4. Competition and Hold-ups

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1. Introduction

A central concern for economists is the extent to which market systems are efficient. In the idealized Arrow-Debreu model of general competitive equilibrium, efficiency follows under mild conditions, notably the absence of externalities. But in recent years, economists have become interested in studying market situations less idealized than in the Arrow-Debreu set-up and in examining the pervasive inefficiencies that may exist. The subject of the present paper, the "hold-up problem", is one example of a situation that is thought to give rise to significant inefficiencies.

The hold-up problem applies when an agent making an investment is unable to receive all the benefits that accrue from the investment. The existence of the problem is generally traced to incomplete contracts: with complete contracts, the inefficiency induced by the failure to capture benefits will not be permitted to persist. In the standard set-up of the problem, investments are chosen before agents interact and contracts can be determined only when agents meet. Prior investments will be a sunk cost and negotiation over the division of surplus resulting from an agreement is likely to lead to a sharing of the surplus enhancement made possible by one agent's investment (Williamson 1985; Grout 1984; Grossman and Hart 1986; Hart and Moore 1988).

What happens if agent interaction is through the marketplace? In an Arrow-Debreu competitive model, complete markets, with price-taking in each market,

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are assumed; if an agent chooses investment *ex-ante*, every different level of investment may be thought of as providing the agent with a different good to bring to the market. If the agent wishes to choose a particular level of investment over some other, and the "buyer" he trades with also prefers to trade with the agent in question, rather than with an "identical" agent with another investment level, then total surplus to be divided must be maximized by the investment level chosen: investment will be efficiently chosen and there is no hold-up problem. In this situation, the existence of complete markets implies that agents know the price that they will receive or pay whatever the investment level chosen: complete markets imply complete contracts.

An unrealistic failure of the Arrow-Debreu set-up is that markets are assumed to exist for every conceivable level of investment, irrespective of whether or not trade occurs in such a market. But without trade, it is far-fetched to assume that agents will believe that they can trade in inactive markets and that a competitive price will be posted in such market.

The purpose of this paper is to investigate the efficiency of investments when the trading pattern and terms of trade are determined explicitly by the interaction of buyers and sellers. To ensure that there are no inefficiencies resulting from market power, a model of Bertrand competition is analyzed where some agents invest prior to trade; however, this does not rule out the dependence of the pattern of outcomes on the initial investment of any agent and the analysis concentrates on the case of a finite number of traders to ensure this possibility. Contracts are the result of competition in the marketplace and we are interested in the degree to which the hold-up problem is mitigated by contracts that result from Bertrand competition. In this regard, it should be said that we shall not permit Bertrand competition in contingent contracts; in our analysis, contracts take the form of an agreement to trade at a particular price. We are thus investigating the efficiency of contracts implied by a simple trading structure rather than attempting explicitly to devise contracts that help address the hold-up problem (Aghion, Dewatripont, and Rey 1994; Nöldeke and Schmidt 1995; Maskin and Tirole 1999; Segal and Whinston 1998, e.g.).

To further tie our hands, we will restrict attention to markets where the

Bertrand competitive outcome is robust to the way that markets are made to clear. Specifically, we assume that buyers and sellers can be ordered by their ability to generate surplus with a complementarity between buyers and sellers. Without investment choices, this set-up gives rise to assortative matching in any reasonable specification of equilibrium. With investment choices, the notion of an ordering of agents may break down though we demonstrate that such an ordering is an equilibrium phenomenon.

With Bertrand competition, there is an asymmetry between buyers and sellers in a market. As a convention, we assume that buyers bid for the right to trade with sellers by naming a price that they wish to receive. There are two asymmetries: one side of the market (here the buyers) bids, and one side of the market (here, again, the buyers) will obtain a contract with a specified return. With this protocol, it is shown that the *ex ante* investments of buyers will be efficient. In essence, a buyer will bid just enough to win the right to trade with a seller and, if he were to have previously enhanced the value of a trade by extra investment, he would have been able to win the right with the same bid, as viewed by the seller, and so receive all the benefits of the extra investment. It turns out that this argument can be applied both when sellers make *ex-ante* investments and when they make *ex-post* investments. With *ex-post* investments, the terms of trade, defined by what buyers receive, are fixed and sellers receive any enhancement to surplus resulting from their investments: sellers' investments will also be efficient. Thus in a world of sequential investments, separated by competitive bidding, the residual rights to the surplus of a trade can switch from the buyer to the seller and both sets of agents make efficient investments.

The classic case of hold-up is thought to arise when both buyers and sellers make *ex-ante* specific investments with any particular investment being of particular value only in a trade involving a particular buyer and seller. In this case we indeed show that sellers' investments are inefficient. However, we show that the extent of the inefficiency is strictly limited. In particular, we show that the overall inefficiency in a market is less than that which could result from an under-investment by one seller in the market with all other sellers making efficient investments. This result holds irrespective of the number of buyers or sellers in the market.

The structure of the paper is as follows. After a discussion of related literature in the next section, Section 3 lays down the basic model and Bertrand competitive equilibria are characterized in Section 4. Section 5 investigates the case of sequential investment and efficiency is demonstrated. Simultaneous *ex-ante* investment is examined in Section 6 and it is demonstrated that the inefficiency of equilibrium is small and can be bounded by an amount independent of the size of the market. Section 7 provides concluding remarks.

2. Related Literature

The literature on the hold-up problem has mainly analyzed the bilateral relationship of two parties that may undertake match specific investments in isolation (Williamson 1985; Grout 1984; Grossman and Hart 1986; Hart and Moore 1988). In other words, these papers identify the inefficiencies that the absence of complete contingent contracts may induce in the absence of any competition for the parties to the match. This literature identifies the institutional (Grossman and Hart 1986; Hart and Moore 1990; Aghion and Tirole 1997) or contractual (Aghion, Dewatripont, and Rey 1994; Nöldeke and Schmidt 1995; Maskin and Tirole 1999; Segal and Whinston 1998) devices that might reduce and possibly eliminate these inefficiencies. We differ from this literature in that we do not alter either the institutional or contractual setting in which the hold-problem arises but rather analyze how competition among different sides of the market may eliminate the inefficiencies associated with such a problem.

The literature on bilateral matching, on the other hand, concentrates on the inefficiencies that arise because of frictions present in the matching process. These inefficiencies may lead to market power (Diamond 1971; Diamond 1982), unemployment (Mortensen and Pissarides 1994) and a class structure (Burdett and Coles 1997; Eeckhout 1999). A recent development of this literature shows how efficiency can be restored in a matching environment thanks to free entry into the market (Roberts 1996; Moen 1997) or Bertrand competition (Felli and Harris 1996). We differ from this literature in that we abstract from any friction in the matching process and focus on the presence of match specific investments before or after the matching process.

A small recent literature considers investments in a matching environment. Some of the papers focus on general investment that may be transferred across matches and identify the structure of contracts that may lead to efficiency (MacLeod and Malcomson 1993; Holmström 1999) or the inefficiencies due to the presence of an exogenous probability that the match will dissolve (Acemoglu 1997). A number of papers consider, instead, specific investments in a matching environment as we do (Acemoglu and Shimer 1998; Cole, Mailath, and Postlewaite 1998; De Meza and Lockwood 1998).

In particular, Acemoglu and Shimer (1998) consider a matching model with frictions. Firms post wage offers before choosing their investments. They obtain efficiency out of the free entry of firms and the fact that wages are announced before investment occurs. We differ in that we do not allow free entry of firms in the economy. As a matter of fact, the finite and discrete number of firms and workers in the market is critical in identifying the specific nature of the investment undertaken by both sides of the market. The mechanism leading to efficiency is therefore quite different in nature: we focus on the ability of Bertrand competition mechanism to achieve efficiency or near-inefficiency.

Cole, Mailath, and Postlewaite (1998) is the paper closest to ours. As in our setting they focus on *ex-ante* match specific investment and analyze efficiency when matches and the allocation of the shares of surplus are in the core of the assignment game. They demonstrate the existence of an equilibrium allocation that induces efficient investments as well as allocations that yield inefficiencies. This is done under a critical assumption. When the numbers of workers (sellers) and buyers (firms) are discrete they are able to pin down an allocation of the matches' surplus yielding efficient investments via a condition defined as 'double-overlapping'. This condition requires the presence of at least two workers (or two firms) with identical innate characteristics; it implies the existence of an immediate competitor for the worker or the firm in each match. In this case, the share of surplus a worker gets is exactly the worker's outside option and efficiency is promoted. In the absence of double-overlapping, investments may not be efficient because indeterminacy arises creating room for under-investment. Such a condition is not needed in our environment since, by specifying the extensive form of market competition as Bertrand competition, we obtain a binding outside option for any value of the workers' and firms' innate characteristics. Notice that double-overlapping is essentially an assumption on the specificity of the investments that both workers and firms choose. If double overlapping holds it means that investment is specific to a small group of workers or firms but among these workers and firms it is general. We do not need this assumption for efficiency.

