An Aggregative Games Approach to Merger Analysis in Multiproduct-Firm Oligopoly*

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Abstract

Using an aggregative games approach, we analyze horizontal mergers in a model of multiproduct-firm price competition with nested CES or nested logit demands. We show that the Herfindahl index provides an adequate measure of the welfare distortions introduced by market power, and that the induced change in the naively-computed Herfindahl index is a good approximation for the market power effect of a merger. We also provide conditions under which a merger raises consumer surplus, and conditions under which a myopic, consumer-surplus-based merger approval policy is dynamically optimal. Finally, we study the aggregate surplus and external effects of a merger.

Keywords: Multiproduct firms, aggregative game, oligopoly pricing, market power, horizontal merger, Herfindahl index.

1 Introduction

Using an aggregative games approach, we provide an analysis of horizontal mergers in a model of multiproduct-firm price competition with nested CES (NCES) or nested multinomial logit (NMNL) demand systems. The paper makes three contributions. First, we show that the Herfindahl index, which plays an important role in antitrust practice, provides an adequate measure of the welfare distortions introduced by market power, and that the induced change in the (naively-computed) Herfindahl index is a good approximation for the market

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power effect of a merger. Second, we provide conditions under which a merger raises consumer surplus, and conditions under which a myopic, consumer-surplus-based merger approval policy is dynamically optimal. Third, we study the aggregate surplus and external effects of a merger.

Almost all mergers involve multiproduct firms. This is reflected in the literature on merger simulation (e.g., Nevo, 2000a) and in the literature on the upward-pricing pressure of mergers (e.g., Werden, 1996; Farrell and Shapiro, 2010; Jaffe and Weyl, 2013), both of which have heavily influenced antitrust practice. Despite this, much of the theoretical literature on horizontal mergers and antitrust, including Farrell and Shapiro (1990), McAfee and Williams (1992), and Nocke and Whinston (2010, 2013), has focused on single-product firms in the homogeneous-goods Cournot setting. An open question is to what extent the insights derived in that earlier literature carry over to more realistic models of price competition with multiproduct firms.[1]

There are several desiderata for a flexible model of horizontal mergers and merger control: First, the underlying demand system should have sound micro-foundations and allow for flexible substitution patterns. Second, the model should allow for arbitrary firm and product heterogeneity (e.g., in terms of marginal costs, qualities, size of product portfolios). Third, the underlying oligopoly game should be tractable and give rise to a unique equilibrium. Fourth, the model should permit rich forms of merger-specific synergies (e.g., marginal cost reductions, quality improvements, new products). Finally, for the model to be useful for antitrust practitioners, its predictions should ideally relate to easily observable sufficient statistics such as firm-level market shares and concentration ratios.

In this paper, we develop a model that, despite its limitations, goes a long way towards satisfying these desiderata. The competitive setting underlying our merger analysis is a game of price competition with multiproduct firms and NCES/NMNL demands. This class of demand systems has discrete/continuous choice micro-foundations and, through its nest structure, allows for substitution patterns that go beyond those of the independence-of-irrelevant-alternatives (IIA) property. Indeed, variants of this class are ubiquitous in the empirical industrial organization literature. We allow quality and marginal costs to differ arbitrarily across products, and firms to own the property rights over arbitrary collections of products.

Another advantage of the NCES/NMNL demand specification is that it gives rise to an aggregative game: Each firm's profit depends on rival firms' prices only through a single-dimensional aggregator. In equilibrium, each firm charges the same markup—the relative markup under NCES demand and the absolute markup under NMNL demand—for each of its products. Moreover, *type aggregation* obtains: All relevant information about a firm's

[1] For instance, Whinston (2007) notes: '[...] the Farrell and Shapiro analysis is based on the strong assumption that market competition takes a form that is described well by the Cournot model, both before and after the merger. [...] There has been no work that I am aware of extending the Farrell and Shapiro approach to other forms of market interaction. The papers that formally study the effect of horizontal mergers on price and welfare in other competitive settings [...] all assume that there are no efficiencies generated by the merger."

product portfolio (the number of products as well as the qualities and marginal costs of the various products) can be summarized in a single-dimensional sufficient statistic—the firm's "type." Building on the aggregative games approach taken in Nocke and Schutz (2018), we show that there exists a unique pricing equilibrium, with intuitive comparative statics. The resulting levels of consumer surplus and aggregate surplus can be expressed as functions of firms' equilibrium market shares.

At the heart of the review of a horizontal merger by an antitrust authority is the Williamson (1968) trade-off between the merger's market power effect (which is due to the internalization of pricing externalities post merger) and its efficiency effect (which is due to potential merger-specific synergies). In our model, merger-specific synergies can take many forms: Some of the marginal costs of the merged firms' products may go down (while those of other may go up); some of the products' qualities may improve (while others may degrade); and the merged entity may offer new products (while possibly withdrawing others). The type aggregation property allows us to refrain from imposing any restrictions on the nature of the synergies as all relevant information can be summarized in the merged firm's post-merger type.

