The Form of Incentive Contracts: Agency with Moral Hazard, Risk Neutrality and Limited Liability *

Joaquín Poblete,† Daniel Spulber‡

January 2011

Abstract

The paper derives the optimal incentive contract in an agency model with moral hazard, risk neutrality, and limited liability. The analysis introduces a critical ratio, which equals the hazard rate of the shock times the marginal rate of technical substitution of the agent’s effort for the shock. The critical ratio indicates the returns to providing incentives for effort in each state of the world. The optimal contract provides incentives for effort in those states when the returns to providing incentives for effort exceed the costs of providing those incentives. When the critical ratio is increasing in the shock, the optimal contract takes the form of debt. When the critical ratio is decreasing in the shock, the optimal contract takes the form of a capped bonus. The properties of the critical ratio are easily determined and hold for a wide range of applied problems in economics and finance. The critical ratio is characterized using a state-space approach and the analysis introduces a corresponding condition in the Mirrlees reduced-form setting. The analysis also provides a framework for comparing moral hazard with adverse selection.

Key Words: Agency, Incentives, Contract, Moral Hazard, Debt, Limited Liability.

---

*We thank the coeditor David Martimort and two anonymous referees for their helpful and insightful comments that greatly improved the paper. We also thank Sandeep Baliga, Marco Ottaviani, Yuk-fai Fong, Yogmin Chen, Tom Gresik and Alessandro Pavan for interesting comments. Poblete is grateful to the Searle Center for Law, Regulation and Economic Growth for research support. Spulber is grateful for the support from a research grant from the Ewing Marion Kauffman Foundation on Entrepreneurship.

†London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom E-mail: j.j.poblete-lavanchy@lse.ac.uk

‡Elinor Hobbs Distinguished Professor of International Business, Management & Strategy, Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL, 60208. E-mail: jems@kellogg.northwestern.edu
1 Introduction

Incentive contracts between principals and agents are fundamental for a wide range of economic and financial transactions. Principals apply agency contracts to provide incentives for employees, managers, insurance buyers, attorneys, sales personnel, business representatives, independent contractors, tenant farmers, and regulated firms.\(^1\) Agency contracts are used in financial agreements between investors and entrepreneurs, banks and borrowers, trustees and beneficiaries, and shareholders and corporate managers. Debt contracts are perhaps the most widely-used form of incentive agreement used in practice because of the importance of bank loans, credit arrangements, corporate bonds, and related financing arrangements. The optimality of debt contracts is an important issue, as emphasized in the extensive discussion in Bolton and Dewatripont (2004). Debt contracts specify a critical threshold such that the agent receives the full marginal returns to effort only when the outcome of the project exceeds the face-value threshold. This contractual form contrasts with financial sharing agreements such as equity and more complex payment schedules based on outcomes. We characterize agency contracts with unobservable effort, risk neutrality, and limited liability. We obtain general conditions under which the optimal contract takes the form of debt.

Our analysis helps to explains the economic forces that motivate the widespread use of debt contracts. We introduce a critical ratio that provides an intuitive and easily-derived condition for characterizing the optimal contract. We model uncertainty explicitly by using a state-space approach, which is a parametric representation that differs from the now-standard approach of working directly with the induced probability distribution on outcomes. The critical ratio is a function of the state variable and the agent’s effort. The critical ratio equals the hazard rate of the random shock multiplied by the marginal rate of technical substitution between the agent’s effort and the random shock. The properties of the critical ratio determine the form of the optimal contract; when the critical ratio is increasing in the shock, the optimal contract takes the form of debt and when the critical ratio is decreasing in the shock, the optimal contract takes the form of a call option. When

\(^1\) The economic model of agency has its origins in labor contracts in economic models of agriculture and industry, particularly sharecropping and piece-rate labor contracts, see Otsuka et al. (1992). Casadesus-Masanell and Spulber (2005) give an overview of the economics of agency and the connection to agency law.
the critical ratio is constant in the shock, the set of optimal contracts contains linear contracts in addition to other contractual forms. The properties of the critical ratio are satisfied in many applied models based on standard forms of the production technology and the probability density of the shock.

The critical ratio is a useful concept because it indicates the returns to providing incentives for effort in each state of the world. In the state-space representation, the outcome of a task performed by the agent depends on the effort of the agent and a random shock. The principal only observes the outcome of the project, so that the outcome is a signal of the unobservable agent effort and shock. There is common knowledge regarding the probability distribution of the shock, the form of the production technology, the agent’s preferences, and the agent’s cost of effort. The critical ratio summarizes the relevant information about the probability distribution of the shock and the form of the production technology. For a given level of effort, the agent knows that higher realizations of the shock will result in better outcomes. Then, for a given level of effort, we can assign a cutoff based on the shadow price to the agent’s incentive compatibility constraint. The critical ratio implies that the optimal contract is within a class of contracts such that the slope of the contract in the outcome is zero when the critical ratio is below the cutoff and the slope of the contract is in the outcome is one when the critical ratio is above cutoff.

The critical ratio suggests how the parties should design the contract so as to motivate agent effort efficiently. When the critical ratio is increasing in the random shock, the agency contract is such that greater incentives for effort are associated with higher realizations of the shock for any given effort level. Therefore, when the critical ratio is increasing in the random shock, it follows that the optimal contract is debt because the agent retains the full marginal product of effort when the outcome of the project clears the threshold equal to the face value of the debt. Conversely, when the critical ratio is decreasing in the random shock, the agency contract is such that greater incentives for effort are associated with lower realizations of the shock. It follows that when the critical ratio is decreasing in the shock, the optimal contract is a bonus because the agent retains the full marginal product of effort when the outcome of the project is less than a cap. When the critical ratio is linear in the random shock, all states are equally efficient and only in that case are linear contracts efficient. When the critical ratio is monotonic, the optimal contract is obtained without additional
regularity assumptions such as implementability.

When the agent’s action is not observable, the parties design the optimal contract to maximize their joint benefits while mitigating moral hazard. Because the agent has limited liability, the principal cannot sell the project to the agent and the parties cannot attain the first-best outcome. Our analysis builds on and extends that of Innes (1990) who shows that with limited liability for a risk-neutral principal and agent, the optimal contract with moral hazard takes the form of debt, see also the analysis of the Innes model in Bolton and Dewatripont (2004). Innes (1990) applies the reduced-form approach in which the probability distribution of outcomes depends on the agent’s choice of effort. The reduced-form approach to agency originates with Mirrlees (1974, 1976) and Holmstrom (1979). We characterize the optimal agency contract with moral hazard by applying the more intuitive state-space approach or parametric representation.

We compare the state-space approach with the reduced-form Mirrlees approach often used in moral hazard models of agency. We show that the critical ratio is increasing in the state variable if and only if the reduced-form distribution of outcomes contingent on the agent’s effort has a hazard rate that is increasing in the agent’s effort. Additionally, we show that our increasing critical ratio condition, or correspondingly the condition that the reduced-form hazard rate for output is decreasing in effort, is less restrictive than assumptions generally used in moral hazard agency models. In particular, we show that the Monotonic Likelihood Ratio Principle (MLRP) implies that the critical ratio is increasing in the state variable but an increasing critical ratio does not imply MLRP. This considerably enlarges the set of production technologies and the set of distributions of the shock that generate optimal agency contracts.

The state-space approach also allows us to compare the moral hazard model of agency with adverse selection models. We show that the sum across agents of the reciprocal of our critical ratio exactly equals information rents in the corresponding adverse selection model. We also show that our increasing critical ratio condition differs from the Spence-Mirrlees single-crossing condition that is standard in adverse selection models. However, when the agent’s effort and the shock are complements in the production of output, there is a fundamental connection between moral hazard and adverse selection. Although the shock, or respectively the agent’s type, are unobservable, it is desir-
able to reward effort in better states because the agent’s effort is more productive in better states. When the agent’s effort and the shock are complements, concavity of the production technology in effort and the shock imply that the increasing marginal rate of technical substitution in our setting and the Spence-Mirrlees single-crossing condition are equivalent. We also show how the critical ratio can be used to characterize contracts with adverse selection by extending Sappington (1990). We demonstrate that with limited liability and risk neutrality, optimal contracts with adverse selection have similarities to debt-style contracts.

The critical ratio introduced here has properties that are readily verified in a wide range of applied problems in economics and finance. The state-space or parametric approach allows us to derive the optimal agency contract based on the underlying forms of risk and the production function. By explicitly addressing uncertainty, the state-space approach corresponds with many economics and finance models with random shocks that affect preferences, initial endowments, technologies, information, exogenous public policies, and states of nature. Parametric uncertainty also is important because it is consistent with empirical analysis in economics and finance. The state-space approach in agency originates in Spence and Zeckhauser (1971), Ross (1973), and Harris and Raviv (1976) who apply a first-order approach with risk-averse agents.

The optimality of debt-style contracts implies that such contracts perform better under uncertainty than other contractual forms. Wealth constraints tend to generate debt-style contracts as emphasized by Innes (1990) and Holmstrom and Tirole (1997). Our results confirm Innes’ conclusion about the optimality of debt-style contracts while generalizing the analysis of optimal contracts and providing new intuition for the results. Debt-style contracts clearly have empirical implications. In practice, debt-style contracts are widely used and correspond to all kinds of compensation agreements with threshold effects. Jewitt et al (2008) consider the effects of imposing minimum and maximum limits on payments that represent limited liability and other constraints and find that payment limits can induce option-type contracts without requiring monotonicity of contracts. In contrast to the present work, they assume that the agent is risk averse and impose various assumptions including MLRP.

