{jj} = 1$. Furthermore, there must be agreement in match $\eta(i)\eta(j)$. If instead we had that $\pi_{\eta(i)\eta(j)} + \pi_{\eta(j)\eta(i)} = 0$, value equation for a player $k \in \{j, \eta(j)\}$ would simplify to

$$V_k = (1-2p)\delta V_k + 2p\delta V_k(j\eta(j)) = (1-2p)\delta V_k + 2p\delta\sigma_k.$$

Thus, $\delta V_j + \delta V_{\eta(j)} < s_{i\eta(j)}$ and the equilibrium would be strongly efficient and not sequential. Thus, $\pi_{\eta(i)\eta(j)} + \pi_{\eta(j)\eta(i)} > 0$. Finally, observe that $\pi_{\eta(i)\eta(j)} = 0$. Otherwise, we would again satisfy condition (15) and thereby fail feasibility as

$$s_{i\eta(i)} - \delta V_i = s_{\eta(j)\eta(i)} - \delta V_{\eta(j)} = \delta V_{\eta(i)},$$

where the first equality would hold by player $\eta(i)$'s indifference, while the latter by player $\eta(j)$'s indifference. Thus, we must have that $\pi_{\eta(i)i} = 1$ for all δ close to 1. This completely pins down the acceptance probabilities up to relabelling, and consequently, for $\pi_{\eta(j)\eta(i)} = q$, value equations (1) reduce to

$$\begin{aligned} s_{\eta(i)\eta(j)} &= \delta V_{\eta(i)} + \delta V_{\eta(j)} \\ V_{\eta(i)} &= (1-p)\delta V_{\eta(i)} + p(s_{i\eta(i)} - \delta V_i) \\ V_{\eta(j)} &= (1-2p)\delta V_{\eta(j)} + 2p\delta V_{\eta(j)}(j\eta(j)) \\ V_i &= (1-p-pq)\delta V_i + pq\delta V_i(ij) + p(s_{i\eta(i)} - \delta V_{\eta(i)}) \\ V_j &= (1-2p-pq)\delta V_j + pq\delta V_j(ij) + 2p\delta V_j(j\eta(j)) \end{aligned} \tag{16}$$

where obviously for any $k, l \in N$, we have that

$$V_k(kl) = \frac{p}{1-\delta+2p\delta} s_{kl}.$$

First observe that $\eta(j)$'s value equation trivially implies that $V_{\eta(j)} \leq V_{\eta(j)}(j\eta(j))$ for all $\delta \leq 1$. Furthermore, j and $\eta(j)$ disagree when $A = N$, $s_{j\eta(j)} \leq \delta V_j + \delta V_{\eta(j)}$, but agree when $A = \{j, \eta(j)\}$, $s_{j\eta(j)} > \delta V_j(j\eta(j)) + \delta V_{\eta(j)}(j\eta(j))$. These observations and the value equation of player j imply that $V_j(ij) \geq V_j(j\eta(j))$ (or else $V_j \leq V_j(j\eta(j))$ which is impossible). Moreover, with equal proposal probabilities, $V_j(ij) \geq V_j(j\eta(j))$ is equivalent to $s_{ij} \geq s_{j\eta(j)}$. By adding this inequality to the inequality defining the core match, $s_{i\eta(i)} + s_{j\eta(j)} > s_{\eta(i)\eta(j)} + s_{ij}$, we further obtain that $s_{i\eta(i)} > s_{\eta(i)\eta(j)}$.

In any sequential LMPE $\lim_{\delta\to 1} q = 0$. Taking limits of value equations (16) as $\delta \to 1$, immediately delivers that

$$\begin{aligned} &\lim_{\delta\to 1} V_{\eta(i)} = s_{\eta(i)\eta(j)} - \sigma_{\eta(j)} && \lim_{\delta\to 1} V_{\eta(j)} = \sigma_{\eta(j)} \\ &\lim_{\delta\to 1} V_i = s_{i\eta(i)} - s_{\eta(i)\eta(j)} + \sigma_{\eta(j)} && \lim_{\delta\to 1} V_j = \sigma_j \end{aligned}$$