The Herfindahl index (HHI) is often used to quantify market power and plays an important role in merger control.[2] Using the outcome under monopolistic competition as the appropriate competitive benchmark in our differentiated-products setting, we show that the Herfindahl index provides an adequate measure of the welfare distortions introduced by market power. Specifically, using a Taylor approximation, we show that the difference in the outcomes of our welfare measures (consumer surplus and aggregate surplus) under oligopoly and monopolistic competition is proportional to the Herfindahl index. Defining the market power effect of a merger as the effect in the absence of merger-specific synergies, we use a Taylor approximation to show that the market power effect on consumer surplus and aggregate surplus is proportional to the naively-computed, merger-induced variation in the Herfindahl index. These results thus provide some justification for the use of the Herfindahl index in antitrust practice.

We also provide an analysis of the consumer surplus effects of mergers. We show that, for any merger, there exists a unique cutoff such that the merger increases consumer surplus if the post-merger type is above that cutoff, and decreases consumer surplus if it is below. As in the homogeneous-goods Cournot model (Farrell and Shapiro, 1990), for a merger to increase consumer surplus it must involve synergies. Moreover, the required synergies are larger the less competitive is the market pre-merger and the larger are the merging parties. This suggests that mergers inducing a larger increase in the naively-computed Herfindahl index should indeed receive additional scrutiny.

Embedding the static pricing game in a dynamic model in which merger opportunities arise stochastically over time and, in every period, firms involved in feasible but not-yet-

[2]For instance, in the U.S. Horizontal Merger Guidelines, the pre-merger Herfindahl index and the "naively-computed" merger-induced change in the Herfindahl index are proposed as indicators of the "likely competitive effects of a merger."

approved mergers have to decide whether to propose their merger, and the antitrust authority has to decide which (if any) of the proposed mergers to approve, we show that a completely myopic merger approval policy is dynamically optimal. This extends the main insight of Nocke and Whinston (2010), derived in a homogeneous-goods Cournot setting, to the case of differentiated-products price competition with NCES or NMNL demands.

Turning to the aggregate surplus effects of mergers, we show that there also exists a post-merger cutoff type above which a merger increases aggregate surplus, and below which it decreases aggregate surplus. That cutoff type is lower than the one for a consumer surplus standard: For a merger to increase aggregate surplus requires fewer synergies than for it to increase consumer surplus.

Building on Farrell and Shapiro (1990)'s analysis of the homogeneous-goods Cournot setting, we also study the external effect of a merger, defined as the sum of the effect on consumer surplus and the non-merging firms' profits. The aggregative properties of our oligopoly model allow us to decompose a merger into a sequence of *infinitesimal* mergers, where, along the sequence, the value of the aggregator increases continuously from its pre-merger to its post-merger equilibrium value. Building on this insight, we show that a consumer-surplus-decreasing merger is more likely to have a positive external effect if the non-merging firms command larger pre-merger market shares and if these pre-merger market shares are more concentrated.[3] We also provide a simple and easily-implementable test to check whether a consumer-surplus-decreasing merger has a positive external effect. That test only requires knowledge of the *pre-merger* market shares and of a demand elasticity parameter.

Our paper is related to several strands of literature. In a diagrammatic analysis of a merger from perfect competition to monopoly, Williamson (1968) was the first to identify the welfare trade-off between the market power effect of a merger and its efficiency effect. Farrell and Shapiro (1990) provide a thorough analysis of this trade-off in a homogeneous-goods Cournot model. They provide a necessary and sufficient condition for a merger to increase consumer surplus, and sufficient conditions for the external effect of a merger to be positive. In a dynamic setting with endogenous merger proposals (and approvals), Nocke and Whinston (2010) study the dynamic optimality of a myopic, consumer-surplus-based merger approval policy in a homogeneous-goods Cournot model. In Sections 4 and 5.2, we extend Farrell and Shapiro (1990) and Nocke and Whinston (2010)'s analysis to the case of differentiated-goods price competition with multiproduct firms.[4,5]

The literature on upward pricing pressure, pioneered by Werden (1996), attempts to operationalize the Williamson (1968) trade-off using information local to the pre-merger

[3]The converse holds if the merger under consideration is consumer-surplus-increasing.

[4]A separate, less-related strand of literature studies the profitability of mergers in the absence of merger-specific synergies (Salant, Switzer, and Reynolds, 1983; Perry and Porter, 1985; Deneckere and Davidson, 1985). Another literature, pioneered by Kamien and Zang (1990), studies the limits of monopolization through mergers in the absence of antitrust policy.