Perhaps most significantly, debt-style contracts perform better than equity-style sharing rules under uncertainty. Economic studies often assume that the contract is a linear sharing rule, appealing to
the dynamic aggregation result of Holmstrom and Milgrom (1987). The optimality of debt contracts addresses an important issue in the vast literature on corporate finance, beginning with the work of Jensen and Meckling (1976). Jensen and Meckling point that debt strengthens incentives relative to equity because the entrepreneur does not share the marginal returns to effort in states where the firm is solvent. Jensen and Meckling observe that debt entails several agency costs; debt encourages the entrepreneur to take excessive risks, debt may lead the entrepreneur to misrepresent to lenders the riskiness of projects, and debt entails bankruptcy and reorganization costs. Jensen and Meckling argue that although the entrepreneur bears the agency costs when selling debt, he takes on debt to take advantage of investment opportunities not attainable due to a limited initial endowment, with the mix of debt and equity reflecting relative agency costs. In addition to the incentive effects of debt for entrepreneurs and owners of firms, debt-style contracts also provide incentives for managers and correspond to call options commonly used in executive compensation. Debt-style contracts also correspond to financial assets including securities and bonds whose features resemble options, see for example Cox and Rubinstein (1985). Real options are an important tool for analyzing investment under uncertainty, see Dixit and Pindyck (1994). The simplicity of debt-style contracts and their optimality in a broad range of environments helps to explain their widespread application.

Bonus contracts are also widely used and often feature a cap on earnings (Healy, 1985, Arya et al. 2007). Jensen (2003) recommends linear bonus incentives without setting targets or imposing caps. Arya et al (2007) find that capping bonuses in compensation plans is a puzzle and suggest that bonus caps can help align incentives between owners and managers. Our analysis shows that with a decreasing critical ratio, bonus caps result from limited liability and shifting incentives to perform to lower realizations of the shock. The optimality of bonus contracts with caps when the critical ratio is decreasing helps to explain the usage of such contracts.

The paper is organized as follows. Section 2 considers the basic model of agency and presents our main assumptions. Section 3 introduces the critical ratio and examines its implications for agency contracts. Section 4 derives the optimal contract and shows the connection of the form of the contract to the properties of the critical ratio. Section 5 discusses the connection between our critical ratio and the Mirrlees reduced-form approach in moral hazard problems and the Spence-Mirrlees condition in adverse selection models, and Section 6 concludes.
Consider two risk-neutral economic actors who enter into a contract. The actor designated as the principal is an investor who provides financial capital and the actor designated as the agent is an entrepreneur who devotes effort to establishing and operating a firm. The project also can represent other situations: the agent performs a service under authority delegated by the principal, the agent is an employee who produces a good within a firm owned by the principal, the agent is a producer who provides a good under a contract from the principal.

The agent provides the production technology for the project and supplies productive effort, \( a \geq 0 \). The agent’s action, \( a \), also represents the cost of effort to the agent. The agent owns a production technology given by

\[
\Pi = \Pi(\theta, a),
\]

where the outcome \( \Pi \) represents output given the agent’s effort, \( a \), and the realization of a random variable, \( \theta \). The principal observes the outcome, \( \Pi \), but cannot observe either the agent’s effort, \( a \), or the state variable, \( \theta \). The agent chooses the action, \( a \), before the realization of the state variable, \( \theta \). The outcome \( \Pi \) can represent revenue or profit resulting from the realization of the state and the agent’s effort. In many applications, the outcome \( \Pi \) is characterized as a signal of the underlying state variable and effort.\(^2\)

The random variable \( \theta \) has a density function \( f(\theta) \) with support \([0, \bar{\theta}]\) and cumulative distribution function \( F(\theta) \). We also allow for the support of \( f(\theta) \) to be \([0, \infty)\). The main part of the discussion features the state-space representation, also referred to as the parametric representation. In a later section, we consider the relationship to the standard reduced-form representation.

The state-space formulation allows specification of the properties of the production technology. Assume that \( \Pi(\theta, a) \) is twice differentiable in \( a \) and \( \theta \), and let \( \Pi_\theta(\theta, a) = \partial \Pi(\theta, a) / \partial \theta \) and \( \Pi_a(\theta, a) = \partial \Pi(\theta, a) / \partial a \). Let \( \Pi(0, a) = 0 \) for ease of presentation.

**Assumption 1** The production technology, \( \Pi(\theta, a) \), is increasing in \( \theta \).

\(^2\) See Spence and Zeckhauser (1971) and Conlon (2009).
This assumption allows us to define $\hat{\theta}(\Pi, a)$ as the shock that satisfies

$$\Pi(\hat{\theta}(\Pi, a), a) = \Pi,$$

(2)

for an effort level $a$, and realization of output, $\Pi$.

Next, we require that the production technology exhibit diminishing marginal returns to effort.

**Assumption 2** The production technology, $\Pi(\theta, a)$, is increasing and weakly concave in $a$.

Assumption 2 guarantees the existence of a first-best effort level

$$a^{FB} = \arg \max_a \int_0^\theta \Pi(\theta, a) f(\theta) d\theta - a.$$

(3)

The contract between the principal and the agent is fully described by a payment from the principal to the agent, $w$. The contract is based only on the outcome, $\Pi$, because the agent’s choice of effort, $a$, and the realization of the random variable $\theta$, are not observable to the principal,

$$w = w(\Pi).$$

(4)

Let $K$ be the cost of establishing the firm, or equivalently $K$ is the cost of providing the project to the agent. The investor’s financial commitment equals the initial investment $K$ and after investing $K$, the principal’s benefit from the project is $\Pi - w$. The entrepreneur’s wealth, or initial endowment, is normalized to zero. The entrepreneur’s liability constraint implies that the payment to the agent must be nonnegative, $w \geq 0$. The entrepreneur’s limited liability prevents the entrepreneur from self-financing the firm, or equivalently rules out the principal selling the task to the agent and achieving the first-best outcome.

For any realization of $\Pi$, the principal’s net benefit is equal to the outcome minus the payment to the agent and minus the firm’s fixed cost,

$$v(w, \Pi) = \Pi - w(\Pi) - K.$$

(5)

The principal’s expected net benefit given the form of the outcome function and the distribution of
the random variable, $\theta$, equals

$$V(w, a) = \int_0^\theta (\Pi(\theta, a) - w(\Pi(\theta, a))) f(\theta) d\theta - K. \quad (6)$$

The contract is individually rational for the principal if $V(w, a) \geq 0$.

The entrepreneur has an opportunity cost $u_0 > 0$. The entrepreneur’s net benefit is given by the payment net of the cost of effort,

$$u(w, a, \Pi) = w(\Pi) - a - u_0. \quad (7)$$

Given the contract, $w$, the agent chooses effort to maximize his expected net benefit,

$$U(w, a) = \int_0^\theta w(\Pi(a, p)) f(p) dp - a - u_0. \quad (8)$$

The agent’s effort is said to satisfy incentive compatibility if it maximizes his expected net benefit, $a \in \arg \max_x U(w, x)$. The contract is individually rational for the agent if his expected net benefit is nonnegative, $U(w, a) \geq 0$. The optimal contract maximizes the expected net benefit of the principal, $V$, subject to incentive compatibility for the agent and individual rationality for the agent.

Following Innes (1990), we require the net benefit of the principal and that of the agent to be monotonic. As Innes explains, making the principal’s net benefit, $v(w, \Pi)$, non-decreasing in $\Pi$ rules out situations in which the parties may subvert the contract. This prevents the principal from sabotaging the project to avoid making payments to the agent after the agent has committed effort. The requirement that the principal’s net benefit is non-decreasing in $\Pi$ also rules out the situation in which the agent borrows money to inflate the outcome of the project and thereby increase his returns based on reported performance. The requirement that the agent’s net benefit, $u(w, \Pi)$, is non-decreasing in $\Pi$ prevents the agent from sabotaging the task to expropriate the principal. The monotonicity requirements rule out forcing contracts that would lead to this type of behavior. We define a feasible contract based on the monotonicity and limited-liability restrictions.

**Definition 1** A contract $w$ is feasible if (a) the principal’s net benefit and the agent’s net benefit, $v(w, \Pi)$ and $u(w, \Pi)$, are non-decreasing in $\Pi$, and (b) the payment to the agent is non-negative, $w \geq 0$. 
Because feasible contracts are monotonic, they are differentiable almost everywhere. Requiring the agent’s net benefit and the principal’s net benefit to be non-decreasing in the outcome implies that \( \Pi' - w(\Pi') \leq \Pi'' - w(\Pi'') \), and \( w(\Pi') \leq w(\Pi'') \) for outcomes \( \Pi' \leq \Pi'' \). Therefore, \( 0 \leq w(\Pi'') - w(\Pi') \leq \Pi'' - \Pi' \), which in the limit implies that \( 0 \leq w'(\Pi) \leq 1 \), where \( w'(\Pi) = dw(\Pi)/d\Pi \).

This also implies that feasible contracts are continuous. It will be shown that the optimization problem is linear in the slope of the contract so that constraints on the slope are necessary for the problem to be well defined. Otherwise, the objective function could be made arbitrarily large, either positively or negatively. The constraints have economic interpretations: the constraint \( w'(\Pi) \geq 0 \) prevents the agent from trying to reduce \( \Pi \) through sabotage, and the constraint \( w'(\Pi) \leq 1 \) prevents the agent from trying to inflate \( \Pi \) through secret borrowing, and also prevents the principal from trying to reduce \( \Pi \) through sabotage so as to avoid the commitment to the contract.

The principal’s problem of choosing an optimal contract subject to feasibility restrictions can be stated as follows,

\[
\max_{w,a} \int_0^{\bar{\theta}} \left[ \Pi(\theta, a) - w(\Pi(\theta, a)) \right] f(\theta) d\theta - K, \tag{9}
\]

subject to

\[
a \in \arg \max_x U(w(\Pi(\theta, x)), x), \tag{10}
\]

\[
U(w, a) \geq 0, \tag{11}
\]

\[
0 \leq w'(\Pi(\theta, a)) \leq 1, \tag{12}
\]

\[
w(\Pi(\theta, a)) \geq 0. \tag{13}
\]

The first constraint is the agent’s incentive compatibility condition for the agent’s choice of effort, \( a \), and the second constraint is the agent’s individual rationality condition. The third set of constraints results from feasibility and limits the slope of the contract. The constraint \( w(\Pi(\theta, a)) \geq 0 \) represents the agent’s limited liability.

