Indicative Bidding in Auctions with Costly Entry*

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Abstract

When selling a business by auction, investment banks frequently use indicative bids – non-binding preliminary bids – to select a limited number of bidders for participation in the auction. We show that if participation in the auction is costly, indicative bids can be informative: a symmetric equilibrium exists in weakly-monotone strategies; but bidder types "pool" over a finite number of bids, so the highest-value bidders are not always selected. We show how equilibrium changes with the number of potential bidders and the participation cost. We also characterize equilibrium play when the number of potential bidders is large, and show that both revenue and bidder surplus are higher than when entry into the auction is unrestricted.

1 Introduction

In an asset sale, buyers often have some initial private information about their willingness to pay, but need to learn more, and incur substantial costs, before making offers. For example, in the sale of financial or business assets, buyers may know the value of possible synergies between the asset and their businesses, but have to perform due diligence before submitting their offers. The legal and accounting costs of inspecting and verifying the quality of the asset often runs into the millions of dollars. The seller of the asset often bears some or all of these entry costs indirectly through their impact on buyer participation and offers, and therefore has an incentive to restrict the number of buyers with whom he negotiates seriously. In these cases, the seller must decide not only how to sell the asset, but also how to select the "right" buyers (those likely to have the highest willingness to pay) to participate in the sale.

One approach to dealing with the seller's selection problem is to auction the right to enter the sale. By bidding for entry rights, buyers undertake costly actions that credibly signal their

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willingness to pay. Fullerton and McAfee (1999) introduced this idea in the context of research tournaments (similar to all-pay auctions), and showed that a properly designed auction for entry rights can efficiently select the most qualified participants for the tournament. For environments like the one we consider – buyers have some initial private information and incur entry costs – Ye (2007) obtained similar results for an auction for the right to enter a subsequent auction. However, in the case of a private sale of financial or business assets, we believe that the use of auctions to restrict entry into a sales mechanism is vulnerable in important ways. One essential feature of any entry rights auction is that buyers commit to pay substantial sums before conducting due dilegence, which places them at risk of shareholder lawsuits if the asset turns out to be worth less than they expected. Even worse, since revenue is based in part on the expected, rather than actual, valuation of the asset, there is a risk of adverse selection among sellers: a rush of entry by sellers with worthless assets could crowd out sellers with legitimate ones.

This paper studies an alternative approach to selecting buyers, based on costless communication. We suppose that the seller simply asks the buyers to indicate how much they expect to be willing to pay for the asset, and permits only the buyers reporting the highest willingness-to-pay to pay the entry cost and participate in the sale. These messages – "indicative bids" – are costless, because they are never paid by the buyers; and they are non-binding, because they do not restrict in any way the real offers that a buyer may subsequently make. In other words, indicative bids are cheap talk: they influence the seller's action in selecting buyers, but do not directly affect the payoffs of either the buyers or the seller. This approach to selling assets is commonly used by investment banks, and appears to work well. An investment bank typically contacts a large number of potential buyers, and gives them an opportunity to submit non-binding proposals of price and other terms of sale. The bank then uses these proposals to select a small number of the most serious buyers, typically between one and three, who then perform due diligence and make offers.[1]

Our main goal is to show whether and how a round of cheap talk in the form of indicative bids can help sellers identify high-value buyers. Clearly, if indicative bids are informative, the seller has every incentive to use them to select entrants to the sale: revenue is highest when the selected buyers are those with the highest willingness to pay. However, the incentives for buyers to honestly report their willingness to pay are less clear. If entry costs are zero, then entry can be thought of as a free option, and we would expect indicative bidding to unravel to all buyers submitting the highest possible bid. (Morever, if entry is costless, the seller has no reason to restrict entry.) But, if entry costs are positive, then buyers' incentives are partly aligned with those of the seller: the seller wants to restrict the number of buyers who incur the entry cost, and buyers want to avoid being selected and paying this cost if they are unlikely to win. Thus, low value buyers will try to separate themselves from high value buyers, by bidding less than the highest possible bid. The

[1] In a typical sale, 15 to 30 potential buyers might be contacted initially with a one-page description of the business for sale, called a"teaser"; five to ten might respond and eventually submit nonbinding proposals. The most serious buyers get face-to-face meetings with the seller, and an opportunity to revise their proposals. Typically no more than three are chosen to gain access to a "data room" containing contracts and financials of the business, perform due diligence, and submit final, binding bids.

main issue is: how much separation can they achieve?

We use a version of the two stage model developed by Ye (2007), who was the first to study indicative bidding, to address this issue. In the first stage, buyers are asked to simultaneously submit indicative bids or opt out. The seller selects the two buyers who sent the highest indicative bids, with ties broken randomly. (If only one buyer opts in, he advances alone to the second round; if all buyers opt out, no sale occurs.) In the second stage, the entrants each pay the entry cost, receive additional information about their private values, and submit binding bids in a second-price auction with no reserve price. We will refer to this mechanism as the indicative bidding mechanism. Ye shows that a fully separating equilibrium fails to exist in this mechanism. We characterize the symmetric equilibria, prove existence, and establish conditions for uniqueness.

Our central result is that indicative bids can yield a partial sorting of buyers based on their initial private information, or type. The key sufficient condition we impose is that entry costs are large enough that a buyer only wants to advance to the second stage if his expected willingness to pay is higher than his opponent's. Given this condition and fixing a countable bid space, we show that the game has a symmetric equilibrium. The equilibrium is a finite partition of the space of buyers' types. There is a finite upper bound on the number of indicative bids that are used, and only the lowest m bids are used with positive probability. Buyers with types in the same element of the partition submit the same indicative bid, and buyers in higher elements submit higher bids. Thus, the equilibrium leads to a partial sorting of the buyers, which helps the seller select high-value buyers with greater likelihood.

How well does the indicative bidding mechanism perform? The natural benchmark is an auction in which entry is unrestricted. Buyers decide on the basis of their private information whether or not to enter the auction, pay the entry cost, learn more about their values, and submit binding bids. When the number of buyers is large, we show that the indicative bidding mechanism always yields both greater revenue and greater buyer surplus than a standard auction with unrestricted entry. Through numerical examples, we find that this appears to hold even when the number of buyers is small.

The main tradeoff between the two mechanisms is that unrestricted entry leads to a greater risk of no sale, while indicative bidding introduces the risk that the most eager buyers fail to advance to the second round due to random tie-breaking. By capping the number of entrants, the indicative bidding mechanism reduces the risk that buyers will have to pay the entry cost when they are likely to lose. This lowers the threshold at which buyers are willing to opt in – increasing the probability that the asset gets sold, as well as the probability that two or more buyers opt in and the seller earns substantial revenue.[2] On the other hand, with indicative bids, the "sorting" of buyers is not perfect, since ties occur with positive probability. This introduces the risk that the "wrong" buyers may be selected, reducing both efficiency and revenue. In all of our numerical examples, the participation effect is larger than the selection effect, and both efficiency and revenue are higher

[2]In a sense, the seller is solving a coordination problem among buyers, reducing both the chance that too many buyers advance and the chance that too few advance.

when indicative bidding is used. As the number of buyers gets large, the selection effect vanishes, because only the highest-type buyers participate, but the participation effect does not vanish. As a result, we can prove that when there are many buyers, both revenue and total surplus are always higher when indicative bids are used. (As noted above, buyer surplus is also higher, but the reason is more subtle.)

This paper connects and makes contributions to two literatures: cheap talk games, and auctions with costly entry. The indicative bidding equilibria are similar to the "cheap talk" equilibria of Crawford and Sobel (1982): indicative bids are monotonic in buyers' initial information, but only a finite number of different bids are used in equilibrium, and different types of buyers "pool" on the same bid. In their seminal paper, Crawford and Sobel show that cheap talk can improve the ex ante payoffs of both parties when the sender has information relevant to the receiver's decision problem. Farrell and Gibbons (1989) and Matthews and Postlewaite (1989) similarly show that cheap talk can be informative prior to bilateral bargaining, and can therefore expand the set of equilibrium payoffs. We show a version of cheap talk can be informative, and expand the payoff set, in a different environment, a setting with one seller and multiple buyers.

Our model also naturally introduces a certain type of receiver commitment into a cheap talk setting, which sharpens the predictions of the model. As Farrell and Gibbons (1989) observe, in standard cheap-talk games, the receiver cannot commit to a choice of outcome as a function of the messages. Instead, the messages derive meaning only from the receiver's interpretation of them and the receiver must act optimally given that interpretation. In our setting, we assume the seller commits both to the rules of the second-stage auction (which is standard) *and* to how he will select entrants based on the indicative bids received. In particular, we assume the seller commits to selecting a fixed number of bidders for the second stage, choosing those who send the highest indicative bids, and breaking ties randomly. This commitment to a monotone selection rule eliminates much of the multiplicity of equilibria that arises in cheap talk games. In particular, it rules out a "babbling" equilibrium, and any equilibrium where adverse off-equilibrium-path beliefs are used to deter unused messages. It also implies that any equilibrium must satisfy a property analogous to the NITS, or *no incentive to separate*, condition introduced by Chen, Kartik and Sobel (2008). The analog to this condition in our model is that high-type buyers must not have an incentive to deviate to messages higher than those used in equilibrium. In our model, this is not an equilibrium refinement, but a requirement for any equilibrium; under some distributional assumptions on signals, it leads directly to uniqueness of the symmetric equilibrium.

The other literature we connect to is the theoretical literature on auctions with costly entry. Much of this literature deals with how the seller should restrict entry when buyers have no private information prior to entry. Cremer, Spiegel, and Zheng (2009) characterize the optimal mechanism, and show that the seller can use entry fees and subsidies to extract all buyers' surplus. In the absence of such payments, Bulow and Klemperer (2009) demonstrate that unrestricted auctions are less efficient than sequential mechanisms, but typically generate more revenue. Lu and Ye (2013) treat the case where buyers' entry costs are private information, but values are learned only upon entry. Earlier papers include McAfee and McMillan (1987), Levin and Smith (1994), Burguet (1996), Menezes and Monteiro (2000), and Compte and Jehiel (2007). Samuelson (1985) was the first to study the impact of entry on auction outcomes in an environment where buyers already know their private values, and entry costs therefore consist of bid preparation costs rather than information acquisition costs. As discussed above, Ye (2007) considers the more general case in which entry involves information updating and where a first-round auction can be used to allocate entry rights into a second-round auction. We differ from the first set of papers by assuming buyers have some private information about their valuation before participation decisions are made; we differ from the last two by using cheap talk to resolve the coordination problem faced by buyers.

The rest of the paper proceeds as follows. Section 2 presents the model. Section 3 characterizes symmetric equilibria, establishes existence of an equilibrium, and gives sufficient conditions for uniqueness and some comparative statics on the number of messages used in equilibrium. We also give a numerical example to illustrate the structure of the equilibrium and some of its features. In Section 4, we compare the equilibrium participation level to the efficient level, and examine whether reserve prices or bidder subsidies can improve effiency or revenues. We also compare the performance of the indicative bidding mechanism to a standard auction with unrestricted entry. Section 5 discusses one key extension, relaxing one of the main assumptions of our bechmark model and extending the mechanism to allow more than two buyers to advance to the second round. Section 6 discusses other possible mechanisms; Section 7 concludes.

2 Model

We begin by describing the environment, our assumptions and the indicative bidding mechanism that the seller uses to sell the asset. Given the mechanism, we then define the strategies and payoffs of the bidders in the game that the bidders face.

2.1 The Indicative Bidding Mechanism

Our model is the same as the "private value updating" model of Ye (2007). There are $N \geq 3$ bidders indexed by i. Each bidder's private value for the asset consists of two components. Specifically, bidder i has private value

$$S_i + T_i$$

for the asset, where S_i and T_i are real-valued random variables. Random variables will be denoted by upper case and realizations by lower case. Bidder i knows s_i initially, but incurs a cost c to learn t_i before bidding. We will refer to c as the bidder's entry cost. In the natural resource setting, it is the cost of gathering additional information about the value of the resource, and in an asset sale, it is the cost of due diligence. A bidder cannot bid without incurring this cost. Each bidder observes his own valuations but not those of the other bidders.

We make the following assumptions about the bidders' valuations.

Assumption 1. *1.* $\{S_i\}_{i\in\{1,2,\ldots,N\}}$ *are independent of* $\{T_i\}_{i\in\{1,2,\ldots,N\}}$

2. S_i *are i.i.d.* $\sim F_S$*, where* F_S *is a massless distribution with support* $[0,1]$ *and density* f_S *bounded away from* 0

3. T_i *are i.i.d.* $\sim F_T$*, where* F_T *is a massless distribution with support in* $\Re^+$

The assumption that $\{S_i\}$ are independent, and independent of $\{T_i\}$, is of course made for tractability. In a natural resource setting, S_i could be related to bidder i's costs of extraction and T_i to the quantity of the resource itself; in an asset sale, S_i would represent bidder i's assessment of the idiosyncratic synergy between the asset and the bidder's existing business, and T_i would be related more to the quality of the asset. Once the support of F_S is assumed to be bounded, assuming it is the unit interval is simply a normalization. The assumptions on T_i – that F_T is nondegenerate and massless and that $\{T_i\}$ are independent – are made for convenience of notation and exposition of proofs; all the results extend if F_T is degenerate, has point masses, or if $\{T_i\}$ are correlated. In fact, another natural version of the model is when $\{T_i\}$ are perfectly correlated, and therefore the same for all bidders. This corresponds to a setting in which T_i represents the component of value that is common to all bidders.

The mechanism we consider consists of two rounds. In the first round, the seller asks potential bidders to submit messages from a set of available messages. One message is "opt out", or decline to participate; the others are "opt-in" messages, and are fully-ordered. The messages are a measure of the bidder's level of interest in acquiring the asset, and we will sometimes refer to them as indicative bids. Potential bidders choose which indicative bids to send based on their first signal S_i. If at least two bidders submit "opt-in" messages, then the seller restricts entry into the auction to the two bidders who sent the highest opt-in messages, with ties broken randomly. In the second round, the two bidders each pay a cost c, learn their second signal T_i, and submit binding bids in a standard second-price auction.[3] If only one bidder opts in, he enters the auction alone, incurs the cost c, and wins at price 0.[4] If all bidders opt out, then the game ends with no sale.

Since S_i is bidder i's private information at the time he decides what indicative bid to send, we refer to S_i as his type. We assume that the bidders selected for the second-round auction play the dominant-strategy equilibrium. Thus, we can define two second-stage payoff functions

$$g_0(s_i) \quad = \quad -c + E_{T_i}\{s_i + T_i\}$$

$$g_1(s_i, s_j) \quad = \quad -c + E_{T_i,T_j} \max\{0, s_i + T_i - s_j - T_j\}$$

where $g_0(s_i)$ is the expected payoff to bidder i with type s_i who advances to the second round alone, and $g_1(s_i, s_j)$ is his expected payoff from advancing against an opponent with type s_j. Note that

[3] We assume a second-price auction for simplicity: since bidding is in dominant strategies, bidders' beliefs conditional on the outcome of indicative bidding do not matter.

[4] We consider the effect of a reserve price in a later section.

g_0 and g_1 are both continuous, and g_1 is a function only of the difference $s_i - s_j$. Also, g_0 is strictly increasing, and g_1 is weakly increasing in s_i everywhere, and strictly increasing when $s_i \geq s_j$.[5]

We maintain the following assumption about the entry cost:

Assumption 2. *1. $c < 1 + E\{T_i\}$, or equivalently, $g_0(1) > 0$*

2. $c > E\{\max\{0, T_i - T_j\}\}$, or equivalently, $g_1(s, s) < 0$ for any s

Assumption 2.1 is simply the requirement that the game is non-trivial: with positive probability, some bidder finds investigating the asset worthwhile. Assumption 2.2 is a much more substantive assumption: it requires costs to be high enough that a bidder only wants to bid against an opponent if he has an initial advantage over him. In the common value case where $T_i = T_j = T$, $g_1(s_i, s_j) = -c + \max\{0, s_i - s_j\}$, and so $g_1(s, s) < 0$ is automatically satisfied. In Section 5, we relax this assumption for the case where N is large.

By continuity, Assumption 2.2 implies that there is some $\epsilon > 0$ such that $g_1(s + \epsilon, s) = 0$. Therefore, if a bidder's opponents were all "bidding truthfully" in the indicative bidding stage, then he would want to shade his bid downwards by at least ϵ, to avoid participation in cases where his payoff would for certain be negative. This ϵ is analogous to the bias between sender and receiver preferences in Crawford and Sobel (1982); an immediate consequence of this is that a fully-revealing equilibrium cannot exist.[6] (In the alternate case where $g_1(s, s) \geq 0$, a fully-revealing equilibrium still generically cannot exist in an indicative bidding game, but the reasoning is different; see Ye (2007).)

Our final assumption is that the set of available messages is countable.

Assumption 3. *The set of messages permitted in the first round is $\{0, 1, 2, \dots, M\}$ for some $M \in \mathbb{Z}^{++} \cup \{\infty\}$, with message 0 being "opt out."*[7]

As we discuss later, symmetric equilibria fail to exist when the set of allowed messages is continuous,[8] or when there is not a lowest opt-in message. Thus, we assume the set of messages is bounded below and either finite or countably infinite, in which case it is without loss to assume it is a subset of the positive integers. Under the above assumptions, the primitives (N, F_S, F_T, c) fully describe the environment facing the seller, and (N, F_S, F_T, c, M) fully define the game facing the bidders.

The indicative bidding game is a cheap talk game with commitment. The messages of the bidders (i.e., senders) are cheap talk: they influence which action the seller (i.e., receiver) takes but, given that action, they do not affect the payoffs of the players. In the standard cheap talk game, the space of messages is unrestricted, and the receiver chooses an action that is his best response

[5] Let $h = s_i - s_j$; the function $\max\{0, h + t_i - t_j\}$ is strictly increasing in h whenever $h + t_i - t_j > 0$, so g_1 is strictly increasing unless h is below the minimum of the support of $T_i - T_j$. Since T_i and T_j have the same support, g_1 is strictly increasing on $\Re^+$.

[6] We thank Navin Kartik for this observation. Note that the "bias" in our model goes in the opposite direction as in Crawford and Sobel: in the classic sender-receiver game, the sender wants to misrepresent his type upwards, while in our setting, a buyer would like to misrepresent his type downwards.

[7] Equivalently, bidders could send messages $\{1, 2, \dots, M\}$ to opt in, and send no message to opt out.

[8] Really, when for any two permitted messages, there is another message between them.

to the messages sent. By contrast, in the indicative bidding game (as is standard in auctions), the seller commits to the mechanism. In particular, he commits to a rule that selects bidders based on the messages they send, and ignores messages outside the set of available messages. Ex interim, once the messages are received, it would typically be in the seller's interest to have more bidders advance or to listen to other messages in determining which bidders advance.

We also implicitly assume that bidders are committed to incur the entry cost if they opt in and are selected. If bidders were able to learn, prior to incurring c, whether they would face a competitor in the second round, it would sometimes be in their interest to opt in initially, but then decline to participate if another bidder opted in; we don't allow this in our model. In real settings, both during and after due diligence, bidders are often kept in the dark as to the number of competitors they face. (Subramanian (2010) offers a couple of funny stories of how this is accomplished.) If bidders don't receive any information about the level of competition, then opting in and backing out after selection would never be a profitable deviation.

2.2 Strategies and Payoffs

Since bidding in the second-round auction is in dominant strategies, we focus on characterizing first-round play. A pure strategy for bidder i is a mapping $\tau_i : [0,1] \to \{0, 1, 2, \ldots, M\}$ from types to messages, and a mixed strategy is a mapping $\sigma_i : [0,1] \to \Delta(\{0, 1, 2, \ldots, M\})$. Let $K_{-i} = (K_1,...K_{i-1}, K_{i+1}, .., K_n)$ denote the vector of (random) messages induced by the strategies of bidder i's opponents and let $H_1, .., H_{n-1}$ denote the ordering, from the largest to the smallest, of the messages in K_{-i}. We will generally let k denote a generic message sent by bidder i, and h a possible realization of H_1 the highest message sent by i's opponents.

Our objective will be to characterize symmetric equilibria. Consequently, we need only define the expected payoff to bidder i when his opponents play a common strategy. Let $v_\tau(s_i, k)$ denote the expected payoff of bidder i given his own type is s_i, he sends message k, and his opponents play a pure strategy τ. This is equal to the probability that bidder i is selected for the second round, times his expected payoff conditional on being selected.

To characterize $v_\tau(s_i, k)$, we must first specify the expected payoff from advancing to the auction. Suppose bidder i is told that he has been selected and that the value of the highest opponent message H_1 is h. Since ties are broken randomly, bidder i's updated belief is that he is advancing against an opponent with type randomly drawn from the set $\tau^{-1}(h)$ according the prior distribution F_S of types. As a result, his expected payoff conditional on entry depends only on the distribution of types sending the highest message, and not on his own message or the messages of opponents who were not selected. We can write this second-round payoff as

$$g_1(s_i, \tau^{-1}(h)) \equiv \frac{\int_{\tau^{-1}(h)} g_1(s_i, s_j) dF_S(s_j)}{\int_{\tau^{-1}(h)} dF_S(s_j)}.$$

The probability that bidder i is selected potentially depends on his own message and the messages sent by multiple opponents; but given his own message k, we can group the different profiles of opponent messages K_{-i} according to the highest opponent message H_1, and write bidder i's interim payoff as

$$
\begin{aligned}
v_\tau(s_i, k) \quad = \quad & \Pr(H_1 = 0) \cdot g_0(s_i) \\
& + \textstyle\sum_{h=1}^{k-1} \Pr(H_1 = h) \cdot g_1(s_i, \tau^{-1}(h)) \\
& + \textstyle\sum_{j=1}^{N-1} \Pr(H_1 = ... = H_j = k) \cdot \frac{2}{j+1} \cdot g_1(s_i, \tau^{-1}(k)) \\
& + \textstyle\sum_{h=k+1}^{M} \sum_{j=2}^{N-1} \Pr(H_1 = h, H_2 = ... = H_j = k) \cdot \frac{1}{j} \cdot g_1(s_i, \tau^{-1}(h)).
\end{aligned}
$$

The first term is the payoff when all other bidders opt out so the highest opponent message is 0; the second term is the payoff when at least one opponent does not opt out and the highest opponent message is below k; the third term is the payoff when j other bidders send message k and the remaining bidders send messages below k; and the last term is the payoff when one opponent sends a message higher than k, $j-1$ opponents send message k, and the rest send messages below k. Note that the second term vanishes in the case $k = 1$, and the last vanishes in the case $k = M$; and of course, opting out gives payoff $v_\tau(s_i, 0) = 0$.

Given a mixed strategy σ, let $\sigma^k(s)$ denote the probability with which a bidder playing σ sends message k given type s. Let supp $\sigma(s) = \{k : \sigma^k(s) > 0\}$ denote the support of $\sigma(s)$. The expected payoff of bidder i given his own type s_i if he sends message k and all of his opponents play the same mixed strategy σ is denoted by $v_\sigma(s_i, k)$ and can be defined as above by replacing $g_1(s_i, \tau^{-1}(h))$ with

$$\widetilde{g}_1(s_i, \sigma, h) \equiv \frac{\int_0^1 g_1(s_i, s_j)\sigma^h(s_j)dF_S(s_j)}{\int_0^1 \sigma^h(s_j)dF_S(s_j)}.$$

As in the case of pure strategies, bidder i's expected payoff conditional upon advancing depends upon the distribution of types who send the highest message and their mixed strategy, but it does not depend on his own message or the messages of bidders who are not selected. This structure is important for the characterization of equilibrium.