[5]A recent literature focuses on the effects of mergers and merger policy on investment and innovation (e.g., Gowrisankaran, 1999; Mermelstein, Nocke, Satterthwaite, and Whinston, 2014; Motta and Tarantino, 2017; Federico, Langus, and Valletti, 2018; Bourreau, Jullien, and Lefouili, 2018).

equilibrium. Werden (1996) considers a merger between two single-product firms and, using pre-merger markups, diversion ratios and prices as primitives, computes the critical level of synergies that makes the merger price-reducing. Farrell and Shapiro (2010) provide guidance on how to implement Werden (1996)'s approach in practice. Jaffe and Weyl (2013) extend this analysis to mergers among multiproduct firms. Using a Taylor approximation around zero upward pricing pressure, they formalize Farrell and Shapiro (2010)'s informal argument that local information on pass-through rates can be combined with upward pricing pressure indices to obtain the likely price effect of a merger. The approximation results we provide in Section 3.3 are of a different nature; those results are obtained around small market shares or around monopolistic competition conduct and relate explicitly the market power effect of a merger to easily-observable concentration ratios. We also derive exact conditions on the consumer surplus and aggregate surplus effects of mergers. Finally, in contrast to the literature on upward pricing pressure, we allow synergies to materialize not only through marginal cost reductions, but also through quality improvements and new products.

The Herfindahl index is a key sufficient statistic in our approximation results in Section 3. In previous work on the homogeneous-goods Cournot model, the Herfindahl index has been shown to provide a measure of an industry's average markup and profitability; see, for instance, Cowling and Waterson (1976), and Belleflamme and Peitz (2010) for a textbook treatment.[6] We are, however, aware of only few results linking the Herfindahl index to industry performance measures in models of differentiated-products industries. In a model with CES preferences and price or quantity competition, Grassi (2017) relates the industry average markup to the Herfindahl index. In Feenstra and Weinstein (2017)'s model with translog preferences, the representative consumer's indirect utility depends on the Herfindahl index both directly, due to translog preferences, and indirectly, due to endogenous markups. To the best of our knowledge, our paper is the first to link explicitly the market power distortion to consumer surplus and aggregate surplus to the Herfindahl index, and to show that the market power effect of a merger is approximately proportional to the naively-computed, merger-induced variation in that index.

The remainder of the paper is organized as follows. In Section 2, we introduce the oligopoly model and solve it using aggregative games techniques. There, we also show that the type aggregation property permits a tractable analysis of mergers in multiproduct-firm oligopoly. Section 3 shows that the Herfindahl index provides an adequate approximation of the welfare distortion from oligopolistic behavior, and that the merger-induced, naively computed variation in the Herfindahl index approximates the market power effect of a merger. Our results on the consumer surplus effects of mergers, in both static and dynamic settings, are derived in Section 4. Section 5 presents our results on the aggregate surplus and external effects of mergers. Section 6 concludes.

[6]Dansby and Willig (1979) show that, in the homogeneous-goods Cournot model, the industry performance gradient index, which measures the rate of potential improvement in aggregate surplus from a small variation in the output vector, is proportional to the square root of the Herfindahl index.

2 Mergers in Multiproduct-Firm Oligopoly

In this section, we present the oligopoly model that will serve as a workhorse throughout the paper. We present the model in Section 2.1. Section 2.2 introduces the important benchmark of monopolistic competition. We solve the oligopoly model using aggregative-games techniques in Section 2.3. Section 2.4 uses the type aggregation property to simplify the treatment of mergers among multiproduct firms.

2.1 The Oligopoly Model

Consider an industry with a set $\mathcal{N}$ of imperfectly substitutable products. Each product belongs to a nest of products; the set of nests is denoted $\mathcal{L}$, a partition of $\mathcal{N}$. Products within the same nest are viewed by consumers as closer substitutes with each other than products in different nests. Specifically, the representative consumer's indirect subutility function is given by

$$V(p) = V_0 \log \left[H^0 + \sum_{l \in \mathcal{L}} \left(\sum_{j \in l} h_j(p_j) \right)^{\beta} \right],$$

where $V_0 > 0$ is a market size parameter, $0 < \beta \leq 1$ is a parameter measuring the substitutability of products within nests relative to that across nests,[7] $H^0 \geq 0$ is a baseline-utility parameter, and

$$h_j(p_j) = \begin{cases} \exp\left(\frac{a_j - p_j}{\lambda}\right) & \text{in the case of NMNL,} \\ a_j p_j^{1-\sigma} & \text{in the case of NCES.} \end{cases}$$

The parameter $a_j > 0$, $j \in \mathcal{N}$, summarizes vertical product characteristics, and will be referred to as the quality of product j; $\sigma > 1$ and $\lambda > 0$ measure the substitutability of products within nests.

Defining the nest- and industry-level aggregators

$$H_l(p_l) = \sum_{j \in l} h_j(p_j), \text{ where } p_l \equiv (p_j)_{j \in l} \ \forall l \in \mathcal{L},$$
$$\text{and } H(p) = H^0 + \sum_{l \in \mathcal{L}} (H_l(p_l))^{\beta}$$

allows us to rewrite the consumer's indirect utility as $V(p) = V^0 \log H(p)$.