Finally, to make sure that the problem has a solution we make the following assumption

**Assumption 3**  *There exists a feasible contract* \( w \) *and an effort level* \( a \) *such that the agent’s incentive compatibility constraint holds and the agent’s and the principal’s individual rationality constraints holds.*
The assumption will hold whenever the financing cost $K$ and the agent’s outside opportunity $u_0$ are small enough.

3 The Critical Ratio

This section provides an intuitive explanation for the optimality of contracts. The general idea is that contracts have two roles; to provide the agent with incentives for performance and to compensate the principal for his investment. The optimal contract is the one that combines these roles most effectively. The trade-off between these two roles is measured by the critical ratio.

Definition 2 The critical ratio $\rho(\theta, a)$ is given by the product of the hazard rate of the shock and the marginal rate of technical substitution (MRTS) of the agent’s effort for the shock,

$$\rho(\theta, a) = \frac{f(\theta)}{1 - F(\theta)} \frac{\Pi_a(\theta, a)}{\Pi_\theta(\theta, a)}.$$  (14)

The critical ratio indicates the expected return to providing incentives for effort to the agent in each state.

The hazard rate of the shock, $\frac{f(\theta)}{1 - F(\theta)}$, measures the likelihood that the shock has a particular value contingent on the shock being greater than or equal to that value. Assumptions on the hazard rate are commonly used in adverse selection models such as auctions and nonlinear pricing. As our later discussion shows, there is a large class of standard distributions that have monotone hazard rates. Hazard-rate conditions as applied in studies of reliability lend themselves to empirical testing and calibration.\(^3\)

The MRTS of agent effort for the shock, $\frac{\Pi_a(\theta, a)}{\Pi_\theta(\theta, a)}$, is the amount that agent effort must be reduced when the shock is increased so as to keep the outcome constant. The MRTS of agent effort for the shock is the ratio of the marginal product of effort to the marginal product of the shock. The MRTS is a common expression in economic models used to indicate the rate at which a firm’s input can be substituted for another input. As our discussion shows, there is a large class of production technologies that have a monotone MRTS. If the output is interpreted as consumer utility, then the

\(^3\) See for example Hall and Van Keilegom (2005).
term indicates the marginal rate of substitution and corresponds to many types of utility functions. The MRTS also plays an important role in agency models with adverse selection, as will be discussed in a later section.

We show that the optimal agency contract with moral hazard depends on the form of the critical ratio, which indicates the expected return to providing incentives for effort to the agent in each state. The ratio measures the relation between incentives and expected payoff when the slope of the contract $w'$ changes. If in some state the critical ratio is higher than in others, that implies that the state provides relatively greater incentives per unit of compensation and therefore the state is efficient at inducing effort from the agent. An efficient contract should therefore have a higher slope in those states with a higher critical ratio. From the principal’s perspective, among all contracts that implement the same effort level, the optimal contract is the one that gives the agent the smallest possible compensation. Since our assumptions require that the slope of the contract is no greater than one and no smaller than zero, the optimal contract from the principal’s standpoint is a contract with a slope equal to one in the states with the highest critical ratio and a slope of zero in the rest.

We begin by showing that among contracts implementing the same effort level, contracts that have a higher slope in states with a higher critical ratio give the principal a higher expected utility provided that some regularity conditions are satisfied. We define a class of contracts such that the slope of the contract is equal to one when the critical ratio is above certain level and the slope is zero when the critical ratio is below that level.

**Definition 3** A contract, $w(\cdot)$, is in the $L$-class if for a given effort level, $a$, there exists $\lambda > 0$ such that $w'(\Pi) = 0$ if $\rho(\theta, a) < \lambda$ and $w'(\Pi) = 1$ if $\rho(\theta, a) > \lambda$.

The first proposition shows that contracts in the $L$-class are optimal for a given level of the agent’s effort. We focus on the principal’s problem of choosing the optimal contract for a given level of the agent’s effort, $a$, that satisfies incentive compatibility. For now, we set aside the agent’s participation constraint, $U(w, a) \geq 0$ and assume that the first order condition approach is valid. The principal’s problem (9) can be written as the choice of a payment from the principal to the agent that maximizes
the principal’s net benefit subject to the agent’s first order incentive compatibility condition and other constraints,

$$\max_w \int_0^\theta [\Pi(\theta, a) - w(\Pi(\theta, a))] f(\theta) d\theta - K,$$

subject to

$$\int_0^\theta w'(\Pi(\theta, a)) \Pi_a(\theta, a) f(\theta) d\theta = 1,$$

as well as $0 \leq w'(\Pi(\theta, a)) \leq 1$ and limited liability, $w(\Pi(\theta, a)) \geq 0$.

**Proposition 1** Given an arbitrary feasible contract, $l(\cdot)$, and an effort level, $a$, that is optimal under $l(\cdot)$, there is an $L-$class contract $w(\cdot)$, such that $a$ satisfies the agent’s first order condition under $w(\cdot)$ and $w(\cdot)$ gives the principal a greater expected net benefit than under the contract $l(\cdot)$, $V(w, a) \geq V(l, a)$.

**Proof:** We show that among all the feasible contracts that satisfy the agent’s first order condition at the effort level, $a$, $L-$class contracts maximize the expected return of the principal. Integrating by parts, the principal’s objective function is equivalent to

$$\max_w \Pi(0, a) - w(\Pi(0, a)) + \int_0^\theta [1 - w'(\Pi(\theta, a))](1 - F(\theta)) \Pi_\theta(\theta, a) d\theta - K. \quad (17)$$

Notice that $\Pi(0, a) = 0$ by assumption. Setting aside the constraints $0 \leq w'(\Pi(\theta, a)) \leq 1$, the Lagrangian of the problem for a relaxed problem with the agent’s incentive compatibility constraint and Lagrange multiplier $\eta$ is

$$\mathcal{L} = -w(\Pi(0, a)) + \int_0^\theta \Pi_\theta(\theta, a)(1 - F(\theta)) d\theta + \int_0^\theta w'(\Pi(\theta, a))[\eta f(\theta) \Pi_a(\theta, a) - (1 - F(\theta)) \Pi_\theta(\theta, a)] d\theta - K. \quad (18)$$

By the agent’s limited liability constraint, $w(\Pi(0, a)) = 0$ is optimal. Notice that the problem is linear in $w'(\Pi(\theta, a))$ so that given the constraints $0 \leq w'(\Pi(\theta, a)) \leq 1$, the solution requires

$$w'(\Pi(\theta, a)) = 0 \text{ if } \rho(\theta, a) < 1/\eta, \text{ and } w'(\Pi(\theta, a)) = 1 \text{ if } \rho(\theta, a) > 1/\eta, \quad (19)$$

for the appropriate choice of $\eta$, where $\rho(\theta, a) = \frac{f(\theta) \Pi_a(\theta, a)}{1 - F(\theta) \Pi_\theta(\theta, a)}$. Letting $\lambda = 1/\eta$, it follows that the contract $w(\cdot)$ is in the $L$-class. □

To understand this result intuitively, suppose that it is the case that the first order approach is valid. Then, given a contract $w(\Pi)$, the agent’s effort is determined by the first order condition (16). The
derivative of the left-hand side of equation (16) with respect to the slope of the compensation $w'(\Pi)$ is $\Pi_\theta f(\theta)$. The higher is $\Pi_\theta f(\theta)$, the less we need to increase the slope of the payoff $w'$ to induce a given effort level. The term $\Pi_\theta f(\theta)$ represents how powerful is a given state in providing incentives. Intuitively, the greater are the likelihood of a state and the marginal return to effort, the more efficient is the state in providing incentives.

An increase in the slope of the payoff function has a direct cost to the principal but provides incentives to the agent. The shadow price $\eta$ indicates the increase in the principal’s objective that is obtained by relaxing the agent’s incentive compatibility constraint. The reciprocal, $\lambda = 1/\eta$, is the additional expenditure on incentives for the agent for an increase in the principal’s objective, so that $\lambda = 1/\eta$ indicates the cost of incentives. The agency contract puts weight on states for which the return on incentives for effort $\rho(\theta, a)$ is greater than the cost of incentives $\lambda = 1/\eta$. For a contract in the L-class, and effort level $a$ such that $\rho(\theta, a) < \lambda$ for some $\theta$ and $\rho(\theta, a) > \lambda$ for some $\theta$, it follows that $\int_0^\bar{\theta} \Pi_a(\theta, a)f(\theta)d\theta > 1$, so that the agent’s effort exhibits moral hazard, $a < a^{FB}$.

The second role of the contract is to provide compensation. The benefit of the contract to the agent is given by,

$$\int_0^\bar{\theta} w(\Pi(\theta, a))f(\theta)d\theta - a.$$ 

We can rewrite the benefit as a function of $w'$ using integration by parts

$$u(w, a) = \int_0^\bar{\theta} w'(\Pi(\theta, a))(1 - F(\theta))\Pi_\theta d\theta - a.$$  \hspace{1cm} (20)

The derivative of the expected payoff with respect to $w'$ is $(1 - F(\theta))\Pi_\theta$. The higher the term $(1 - F(\theta))\Pi_\theta$ is, the less we need to increase the slope of the payoff $w'$ to provide a given compensation level to the agent. The term $(1 - F(\theta))\Pi_\theta$ represents how efficient a state is in providing compensation. Intuitively the lower the revenue and the faster the revenue increases in the state of nature $\theta$, the more efficient the state is in providing a compensation to the agent.