3 Characterizing Equilibrium

Our solution concept is perfect Bayesian equilibrium, and we focus on symmetric equilibria.[9] We first provide a characterization of symmetric equilibria. We then establish that, for a given set of primitives (N, c, F_S, F_T), and number of available opt-in messages M, a symmetric equilibrium exists. We provide a sufficient condition for uniqueness and show that condition is satisfied if F_S is uniform. We end this section with a numerical example that illustrates the construction and

[9]Like many entry games, this game may also have many asymmetric equilibria. For example, if $s_i \sim i.i.d.\, U[0, 1]$, $t_i \sim i.i.d.\, U[0, 5]$ and $c = 2$, then $g_0(0) > 0 > g_1(1, 0)$, so any bidder finds it profitable to enter alone but unprofitable to enter against any opponent; in that case, for any j, the strategies "bidder j enters regardless of s_j; bidders $i \neq j$ never enter" is an equilibrium.

properties of the equilibrium.

3.1 Necessary and Sufficient Conditions for Equilibrium

We begin by establishing several properties common to all symmetric equilibria, including mixed strategy equilibria.

Lemma 1. *In any symmetric equilibrium, strategies are weakly monotone increasing.*

Since we allow mixed strategies, monotonicity here means that if $k' > k$ and $k' \in \operatorname{supp} \sigma(s)$, then $k \notin \operatorname{supp} \sigma(s')$ for $s' > s$. The details of the proof of Lemma 1 are in the appendix. It consists of showing that, for any $k' > k$, $v_\sigma(\cdot, k') - v_\sigma(\cdot, k)$ is strictly single-crossing from below, since in that case,

$$v_\sigma(s, k') - v_\sigma(s, k) \quad \geq \quad 0 \quad \longrightarrow \quad v_\sigma(s', k') - v_\sigma(s', k) \quad > \quad 0$$

and so $k' \in \operatorname{supp} \sigma(s)$ implies $k \notin \operatorname{supp} \sigma(s')$. If the probability of advancing at the higher message k' is greater than at k, then the single-crossing property follows directly from the properties of g_0 and g_1. The more subtle part of the proof is showing that in equilibrium, there cannot be multiple messages k and k' giving the same probability of advancing, since this would potentially allow for non-monotonicities. However, we show that if $\operatorname{supp} \sigma(s)$ and $\operatorname{supp} \sigma(s')$ contain distinct messages giving the same probability of advancing, then all bidders with types in the interval (s, s') must use *only* messages giving that same probability of advancing, and then show that this leads to a contradiction.

Our next lemma establishes that the number of messages used in any symmetric equilibrium is finite and that no messages are "skipped": if two messages are used in equilibrium, then any message between them must also be used.

Lemma 2. *Fix $M \in \mathbb{Z}^{++} \cup \{\infty\}$, and let $\{0, 1, 2, \ldots, M\}$ be the set of available messages. In any symmetric equilibrium, the set of messages played with positive probability is $\{0, 1, 2, \ldots, m\}$ for some finite $m \geq 1$.*

The details of the proof are in the appendix, but the intuition is roughly this. Lemma 1 requires that higher types play higher messages. Consider two consecutive message k and $k+1$. By sending $k+1$ instead of k, a bidder increases his chance of being selected only when his highest opponent sent message k or higher. (Otherwise, he would be selected for sure by sending either message.) Consider the lowest bidder type to send message $k+1$ in equilibrium. Since he gets a negative payoff from advancing against an opponent who sent message $k+1$ or higher, for him to send message $k+1$, he must strictly prefer to be selected against an opponent who sent message k. This implies a minimum "width" of at least ϵ on the set of types sending each "interior" message, where ϵ solves $g_1(s+\epsilon, s) = 0$. This also leads to the "no skipped message" result: if message $k+1$ was not used, a bidder who is close to indifferent between messages k and $k+2$ strictly prefers to be selected when multiple opponents sent message k, but strictly prefers not to be chosen when two or more opponents sent message $k+2$ or higher; message $k+1$ would therefore be a profitable deviation.

The two lemmas establish two conditions that are necessary for a symmetric equilibrium – monotonicity of the strategy, and finiteness (and consecutiveness) of the messages in its support. Next, we show that these two conditions, combined with two more, are both necessary and sufficient for symmetric equilibria. In stating the lemma, we focus on the case of pure strategy symmetric equilibria for ease of notation; suitably modified to allow for mixing by the indifferent types, the same conditions turn out to be necessary and sufficient for *any* symmetric equilibrium.

Lemma 3. *Fix $M \in \mathbb{Z}^{++} \cup \{\infty\}$, and let $\{0, 1, 2, \ldots, M\}$ be the set of available messages. Let $\tau : [0,1] \to \{0, 1, \ldots, M\}$ be a first-round (pure) strategy. "Everyone playing τ" constitutes a symmetric equilibrium* if and only if *there is some finite m, $1 \leq m \leq M$, and some set of numbers*

$$0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1} < \alpha_m = 1$$

such that:

1. *$\tau(s_i) = 0$ for $s_i < \alpha_0$, and $\tau(s_i) = k$ for $s_i \in (\alpha_{k-1}, \alpha_k)$ for each $k \in \{1, \ldots, m\}$*
2. *$\tau(\alpha_k) \in \{k, k+1\}$ for $k \in \{0, 1, 2, \ldots, m-1\}$, and $\tau(1) \in \{m, m+1, m+2, \ldots, M\}$*
3. *$v_\tau(\alpha_k, k+1) = v_\tau(\alpha_k, k)$ for $k \in \{0, 1, 2, \ldots, m-1\}$*
4. *either $m = M$ or $v_\tau(1, m+1) \leq v_\tau(1, m)$, and $\tau(1) = m$ if $v_\tau(1, m+1) < v_\tau(1, m)$*[10]

The first two conditions are just a restatement of Lemmas 1 and 2: that symmetric equilibrium strategies must be monotonic and use messages $\{0, 1, \ldots, m\}$ for some m. The third condition is that the threshold types – those bidders at the boundary between sending two messages – must be indifferent among the two. If $m < M$, then bidders have a credible way to signal that they are high-type and guarantee selection; thus, the final condition requires that unless $m = M$, such deviations to unused messages must not be profitable. This condition is an analogue of the No Incentive To Separate (NITS) condition introduced by Chen, Kartin and Sobel (2008). In their setting, NITS is the condition that the lowest-type sender would not choose to reveal his type truthfully if he could. In their model, this is an equilibrium refinement, and selects a unique equilibrium – the one with the maximal number of messages. In our setting, this is not a refinement, but a condition that must be satisfied in any equilibrium – as we will see below, this will sometimes, but not always, lead to uniqueness of equilibrium.[11]

[10]The case $v(1, m+1) = v(1, m)$ requires a non-generic set of primitives, so generically, exactly m opt-in messages will be used in equilibrium. However, we will refer to $\{0, 1, 2, \ldots, m\}$ as the messages played by types $s_i \in [0, 1)$, or the messages played with positive probability, to accomodate the non-generic case where $m < M$, $v(1, m+1) = v(1, m)$, and $\tau(1) > m$.

[11]In a standard sender-receiver setting, NITS is a condition on the lowest type, while in our model, the condition is on the highest type; as noted above, the direction in which the sender/bidder would like to misrepresent his type goes in opposite directions in the two models.

Proof of Lemma 3. *Necessity.* Given any symmetric equilibrium τ, define $\alpha_k = \sup\{s_i : \tau(s_i) = k\}$ for each $k \in \{0, 1, \dots, m\}$; conditions 1 and 2 follow from Lemma 1. Since g_0 and g_1 are both continuous in s_i, $v_\tau(\cdot, k)$ is continuous for each k, and so $v_\tau(s_i, k+1) - v_\tau(s_i, k)$ is continuous in s_i; since it must be weakly negative just below α_k and weakly positive just above α_k, it must be zero at $s_i = \alpha_k$. For the last condition, if $M > m$ and $v_\tau(1, m+1) > v_\tau(1, m)$, then by continuity, message $m+1$ is a profitable deviation for bidders with types just below 1.

Sufficiency. Fix a strategy τ satisfying the four conditions of Lemma 3. Combined with $v_\tau(\alpha_k, k+1) - v_\tau(\alpha_k, k) = 0$ (the third condition), single-crossing differences means that bidder i strictly prefers message $k+1$ to k if and only if $s_i > \alpha_k$, and strictly prefers k to $k+1$ if and only if $s_i < \alpha_k$. By transitivity, if $s_i \geq \alpha_{k-1}$, bidder i weakly prefers message k to any lower message, and if $s_i \leq \alpha_k$, i weakly prefers k to any higher message up to m. Conditions 1 and 2 of the lemma, then, imply that for bidders with types $s_i < 1$, τ selects a best-response among $\{0, 1, 2, \dots, m\}$. If $M = m$, there are no other options; if $M > m$, then by the fourth condition, $v_\tau(1, m+1) - v_\tau(1, m) \leq 0$, and therefore $v_\tau(s_i, m+1) - v_\tau(s_i, m) < 0$ (single-crossing differences), so no bidder gains by deviating to a message above m;[12] so τ selects a best-response for all bidders with $s_i < 1$.

Finally, for $s_i = 1$, $1 > \alpha_{m-1}$ implies a bidder with type 1 prefers message m to any lower message. If $v_\tau(1, m+1) = v_\tau(1, m)$, he is indifferent among all messages m and higher, and if $v_\tau(1, m+1) < v_\tau(1, m)$, the last condition says τ selects only message m. Thus, τ is always a best-response to τ, and therefore a symmetric equilibrium. □

Lemma 3 provides a complete characterization of pure strategy symmetric equilibria. Any symmetric equilibrium is a finite partition of the type space, in which bidders with types in the same interval send the same message, and bidders with types in higher intervals send higher messages. The messages used are 0 to m; each threshold type α_k is indifferent between messages k and $k+1$, so he can send either message.

Note that if $M = m$, then all available messages are being used in equilibrium. But when M is sufficiently large (or infinite), only a subset $m < M$ of the available messages are used. It may seem counterintuitive that the highest type is not willing to separate himself by sending a higher message if one is available; after all, he is certain to be selected if he sends the higher message. But the increase in probability of being selected arises solely from breaking ties against opponents who are sending the highest equilibrium message m, and are therefore in the same interval as the highest type. As more messages are used, this interval gets sufficiently small that the highest type's payoff against a randomly chosen opponent drawn from this interval ($g_1(1, [\alpha_{m-1}, 1])$) is negative. When this is the case, he will not want to separate himself by sending a higher message.

The indifference of threshold types introduces a trivial kind of multiplicity in pure strategy symmetric equilibria: if one equilibrium exists, we can change the strategies of one or more threshold types to generate more equilibria. This type of multiplicity is uninteresting – since threshold types $\{\alpha_k\}$ occur with probability 0, their behavior in equilibrium does not affect expected payoffs. The indifference also allows for symmetric mixed strategy equilibria, since bidders with types $s_i = \alpha_k$

[12]Since messages above m are played with zero probability, they all give the same payoff as message $m+1$.

could mix between k and $k+1$, and in the nongeneric case where $v(1, m+1) = v(1, m)$, bidders with type $s_i = 1$ could mix among all messages above m. Lemmas 1 and 2, however, imply that all symmetric mixed strategy equilibria take the same basic form as the pure-strategy equilibria: they are partitions in which only the threshold types use mixed strategies. Thus, suitably modified to allow for mixed strategies and mixing by threshold types, the conditions of Lemma 3 are necessary and sufficient for *all* symmetric equilibria.

We turn next to the issue of existence. Our approach is constructive: we develop an algorithm that first determines the value of m and then generates a partition $\alpha = (\alpha_0, \ldots, \alpha_{m-1})$ that satisfies the indifference and boundary conditions of Lemma 3. In order to do this, we will need to express each of these necessary and sufficient conditions in terms of the partition α defined by the strategy τ.

Indifference of Bidders With Type α_0

We begin with the condition $v_\tau(\alpha_0, 1) = v_\tau(\alpha_0, 0) = 0$. With the strategy τ of the other $N-1$ bidders satisfying the first condition of Lemma 3, we know that (modulo indifferent types) $\tau^{-1}(k) = [\alpha_{k-1}, \alpha_k]$. Since threshold types are zero-probability and therefore don't affect other bidders' expected payoffs, we can rewrite the expected payoff $v(s_i, 1)$ in terms of the thresholds $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$, as

$$\begin{aligned} v_\tau(s_i, 1) &= (F_S(\alpha_0))^{N-1} g_0(s_i) \\ &\quad + \left(\sum_{j=1}^{N-1} \binom{N-1}{j} (F_S(\alpha_1) - F_S(\alpha_0))^j (F_S(\alpha_0))^{N-1-j} \tfrac{2}{j+1} \right) g_1(s_i, [\alpha_0, \alpha_1]) \\ &\quad + \left(\sum_{j=1}^{N-1} (N-1)(1 - F_S(\alpha_1)) \binom{N-2}{j-1} (F_S(\alpha_1) - F_S(\alpha_0))^{j-1} (F_S(\alpha_0))^{N-1-j} \tfrac{1}{j} \right) g_1(s_i, [\alpha_1, 1]) \end{aligned}$$

(Analogous to $g_1(s_i, \tau^{-1}(h))$ defined above, $g_1(s_i, [a, b])$ is the expected value of $g_1(s_i, S_j)$, conditional on S_j drawn from the prior distribution F_S restricted to the interval $[a, b]$. Since f_S is massless, it doesn't matter whether the endpoints are included or excluded from the expectation.) Note that this is a function only of s_i, α_0 and α_1, not of the higher thresholds $\{\alpha_2, \ldots, \alpha_{m-1}\}$ or other features of τ. We can therefore define $\delta(\alpha_0, \alpha_1)$ as the right-hand side of the above equation, evaluated at $s_i = \alpha_0$. Plugging in $s_i = \alpha_0$ and rearranging to eliminate the summation signs, this simplifies to

$$\begin{aligned} \delta(\alpha_0, \alpha_1) &= F_S(\alpha_0)^{N-1} g_0(\alpha_0) \\ &\quad + \left(\tfrac{2}{N} \tfrac{F_S(\alpha_1)^N - F_S(\alpha_0)^N}{F_S(\alpha_1) - F_S(\alpha_0)} - 2F_S(\alpha_0)^{N-1} \right) g_1(\alpha_0, [\alpha_0, \alpha_1]) \\ &\quad + (1 - F_S(\alpha_1)) \tfrac{F_S(\alpha_1)^{N-1} - F_S(\alpha_0)^{N-1}}{F_S(\alpha_1) - F_S(\alpha_0)} g_1(\alpha_0, [\alpha_1, 1]) \end{aligned}$$

(The algebra is shown in a separate technical appendix.) By construction, we have the equivalence relation

$$v_\tau(\alpha_0, 1) = 0 \iff \delta(\alpha_0, \alpha_1) = 0.$$

Indifference of Bidders With Type α_k

Next, we consider the indifference condition $v_\tau(\alpha_k, k+1) - v_\tau(\alpha_k, k) = 0$. Rather than writing out $v_\tau(s_i, k+1)$ and $v_\tau(s_i, k)$ and subtracting, it is simpler to directly calculate their difference, by considering all the scenarios of opponent types under which message $k+1$ gives a strictly higher probability of being selected than message k. This event occurs any time the second-highest opponent message is either k or $k+1$, and these scenarios can be grouped into four categories:

- those where $j \geq 2$ of i's opponents send message k and the rest send lower messages, so that sending message $k+1$ increases the likelihood of being selected from $\frac{2}{j+1}$ to 1
- those where one of i's opponents sends a message above k, $j-1 \geq 1$ send message k, and the rest send lower messages, in which case message $k+1$ increases the likelihood from $\frac{1}{j}$ to 1
- those where $j \geq 2$ opponents send message $k+1$ and the rest send lower messages, so that message $k+1$ increases the likelihood from 0 to $\frac{2}{j+1}$
- those where one opponent sends a message higher than $k+1$, $j-1 \geq 1$ send message $k+1$, and the rest send lower messages, so message $k+1$ increases the likelihood of advancing from 0 to $\frac{1}{j}$

For each scenario, we multiply the probability of the scenario occurring, times the increase in the probability of advancing from sending message $k+1$ rather than k, times the expected payoff from advancing in that scenario. For an arbitrary value of s_i, this gives

$$\begin{aligned}
v_\tau(s_i, k+1) - v_\tau(s_i, k) \quad = \\
& \textstyle\sum_{j=2}^{N-1} \binom{N-1}{j} (F_S(\alpha_k) - F_S(\alpha_{k-1}))^j (F_S(\alpha_{k-1}))^{N-1-j} (1 - \frac{2}{j+1}) g_1(s_i, [\alpha_{k-1}, \alpha_k]) \\
& + \textstyle\sum_{j=2}^{N-1} (N-1)(1 - F_S(\alpha_k)) \binom{N-2}{j-1} (F_S(\alpha_k) - F_S(\alpha_{k-1}))^{j-1} (F_S(\alpha_{k-1}))^{N-1-j} (1 - \frac{1}{j}) g_1(s_i, [\alpha_k, 1]) \\
& + \textstyle\sum_{j=2}^{N-1} \binom{N-1}{j} (F_S(\alpha_{k+1}) - F_S(\alpha_k))^j (F_S(\alpha_k))^{N-1-j} \frac{2}{j+1} g_1(s_i, [\alpha_k, \alpha_{k+1}]) \\
& + \textstyle\sum_{j=2}^{N-1} (N-1)(1 - F_S(\alpha_{k+1})) \binom{N-2}{j-1} (F_S(\alpha_{k+1}) - F_S(\alpha_k))^{j-1} (F_S(\alpha_k))^{N-1-j} \frac{1}{j} g_1(s_i, [\alpha_{k+1}, 1])
\end{aligned}$$

Note that this is a function only of s_i, α_{k-1}, α_k, and α_{k+1}; the other thresholds $\alpha_0, \alpha_1, \ldots, \alpha_{k-2}$ and $\alpha_{k+2}, \ldots, \alpha_{m-1}$ affect $v_\tau(s_i, k+1)$ and $v_\tau(s_i, k)$ individually, but drop out of the difference. We can therefore define $\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ as the right-hand side of this equation evaluated at $s_i = \alpha_k$.

Plugging in, rewriting $g_1(\alpha_k, [\alpha_k, 1])$ as the appropriate weighted average of $g_1(\alpha_k, [\alpha_k, \alpha_{k+1}])$ and $g_1(\alpha_k, [\alpha_{k+1}, 1])$, and simplifying, we can write Δ as

$$\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) \quad = \quad c_1 \cdot g_1(\alpha_k, [\alpha_{k-1}, \alpha_k]) \quad + \quad c_2 \cdot g_1(\alpha_k, [\alpha_k, \alpha_{k+1}]) \quad + \quad c_3 \cdot g_1(\alpha_k, [\alpha_{k+1}, 1]),$$

where the coefficients

$$\begin{aligned} c_1 &= F_S(\alpha_k)^{N-1} + F_S(\alpha_{k-1})^{N-1} - \tfrac{2}{N}\tfrac{F_S(\alpha_k)^N - F_S(\alpha_{k-1})^N}{F_S(\alpha_k) - F_S(\alpha_{k-1})}, \\ c_2 &= \tfrac{2}{N}\tfrac{F_S(\alpha_{k+1})^N - F_S(\alpha_k)^N}{F_S(\alpha_{k+1}) - F_S(\alpha_k)} - 2F_S(\alpha_k)^{N-1} - (F_S(\alpha_{k+1}) - F_S(\alpha_k))\tfrac{F_S(\alpha_k)^{N-1} - F_S(\alpha_{k-1})^{N-1}}{F_S(\alpha_k) - F_S(\alpha_{k-1})}, \\ c_3 &= (1 - F_S(\alpha_{k+1}))\tfrac{F_S(\alpha_{k+1})^{N-1} - F_S(\alpha_k)^{N-1}}{F_S(\alpha_{k+1}) - F_S(\alpha_k)} - (1 - F_S(\alpha_{k+1}))\tfrac{F_S(\alpha_k)^{N-1} - F_S(\alpha_{k-1})^{N-1}}{F_S(\alpha_k) - F_S(\alpha_{k-1})} \end{aligned}$$

are all positive. (Once again, this algebra is shown in a separate technical appendix.) By construction, we have the equivalence relation

$$v_\tau(\alpha_k, k+1) = v_\tau(\alpha_k, k) \quad \Longleftrightarrow \quad \Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = 0.$$

Lack of Profitable Deviations to Unused Messages

Finally, we consider the constraint that if some (high) messages are not used in equilibrium, bidders with type $s_i = 1$ cannot benefit by deviating to them. By sending such a message, say $m+1$, bidder i would be selected for the second round for sure; by sending message m, he would still be selected for sure, *except* when two or more other bidders also sent message m, in which case he would be selected with probability less than 1. Thus, the only scenarios where message $m+1$ gives a higher probability of advancing than message m are scenarios where, upon advancing, i faces an opponent who sent message m, i.e., an opponent with type above α_{m-1}. Thus, $v_\tau(s_i, m+1) - v_\tau(s_i, m)$ has the same sign as $g_1(s_i, [\alpha_{m-1}, 1])$. Since $g_1(1, s)$ is strictly decreasing in s, its expected value over $s \in [\underline{s}, 1]$ is strictly decreasing in $\underline{s}$. Therefore, define $\overline{\alpha}$ as the solution to

$$g_1(1, [\overline{\alpha}, 1]) \quad = \quad 0$$

when such a solution exists on $[0, 1)$, and as $-\infty$ if no such solution exists, i.e., if $g_1(1, [0, 1]) < 0$. Then we obtain the following equilvalence relation:

$$v_\tau(1, m+1) \leq v_\tau(1, m) \quad \Longleftrightarrow \quad g_1(1, [\alpha_{m-1}, 1]) \leq 0 \quad \Longleftrightarrow \quad \alpha_{m-1} \geq \overline{\alpha}.$$

The functions $\delta(\alpha_0, \alpha_1)$ and $\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ and the parameter $\overline{\alpha}$ provide local conditions that the thresholds must satisfy. We will use them to construct a symmetric equilibrium. In order to do so, we first establish some useful properties of δ and Δ:

Lemma 4. *$\delta(\cdot,\cdot)$ and $\Delta(\cdot,\cdot,\cdot)$ are both continuous in each of their arguments, and for any $0 \leq \lambda < \lambda' < \lambda'' \leq 1$,*

1. *$\delta(0, \lambda') < 0$*
2. *If $\Delta(0, \lambda', \lambda'') \geq 0$ then $\delta(\lambda', \lambda'') > 0$*
3. *$\Delta(\lambda, \lambda', \lambda'') < 0$ if $\lambda' - \lambda < \epsilon$, where ϵ is defined by $g_1(\lambda + \epsilon, \lambda) = 0$*
4. *$\Delta(\cdot, \lambda', \lambda'')$ is strictly single-crossing from above*

Proofs are in the appendix. The first three properties follow straightforwardly from the definition of the functions and the properties of g_1. The last property is more difficult to establish and is crucial to the existence result. It implies that, fixing the higher thresholds λ' and λ'', there is at most a single value for the lowest threshold λ that solves $\Delta(\lambda, \lambda', \lambda'') = 0$. We will therefore define[13]

$$\alpha^*(\lambda', \lambda'') = \begin{cases} \text{the unique solution to } \Delta(\lambda, \lambda', \lambda'') = 0 \text{ on } [0, \lambda') \text{ if a solution exists} \\ -\infty \text{ otherwise} \end{cases}$$

Since Δ is continuous in all three arguments and strictly single-crossing in its first argument, α^* is continuous in both its arguments.