Finally De Meza and Lockwood (1998) analyze a matching environment in which both sides of the market can undertake match specific investments but focus on a setup that delivers inefficient investments. As a result the presence of asset ownership and asset trading may enhance welfare as in Grossman and Hart (1986). They focus on whether one would observe asset trading before or after investment and match formation. In our setting, given that we obtain efficiency or near-efficiency, we do not need to explore the efficiency enhancing role of asset ownership. However, we do explore the different efficiency properties of an environment in which firms undertake investments both before and/or after matches are formed. The difference is that we take this timing to be exogenously rather than endogenously determined. As we argue in the Conclusions, Section 7 below, the only large inefficiency that could arise in a general model in which both workers and firms can undertake *ex-ante* and *ex-post* investments is generated by the *ex-post* investments of the agents on the side of the market whose remunerations are established in the employment contract.

3. The Framework

We consider a simple matching model: S workers match with T firms, we assume that the number of workers is higher than the number of firms $S > T$.[2] Each firm is assumed to match only with one worker. Workers and firms are labelled, respectively, $s = 1, \dots, S$ and $t = 1, \dots, T$. Both workers and firms can make match specific investments, denoted respectively x_s and y_t, incurring costs $C(x_s)$ respectively $C(y_t)$.[3] The cost function $C(\cdot)$ is strictly convex and $C(0) = 0$. The

[2] In a companion paper we also analyze the case in which $S \leq T$ (Felli and Roberts 1999).

[3] For simplicity we take both cost functions to be identical, none of our results depending on this assumption. If the cost functions were type specific we would require the marginal costs

surplus of each match is then a function of the identities of the worker and the firm involved and of both specific investments $v(t, s, y_t, x_s)$.

We assume *positive assortative matching*; in other words, the lower is the label of both the worker and the firm in the match the higher is the surplus generated by the match:[4] $v_1(t, s, y_t, x_s) < 0$, $v_2(t, s, y_t, x_s) < 0$ and the increase in surplus generated by a lower identity of the worker or the firm in the match increases if the identity of the partner decreases as well: $v_{12}(t, s, y_t, x_s) > 0$.[5] Notice that this assumption gives a particular meaning to the term specific investments we used for x_s and y_t. Indeed, in our setting the investments x_s and y_t have a use and value in matches other than (s, t); however, these values decrease with the identity of the partner implying that at least one component of this value is specific to the match in question, since we consider a discrete number of firms and workers.

We also assume that the surplus of each match is increasing and concave in the match specific investments — $v_3 > 0$, $v_4 > 0$, $v_{33} \leq 0$, $v_{44} \leq 0$ and $(v_{33}v_{44} - v_{34}^2) \geq 0$ — and the two specific investments x_s and y_t are complementary: $v_{34}(t, s, y_t, x_s) > 0$.[6] Finally, we assume complementarity between the quality of the match — whether it improves in the worker's or the firm's type — and both specific investments x_s and y_t. This assumption takes the form of a negative cross-partial derivative of the match surplus function with respect to the identity of the party to the match and either partner's specific investment: $v_{l,k}(t, s, y_t, x_s) < 0$ for every $l \in \{1, 2\}$ and $k \in \{3, 4\}$.[7]

to increase with the identity of the worker or the firm.

[4] For convenience we denote with $v_l(\cdot,\cdot,\cdot,\cdot)$ the partial derivative of the surplus function $v(\cdot,\cdot,\cdot,\cdot)$ with respect to the l-th argument and with $v_{lk}(\cdot,\cdot,\cdot,\cdot)$ the cross-partial derivative with respect to the l-th and k-th argument or the second-partial derivatives if $l = k$.

[5] The surplus function is assumed to be defined for all positive real numbers (s, t) even though they take on integer values in the economy under consideration. This device allows simplicity in defining terms and permits us to compare economies where agents are more or less substitutable with each other.

[6] Notice that our assumption of concavity of the surplus function $v(\cdot,\cdot,\cdot,\cdot)$ with respect to the two investments y_t and x_s implies a restriction on the degree of complementarity of these two investments.

[7] As established in Milgrom and Roberts (1990), Milgrom and Roberts (1994) and Edlin and Shannon (1998) our results can be derived with much weaker assumptions on the smoothness and concavity of the surplus function $v(\cdot,\cdot,\cdot,\cdot)$ in the two investments x_s and y_t.

Denote, for convenience, each match surplus, net of the firm's investment cost as

$$w(t, s, y_t, x_s) = v(t, s, y_t, x_s) - C(y_t). \tag{4.1}$$

Our assumption of concavity of the surplus function $v(\cdot, \cdot, \cdot, \cdot)$ with respect to firm's and worker's investments imply the appropriate concavity properties for the net surplus function $w(\cdot, \cdot, \cdot, \cdot)$ as defined in (4.1).

In Section 6 below we need stronger assumptions on the responsiveness of firms' investments to both the workers and firms identity and investment. These assumptions, labelled "responsive complementarity", can be described as follows:

$$\frac{\partial}{\partial s}\left(-\frac{w_{13}}{w_{33}}\right) > 0, \qquad \frac{\partial}{\partial y}\left(-\frac{w_{13}}{w_{33}}\right) < 0 \tag{4.2}$$

To be able to interpret these conditions, we first need to define the socially optimal investment choice when firm t matches with worker s. This investment level, denoted $y(t, s, x)$, is the solution to the following problem:

$$y(t, s, x) = \underset{y}{\text{argmax}}\, w(t, s, y, x) \tag{4.3}$$

and is implicitly defined by the following first order condition:

$$w_3(t, s, y(t, s, x), x) = 0 \tag{4.4}$$

Differentiating $y(t, s, x)$ with respect to t gives

$$\frac{\partial y}{\partial t} = -\frac{w_{13}}{w_{33}} \tag{4.5}$$

so that (4.2) says that increases in s and decreases in y make investment more responsive to the type of the investor.[8] Responsiveness complementarity, and

[8] As y is a function of s and t, it is not freely variable. However, if investment is subject to a supplementary cost of p per unit then first-order conditions give $w_3 = p$ and an interpretation of (4.2) is that it is a responsiveness condition in compensated terms where p changes to induces the appropriate change in y.

the other conditions that we have imposed, is satisfied by a standard iso-elastic specification of the model.

We analyze two different timings of our model.

We start with a situation in which, first, each worker chooses his match-specific investment, then workers Bertrand compete for the firms so as to determine the equilibrium matches and, at the same, time the share of the match surplus accruing to each party to the match. Firms then choose their match-specific investment so as to maximize their profits. This timing is analyzed in Section 5 below.

We then proceed to analyze (Section 6 below) the situation in which both workers and firms choose their match-specific investment before competition occurs and equilibrium matches and shares of surplus are determined. Notice that in this case, given the absence of uncertainty, both workers and firms can perfectly foresee the match they will end up with in equilibrium.

We assume the following extensive forms of the Bertrand competition game in which the T firms and the S workers engage. Workers Bertrand compete for firms. All workers simultaneously and independently make wage offers to every one of the T firms. Notice that we allow workers to make offers to more than one, possibly all firms. Each firm observes the offers she receives and decides which offer to accept. For sake of simplicity, we assume that this decision is taken sequentially in order of efficiency.[9] In other words the most efficient firm, labelled 1, decides first which offer to accept. This commits the worker selected to work for firm 1 and automatically withdraws all offers this worker made to other firms. All other firms and workers observe this decision and then firm 2 decides which offer to accept. This process is repeated until firm T decides which offer to accept. Notice that since $S > T$ even firm T, the last firm to decide, can potentially choose among multiple offers.[10]

[9] The equilibrium characterization we derive in the next section (Propositions 1 and 2 below) applies also to the case in which firms decide sequentially which bid to accept following any other order. In particular, the characterization applies also to the case in which the order in which firms decide is randomly determined after the workers have submitted their bids, provided that this randomization is independent of the bids submitted.