The next section formally derives the optimal contract. The analysis is more general than the approach taken in Proposition 1. We verify that L-class contracts are optimal without requiring the first-order approach taken in Proposition 1, which may not be valid, and also taking into account the agent’s participation constraint, which may be binding. We formally derive the optimal contract.
when the critical ratio is increasing, decreasing, or constant.

4 The Optimal Contract

This section examines the optimal principal-agent contract and characterizes the form of the contract. The optimality of the contract and its form depend on the monotonicity of the critical ratio. Propositions 2 and 3 are important because they specify the form of the optimal contract in an agency model with moral hazard, risk neutrality and limited liability. The underlying assumptions governing uncertainty and the production function are readily verified and fit with a wide range of standard models in economics and finance. It should be emphasized that the contractual forms apply to any type of incentive contract with moral hazard. This includes situations that do not involve financial obligations, including regulation, procurement, managerial incentives, and incentives for employee effort.

4.1 The Form of the Optimal Contract

The optimal contract between the principal and the agent involves choosing the payment function and the effort level that maximize the principal’s expected net benefit over the set of feasible contracts. Our first result in this section shows that when the critical ratio is increasing in \( \theta \) the optimal contract is debt. The face value of the debt specified by the contract is a cut-off level, \( r \), such that the agent receives no payment when the outcome is below the cut-off and the agent receives the difference between the outcome and the cut-off value otherwise. When the agency contract takes the form of debt, the agent receives

\[
 w(\Pi) = \max \{ \Pi - r, 0 \},
\]

(21)

for some \( r \geq 0 \). When the agency contract takes the form of debt, the principal receives the outcome net of the payment, which is the minimum of the outcome and the cut-off level,

\[
 \Pi - w(\Pi) = \min \{ \Pi, r \},
\]

(22)
for some $r \geq 0$. With a debt-style contract, the principal can be viewed as the lender and the agent as the borrower. The agent is the residual claimant, receiving returns only when the debt has been paid, with the payment to the agent equal to the outcome net of the debt.

Debt-style incentive contracts also can be viewed as call options for which the agent as the buyer and the principal is the seller of the option. Under a call option contract of the type used in financial markets, the buyer has the right but not the obligation to buy a commodity or financial asset from the seller at a particular time and at a strike price $r$. The buyer will choose to exercise a call option when the market price rises above the strike price specified in the contract. The buyer then benefits by earning the difference between the market price and the strike price. If we interpret the outcome $\Pi$ as the market price of a security and the cut-off level $r$ as the strike price, the contract gives the agent the value of the call on its expiration date. The agent as the owner of the call option has an incentive to increase effort so as to increase the returns obtained when the market price exceeds the strike price. Call options based on a firm’s stock are often used in executive compensation because they give managers an incentive to increase the price of the firm’s stock above the strike price.

When the critical ratio $\rho(\theta, a)$ is increasing in $\theta$, the agency contract associates greater incentives for effort with higher outcomes. For any given level of effort, higher outcomes result from higher realizations of the shock. In moral hazard agency models without limited liability on the agent side or constraints on transfers to the principal, the optimal contract provides incentives that place weight on some realizations of the outcome. When the parties to the contract are risk-neutral, the incentive weight must be arbitrarily large on a measure zero of outcomes, which would create existence and characterization problems. In contrast, the agency model presented here and by Innes (1990) features limited liability, and monotonic requirements which impose lower and upper bounds on transfers. These bounds lead to smoothing such that contractual incentives place weight on a positive measure of outcomes. In our setting, an increasing critical ratio means that the optimal contract puts as much weight as possible on the highest realization of the outcome, then puts as much weight as possible on the next highest realization of the outcome, and so on. The process continues until the relevant participation constraint is binding or the benefit of providing incentives is outweighed by the costs. As a consequence, the contract generates a threshold for outcomes such
that the agent receives all of the output in excess of the threshold. Therefore, an increasing critical ratio leads to an optimal contract in the form of debt.

**Proposition 2** If the critical ratio \( \rho(\theta, a) \) is increasing in \( \theta \), then there exists an optimal contract \( w \) that takes the form of debt, \( w(\Pi) = \max\{\Pi - r, 0\} \) for some \( r \geq 0 \).

The proof, which is given in the Appendix, is a modified version of the proof of Innes’ (1990) Proposition 1. We do not need to impose regularity conditions such as implementability. We show that if there is a candidate for the optimal contract \( w(\cdot) \), then there is a debt contract, \( w^D(\cdot) \), such that the agent’s globally optimal effort level under \( w^D(\cdot) \) is no less than the globally optimal effort level under \( w(\cdot) \), so that everyone is at least as well off under \( w^D(\cdot) \). We derive results analogous to Innes’ Lemmas 1 and 2. The main difference is that Innes assumes MLRP whereas we show that an increasing critical ratio is sufficient to obtain the optimal contract. Because we assume an increasing critical ratio rather than MLRP, Proposition 2 extends Innes’ analysis. As will be shown in a later section, an increasing critical ratio is a considerably weaker condition than MLRP.

Our next result shows that when the critical ratio is decreasing in \( \theta \), the optimal contract is a bonus. The bonus has a cap, \( r \), such that the agent receives no payment when the outcome is above the cap and the agent receives all of the output below the cap. When an agency contract takes the form of a bonus, the agent receives the minimum of the outcome and the cut-off level, \( r \),

\[
w(\Pi) = \min\{\Pi, r\}, \tag{23}
\]

for some \( r \geq 0 \). When the agency contract takes the form of a bonus, the principal’s net return can be viewed as a debt to the agent,

\[
\Pi - w(\Pi) = \max\{\Pi - r, 0\}, \tag{24}
\]

for some \( r \geq 0 \). With a bonus contract, the principal is the residual claimant, receiving returns only when the cap is exceeded with the principal’s earnings equal to the outcome net of the critical value. Also, the bonus contract can be viewed as the principal owning a call option on the outcome with the bonus cap as the strike price.

\[^4\] We thank a referee for suggesting this intuition.
The next proposition shows that bonus contracts are optimal when \( \rho(\theta, a) \) is decreasing in \( \theta \). The bonus is capped because a decreasing critical ratio implies that it is desirable to associate more incentives for performance with lower realizations of the outcome. Higher outcomes result from higher realizations of the shock for any given level of effort. Again, limited liability for the agent constrains the weight that can be put on any realization of the outcome. The contract puts as much weight as possible on the lowest realization of the outcome, then puts as much weight as possible on the next lowest realization of the outcome, and so on. The optimal contract generates a threshold for outcomes such that the agent receives all of the output below the threshold. Therefore, a decreasing critical ratio leads to an optimal contract that is a bonus with a cap.

**Proposition 3**  
*If the critical ratio \( \rho(\theta, a) \) is decreasing in the shock \( \theta \), then there exists an optimal contract \( w \) that takes the form of a bonus with a cap, \( w(\Pi) = \min\{\Pi, r\} \) for some \( r > 0 \).*

The proof of Proposition 3 is given in the Appendix. When the critical ratio is decreasing, the unique L-class contract \( w(\Pi) = \min\{\Pi, r\} \) is concave in \( \Pi \), and therefore the first order condition approach is valid. If the agent’s outside opportunity is sufficiently low, then the agent’s participation constraint is not binding and proposition 3 is just a corollary of proposition 1. The proof on the appendix shows that bonus contracts are optimal also in the case that the agent’s participation constraint binds.

From Propositions 1, 2 and 3, at the agent’s equilibrium effort \( a \) and the cut-off \( r \) the reciprocal of the critical ratio equals the shadow price on the agent’s incentive compatibility contraint,

\[
\frac{1}{\rho(\hat{\theta}(r, a), a)} = \eta. \tag{25}
\]

This follows from the continuity of the critical ratio and of the production relationship. Note that \( \hat{\theta}(\Pi, a) \) is increasing in \( \Pi \). When the critical ratio is increasing, \( 1/\rho(\hat{\theta}(\Pi, a), a) \) is decreasing in \( \Pi \), so that it equals the shadow price at the face value of the debt \( r \). When the critical ratio is decreasing, \( 1/\rho(\hat{\theta}(\Pi, a), a) \) equals the shadow price at the cap on the bonus \( r \).

Linear contracts are often used in practice, taking the form of sharecropping in agriculture, piece rates in manufacturing, cost sharing in procurement, sales commissions, and other types of proportional reward sharing between principals and agents. Linear contracts are often assumed in studies.
of agency in economics and finance.

**Corollary 1** If the critical ratio \( \rho(\theta, a) \) is constant in \( \theta \), then there exists an optimal linear contract \( w \).

Observe that when the critical ratio \( \rho(\theta, a) \) is constant all contracts belong to the L-class. By Proposition 1, any two contracts that implement the same effort level will give the principal and the agent the same expected utility. Since linear contracts implement any effort level in \([0, a^{FB}]\), we can restrict attention to linear contracts. Therefore, an optimal linear contract always exists. Even though there always exists an optimal linear contract, notice that linear contracts perform just as well as any other contract that implements the same effort level.

### 4.2 The Monotonicity of the Critical Ratio

The state-space presentation allows us to consider the applicability of specific probability distributions of the shock and specific functional forms of the production technology. We now show that we can verify the monotonicity of the critical ratio for a large class of standard probability distributions and standard production technologies. Because the critical ratio is the product of the hazard rate of the shock and the MRTS, it is sufficient to consider the monotonicity of the two components separately. Observe, for example, that the critical ratio is increasing if one component is increasing and the other component is either increasing or constant.