Applying Roy's identity, we obtain the demand for product i in nest l:

$$\begin{aligned} D_i(p) &= V_0 \beta \frac{-h_i'(p_i)}{h_i(p_i)} \frac{h_i(p_i)}{H_l(p_l)} \frac{H_l(p_l)^{\beta}}{H(p)} \\ &= V_0 \beta \frac{-h_i'(p_i)}{H_l(p_l)^{1-\beta} H(p)}. \end{aligned} \tag{1}$$

[7] If $\beta = 1$, the nest structure is irrelevant.

As shown in Nocke and Schutz (2018), demand system (1) can alternatively be derived from discrete/continuous choice.[8] With such a micro-foundation, $V_0\beta$ is the total number of consumers, H_l^β/H is the probability that a given consumer chooses nest l, h_i/H_l is the probability that a consumer picks product i conditional on having chosen nest l, and $-h_i'/h_i$ is the number of units of product i a consumer purchases conditional on having chosen product i.[9] Moreover, $(\log H^0)/\beta$ is the value of the outside option. In the remainder of the paper, we normalize V_0 to 1.

There is a set $\mathcal{F}$ of firms, which we assume to be a partition of $\mathcal{L}$. That is, each firm is assumed to have property rights over the production of all products within one or more nests.[10] Each product $i \in \mathcal{N}$ has constant marginal cost of production $c_i > 0$.

The economic environment can thus be summarized by the tuple $(\mathcal{N}, \mathcal{L}, \mathcal{F}, (a_j)_{j\in\mathcal{N}}, (c_j)_{j\in\mathcal{N}})$ along with nest parameter β, and elasticity parameters σ under NCES and λ under NMNL. The profit of firm $f \in \mathcal{F}$ is given by

$$\Pi^f = \sum_{l\in f}\sum_{i\in l}(p_i - c_i)D_i(p).$$

Firms compete by simultaneously setting the prices of all of their products. We seek the Nash equilibrium of this multiproduct-firm pricing game. Aggregate surplus is the sum of consumer surplus, $\log H$, and industry-level profits, $\sum_{f\in\mathcal{F}}\Pi^f$.

Firms' market shares will play an important role in our analysis. We define the market share of firm f as

$$s^f = \sum_{l\in f}\frac{(H_l)^\beta}{H}.$$

In the discrete/continuous choice micro-foundation mentioned above, s^f corresponds to the probability that any given consumer chooses one of firm f's products. Moreover, s^f is equal to firm f's market share in volume under NMNL, and to firm f's market share in value under NCES.[11] In both cases, the firms' market shares add up to $1 - H^0/H$, where H^0/H is the market share of the outside option, as is standard in the literature on demand estimation

[8] Anderson, de Palma, and Thisse (1987) are the first to provide a micro-foundation for the non-nested CES demand system.

[9] Under NMNL demand, $-h_i'/h_i$, the conditional demand for product i, is constant and equal to $1/\lambda$; under NCES demand, it is equal to $(\sigma-1)/p_i$.

[10] This assumption is for tractability. If we allowed for the possibility that different products within the same nest are offered by different firms, and firms owned products in more than one nest, we would be unable to use the aggregative games approach explored in this paper.

[11] Under NMNL, firm f's market share in volume is given by

$$\frac{\lambda}{\beta}\sum_{l\in f}\sum_{j\in l}D_j(p) = \lambda\frac{\sum_{l\in f}H_l^{\beta-1}\left(\sum_{j\in l}(-h_j')\right)}{H} = \frac{\sum_{l\in f}H_l^{\beta-1}\left(\sum_{j\in l}h_j\right)}{H} = s^f.$$

(see, e.g., Berry, Levinsohn, and Pakes, 1995; Nevo, 2001).[12]

2.2 The Monopolistic Competition Benchmark

Before analyzing the oligopoly model, it is instructive to consider first the monopolistic competition benchmark. Under monopolistic competition, firms do not internalize the impact of their behavior on the industry aggregator H, i.e., they behave as if $\partial H/\partial p_i = 0$ for every $i \in \mathcal{N}$.

Under this behavioral assumption, the first-order condition of profit maximization for product $i \in n \in f$ is given by

$$\frac{H_n^{\beta-1}}{H}\left(-h_i' - (p_i - c_i)h_i'' + (1-\beta)\frac{\partial H_n}{\partial p_i}\frac{\sum_{j\in n}(p_j - c_j)h_j'}{H_n}\right) = 0,$$

which can be rewritten as

$$\frac{p_i - c_i}{p_i}\frac{p_i h_i''}{-h_i'} = 1 + (1-\beta)\frac{\sum_{j\in n}(p_j - c_j)(-h_j')}{H_n}. \tag{2}$$

If $\beta = 1$ (i.e., in the absence of nests), we immediately obtain that firm f sets the Lerner index of product i equal to the reciprocal of the perceived price elasticity of demand. Under CES demand, that elasticity is equal to σ; under MNL demand, it is equal to p_i/λ.

If $\beta < 1$, firm f internalizes self-cannibalization effects within its own nests, and it optimally sets a Lerner index that exceeds that in the absence of nests. Note that, if $\beta < 1$, $p_i h_i''/(-h_i')$ is no longer the perceived price elasticity of demand; instead, it is equal to the perceived price elasticity of product i when firm f ignores the impact of p_i on the nest-level aggregator H_n (i.e., when ignoring self-cannibalization effects).