3.2 Equilibrium Existence

We have two possible sets of sufficient conditions for equilibrium existence, depending on whether the number of available messages acts as a binding constraint on bidders:

- If we find an $m \geq 1$ and an $\alpha = (\alpha_0, \ldots, \alpha_{m-1})$ such that $\delta(\alpha_0, \alpha_1) = 0$, $\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = 0$ for each $k \in \{1, 2, \ldots, m-1\}$, and $\alpha_{m-1} \geq \overline{\alpha}$, then for any $M \geq m$, any τ satisfying the support conditions of Lemma 3 for that α is a symmetric equilibrium.
- If we find an $m \geq 1$ and an $\alpha = (\alpha_0, \ldots, \alpha_{m-1})$ such that $\delta(\alpha_0, \alpha_1) = 0$, $\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = 0$ for each $k \in \{1, 2, \ldots, m-1\}$, and $\alpha_{m-1} < \overline{\alpha}$, then for $M = m$, any τ satisfying the support conditions of Lemma 3 for that α is a symmetric equilibrium.

The first set of conditions covers the case where M is sufficiently large that m is determined endogenously by the condition that the highest type weakly prefers sending message m to a higher message. The second set covers the constrained case where the highest type would like to separate

[13] If $\lambda' = \lambda''$ and $\Delta(0, \lambda', \lambda'') \geq 0$, then $\Delta(\lambda, \lambda', \lambda'') = 0$ has two solutions, one with $\lambda \in [0, \lambda')$ and the other at $\lambda = \lambda'$. The latter corresponds to the trivial case where two messages give identical payoffs because neither is used by any other bidder. By the "no skipped messages" property established above, we know this will never happen in equilibrium; we therefore ignore this solution, and define $\alpha^*(\lambda', \lambda'')$ to be the unique solution on $[0, \lambda')$, and $-\infty$ if no such solution exists. Note that by property 3 of Lemma 4, $\alpha^*(\lambda', \lambda'') \leq \lambda' - \epsilon$ when it exists, so it can't "approach" λ' and then vanish discontinuously.

himself from opponents who are sending message m but cannot do so because there are no more messages available, i.e., $m = M$.

Our next task is to show that we can find such solutions α to the indifference equations. As noted in the proof of Lemma 2, no equilibrium exists with $m > 2\lceil\frac{1}{\epsilon}\rceil$. For fixed $m \in \{1, 2, \ldots, 2\lceil\frac{1}{\epsilon}\rceil\}$, the existence problem can be described as follows. We parameterize the solution $\alpha(t)$ to the $m-1$ indifference equations $\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = 0$ by setting $\alpha_m(t) = 1$, $\alpha_{m-1}(t) = 1-t$, and iteratively defining $\alpha_{m-k}(t) = \alpha^*(\alpha_{m-k+1}(t), \alpha_{m-k+2}(t))$. Any equilibrium with m opt-in messages must correspond to a value of t such that $(\alpha_{m-1}(t), \ldots, \alpha_0(t))$ all exist, and $\delta(\alpha_0(t), \alpha_1(t)) = 0$. Thus, varying t from 0 to 1 will reveal all symmetric equilibria with m messages, if any exist. An equilibrium will fail to exist if the solution to $\Delta(s, \alpha_{m-k+1}(t), \alpha_{m-k+2}(t)) = 0$ is negative at some $k \leq m$ (i.e., $\alpha_{m-k}(t) = -\infty$). The other possiblity is that a solution to the indifference conditions exists but m is less than M and the boundary condition $\alpha_{m-1} \geq \overline{\alpha}$ is not satisfied. Thus, the issue of existence reduces to the following question: is there an m such that a non-negative solution to the indifference and boundary conditions exists?

Our proof is by construction. We will show that that there exists an M^* such that (i) if $M \geq M^*$, then a solution to the indifference equations that satisfies the boundary condition exists when $m = M^*$, and (ii) if $M < M^*$, then a solution exists when $m = M$.

We first determine M^*. Let $\beta = (\beta_0, .., \beta_{M^*})$ denote thresholds working down "from the top" as follows:

1. $\beta_0 = 1$, $\beta_1 = 1$, and $\beta_2 = \overline{\alpha}$ as defined above. Here we set the second highest threshold equal to 1. (The highest threshold is equal to 1 by definition.) Given this choice, $\Delta(\beta_2, \beta_1, \beta_0) = 0$. (To see why, note that $g_1(1, [\beta_2, 1]) = 0$ by definition of $\beta_2 = \overline{\alpha}$, and that all other terms in $\Delta(\beta_2, \beta_1, \beta_0)$ contain positive powers of either $1 - F_S(\beta_1) = 0$ or $F_S(\beta_0) - F_S(\beta_1) = 0$. Thus, $\overline{\alpha} = \alpha^*(\beta_1, \beta_0)$.)

2. Define $\beta_3 = \alpha^*(\beta_2, \beta_1)$, $\beta_4 = \alpha^*(\beta_3, \beta_2)$, etc., as long as these solutions exist (and are non-negative). Since (by part 3 of Lemma 4) each solution requires $\beta_k - \beta_{k+1} \geq \epsilon$, there will be only a finite number. Let $\overline{M}$ be the index of the last solution, i.e., the number such that $\beta_{\overline{M}} \geq 0$ but $\Delta(0, \beta_{\overline{M}}, \beta_{\overline{M}-1}) < 0$ so $\Delta(s, \beta_{\overline{M}}, \beta_{\overline{M}-1}) = 0$ has no non-negative solution.

3. Calculate $\delta(\beta_{\overline{M}}, \beta_{\overline{M}-1})$. If it is strictly positive, set $M^* = \overline{M}$, and if it is weakly negative, set $M^* = \overline{M} - 1$.

Note from the construction that $M^* \geq 1$.[14] Also note that by construction, M^* is the largest integer such that β is a solution to $\{\Delta(\beta_{k+1}, \beta_k, \beta_{k-1}) = 0\}_{k=1,2,\ldots,M^*-1}$ and $\delta(\beta_{M^*}, \beta_{M^*-1}) > 0$. The argument is as follows. If $M^* = \overline{M}$, then $\delta(\beta_{M^*}, \beta_{M^*-1}) = \delta(\beta_{\overline{M}}, \beta_{\overline{M}-1}) > 0$. If $M^* = \overline{M} - 1$, then $\beta_{\overline{M}} = \beta_{M^*+1}$ exists, so

$$0 = \Delta(\beta_{M^*+1}, \beta_{M^*}, \beta_{M^*-1}) \leq \Delta(0, \beta_{M^*}, \beta_{M^*-1}),$$

[14] If $g_1(1, [0, 1]) < 0$, so that $\overline{\alpha} = -\infty$ and β_2 does not exist, then $\overline{M} = 1$, and $\delta(\beta_{\overline{M}}, \beta_{\overline{M}-1}) = \delta(\beta_1, \beta_0) = \delta(1, 1) = g_0(1) > 0$, so $M^* = \overline{M} = 1$. If $\underline{\alpha} \geq 0$ so β_2 does exist, then $M^* \geq \overline{M} - 1 \geq 1$.

where the inequality follows from condition 4 of Lemma 4. Applying condition 2 of 4 implies that $\delta(\beta_{M^*}, \beta_{M^*-1}) > 0$. Finally, note that $\delta(\beta_{M^*}, \beta_{M^*-1}) > 0$ implies $\beta_{M^*} > 0$.

With M^* defined in this way, we get the following result:

Theorem 1. *Fix the primitives of the environment – N, F_S, F_T, and c – and define M^* as above. Fix the number of available opt-in messages $M \in \mathbb{Z}^{++} \cup \{\infty\}$. Let $m = \min\{M^*, M\}$. A symmetric pure strategy equilibrium exists in which the set of messages used is $\{0, 1, 2, \ldots, m\}$.*

Proof. *The case of $m = 1$.* Sacrificing brevity for (hopefully) clarity, we will treat the $m = 1$ case first. We will show that an α_0 exists such that $\delta(\alpha_0, 1) = 0$. If $M^* > 1$, then $1 = m = \min\{M, M^*\}$ implies $M = m = 1$, and there is no additional condition that must be satisfied. If $M^* = 1$, we will show in addition that such an α_0 exists with $\alpha_0 \geq \overline{\alpha}$, so that $v_\tau(1, 2) \leq v_\tau(1, 1)$ holds as well in case $M > 1$. We can then define τ by $\tau(s_i) = 0$ for $s_i \leq \alpha_0$ and 1 for $s_i > \alpha_0$, and the sufficient conditions of Lemma 3 are all satisfied.

Plugging $\alpha_1 = 1$ into the statement for δ, note that

$$\delta(t, 1) \quad = \quad (F_S(t))^{N-1} g_0(t) + \left(\sum_{j=1}^{N-1} \binom{N-1}{j} (1 - F_S(t))^j (F_S(t))^{N-1-j} \frac{2}{j+1} \right) g_1(t, [t, 1])$$

which, as noted in Lemma 4, is continuous in t. At $t = 0$, the first term vanishes, as do the $j < N-1$ terms of the sum, but the $j = N-1$ term does not vanish, leaving $\delta(0, 1) = \frac{2}{N} g_1(0, [0, 1]) < 0$. On the other hand, at $t = 1$, the entire second term vanishes, and $\delta(1, 1) = g_0(1) > 0$. By continuity, then, $\delta(t, 1)$ must cross 0 at some value of t; we set α_0 equal to that solution.

If $M^* = 1$, we must show in addition that a solution exists with $\alpha_0 \geq \overline{\alpha}$. Note that if $M^* = 1$, then either the construction of $\beta_0, \beta_1, \ldots$ terminated at $\overline{M} = 1$, meaning β_2 did not exist; or it terminated at $\overline{M} = 2$, but $\delta(\beta_2, \beta_1) \leq 0$. In the former case, $\overline{\alpha} = -\infty$, meaning $g_1(1, [0, 1]) < 0$; so $g_1(1, [\alpha_0, 1]) < 0$ at any α_0. In the latter case, $\delta(\beta_2, \beta_1) = \delta(\overline{\alpha}, 1) \leq 0$; since $\delta(1, 1) > 0$, $\delta(t, 1) = 0$ must have a solution on $[\overline{\alpha}, 1]$. Thus, if $M^* = 1$, an $\alpha_0 \geq \overline{\alpha}$ exists satisfying $\delta(\alpha_0, 1) = 0$.

The case of $m = M$. The logic for the $m > 1$ case is similar, just with more moving parts. Suppose that $1 < M \leq M^*$, so $m = M$. We will construct a set of thresholds α satisfying $\{\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = 0\}_{k=1,2,\ldots,m-1}$ and $\delta(\alpha_0, \alpha_1) = 0$. Since the last condition of Lemma 3 is satisfied when $m = M$, any strategy τ satisfying the support conditions is then an equilibrium.

For $t \in [0, 1]$, define $\alpha_m(t) = 1$ and $\alpha_{m-1}(t) = 1 - t$.

Define $\alpha_{m-2}(t) = \alpha^*(\alpha_{m-1}(t), \alpha_m(t))$, and let $[0, \bar{t}_{m-2}]$ be the largest interval on which $\alpha_{m-2}(t)$ is well-defined and non-negative. By continuity, note that $\alpha_{m-2}(\bar{t}_{m-2}) = 0$. Also, $\alpha_{m-2}(0) = \beta_2 > 0$, so again by continuity, $\bar{t}_{m-2} > 0$.

Similarly, define $\alpha_{m-3}(t) = \alpha^*(\alpha_{m-2}(t), \alpha_{m-1}(t))$, and $[0, \bar{t}_{m-3}]$ the largest interval on which it is well-defined and non-negative. Since $\alpha_{m-3}(t) < \alpha_{m-2}(t) - \epsilon$, $\bar{t}_{m-3} < \bar{t}_{m-2}$; and since $\alpha_{m-3}(0) = \beta_3 > 0$, $\bar{t}_{m-3} > 0$.

Continue to iteratively define $\alpha_{k-1}(t) = \alpha^*(\alpha_k(t), \alpha_{k+1}(t))$, well-defined and continuous on $[0, \bar{t}_{k-1}]$ with $\alpha_{k-1}(\bar{t}_{k-1}) = 0$, down to $\alpha_0(t)$. Since $\alpha_0(0) = \beta_m \geq \beta_{M^*} > 0$, $\bar{t}_0 > 0$.

Now consider $\delta(\alpha_0(t), \alpha_1(t))$ as a function of t. It is well-defined on $[0, \bar{t}_0]$. It is positive at $t = 0$. ($\delta(\alpha_0(0), \alpha_1(0)) = \delta(\beta_m, \beta_{m-1})$; if $m = M^* = \overline{M}$, then $\delta(\beta_{\overline{M}}, \beta_{\overline{M}-1}) > 0$ from the construction of M^*; if $m < \overline{M}$, then β_{m+1} exists, which means $\Delta(0, \beta_m, \beta_{m-1}) \geq 0$ and therefore $\delta(\beta_m, \beta_{m-1}) > 0$.) It is negative at $t = \bar{t}_0$, because $\alpha_0(\bar{t}_0) = 0 < \alpha_1(\bar{t}_0)$ and $\delta(0, s') < 0$. And it is continuous, because $\alpha_0(t)$ and $\alpha_1(t)$ are both continuous on $[0, \bar{t}_0]$ and δ is continuous in both its arguments. Thus, at some t, it must equal zero; let t^* be a solution to $\delta(\alpha_0(t), \alpha_1(t)) = 0$.

Now, let $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1}) = (\alpha_0(t^*), \alpha_1(t^*), \ldots, \alpha_{m-1}(t^*))$. By construction, $\alpha_{k-1} = \alpha^*(\alpha_k, \alpha_{k+1})$, or $\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = 0$, for each $k = 1, 2, \ldots, m-1$. By construction, $\delta(\alpha_0, \alpha_1) = 0$. And by assumption, $m = M$. So if we define τ by $\tau(s_i) = 0$ for $s_i \leq \alpha_0$ and $\tau(s_i) = k$ for $s_i \in (\alpha_{k-1}, \alpha_k]$ ($k \in \{1, 2, \ldots, m\}$), all the conditions of Lemma 3 are satisfied and τ is therefore a symmetric equilibrium.

The case $m = M^* < M$. If $m = M^* < M$, we need to ensure that α also satisfies $v_\tau(1, m+1) \leq v_\tau(1, m)$, or $\alpha_{m-1} \geq \overline{\alpha}$. Since $\alpha_{m-1}(t) = 1 - t$, this is equivalent to $\alpha_{m-1} \geq \alpha_{m-1}(1 - \overline{\alpha})$. If we can show that the first solution t^* to $\delta(\alpha_0(t), \alpha_1(t)) = 0$ satisfies $t^* \leq 1 - \underline{\alpha} = 1 - \beta_2$, then $v_\tau(1, m+1) \leq v_\tau(1, m)$ is satisfied and that α leads to an equilibrium.

To see this, suppose first that $\bar{t}_0 \leq 1 - \beta_2$. In that case, $\alpha_0(t)$ hits 0 before $t = 1 - \beta_2$, so $\delta(\alpha_0(t), \alpha_1(t))$ must cross 0 at or before $t = 1 - \beta_2$, so the first solution t^* gives $\alpha_{m-1} \geq \beta_2 = \overline{\alpha}$ and we're done.

On the other hand, if $\bar{t}_0 > 1 - \beta_2$, then $\alpha_0(t)$ and $\alpha_1(t)$ are both well-defined and continuous on $[0, 1-\beta_2]$, but we will show that $\delta(\alpha_0(1-\beta_2), \alpha_1(1-\beta_2)) < 0$. To see why, recall that $m = M^*$. Since $\alpha_m(1-\beta_2) = 1 = \beta_1$ and $\alpha_{m-1}(1-\beta_2) = \beta_2$, $\alpha_k(1-\beta_2) = \beta_{m+1-k}$, so $\alpha_0(1-\beta_2) = \beta_{m+1} = \beta_{M^*+1}$ and $\alpha_1(1-\beta_2) = \beta_{M^*}$. Since β_{M^*+1} exists, $M^* = \overline{M} - 1$, which means (by the construction of M^*) that $\delta(\beta_{M^*+1}, \beta_{M^*}) = \delta(\beta_{\overline{M}}, \beta_{\overline{M}-1}) \leq 0$. (Otherwise M^* would have been set equal to $\overline{M}$ rather than $\overline{M} - 1$.) Thus, $\delta(\alpha_0(t), \alpha_1(t))$ is continuous on $[0, 1 - \beta_2]$ and weakly negative at $1 - \beta_2$, so the first time it crosses 0 must be at $t^* \leq 1 - \beta_2 = 1 - \overline{\alpha}$ as we needed. □

Theorem 1 establishes existence of at least one symmetric equilibrium in which $m = \min\{M, M^*\}$ opt-in messages are used. However, other symmetric equilibria can exist. In particular, for a given number of messages m, it may be possible to have multiple sets of thresholds $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1}) \neq (\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{m-1})$ satisfying the indifference conditions, and therefore leading to different equilibria. There could also be symmetric equilibria using different numbers of messages. The potential multiplicity can be an issue, but there is a sense in which it is a tractable problem. Once the primitives $\{F_S, F_T, N, c\}$ of the environment are known, finding all symmetric equilibria via exhaustive search is computationally feasible, as it reduces to a finite number of one-dimensional searches. Thus, once the environment is fully understood, calculation of all symmetric equilibria is possible, and uniqueness can be verified or falsified for the particular environment.

We have not yet imposed any restrictions on the shape of the density of types. In the next section, we will explore additional conditions of F_S which ensure that the symmetric equilibrium is *essentially unique,* that is, all symmetric equilibria use the same number of messages and the same set of thresholds, and therefore give the same outcome with probability 1.

3.3 When Is Equilibrium Essentially Unique

The sufficient conditions for uniqueness will require more structure on Δ and δ, the functions that give the change in payoffs from sending message $k+1$ rather than k. Basically, these functions must satisfy additional single-crossing properties, but in appropriately-transformed variables in "probability space" rather than the space of types. To present these conditions, we change variables and restate Δ as a function of the variables

$$\begin{aligned} x &= F_S(\alpha_k) - F_S(\alpha_{k-1}), \\ y &= F_S(\alpha_{k+1}) - F_S(\alpha_k), \text{ and} \\ z &= 1 - F_S(\alpha_{k+1}). \end{aligned}$$

Note that these are the probabilities, respectively, that a given bidder send message k, message $k+1$, and any message higher than $k+1$. We then define the analogue of the function Δ as

$$\tilde{\Delta}(x, y, z) \equiv \Delta(F_S^{-1}(1-x-y-z), F_S^{-1}(1-y-z), F_S^{-1}(1-z))$$

and obtain the equivalence relationships

$$\begin{gathered} \tilde{\Delta}(F_S(\alpha_k) - F_S(\alpha_{k-1}), F_S(\alpha_{k+1}) - F_S(\alpha_k), 1 - F_S(\alpha_{k+1})) = 0 \\ \Updownarrow \\ \Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = 0 \\ \Updownarrow \\ v_\tau(\alpha_k, k+1) = v_\tau(\alpha_k, k). \end{gathered}$$

Similarly, for $\tilde{\delta}$, we will let $y = F_S(\alpha_1) - F_S(\alpha_0)$ and $z = 1 - F_S(\alpha_1)$, and define

$$\tilde{\delta}(y, z) = \delta(F_S^{-1}(1-y-z), F_S^{-1}(1-z))$$

which yields the equivalence relations

$$\begin{gathered} \tilde{\delta}(F_S(\alpha_1) - F_S(\alpha_0), 1 - F_S(\alpha_1)) = 0 \\ \Updownarrow \\ \delta(\alpha_0, \alpha_1) = 0 \\ \Updownarrow \\ v_\tau(\alpha_0, 1) = 0. \end{gathered}$$

The sufficient conditions for uniqueness of symmetric equilibrium are that the new functions $\tilde{\Delta}$ and $\tilde{\delta}$ are single-crossing in each of their arguments. The point of the change-of-variables is to transform what is "held fixed" when one variable is changed. A change in any argument of Δ

consists of changing one threshold holding fixed the other two thresholds; a change in any argument of $\tilde{\Delta}$ corresponds to a change in the probability of one message, holding fixed the probabilities of certain other messages. The transformation is not necessary to prove single-crossing for the first argument because a decrease in α_{k-1} is essentially the same as an increase in the probability of message $k-1$. But it does make a difference for the second and third arguments. For example, when we increase α_k (the middle argument of Δ), we are simultaneously increasing the probability of message $k-1$ and decreasing the probability of message k. By contrast, when we change the probability of message k (the middle argument of $\tilde{\Delta}$), we are simultaneously decreasing α_{k-1} and α_k so that the probability weight on the interval $[\alpha_{k-1}, \alpha_k]$ remains constant.

Before we demonstrate what the new sufficient conditions buy us, we will show that they are not implausible, as they hold for at least one common distribution.

Lemma 5. *If F_S is the uniform distribution on $[0,1]$, then $\tilde{\Delta}(x,y,z)$ and $\tilde{\delta}(y,z)$ are both strictly single-crossing from above in y and z.*

The proof is in the appendix. While these single-crossing conditions may hold more generally, they are difficult to verify for other distributions. Thus, we think of the uniform case as a benchmark for what can be shown under stronger assumptions.

Recall (Lemma 4) that $\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ is strictly single-crossing from above in α_{k-1}; this is equivalent to $\tilde{\Delta}(x,y,z)$ being strictly single-crossing from below in x. Thus, we can define

$$x^*(y,z) = \begin{cases} \text{the unique solution to } \tilde{\Delta}(x,y,z) = 0 \text{ on } (0, 1-y-z] \text{ if a solution exists} \\ -\infty \text{ if } \tilde{\Delta}(1-y-z, y, z) < 0. \end{cases}$$

Here $x^*(y,z)$ is the analogue of $\alpha^*(\lambda', \lambda'')$ and, as in that case, it is continuous in both arguments. The single-crossing conditions of $\tilde{\Delta}$ in y and z impose some additional structure on x^* :

Lemma 6. *If $\tilde{\Delta}(x,y,z)$ is strictly single-crossing from above in y and z, then $x^*(y,z)$ is strictly increasing in both of its arguments.*

Proof. Let $y' > y$ and fix z; let $x = x^*(y,z)$ and $x' = x^*(y',z)$. If $x' \leq x$, then

$$\tilde{\Delta}(x,y',z) \quad < \quad \tilde{\Delta}(x,y,z) \quad = \quad 0 \quad = \quad \tilde{\Delta}(x',y',z) \quad \leq \quad \tilde{\Delta}(x,y',z)$$

The inner equalities follow from the definition of x^*, the inequalities hold by strict single-crossing, with the last one giving a contradiction. The identical argument works for z as well. □

The next lemma uses this result to establish the pieces we will need for equilibrium uniqueness. We rule out equilibria using more than M^* messages; nontrivial multiplicity of equilibrium using the same number $m \leq M^*$ of messages; and we establish that when $m < M^*$, the unique solution to the indifference conditions is only an equilibrium if $m = M$.