Consider first the hazard rate of the shock, \( \frac{f(\theta)}{1-F(\theta)} \). The hazard rate is either constant, increasing, or decreasing for many common distributions. For example, the uniform distribution defined on a bounded interval has a hazard rate that is increasing in \( \theta \). The exponential distribution \( f(\theta) = \varphi \exp(-\varphi \theta), \varphi > 0 \), defined on \([0, \infty)\), has a constant hazard rate. For \( \alpha > 1 (\alpha < 1) \), the gamma distribution \( f(\theta) = \varphi (\varphi \theta)^{\alpha-1} \exp(-\varphi \theta) / \Gamma(\alpha), \) on \([0, \theta]\), \( \varphi > 0 \), has an increasing (decreasing) hazard rate. Also, for \( \alpha > 1 (\alpha < 1) \), the Weibull distribution \( f(\theta) = \varphi \alpha \theta^{\alpha-1} \exp(-\varphi \theta^\alpha), \) on \([0, \theta]\), \( \varphi > 0 \), has an increasing (decreasing) hazard rate, see Barlow et al. (1963). The Pareto distribution \( f(\theta) = \varphi \theta^{\alpha} / ((\theta)^{\alpha+1}), \varphi > 0 \), defined on \([\theta_0, \infty)\), has a hazard rate that is decreasing in \( \theta \). The Pareto distribution, or power-law distribution, is widely used in finance and in network economics, see

Next, consider the monotonicity of the MRTS, \( \frac{\Pi_\alpha(\theta, a)}{\Pi_\theta(\theta, a)} \), in the shock \( \theta \). The MRTS is increasing in the shock if effort and the shock are complements, \( \Pi_{\alpha\theta} > 0 \) and the function \( \Pi(\theta, a) \) is concave in the shock, \( \Pi_{\theta\theta} \leq 0 \),

\[
\frac{d}{d\theta} \frac{\Pi_\alpha(\theta, a)}{\Pi_\theta(\theta, a)} = \frac{\Pi_{\theta a}(\theta, a)\Pi_\theta(\theta, a) - \Pi_{\theta\theta}(\theta, a)\Pi_\alpha(\theta, a)}{(\Pi_\theta(\theta, a))^2} > 0.
\]

The complementarity condition is implied by the increasing differences condition that is used monotone comparative statics analysis, see Topkis (1998) and Milgrom and Shannon (1994).

Stiglitz’ (1974) classic model of sharecropping assumes that there is a multiplicative production function, \( \Pi(\theta, a) = \theta Q(a) \), where \( Q(a) \) is increasing and concave. The multiplicative production technology satisfies our assumptions 1 and 2 and gives an MRTS equal to \( \Pi_{\alpha}(\theta, a)/\Pi_{\theta}(\theta, a) = \theta Q'(a)/Q(a) \), which is increasing in \( \theta \). Such multiplicative outcome functions are used in many types of economic models in which the random state variable represents prices, taxes, subsidies, technology parameters, discount rates, depreciation rates, failure rates, labor, and natural shocks such as weather and demographic effects.

The Cobb-Douglas production function, \( \Pi(\theta, a) = \zeta \theta^\beta a^\gamma \), satisfies our assumptions 1 and 2 and yields an increasing MRTS. The MRTS for the Cobb-Douglas production function is \( \Pi_{\alpha}(\theta, a)/\Pi_{\theta}(\theta, a) = (\gamma/\beta)(\theta/a) \), which is increasing in \( \theta \) for any positive \( \zeta, \gamma, \) and \( \beta \). Any increasing and concave function \( H \) of the product of the agent’s effort and the shock, where \( \Pi(\theta, a) = H(\theta a) \) and \( H(0) = 0 \), satisfies our assumptions 1 and 2 and gives an MRTS that is increasing in \( \theta \). Also, the additive production function \( \Pi(\theta, a) = A(\theta) + H(\theta)Q(a) \), where \( A, H \) and \( Q \) are increasing and concave also satisfies our assumptions 1 and 2 and gives an MRTS that is increasing in \( \theta \).

The additive production function \( \Pi(\theta, a) = A\theta + Q(a) \), where \( Q \) is an increasing and concave function and \( A > 0 \), also satisfies our assumptions 1 and 2 and gives an MRTS that is constant in \( \theta \). Also, any increasing and concave function \( H \) of the sum of the agent’s effort and the shock, \( \Pi(\theta, a) = H(\theta + a) \), satisfies our assumptions 1 and 2 and gives a constant MRTS. The analysis then must be adjusted for \( \Pi(0, a) > 0 \). Either of these two production functions and an exponentially distributed hazard rate are sufficient for the set of optimal contracts to contain linear contracts.
5 Discussion

In agency models with moral hazard, the principal observes the outcome that results from the agent’s effort and random shocks. The standard approach in the literature examines the probability distribution over outcomes induced by the agent’s effort. Our discussion thus far has emphasized the state-space representation, although the results can be obtained using the Mirrlees reduced-form approach. Conlon (2009) provides a detailed analysis of the relation between the two representations. This section compares our assumptions with those of standard agency models with moral hazard. Also, this section compares our assumptions with those in agency models with adverse selection.

5.1 Comparison with the Standard Moral Hazard Agency Model

The model can be expressed in terms of the reduced form distribution as is now standard in the moral hazard literature. Observe first that for the outcome $\Pi$ to be less than or equal to $\Pi$, it must be the case that the state $\theta$ is less than or equal to $\hat{\theta}(\Pi, a)$. Let $G(\Pi, a)$ be the induced distribution for the outcome, $\Pi$, given action, $a$,

$$G(\Pi, a) = F(\hat{\theta}(\Pi, a)).$$

The increasing critical ratio has an equivalent condition in the reduced form.

**Definition 4** The reduced form distribution of the outcome, $G(\Pi, a)$, satisfies the decreasing hazard rate condition (DHRC) when the hazard rate for output in the reduced form is decreasing in the agent’s effort, $a$,

$$\frac{\partial}{\partial a} \frac{g(\Pi, a)}{1 - G(\Pi, a)} < 0.$$  

We now show that the critical ratio, $\rho(\theta, a)$, is increasing in the shock, $\theta$, if and only if the reduced form distribution of the outcome, $G(\Pi, a)$, satisfies DHRC.

Because the production technology, $\Pi(\theta, a)$, is increasing in $\theta$, we can express the critical ratio as a
function of \( \Pi \). From the identity 
\[
\frac{\partial \hat{\theta}(\Pi, a)}{\partial a} = -\frac{\Pi_a(\theta, a)}{\Pi_\theta(\theta, a)}.
\] (26)

Assumption 2 implies that the induced distribution of \( \Pi \) in the reduced-form setting satisfies first-order stochastic dominance in \( a \),
\[
G_a(\Pi, a) = f(\theta)\frac{\partial \hat{\theta}(\Pi, a)}{\partial a} = -f(\theta)\frac{\Pi_a(\theta, a)}{\Pi_\theta(\theta, a)} < 0.
\] (27)

This immediately gives a representation of the critical ratio in terms of the induced distribution of output,
\[
\rho(\hat{\theta}(\Pi, a), a) = -\frac{G_a(\Pi, a)}{1 - G(\Pi, a)}.
\] (28)

Differentiating (28) with respect to \( \Pi \) gives
\[
\frac{\partial}{\partial \Pi} \rho(\hat{\theta}(\Pi, a), a) = -\left( \frac{g_a(\Pi, a)}{g(\Pi, a)} + \frac{G_a(\Pi, a)}{1 - G(\Pi, a)} \right) \left( \frac{g(\Pi, a)}{1 - G(\Pi, a)} \right).
\] (29)

We can rewrite this as
\[
\frac{\partial}{\partial \Pi} \rho(\hat{\theta}(\Pi, a), a) = -\left( \frac{\partial \ln \frac{g(\Pi, a)}{1 - G(\Pi, a)}}{\partial a} \right) \left( \frac{g(\Pi, a)}{1 - G(\Pi, a)} \right).
\] (30)

Because the hazard rate is positive, (30) implies that the critical ratio is increasing in \( \Pi \) if and only if
\[
\frac{\partial}{\partial a} \frac{g(\Pi, a)}{1 - G(\Pi, a)} < 0,
\] which is the DHRC. Noting that \( \partial \hat{\theta}(\Pi, a)/\partial \Pi = 1/\Pi_\theta \), observe from (30) that
\[
\frac{\partial}{\partial \Pi} \rho(\hat{\theta}(\Pi, a), a) = \frac{\rho_\theta(\hat{\theta}(\Pi, a), a)}{\Pi_\theta(\hat{\theta}(\Pi, a), a)}.
\] (31)

Because \( \Pi_\theta > 0 \), it follows from (30) that the critical ratio is increasing in \( \theta \), \( \rho_\theta(\theta, a) > 0 \), if and only if \( \frac{\partial}{\partial \Pi} \rho(\hat{\theta}(\Pi, a), a) \) is positive, which is the case if and only if DHRC holds. Therefore, the critical ratio is increasing in the shock \( \theta \) if and only if DHRC holds.

Debt therefore is an optimal contract if the production function satisfies our assumptions 1 and 2 and if the reduced form distribution of output satisfies (DHRC). DHRC means that greater effort by the agent reduces the likelihood of an outcome \( \Pi \) conditional on the outcome being no less than \( \Pi \). Debt is an optimal contract because it bases its rewards on good states, that is, above a threshold. Greater agent effort increases the likelihood of the outcome exceeding a threshold,
\[
\partial (1 - G(\Pi, a))/\partial a = -G_a(\Pi, a) > 0,
\] which affects the hazard rate.
We can compare the increasing critical ratio property with the standard Monotone Likelihood Ratio Property (MLRP) assumption. Innes (1990) uses the Mirrlees reduced-form approach and assumes MLRP. The MLRP condition can be stated as

\[
\frac{\partial}{\partial \Pi} \left[ \frac{g_a(\Pi, a)}{g(\Pi, a)} \right] > 0.
\]

It can be shown that whenever the MLRP condition holds, the critical ratio is increasing in \( \theta \), although the converse is not true. This shows that the critical ratio, rather than the Likelihood Ratio, determines the form of the optimal contract.