[10] An alternative extensive form of the Bertrand competition game that would lead to the same equilibrium characterization can be described as follows. All workers submit simultaneously

We look for the *trembling-hand-perfect* equilibrium of such a game.

4. Competition

4.1. Equilibrium Characterization when Investments are Sequential

In this section we solve the extensive form of the Bertrand competition game, as described in Section 3 above, in which workers and firms engage so as to determine the equilibrium matches and the share of the match surplus accruing to each party to a match.

The solution differs depending on which timing of the model we consider.

Consider first the case in which each firm chooses her match specific investment only after the Bertrand competition game has assigned each worker to a firm and determined the share of surplus each firm and worker receives. In such a case the surplus that is shared in each potential match is described in (4.1) above. Indeed, in this case, since the firm's investment has yet to be chosen, the cost of the firm's investment will be deducted from the match surplus: Bertrand competition among the workers determines the share of the surplus, net of the firm's investment cost, that the worker and firm in the match receive, as defined in (4.1).

Moreover, given the timing, each firm's choice of investment y necessarily depends on the particular match in which the firm in question is involved, say (t, s), and the worker's *ex-ante* investment choice x. If, as we will show in Proposition 3 below, firm t chooses her investment efficiently this level of investment $y(t, s, x)$ is the one that solves (4.3) above and it is implicitly defined in (4.4). We can now define a reduced form of the net surplus function in (4.1) as

$$\omega(t, s, x) = w(t, s, y(t, s, x), x). \tag{4.6}$$

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and independently offers to all firms. Firms simultaneously and independently decide which offer to accept. If a worker's offer is accepted by one firm only the worker is committed to work for that firm. If instead the same worker offer is accepted by more than one firm then the bidding process is repeated among the firms and workers who are not committed to a match yet. This process continues until all firms are matched.

Notice that, using the definition of (4.6), (4.4), and the concavity and complementarity properties of the net surplus function $w(\cdot,\cdot,\cdot,\cdot)$ we can show that: $\omega_1 < 0$, $\omega_2 < 0$, $\omega_{12} > 0$, $\omega_{13} < 0$, $\omega_{23} < 0$, $\omega_{33} \leq 0$.

To be able to provide a characterization of the Bertrand competition subgame played by the workers and firms we need to assess the relative size of the workers' investment choices x_s, $s = 1, \ldots, S$. For different values of these investment choices we expect to see different equilibrium matches and different equilibrium shares of the surplus accruing to firms and workers. In particular it is possible that, for example, a worker with a high identity k chooses a high investment level in order to be able to match with a firm with a low identity h and at the same time the worker with the low identity h chooses a low investment since he foresees he will be matched with the firm with low identity k. Notice that if this situation occurs we observe a pair of equilibrium matches that do not exhibit positive assortative matching. One way to proceed is to solve for the Bertrand competition subgame for every possible configuration of the relative size of the workers' investment choices.

We proceed, instead, by making an excursion into a worker's optimal investment choice so as to solve the Bertrand competition game only for the optimal levels of the workers' investment choices. We then come back to the workers' investment choices in the following section to characterize the efficiency properties of these investments.

Consider the match between worker s and firm t. If, as we show in Proposition 3 below, worker s's investment choice is efficient then this level of investment $x(t, s)$ solves the following problem:

$$\tilde{x}(t, s) = \underset{x}{\text{argmax}}\, \omega(t, s, x) - C(x), \tag{4.7}$$

and it is implicitly defined by the following first order condition:

$$\omega_3(t, s, \tilde{x}(t, s)) = \frac{dC(\tilde{x}(t, s))}{dx} \tag{4.8}$$

Denote $\nu(t, s)$ to be the net surplus function computed for any worker's optimal investment level $\tilde{x}(t,s)$:

$$\nu(t,s) = \omega(t,s,\tilde{x}(t,s,)) - C(\tilde{x}(t,s)) \tag{4.9}$$

The characterization of the equilibrium of the Bertrand competition subgame as described above can then be summarized in the next proposition.

Proposition 1. *The unique equilibrium of the Bertrand competition subgame in the case of sequential investments is such that the firm labelled t matches with the worker labelled t.*

The share of the surplus that each worker and each firm receive are:

$$\pi^W(t,t,x_t) = \sum_{h=t}^{T} [\omega(h,h,x_h) - \omega(h,h+1,x_{h+1})] \tag{4.10}$$

$$\begin{aligned} \pi^F(t,t,x_t) &= \omega(t,t+1,x_{t+1}) - \\ &- \sum_{h=t+1}^{T} [\omega(h,h,x_h) - \omega(h,h+1,x_{h+1})] \end{aligned} \tag{4.11}$$

Proof: We first assume that the workers' optimal investment choices are such that $x_s > x_{s+1}$, for every $s = 1, \ldots, S-1$, and we construct the equilibrium matches and shares of surplus. We then show that there does not exist any other equilibrium that does not exhibit positive assortative matching. We finally show that positive assortative matching implies $x_s > x_{s+1}$.

We characterize the equilibrium proceeding by induction. Denote by t the class of subgames that starts with firm t having to choose among the submitted bids. These subgames differ depending on the bids previously accepted by firms $1, \ldots, t-1$. We first solve for the equilibrium of the T-th (the last) subgame in which all firms but firm T have selected a worker's bid.

Without loss in generality, we take $S = T+1$. This subgame is then a simple decision problem for firm T that has to choose between the bids submitted by the two remaining workers. Denote $\alpha(T)$ and $\alpha(T+1)$ the identities of these two workers such that $\alpha(T) < \alpha(T+1)$ and $B_{\alpha(T)}$, respectively $B_{\alpha(T+1)}$, their bids. Firm T clearly chooses the highest of these two bids.

Worker $\alpha(T+1)$ generates net surplus $\omega(T, \alpha(T+1), x_{\alpha(T+1)})$ if selected by firm T while worker $\alpha(T)$ generates net surplus $\omega(T, \alpha(T), x_{\alpha(T)})$ if selected. This implies that $\omega(T, \alpha(T+1), x_{\alpha(T+1)})$ is worker $\alpha(T+1)$'s maximum willingness to bid while $\omega(T, \alpha(T), x_{\alpha(T)})$ is worker $\alpha(T)$'s maximum willingness to bid. Notice that from $x_s > x_{s+1}$, $\omega_3 > 0$ and $\omega_2 < 0$ we have:

$$\omega(T, \alpha(T), x_{\alpha(T)}) > \omega(T, \alpha(T+1), x_{\alpha(T+1)}).$$

Worker $\alpha(T)$ therefore submits a bid equal to the minimum necessary to outbid worker $\alpha(T+1)$. In other words the equilibrium bid of worker $\alpha(T)$ coincides with the equilibrium bid of worker $\alpha(T+1)$: $B_{\alpha(T)} = B_{\alpha(T+1)}$. Worker $\alpha(T+1)$, on his part, has an incentive to deviate and outbid worker $\alpha(T)$ for any bid $B_{\alpha(T)} < \omega(T, \alpha(T+1), x_{\alpha(T+1)})$. Therefore the unique equilibrium is such that both workers' equilibrium bids are:

$$B_{\alpha(T)} = B_{\alpha(T+1)} = \omega(T, \alpha(T+1), x_{\alpha(T+1)})$$

while the equilibrium match is the one between firm T and worker $\alpha(T)$.[11] Notice that on the equilibrium path $\alpha(T) = T$ and $\alpha(T+1) = T+1$.

We now move to the t-th subgame, $(t < T)$. In this case firm t has to choose among the potential bids of the remaining $(T - t + 2)$ workers labelled $\alpha(t), \ldots, \alpha(T+1)$, where $\alpha(t) < \ldots < \alpha(T+1)$. Our induction hypothesis is that the continuation equilibria of the following $j = t+1, \ldots, T$ subgames are

[11] This is just one of a whole continuum of subgame perfect equilibria of this simple Bertrand game *but* the unique trembling-hand-perfect equilibrium. Trembling-hand-perfection is here used in a completely standard way to insure that worker $\alpha(T+1)$ does not choose an equilibrium bid (not selected by firm T) in excess of his maximum willingness to pay.

such that worker $\alpha(j)$ matches with firm j and the equilibrium payoffs are:

$$\hat{\pi}_j^W = \sum_{h=j}^{T} \left[\omega(h, \alpha(h), x_{\alpha(h)}) - \omega(h, \alpha(h+1), x_{\alpha(h+1)})\right] \tag{4.12}$$

$$\begin{aligned} \hat{\pi}_j^F = {} & \omega(j, \alpha(j+1), x_{\alpha(j+1)}) - \\ & \sum_{h=j+1}^{T} \left[\omega(h, \alpha(h), x_{\alpha(h)}) - \omega(h, \alpha(h+1), x_{\alpha(h+1)})\right]. \end{aligned} \tag{4.13}$$

Firm t clearly chooses the highest bid she receives.