Lemma 7. *If $\tilde{\Delta}(x, y, z)$ and $\tilde{\delta}(y, z)$ are both strictly single-crossing from above in y and z, then:*

1. *For a given $m \leq M^*$, the solution α to the indifference conditions $\delta(\alpha_0, \alpha_1) = 0$ and $\{\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = 0\}_{k=1,2,\ldots,m-1}$ is unique.*

2. *For a given $m > M^*$, no solution exists.*

3. *For $m < M^*$, at the solution α to the indifference conditions, $\alpha_{m-1} < \overline{\alpha}$, so the final necessary condition of Lemma 3 is only satisfied if $M = m$.*

Proof. The common element of all three parts of the proof is the effect of the monotonicity of x^* shown in Lemma 5. Suppose that $(\alpha_0^1, \alpha_1^1, \ldots, \alpha_{m-1}^1, \alpha_m^1)$ and $(\alpha_0^2, \alpha_1^2, \ldots, \alpha_{m-1}^2, \alpha_m^2)$ both satisfy the indifference condition $\Delta(\alpha_{k-1}^i, \alpha_k^i, \alpha_{k+1}^i) = 0$ for $k = 1, 2, \ldots, m-1$, and suppose that $\alpha_m^1 = \alpha_m^2 = 1$ and $\alpha_{m-1}^1 > \alpha_{m-1}^2$. Then

$$\begin{array}{rcl}
F_S(\alpha_m^1) - F_S(\alpha_{m-1}^1) & < & F_S(\alpha_m^2) - F_S(\alpha_{m-1}^2) \\
& \downarrow & \\
x^*(F_S(\alpha_m^1) - F_S(\alpha_{m-1}^1), 1 - F_S(\alpha_m^1)) & < & x^*(F_S(\alpha_m^2) - F_S(\alpha_{m-1}^2), 1 - F_S(\alpha_m^2)) \\
& \downarrow & \\
F_S(\alpha_{m-1}^1) - F_S(\alpha_{m-2}^1) & < & F_S(\alpha_{m-1}^2) - F_S(\alpha_{m-2}^2) \\
& \downarrow & \\
\alpha_{m-2}^1 & > & \alpha_{m-2}^2
\end{array}$$

Repeating the same argument, we get $F_S(\alpha_{m-2}^1) - F_S(\alpha_{m-3}^1) < F_S(\alpha_{m-2}^2) - F_S(\alpha_{m-3}^2)$, and therefore $\alpha_{m-3}^1 > \alpha_{m-3}^2$. By induction, we can show that $\alpha_{m-1}^1 > \alpha_{m-1}^2$ implies $\alpha_k^1 > \alpha_k^2$ for every k, as well as $F_S(\alpha_{k+1}^1) - F_S(\alpha_k^1) < F_S(\alpha_{k+1}^2) - F_S(\alpha_k^2)$ for every k: a smaller top interval $[\alpha_{m-1}, 1]$ "propagates downward" to give narrower lower intervals $[\alpha_{k-1}, \alpha_k]$, as measured by the probability $F_S(\alpha_k) - F_S(\alpha_{k-1})$, and therefore higher thresholds α_k all the way down.

By decreasing the probability weight on both intervals $[\alpha_0, \alpha_1]$ and $[\alpha_1, 1]$, such a change also leads to a higher value of $\delta(\alpha_0, \alpha_1)$, at least in the single-crossing sense, via the single-crossing conditions assumed for $\tilde{\delta}$. This, combined with the definition of M^* in the construction above, allows us to prove each of the three parts of Lemma 7 in a straightforward way; the details of each case are in the appendix. □

The main economic results that follow from the additional single crossing conditions are given in Theorem 2. Part 1 states the uniqueness result, and part 2 establishes several comparative static results on the number of messages used in equilibrium.

Theorem 2. *Fix N, F_S, F_T, and c, and calculate M^* as above. Fix M the number of opt-in messages. If $\tilde{\Delta}(x, y, z)$ and $\tilde{\delta}(y, z)$ are both strictly single-crossing from above in y and z, then:*

1. *Symmetric equilibrium is essentially unique,*[15] *and exactly $m = \min\{M^*, M\}$ opt-in messages are used with positive probability*

2. *M^* is weakly decreasing in N and c, and as long as Assumption 2 continues to hold, M^* weakly increases when a mean-preserving spread is applied to F_T*

Proof. Part 1 follows directly from Lemma 7. If $m < \min\{M, M^*\}$ and α solves the indifference conditions, then $\alpha_{m-1} < \overline{\alpha}$ and $m < M$, so the last necessary condition of Lemma 3 fails. By definition, m can't be greater than M, and if $m > M^*$, no α solves the indifference conditions. And for $m = \min\{M, M^*\}$, there is exactly one α solving the indifference conditions, which is therefore the unique set of equilibrium thresholds.

Part 2 is proved in the appendix, but much of the intuition can be gained from considering the effect of a decrease in the entry cost c. Fixing the set of entrants, this increases the second-round payoffs g_0 and g_1. Since Δ is a weighted sum of $g_1(s_i, s_j)$ terms, a decrease in c therefore increases the value of $\Delta(\lambda, \lambda', \lambda'')$ at each point $(\lambda, \lambda', \lambda'')$. Given the single-crossing property in the first argument of Δ, this implies that given (λ', λ''), a decrease in c increases the solution to $\Delta(\,\cdot\,, \lambda', \lambda'') = 0$, $\alpha^*(\lambda', \lambda'')$. In essence, this means that as we construct our sub-intervals $[\alpha_{k-1}, \alpha_k]$ downward from the top, each sub-interval is narrower; since M^* is basically the count of how many sub-intervals fit into the type space $[0, 1]$, M^* increases with a decrease in c. The effects of F_T and N on M^* are somewhat less transparent, but basically analogous; see the appendix for proofs. □

The comparative static results above focus on M^*, which we think of informally as a proxy for how much information is conveyed through indicative bids, and therefore how effectively the seller can sort the highest-type bidders. Similar results could be proven about how the equilibrium partition α responds to changes in primitives. However, we have not been able to prove general results about how these changes affect equilibrium payoffs. In what follows, we explore this issue – the relationship between environmental primitives and equilibrium payoffs – through numerical examples, and then through limit results.

3.4 A Numerical Example

Next, we use a numerical example to illustrate the construction and properties of the symmetric equilibrium. Throughout the example, we will let F_S be the uniform distribution on $[0, 1]$, and F_T be the exponential distribution (i.e., $f_T(t) = \lambda e^{-\lambda t}$), with parameter $\lambda = 12$. Since F_S is uniform, the additional conditions of Theorem 2 hold, and therefore for a given M, symmetric equilibrium will be essentially unique. To make the numbers easier to read, in the tables we present, we multiply everything by 100.

[15] Given M, all symmetric equilibria correspond to the same m and same α, and therefore differ only in the strategies of the threshold types $\{\alpha_0, \alpha_1, \ldots, \alpha_{m-1}\}$ and possibly 1, all of which occur with probability 0.

For our baseline case, we let $N = 4$ and $c = 0.05$. Note that when F_T is the exponential distribution, $T_i - T_j$ follows a Laplace distribution, allowing us to calculate

$$g_1(s,s) \quad = \quad -c + \frac{1}{2\lambda} \quad = \quad -0.05 + \frac{1}{24} \quad < \quad 0$$

so Assumption 2 holds.

First, we calculate M^*. We begin by setting $\beta_0 = \beta_1 = 1$. We then set

$$\beta_2 \quad = \quad \alpha^*(1,1) \quad \approx \quad 0.9700$$

since $\Delta(0.9700, 1, 1) = 0$. We similarly calculate

$$\begin{aligned} \beta_3 &\quad = \quad \alpha^*(\beta_2, \beta_1) \quad \approx \quad 0.8826 \\ \beta_4 &\quad = \quad \alpha^*(\beta_3, \beta_2) \quad \approx \quad 0.6776 \\ \beta_5 &\quad = \quad \alpha^*(\beta_4, \beta_3) \quad \approx \quad 0.2651 \end{aligned}$$

There is no solution to $\Delta(s, \beta_5, \beta_4) = 0$, so β_6 does not exist and $\overline{M} = 5$. We then calculate $\delta(\beta_5, \beta_4) \approx -0.0147$; since this is negative, $M^* = \overline{M} - 1 = 4$.

Next, we show the construction of equilibrium thresholds $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ for the case where $m = 4$ opt-in messages are used. Figure 1 illustrates the construction, with t on the x-axis going from 0 to 1. The blue lines show $\alpha_4(t) = 1$, $\alpha_3(t) = 1 - t$, $\alpha_2(t) = \alpha^*(\alpha_3(t), \alpha_4(t))$, and so on. The red line shows the function $\delta(\alpha_0(t), \alpha_1(t))$. As expected, this is positive at $t = 0$, and crosses 0 once, at $t = t^* \approx 0.0197$. This gives the equilibrium thresholds $\alpha = (\alpha_0(t^*), \alpha_1(t^*), \alpha_2(t^*), \alpha_3(t^*)) \approx (0.3849, 0.7389, 0.9103, 0.9803)$.

We can repeat the same construction for each $m < M^*$. For $m = 3$, for example, we would set $\alpha_3(t) = 1$, $\alpha_2(t) = 1 - t$, and then $\alpha_1(t) = \alpha^*(\alpha_2(t), \alpha_3(t))$ and $\alpha_0(t) = \alpha^*(\alpha_1(t), \alpha_2(t))$. We would then calculate $\delta(\alpha_0(t), \alpha_1(t))$, and find that it is positive at $t = 0$ and crosses 0 once, at $t^* \approx 0.0888$. This gives thresholds $(\alpha_0(t^*), \alpha_1(t^*), \alpha_2(t^*)$ solving each of the indifference conditions for $m = 3$. We do likewise for $m = 2$ and $m = 1$. Table 1 shows the resulting thresholds, as well as the resulting equilibrium payoffs.

Note that "revenue," "bidder surplus," and "total surplus" all refer to *ex ante* levels, taken in expectation over the realized types of the bidders. We should emphasize that Table 1 does not show multiple equilibria of the same game, but unique equilibria of different games, depending on the seller's choice of M. For $m = M^* = 4$, $\alpha_3 \approx 0.9803 > 0.9700 = \overline{\alpha}$; so these thresholds form an equilibrium for any value of $M \geq 4$. For $m = 3$, however, $\alpha_{m-1} < 0.9700$, so the strategy with these thresholds is only an equilibrium when $M = 3$ – that is, when the seller limits the game to three opt-in messages. Likewise, the thresholds for $m = 2$ are only an equilibrium when only $M = 2$ opt-in messages are available, and the thresholds with $m = 1$ only when $M = 1$.

There are several things worth noting about this example.

Figure 1: Construction of α when $m = 4$

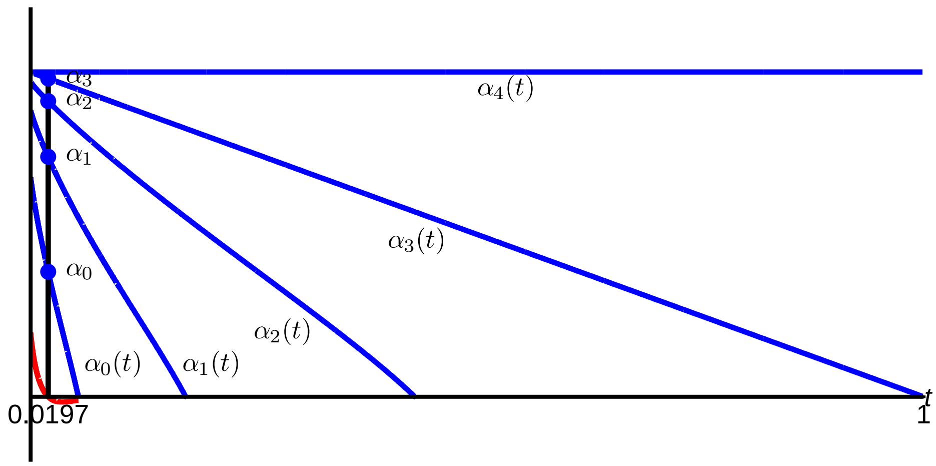

Table 1: "Base case" example: $N = 4$, $c = 0.05$, $\lambda = 12$, various numbers of messages available

Opt-in messages available (M)	1	2	3	≥ 4
α_4				100.00
α_3			100.00	98.03
α_2		100.00	91.12	91.03
α_1	100.00	74.58	73.90	73.89
α_0	40.90	38.58	38.50	38.49
Revenue	55.04	59.29	59.52	59.52
Bidder Surplus	21.43	19.61	19.51	19.50
Total Surplus	76.47	78.89	79.02	79.02

Revenue and total surplus are both increasing in the number of messages available. This is consistent across all parameterizations of the example that we have done, and we believe it to be true generally, although we have not been able to prove it. The intuition for revenue is clear: when M is higher, α_0 is lower, so the seller is less likely to get zero revenue due to fewer than two bidders opting in; and since the intervals $[\alpha_{k-1}, \alpha_k]$ are compressed with more messages, the bidders sort more effectively, and the selected bidders are more likely to be those with the two highest signals, increasing revenue. The intuition for total surplus is a bit less unambiguous: a lower participation threshold α_0 means more participation costs incurred, which offsets some of the efficiency gains from more reliably selecting the highest-type bidders. Still, in every example we've done, total surplus has been increasing in M.

Bidder surplus is decreasing in the number of messages available. Bidders benefit from a higher opt-in threshold, since they benefit disproportionately from being the only participant in the second round. Bidders also benefit from wider intervals (less perfect sorting), since they benefit from facing the "wrong" opponent (one with a lower type) when they advance. This seems to outweigh the cost of a bidder with a high type being less certain to advancing due to wider intervals. This too is consistent across all of our numerical examples.

When $m = M^*$, there is finer separation at the top of the type space. Table 1 shows that when $M \geq M^* = 4$, there is finer separation among high types than low types: the intervals over which bidder types pool are narrower at the top and get wider as types get lower,

$$1 - \alpha_3 \quad < \quad \alpha_3 - \alpha_2 \quad < \quad \alpha_2 - \alpha_1 \quad < \quad \alpha_1 - \alpha_0$$

In fact, this will always be true when F_S is the uniform distribution and $m = M^*$.[16] However, even in the uniform case, it is not always true when $m = M < M^*$.

As noted above, in this example, both revenue and total surplus are increasing in M, the number of opt-in messages allowed by the seller. We have not been able to prove this as a general result, but it consistently holds across every numerical example we have considered. Thus, it appears that the seller has no incentive to restrict the set of messages used, and is best served by letting the message set be the positive integers, in essence "selecting" the equilibrium with M^* opt-in messages. Thus, we think it is natural to focus on the "M^*-message equilibrium," which we know always exists when M is large enough to not be a binding constraint.

Of course, as the primitives of the environment change, M^* changes; and since M^* takes only integer values, any change in M^* will be discontinuous. Thus, in some sense, equilibrium changes discontinuously as primitives cross certain thresholds. However, perhaps surprisingly, the payoff-relevant parts of the equilibrium do not change discontinuously at these thresholds. This is because, as primitives approach a level at which M^* changes, the top threshold α_{M^*-1} approaches 1, and the top interval $[\alpha_{M^*-1}, 1]$ becomes so small as to be payoff-irrelevant, and thus has no meaningful effect when it finally disappears. We can illustrate this using our existing example, examining the changes in equilibrium outcomes as c increases. There is a value of entry costs c, just above 0.0594, at which M^* drops from 4 to 3. Table 2 shows the equilibrium thresholds, and equilibrium payoffs, at various values of c close to this threshold. We can see that the lower thresholds $(\alpha_0, \alpha_1, \alpha_2)$, and expected payoffs, all change continuously as c crosses the threshold and M^* drops.

[16] More generally, $F_S(\alpha_k) - F_S(\alpha_{k-1})$ will be decreasing in k whenever $m = M^*$ and the additional single-crossing conditions used in Lemma 7 hold. Given the monotonicity of x^* when these conditions hold, the result is easily proved by induction when $1 - F_S(\alpha_{m-1}) < F_S(\alpha_{m-1}) - F_S(\alpha_{m-2})$ holds as the base case. When $m = M^*$, $1 - F_S(\alpha_{m-1}) \leq x^*(0,0)$ (this is equivalent to $\alpha_{m-1} \geq \overline{\alpha}$, or $g_1(1, [\alpha_{m-1}, 1]) \leq 0$), while $F_S(\alpha_{m-1}) - F_S(\alpha_{m-2}) = x^*(1 - F_S(\alpha_{m-1}), 0) > x^*(0,0)$. However, when $m < M^*$, $1 - F_S(\alpha_{m-1}) > x^*(0,0)$; so the base case, and therefore the result, may not hold when $m < M^*$.

Table 2: Change in equilibrium as c changes, with $N = 4$, $\lambda = 12$, and $m = M^*$

Entry cost (c)	0.0593	0.0594	0.0595	0.0596
M^*	4	4	3	3
α_4	100.00	100.00		
α_3	99.98	*100.00	100.00	100.00
α_2	94.05	94.08	94.11	94.14
α_1	77.54	77.57	77.61	77.65
α_0	41.14	41.17	41.20	41.22
Revenue	58.03	58.02	58.00	57.98
Bidder Surplus	19.13	19.12	19.12	19.12
Total Surplus	77.16	77.14	77.12	77.10

* When $c = 0.0594$, $\alpha_3 \approx 99.9985$, which rounds to 100.00.

4 Welfare and Revenue

In this section, we evaluate the performance of the indicative bidding mechanism. We first consider whether participation levels within the mechanism are optimal and, if not, whether the seller can use reserve prices or entry subsidies to improve efficiency and/or revenues. We then compare the payoffs of the indicative bidding mechanism to that of an ascending auction in which the seller does not restrict entry. Throughout this section, we will focus on the "M^*-message equilibrium" – the equilibrium where the seller does not artificially restrict the message space.

4.1 Optimal Participation

A reserve price reduces the expected payoff from advancing, which in turn has the effect of shifting up the partition. Subsidies to entrants have the opposite effect. These shifts in the partition make it difficult to sign the impact of these policies on expected revenues and total surplus. The main reason is that they affect the amount of sorting that occurs and hence the likelihood that the seller selects the "wrong" bidders. We will refer to this cost as the *selection* effect. However, as we shall show below, when N is large, the selection effect becomes small, and this fact allows us to obtain general results.

We use our example from before to illustrate the issue. Table 3 reports how equilibrium thresholds and payoffs change as N increases.

Two features of the table stand out. One is that M^*, the number of opt-in messages used in equilibrium, is weakly decreasing in N; it falls from 5 to 1 as N increases from 3 to 200. Thus, sorting grows coarser as N increases and, when N is sufficiently large, the only two messages used in equilibrium are opt in and opt out. The second is that α_0, the probability of opting out, increases with N; it rises from .296 when $N = 3$ to .974 when $N = 200$. Recall that a bidder with type α_0 earns a positive expected payoff only if he is the sole bidder selected. Since the likelihood of this event falls with more bidders, the probability of an individual bidder opting in decreases with N.

Table 3: Effect on "unrestricted" equilibrium of variation in N, with $c = 0.05$ and $\lambda = 12$

Potential Bidders (N)	3	4	5	7	10	20	50	200
M^*	5	4	4	4	3	3	2	1
α_5	100.00							
α_4	99.94	100.00	100.00	100.00				
α_3	96.79	98.03	98.78	99.59	100.00	100.00		
α_2	87.80	91.03	93.11	95.59	97.45	99.47	100.00	
α_1	67.36	73.89	78.43	84.27	89.09	95.06	98.59	100.00
α_0	29.59	38.49	45.48	55.66	65.40	79.88	90.82	97.40
Revenue	50.50	59.52	65.63	73.42	79.98	88.77	95.20	99.50
Bidder Surplus	23.79	19.50	16.63	12.98	9.95	5.96	3.12	1.24
Total Surplus	74.29	79.02	82.25	86.40	89.92	94.73	98.32	100.74

Thus, in the example, when N is large, most bidders opt out and the only bidders opting in are the very high types. It is in this sense that the selection effect can get small as N becomes large.

The other key thing to note is that as N increases, the individual participation rate gets small, but the aggregate participation level increases. We refer to this as the *participation effect.* For each N, the number of bidders opting in is distributed binomial with parameter $1 - F_S(\alpha_0)$. The values of α_0 reported in Table 4 imply that both the probability of a sale (i.e., at least one bidder opts in) and the probability that the seller earns revenue from the sale (i.e., at least two bidders opt in) increase with N. The former is good for efficiency and the latter for revenue. The participation effect is sufficiently strong that both revenue and total surplus increase with N, while bidder surplus declines with N.

We turn next to generalizing the results of the example. Lemma 8 establishes that, as the number of potential bidders gets large, only messages 0 and 1 are played in any symmetric equilibrium of the indicative bidding mechanism.

Lemma 8. *Fix F_S, F_T, and c. There exists an N^* such that if $N > N^*$, $m = 1$ in any symmetric equilibrium.*

The proof is in the appendix. For intuition, suppose an equilibrium existed with $m = 2$, so that $\alpha_0 < \alpha_1 < 1 = \alpha_2$. The threshold type α_1 is indifferent between sending message 1 or message 2: he loses on average against types who send message 2, and therefore must gain on average against the types that send message 1. This means $\alpha_1 - \alpha_0 > \epsilon$, so α_0 is bounded away from 1. Consider, then, a bidder with type α_0, who is indifferent between opting out and sending message 1. If he sends message 1, he makes money when he is the only bidder to opt in, and loses money when he is not the only bidder to opt in but he gets selected anyway. As N grows, the probabilities of both these events go to 0; but we show that the former probability goes to zero faster than the latter, generating a contradiction. Note that we do not impose any restrictions on F_S other than the conditions of Assumption 1. Thus, for large N, Lemma 8 rules out symmetric equilibria with different numbers of messages; however, multiple equilibria with $m = 1$ and different values of α_0 remain possible.

When N is sufficiently large, the equilibrium is characterized by the NITS condition and a single indifference condition, $\delta(\alpha_0, 1) = 0$. The next lemma shows that, as N goes to infinity, these conditions imply that the participation threshold α_0 goes to 1, but at a rate such that the probability that no one bids converges to a limit that lies between 0 and 1. We derive an "almost" closed form expression for this limit that can be computed from primitives.

Lemma 9. *Fix F_S, F_T, and c. Then, as $N \to \infty$, $\alpha_0(N) \to 1$ and*

$$\lim_{N\to\infty} F_S(\alpha_0(N))^N \quad = \quad Q^{-1}\left(\frac{-2g_1(1,1)}{g_0(1) - 2g_1(1,1)}\right)$$

where $Q : [0,1] \to [0,1]$ is defined as

$$Q(\lambda) = \frac{-\lambda \ln(\lambda)}{1-\lambda}.$$

for $\lambda \in (0,1)$.

The calculations are in the appendix. Lemma 9 establishes that, in the limit, all bidders who opt in have signals approximately equal to 1, and the probability that all bidders opt out is bounded away from 0 and 1. Thus, as N goes to infinity, the selection effect vanishes, but the participation effect does not.

Equilibrium payoffs depend on the probabilities of three events: the probability of no sale, probability of a sale with zero revenue, and the probability of a sale with revenue. Lemma 9 gives the limit probabilitiy of the first event, which we label

$$\phi \quad \equiv \quad Q^{-1}\left(\frac{-2g_1(1,1)}{g_0(1) - 2g_1(1,1)}\right).$$

This can be used to derive the other two limit probabilities. Since

$$\lim_{N\to\infty} N(1 - F_S(\alpha_0)) \quad = \quad \lim_{N\to\infty} (-\ln(F_S(\alpha_0)^N)) \quad = \quad -\ln(\phi),$$

the probability that only one bidder opts in is

$$\lim_{N\to\infty} N(1 - F_S(\alpha_0))F_S(\alpha_0)^{N-1} = -\phi \ln(\phi),$$

and the probability that two or more bidders opt in is $1 - \phi + \phi \ln(\phi)$. These results basically follow from the fact that, fixing the primitives, as $N \to \infty$, the equilibrium distribution of the number of bidders opting in converges to a Poisson distribution with parameter $-\ln \phi$.

The next lemma gives closed form expressions for equilibrium payoffs in the limit.

Lemma 10. *Fix F_S, F_T, and c. In the limit as $N \to \infty$, expected revenue is*

$$R = (1 - \phi + \phi \ln(\phi))(g_0(1) - g_1(1,1)),$$

total surplus is

$$W = -\phi \ln(\phi) g_0(1) + (1 - \phi + \phi \ln(\phi))(g_0(1) + g_1(1,1)),$$

and bidder surplus is 0.