Following Nocke and Schutz (2018), we call the left-hand side of equation (2) the ι-markup on product i. As the right-hand side is the same for every $i \in n$, firm f charges the same ι-markup, $\tilde{\mu}_n > 1$, for each product i in nest n. Under NCES demand, this implies that the Lerner index of product i is equal to $\tilde{\mu}_n/\sigma$, whereas under NMNL demand, the absolute markup $p_i - c_i$ is equal to $\tilde{\mu}_n\lambda$.

Using the common ι-markup property within nest n, the sum on the right-hand side of

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whereas, under NCES, its market share in value is

$$\frac{1}{\beta(\sigma-1)}\sum_{l\in f}\sum_{j\in l} p_j D_j(p) = \frac{1}{\sigma-1}\frac{\sum_{l\in f} H_l^{\beta-1}\left(\sum_{j\in l} p_j(-h_j')\right)}{H} = \frac{\sum_{l\in f} H_l^{\beta-1}\left(\sum_{j\in l} h_j\right)}{H} = s^f.$$

[12]If $\sum_{f\in\mathcal{F}} s^f < 1$, then our definition of market shares does not coincide with that used by antitrust practitioners. See Nevo (2000b) for guidance on how to compute the market share of the outside option, and therefore how to implement our definition in practice.

equation (2) can be written as:

$$\sum_{j\in n}(p_j - c_j)(-h'_j) = \sum_{j\in n} \frac{p_j - c_j}{p_j} \frac{h''_j}{-h'_j} \frac{(h'_j)^2}{h''_j} = \tilde{\mu}_n \sum_{j\in n} \frac{(h'_j)^2}{h''_j} = \tilde{\alpha}\tilde{\mu}_n \sum_{j\in n} h_j = \tilde{\alpha}\tilde{\mu}_n H_n, \tag{3}$$

where $\tilde{\alpha} = (\sigma - 1)/\sigma < 1$ under NCES demand and $\tilde{\alpha} = 1$ under NMNL demand. Equation (2) therefore boils down to

$$\tilde{\mu}_n = \frac{1}{1 - \tilde{\alpha}(1-\beta)} \equiv \mu^{\text{mc}}. \tag{4}$$

As μ^{mc} does not depend on the identity of nest n nor on the identity of firm f, the monopolistically competitive ι-markup μ^{mc} is the same for each product $i \in \mathcal{N}$.

From equation (4), we obtain:

$$\frac{p_i - c_i}{p_i} = \frac{1}{\sigma - (\sigma - 1)(1-\beta)} \qquad \text{(under NCES demand)},$$

$$p_i - c_i = \frac{\lambda}{\beta} \qquad \text{(under NMNL demand)}.$$

2.3 Equilibrium Analysis

We now turn to the equilibrium analysis of our multiproduct-firm pricing game. This requires adapting the aggregative-games approach taken in Nocke and Schutz (2018, Section 5), where each firm is restricted to own only a single nest.

The first-order condition of profit maximization for product i in nest n owned by firm f is given by

$$\frac{H_n^{\beta-1}}{H} \left(-h'_i - (p_i - c_i)h''_i + (1-\beta) \frac{\partial H_n}{\partial p_i} \frac{\sum_{j\in n}(p_j - c_j)h'_j}{H_n} + \frac{H_n^{1-\beta}}{H} \frac{\partial H}{\partial p_i} \sum_{l\in f} H_l^{\beta-1} \sum_{j\in l}(p_j - c_j)h'_j \right) = 0.$$

The last term on the left-hand side of the equation, which is absent under monopolistic competition, captures the impact of the price change through the aggregator H. Simplifying and rearranging terms, we obtain

$$\frac{p_i - c_i}{p_i} \frac{p_i h''_i}{-h'_i} = 1 + (1-\beta) \frac{\sum_{j\in n}(p_j - c_j)(-h'_j)}{H_n} + \beta \frac{1}{H} \sum_{l\in f} H_l^{\beta-1} \sum_{j\in l}(p_j - c_j)(-h'_j). \tag{5}$$

Despite the additional term on the right-hand side of the equation, we obtain the common ι-markup property within nest n, i.e.,

$$\frac{p_i - c_i}{p_i} \frac{p_i h''_i}{-h'_i} = \tilde{\mu}_n$$

for every $i \in n$.