To conclude the argument observe that player $\eta(i)$ always possesses a deviation that sets $q = 0$ (namely rejecting any offer from $\eta(j)$ when $A = N$). If so, i's and $\eta(i)$'s value functions reduce to

$$\begin{aligned} \hat{V}_i &= (1-p)\delta \hat{V}_i + p(s_{i\eta(i)} - \delta \hat{V}_{\eta(i)}) \\ \hat{V}_{\eta(i)} &= (1-p)\delta \hat{V}_{\eta(i)} + p(s_{i\eta(i)} - \delta \hat{V}_i) \end{aligned}$$

and $\eta(i)$ secures a payoff $\hat{V}_{\eta(i)} = \frac{p}{1-\delta+2p\delta} s_{i\eta(i)} \to \sigma_{\eta(i)}$. For $q > 0$ to be an equilibrium for all close to 1 such a deviation cannot be profitable. Thus, $V_{\eta(i)} \geq \hat{V}_{\eta(i)}$ for all δ close to 1 and

$$\lim_{\delta\to 1} V_{\eta(i)} = s_{\eta(i)\eta(j)} - \sigma_{\eta(j)} = s_{\eta(i)\eta(j)} - (s_{j\eta(j)}/2) \geq s_{i\eta(i)}/2 = \sigma_{\eta(i)} = \lim_{\delta\to 1} \hat{V}_{\eta(i)}.$$

This implies that $2s_{\eta(i)\eta(j)} \geq s_{j\eta(j)} + s_{i\eta(i)}$, which by efficiency and uniqueness of the core immediately implies that $s_{\eta(i)\eta(j)} > s_{ij}$. This concludes the proof as

$$s_{i\eta(i)} > s_{\eta(i)\eta(j)} > s_{ij} \geq s_{j\eta(j)},$$

and therefore $\eta(i) = a$, $i = b$, $j = c$, and $\eta(j) = d$ because the invoked labelling conventions.

To establish the final part of the result, we first find necessary conditions for the existence of a sequential LMPE, and then show that these conditions are also sufficient. Recall that the previous part of the proof establishes that a sequential LMPE exists only if

$$s_{ab} > s_{ad} > s_{bc} \geq s_{cd}. \tag{17}$$

For the proposed strategy profile to be an equilibrium c and d weakly prefer to delay instead of offering to each other and so $\delta V_c + \delta V_d \geq s_{cd}$ for all δ sufficiently close to 1. By solving value equations (16) it is possible to show that

$$\lim_{\delta\to 1} \frac{\delta(V_c + V_d) - s_{cd}}{1-\delta} = \frac{s_{cd}(s_{bc} - s_{cd}) + 2s_{ad}(s_{bc} + s_{cd}) - s_{ab}(s_{bc} + 3s_{cd})}{2p[2(s_{ab} - s_{ad}) - (s_{bc} - s_{cd})]} \tag{18}$$

Thus, $\lim_{\delta\to 1} \delta(V_c + V_d) = s_{cd}$. The strategy is consistent with equilibrium behavior only if $\delta(V_c + V_d)$ to converges to s_{cd} from above. If $s_{bc} = s_{cd}$ then the right hand side of equation (18) reduces to $-s_{cd}/p < 0$ which is not consistent with equilibrium behavior. Thus, $s_{bc} > s_{cd}$. Next observe that the denominator in equation (18) must be positive since $s_{ab} - s_{ad} > 0$ by (17) and since $s_{ab} - s_{ad} > s_{bc} - s_{cd}$ by definition of the core. Thus, as the denominator is always positive, equation (18) is satisfied if and only if the numerator is also positive. This requires that

$$\frac{s_{bc} - s_{cd}}{s_{ab} - s_{ad}} \geq 2\frac{s_{bc} + s_{cd}}{s_{ab} + s_{cd}}. \tag{19}$$

The first part of the proof also establishes that a strategy is consistent with weak efficiency only if $2s_{ad} \geq s_{ab} + s_{cd}$. However, if $s_{ad} = (s_{ab} + s_{cd})/2$, by substituting s_{ad} in (19) one obtains

$$\frac{s_{bc} - s_{cd}}{s_{ab} - s_{cd}} \geq 4\frac{s_{bc} + s_{cd}}{s_{ab} + s_{cd}},$$

which with some rearrangement in turn implies that

$$0 \geq 3\left(s_{ab} - s_{cd}\right)\left(s_{bc} + s_{cd}\right) + 2s_{cd}\left(s_{ab} - s_{bc}\right),$$

which cannot be by (17). Thus, $2s_{ad} > s_{ab} + s_{cd}$. Combining the above inequalities establishes that

$$s_{ab} > s_{ad} > (s_{ab} + s_{cd})/2 > s_{bc} > s_{cd}.$$

This establishes why condition (6) is necessary for the existence of a sequential LMPE.