Worker $\alpha(t)$'s maximum willingness to bid is then exactly the net surplus generated by the match of $\alpha(t)$ and firm t: $\omega(t, \alpha(t), x_{\alpha(t)})$. Worker $\alpha(j)$'s maximum willingness to bid for firm t is instead

$$\omega(t, \alpha(j), x_{\alpha(j)}) - \hat{\pi}_j^W.$$

Indeed, worker $\alpha(j)$ is willing to pay the surplus he will be able to generate if matched with firm t in excess of the payoff $\hat{\pi}_j^W$ he can guarantee himself, from our induction hypothesis, by not competing for firm t and moving to subgame j the only one in which his bid will be selected. Further, from $x_s > x_{s+1}$, $\omega_1 < 0$, $\omega_2 < 0$, $\omega_3 > 0$ and the positive assortative matching assumption $\omega_{12} > 0$ we obtain:

$$\omega(t, \alpha(t), x_{\alpha(t)}) > \omega(t, \alpha(j), x_{\alpha(j)})$$

and

$$\omega(t, \alpha(j), x_{\alpha(j)}) - \hat{\pi}_j^W > \omega(t, \alpha(j+1), x_{\alpha(j+1)}) - \hat{\pi}_{j+1}^W$$

These inequalities allow us to conclude that worker $\alpha(t)$ is the one with the highest willingness to bid followed by worker $\alpha(t+1)$ and so on in increasing order of worker's identity. Therefore, using an argument symmetric to the one presented in the analysis of the T-th subgame, we conclude that the equilibrium match is the one between firm t and worker $\alpha(t)$ while the equilibrium bids of worker $\alpha(t)$ and $\alpha(t+1)$ are

$$B_{\alpha(t)} = B_{\alpha(t+1)} = \omega(t, \alpha(t+1), x_{\alpha(t+1)}) - \hat{\pi}^W_{t+1}.$$

Notice that on the equilibrium path $\alpha(t) = t$ and $\alpha(j) = j$. Therefore (4.12) and (4.13) coincide with (4.10) and (4.11).

We still need to prove that there does not exist any other equilibrium such that $x_h < x_k$ where $k > h$. As discussed above, the only situation in which this investment levels may be observed in equilibrium is such that a worker with high identity k makes a high investment with the purpose of being matched with a firm with a low identity h while the worker with the low identity h makes a low investment since he foresees that he will be matched with a firm with a high identity k. The equilibrium matches are then (k, h) and (h, k), which clearly do not exhibit positive assortative matching. We therefore show first that given our assumptions on the surplus function $v(\cdot, \cdot, \cdot, \cdot)$ and the structure of the Bertrand competition game there does not exist an equilibrium that does not exhibit positive assortative matching. We then proceed to show that when positive assortative matching is observed in equilibrium the investment levels are such that $x_s > x_{s+1}$, for every $s = 1, \ldots, S-1$.

We proceed by contradiction and assume that there exists an equilibrium with a pair of firms t and $t+1$ and workers $\alpha(t)$ and $\alpha(t+1)$ such that the equilibrium matches are $(t, \alpha(t+1))$ and $(t+1, \alpha(t))$. In any equilibrium, firms choose the highest bid and at least two workers must make such a bid. Otherwise, the highest bid worker would deviate and lower his bid. Assume that the equilibrium remuneration paid by firm t is $R(t)$. In equilibrium, worker $\alpha(t)$ must have no incentive to change his investment, outbid worker $\alpha(t+1)$ and match with firm t. This implies that:

$$\nu(t, \alpha(t)) - [\nu(t, \alpha(t+1)) - R(t)] < R(t+1). \tag{4.14}$$

Recall that $R(t+1)$ is $\alpha(t)$'s equilibrium payoff and $[\nu(t, \alpha(t+1)) - R(t)]$ is worker $\alpha(t+1)$'s equilibrium bid.

Similarly, worker $\alpha(t+1)$ has no incentive to change his investment and outbid worker $\alpha(t)$ so as to match with firm $t+1$. This implies

$$\nu(t+1, \alpha(t+1)) - [\nu(t+1, \alpha(t)) - R(t+1)] < R(t). \tag{4.15}$$

From (4.14) and (4.15) we obtain:

$$\nu(t, \alpha(t+1)) - \nu(t, \alpha(t)) \geq \nu(t+1, \alpha(t+1)) - \nu(t+1, \alpha(t))$$

This contradicts $\nu_{12} > 0$, $\nu_1 < 0$, $\nu_2 < 0$ which is implied by definition (4.9) and our positive assortative matching hypothesis $v_{12} > 0$, $v_1 < 0$, $v_2 < 0$.

Now that we ruled out equilibria that are not consistent with positive assortative matching we can establish the relative size of workers' equilibrium investment choices. From (4.8) we get

$$\frac{\partial x(t,s)}{\partial t} = -\frac{\omega_{31}}{\omega_{33}} < 0, \qquad \frac{\partial x(t,s)}{\partial s} = -\frac{\omega_{32}}{\omega_{33}} < 0$$

These inequalities and the fact that in equilibrium only positive assortative matches occur conclude the proof. ■

Notice that Proposition 1 provides us with a number of properties of the competition among workers for the matches.

In particular, it is important to notice that the equilibrium allocation of the Bertrand game is *efficient*. In other words a central planner would choose exactly the same matches as the ones observed in equilibrium. Notice that the efficiency of the allocation that matches a t worker with a t firm follows from our assumption of positive assortative matching, in particular from $v_{12}(s, t, x_s, y_t) > 0$ and the complementarity and concavity properties of the surplus function $v(\cdot, \cdot, \cdot, \cdot)$.

Further notice that the worker's equilibrium payoff $\pi^W(t, t, x_t)$ is the sum of the net social surplus, as in (4.6), and an expression $\mathcal{W}_t$ that does not depend on worker t's match specific investment x_t:

$$\pi^W(t, t, x_t) = \omega(t, t, x_t) + \mathcal{W}_t. \tag{4.16}$$

Similarly, the firm's equilibrium payoff $\pi^F(t, t, y_t)$ is the sum of the surplus generated by the match of firm t with worker $(t+1)$ — an inefficient match if it occurs — and an expression $\mathcal{P}_t$ that does not depend on firm t's match-specific investment y_t:

$$\pi^F(t,t,x_t) = \omega(t,t+1,x_{t+1}) + \mathcal{P}_t. \tag{4.17}$$

These properties will play a crucial role when we analyze the efficiency of the workers and firms' investments.

4.2. Equilibrium Characterization when Investments are Non-Sequential

Let us consider now the case in which both firms and workers choose a match-specific investment before the Bertrand competition subgame that determines the equilibrium match and the share of surplus accruing to each party to a match. The difference with the analysis presented above is minimal. The only difference is in the fact that investments are sunk once workers and firms get to the Bertrand competition subgame. Therefore the surplus shared in this subgame is the gross surplus $v(t,s,y_t,x_s)$ and at the time of the Bertrand competition game both investments x_s and y_t are given.

To solve for the equilibrium of the Bertrand competition game, we need to establish the relative size of the investments choices of both the workers x_s and the firms y_t. In contrast to what we did in Section 4.1, we cannot here consider only the subgames of the Bertrand competition game in which these investments are optimally chosen. However, in the second step of the proof we do restrict attention to subgames associated with optimally chosen workers' investment choices. Once again, we need to make an excursion into the workers' optimal investment decision.