**Proposition 4:** (i) MLRP implies that the critical ratio \( \rho(\theta, a) \) is increasing in the shock \( \theta \). (ii) MLRP is not necessary for the critical ratio \( \rho(\theta, a) \) to be increasing in the shock \( \theta \).

The proof is given in the appendix. Part (i) of the proposition was suggested by a referee. A related application of MLRP is given in Kim (1997), see also Park (1995). The condition that the critical ratio is increasing in the shock, or equivalently, that the reduced-form hazard rate of output is decreasing in the agent’s effort, is a weaker requirement than MLRP.

As noted previously, our analysis does not require any regularity assumptions, such as implementability.\(^5\) As is well known, such sufficiency conditions are very difficult to satisfy in practice.\(^6\) Because our assumptions do not require either MLRP or implementability conditions, this considerably enlarges the class of applicable production technologies and class of distributions of the shock. Our approach therefore significantly enlarges the class of applicable reduced-form outcome distributions.

\(^5\) One type of implementability assumption is the Convex Distribution Function Condition (CDFC). CDFC states that effort improves the distribution of output, \( G_a(\Pi, a) < 0 \), although at a decreasing rate, \( G_{aa}(\Pi, a) < 0 \). The analysis in Innes (1990) does not require implementability or CDFC.

\(^6\) The MLRP and CDFC conditions in combination greatly restrict the class of distributions and production functions. In particular, Jewitt (1988) observes that few distributions satisfy both the MLRP and CDFC conditions. One distribution was provided by Rogerson (1985) (attributed to Steve Matthews) and later two classes of differentiable examples were provided by Licalzi and Spaeter (2003).
5.2 Comparison with the Adverse Selection Agency Model

Examining the interaction between random shocks and the agent’s effort reveals similarities between moral hazard (hidden action) and adverse selection (hidden information) models.\(^7\) We will show that information rents in agency with adverse selection are equal to the reciprocal of our critical ratio. The agent has private information about his action in a moral hazard model and the agent in a hidden information model has private information about his type in an adverse selection model. The key similarity between the two models is the correspondence between the random shock in a moral hazard model and the agent’s private information in an adverse selection model, see Spence and Zeckhauser (1971).\(^8\)

Optimal contract design with a risk-neutral agent and principal and limited liability has been studied as an adverse selection problem by Sappington (1983) in a discrete-types framework. Invert \(\Pi = \Pi(\theta, a)\) to obtain a cost of effort \(a = C(\Pi, \theta)\) needed to achieve output \(\Pi\) in state \(\theta\). The agent’s net benefit with separability in the transfer payment is \(w - C(\Pi, \theta)\). The standard Spence-Mirrlees single-crossing condition and differentiability requires \(-C_{\Pi \theta}(\Pi, \theta) > 0\), see for example Sappington (1983). In our setting, the Spence-Mirrlees condition is equivalent to

\[
-C_{\Pi \theta}(\Pi, \theta) = \frac{\Pi_{\theta a}(\theta, a)\Pi_a(\theta, a) - \Pi_{aa}(\theta, a)\Pi_\theta(\theta, a)}{(\Pi_a(\theta, a))^3} > 0.
\]

The Spence-Mirrlees single-crossing condition therefore reduces to

\[
\frac{\partial}{\partial a} \frac{\Pi_\theta(\theta, a)}{\Pi_a(\theta, a)} > 0.
\]

This contrasts with the increasing MRTS condition,

\[
\frac{\partial}{\partial \theta} \frac{\Pi_a(\theta, a)}{\Pi_\theta(\theta, a)} > 0.
\]

Some of the earlier examples satisfy both the increasing MRTS condition and the Spence-Mirrlees

\(^7\) For adverse selection models of agency, see for example Baron and Myerson (1982), Vogelsang (2002).
\(^8\) The hazard rate plays a similar role in moral hazard and adverse selection. In adverse selection this rate represents the trade off between providing incentives to the agent of type \(\theta\) and the cost of providing payments to agents with higher types. In the moral hazard setting, the hazard rate shows the trade off between providing the agent with incentives for the case when the state of nature turns out to be \(\theta\) and the fact that the payment in higher states will increase.
condition. These include the Cobb-Douglas technology, the multiplicative production technology, 
\( \Pi(\theta, a) = \theta Q(a) \), and the additive technology, 
\( \Pi(\theta, a) = A(\theta) + H(\theta)Q(a) \), where \( A, H \) and \( Q \) are increasing and concave functions.

Complementarity of effort and the state means that the marginal return to effort is greater in better 
states in either model. Although the state, or the agent’s type are unobservable, the principal would 
prefer to provide incentives for effort in better states in the moral hazard model and to higher-type 
agents in the adverse selection model. The Spence-Mirrlees condition holds if the agent’s effort 
and the shock are complements, \( \Pi_{a\theta}(\theta, a) > 0 \), and the production technology is weakly concave 
in effort, \( \Pi_{aa}(\theta, a) \leq 0 \). Recall that in our framework, the increasing MRTS condition holds if the 
agent’s effort and the shock are complements, \( \Pi_{a\theta}(\theta, a) > 0 \), and the production technology is 
weakly concave in the shock, \( \Pi_{\theta\theta}(\theta, a) \leq 0 \).

To illustrate the basic financing model with adverse selection, we informally consider the following 
model based on Sappington (1983). We generalize his setting to allow a continuum of agent types \( \theta \) and let our assumptions 1 and 2 hold. As is standard in adverse selection problems, the agent 
oberves the state variable, \( \theta \), before choosing effort. Then, the agent’s effort induced by the contract 
will be state dependent, \( a = a(\theta) \). We now consider the choice of the optimal agency contract with 
limited liability in comparison to the contract with moral hazard.

With adverse selection, the agent’s net benefit equals \( B(\theta) = w(\Pi(\theta, a(\theta))) - a(\theta) - u_0 \). The principal’s problem of choosing an optimal contract subject to feasibility restrictions can be stated as follows,

\[
\max_{w,a} \int_0^\theta \left[ \Pi(\theta, a(\theta)) - w(\Pi(\theta, a(\theta))) \right] f(\theta) d\theta - K,
\]

subject to \( B(\theta) \geq 0 \) and

\[
w'(\Pi(\theta, a(\theta)))\Pi_a(\theta, a(\theta)) = 1,
\]

\[
w(\Pi(\theta, a(\theta))) \geq 0.
\]

The optimal contract that solves the principal’s problem involves a binding individual rationality 
constraint for the lowest-type agent, \( B(0) = 0 \). Incentive compatibility and the envelope theorem 
imply that \( B'(\theta) = w'(\Pi(\theta, a(\theta)))\Pi_{a\theta}(\theta, a(\theta)) \). Integrating \( B'(\theta) \), individual rationality and incentive
compatibility imply
\[ B(\theta) = \int_0^\theta w'(\Pi(z, a(z)))\Pi_\theta(z, a(z))dz = \int_0^\theta \Pi_\theta(z, a(z)) \Pi_\alpha(z, a(z))dz. \]
This is the information rent for a type-\(\theta\) agent, where \(a(\theta)\) is the agent effort induced by the contract \(w(\Pi)\). Information rents are greater for higher-type agents.\(^9\) Summing information rents across agents and integrating by parts gives total information rents,
\[ \int_0^\theta B(\theta)f(\theta)d\theta = \int_0^\theta \frac{1}{\rho(\theta, a(\theta))}f(\theta)d\theta. \]
Therefore, with adverse selection, total information rents equal the sum across agents of the reciprocal of our critical ratio, \(1/\rho(\theta, a(\theta))\).

From the definition of \(B(\theta)\), the agency contract is \(w(\Pi(\theta, a(\theta))) = B(\theta) + a(\theta) + u_0\). Substituting for the agency contract, the principal’s objective can be written as a function of the agent’s effort. Therefore, the principal chooses each type of agent’s action to solve the following optimization problem,
\[ \max_a \int_0^\theta \left[ \Pi(\theta, a(\theta)) - a(\theta) - \frac{1}{\rho(\theta, a(\theta))} \right] f(\theta)d\theta - K - u_0. \]
The virtual net benefit equals \(J(\theta, a(\theta)) = \Pi(\theta, a(\theta)) - a(\theta) - 1/\rho(\theta, a(\theta))\). To apply the first-order approach with adverse selection, let \(f(\theta)\) and \(\Pi(\theta, a(\theta))\) be such that the Spence-Mirrlees condition holds as well as \(J_{aa}(\theta, a) < 0\) and \(J_{a\theta}(\theta, a) > 0.\)\(^{10}\)

The virtual net benefit \(J(\theta, a(\theta))\) equals output minus the cost of effort and minus the reciprocal of the critical ratio. The reciprocal of the critical ratio reflects the change in the net benefit needed

---

\(^9\) See Julien (2000) for a general discussion of information rents in adverse selection models.