Proof. The proof is informative, so we include it in the text. Define

$$R(s_1, s_2) = E_{T_1,T_2}[\min\{s_1 + T_1, s_2 + T_2\}]$$

as the seller's expected revenue conditional on selecting two bidders whose types are (s_1, s_2). In the limit, all bidders who opt in have signals $s_i \approx 1$, so revenue to the seller is $R(1,1)$ whenever two or more bidders opt in, and 0 otherwise. Similarly, in the limit, total surplus is $g_0(1)$ when one bidder opts in, and $R(1,1) + 2g_1(1,1)$ when two or more bidders opt in. The probabilities of these events are $-\phi \ln(\phi)$ and $(1 - \phi + \phi \ln(\phi))$ respectively. Therefore, to establish the statements for R and W given above, we need to show that $R(1,1) = g_0(1) - g_1(1,1)$.

The argument exploits the fact that a bidder's entry decision in a second-price auction does not impose a net externality on the other players in the game.[17] Suppose two bidders $i = 1, 2$ are selected who have ex post values $v_i = s_i + t_i$. Then, regardless of bidder 1's bidding decision, the sum of the seller's revenue and bidder 2's payoff is $v_2 - c$: if bidder 1 does not to bid, then revenue is 0 and bidder 2's payoff is $v_2 - c$; if bidder 1 bids $b_1 < v_2$, then revenue is b_1 and bidder 2's payoff is $v_2 - b_1 - c$; and finally, if bidder 1 bids $b_1 > v_2$, then revenue is v_2 and bidder 2's payoff is $-c$. Thus, the net payoff to the other players in the game is independent of bidder 1's bidding decision for every realization of the signals. Taking expectations over T_1 and T_2, it then follows that the net expected payoff to the other players is also independent of whether bidder 1 opts in or out, or $R(s_1, s_2) + g_1(s_2, s_1) = g_0(s_2)$; letting $s_1 = s_2 = 1$, this means $R(1,1) + g_1(1,1) = g_0(1)$.

To derive the limit of bidder surplus (BS), we subtract limit revenue from limit total surplus:

$$\begin{aligned} BS = W - R &= -\phi \ln(\phi) g_0(1) + (1 - \phi + \phi \ln(\phi))(2g_1(1,1)) \\ &= -\phi \ln(\phi)(g_0(1) - 2g_1(1,1)) - (1 - \phi)(-2g_1(1,1)) \\ &\propto \frac{-\phi \ln(\phi)}{1 - \phi} - \frac{-2g_1(1,1)}{g_0(1) - 2g_1(1,1)} \\ &= 0 \end{aligned}$$

where the final equality follows from Lemma 9. □

We can now state our main result on the efficiency of participation levels in the indicative

[17] Levin and Smith (1994) use this argument to show that the entry level in a private value English auction with costly and unrestricted entry is efficient. By analogous arguments, with private values, a VCG mechanism leads to efficient amount of information acquisition (Bergemann and Valimaki (2002)), and the efficient amount of value-enhancing ex-ante investment (Arozamena and Cantillon (2004)).

bidding mechanism.

Theorem 3. *Fix F_S, F_T, and c. When N is sufficiently large, participation in any symmetric equilibrium is below the socially efficient level.*

The proof consists of showing that, holding fixed g_0 and g_1, W is strictly decreasing in ϕ at the equilibirum level.[18] The calculations are in the appendix. The inefficiency result may seem surprising since, in the limit, the second stage of the indicative bidding mechanism is essentially the same as in Levin and Smith (1994) and they show that entry in their model is efficient. The difference is that, in Levin and Smith, participation is equivalent to entry whereas, in the indicative bidding mechanism, participation only makes a bidder eligible to enter. Thus, in our setting, besides possibly being selected and then possibly winning the auction or altering the price, a bidder's participation decision has another effect: he may "displace" another bidder, who would have been selected in his absence. When N is large enough for Lemma 8 to hold, then this "displaced" bidder would have had a negative expected payoff in the second round, and so the externality from a bidder's participation is positive, which implies that participation is below the efficient level.

We have shown that, as N gets large, there remains a nonvanishing chance that exactly one bidder opts in, and buys the asset at price 0. Thus, it might seem that, even in the limit, the seller could increase revenue by using a reserve price. However, Theorem 3 implies that this intuition is wrong. When N is large, bidder surplus is effectively zero, so ex ante, the seller is capturing all of the surplus. A reserve price would further depress participation, leading to lower total surplus and lower revenue. On the other hand, subsidizing participation would increase participation, thereby increasing total surplus, and because N is large, the seller would capture this additional surplus:

Corollary 1. *Fix F_S, F_T, and c. When N is sufficiently large,*

1. *a small entry subsidy would strictly increase both revenue and total surplus;*

2. *any positive reserve price would strictly reduce both revenue and total surplus.*

Proof. If we consider a reserve price that only binds when there is just one entrant, or a subsidy paid to entering bidders, these do not change total surplus ex post; thus, the only effect on total surplus is through the participation level. Since participation is below the socially efficient level when N is large, a subsidy would increase it, increasing total surplus, while a reserve price would decrease it, decreasing total surplus. Since $W'(\phi)$ is *strictly* negative at the equilibrium participation level, these effects are "first-order", and therefore dominate any changes in bidder surplus for large but finite N, so the revenue effect follows from the welfare effect. A reserve price large enough to preclude sales would reduce ex post efficiency directly as well as depressing participation, with both effects reducing total surplus and therefore revenue. □

[18] For simplicity, we prove the result using the limit payoffs. However, it holds more generally when N is finite but sufficiently large that $M^* = 1$.

4.2 Comparison to Auctions with Unrestricted Entry

A natural benchmark for evaluating the performance of the indicative bidding game is the ascending auction in which the seller does not ration entry, but allows bidders to choose whether or not to enter. We assume that the timing is the same: bidders learn s_i; they then decide simultaneously whether to enter; and those bidders who choose to enter incur the cost c, learn their second signal t_i, and bid in the auction. This is a hybrid version of the entry games studied by Levin and Smith (1994) and by Samuelson (1985). In Levin and Smith, there is no s_i and so bidders have no private information when deciding whether to enter and play a symmetric mixed strategy equilibrium; and in Samuelson, there is no t_i, so bidders know their valuations when they decide whether to enter and play a symmetric cutoff equilibrium.

For $n \geq 1$, we extend our earlier notation by defining $g_n(s_i, [\underline{s}, 1])$ to be the expected payoff to a bidder with type s_i from entering an $n+1$-bidder English auction against n opponents with types that are *i.i.d.* draws from the distribution F_S truncated to the interval $[\underline{s}, 1]$. This means that

$$g_n(s_1, [\underline{s}, 1]) = -c + E_{S_{-1} \in [\underline{s},1]^n} E_{T_1, T_{-1}} \max \left\{ 0, s_1 + T_1 - \max_{j>1} \{s_j + T_j\} \right\}$$

where the outer expectation is taken in each variable with respect to the prior distribution F_S, updated for each $s_j \in [\underline{s}, 1]$. It is not hard to show in our setting (with Assumption 1) that the auction with unrestricted entry has an essentially-unique symmetric equilibrium, where bidders enter when their type $s_i \in (\gamma, 1]$, where γ solves

$$0 = \sum_{n=0}^{N-1} \binom{N-1}{n} F_S(\gamma)^{N-1-n} (1 - F_S(\gamma))^n g_n(\gamma, [\gamma, 1]) \ .$$

We will refer to γ as the entry threshold. As in the indicative bidding game, when $N \to \infty$, $\gamma \to 1$, $F_S(\gamma)^N \to q \in (0, 1)$, and the distribution of the number of bidders who enter converges to a Poisson distribution with parameter $-\ln q$.

The main tradeoff between the two mechanisms is the cost of not selling the asset versus the cost of not selecting the bidders with the highest types. In the auction with unrestricted entry, too many bidders can enter. By capping the number of entrants at two, the indicative bidding mechanism reduces the risk that bidders will have to pay entry costs when they are likely to lose the auction. As a result, their willingness to participate increases, and the likelihood of the asset being sold rises. The potential cost of using the indicative bidding mechanism is that ties can occur and, in randomly choosing which bidders to advance, the seller sometimes ends up not selecting the bidders with the two highest valuations. By contrast, the auction with unrestricted entry always allocates the asset to the bidder with the highest valuation, at a price equal to the second-highest valuation.

To illustrate these tradeoffs, we return to our example from before, with F_S the uniform distribution on $[0, 1]$, F_T the exponential distribution with parameter $\lambda = 12$, and $c = 0.05$. We can calculate the equilibrium entry threshold γ for the unrestricted auction, and based on that, calculate expected equilibrium payoffs. Table 4 compares these payoffs to our earlier results on the payoffs under indicative bidding for various values of N. Note that the results for unrestricted entry were calculated by simulation, and are subject to a small amount of simulation error.

In this example, α_0 is always less than γ, that is, the opt-in threshold for indicative bidding is always lower than the entry threshold for the auction with unrestricted entry. This implies that, for each N, the number of bidders opting in under indicative bidding first-order stochastically dominates the number of bidders entering under unrestricted entry. Hence, the probability of a sale and the probability of two or more bidders is higher under indicative bidding. The former is good for efficiency and the latter for revenue. Of course, there is still a tradeoff: since entry into the second round of the indicative bidding mechanism is capped at two bidders, the expected revenue conditional on at least two entrants is lower than it would be with unrestricted entry, and the seller may select the "wrong" bidders, reducing both efficiency and revenues. Nonetheless, the participation effect appears to consistently dominate the selection effect: for each N, revenue, bidder surplus and total surplus are all higher under indicative bidding.

Table 4: Comparing indicative bidding with $m = M^*$ to unrestricted entry for various N

Indicative Bidding

Potential Bidders (N)	3	4	5	7	10	20	50	200
α_0 ("opt-in" threshold)	29.59	38.49	45.48	55.66	65.40	79.88	90.82	97.40
Revenue	50.50	59.52	65.63	73.42	79.98	88.77	95.20	99.50
Bidder Surplus	23.79	19.50	16.63	12.98	9.95	5.96	3.12	1.24
Total Surplus	74.29	79.02	82.25	86.40	89.92	94.73	98.32	100.74

Unrestricted Entry

Potential Bidders (N)	3	4	5	7	10	20	50	200
γ (entry threshold)	33.56	44.65	52.68	63.33	72.53	84.84	93.39	98.25
Revenue	50.04	58.24	63.58	70.15	75.65	83.22	89.27	93.28
Bidder Surplus	22.91	18.40	15.55	12.15	9.39	5.76	2.87	0.80
Total Surplus	72.95	76.63	79.12	82.30	85.03	88.98	92.14	94.08

The dominance of the indicative bidding mechansim depends upon the message space being unconstrained. A comparison with the payoffs reported in Table 1 for $N = 4$ reveals that the revenue and surplus ranking between the two mechanisms reverses when the number of opt-in messages is constrained to be one. The gain from coordinating entry through a cap on entrants is lower at $m = 1$ than at $m = M^*$, because coaser sorting of bidder types increases the expected cost of participating and raises the participation threshold. Thus, α_0 increases from 38.49 to 40.90, but it is still less than $\gamma = 44.65$ so the participation effect still favors the indicative bidding mechanism. However, the cost of selecting the "wrong" bidders is much higher at $m = 1$ than at $m = M^*$, since the likelihood of ties increases as fewer messages are used. In this case, the selection effect dominates the participation effect, at least in terms of revenue and total surplus.

In all the numerical examples we have run, indicative bidding with $m = M^*$ consistently Paretodominates unrestricted entry. Although we have not found a counterexample (a set of primitives where unrestricted entry outperforms indicative bidding with M^* messages), we have not been able to prove that this will always be true. However, as N gets large, all bidders who send opt-in messages have signals that are approximately equal to 1, so the cost of advancing the "wrong" bidders gets very small. This leads to the following result:

Theorem 4. *When N is sufficiently large, the indicative bidding mechanism gives strictly more participation, higher revenue, and greater bidder surplus than an auction with unrestricted entry.*

The proof is in the appendix. The intuition for the participation result is relatively straightforward. In either mechanism, when N is large, bidders are trading off the benefit of potentially entering as the lone second-round bidder (and getting payoff $g_0(1) > 0$), versus the cost of potentially entering along with other bidders (and getting payoff $g_1(1,1) < 0$ or worse). In the indicative bidding mechanism, however, the latter risk is partly mitigated: in the event that a bidder opts in and would not be alone, there is some chance he will not be selected for the second round, and even if he is, he'll face at most one opponent. Thus, in order for indifference to hold at the equilibrium threshold in both games, the participation threshold α_0 in the indicative bidding game must be lower (induce more participation) than the entry threshold γ in the unrestricted auction. (Theorem 4 establishes that this holds for large N, but it can be shown to hold for small N as well.) We then show that the indicative bidding game yields higher total surplus than the unrestricted auction, for two reasons: first, because when more than two bidders opt in, only two advance to the second round, so total second-round expected payoffs are higher; and second, because it induces more participation, and our previous Theorem showed that increasing the participation level increases total surplus. Since the limiting bidder surplus is 0 in both mechanisms, the revenue result follows. Finally, while bidder surplus is 0 in the limit in both games, we show that it is of order $\frac{1}{N}$, and calculate the leading term, and prove that it is greater in the indicative bidding mechanism.

In the limit, as N gets large, the auction with unrestricted entry converges to the game studied by Levin and Smith (1994), where bidders have no initial private information. Levin and Smith note that equilibrium entry in their setting is at the efficient level *for uncoordinated entry*, but that bidders' failure to coordinate still creates an inefficiency. That is, the fact that sometimes nobody enters, and sometimes too many bidders enter, leads to outcomes which could be improved by capping entry at a fixed number. Indicative bidding gives one way to do this; Theorem 4 shows that it does indeed dominate unrestricted entry.

5 Extension: Smaller Entry Costs and More Bidders

In this section, we discuss relaxing two important limitations of the primary model. The first limitation is the restriction that only two bidders advance to the second round. The second limitation is the assumption that entry costs are sufficiently large that bidders do not want to advance against an opponent with the same or higher type. It is natural to think about relaxing both of these assumptions together.

For technical reasons, a complete analysis of the more general model is beyond the scope of this paper. Recall that the indifference condition characterizing each threshold can be expressed as

$$\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) \quad = \quad c_1 g_1(\alpha_k, [\alpha_{k-1}, \alpha_k]) \quad + \quad c_2 g_1(\alpha_k, [\alpha_k, \alpha_{k+1}]) \quad + \quad c_3 g_1(\alpha_k, [\alpha_{k+1}, 1]).$$

When $g_1(s, s) < 0$, each of the terms has unambiguous sign: the last two are always negative, so whenever $\Delta \approx 0$, the first must be positive. Knowing these signs allows us to establish single-crossing properties, leading to the constructive proof of equilibrium existence and, under stronger assumptions, uniqueness. If on the other hand $g_1(s, s) > 0$, then the middle term would have indeterminate sign, making it difficult to prove the same single-crossing properties for Δ. Further, with $g_1(1, 1) > 0$, it would not necessarily be true that $\Delta(\alpha_k, \alpha_k, \alpha_{k+1}) < 0$, and therefore our construction of thresholds sequentially "from the top" would not be guaranteed to work.[19] Finally, if more bidders advance, say three bidders, then the indifference condition analogous to Δ would consist of six terms (since, for example, the expected payoff from advancing against one opponent with type in (α_{k-1}, α_k) and one in $(\alpha_{k+1}, 1)$ would have to be represented), again making the single-crossing properties much harder to prove. This prevents us from using the techniques in the earlier sections to prove existence and characterize equilibrium.

However, if we consider the limit as N gets large and assume a symmetric equilibrium exists, we can characterize limit outcomes in any symmetric equilibrium. Recall that g_n was defined above by

$$g_n(s_1, [\underline{s}, 1]) = -c + E_{S_{-1} \in [\underline{s}, 1]^n} E_{T_1, T_{-1}} \max \left\{ 0, s_1 + t_1 - \max_{j>1} \{s_j + t_j\} \right\}$$

as the expected payoff to a bidder with type s_1 in an auction against n opponents, with types drawn from the prior distribution restricted to the interval $[\underline{s}, 1]$. The last two parts of Assumption 1 were that $g_0(1, \cdot) > 0 > g_1(1, 1)$. We now relax this assumption. Assume the generic case where there is no n with $g_n(1, 1)$ exactly equal to 0, and define n^* as the unique value such that

$$g_{n^*-1}(1, 1) \quad > \quad 0 \quad > \quad g_{n^*}(1, 1)$$

Note that $n^* - 1$ is the maximal number of "strong" bidders that another strong bidder would be willing to compete against. (A bidder with type $s_i = 1$ earns positive payoffs in an n^*-bidder auction, regardless of the opponents he faces, but earns negative payoffs in an $n^* + 1$-bidder auction if his opponents are sufficiently strong.) n^* is also the efficient number of entrants if all potential entrants have signals equal to 1; in our baseline model, n^* was one.

Now modify the indicative bidding game such that the n bidders sending the highest messages advance to the second round, with ties broken randomly. As N gets large, there are two cases to

[19] We conjecture that we might be able to re-express the indifference condition as a function of $F_S(\alpha_{k-1})$, $F_S(\alpha_k) - F_S(\alpha_{k-1})$, and $F_S(\alpha_{k+1}) - F_S(\alpha_k)$, prove analogous single-crossing conditions on the resulting function, and then construct equilibrium "up from the bottom" rather than "down from the top". With $g_1(s, s) > 0$, there is also the additional question of whether the opt-out message is used in equilibrium – if the bottom interval is narrow enough, even bidders with types $s_i \approx 0$ would be willing to opt in and pool with the bidders in (α_0, α_1), so it's possible that $\alpha_0 = 0$ and message 0 is not used in equilibrium.

consider.

If $g_{n-1}(1,1) < 0$ – or equivalently, $n > n^*$ – then we are in the analogue of our baseline model from earlier: a bidder only wants to advance if he has an ex-ante advantage over some or all of his competitors or fewer than the maximum opted in. In that case, as N grows, the results will look familiar: regardless of M, only messages 0 and 1 will be used in equilibrium, and the threshold for opting in will go to 1.

On the other hand, if $g_{n-1}(1,1) > 0$, or $n \leq n^*$, then bidders with types close to 1 prefer entry regardless of their opponents. In that case, if M were infinite, symmetric equilibrium would fail to exist. If M is finite, then a nonvanishing interval of bidders with types $s_i \in (\alpha_{M-1}, 1]$ will pool on the highest available message M. While we cannot fully characterize equilibrium play in this case, we can still calculate revenue and bidder surplus in any symmetric equilibrium, since in the limit, they depend only on the cutoff α_{M-1}.

Suppose that for $n \leq n^*$ (so $g_n(1,1) > 0$), $g_n(s, [s,1]) = 0$ is increasing in s.[20] Let $\underline{\alpha}^{(n)}$ denote the unique solution to

$$g_n(\underline{\alpha}^{(n)}, [\underline{\alpha}^{(n)}, 1]) = 0$$

or 0 when no such solution exists (when $g_n(0, [0,1]) > 0$).

Theorem 5. *Consider the modified indicative-bidding game where $n \geq 2$ bidders advance to the second round. Assume M is finite. In any symmetric equilibrium:*

1. *If $n \leq n^*$, all bidders with types $s_i \geq \underline{\alpha}^{(n)}$ send the highest available message M. As N grows, the set of bidder types sending message M goes to $[\underline{\alpha}^{(n)}, 1]$.*

2. *If $n > n^*$, when N is large, messages 0 and 1 are the only messages used. As N grows, the opt-in threshold α_0 goes to 1, and $F_S(\alpha_0)^N$ goes to an interior limit in $(0,1)$*

Equilibrium looks very different, depending on whether n is above or below n^*. When $n \leq n^*$, there is no chance of underparticipation – with probability 1, more than n bidders will opt in. However, there is inefficiency due to the wrong bidders being chosen: with N large, there are guaranteed to be bidders with types very close to 1, but the bidders who advance are chosen at random among all bidders with types in $[\underline{\alpha}^{(n)}, 1]$. When $n > n^*$, on the other hand, there is no problem with the wrong bidders entering: all bidders advancing have types very close to 1. However, there is a coordination problem: even though with large N there are guaranteed to be bidders with types close to 1, there is still a positive probability that all bidders opt out, or that the number opting in is less than the social optimum n^*.

It is also straightforward to see how these inefficiencies change with n. For n below n^*, as n increases, the added bidders are welfare-positive, and the entry interval $[\alpha_{M-1}, 1]$ shrinks, reducing the inefficiency from "wrong" entry, so total surplus rises. Both these changes increase revenue as well, so revenue is increasing in n below n^*.

[20] This holds automatically when F_S is the uniform distribution, and should hold for most standard-looking distributions. However, it need not always hold. For example, if F_S were bimodal, and put most probability weight near $\frac{1}{3}$ and $\frac{2}{3}$, then $g_n(s, [s,1])$ would likely be higher for s just below $\frac{1}{3}$ than for s just above $\frac{1}{3}$.

When n is above n^*, as n *decreases*, the additional bidders being "sent home" when more than n opt in were expecting negative payoffs, so eliminating them increases surplus; and the entry threshold falls, increasing entry towards the efficient level and increasing total surplus. In the limit, bidder surplus is 0 whenever $n > n^*$, so the seller captures the increase in total surplus as increased revenue. Summing up:

Theorem 6. *In the limit as N gets large, revenue and total surplus are increasing in n when $n \leq n^*$, and decreasing in n when $n > n^*$.*

This suggests that for either revenue or efficiency, the only two candidates for the "best" mechanism of this type are $n = n^*$ and $n = n^* + 1$. Depending on the primitives of the environment, however, either one of these could be more efficient and higher-revenue. Recall that by definition, $g_{n^*-1}(1,1) > 0 > g_{n^*}(1,1)$. If $g_{n^*-1}(1,1)$ is close to 0, then the limit entry threshold $\underline{\alpha}^{(n^*-1)}$ will be close to 1; so the inefficiency due to the wrong bidders being chosen will be small. Bidder surplus will be small as well (since all bidders are close to marginal when $\underline{\alpha}^{(n)}$ is close to 1), so the n^*-bidder game is likely to be both more efficient and higher revenue than the $n^* + 1$-bidder game. On the other hand, when $g_{n^*}(1,1)$ is close to 0, the probability that not enough bidders enter will be small,[21] so the efficiency and revenue losses due to mis-coordination among bidders will be small, and the $n^* + 1$-bidder game will likely be more efficient and higher-revenue than the n^*-bidder game.

6 Discussion

6.1 Pure Cheap Talk

As discussed above, we assume the seller commits to an ordered message space and a decision rule in which he selects the bidders who send the highest opt-in messages, with a cap of two bidders and ties broken randomly. Under additional assumptions (such as F_S uniform), this commitment can lead to uniqueness of symmetric equilibrium – for example, unlike in a pure cheap-talk game, there is no babbling equilibrium, since "treating all messages the same" is not permitted behavior for the seller.[22] However, instead of "hard-coding" the messages into the mechanism, we could alternatively envision a pure cheap-talk implementation of this game: the bidders send opt-in messages, and then the seller decides which bidders advance to the auction. The seller commits to a cap of two bidders, but does not commit to how he will respond to the messages that the bidders send. This is the more precise analogue to the model of Crawford and Sobel (1982) – the

[21] Recall that in the baseline model with $n = 2$, the limit probability that all bidders opt out is $\phi = Q^{-1}\left(\frac{-2g_1(1,1)}{g_0(1)-2g_1(1,1)}\right)$; as $g_1(1,1)$ goes to 0, ϕ goes to 0 as well, so the probability that at least two bidders opt in goes to 1.