Consider the match between worker s and firm t. If, as we prove in this case in Lemma 1 below, worker s's investment choice is constrained efficient then the worker's optimal level of investment $\hat{x}(t,s,y)$ solves the following problem:

$$\hat{x}(t,s,y) = \underset{x}{\operatorname{argmax}}\, v(t,s,y,x) - C(x), \tag{4.18}$$

and it is implicitly defined by the following first order condition:

$$v_4(t,s,y,\hat{x}(t,s,y)) = \frac{dC(\hat{x}(t,s,y))}{dx}. \tag{4.19}$$

Denote $\mu(t,s,y)$ the surplus function computed for any worker's optimal investment level $\hat{x}(t,s,y)$:

$$\mu(t,s,y) = v(t,s,y,\hat{x}(t,s,y)) - C(\hat{x}(t,s,y)) \tag{4.20}$$

For the same match (t,s) we can define also the *efficient best reply* $\hat{y}(t,s,x)$ of firm t to every level of the worker's investment choice x if firm t chooses its investment so as to maximize the net surplus $w(t,s,y,x)$:

$$\hat{y}(t,s,x) = \underset{y}{\operatorname{argmax}}\, w(t,s,y,x). \tag{4.21}$$

This best reply is implicitly defined by the following first order condition:

$$w_3(t,s,\hat{y}(t,s,x),x) = 0. \tag{4.22}$$

The characterization of the equilibrium matches and payoffs is then presented in the following proposition.

Proposition 2. *The equilibrium of the Bertrand competition subgame in the case of non-sequential investments is such that the firm labelled t matches with the worker labelled t.*

The share of the surplus that each worker and each firm receive are:

$$\Pi^W(t,t,y_t,x_t) = \sum_{h=t}^{T} \left[v(h,h,y_h,x_h) - v(h,h+1,y_h,x_{h+1}) \right] \tag{4.23}$$

$$\Pi^F(t,t,y_t,x_t) = v(t,t+1,y_t,x_{t+1}) - \sum_{h=t+1}^{T} \left[v(h,h,y_h,x_h) - v(h,h+1,y_h,x_{h+1}) \right]. \tag{4.24}$$

Proof: We proceed by induction. Consider the T-th (the last) subgame in which all firms but firm T have selected a worker's bid.

Let $S = T + 1$. We use the same notation as in the proof of Proposition 1. Firm T clearly chooses the highest between the two remaining bids: $B_{\alpha(T)}$ and $B_{\alpha(T+1)}$.

Worker $\alpha(T+1)$'s maximum willingness to bid is then $v(T, \alpha(T+1), y_T, x_{\alpha(T+1)})$ while worker $\alpha(T)$'s maximum willingness to bid is $v(T, \alpha(T), y_T, x_{\alpha(T)})$. We need to show that for any level of firm T investment y_T there does not exists an equilibrium of this subgame such that worker $\alpha(T + 1)$ outbids worker $\alpha(T)$.

By way of contradiction assume that the equilibrium in which firm T matches with worker $\alpha(T + 1)$ exists. If this is the equilibrium match, worker $\alpha(T)$ will invest accordingly. In other words, from (4.20), worker $\alpha(T)$'s maximum willingness to pay for the match with firm T is $\mu(T, \alpha(T), y_T)$. Worker $\alpha(T + 1)$'s equilibrium payoff is such that:

$$\mu(T, \alpha(T + 1), y_T) - \mu(T, \alpha(T), y_T) \geq 0.$$

Notice that the latter inequalities contradicts $\mu_2 < 0$ that follows from (4.20), (4.19) and:

$$\mu_2(t, s, y) = v_2(t, s, \hat{x}(t, s, y), y) < 0.$$

Worker $\alpha(T)$ therefore matches with firm T and submits a bid equal to the minimum necessary to outbid worker $\alpha(T+1)$: $B_{\alpha(T)} = B_{\alpha(T+1)}$. Worker $\alpha(T+1)$, on his part, has an incentive to deviate and outbid worker $\alpha(T)$ for any bid $B_{\alpha(T)} < v(T, \alpha(T+1), y_T, x_{\alpha(T+1)})$. Therefore the unique equilibrium is such that both workers' equilibrium bids are:

$$B_{\alpha(T)} = B_{\alpha(T+1)} = v(T, \alpha(T + 1), y_T, x_{\alpha(T+1)}).$$

On the equilibrium path $\alpha(T) = T$ and $\alpha(T + 1) = T + 1$. Hence (4.10) and (4.11) hold for firm T and worker T. Furthermore in equilibrium worker $T + 1$ does not match with any firm and receives a zero payoff.

Consider now the t-th subgames, $(t < T)$. Firm t has to choose among the potential bids of the remaining $(T - t + 2)$ workers labelled $\alpha(t), \ldots, \alpha(T+1)$, where $\alpha(t) < \ldots < \alpha(T+1)$. Our induction hypothesis is that the continuation equilibria of the following $j = t+1, \ldots, T$ subgames are such that worker $\alpha(j)$ matches with firm j and the equilibrium payoffs are:

$$\hat{\Pi}_j^W = \sum_{h=j}^{T} \left[v(h, \alpha(h), y_h, x_{\alpha(h)}) - v(h, \alpha(h+1), y_h, x_{\alpha(h+1)}) \right] \tag{4.25}$$

$$\begin{aligned} \hat{\Pi}_j^F &= v(j, \alpha(j+1), y_j, x_{\alpha(j+1)}) - \\ &- \sum_{h=j+1}^{T} \left[v(h, \alpha(h), y_h, x_{\alpha(h)}) - v(h, \alpha(h+1), y_h, x_{\alpha(h+1)}) \right]. \end{aligned} \tag{4.26}$$

Firm t chooses the highest bid she receives.

We therefore need to show that for any set of $(T - t + 2)$ unmatched workers $\{\alpha(t), \ldots, \alpha(T+1)\}$, worker $\alpha(t)$ is the worker with the highest willingness to pay for firm t and worker $\alpha(t+1)$ is the worker with the second highest willingness to pay for firm t.

For this purpose, assume that at stage t worker α^* is matched with firm t. Notice that at stage $t+1$ the induction hypothesis identifies the remaining equilibrium matches. Let $\alpha(j)$ be the worker with the second highest willingness to pay for a match with firm t.

We proceed in two steps. We first show that necessarily $\alpha(j)$ is the worker with the lowest identity (the highest innate ability) among the $\alpha(t+1), \ldots, \alpha(T+1)$, workers. We then show that $\alpha^* = \alpha(t)$ and that $\alpha(t+1)$ is the worker with the second highest willingness to pay for the match with firm t and hence is the 'runner-up' for this match.

Step 1. $\alpha(j) = \alpha(t+1)$.

Assume by way of contradiction that this is not the case. By the induction hypothesis, we know that $\alpha(j)$'s maximum willingness to pay for the match with firm t is:

$$B_{\alpha(j)} = v(t, \alpha(j), y_t, x_{\alpha(j)}) - \hat{\Pi}^W_{\alpha(j)}.$$

Consider now worker $\alpha(j-1) \neq \alpha^*$ we get:

$$\begin{aligned} B_{\alpha(j)} - B_{\alpha(j-1)} &= v(t, \alpha(j), y_t, x_{\alpha(j)}) - \hat{\Pi}^W_{\alpha(j)} \\ &\quad - \left[v(t, \alpha(j-1), y_t, x_{\alpha(j)}) - \hat{\Pi}^W_{\alpha(j-1)} \right] \\ &= v(t, \alpha(j), y_t, x_{\alpha(j)}) - v(t, \alpha(j-1), y_t, x_{\alpha(j-1)}) \quad (4.27) \\ &\quad -v(j-1, \alpha(j), y_{j-1}, x_{\alpha(j)}) \\ &\quad -v(j-1, \alpha(j-1), y_{j-1}, x_{\alpha(j-1)}) \end{aligned}$$

Since, by assumption, worker $\alpha(j)$ has the second highest willingness to pay for the match with firm t it must be the case that

$$\hat{y}(t, \alpha(j), x_{\alpha(j)}) > y(j-1, \alpha(j), x_{\alpha(j)}), \tag{4.28}$$

where $\hat{y}(\cdot, \cdot, \cdot)$ is a firm's efficient best reply as defined in (4.21). From condition (4.27), (4.28) and $x_{\alpha(j)} < x_{\alpha(j-1)}$ — implied by (4.28) — we conclude that

$$B_{\alpha(j-1)} > B_{\alpha(j)}.$$

This is a contradiction to the fact that worker $\alpha(j)$ has the second highest willingness to pay for the match with firm t.

Step 2. Worker $\alpha(t)$ has the highest and worker $\alpha(t+1)$ the second highest willingness to pay for the match with firm t among the $T-t+2$ unmatched workers.