\(^{10}\) Consider the nonlinear pricing study by Maskin and Riley (1984). Their assumptions (1984, Proposition 4) generate second-order conditions for the firm’s choice of a nonlinear pricing schedule with adverse selection. The consumer’s utility function can be interpreted as a production function, \(\Pi = \Pi(\theta, a)\), where the agent’s type is a utility parameter, \(\theta\), and the agent’s action is consumption, \(a\). Their assumptions, while much more extensive, include the following restrictions, \(\Pi_a(\theta, a) > 0\), \(\Pi_{aa}(\theta, a) < 0\), \(\Pi_{a\theta}(\theta, a) > 0\), \(\Pi_{\theta}(\theta, a) > 0\), and \(\Pi_{\theta\theta}(\theta, a) < 0\). Maskin and Riley’s (1984) assumptions imply that the marginal rate of substitution is increasing in the shock. To obtain a fully-separating equilibrium, Maskin and Riley (1984, equation 23) further assume that the hazard rate of agent types, \(f(\theta)/(1-F(\theta))\), does not decline too rapidly by requiring that the distribution of agent types satisfies \(2(f(\theta))^2 + (1-F(\theta))f'(\theta) > 0\). This is a weaker condition than requiring that the hazard rate of agent types be increasing, that is \((f(\theta))^2 + (1-F(\theta))f'(\theta) > 0\). Their assumptions, with strengthening to obtain an increasing hazard rate for the shock, imply that the critical ratio \(\rho(\theta, a)\) in our setting is increasing in \(\theta\).
to compensate for information rents. The Spence-Mirrlees single-crossing condition implies that the reciprocal of the critical ratio $\rho(\theta, a)$ is increasing in the agent’s effort,

$$\frac{\partial}{\partial a} \frac{1}{\rho(\theta, a)} = \frac{1 - F(\theta)}{f(\theta)} \frac{\partial}{\partial a} \frac{\Pi_a(\theta, a)}{\Pi_a(\theta, a)} > 0.$$ 

The agent’s effort $a(\theta)$ that solves the principal’s optimization problem is either zero, or if it is positive it is increasing and solves the first-order condition,

$$\Pi_a(\theta, a(\theta)) - 1 = \left[ \frac{\partial}{\partial a} \frac{1}{\rho(\theta, a)} \right]_{a=a(\theta)}.$$

When the first-order condition holds, the Spence-Mirrlees single-crossing condition implies that the marginal product of effort is greater than the marginal cost of effort, so that effort is inefficient.

Suppose that the increasing critical ratio condition is satisfied, that is, the critical ratio $\rho(\theta, a)$ is increasing in the type parameter $\theta$ for a given level of effort, $a$. The increasing critical ratio condition implies that the distortion in virtual net benefits decreases for higher-type agents for a given effort level. In equilibrium, however, higher types of agents exert more effort, which increases the information rent effect on the virtual net benefit because the critical ratio is decreasing in effort $a$. The concavity of the virtual net benefit in effort means that the marginal net benefit is decreasing in effort for a given agent type. It can be shown that the optimal contract results in no effort by the lowest types of agents, $[0, \theta^*]$. The optimal contract generates positive effort for higher types, $[\theta^*, \overline{\theta}]$. The right-hand side of the first-order condition is zero in the highest state so that effort is efficient in the highest-type agent.

The optimal contract with adverse selection is constant for low types, $w'(\Pi) = 0$ on $[0, \Pi(\theta^*, a(\theta^*))]$. Recall from the first-order condition that $\Pi_a(\theta, a(\theta)) > 1$. Because the agent chooses effort such that $w'(\Pi) = 1/\Pi_a(\theta, a(\theta))$, it follows that the optimal contract is increasing for high types, $0 < w'(\Pi) < 1$ on $[\Pi(\theta^*, a(\theta^*)), \Pi(\theta, a(\theta))]$, and $w'(\Pi) = 1$ at $\Pi(\theta, a(\theta))$. The contract with adverse selection is therefore similar in shape to a debt-style contract because it offers a fixed payment to the agent for outcomes below a threshold and payments that are increasing in the outcome when outcomes are above that threshold. Therefore, given limited liability and risk neutrality, the optimal contract under adverse selection shares features with the debt-style contract under moral hazard.\footnote{Stiglitz and Weiss (1981) point out that incentive schedules may dominate debt contracts when there is}
6 Conclusion

We develop a basic model of agency with unobservable effort, risk neutrality, and limited liability. We introduce and characterize a critical ratio that determines the form of the optimal contract. This ratio is equal to the hazard rate of the shock times the MRTS of agent effort for the shock. When the critical ratio is increasing in the random shock, higher states are better for providing incentives so that the optimal contract is debt. When the critical ratio is strictly decreasing in the random shock, lower states are more efficient at providing incentives and the optimal contract is a bonus with a cap. When the critical ratio is linear in the random shock, all states are equally efficient and only in that case are linear contracts efficient. Moral hazard inefficiencies arise as a consequence of the agent’s limited liability. Applying these types of conditions in a moral hazard setting yields a characterization of the optimal contract based on assumptions that can be easily verified and frequently occur in economics and finance applications.

The result that the optimal contract takes the form of debt when monotonicity holds, has far reaching implications. Debt-style contracts help to explain the use of performance targets and rewards in a wide variety of economic situations. Such incentive contracts have critical performance levels rather than more complex performance schedules. Debt-style contracts are optimal for sharecropping contracts, employee performance contracts, procurement contracts, and regulatory incentives. Moreover, debt-style contracts are at the heart of financial contracts and performance rewards for entrepreneurs and managers. Debt contracts are simple to design and apply, and have standardized legal formats, thus mitigating transaction costs. Debt contracts have important properties that facilitate market pricing and trading. Financial markets have extensive experience in pricing and trading debt, options, and similar assets.

The monotonicity of the critical ratio holds for a very wide range of applications in economics and finance. Our approach should help researchers to derive optimal contracts within many economic adverse selection. Gale and Hellwig show that the optimal contract with adverse selection is debt based on revelation of information arguments, see also Myers and Majluf (1984). Gale and Hellwig (1985) consider a production function $\Pi(\theta, a)$ and assume that it is increasing and concave in the agent’s action and satisfies complementarity.
and financial models. Our assumptions allow for tractable economic analysis that indicates how the form of optimal contracts depends on underlying uncertainty, technology, and wealth effects. The monotone critical ratio condition is less restrictive than the monotonic likelihood ratio property in the reduced form setting. We introduce the decreasing hazard rate condition in the reduced form setting that provides a necessary and sufficient condition for the increasing critical ratio condition.

Our analysis emphasizes explicit uncertainty by considering shocks to the output function. This offers some insights in comparison to the Mirrlees reduced-form approach to contract design. With explicit uncertainty, the contract can be viewed more readily as a method of modifying the pattern of outcomes across states of the world. This recalls the common observation in finance that options are a method of obtaining a pattern of returns that could not be obtained with the underlying stock (Cox and Rubinstein, 1985, p. 45). Debt contracts provide incentives to the agent in higher states and option contracts provide incentives to the agent in lower states.

An important aspect of our analysis is that it identifies a close connection between moral hazard and adverse selection. The uncertainty in the moral hazard setting corresponds to the unobservable type in the adverse selection setting. In the moral hazard setting, the agent anticipates his future type when choosing effort and in the adverse selection setting, the agent knows his type when choosing effort. The increasing MRTS condition, which implies the increasing critical ratio condition when the hazard rate is constant or increasing in the shock, differs from the Spence-Mirrlees single-crossing condition in adverse selection model. Despite the fundamental differences between moral hazard and adverse selection, the optimal contract in both settings offers a fixed payment to the agent for outcomes below a threshold and payments that are increasing in the outcome for outcomes above that threshold. Therefore, in agency with limited liability and risk neutrality, the form of incentive contracts is similar under moral hazard and adverse selection.

References


Appendix

Proof of Proposition 2. The proof closely follows that of Innes (1990), although he considers the dual problem of maximization of the agent’s net benefit. We show that an increasing critical ratio implies results similar to Lemmas 1 and 2 in Innes (1990) and then argue that this is enough to show the optimality of debt.

Lemma 1 (Innes): Consider a payoff function to the agent \( w(\Pi) \) and a debt contract \( D(\Pi, r) = \max\{\Pi - r, 0\} \). Then \( \partial E\{D(\Pi(\theta, a), r) - w(\Pi(\theta, a))\}/\partial a > 0 \) if either (a) \( E\{D(\Pi(\theta, a), r) - w(\Pi(\theta, a))\} = 0 \), or (b) \( E\{D(\Pi(\theta, a), r) - w(\Pi(\theta, a))\} \geq 0 \).

Proof. Given an arbitrary contract \( b \) notice that \( \partial E\{b(\Pi(\theta, a)) - w(\Pi(\theta, a))\}/\partial a \) can be written as

\[
\int_0^\beta (b'(\Pi(\theta, a)) - w'(\Pi(\theta, a)))\Pi_a(\theta, a)f(\theta)d\theta. \tag{32}
\]

a) If \( E\{b(\Pi(\theta, a)) - w(\Pi(\theta, a))\} = 0 \), then integrating by parts exactly as in Proposition 1 gives

\[
b((\Pi(0, a)) - w((\Pi(0, a)) + \int_0^\beta (b'(\Pi(\theta, a)) - w'(\Pi(\theta, a)))\Pi_a(\theta, a)(1 - F(\theta))d\theta = 0. \tag{33}
\]

We now look for the contract \( b \) that maximizes (32) subject to (33). The Lagrangian for this problem is

\[
\mathcal{L} = \max_b \int_0^\beta (b'(\Pi(\theta, a)) - w'(\Pi(\theta, a)))[f(\theta)\Pi_a(\theta, a) - \lambda(1 - F(\theta))\Pi_a(\theta, a)]d\theta \tag{34}
\]

\[-\lambda[b(\Pi(0, a)) - w(\Pi(0, a))].
\]

The problem is linear in \( b' \). If we also restrict \( b' \) to be in the unit interval to preserve monotonicity, then it is clear that the solution requires \( b(\Pi(0, a)) = 0 \) and
\[ b'(\Pi(\theta, a)) = 0 \text{ if } \rho(\theta, a) < \lambda \]
\[ b'(\Pi(\theta, a)) = 1 \text{ if } \rho(\theta, a) > \lambda \]

for the appropriate choice of \( \lambda \), where \( \rho(\theta, a) = \frac{f(\theta)}{1-F(\theta)} \frac{h_a}{h} \). Because, by assumption, the critical ratio is increasing, the contract that maximizes the derivative is debt, more precisely, it is the debt contract that satisfies \( E\{D(\Pi(\theta, a), r) - w(\Pi(\theta, a))\} = 0 \). To show that \( \partial E\{D(\Pi(\theta, a)) - w(\Pi(\theta, a))\}/\partial a > 0 \), notice that \( w(\cdot) \) is feasible in the problem, it gives a value of 0, and yet it is not optimal.