[22] This is also what led us to restrict the message set to the positive integers. In the proof of necessity in Lemma 3, we showed that if two messages are used in equilibrium, there cannot be another unused message between them, since some bidders would find that message to be a profitable deviation. Since only finitely many messages can be used in equilibrium, this rules out *any* symmetric equilibrium if the set of available messages is a continuum or, more generally, any set which is dense within a sub-interval of the real line.

messages have no exogenous meaning, just what the seller infers from them, and the seller's beliefs are part of the equilibrium. The seller, of course, maximizes revenue by selecting the bidders with the highest signals; in a monotonic equilibrium, he does this by selecting the bidders who sent the highest messages. However, with pure cheap talk, like in Crawford and Sobel, we will get a multiplicity of equilibria.

Suppose primitives are such that $M^* > 1$, and suppose that the message space is $\Re^+$, with 0 still being the opt-out message. Pick any m between 1 and M^*, and let $(\alpha_0, \ldots, \alpha_{m-1})$ be the thresholds corresponding to the equilibrium of the baseline model with $M = m$. Consider the following first-round strategy for bidders:

- If $s_i \leq \alpha_0$, send message 0 (opt out)
- If $s_i \in (\alpha_{k-1}, \alpha_k]$ ($k \in \{1, \ldots, m\}$), send message k

and the following beliefs and strategy for the seller:

- If a bidder sent message k for some $k \in \{1, \ldots, m\}$, believe his type is drawn from the distribution F_S truncated to the interval $(\alpha_{k-1}, \alpha_k]$
- If a bidder sent any other message, believe his type is in the interval $(\alpha_0, \alpha_1]$ (i.e., interpret any off-equilibrium-path message as being identical to message 1)
- If two or more bidders opted in, select the two you believe have the highest expected type, breaking ties randomly

On-the-equilibrium-path beliefs are consistent with Bayes' Law; the seller is maximizing expected payoff subject to his beliefs; the bidders face the same problem as in the baseline game; and any deviation to an off-equilibrium-path message is the same as deviating to message 1, and is therefore not profitable for any bidder not already sending that message. Thus, this is an equilibrium. But since this holds for any $m \leq M^*$, there are multiple symmetric equilibria when $M^* > 1$.

All of these equilibria would fail Farrell's (1993) condition of neologism-proofness; around each threshold α_k would be an interval of types which is self-signaling. (The logic is the same as an "in-between" message being a profitable deviation in the baseline game.) As discussed above, the "no incentive to separate" (NITS) condition introduced by Chen, Kartik and Sobel (2008) is exactly analogous to our condition $v_\tau(1, m+1) \leq v_\tau(1, m)$, which need not hold in a pure cheap talk equilibrium but could now be used (like in Chen, Kartik and Sobel) as an equilibrium refinement; out of all these equilibria, it would select the most informative (the one with M^* messages).

6.2 Why Not Auction Entry Rights?

Ye (2007), after noting that non-binding bids can never perfectly select the bidders with the highest signals, proposes instead using binding first-round bids to auction the right to participate in the second round. In our setting, with the efficient number of entrants being one, this could be done by simply auctioning off, in the first round, the right to claim the good for free in the second round – effectively, holding the "real" auction prior to due diligence. Such an auction would indeed have a symmetric, monotonic equilibrium, and would always lead to the efficient entry outcome – the bidder with the highest initial signal being the only one to incur the entry cost.

However, we feel such a mechanism would be vulnerable in a particular way. Rather than the model in this paper, consider a variation where a buyer's value is $s_i \cdot V$, where s_i is bidder i's private information and V is the objective quality of the good, perfectly observed by anyone performing due diligence. If the distribution of V was fixed, then a one-winner "entry rights" auction would result in the efficient outcome (the bidder with the highest s_i winning the right to the prize), with bidders bidding based on the expected value $E(V)$. However, if the seller knew V, then high-value sellers would be unhappy accepting a price based on the ex-ante distribution of V, and would want to change to a mechanism where the price was determined after V was learned. Worse, if worthless assets could easily be generated, there could be a rush of entry by sellers with worthless goods, hoping to capitalize on prices based on $E(V)$; sellers with legitimate assets would have an even stronger incentive to distinguish themselves. A mechanism using indicative bids avoids this problem: since no money is committed until after due diligence, sellers with worthless goods would have no incentive to enter.

(Two recent papers have raised other possible concerns about the use of entry rights auctions. Kagel, Pevnitskaya and Ye (2008) find that entry rights auctions underperform in an experimental setting, because bidders initially overbid in the first-round auction, leading to excessive bankruptcies in their experiment. Bhattacharya, Roberts and Sweeting (2012) point out that an entry rights auction may be less efficient than a standard auction with unrestricted entry if entry costs are high and bidders are risk-averse. They do find, however, that an entry rights auction is both more efficient and higher-revenue (lower-cost in the procurement setting they look at) given the estimated parameters of the empirical environment they study.)

6.3 More On Bidder Subsidies

Ye (2007) also observed that, in the case where F_T is degenerate (his *game without value updating*), by exactly subsidizing the participation cost c for the bidders selected to advance to the second round, a seller could indeed create a game with a symmetric, strictly monotone equilibrium, allowing him to always select the two highest-type bidders for the second round. (Conditional on being the marginal entrant, a bidder expects zero profits in the auction, since he has the second-highest signal, but has his costs covered, so he is indifferent about whether to enter and therefore willing to reveal his exact signal truthfully.) In fact, when c is sufficiently small, such a strategy leads to higher revenue than indicative bidding without a subsidy, although as c grows, this ordering reverses. (In the limit where N is large, the increase in revenue from subsidizing two bidders, instead of using indicative bids, is positive but small while c is small, and becomes strongly negative as c grows.) However, we are not aware of a simple analogue to this strategy for the case where F_T is non-degenerate.

7 Conclusion

We have shown that indicative bidding can be informative, and often leads to greater efficiency and higher revenue than an auction with unrestricted entry, particularly when the number of bidders is large. We have shown this within a fairly stylized model – additive and independent signals, a single round of simultaneous messages, and a particular choice of second-stage mechanism. But we expect the general insight to hold in a wider range of trading environments and communication protocols: when there are several buyers and entry costs are high, a seller may benefit from using indicative bids to "thin the field," and then either bargain with or hold an auction among a smaller number of buyers.

Proofs Omitted From Text

Recall that for a mixed strategy σ, we let $\operatorname{supp}\sigma(s_i)$ denote the support of $\sigma(s_i)$; for a set $S \subseteq [0,1]$, we will let $\operatorname{supp}\sigma(S) = \cup_{s\in S}\operatorname{supp}\sigma(s)$.

A.1 Proof of Lemma 1

We want to show that if σ is a symmetric equilibrium strategy, then for any two types $s' > s$ and messages $k' > k$, if $k' \in \operatorname{supp}\sigma(s)$, then $k \notin \operatorname{supp}\sigma(s')$. It suffices to show that for $k' > k$, $v_\sigma(\cdot, k') - v_\sigma(\cdot, k)$ is strictly single-crossing from below, since in that case,

$$v_\sigma(s,k') - v_\sigma(s,k) \quad \geq \quad 0 \quad \longrightarrow \quad v_\sigma(s',k') - v_\sigma(s',k) \quad > \quad 0$$

and so $k' \in \operatorname{supp}\sigma(s)$ implies $k \notin \operatorname{supp}\sigma(s')$.

Case 1: k' and k give different probabilities of advancing. We begin by showing that single-crossing holds when the two messages k and $k' > k$ give different probabilities of being selected. As in the text, let K_{-i} denote the messages sent by bidder i's opponents; and let $K_{-i} = 0$ indicate the event that all of i's opponents opt out.

Case 1.a: $k = 0$. If $k = 0$, then $v_\sigma(s_i, k') - v_\sigma(s_i, k) = v_\sigma(s_i, k')$, which we can write as

$$\begin{aligned} v_\sigma(s_i,k') &= \Pr(K_{-i}=0)g_0(s_i) \\ &\quad + \Pr(K_{-i}\neq 0)E_{K_{-i}\neq 0}\left\{\Pr(i \text{ is selected}|k',K_{-i})E_{s_j|i \text{ selected},\, k',K_{-i}}g_1(s_i,s_j)\right\} \end{aligned}$$

where the expectation over s_j is taken over the distribution of types who sent the highest opposing message given K_{-i}.

As noted in the text, $g_0(s_i)$ is strictly increasing in s_i, and $g_1(s_i, s_j)$ is weakly increasing in s_i everywhere and strictly increasing when $s_i > s_j$. Thus, $v_\sigma(s_i, k')$ is weakly increasing in s_i. By assumption, message k' causes bidder i to advance with positive probability, which means that either $\Pr(K_{-i} = 0) > 0$, or for some K_{-i} which occur with positive probability, $\Pr(i \text{ is selected } |k', K_{-i}) > 0$. In the former case, v_σ is strictly increasing in s_i. In the latter case, v_σ is strictly increasing in s_i unless for every K_{-i} which occurs under σ with positive probability and for which $\Pr(i \text{ is selected}|k', K_{-i}) > 0$, bidder i would face an opponent with type $s_j \geq s_i$. But in that case, $g_1(s_i, s_j) < 0$ everywhere it appears with positive weight, so $v_\sigma < 0$. So either way, $v_\sigma(s_i, k')$ is strictly increasing in s_i except when it is strictly negative, so it is strictly single-crossing in s_i from below.

Case 1.b: $k \geq 1$. Note first that conditional on the profile of messages K_{-i} sent by his opponents, the expected payoff to bidder i if he advances, $E_{s_j|i \text{ selected},\, k,\, K_{-i}}g_1(s_i, s_j)$, does not depend on the message he himself sends. This is because the distribtion of s_j he faces is simply the distribution of types among his opponents who sent the highest message.[23] For $k' > k \geq 1$, this allows us to express the difference $v_\sigma(s_i, k') - v_\sigma(s_i, k)$ as

$$v_\sigma(s_i, k') - v_\sigma(s_i, k) = \\ E_{K_{-i}} \left\{ [\Pr(i \text{ is selected}|k', K_{-i}) - \Pr(i \text{ is selected}|k, K_{-i})] \, E_{s_j|i \text{ selected}, K_{-i}} g_1(s_i, s_j) \right\}$$

As above, since $g_1(s_i, s_j)$ is strictly increasing in s_i except when $s_i \leq s_j$, the difference $v_\sigma(s_i, k') - v_\sigma(s_i, k)$ is strictly increasing in s_i unless at every K_{-i} which occurs with positive probability and under which k' gives a strictly higher probability of advancing than k, bidder i's opponent would have a type $s_j \geq s_i$ with probability 1. But again, that would imply $v_\sigma(s_i, k') - v_\sigma(s_i, k)$ strictly negative. So the difference is weakly increasing everywhere, and strictly increasing except where strictly negative; thus, strictly single-crossing.

Case 2: k and k' give the same probability of advancing. All that is left to rule out, then, are non-monotonicities where $k' \in \operatorname{supp} \sigma(s)$, $k \in \operatorname{supp} \sigma(s')$, and k and k' give bidder i exactly the same probability of advancing to the second round. We will show this is impossible.

If this were the case, then for all $s'' \in (s, s')$, $\operatorname{supp} \sigma(s'')$ can only contain messages giving that same probability of advancing. (Since $s'' < s'$, if a bidder with type s'' ever sent a message k'' with strictly higher probability of advancing than k, this would violate strict single-crossing of $v_\sigma(\cdot, k'') - v_\sigma(\cdot, k)$; if a bidder with type $s'' > s$ sent a message k'' with strictly lower probability of advancing than k', it would violate strict single-crossing of $v_\sigma(\cdot, k') - v_\sigma(\cdot, k'')$.) Thus, let $\mathcal{M} = \operatorname{supp} \sigma((s, s'))$, and note that every message in $\mathcal{M}$ must give the same probability of advancing. For a message $k'' \in \mathcal{M}$, let $p(k'') = \int_s^{s'} \sigma^{k''}(\lambda) dF_S(\lambda)$ be the probability with which message k'' gets played by a bidder with type in (s, s'). Since (s, s'') has positive measure and $\mathcal{M}$ has at most countably many messages, there is some message $k^* \in \mathcal{M}$ such that $p(k^*) > 0$.

This means there is probability at least $p(k^*)^{N-1} > 0$ that every bidder other than bidder i sends message k^*. This in turn means that another message $k'' \neq k^*$ cannot give the same probability of advancing: if $k'' > k^*$, message k'' advances for sure when all other bidders send k^*, and if $k'' < k^*$, k'' advances with probability 0, while k^* advances with probability $\frac{2}{N}$. This means that if $k \neq k'$, messages k and k' cannot both give the same probability of advancing as k^*, giving a contradiction.

[23] *Unconditional* on K_{-i}, the distributions of opponents a bidder faces can indeed depend on his own message. For a simple example, focus on bidder 1, and suppose he has two opponents: bidder 2 always sends message 1, and bidder 3 sends message 2 when $s_3 > \frac{1}{2}$ and 0 otherwise. If bidder 1 sends message 2, he advances for sure, and the distribution of his opponent's type is $\frac{1}{2}U[\frac{1}{2}, 1] + \frac{1}{2}U[0, 1]$. If bidder 1 sends message 1, however, he is twice as likely to advance when bidder 3 is weak as when he is strong, so the distribution of his opponent's type when he advances is $\frac{1}{3}U[\frac{1}{2}, 1] + \frac{2}{3}U[0, 1]$. However, *conditional on K_{-i}* and advancing, his opponent's distribution is the same regardless of which message he sent: when bidder 3 sends message 2, 1's opponent's distribution is $U[\frac{1}{2}, 1]$ whenever he advances, and when bidder 3 sends message 0, it is $U[0, 1]$. Thus, for a given K_{-i}, a bidder's own message can change his probability of advancing, but not his expected payoff when he advances.

A.2 Proof of Lemma 2

Finiteness of $\operatorname{supp}\sigma([0,1))$. By assumption, $g_1(s,s) < 0$ and g_1 is continuous; let $\epsilon > 0$ solve $g_1(s+\epsilon, s) = 0$. We will show that for any interval $[s, s+\epsilon)$, $\operatorname{supp}\sigma([s, s+\epsilon))$ contains no more than two messages, which limits the number of messages used in equilibrium (other than by bidders with type $s_i = 1$) to at most $2\lceil \frac{1}{\epsilon} \rceil$.

Toward contradiction, suppose that three messages $k_1 < k_2 < k_3$ are all played by bidders in that interval, and that k_2 and k_3 give different probabilities of advancing.[24] By monotonicity of strategies, since k_1 is sent by some bidder with type $s_i \geq s$, all bidders sending message k_2 or higher have types greater than s. Now consider the scenarios where sending message k_3 gives bidder i a higher probability of advancing than sending message k_2. This can only occur when the second-highest message sent by i's opponents is at least k_2 (otherwise i would have advanced for sure), which means the highest message of i's opponents must also be at least k_2; which means whenever sending k_3 gives i a higher probability of advancing, he advances against an opponent with type $s_j > s$. But since $s_i < s + \epsilon$, this means $g_1(s_i, s_j) < g_1(s+\epsilon, s) \leq 0$ in all the cases where sending the higher message caused i to advance. But then bidder i prefers not to advance in all those cases, so sending k_2 is strictly better than k_3, contradicting k_3 being played.

Specific Set of Messages Played In Equilibrium. First, note that the opt-out message must be played with positive probability. If not, it would be impossible to advance alone, and all messages used in equilibrium would give a positive probability of advancing; together, these would imply that bidders with signals close to 0 earn negative payoffs, when they could deviate to message 0 and get 0.

What remains to be shown is that no messages are "skipped" in equilibrium. Let $\mathcal{K}$ be the set of messages that are sent with positive probability. Suppose there were messages $k < k' < k''$, with $k, k'' \in \mathcal{K}$ but $k' \notin \mathcal{K}$. Suppose k and k'' are adjacent in $\mathcal{K}$, i.e., there are no messages $k''' \in \mathcal{K}$ with $k < k''' < k''$. (Since we just showed $\mathcal{K}$ is finite, this is without loss.) By monotonicity, there must be some type s such that bidders below type s use message k or lower, and bidders above type s use k'' or higher; by continuity, bidders with type s must be indifferent between k and k''.

From bidder i's point of view, sending message k'' instead of message k increases the probability of advancing in two scenarios: when his second-highest competitor sends message k'', and when his second-highest competitor sends message k. If the second-highest opposing message is k'', the highest is at least k''; by monotonicity, this means that when i advances in this scenario, he is up against a stronger opponent with $s_j \geq s = s_i$, and earns negative expected payoff. This means that if he is indifferent between the two messages, he must earn strictly positive payoff in expectation in the second scenario, when the second-highest competitor sends k. By deviating to message k', he would get the best of both worlds: always advancing when his second-highest competitor

[24] Let k_1 be the lowest message in $\operatorname{supp}\sigma([s, s+\epsilon))$, and let s^* be the highest type bidder who plays k_1. Since the top of the interval is open, $s^* < s + \epsilon$. If two higher messages cannot be found giving different probabilities of advancing, then all bidders with types in $(s^*, s+\epsilon)$ send messages with the same probability of advancing; as shown above, this would mean they must all send the same message, in which case $|\operatorname{supp}\sigma([s, s+\epsilon))| = 2$.

sent message k, but never advancing when he sent k'', making k' a profitable deviation. So no messages are "skipped" in equilibrium, and (from above) a finite number are played, including 0; so $\mathcal{K} = \{0, 1, \ldots, m\}$ for some finite m.

(Proof of Lemma 3 is in text)

A.3 Proof of Lemma 4

As noted above, g_0 is continuous, and g_1 is continuous in both its arguments, leading to $g(\lambda, [\lambda', \lambda''])$ being continuous in all its arguments as well. Continuity of δ and Δ follow from their definitions.

Part 1: $\delta(0, \lambda') < 0$. When we plug $\alpha_0 = 0$ into the expression for $\delta(\alpha_0, \alpha_1)$ in the text, every term with a positive power of $F_S(\alpha_0)$ drops out – which is every term but the $j = N - 1$ terms in the two sums. This leaves

$$\delta(0, \lambda') \quad = \quad F_S(\lambda')^{N-1} \tfrac{2}{N} g_1(0, [0, \lambda']) + (N-1)(1 - F_S(\lambda'))F_S(\lambda')^{N-2} \tfrac{1}{N-1} g_1(0, [\lambda', 1]) \quad < \quad 0.$$

Part 2: $\Delta(0, \lambda', \lambda'') \geq 0 \to \delta(\lambda', \lambda'') > 0$. Plugging $(\alpha_0, \alpha_1) = (\lambda', \lambda'')$ into δ and $(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = (0, \lambda', \lambda'')$ into Δ, the last two lines of δ and Δ are identical and drop out of the difference, and all but the $j = N - 1$ terms of the first two lines of $\Delta(0, \lambda', \lambda'')$ vanish, leaving

$$\begin{aligned} \delta(\lambda', \lambda'') - \Delta(0, \lambda', \lambda'') \quad = \quad & F_S(\lambda')^{N-1} \left(g_0(\lambda') - g_1(\lambda', [0, \lambda']) \right) \\ & + F_S(\lambda')^{N-1} \tfrac{2}{N} g_1(\lambda', [0, \lambda']) + (1 - F_S(\lambda'))F_S(\lambda')^{N-2} g_1(\lambda', [\lambda', 1]) \end{aligned}$$

The first line is strictly positive – a bidder always prefers to advance alone than against a weak opponent, and $\lambda' > 0$ so $F_S(\lambda') > 0$. The second line is strictly positive whenever $\Delta(0, \lambda', \lambda'') \geq 0$: the first term is $\frac{2}{N-2}$ times the only positive term in the expression for $\Delta(0, \lambda', \lambda'')$, and the second is $\frac{1}{N-2}$ times one (of many) negative terms in $\Delta(0, \lambda', \lambda'')$. So if $\frac{2}{N-2}\Delta(0, \lambda', \lambda'') \geq 0$, $\delta(\lambda', \lambda'') - \Delta(0, \lambda', \lambda'') > 0$, giving the result.

Part 3: $\Delta(\lambda, \lambda', \lambda'') < 0$ *if* $\lambda' - \lambda < \epsilon$. Note that $g_1(\lambda', [\lambda', 1])$, $g_1(\lambda', [\lambda', \lambda''])$, and $g_1(\lambda', [\lambda'', 1])$ are all strictly negative, and have non-vanishing coefficients if $\lambda < \lambda' < \lambda''$. Thus, $\Delta(\lambda, \lambda', \lambda'') \geq 0$ requires $g_1(\lambda', [\lambda, \lambda']) > 0$, which requires $g_1(\lambda', \lambda) > 0$, which requires (by definition) $\lambda' > \lambda + \epsilon$.

Part 4: $\Delta(-, \lambda', \lambda'')$ *strictly single-crossing from above.* As noted in the text, $\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ can be written as

$$\Delta(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) \quad = \quad c_1 E_1 \quad + \quad c_2 E_2 \quad + \quad c_3 E_3$$

where

$$E_1 = g_1(\alpha_k, [\alpha_{k-1}, \alpha_k]), \quad E_2 = g_1(\alpha_k, [\alpha_k, \alpha_{k+1}]), \quad E_3 = g_1(\alpha_k, [\alpha_{k+1}, 1])$$

and expressions for c_1, c_2 and c_3 are given in the text. It is straightforward to show that c_1, c_2, and c_3 are all continuous and differentiable in α_{k-1}. E_1 is also continuous and differentiable in α_{k-1}, and E_2 and E_3 don't vary with α_{k-1}, so Δ is differentiable with respect to α_{k-1}, with (letting primes denote partial derivatives with respect to α_{k-1})

$$\Delta' \;=\; c_1 E_1' + c_1' E_1 + c_2' E_2 + c_3' E_3$$

Adding and subtracting $\frac{c_1'}{c_1}\Delta$ gives

$$\begin{aligned}\Delta' &= c_1 E_1' + c_1' E_1 + c_2' E_2 + c_3' E_3 - \frac{c_1'}{c_1}(c_1 E_1 + c_2 E_2 + c_3 E_3) - \frac{c_1'}{c_1}\Delta \\ &= c_1 E_1' + c_2\left(\frac{c_2'}{c_2} - \frac{c_1'}{c_1}\right) E_2 + c_3\left(\frac{c_3'}{c_3} - \frac{c_1'}{c_1}\right) E_3 - \frac{c_1'}{c_1}\Delta\end{aligned}$$

As long as $\alpha_{k-1} < \alpha_k$, $c_1 > 0$, and $E_1' < 0$, so the first term is strictly negative. We show in the technical appendix that $\frac{c_2'}{c_2} \geq \frac{c_1'}{c_1}$ and $\frac{c_3'}{c_3} \geq \frac{c_1'}{c_1}$; since $c_2, c_3 \geq 0$ and $E_2, E_3 < 0$, this makes the second and third terms weakly negative. When $\Delta = 0$, the final term drops out, leaving $\Delta' < 0$ when $\Delta = 0$, which is enough to establish strict single-crossing.