By step 1 we simply need to show that worker $\alpha(t)$ is matched in equilibrium with firm t. By way of contradiction, assume that this is not the case: worker $\alpha(j)$, $j > t$, is matched with firm t. Then, by the induction hypothesis, worker $\alpha(t)$ will be matched in equilibrium with firm $t+1$. Further, by step 1, worker $\alpha(t)$

is the runner-up for the worker $\alpha(j)$ that is matched with firm t. In other words the equilibrium matches are: $(t, \alpha(j))$, and $(t+1, \alpha(t))$. Worker $\alpha(t)$ equilibrium payoff in the $t+1$ subgame is then $\mu(t+1, \alpha(t), y_{t+1}) - B^{t+1}$ where B^{t+1} is the equilibrium bid of the runner-up for the match with firm $t+1$. Therefore worker $\alpha(t)$'s maximum willingness to pay for the match with firm t is

$$\mu(t, \alpha(t), y_t) - \mu(t+1, \alpha(t), y_{t+1}) + B^{t+1}$$

This is the amount worker $\alpha(j)$ needs to bid so as to match in equilibrium with firm t. Worker $\alpha(j)$'s payoff is then:

$$\mu(t, \alpha(j), y_t) - \mu(t, \alpha(t), y_t) + \mu(t+1, \alpha(t), y_{t+1}) - B^{t+1}. \tag{4.29}$$

For $(t, \alpha(j))$ to be an equilibrium match it must not be profitable for worker $\alpha(j)$ to deviate, and match with firm $t+1$. If he deviates then he needs to pay B^{t+1}, since the deviation would not affect the willingness to pay and identity of the runner-up for firm $t+1$, and receive payoff:

$$\mu(t+1, \alpha(j), y_{t+1}) - B^{t+1}. \tag{4.30}$$

The payoff in (4.29) is higher then the payoff in (4.30) if and only if:

$$\mu(t, \alpha(j), y_t) - \mu(t, \alpha(t), y_t) \geq \mu(t+1, \alpha(j), y_{t+1}) - \mu(t+1, \alpha(t), y_{t+1}). \tag{4.31}$$

Condition (4.31) is a contradiction of $\mu_1 < 0$, $\mu_2 < 0$, $\mu_{12} > 0$ and the fact that:

$$y_t = \hat{y}(t, \alpha(t), x_{\alpha(t)}) > y_{t+1} = \hat{y}(t+1, \alpha(t+2), x_{\alpha(t+2)}).$$

The result is then that

$$\hat{\Pi}_t^W = v(t, \alpha(t), y_t, x_{\alpha(t+1)}) - \Pi_{t+1}^W. \tag{4.32}$$

On the equilibrium path $\alpha(t) = t$ and $\alpha(t+1) = t+1$. Using recursively (4.32), we then obtain (4.25) and (4.24). ■

Notice that the same efficiency properties we discuss in relation to Proposition 1 hold in this case as well.

As in the sequential investment case, the worker's equilibrium payoff $\Pi^W(t, t, y_t, x_t)$ is equal to the sum of the social surplus, $v(t, t, y_t, x_t)$ and an expression W_t that does not depend on worker t's match specific investment x_t:

$$\Pi^W(t, t, y_t, x_t) = v(t, t, y_t, x_t) + W_t. \tag{4.33}$$

Similarly, the firm's equilibrium payoff $\Pi^F(t, t, y_t, x_t)$ is the sum of the surplus generated by the (inefficient) match of firm t with worker $(t+1)$ and an expression P_t that does not depend on firm t's match-specific investment y_t:

$$\Pi^F(t, t, y_t, x_t) = v(t, t + 1, y_t, x_{t+1}) + P_t. \tag{4.34}$$

5. Sequential Investment

In this section we analyze the investment decision of firms and workers in the case in which specific investments are chosen sequentially. In particular firms choose their investment only after the Bertrand competition game that determines both the matches and the shares of surplus accruing to every party to a match.

The main result we present is that the investment choice is efficient despite of the fact that these investments are — at least in part — match specific and that the competition game gives rise to outcomes where each worker and each firm does not capture the full return from his/her investment decision. The rationale behind this result can be described as follows. In a dynamic setting, such as the one we consider, it is possible for both parties to a match to be residual claimants of the match surplus at different times. This is what happens in this sequential case for workers and firms. As discussed in the Introduction, we take contracts to be simple agreements that specify the constant (non-contingent) worker's remuneration. Once each worker's compensation, determined by the Bertrand competition game, is specified in the contract, each firm is residual claimant of any return of her investment choice in excess of the remuneration she has promised the worker. At the same time, when matches and remunerations are determined, workers Bertrand compete for firms in an environment in which the match surplus, and hence each worker's willingness to bid, differ across possible matches. The result is then that, in equilibrium, each worker's remuneration is the difference between the surplus generated in the match and the bid that the immediate competitor submits for the firm involved in the match. Therefore when choosing investment, before the Bertrand competition game, each worker is the residual claimant of the returns from his investment in excess of the competitor bid that does not depend on this investment. In this way the marginal incentives of both workers and firms to invest are efficient.

We proceed to solve our extensive form game backward. We start therefore from the firms' investment choices. Consider firm t and let (t, s) be the match in which this firm is involved. Let x be the worker's investment choice, clearly both the match and the worker's investment are given when firm t chooses her investment. Further denote $\tilde{y}$ the level of the firm's investment foreseen by all the workers and firms in the early stages of the game when worker s chooses his investment and when the equilibrium match is determined by the Bertrand competition game. Denote $\pi^W(t, s, \tilde{y}, x)$ the worker's payoff determined by the Bertrand competition game and stated in the contract. Firm t's investment $y(t, s, x)$ is then the solution to the following maximization problem:

$$y(t, s, x) = \underset{y}{\operatorname{argmax}}\, w(t, s, y, x) - \pi^W(t, s, \tilde{y}, x). \tag{4.35}$$

Notice that worker s's payoff $\pi^W(s, t, x_s, \tilde{y})$ is independent of firm t's investment at this stage of the game. This confirms that the firm is the residual claimant of the match surplus in excess of the worker's given payoff. Firm t's investment $y(s, t, x)$ is implicitly defined by the following first order condition:

$$w_3(t, s, y(t, s, x), x) = 0. \tag{4.36}$$

Notice that this first order condition coincides with (4.8) therefore:
$w(t, s, y(t, s, x), x) = \omega(t, s, x)$.

We can now move back to the worker's investment choice that precedes the Bertrand competition game, as analyzed in Section 4 above. Proposition 1 characterizes the equilibrium bids and the payoffs to the worker and the firm for every match. Under this assumption, therefore, the payoff to worker t when choosing his investment x_t is given by (4.10), restated in (4.16) above. In other words worker t's investment choice x_t is the solution to the following problem:

$$x_t = \underset{x}{\operatorname{argmax}}\, \omega(t,t,x) + \mathcal{W}_t - C(x) \tag{4.37}$$

The solution x_t to this problem is implicitly defined by the following first order conditions:

$$\omega_3(t,t,x) = w_4(t,t,y(t,t,x_t),x_t) = \frac{dC(x_t)}{dx_t}. \tag{4.38}$$

We characterize now the efficiency properties of both worker t's and firm t's investments. The efficient levels of investments x_t^* and y_t^* are defined by the solution to the following social planner's problem:

$$\max_{x,y} w(t,t,y,x) - C(x) \tag{4.39}$$

These efficient levels are implicitly and uniquely defined by the following pair of first order conditions:

$$w_4(t,t,y_t^*,x_t^*) = \frac{dC(x_t^*)}{dx_t} \tag{4.40}$$

$$w_3(t,t,y_t^*,x_t^*) = 0 \tag{4.41}$$

We have now all the elements to prove the main result of our analysis of the sequential investments model: namely that each worker's and firm's investment choice is efficient. This result is stated in the following proposition.

Proposition 3. *The equilibrium of the sequential investment model is such that all workers' investment choices x_s, $s = 1,\ldots,S$, and firms' investment choices y_t, $t = 1,\ldots,T$, are efficient.*

Proof: The result is proved by a simple comparison of (4.38) with (4.40) and of (4.36) with (4.41). ■

We conclude this section with two observations. Notice, first, that Proposition 3 holds even in the case in which $T = 1$ and $S = 2$. In other words, we do not need a large competitive market to guarantee that investment choices by the firms and the workers are efficient. Indeed, what guarantees efficiency is not the size of the market but the fact that at different times both the workers and the firms are residual claimants of the match surplus and hence have efficient marginal incentives to invest.