Using the common ι-markup property within each nest l as well as equation (3), equation (5) can be rewritten as

$$\tilde{\mu}_n \left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \tilde{\alpha}\beta \frac{1}{H} \sum_{l \in f} \tilde{\mu}^l H_l^\beta, \tag{6}$$

which immediately implies that $\tilde{\mu}_n = \tilde{\mu}_{n'} \equiv \tilde{\mu}^f$ for every $n, n' \in f$. Firm f therefore applies the same ι-markup $\tilde{\mu}^f$ to all the products in its portfolio. Using this common ι-markup property, both within and across nests, equation (6) simplifies to

$$\tilde{\mu}^f \left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \tilde{\alpha}\beta\tilde{\mu}^f \frac{\sum_{l \in f} H_l^\beta}{H} = 1 + \tilde{\alpha}\beta\tilde{\mu}^f s^f. \tag{7}$$

Define the elasticity measure $\alpha \equiv \tilde{\alpha}\beta/(1 - \tilde{\alpha}(1-\beta))$, and note that $\alpha < 1$ under NCES and $\alpha = 1$ under NMNL. Using equation (7), we can decompose firm f's ι-markup as follows:

$$\tilde{\mu}^f = \underbrace{\frac{1}{1 - \tilde{\alpha}(1-\beta)}}_{=\mu^{\text{mc}}} \underbrace{\frac{1}{1 - \alpha s^f}}_{\equiv \mu^f}.$$

That is, under oligopoly, firm f's ι-markup $\tilde{\mu}^f$ is the product of the monopolistically competitive ι-markup μ^{mc} and a market power factor, the normalized markup $\mu^f > 1$. This decomposition reveals that firms with larger market shares have more market power, and therefore set higher ι-markups.

Equations (3) and (7) yield a simple formula for firm f's equilibrium profit:

$$\Pi^f = \tilde{\alpha}\beta\tilde{\mu}^f s^f = \mu^f - 1. \tag{8}$$

Next, we express firm f's market share as a function of the industry-level aggregator H and firm f's normalized markup μ^f. Under NCES,

$$\begin{aligned} s^f &= \frac{1}{H} \sum_{l \in f} \left(\sum_{j \in l} a_j \left(\frac{\sigma}{\sigma - \tilde{\mu}^f} c_j \right)^{1-\sigma} \right)^\beta, \\ &= \frac{1}{H} \underbrace{\sum_{l \in f} \left(\sum_{j \in l} a_j c_j^{1-\sigma} \right)^\beta}_{\equiv T^f} \left(1 - (1-\tilde{\alpha})\tilde{\mu}^f\right)^{\frac{\tilde{\alpha}\beta}{1-\tilde{\alpha}}}, \\ &= \frac{T^f}{H} \left(1 - (1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

Under NMNL,

$$s^f = \frac{1}{H} \sum_{l \in f} \left(\sum_{j \in l} \exp\left(\frac{a_j - c_j}{\lambda} - \tilde{\mu}^f \right) \right)^{\beta} = \frac{1}{H} \underbrace{\sum_{l \in f} \left(\sum_{j \in l} \exp\left(\frac{a_j - c_j}{\lambda} \right) \right)^{\beta}}_{\equiv T^f} \exp(-\mu^f).$$

We call T^f firm f's type. As we shall see below, that uni-dimensional sufficient statistic aggregates all the relevant information about firm f's product portfolio—the *type aggregation property*.

The above analysis implies that, if H is an equilibrium aggregator level, then firm f's markup and market share μ^f and s^f jointly solve the following system of equations:

$$\mu^f = \frac{1}{1 - \alpha s^f}, \tag{9}$$

$$s^f = \begin{cases} \frac{T^f}{H} \left(1 - (1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}} & \text{in the case of NCES,} \\ \frac{T^f}{H} \mathrm{e}^{-\mu^f} & \text{in the case of NMNL.} \end{cases} \tag{10}$$

Nocke and Schutz (2018, Section 5) show that this system has a unique solution ($m(T^f/H)$, $S(T^f/H)$). We call $m(T^f/H)$ and $S(T^f/H)$ the firm's *markup fitting-in function* and *market-share fitting-in function*, respectively. Both fitting-in functions are increasing, $m' > 0$ and $S' > 0$, i.e., a firm that has a higher type and operates in a less competitive environment (lower H) sets a higher markup and commands a higher market share; moreover, the range of S is the entire interval $(0, 1)$. Using equation (8), we obtain the *profit fitting-in function* $\pi(T^f/H) = m(T^f/H) - 1$.

The equilibrium aggregator level is pinned down by the equilibrium condition

$$\frac{H^0}{H} + \sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}\right) = 1, \tag{11}$$

which says that market shares add up to unity. The continuity and monotonicity properties of S along with the fact that S has full range imply that equation (11) has a unique solution, establishing equilibrium existence and uniqueness.

We summarize these insights in the following proposition:

Proposition 1. *The multiproduct-firm pricing game has a unique equilibrium. The equilibrium aggregator level H^* is the unique solution of equation* (11)*. In equilibrium, firm $f \in \mathcal{F}$ sets a markup of $m(T^f/H^*)$, commands a market share of $S(T^f/H^*)$, and earns a profit of $\pi(T^f/H^*)$.*

Proof. The only thing left to prove is that first-order conditions are necessary and sufficient for global optimality. This is done in Appendix A. □

The following proposition, which follows immediately from Nocke and Schutz (2018), provides intuitive comparative statics:

Proposition 2 (Nocke and Schutz, 2018, Proposition 6). *An increase in T^f raises firm f's equilibrium markup $m(T^f/H^*)$, market share $S(T^f/H^*)$, and profit $\pi(T^f/H^*)$, reduces firm $g \neq f$'s equilibrium markup $m(T^g/H^*)$, market share $S(T^g/H^*)$, and profit $\pi(T^g/H^*)$, and raises consumer surplus and aggregate surplus.*

The Monopolistic Competition Limit. In the monopolistic competition outcome studied in Section 2.2, each firm f sets a normalized markup μ^f of one. In the oligopoly model studied here, this outcome arises in the limit as firms' market shares tend to zero, that is, when firms become atomless. Such a limiting outcome can be obtained by infinitely replicating the population of firms, or by making the value of the outside option, H^0, go to infinity.