(Proof of Theorem 1 is in text)

A.4 Proof of Lemma 5

Changing variables as in the text, we continue to write

$$\tilde{\Delta}(x,y,z) \;=\; c_1 E_1 \;+\; c_2 E_2 \;+\; c_3 E_3$$

but now with

$$\begin{aligned}E_1 &= g_1(F_S^{-1}(1-y-z), [F_S^{-1}(1-x-y-z), F_S^{-1}(1-y-z)]) \\ E_2 &= g_1(F_S^{-1}(1-y-z), [F_S^{-1}(1-y-z), F_S^{-1}(1-z)]) \\ E_3 &= g_1(F_S^{-1}(1-y-z), [F_S^{-1}(1-z), 1])\end{aligned}$$

$$\begin{aligned}c_1 &= (1-y-z)^{N-1} + (1-x-y-z)^{N-1} - \tfrac{2}{N}\tfrac{(1-y-z)^N - (1-x-y-z)^N}{x} \\ c_2 &= \tfrac{2}{N}\tfrac{(1-z)^N - (1-y-z)^N}{y} - 2(1-y-z)^{N-1} - y\tfrac{(1-y-z)^{N-1} - (1-x-y-z)^{N-1}}{x} \\ c_3 &= z\tfrac{(1-z)^{N-1} - (1-y-z)^{N-1}}{y} - z\tfrac{(1-y-z)^{N-1} - (1-x-y-z)^{N-1}}{x}\end{aligned}$$

If F_S is the uniform distribution, then

$$E_1 \;=\; g_1(1-y-z, [1-x-y-z, 1-y-z]) \;=\; \tfrac{1}{x}\int_{1-x-y-z}^{1-y-z} g_1(1-y-z, s)ds$$

Since $g_1(s_i, s_j)$ depends only on $s_i - s_j$, think of it as a function of one variable $h(s) = E_{T_i,T_j}\max\{0, s + T_i - T_j\}$; then

$$E_1 \;=\; \tfrac{1}{x}\int_{1-x-y-z}^{1-y-z} h(1-y-z-s)ds \;=\; \tfrac{1}{x}\int_0^x h(s)ds$$

and therefore does not depend on y or z. Similarly,

$$E_2 \quad = \quad \frac{1}{y}\int_{1-y-z}^{1-z} h(1-y-z-s)ds \quad = \quad \frac{1}{y}\int_{-y}^{0} h(s)ds$$

does not depend on x or z, and is decreasing in y; and

$$E_3 \quad = \quad \frac{1}{z}\int_{1-z}^{1} h(1-y-z-s)ds \quad = \quad \frac{1}{z}\int_{-y-z}^{-y} h(s)ds$$

does not depend on x, and is decreasing in both y and z.

The coefficients c_1, c_2, and c_3, and the expected payoff terms E_1, E_2, and E_3, are all differentiable in x, y, and z. Differentiating, and letting y and z superscripts denote partial derivatives with respect to y and z, respectively, and adding and subtracting $\frac{c_1^y}{c_1}\tilde{\Delta}(x,y,z)$ in the first line and $\frac{c_1^z}{c_1}\tilde{\Delta}(x,y,z)$ in the second, gives

$$\begin{aligned}
\frac{\partial\tilde{\Delta}(x,y,z)}{\partial y} \quad &= \quad c_2\left(\frac{c_2^y}{c_2}-\frac{c_1^y}{c_1}\right)E_2 + c_3\left(\frac{c_3^y}{c_3}-\frac{c_1^y}{c_1}\right)E_3 + c_2E_2^y + c_3E_3^y + \frac{c_1^y}{c_1}\tilde{\Delta}(x,y,z) \\
\frac{\partial\tilde{\Delta}(x,y,z)}{\partial z} \quad &= \quad c_2\left(\frac{c_2^z}{c_2}-\frac{c_1^z}{c_1}\right)E_2 + c_3\left(\frac{c_3^z}{c_3}-\frac{c_1^z}{c_1}\right)E_3 + c_3E_3^z + \frac{c_1^z}{c_1}\tilde{\Delta}(x,y,z)
\end{aligned}$$

In the technical appendix, we show that $\frac{c_2^z}{c_2} \geq \frac{c_1^z}{c_1}$, $\frac{c_3^y}{c_3} \geq \frac{c_1^y}{c_1}$, $\frac{c_2^z}{c_2} \geq \frac{c_1^z}{c_1}$, and $\frac{c_3^z}{c_3} \geq \frac{c_1^z}{c_1}$; since E_2 and E_3 are negative, this means the first two terms in both expressions are negative. Since E_2^y, E_3^y, and E_3^z are all negative, when $\tilde{\Delta}=0$, both $\frac{\partial\tilde{\Delta}}{\partial y}$ and $\frac{\partial\tilde{\Delta}}{\partial z}$ are negative; we also show the former is strictly negative when $x>0$, and the latter strictly negative when $z>0$, which is sufficient to get strict single-crossing where we need it.

Similar steps (shown in the technical appendix) establish that $\tilde{\delta}$ is also strictly single-crossing from above in both its arguments when F_S is the uniform distribution.

(Proof of Lemma 6 is in text)

A.5 Proof of Lemma 7

Uniqueness of α given $m \leq M^*$: Fix m, and suppose two sets of thresholds $\alpha^i = (\alpha_0^i, \ldots, \alpha_{m-1}^i)$, $i=\{1,2\}$, satisfied the indifference conditions $\left\{\Delta(\alpha_{k-1}^i, \alpha_k^i, \alpha_{k+1}^i)\right\}_{k=1,2,\ldots,m-1}$ and $\delta(\alpha_0^i,\alpha_1^i)=0$. If $\alpha_{m-1}^1 > \alpha_{m-1}^2$, then by the monotonicity property discussed in the text, $F_S(\alpha_1^1)-F_S(\alpha_0^1) < F_S(\alpha_1^2)-F_S(\alpha_0^2)$ and $1-F_S(\alpha_1^1) < 1-F_S(\alpha_1^2)$. But if $\tilde{\delta}$ is strictly single-crossing from above in both its arguments, then

$$\begin{array}{rcl}
\tilde{\delta}(F_S(\alpha_1^1)-F_S(\alpha_0^1), 1-F_S(\alpha_1^1)) & = & 0 \\
\downarrow & & \\
\tilde{\delta}(F_S(\alpha_1^2)-F_S(\alpha_0^2), 1-F_S(\alpha_1^2)) & < & 0
\end{array}$$

so α^1 and α^2 cannot both solve the last indifference condition $\delta(\alpha_0^i,\alpha_1^i)=0$. If $\alpha_{m-1}^1 < \alpha_{m-1}^2$, the reverse argument holds. This leaves only $\alpha_{m-1}^1 = \alpha_{m-1}^2$; but since the α^* function is uniquely defined, this would imply $\alpha^1_{m-2} = \alpha^2_{m-2}$, $\alpha^1_{m-3} = \alpha^2_{m-3}$, and so on, and uniqueness would not be violated.

Non-existence of α *for* $m > M^*$. Suppose $m > M$ and a solution to the indifference conditions exists. Now, $\alpha_m = 1 = \beta_0$ and $\alpha_{m-1} < 1 = \beta_1$. By the same monotonicity argument, we get $F_S(\alpha_{m-k}) - F_S(\alpha_{m-k-1}) > F_S(\beta_k) - F_S(\beta_{k+1})$, and $\alpha_{m-k} < \beta_k$, for $k = 1, 2, \ldots, m-1$. But then by single-crossing of $\tilde{\delta}$,

$$0 \;=\; \delta(\alpha_0, \alpha_1) \;=\; \tilde{\delta}(F_S(\alpha_1) - F_S(\alpha_0), 1 - F_S(\alpha_1))$$

implies

$$0 \;<\; \tilde{\delta}(F_S(\beta_{m-1}) - F_S(\beta_m), 1 - F_S(\beta_{m-1})) \;=\; \delta(\beta_m, \beta_{m-1})$$

since both arguments are lower. However, as noted above, the definition of M^* implied that it was the highest value of $\tilde{m}$ such that $(\beta_1, \beta_2, \ldots, \beta_{\tilde{m}})$ are all positive and $\delta(\beta_{\tilde{m}}, \beta_{\tilde{m}-1}) > 0$; since $m > M^*$, this gives a contradiction.

Necessity of $m = M$ *when* $m < M^*$. From the construction of M^*, for any $m < M^*$, $(\beta_1, \ldots, \beta_{m+1})$ are all positive, and $\delta(\beta_{m+1}, \beta_m) > 0$. (If $m = M^* - 1$, this is because $\delta(\beta_{M^*}, \beta_{M^*-1}) > 0$; if $m < M^* - 1$, this is because $\beta_{m+2} = \alpha^*(\beta_{m+1}, \beta_m)$ exists and is non-negative, so $\Delta(0, \beta_{m+1}, \beta_m) \geq 0$, implying $\delta(\beta_{m+1}, \beta_m) > 0$ by part 2 of Lemma 4.)

So now suppose that our claim was violated: that at some solution to the indifference conditions, $m < M^*$ and $\alpha_{m-1} \geq \overline{\alpha} = \beta_2$. Once again, iteratively applying the monotonicity of x^* leads to $F_S(\alpha_1) - F_S(\alpha_0) \leq F_S(\beta_m) - F_S(\beta_{m+1})$ and $\alpha_1 \geq \beta_m$. Then

$$\begin{aligned} 0 \;=\; \delta(\alpha_0, \alpha_1) &\;=\; \tilde{\delta}(F_S(\alpha_1) - F_S(\alpha_0), 1 - F_S(\alpha_1)) \\ &\;\geq\; \tilde{\delta}(F_S(\beta_m) - F_S(\beta_{m+1}), 1 - F_S(\beta_m)) \;=\; \delta(\beta_{m+1}, \beta_m) \end{aligned}$$

by single-crossing of $\tilde{\delta}$, giving a contradiction.

A.6 Proof of Theorem 2

Part 1 (essential uniqueness of equilibrium) is proved in the text. To prove Part 2, we first show how changes in c, the dispersion of T_i, or N affect Δ and δ. Let δ, Δ, x^*, M^*, and $(\beta_0, \beta_1, \ldots, \beta_{M^*})$ correspond to some initial set of primitives, and let δ', Δ', $(x^*)'$, $(M^*)'$, and $(\beta'_0, \beta'_1, \ldots)$ correspond to a set of primitives which have been modified by either decreasing c, applying a mean-preserving spread to F_T, or decreasing N. Recall that, for $k > 0$, the difference in payoffs of sending message $k+1$ rather than k can be expressed as

$$\Delta(\lambda, \lambda', \lambda'') \;=\; c_1 g_1(\lambda', [\lambda, \lambda']) \;+\; c_2 g_1(\lambda', [\lambda', \lambda'']) \;+\; c_3 g_1(\lambda', [\lambda'', 1])$$

and for $k = 0$, it can be expressed as

$$\delta(\lambda, \lambda') = d_1 g_0(\lambda) + d_2 g_1(\lambda, [\lambda, \lambda']) + d_3 g_1(\lambda, [\lambda', 1]),$$

where the coefficients $\{c_1, c_2, c_3, d_1, d_2, d_3\}$ represent the increases in the likelihood of advancing and are therefore nonnegative.

Now suppose c decreases. By definition, g_0 and g_1 are strictly decreasing in c, so both Δ and δ will increase when c decreases. A mean-presevering spread (MPS) of F_T has no impact on g_0, since it depends only $E(T_i)$, but it increases g_1. To see why, note that

$$g_1(s_i, s_j) = E_{T_i}[E_{T_j}[\max\{0, s_i - s_j + T_i - T_j\}]].$$

The inner expectation is the expected value of a convex function of T_j, and therefore increases when a MPS is applied to the distribution of T_j. The outer expectation is the expected value of a convex function of T_i, and therefore increases when a MPS is applied to the distribution of T_i. Again an increase in g_1 means an increase in δ and Δ. Finally, a decrease in N has no impact on g_0 or g_1, but does affect the weights on these functions. We show in the technical appendix that δ and Δ are single-crossing from above in N. Thus, each of the three changes – a decrease in c, a mean-preserving spread of F_T, or a decrease in N – "increase" Δ and δ, at least in the sense of single-crossing: $\Delta(\lambda, \lambda', \lambda'') \geq 0$ implies $\Delta'(\lambda, \lambda', \lambda'') \geq 0$, and $\delta(\lambda, \lambda') > 0$ implies $\delta'(\lambda, \lambda') > 0$.

Importantly, the former also implies that $(x^*)'(\lambda', \lambda'') \leq x^*(\lambda', \lambda'')$. This is because by definition, $\Delta(x^*(\lambda', \lambda''), \lambda', \lambda'') = 0$, which implies that $\Delta'(x^*(\lambda', \lambda''), \lambda', \lambda'') \geq 0$. Since Δ' is single-crossing from below in its first argument, the solution to $\Delta'(\cdot, \lambda', \lambda'') = 0$ must therefore be weakly lower than $x^*(\lambda', \lambda'')$, i.e., $(x^*)'(\lambda', \lambda'') \leq x^*(\lambda', \lambda'')$.

Thus, $1 - F_S(\beta_2') = (x^*)'(0, 0) \leq x^*(0, 0) = 1 - F_S(\beta_2)$, and therefore $\beta_2' \geq \beta_2$. From there,

$$F_S(\beta_2') - F_S(\beta_3') \leq F_S(\beta_2) - F_S(\beta_3)$$

and therefore $\beta_3' \geq \beta_3$, and so on. This means that if $\beta_2, \beta_3, \ldots, \beta_{M^*}$ are all positive, then $\beta_2', \beta_3', \ldots, \beta_{M^*}'$ are all positive as well, with $F_S(\beta_{M^*-1}') - F_S(\beta_{M^*}') \leq F_S(\beta_{M^*-1}) - F_S(\beta_{M^*})$ and $\beta_{M^*-1}' \geq \beta_{M^*-1}$.

The definition of M^* required that $\delta(\beta_{M^*}, \beta_{M^*-1}) > 0$, or

$$\tilde{\delta}(F_S(\beta_{M^*-1}) - F_S(\beta_{M^*}), 1 - F_S(\beta_{M^*-1})) > 0$$

The fact that $\delta(\cdot, \cdot) > 0$ implies $\delta'(\cdot, \cdot) > 0$ means that

$$\tilde{\delta}'(F_S(\beta_{M^*-1}) - F_S(\beta_{M^*}), 1 - F_S(\beta_{M^*-1})) > 0;$$

single-crossing of $\tilde{\delta}$ then gives

$$\tilde{\delta}'(F_S(\beta'_{M^*-1}) - F_S(\beta'_{M^*}), 1 - F_S(\beta'_{M^*-1})) > 0,$$

or $\delta'(\beta'_{M^*}, \beta'_{M^*-1}) > 0$, ensuring that $(M^*)' \geq M^*$.

A.7 Proof of Lemma 8

To prove the result, we will show that when N is sufficiently large, no α can simultaneously satisfy the indifference conditions $\delta(\alpha_0, \alpha_1) = 0$ and $\Delta(\alpha_0, \alpha_1, \alpha_2) = 0$, ruling out equilibria in which two or more opt-in messages are played.

First, property 3 of Lemma 4 implies that if $\Delta(\alpha_0, \alpha_1, \alpha_2) = 0$, then $\alpha_1 - \alpha_0 \geq \epsilon$, where ϵ was defined earlier by $g_1(s + \epsilon, s) = 0$. We assumed earlier that F_S had a density bounded away from 0; let $\underline{f_S}$ denote its lower bound, so that $\alpha_1 - \alpha_0 \geq \epsilon$ implies $F_S(\alpha_1) - F_S(\alpha_0) \geq \epsilon \underline{f_S}$.

Second, recall from earlier that

$$\begin{aligned}
\delta(\alpha_0, \alpha_1) \quad &= \quad F_S(\alpha_0)^{N-1} g_0(\alpha_0) \\
&\quad + \left(\textstyle\sum_{j=1}^{N-1} \binom{N-1}{j} (F_S(\alpha_1) - F_S(\alpha_0))^j F_S(\alpha_0)^{N-1-j} \frac{2}{j+1} \right) g_1(\alpha_0, [\alpha_0, \alpha_1]) \\
&\quad + \left(\textstyle\sum_{j=1}^{N-1} (N-1)(1 - F_S(\alpha_1)) \binom{N-2}{j-1} (F_S(\alpha_1) - F_S(\alpha_0))^{j-1} F_S(\alpha_0)^{N-1-j} \frac{1}{j} \right) g_1(\alpha_0, [\alpha_1, 1])
\end{aligned}$$

Note that each of the last two lines is negative, that $g_0(\alpha_0) \leq g_0(1)$, and that $g_1(\alpha_0, [\alpha_0, \alpha_1]) \leq g_1(\alpha_0, \alpha_0) = g_1(1, 1)$. Thus, if we want just an upper bound on $\delta(\alpha_0, \alpha_1)$, we can drop the last line, and all but the $j = 1$ term from the second-to-last, and replace the arguments of g_0 and g_1, giving

$$\begin{aligned}
\delta(\alpha_0, \alpha_1) \quad &\leq \quad F_S(\alpha_0)^{N-1} g_0(1) + \binom{N-1}{1} (F_S(\alpha_1) - F_S(\alpha_0)) F_S(\alpha_0)^{N-2} g_1(1, 1) \\
&= \quad (N-1) F_S(\alpha_0)^{N-1} g_0(1) \left[\tfrac{1}{N-1} + \tfrac{F_S(\alpha_1) - F_S(\alpha_0)}{F_S(\alpha_0)} \tfrac{g_1(1,1)}{g_0(1)} \right]
\end{aligned}$$

Thus, if $\delta(\alpha_0, \alpha_1) = 0$ holds, then the term above in square brackets must be nonnegative; since $g_1(1, 1) < 0$, this requires

$$\tfrac{1}{N-1} \quad \geq \quad \tfrac{F_S(\alpha_1) - F_S(\alpha_0)}{F_S(\alpha_0)} \tfrac{|g_1(1,1)|}{g_0(1)} \quad \geq \quad \epsilon \underline{f_S} \tfrac{|g_1(1,1)|}{g_0(1)}$$

But $\epsilon \underline{f_S} \frac{|g_1(1,1)|}{g_0(1)}$ depends only on F_S, F_T, and c, not on N. So fixing the rest of the environment, if

$$N \quad > \quad 1 + \frac{1}{\epsilon \underline{f_S}} \frac{g_0(1)}{|g_1(1, 1)|}$$

then all symmetric equilibria use only messages 0 and 1 and $\alpha_1 = 1$.

A.8 Proof of Lemma 9

In equilibrium, the number of bidders who participate in the indicative bidding mechanism has a binomial distribution with parameters $p = 1 - F_S(\alpha_0)$ and N. As N goes to infinity, we show that $\alpha_0 \to 1$, so $p \to 0$; but it goes to 0 at a rate such that $N \cdot p$ has a finite, nonzero limit. In other words, the distribution of the number of bidders who participate converges to the Poisson distribution with some parameter ρ. In what follows, we focus on the limiting value of the probability that no one participates, which is given by $\phi = e^{-\rho}$.

The indifference condition determining α_0 simplifies to

$$0 \quad = \quad \delta(\alpha_0, 1) \quad = \quad F_S(\alpha_0)^{N-1} g_0(\alpha_0) + \left(\tfrac{2}{N} \tfrac{1-F_S(\alpha_0)^N}{1-F_S(\alpha_0)} - 2F_S(\alpha_0)^{N-1} \right) g_1(\alpha_0, [\alpha_0, 1])$$

(Since $\alpha_1 = 1$, the third term in $\delta(\alpha_0, 1)$ drops out; restating the coefficients on the terms involving g_0 and g_1 to eliminate summation terms is shown in the technical appendix.) Abusing notation slightly, we fix the rest of the primitives but allow N to vary, letting α_0 vary with N (but suppressing this dependence). If $F_S(\alpha_0)^N \to 0$, the middle term

$$\frac{2}{N} \frac{1 - F_S(\alpha_0)^N}{1 - F_S(\alpha_0)} \quad \geq \quad F_S(\alpha_0)^{\frac{N-1}{2}}$$

vanishes more slowly than the rest of the expression, so the negative term dominates and $\delta_0(\alpha_0, 1) < 0$. Thus, as N grows, the probability of all bidders opting out is bounded away from 0, which implies that the opt-in threshold α_0 has to approach 1. Similarly, it's not hard to show that as N grows, if $F_S(\alpha_0)^N \to 1$, the coefficient on g_1 vanishes, giving $\delta(\alpha_0, 1) \to g_0(1) > 0$. Therefore, α_0 needs to go to 1 at a speed such that the limit of $F_S(\alpha_0)^N$ lies in $(0, 1)$.

We now calculate this limit. After some manipulation, we can rewrite the condition $\delta(\alpha_0, 1) = 0$ as

$$2F_S(\alpha_0) \tfrac{-g_1(\alpha_0, [\alpha_0, 1])}{g_0(\alpha_0) - 2g_1(\alpha_0, [\alpha_0, 1])} \quad = \quad N(1 - F_S(\alpha_0)) \tfrac{F_S(\alpha_0)^N}{1 - F_S(\alpha_0)^N}.$$

The limit of the term on the left hand side of this equation is

$$\lim_{N\to\infty} \tfrac{-2F_S(\alpha_0) g_1(\alpha_0, [\alpha_0, 1])}{g_0(\alpha_0) - 2g_1(\alpha_0, [\alpha_0, 1])} \quad = \quad \tfrac{-2g_1(1,1)}{g_0(1) - 2g_1(1,1)}.$$

To calculate the limit of the term on the right hand side, we use the fact that

$$\lim_{N\to\infty} \frac{-\ln(F_S(\alpha_0)^N)}{N(1 - F_S(\alpha_0))} \quad = \quad 1$$ [25]

to establish that

$$\lim_{N\to\infty} N(1 - F_S(\alpha_0)) \tfrac{F_S(\alpha_0)^N}{1 - F_S(\alpha_0)^N} \quad = \quad \lim_{N\to\infty} (-\ln(F_S(\alpha_0)^N)) \tfrac{F_S(\alpha_0)^N}{1 - F_S(\alpha_0)^N}$$

[25] $\frac{-\ln(F_S(\alpha_0)^N)}{N(1-F_S(\alpha_0))} = \frac{-N\ln(F_S(\alpha_0))}{N(1-F_S(\alpha_0))} = \frac{-\ln(F_S(\alpha_0))}{1-F_S(\alpha_0)}$, which by l'Hôpital's Rule has limit $\frac{-1/F_S(\alpha_0)}{-1} = 1$ as $F_S(\alpha_0) \to 1$.

and therefore

$$\lim_{N\to\infty}(-\ln(F_S(\alpha_0)^N)\tfrac{F_S(\alpha_0)^N}{1-F_S(\alpha_0)^N} \quad = \quad \tfrac{-2g_1(1,1)}{g_0(1)-2g_1(1,1)}$$

As in the text, let $Q(\lambda) = \frac{-\lambda\ln(\lambda)}{1-\lambda}$ for $\lambda \in (0,1)$, and (by continuity) by $Q(0) = 0$ and $Q(1) = 1$. Note that Q is strictly increasing and onto, and therefore invertible. Applying Q^{-1} to both limit expressions, we obtain

$$\lim_{N\to\infty} F_S(\alpha_0)^N \quad = \quad Q^{-1}\left(\tfrac{-2g_1(1,1)}{g_0(1)-2g_1(1,1)}\right).$$

(Proof of Lemma 10 is in text)

A.9 Proof of Theorem 3

Since we are contemplating different participation levels, let $\tilde{\phi}$ denote any hypothetical participation level (limiting probability of no entry), and as in the text, let ϕ refer to the equilibrium level. Following the logic in Lemma 10, limit total surplus given any participation level $\tilde{\phi}$ is

$$W(\tilde{\phi}) \quad = \quad -\tilde{\phi}\ln(\tilde{\phi})g_0(1) + (1-\tilde{\phi}+\tilde{\phi}\ln(\tilde{\phi}))(g_0(1)+g_1(1,1))$$

and therefore, differentiating and simplifying,

$$W'(\tilde{\phi}) \quad = \quad -g_0(1) + \ln(\tilde{\phi})g_1(1,1)$$

Since $g_1(1,1) < 0$, this is strictly decreasing in $\tilde{\phi}$; thus, if we can show it is negative at $\tilde{\phi} = \phi$, then it is negative above that. Thus, it only remains to show that $-g_0(1) + \ln(\phi)g_1(1,1) < 0$, which is equivalent to

$$-g_0(1) + \ln\left(Q^{-1}\left(\tfrac{-2g_1(1,1)}{g_0(1)-2g_1(1,1)}\right)\right)g_1(1,1) \quad < \quad 0.$$

To show that this does indeed hold, let $\xi = \frac{g_0(1)}{-g_1(1,1)} > 0$, and divide the last equation by $-g_1(1,1) > 0$; the result we want is then equivalent to

$$-\xi - \ln\left(Q^{-1}\left(\tfrac{2}{\xi+2}\right)\right) \quad < \quad 0$$

$$Q^{-1}\left(\tfrac{2}{\xi+2}\right) \quad > \quad e^{-\xi}$$

$$\tfrac{2}{\xi+2} \quad > \quad Q(e^{-\xi}) \quad = \quad \tfrac{\xi e^{-\xi}}{1-e^{-\xi}}$$

$$2 - 2e^{-\xi} \quad > \quad \xi^2 e^{-\xi} + 2\xi e^{-\xi}$$

$$2e^{\xi} \quad > \quad 2 + 2\xi + \xi^2$$

The right-hand side is just the first three terms of the Taylor expansion of the left, and the inequality therefore holds strictly for $\xi > 0$. Thus, at or above the equilibrium participation threshold, total surplus is strictly decreasing in the participation threshold, with nonzero derivative.