Further, notice that the same efficiency properties do not hold in the presence of only one worker, labelled s, and one firm, labelled t. In this case Bertrand competition does not help in determining how the worker and the firm share the surplus of the match: they are in a situation of bilateral monopoly. Assume therefore that the worker gets a proportion γ of the net surplus of the match, as defined in (4.1), while the firm gets a proportion $(1-\gamma)$. The firm then chooses a constrained efficient level of investment $y(x)$ implicitly defined by

$$w_3(t, s, y(x), x) = 0$$

The worker, on his part, chooses an inefficiently low level of investment $\hat{x}$ defined by the following condition:

$$\gamma w_4(t, s, y(\hat{x}), \hat{x}) = \frac{dC(\hat{x})}{dx}.$$

Notice that the worker's investment $\hat{x}$ is optimal only in the case $\gamma = 1$.

There is a sense, however, in which the case we just described is a very special one. Consider the model in which the number of firms is the same as the number of workers, $S = T$. In this case the inefficiency generated by the under-investment of firm T and the constrained efficient under-investment of worker T are the only inefficiencies present. The rest of the firms and workers will invest efficiently as in (4.36) and (4.38). In other words, our result, and in particular the fact that both sides of the market are residual claimants at different times, still holds for firms $1, \ldots, T-1$ and workers $1, \ldots, T-1$. Notice that this implies that the inefficiency generated by the under-investment of firm T and worker T is associated with the least productive match in the market, and it is therefore a smaller inefficiency than that which results from any other match placed in similar circumstances.

6. The Inefficiencies of Non-Sequential Investment

In Section 5 above we have argued that the agents on the side of the market that is responsible for bidding for matches in the Bertrand competition game make constrained efficient *ex-ante* investments.[12] In our model, these are the workers. This section analyses the potential inefficiencies that arise if firms also make *ex-ante* investments that precede the Bertrand competition game. We show that in this case while the workers' investment choices are constrained efficient the firms' investment choices are inefficient. However, the inefficiency created by the T firms' *ex-ante* under-investment (if $T \geq 2$) is less that the inefficiency created by the under-investment of the single best firm, 1, matching in isolation with the worst worker, T. In other words, the aggregate inefficiency is tempered by market competition.

We start from the analysis of the efficiency properties of the worker's investment choice.

As proved in Proposition 2 above worker t matches with firm t $(t = 1, \ldots, T)$ and worker t's share of the surplus is the sum of the match surplus and an expression that is independent of the worker t's investment x_t, as in (4.33). This implies that for any given value of firm t's investment choice $\bar{y}$ the worker chooses investment so as to solve the following problem:

$$x_t = \underset{x}{\operatorname{argmax}}\, v(t, t, \bar{y}, x) + W_t - C(x). \tag{4.42}$$

The solution is implicitly defined by the following first order condition:

$$w_4(t, t, \bar{y}, x(t, t, \bar{y})) = \frac{dC(x(t, t, \bar{y}))}{dx}; \tag{4.43}$$

[12] The constraint on the workers' investment choices is represented by the firms' investment choices that affect directly the marginal returns of the workers' choices.

where $x(t,t,\bar{y})$ is worker t's reaction function. Notice that assumptions $w_{44} \leq 0$ and $w_{34} > 0$ imply that $x(t,t,\bar{y})$ is strictly monotonic. Condition (4.43) allows us to prove the following lemma.

Lemma 1. *The worker's investment choice is constrained efficient.*

Proof: Let firm t's investment choice be given at the level $\bar{y}$. The worker's efficient level of investment is given by the solution to the central planner's problem (4.39) where we substitute y_t^* with $\bar{y}$. This solution is implicitly defined by (4.40) where we need to substitute, once again, y_t^* with $\bar{y}$. Comparison of (4.40) with (4.43) concludes the proof. ■

As the constrained efficiency of workers' investment decisions is unaffected by the timing of firms' investments, we can suppress the investment decision of the worker in the remainder of this section: the value created by a match between a workers of type s and a firm of type t who chooses an investment y is then given by $v(t,s,y)$. The net surplus of the match is then

$$w(t,s,y) = v(t,s,y) - C(y) \tag{4.44}$$

As above we concentrate on the positive assortative matching case where types and investments are all complementary with each other so ensuring all cross-partial derivatives of $w(\cdot,\cdot,\cdot)$ are positive. For our main result we need the "responsive complementarity" assumptions as stated in (4.2) above.

From Proposition 2 we know that the return to firm t is given, from (4.34), by:

$$\Pi^F(t,t,y_t) = w(t,t+1,y_t) + P_t \tag{4.45}$$

where P_t depends upon investments made by firms of a higher identity than t. If firm t must make an *ex-ante* investment then, recognising that competition will follow leading to the return given by (4.45), y_t will be chosen to maximize (4.45) and we have:[13]

$$y_t = \underset{y}{\text{argmax}}\, w(t, t+1, y) \tag{4.46}$$

On the other hand, efficiency calls for the maximization of total surplus as in (4.39). As the surplus from the match between firm t and worker t is $w(t, t, y_t)$, efficiency requires an investment of y_t^* satisfying

$$y_t^* = \underset{y}{\text{argmax}}\, w(t, t, y). \tag{4.47}$$

The inefficiency of *ex-ante* investment by all firms is therefore given by

$$L = \sum_1^T w(t, t, y_t^*) - \sum_1^T w(t, t, y_t) \tag{4.48}$$

How large is this loss L? First, notice that the difference between y_t^* and y_t is approximately proportional to the difference in characteristics between worker t and $t+1$ (given that y is differentiable in s). On the other hand, as y_t^* solves (4.47) , the difference between $w(t, t, y_t)$ and $w(t, t, y_t^*)$ will be approximately proportional to the *square* of the difference between y_t and y_t^* which will be small if worker t and worker $t+1$ have similar characteristics. To give an example of how this affects L, consider a situation where the characteristics of a worker are captured by a real number c with workers 0 through T having characteristics which are evenly spaced between $\bar{c}$ and $\underline{c}$. How is L affected by the size of the market T? The difference between y_t^* and y_t is approximately proportional to $(\bar{c} - \underline{c})/T$ and the difference between $w(t, t, y_t)$ and $w(t, t, y_t^*)$ will be approximately proportional to $\left[(\bar{c} - \underline{c})/T\right]^2$. Summing over t then gives a total loss L that is proportional to $(\bar{c} - \underline{c})^2/T$: in large markets the aggregate inefficiency created by *ex-ante* investment will be arbitrarily small.[14]

[13] Notice that if the argument x is not suppressed in the function $w(\cdot, \cdot, \cdot, \cdot)$ then (4.46) below defines firm t's reaction function $y(t, t, x)$. Our complementarity and concavity assumption on the surplus function imply that $y(t, t, x)$ is strictly monotonic in x and the Nash equilibrium of this investment game is unique.

[14] See Kaneko (1982).

This is a result that changes the degree of specificity of the workers' investment choices. Increasing the number and hence the density of firms evenly spaced in the interval $[\overline{c}, \underline{c}]$ is equivalent to introducing firms with closer and closer characteristics. This is equivalent to reducing the loss in productivity generated by the match of a worker that made a given investment with the firm that is immediately below in characteristics levels. Hence, there is a sense in which this result is not fully satisfactory since we know that if the worker's investment is general in nature the firms' investment choices are efficient.

Therefore, in the rest of this section, we identify an upper-bound on the aggregate inefficiency present in the economy that is independent of the number of firms and does not alter the specificity of the workers investment choices. Whatever the size of T, it is possible to get a precise upper-bound on the loss L. Indeed, the inefficiency created by the firms' *ex-ante* under-investment is less than that which could be created by the under-investment of only one firm (the best 1) in a match with a worker (the worst T).