Firm Conduct. Some of the approximation results derived in Section 3 will require bridging the gap between monopolistic competition conduct and fully-fledged "Bertrand-Nash" conduct. Specifically, let $\theta \in [0,1]$ be a conduct parameter, and assume that each firm believes that the impact of p_i, $i \in \mathcal{N}$, on the aggregator is $\theta \partial H/\partial p_i$ instead of $\partial H/\partial p_i$, i.e., firms internalize their impact on the aggregator only to a certain extent.[13] Under those conjectures, the first-order condition for product $i \in n \in f$ becomes

$$\frac{H_n^{\beta-1}}{H}\Bigg(-h_i' - (p_i - c_i)h_i'' + (1-\beta)\frac{\partial H_n}{\partial p_i}\frac{\sum_{j\in n}(p_j - c_j)h_j'}{H_n} + \theta \times \frac{H_n^{1-\beta}}{H}\frac{\partial H}{\partial p_i}\sum_{l\in f} H_l^{\beta-1}\sum_{j\in l}(p_j - c_j)h_j'\Bigg) = 0.$$

Proceeding as above, we find that firm f's markup and market share jointly solve the following system of equations

$$\mu^f = \frac{1}{1-\theta\alpha s^f},$$
$$s^f = \begin{cases} \frac{T^f}{H}\left(1-(1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}} & \text{in the case of NCES,} \\ \frac{T^f}{H}\,\mathrm{e}^{-\mu^f} & \text{in the case of NMNL,} \end{cases}$$

which pins down the fitting-in functions $m(T^f/H,\theta)$ and $S(T^f/H,\theta)$. The profit fitting-in function is given by $\pi(T^f/H,\theta) = \alpha m(T^f/H,\theta)S(T^f/H,\theta)$. The equilibrium aggregator

[13] Our treatment of firm conduct is closely related to the classical approach under quantity competition with homogeneous products surveyed by Bresnahan (1989). In Bresnahan (1989), a firm conjectures that the price sensitivity of the inverse demand function $P(\cdot)$ is $\theta P'(\cdot)$ instead of $P'(\cdot)$. The special cases of perfect competition and Cournot-Nash conducts arise respectively when $\theta = 0$ and $\theta = 1$.

level $H^*(\theta)$ uniquely solves the equation

$$\frac{H^0}{H} + \sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}, \theta\right) = 1.$$

It is easy to see that $H^*(\theta)$, $m(\cdot, \theta)$, $S(\cdot, \theta)$, and $\pi(\cdot, \theta)$ all tend to their value under monopolistic competition as θ tends to 0, and to their value under fully-fledged oligopoly as θ tends to 1.[14]

2.4 Modeling Mergers

Consider a merger between the firms $\mathcal{M} \subsetneq \mathcal{F}$, and let $\mathcal{O} \equiv \mathcal{F} \setminus \mathcal{M}$ be the set of non-merging firms—the outsiders. The post-merger economic environment can be summarized by the tuple $(\overline{\mathcal{N}}, \overline{\mathcal{L}}, \overline{\mathcal{F}}, (\overline{a}_j)_{j \in \overline{\mathcal{N}}}, (\overline{c}_j)_{j \in \overline{\mathcal{N}}})$ along with the pre-merger nest parameter β, and the pre-merger elasticity parameters σ under NCES and λ under NMNL.

We assume that the merger does not directly affect the outsiders. Formally, this means that: For every $f \in \mathcal{O}$ and $l \in f$, the nest l is contained in $\overline{\mathcal{N}}$; for every $f \in \mathcal{O}$ and $l \in f$, the nest l belongs to $\overline{\mathcal{L}}$; for every $i \in l \in f \in \mathcal{O}$, we have $\overline{a}_i = a_i$ and $\overline{c}_i = c_i$. These assumptions imply that the post-merger type of each outsider $f \in \mathcal{O}$ is equal to its pre-merger type, T^f.