(Proof of Corollary 1 is in the text)

A.10 Proof of Theorem 4

Part 1: More participation. For $n \geq 1$ and any s, $g_n(s, [s, 1]) \leq g_1(s, [s, 1]) \leq g_1(1, 1)$. So

$$
\begin{aligned}
0 &= \textstyle\sum_{n=0}^{N-1} \binom{N-1}{n} F_S(\gamma)^{N-1-n}(1 - F_S(\gamma))^n g_n(\gamma, [\gamma, 1]) \\
&\leq F_S(\gamma)^{N-1} g_0(1) + \textstyle\sum_{n=1}^{N-1} \binom{N-1}{j} F_S(\gamma)^{N-1-n}(1 - F_S(\gamma))^n g_1(1, 1) \\
&= F_S(\gamma)^{N-1} g_0(1) + (1 - F_S(\gamma)^{N-1}) g_1(1, 1)
\end{aligned}
$$

which in turn implies

$$F_S(\gamma)^{N-1} \quad \geq \quad \tfrac{-g_1(1,1)}{g_0(1) - g_1(1,1)}$$

Since the right-hand side is strictly positive, as N grows, this requires $F_S(\gamma) \to 1$, and therefore

$$\lim_{N\to\infty} F_S(\gamma)^N = \lim_{N\to\infty} F_S(\gamma)^{N-1} \geq \frac{-g_1(1, 1)}{g_0(1) - g_1(1, 1)}.$$

We will show that this implies $\lim_{N\to\infty} F_S(\gamma)^N > \lim_{N\to\infty} F_S(\alpha_0)^N$, which implies that the entry threshold for the unrestricted auction is greater than the participation threshold for the indicative bidding game.

Proof is by contradiction. Suppose instead that $\lim_{N\to\infty} F_S(\gamma)^N \leq \lim_{N\to\infty} F_S(\alpha_0)^N$. Then

$$\tfrac{-g_1(1,1)}{g_0(1)-g_1(1,1)} \quad \leq \quad \lim_{N\to\infty} F_S(\gamma)^N \quad \leq \quad \lim_{N\to\infty} F_S(\alpha_0)^N \quad = \quad Q^{-1}\left(\tfrac{-2g_1(1,1)}{g_0(1)-2g_1(1,1)}\right)$$

Letting

$$\xi \quad \equiv \quad \frac{g_0(1)}{-g_1(1, 1)} \quad > \quad 0$$

and applying Q (strictly increasing) to both sides of the outermost inequality gives

$$Q\left(\tfrac{1}{\xi+1}\right) \quad \leq \quad \tfrac{2}{\xi+2}$$

Some algebra shows $Q(\frac{1}{\xi+1}) = \frac{\ln(\xi+1)}{\xi}$, and $\lim F_S(\gamma)^N \leq \lim F_S(\alpha_0)^N$ is therefore equivalent to

$$(\xi + 2)\ln(\xi + 1) \quad \leq \quad 2\xi$$

If we let $\varphi(\xi) \equiv (\xi+2)\ln(\xi+1)-2\xi$, however, $\varphi(0) = 0$, $\varphi'(0) = 0$, and $\varphi''(\xi) = \frac{1}{\xi+1} - \frac{1}{(\xi+1)^2} > 0$ for $\xi > 0$, and therefore $\varphi(\xi) > 0$ for $\xi > 0$, giving a contradiction. Thus, for any $g_0(1) > 0 > g_1(1, 1)$, $\lim_{N\to\infty} F_S(\gamma)^N > \lim_{N\to\infty} F_S(\alpha_0)^N$, so all bidders opting out is strictly more likely in the auction with unrestricted entry than in the indicative bidding mechanism.[26]

Part 2: Higher Revenue. We showed above that in the limit as N goes to infinity, ex-ante bidder surplus is 0, so the seller captures all the surplus. Thus, it suffices to show that total surplus is higher in the indicative bidding game than in the unrestricted auction.

To show this, consider a social planner who starts with the unrestricted auction at its equilibrium participation level, and moves to the indicative bidding game in two steps:

1. In step 1, switch from unrestricted entry to indicative bidding, but holding the participation level constant

2. In step 2, change the participation level from its old level to the new equilibrium level

We will show that each step increases total surplus.

When N is large, equilibrium of the indicative bidding game uses only messages 0 and 1; so the only change in step 1 is that when more than two bidders opt in, two are chosen at random for the second round instead of all of them advancing. The "no net externality" result discussed in the text still holds when there are more than two bidders, so when a bidder is "sent home" rather than being allowed to participate in the auction, the combined payoffs to the seller and the remaining bidders does not change, so the change in total surplus is simply the negative of the expected payoff that bidder would have received. When N is large, all entering bidders have types close to 1, and so conditional on multiple bidders advancing to the second round, each one has a negative expected payoff; thus, dismissing bidders in this way strictly increases total surplus. So step 1 strictly increases total surplus.

As for the second step, we showed above that when the entry threshold is at or above its equilibrium level, total surplus is strictly decreasing in the entry threshold. And we showed that the equilibrium threshold is lower in the indicative bidding game than in the unrestricted auction. Thus, step 2 involves lowering the entry threshold from a higher level, down to its equilibrium level; which we showed above strictly increases total surplus.

Thus, switching from an unrestricted auction to indicative bidding strictly increases total surplus when N is sufficiently large; since bidder surplus is 0, it strictly increases revenue as well.

Part 3: Greater Bidder Surplus. Since limit bidder surplus is 0 in either game, we want to show this result for N large but finite (and therefore $\alpha_0 < 1$ and $\gamma < 1$), not at the actual limit. In the game with indicative bids, bidders with types $s_i < \alpha_0$ get zero payoff, and bidders with types $s_i \geq \alpha_0$ get payoff

$$v_\tau(s_i, 1) \quad = \quad F_S(\alpha_0)^{N-1} g_0(s_i) + \left(\frac{2}{N} \frac{1-F_S(\alpha_0)^N}{1-F_S(\alpha_0)} - 2F_S(\alpha_0)^{N-1} \right) g_1(s_i, [\alpha_0, 1])$$

[26] As noted earlier, if $F_S(\alpha_0)^N \to \phi$, $N(1-F_S(\alpha_0))F_S(\alpha_0)^{N-1} \to -\phi \ln(\phi)$; and similarly, $N(1-F_S(\gamma))F_S(\gamma)^{N-1} \to -q \ln(q)$ if $q = \lim F_S(\gamma)^N$. $-\lambda \ln(\lambda) + \lambda$ is increasing in λ, so $q > \phi$ implies $-q \ln(q) + q > -\phi \ln(\phi) + \phi$: the probability that zero or one bidder enters (and therefore the probability of zero revenue) is also higher in the auction with unrestricted entry.

Since $v_\tau(\alpha_0, 1) = 0$, we can write $v_\tau(s_i, 1)$ as $v_\tau(s_i, 1) - v_\tau(\alpha_0, 1)$, so

$$\begin{aligned} v_\tau(s_i,1) &= F_S(\alpha_0)^{N-1}(g_0(s_i) - g_0(\alpha_0)) + \left(\tfrac{2}{N}\tfrac{1-F_S(\alpha_0)^N}{1-F_S(\alpha_0)} - 2F_S(\alpha_0)^{N-1}\right)(g_1(s_i,[\alpha_0,1]) - g_1(\alpha_0,[\alpha_0,1])) \\ &= (s_i-\alpha_0)\left[F_S(\alpha_0)^{N-1}\tfrac{g_0(s_i)-g_0(\alpha_0)}{s_i-\alpha_0} + \left(\tfrac{2}{N}\tfrac{1-F_S(\alpha_0)^N}{1-F_S(\alpha_0)} - 2F_S(\alpha_0)^{N-1}\right)E_{s_j\in[\alpha_0,1]}\tfrac{g_1(s_i,s_j)-g_1(\alpha_0,s_j)}{s_i-\alpha_0}\right] \end{aligned}$$

with the expectation over s_j taken with respect to F_S truncated to $[\alpha_0, 1]$.

Now, $g_0(s) = -c+s+E(t_i)$, so $\frac{g_0(s_i)-g_0(\alpha_0)}{s_i-\alpha_0} = 1$. As for $g_1(s_i, s_j) = -c+E\max\{0, s_i-s_j+t_i-t_j\}$, note that $\max\{0, s_i - s_j + t_i - t_j\}$ (at a particular realization of t_i and t_j, not in expectation) is differentiable with respect to s_i almost everywhere, with derivative 1 when $s_i - s_j + t_i - t_j > 0$ and derivative 0 when $s_i - s_j + t_i - t_j < 0$. This means $g_1(s_i, s_j)$ has derivative equal to the probability that $t_i - t_j > -s_i + s_j$. Now, $t_i - t_j$ has a nondegernate, massless distribution which is symmetric about 0, so when s_i and s_j are close together, $g_1(s_i, s_j)$ has derivative close to $\frac{1}{2}$. In the limit, as $\alpha_0 \to 1$, $(s_i, s_j) \to (1, 1)$, so

$$v_\tau(s_i,1) \approx (s_i-\alpha_0)\left[F_S(\alpha_0)^{N-1} + \left(\tfrac{2}{N}\tfrac{1-F_S(\alpha_0)^N}{1-F_S(\alpha_0)} - 2F_S(\alpha_0)^{N-1}\right)\tfrac{1}{2}\right] = (s_i-\alpha_0)\tfrac{1-F_S(\alpha_0)^N}{N(1-F_S(\alpha_0))}$$

The ex-ante expected surplus of a single bidder is then

$$E_{s_i}\max\{0, v_\tau(s_i,1)\} = \int_0^{\alpha_0} 0 f_S(s)ds + \int_{\alpha_0}^1 v_\tau(s,1)f_S(s)ds$$

Since f_S was assumed to be continuous, in the limit as $\alpha_0 \to 1$, this goes to

$$\int_{\alpha_0}^1 (s_i-\alpha_0)\tfrac{1-F_S(\alpha_0)^N}{N(1-F_S(\alpha_0))}f_S(1)ds = \tfrac{1}{2}(1-\alpha_0)^2\tfrac{1-F_S(\alpha_0)^N}{N(1-F_S(\alpha_0))}f_S(1) = \tfrac{1}{2N}(1-\alpha_0)(1-F_S(\alpha_0)^N)$$

with the last equality coming because $1 - F_S(\alpha_0) = \int_{\alpha_0}^1 f_S(s)ds = (1 - \alpha_0)f_S(1)$ in the limit. Summing over all N bidders, then, the combined expected surplus of all bidders in the indicative-bidding game is $U_I = \frac{1}{2}(1 - \alpha_0)(1 - F_S(\alpha_0)^N)$. This is equal to 0 in the limit (since $\alpha_0 \to 1$), but if we multiply by N to calculate the leading term, and once again note that $1 - \alpha_0 = \frac{1-F_S(\alpha_0)}{f_S(1)}$,

$$N \cdot U_I \to \tfrac{1}{2f_S(1)}N(1-F_S(\alpha_0))(1-F_S(\alpha_0)^N) \to \tfrac{1}{2f_S(1)}(-\ln(\phi))(1-\phi)$$

where $\phi = \lim F_S(\alpha_0)^N$ as before.

To compare, we next do the same calculation for the auction with unrestricted entry. Let γ denote the entry threshold; the expected payoff to a bidder with type $s_i \geq \gamma$ is

$$\sum_{n=0}^{N-1}\binom{N-1}{n}F_S(\gamma)^{N-1-n}(1-F_S(\gamma))^n g_n(s_i,[\gamma,1])$$

which is 0 at $s_i = \gamma$ and can therefore be rewritten as

$$\begin{aligned}
&\textstyle\sum_{n=0}^{N-1} \binom{N-1}{n} F_S(\gamma)^{N-1-n}(1-F_S(\gamma))^n \left(g_n(s_i,[\gamma,1]) - g_n(\gamma,[\gamma,1])\right) \\
= \quad & (s_i-\gamma) \textstyle\sum_{n=0}^{N-1} \binom{N-1}{n} F_S(\gamma)^{N-1-n}(1-F_S(\gamma))^n \frac{g_n(s_i,[\gamma,1])-g_n(\gamma,[\gamma,1])}{s_i-\gamma}
\end{aligned}$$

As $\gamma \to 1$, and therefore $s_i \to 1$ as well, the derivative of g_n with respect to its first argument goes to the probability that T_i is the greatest of the $\{T_j\}$, which by symmetry is $\frac{1}{n+1}$ when there are n other bidders; so the expected payoff is

$$(s_i-\gamma) \textstyle\sum_{n=0}^{N-1} \binom{N-1}{n} F_S(\gamma)^{N-1-n}(1-F_S(\gamma))^n \frac{1}{n+1} \quad = \quad (s_i-\gamma)\frac{1}{1-F_S(\gamma)}\frac{1}{N}\left(1-F_S(\gamma)^N\right)$$

(Equality holds because $\binom{N-1}{n}\frac{1}{n+1} = \frac{(N-1)!}{(n+1)!(N-(n+1))!} = \frac{1}{N}\binom{N}{n+1}$, and $\sum_{n=1}^{N}\binom{N}{n}x^{N-n}(1-x)^n = 1$.) Taking the expectation over s_i, with payoff 0 for types $s_i < \gamma$, gives a single bidder's ex-ante expected payoff as

$$\begin{aligned}
&\textstyle\int_0^\gamma 0 f_S(s)ds + \int_\gamma^1 (s-\gamma)\frac{1}{1-F_S(\gamma)}\frac{1}{N}\left(1-F_S(\gamma)^N\right) f_S(s)ds \\
\approx \quad & \tfrac{1}{2}(1-\gamma)^2 \frac{1}{1-F_S(\gamma)}\frac{1}{N}\left(1-F_S(\gamma)^N\right) f_S(1) \\
\approx \quad & \tfrac{1}{2}(1-\gamma)\frac{1}{N}\left(1-F_S(\gamma)^N\right)
\end{aligned}$$

so the combined surplus of all bidders is $U_A = \frac{1}{2}(1-\gamma)\left(1-F_S(\gamma)^N\right)$. Rewriting $1-\gamma$ as $\frac{1-F_S(\gamma)}{f_S(1)}$, multiplying by N, and letting q denote the limit of γ^N, we get

$$\begin{aligned}
\lim N \cdot U_A &= \lim \frac{1}{2f_S(1)} N(1-F_S(\gamma))(1-F_S(\gamma)^N) \\
&= \lim \frac{1}{2f_S(1)} (-\ln(F_S(\gamma)^N))(1-F_S(\gamma)^N) \\
&= \frac{1}{2f_S(1)}(-\ln(q))(1-q)
\end{aligned}$$

The function $-(1-h)\ln(h)$ is strictly decreasing in h, and we showed above that $\phi = \lim F_S(\alpha_0)^N < \lim F_S(\gamma)^N = q$, so

$$\lim N \cdot U_I \quad = \quad -\frac{1}{2f_S(1)}(1-\phi)\ln(\phi) \quad > \quad -\frac{1}{2f_S(1)}(1-q)\ln(q) \quad = \quad \lim N \cdot U_A$$

so bidder surplus is strictly higher with indicative bidding when N is large but finite.

A.11 Proof of Theorem 5

For ease of exposition, we will show this for the case where $s_i \sim U[0,1]$, so that $F_S(h) = h$. Subject to the assumption (made in the text) that $g_n(h,[h,1]) = 0$ has at most one solution, the results all extend easily to non-uniform F_S.

Part 1: $n \leq n^*$. First, suppose $\underline{\alpha}^{(n)} = 0$, or $g_{n-1}(0, [0,1]) \geq 0$. This of course also implies that a bidder with type higher than his opponents earns positive expected payoffs; so if message M was not used, bidders with types close to 1 would deviate to it. Like in the baseline model, expected payoffs have strict single-crossing differences in message and type, so symmetric equilibrium must be monotonic. So in any symmetric equilibrium where more than one message is used, there must be some $\alpha > 0$ such that bidders above α send M and bidders below α do not. But if $g_{n-1}(\alpha, [\alpha, 1]) > 0$ for any positive α, bidders with types slightly below α would prefer to deviate to message M; so any symmetric equilibrium must have the entire interval $(0, 1]$ sending message M.

So now assume $\underline{\alpha}^{(n)} > 0$, and suppose $M = 1$ (only one opt-in message is available). If a symmetric equilibrium exists, the threshold α must solve

$$0 \quad = \quad \sum_{j=0}^{n-2} \binom{N-1}{j} \alpha^{N-1-j}(1-\alpha)^j g_j(\alpha, [\alpha, 1]) + \sum_{j=n-1}^{N-1} \binom{N-1}{j} \alpha^{N-1-j}(1-\alpha)^j \tfrac{n}{j+1} g_{n-1}(\alpha, [\alpha, 1])$$

If $\alpha \geq \underline{\alpha}^{(n)}$, then $g_{n-1}(\alpha, [\alpha, 1]) \geq 0$ and $g_j(\alpha, [\alpha, 1]) > 0$ for $j < n$, so the right-hand side is strictly positive; so $\alpha < \underline{\alpha}^{(n)}$ at any equilibrium. (At $\alpha = 0$, only the $j = N - 1$ term survives, so if $\underline{\alpha}^{(n)} > 0$, the right-hand side is strictly negative at $\alpha = 0$ and by continuity, an equilibrium exists.)

Next, we show $\alpha \to \underline{\alpha}^{(n)}$ as $N \to \infty$. With some algebra, we can rearrange the indifference condition and multiply by N to get

$$0 \quad = \quad N \sum_{j=0}^{n-2} \binom{N-1}{j} \alpha^{N-1-j}(1-\alpha)^j g_j(\alpha, [\alpha, 1]) + \tfrac{n}{1-\alpha} \left[1 - \sum_{j=0}^{n-1} \binom{N}{j} \alpha^{N-j}(1-\alpha)^j\right] g_{n-1}(\alpha, [\alpha, 1])$$

Each term in both summations is of order at most $N^{n-1}\alpha^{N-(n-1)}$; we already know $\alpha \leq \underline{\alpha}^{(n)} < 1$, so α is bounded away from 1, so $N^{n-1}\alpha^{N-(n-1)}$ goes to 0 as N grows. So as N grows, all terms but $\frac{n}{1-\alpha} g_{n-1}(\alpha, [\alpha, 1])$ vanish; so for equality, we need $g_{n-1}(\alpha, [\alpha, 1]) \to 0$, or $\alpha \to \underline{\alpha}^{(n)}$.

For $M > 2$, the argument is similar. Message M must be used, or it would be a profitable deviation for bidders with types close to 1. Let α be the cutoff between bidders using M and any lower message. If $\alpha > \underline{\alpha}^{(n)}$, then $g_{n-1}(\alpha, [\alpha, 1]) > 0$, and bidders with types slightly below α will deviate to message M. For any $\alpha < \underline{\alpha}^{(n)}$, as N grows, the events in the marginal bidder's expected payoff where at least $n-1$ other bidders have types above α dominate the other events, so expected payoff is negative; so $\alpha \to \underline{\alpha}^{(n)}$. We can't characterize exactly what happens below α, but it's not relevant for payoffs, since as N grows, the probability goes to 1 that at least n bidders have types above α.

Part 2: $n > n^*$. This is the generalization of the "small rents" case $g_1(1, 1) < 0$ when $n = 2$. First, if at least three messages were used in equilibrium, the width of the interval of types sending the second-highest message must be bounded away from zero. (The bidders sending the highest message must strictly prefer to advance against $n-1$ opponents sending the second-highest message; if the interval of bidders sending the second-highest were vanishingly small, bidders just above that interval would be getting payoff equal to $g_{n-1}(1, 1) < 0$ in that scenario.) This means if three or more messages are used, α_0 is bounded away from 1; but then as N grows, the probability fewer than n bidders opt in goes to 0, and so bidders with types just above α_0 earn negative payoffs.

Once we know only two messages get used, the indifference condition (as in the above case) is

$$0 = N\sum_{j=0}^{n-2}\binom{N-1}{j}\alpha^{N-1-j}(1-\alpha)^j g_j(\alpha,[\alpha,1]) + \frac{n}{1-\alpha}\left[1-\sum_{j=0}^{n-1}\binom{N}{j}\alpha^{N-j}(1-\alpha)^j\right] g_{n-1}(\alpha,[\alpha,1])$$

As argued above, as N grows, if α were bounded away from 1, all terms would vanish but $\frac{n}{1-\alpha}g_{n-1}(\alpha,[\alpha,1]) \leq \frac{n}{1-\alpha}g_{n-1}(1,1) < 0$, so $\alpha \to 1$. If $\alpha^N \to 1$, only the first term, $N\alpha^{N-1}g_0(\alpha)$, survives, which is strictly positive. If $\alpha^N \to 0$, the $\frac{n}{1-\alpha}g_{n-1}(\alpha,[\alpha,1])$ term once again dominates, which is negative. Thus, α^N goes to some limit in $(0,1)$.

A.12 Proof of Theorem 6

First, let $n < n'^*$, and consider a change in the number of advancing bidders from n to n'. Make the transition in two steps: first, increase the participation threshold from the old equilibrium level $\underline{\alpha}^{(n)}$ to what will be the new equilibrium level $\underline{\alpha}^{(n')} > \underline{\alpha}^{(n)}$; and second, increase the number of advancing bidders from n to n' with the entry threshold already at $\alpha_{(n')}$. The first step increases total surplus: for large N, there is still plenty of participation, but now entrants have stochastically higher types. The second step also increases total surplus: with only bidders above $\underline{\alpha}^{(n')}$ participating, the efficient number of participants is at least n'. (With n' participants, even marginal types break even, so inframarginal types earn positive payoffs, so the move from n to n' participants is strictly surplus-positive.) Both steps also strictly increase seller's revenue.

Second, let $n > n'^*$, and consider a decrease from n to n' in the number of advancing bidders, in two steps: first changing the number selected without changing the participation threshold, and second, increasing the participation threshold to the new equilibrium level. Similar to the arguments in the proof of Theorem 4, both steps increase total surplus: the first because any bidders newly sent home would have expected negative payoff (and their dismissal imposes no net externality on the other players); and the second because the equilibrium entry level is still inefficiently low (because incremental entrants impose positive externalities, since they only impose a net externality if they "crowd out" another bidder, who would have expected a negative payoff). Since $\alpha \to 1$ in the limit, bidder surplus is zero (since marginal entrants are indifferent and as $\alpha \to 1$, there are no inframarginal entrants), so an increase in total surplus also increases revenue.

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