Proposition 4. *Assume that there are at least two firms ($T \geq 2$). Let M be the efficiency loss resulting from firm 1 choosing an investment level given by $\tilde{y} = \text{argmax}_y\, w(1, T+1, y)$:*

$$M = w(1, 1, y_1^*) - w(1, 1, \tilde{y}). \tag{4.49}$$

If types and investments are complementary (in a sense that second cross-partial derivatives are positive and (4.2) is satisfied) then

$$L < M. \tag{4.50}$$

Proof: If $y(t, s)$ is the efficient investment level when worker of type s is matched with a firm of type t then L and M can be written as

$$L = \sum_1^T w(t, t, y(t, t)) - \sum_1^T w(t, t, y(t, t+1)) \tag{4.51}$$

$$M = \sum_{1}^{T} w(1,1,y(1,t)) - \sum_{1}^{T} w(1,1,y(1,t+1)) \tag{4.52}$$

so that

$$\begin{aligned} M - L = & \sum_{1}^{T} \Big\{ \Big[w(1,1,y(1,t)) - w(t,t,y(t,t)) \Big] \\ & - \Big[w(1,1,y(1,t+1)) - w(t,t,y(t,t+1)) \Big] \Big\} \end{aligned} \tag{4.53}$$

Define a function f as

$$f_t(\alpha, \beta) = w(t-\beta, t-\beta, y(t-\beta, t+\alpha)) \tag{4.54}$$

so that (4.53) becomes

$$M - L = \sum_{t=1}^{T} \{[f_t(0,t-1) - f_t(0,0)] - [f_t(1,t-1) - f_t(1,0)]\} \tag{4.55}$$

From (4.55), it is clear that, as $T > 1$, each bracketed term in the summation will be positive with some strictly positive if

$$\frac{\partial^2 f_t}{\partial\alpha\partial\beta} < 0 \tag{4.56}$$

which, using (4.54), corresponds to

$$d = -y_2 \ (w_{23} + w_{13} + w_{33} \ y_1) - w_3 \ y_{12} < 0 \tag{4.57}$$

with each derivative on the right-hand-side of (4.57) being evaluated at $(t-\beta, t-\beta, y)$ where y is evaluated at $(t-\beta, t+\alpha)$.

To investigate the actual sign of d, we must investigate the function $y(t,s)$ which is defined by (4.4). Differentiating (4.4) and denoting the evaluation of each derivative w_{ij} at $(t-\beta, t+\alpha, y(t-\beta, t+\alpha))$ with $\hat{w}_{ij}$ gives

$$y_2 = -\left(\frac{\hat{w}_{23}}{\hat{w}_{33}}\right) \tag{4.58}$$

$$y_1 = -\left(\frac{\hat{w}_{13}}{\hat{w}_{33}}\right) \tag{4.59}$$

$$y_{12} = -\frac{1}{\hat{w}_{33}}\left[\hat{w}_{312} - \frac{\hat{w}_{323}\hat{w}_{13}}{\hat{w}_{33}} - \frac{\hat{w}_{133}\hat{w}_{23}}{\hat{w}_{33}} + \frac{\hat{w}_{333}\hat{w}_{23}\hat{w}_{13}}{\hat{w}_{33}^2}\right] \tag{4.60}$$

Using (4.58), (4.59) and (4.60) in (4.57) gives

$$d = \frac{\hat{w}_{23}}{\hat{w}_{33}}\left[w_{23} \quad + \quad w_{13} - \frac{w_{33}\hat{w}_{13}}{\hat{w}_{33}}\right] + \frac{w_3}{\hat{w}_{33}}\left[\hat{w}_{312} - \frac{\hat{w}_{233}\hat{w}_{13}}{\hat{w}_{33}} - \frac{\hat{w}_{133}\hat{w}_{23}}{\hat{w}_{33}} + \frac{\hat{w}_{333}\hat{w}_{23}\hat{w}_{13}}{\hat{w}_{33}^2}\right]$$

Taking the first bracketed term, the responsive complementarity assumption, (4.2) above, gives

$$\frac{w_{13}}{w_{33}} > \frac{\hat{w}_{13}}{\hat{w}_{33}} \tag{4.61}$$

so the first term is negative (recall that $w_{23} < 0$ and $w_{33} < 0$). Taking the second bracketed term, (4.2) again implies that

$$\hat{w}_{312} - \hat{w}_{233}\frac{\hat{w}_{13}}{\hat{w}_{33}} > 0 \tag{4.62}$$

and

$$\hat{w}_{133} - \hat{w}_{333}\frac{\hat{w}_{13}}{\hat{w}_{33}} < 0 \tag{4.63}$$

so that the term in brackets is positive and, as w is evaluated at a point of under-investment, we have $w_3 > 0$ which together with $\hat{w}_{33} < 0$ ensures that the second term in (4.61) is also negative. Thus d is negative: every term in the summation of (4.53) is positive and so $M > L$: the overall efficiency loss in the market is less than that which is possible by the under-investment of a single firm. ■

Notice that from Lemma 1, a similar result would hold when the argument x is not dropped from the function $w(\cdot,\cdot,\cdot,\cdot)$. Of course, in such a case, the inefficiencies will be exacerbated by the worker's inefficient — though constrained efficient — best reply to the firm's inefficient investment choice.

The intuition of Proposition 4 can be described as follows. As a result of the Bertrand competition game firms have incentive to invest in match specific investments with the purpose of improving their outside option: the maximum willingness to pay of the immediate competitor for the worker they match with. This implies that the under-investment of each firm is relatively small. The total inefficiency is then obtained by aggregating these relatively small under-investments. Given the decreasing returns to investment and the assumptions on how optimal firms' investments change across different matches, the sum of the loss in surplus generated by these almost optimal investments is clearly dominated by the loss in surplus generated by the unique under-investment of the best firm matched with the worst worker. Indeed, the firm's investment choice in the latter case is very far from the optimal level (returns from a marginal increase of investment are very high).

7. Concluding Remarks

When both sides to a market can undertake match specific investments Bertrand competition between these sides (workers and firms) for matches may help solve the hold-up problems generated by the absence of fully contingent contracts. In this paper, we have shown two results, quite different in their nature. When workers' investments precede Bertrand competition that, in turn, precede firms' investments, efficiency can be achieved.

Inefficiencies are present, instead, when both workers and firms choose their investments before Bertrand competition. However, in this case we show that the aggregate inefficiency due to firms' under-investments is low in the sense that is bounded above by the inefficiency that would be induced by the sole under-investment of the best firm matched with the worst worker.

We believe that these results can be generalized in a number of ways. In particular, the efficiency of the sequential investment model does not depend on the assortative nature of the matching process. If matching is not assortative, but the market guarantees that efficient matches will form, the same result we presented in Proposition 3 above applies. Indeed, it is still possible, in this non-assortative framework, for both sides of the market to be residual claimants at different point in time. Notice that, although the flavour of the result should persist in more general models, assortative matching is critical for the second result we derived in Proposition 4 above.

The results we derived in our analysis are also helpful in shedding light on a general environment in which both sides of the market undertake specific investments, both before and after market competition. Indeed, in our analysis two rather different asymmetries play a critical role.

In the first place, Bertrand competition implies that the incentives of one side of the market to make *ex-ante* efficient investments are correct. Indeed, the side of the market that Bertrand competes is residual claimant of the match surplus in excess of the bid needed to outbid its immediate competitor. This result is independent of the degree of competition generated by the immediate competitor; in other words, it is independent of the degree of specificity of the investment undertaken.

The other side of the market undertakes, instead, inefficient investments. This is because parties on this side of the market are paid the maximum willingness to pay of the immediate competitor for the match in which they are involved. Since we consider match specific investments and a discrete number of parties on both sides of the market, this willingness to pay differs from the surplus generated in the match. However, even these parties have an incentive to invest since their investment affects, although not to an efficient degree, this willingness to pay and hence their remuneration. Of course, now the degree of specificity matters. However, if matching is assortative, the aggregate inefficiency generated by this under-investment is small in the sense we presented in Proposition 4 above. In other words, the incentives of both sides of the market to undertake *ex-ante* investments are either efficient or near-efficient (inefficiencies are small) and the hold-up problem can be relegated to the position of a minor ripple to the smooth and efficient operation of competitive markets.

The other asymmetry that plays a critical role concerns the content of the simple contract the parties write at the end of the Bertrand competition game. If this contract specifies the remuneration of one party to the match, the other party is now residual claimant of the match surplus in excess of the amount she promised to the match partner in the contract. Therefore this party's incentive to undertake match specific investments are efficient. The same is not true for the other party whose incentives to undertake investments are fully blunted. Notice that, in the absence of fully contingent contracts, this is the main source of inefficiency present in this general environment and it is the inefficiency with which institutional mechanisms (such as private ownership) or contractual devices (such as options to own) should be concerned.

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