To show that condition (6) is sufficient, verify that no player can have a profitable deviation given the agreement probabilities pinned down in the first part of the proof. First observe that c and b prefer delaying to offering to each other as

$$\lim_{\delta \to 1} \delta(V_b + V_c) = s_{ab} - s_{ad} + \sigma_d + \sigma_c = s_{ab} + s_{cd} - s_{ad} > s_{bc},$$

where the last inequality holds by the uniqueness of the efficient match. By construction, d is indifferent about offering to a or delaying. Given (6), players c and d weakly prefer delaying to offering to each other as argued in the earlier part of the proof. Given (6), players a and b weakly prefer offering to each other than delaying as

$$\lim_{\delta \to 1} \frac{\delta(V_a + V_b) - s_{ab}}{1 - \delta} = \frac{s_{cd} - 2s_{ad}}{2p} < 0,$$

which implies that $\delta V_a + \delta V_b \leq s_{ab}$ for all δ close to 1. Thus, for sufficiently high δ, a and b prefer offering to each other than delaying. As we have already established that b prefers delaying to offering to c, b's optimal offer strategy is to offer to a with probability 1 for all δ close to 1. Given (6), player a prefers offering to b than offering to d as

$$\lim_{\delta \to 1} \frac{s_{ab} - \delta V_b - s_{ad} - \delta V_d}{1 - \delta} = \frac{2s_{ad} - s_{cd}}{2p} > 0.$$

Thus, it is optimal for a to offer to d with probability 1. Finally, mixing probabilities are consistent with a weakly efficient LMPE as the probability that d and a agree converges to zero from above by

$$\lim_{\delta \to 1} \frac{q}{1 - \delta} = \frac{2(2s_{ad} - s_{ab} - s_{cd})}{p(2s_{ab} - 2s_{ad} - s_{bc} + s_{cd})} > 0,$$

where the inequality holds as the numerator is positive by $2s_{ad} > s_{ab} + s_{cd}$, while the denominator is positive by $s_{ab} - s_{ad} > 0$ and $s_{ab} - s_{ad} > s_{bc} - s_{cd}$. All players thus best respond for δ close to 1, and condition (6) is indeed sufficient for the existence of a sequential LMPE. ■

Proof of Proposition 11. By Theorem 4, a sufficient condition for the existence of an efficient and thus assortative MPE is that there exist no worker i and firm j having a weakly profitable pairwise deviation when receiving their Rubinstein payoffs. As the efficient match is assortative, the core match of worker i is firm i. Thus, there is an assortative MPE if for all $i \neq j$

$$\frac{q_i}{p_i + q_i} S(i,i) + \frac{p_j}{p_j + q_j} S(j,j) > S(i,j).$$

If $w = f$, no agent is unmatched in the efficient match. Along with the condition that $p_i = q_i = p$, the above expression simplifies to

$$S(i,i) + S(j,j) > 2S(i,j) = S(i,j) + S(j,i), \tag{20}$$

where the equality follows from the condition that $S(i,j) = S(j,i)$. The existence of an assortative MPE then follows, as condition (20) is necessarily satisfied by the increasing differences assumption C3.

By Theorem 4, a necessary condition for the existence of an assortative MPE is that there exists no $i \neq j$ such that

$$\frac{q_i}{p_i + q_i} S(i,i) + \frac{p_j}{p_j + q_j} S(j,j) < S(i,j). \tag{21}$$

To establish necessity, we will a continuous map S for which this condition holds for some $i \neq j$, thereby proving that there is no assortative MPE. First, suppose without loss of generality that $w < f$. Setting $i = w$ and $j = w + 1$, and using the fact that $S(w+1, w+1) = 0$, condition (21) simplifies to

$$\frac{q_w}{p_w + q_w} S(w,w) < S(w, w+1).$$

However, this condition fails whenever S is such that $S(w,w) - S(w,w+1) < \frac{p_w}{p_w + q_w} S(w,w)$, which is possible while satisfying C1, C2, and C3.

Next, suppose that $w = f$. Without loss of generality, let $i < j$ and

$$\alpha = \frac{q_i}{p_i + q_i} < \frac{q_j}{p_j + q_j} = 1 - \beta.$$

By condition (21) there is then no assortative MPE if

$$\alpha S(i,i) + \beta S(j,j) < S(i,j), \text{ which implies}$$
$$\alpha[S(i,i) - S(i,j)] + \beta[S(j,j) - S(i,j)] < [1 - \alpha - \beta] S(i,j).$$

This condition is met whenever S is such that $S(i,i) = S(i,j) + 2\varepsilon = S(j,j) + 3\varepsilon$, since

$$\alpha 2\varepsilon + \beta\varepsilon < [1 - \alpha - \beta] S(i,j)$$

for all ε close to 0, as $1 - \alpha - \beta > 0$.

To conclude, suppose that $w = f$, and that $p_k = q_k = p$ for all $k \leq \max\{w, f\}$. Without loss of generality, assume also that that $S(j, i) < S(i, j)$ and that $i < j$. Equation (21) then simplifies to

$$S(i,i) + S(j,j) < 2S(i,j), \text{ which implies}$$
$$[S(i,i) - S(i,j)] - [S(j,i) - S(j,j)] < S(i,j) - S(j,i).$$

This condition is met whenever $S(k,i) - S(k,j) \leq \varepsilon$, as the LHS is necessarily bounded by ε by assumption, while the RHS can be bounded away from zero. ■