(b) Repeat the same approach in (a) assuming \( E\{b(\Pi(\theta, a)) - w(\Pi(\theta, a))\} = M \) for some \( M > 0 \).

Given an arbitrary contract \( w(\cdot) \) that implements \( a^{ND} \), let \( D(\Pi, r_0) \) be the debt contract such that \( E\{D(\Pi(\theta, a_0), r_0)\} = E\{w(\Pi(\theta, a_0))\} \). Let \( a(r_0) \) be the highest effort level that satisfies the incentive compatibility constraint with the debt contract \( D(\Pi, r_0) \).

The next lemmas closely follow Innes (1990).

**Lemma 2 (Innes):** \( a(r_0) \geq a^{ND} \).

**Proof.** Define \( \phi(\Pi) = D(\Pi, r_0) - w(\Pi) \). The following properties are sufficient to prove the lemma:

1. \( E\{\phi(\Pi(\theta, a))\} = 0 \) at \( a = a^{ND} \), and
2. \( E\{\phi(\Pi(\theta, a))\} \leq 0 \) for all \( a < a^{ND} \). Property (1) follows from the definition of \( D(\Pi, r_0) \). To prove Property (2), suppose to the contrary that at some \( a < a^{ND} \) \( E\{\phi(\Pi(\theta, a))\} > 0 \). Then, we can invoke Lemma 1 to show that at \( a = a^{ND} \) it must also be the case that \( E\{\phi(\Pi(\theta, a))\} > 0 \) which contradicts the definition of \( \phi(\Pi(\theta, a)) \).

**Lemma 3 (Innes):** If there is an optimal contract, then there must exist an optimal debt contract.

**Proof.** Suppose to the contrary that the optimal contract is \( \hat{w}(\cdot) \) and implements an effort level \( \hat{a} \). By definition, the debt contract \( r_0 \) gives both the agent and the principal the same payments as \( \hat{w}(\cdot) \) under the effort level \( \hat{a} \). By revealed preferences, the debt contract \( r_0 \) is not worse for the agent. By Lemma 2, \( a(r_0) \geq \hat{a} \) and since the expected utility of the principal is increasing in the effort level \( a \), the debt contract \( r_0 \) is no worse for the principal than the contract \( w(\cdot) \).

We now show that an optimal debt contract exists. The expected utility of the agent with a debt
contract with face value \( r \) is given by

\[
U(r) = \max_a \int_0^\bar{\theta} \max\{\Pi(\theta, a) - r, 0\} f(\theta) d\theta. \tag{36}
\]

The function \( U(r) \) is continuous and decreasing in \( r \) and it has increasing differences in \( \{a, -r\} \). By standard monotone comparative statics arguments, \( a(r) \) is decreasing in \( r \). The expected utility of the principal can be written as

\[
V(r) = \int_0^\bar{\theta} \min\{\Pi(\theta, a(r)), r\} f(\theta) d\theta \tag{37}
\]

A feasible debt contract exists by assumption 3 and \( \Pi(0, a) = 0 \). The principal’s expected utility is not necessarily continuous in \( r \) because the function \( a(r) \) is not necessarily continuous in \( r \). However, we know that \( a(r) \) is decreasing in \( r \) and hence \( V(r) \) is decreasing at any discontinuity. Define the modified function \( \tilde{V}(r) \) as follows:

\[
\tilde{V}(r) = \max_{r \leq r} V(r)
\]

It is easy to check that \( \tilde{V}(r) \) is continuous and increasing in \( r \). Consider the following problem:

\[
\max_r \tilde{V}(r) \text{ subject to } U(r) \geq 0.
\]

The problem has a solution because we are maximizing a continuous function and the set of debt contracts that satisfy \( U(r) \geq 0 \) is closed. We can always restrict attention to a compact set of debt contracts with \( r < \tau \) for some \( \tau \) positive since \( U(r) \) is decreasing in \( r \) and converges to \(-u_0\) and therefore high levels of debt will not satisfy the agent’s participation constraint. Let the solution of this problem to be \( \tilde{V}^* \). Let \( r^* \) be the lowest \( r \) such that \( \tilde{V}(r) = \tilde{V}^* \). Note that \( r^* \) is well defined because \( \tilde{V} \) is a continuous function and by definition, we have \( V(r^*) = \tilde{V}(r) = \tilde{V}^* \). Finally, we claim that a contract with face value \( r^* \) is an optimal debt contract. Suppose not, then there exists \( r \) so that \( V(r) > V(r^*) \) and \( U(r) \geq 0 \), but this implies that \( \tilde{V}(r) \geq V(r) > \tilde{V}(r^*) \), which contradicts the optimality of \( r^* \) in the modified problem. □

This establishes Proposition 2.

**Proof of Proposition 3.** Consider bonus contract with cap \( r \). The expected utility of the agent
can be written as

\[ U(r,a) = \int_0^\theta \min\{\Pi(\theta, a), r\} f(\theta) d\theta - a - u_0. \]  

(38)

The expected utility \( U(r,a) \) is increasing in \( r \) and concave in \( a \). Therefore, the first-order condition is necessary and sufficient for optimality. Because for each \( r \) there is a unique optimum effort level, the theorem of the maximum implies that \( a(r) \) is a continuous function. Because \( \partial^2 U/\partial r \partial a > 0 \), it is also the case that \( a(r) \) is increasing in \( r \) by standard monotone comparative static arguments.

The expected utility of the principal can be written as

\[ V(r,a) = \int_0^\theta \max\{\Pi(\theta, a) - r, 0\} f(\theta) d\theta - K. \]  

(39)

Such a feasible contract exists by assumption 3 and \( \Pi(0,a) = 0 \). To prove the proposition, suppose to the contrary that the optimal contract is \( \hat{w}(.\cdot) \) and implements an effort level \( \hat{a} \). Consider the bonus contract with cap \( \tilde{r} \) that at \( \hat{a} \) satisfies

\[ \partial U(\tilde{r}, a)/\partial a = 0. \]

Because the agent’s problem is concave, it is clear that \( \hat{a} \) is incentive compatible with the bonus contract with cap \( r \). In this case, only bonus contracts are in the L-class and then Proposition 1 implies that \( V(\tilde{r}, \hat{a}) > V(w, \hat{a}) \). Because total surplus is the same, this also implies that \( U(\hat{a}, \tilde{r}) < U(\hat{a}, w) \). By continuity, we can increase \( r \) until \( U(a(r^*), r^*) = U(\hat{a}, w) \). We have already argued that \( a(r) \) is an increasing function and therefore \( a(r^*) > a(\tilde{r}) \geq \hat{a} \). Notice that because more effort is being exerted under the bonus contract \( r^* \), more surplus is created. Finally notice that because the agent is not better off, it must be the case that the principal is better off under the bonus contract \( r^* \). So, we have shown that if there exists an optimal contract, it must be a bonus contract with a cap.

To show existence of an optimal contract, note that we can restrict attention to a compact subset of those contracts. To show this, define \( \overline{r} \) as the unique value that satisfies \( V(a^{FB}, \overline{r}) = 0 \). Since \( a(r) < a^{FB} \) for every \( r \), a contract with \( r > \overline{r} \) will never be optimal for the principal and thus we can restrict attention to contracts where \( r \in [0, \overline{r}] \).

**Proof of Proposition 4.** (i) The effect of effort on the reduced-form distribution can be written
as

\[ G_a(\Pi, a) = -\int_{\Pi} g_a(\pi, a) g(\pi, a) d\pi \]

MLRP implies that

\[ -\int_{\Pi} g_a(\pi, a) g(\pi, a) d\pi < -\frac{g_a(\Pi, a)}{g(\Pi, a)} \int_{\Pi} g(\pi, a) d\pi = -\frac{g_a(\Pi, a)}{g(\Pi, a)} [1 - G(\Pi, a)]. \]

Therefore, \( G_a(\Pi, a)/[1 - G(\Pi, a)] + g_a(\Pi, a)/g(\Pi, a) < 0 \), which holds if and only if the DHRC condition holds. (See equation 29). Therefore, MLRP implies DHRC, or equivalently that the critical ratio is increasing in \( \theta \). (ii) Suppose, for example, that the technology is multiplicative, \( \Pi(\theta, a) = \theta Q(a) \), where \( Q(a) \) is concave. Let the probability density be given by \( f(\theta) = \gamma + \alpha \theta \) with \( 0 < \alpha < 1 \). Then, the critical ratio \( \rho(\theta, a) \) is increasing in the shock \( \theta \). The induced probability distribution over outcomes is \( g(\Pi|a) = f(\bar{\theta}(\Pi, a)) = f(\Pi/Q(a)) \), so that

\[ \frac{g_a(\Pi|a)}{g(\Pi|a)} = -\frac{f'(\bar{\theta}(\Pi, a))}{f(\bar{\theta}(\Pi, a))} \frac{Q'(a)}{Q(a)} = -\frac{f'(\Pi/Q(a))}{f(\Pi/Q(a))} Q'(a)/Q^2(a). \]  

(40)

Suppose that MLRP holds, that is,

\[ \frac{\partial}{\partial \Pi} \left( \frac{g_a(\Pi|a)}{g(\Pi|a)} \right) = -\left( \frac{f'(\theta)}{f(\theta)} + \left[ \theta \frac{f''(\theta)}{f^2(\theta)} \right] \right) \frac{Q'(a)}{Q(a)^2} > 0 \]

So, MLRP implies that

\[ f'(\theta) + \theta f''(\theta) - \theta (f'(\theta))^2 / f(\theta) < 0 \]

From the form of the probability density, this inequality can be expressed as

\[ \alpha - \theta (\alpha)^2 / (\gamma + \alpha \theta) < 0 \]

As \( \theta \) approaches zero, this expression implies \( \alpha < 0 \), which is a contradiction, so MLRP does not hold. \( \square \)