The merged firm M is defined as $M = \overline{\mathcal{L}} \setminus \left(\bigcup_{f \in \mathcal{O}} \bigcup_{l \in f} \{l\}\right)$. The post-merger set of firms is therefore $\overline{\mathcal{F}} = \{M\} \cup \mathcal{O}$. We allow for the possibility that the merger affects the merging firms' set of products by adding or dropping products (including entire nests) as well as the marginal costs and qualities of their pre-existing products. Formally, this means that we do not impose any condition on the relationship between the merging firms' pre-merger products, $\left(\bigcup_{f \in \mathcal{M}} \bigcup_{l \in f} \{l\}\right)$, and the merged firm's post-merger products, M, implying no restriction on the relationship between the merged firm's type, T^M, and the merger partners' pre-merger types, $(T^f)_{f \in \mathcal{M}}$.

Our aggregative-games tools and the type aggregation property deliver important benefits in terms of tractability, in that they allow us to view a merger as an event that turns the pre-merger type vector $(T^f)_{f \in \mathcal{F}}$ into $(T^M, (T^f)_{f \in \mathcal{O}})$ (rather than an event that turns the pre-merger economic environment $(\mathcal{N}, \mathcal{L}, \mathcal{F}, (a_j)_{j \in \mathcal{N}}, (c_j)_{j \in \mathcal{N}})$ into $(\overline{\mathcal{N}}, \overline{\mathcal{L}}, \overline{\mathcal{F}}, (\overline{a}_j)_{j \in \overline{\mathcal{N}}}, (\overline{c}_j)_{j \in \overline{\mathcal{N}}})$).

A special case of interest arises when the merger does not involve any synergies, so that $M = \left(\bigcup_{f \in \mathcal{M}} \bigcup_{l \in f} \{l\}\right)$, $\overline{a}_j = a_j$ and $\overline{c}_j = c_j$ for all $j \in l \in M$, implying that $T^M = \sum_{f \in \mathcal{M}} T^f$. We say that the merger involves synergies if $T^M > \sum_{f \in \mathcal{M}} T^f$.

[14] Our results on mergers in Section 4 and 5 are all stated and proved in the case of Bertrand-Nash conduct ($\theta = 1$). Our aggregative-games techniques can easily be applied to generalize those results to the case where $\theta \in (0, 1]$.

3 Market Shares as Sufficient Statistics for Industry Performance

In antitrust practice, the Herfindahl index (HHI), defined as

$$\text{HHI}\left((s^f)_{f\in\mathcal{F}}\right) \equiv \sum_{f\in\mathcal{F}} (s^f)^2,$$

is often used to gauge the extent of market power in an industry as well as the potential market power effect of a merger (see, e.g., the 2010 U.S. Horizontal Merger Guidelines). The presumption is that there is more market power in industries where the Herfindahl index is larger, and the market power effect tends to be larger when (i) the pre-merger Herfindahl index is larger and (ii) the merger-induced increase in the Herfindahl index is larger.

In this section, we provide theoretical support for this presumption. Specifically, we derive approximation results that show that this index is an adequate measure of the welfare distortions introduced by market power. Using similar approximation techniques, we also show that the naively-computed change in the Herfindahl index induced by a merger is an appropriate measure of the market power effect of the merger. To prove these results, we first relate measures of industry performance to the equilibrium market share vector. Such an analysis is useful for antitrust practice as market shares are often readily available while firms' types are not.

3.1 Market Shares and Welfare

Let $(s^f)_{f\in\mathcal{F}}$ be the profile of equilibrium market shares. Assume that consumers have access to an outside option ($H^0 > 0$), so that $\sum_{f\in\mathcal{F}} s^f < 1$. Equation (11) implies that the equilibrium aggregator level H^* is equal to $H^0/(1-\sum_{f\in\mathcal{F}} s^f)$. As shown in Anderson and Nocke (2014), this implies that consumer surplus can be written as a function of market shares:[15]

$$\text{CS}\left((s^f)_{f\in\mathcal{F}}\right) = \log H^0 - \log\left(1 - \sum_{f\in\mathcal{F}} s^f\right).$$

As firm f's equilibrium profit is $\mu^f - 1$ and $\mu^f = 1/(1-\alpha s^f)$, the same holds for aggregate surplus:

$$\text{AS}\left((s^f)_{f\in\mathcal{F}}\right) = \log H^0 - \log\left(1 - \sum_{f\in\mathcal{F}} s^f\right) + \sum_{f\in\mathcal{F}} \frac{\alpha s^f}{1-\alpha s^f}.$$

Note that both consumer surplus and aggregate surplus are increasing in the vector of market shares. Moreover, by convexity of $s \mapsto s/(1-\alpha s)$, a mean-preserving spread of

[15] See Armstrong and Vickers (forthcoming) for a treatment of the related concept of consumer surplus as a function of quantities.

market shares would leave consumer surplus unchanged but raise industry profit and therefore aggregate surplus.[16]

3.2 The Herfindahl Index as a Measure of Market Power

We now argue that the Herfindahl index provides an adequate measure of the consumer surplus and aggregate surplus distortions stemming from oligopolistic behavior. As a benchmark for the hypothetical situation without market power, we use the equilibrium outcome under monopolistic competition. In the context of differentiated goods, monopolistic competition is arguably a more appealing benchmark than perfect competition: If goods are homogeneous and firms compete either in prices or quantities, then the equilibrium outcome converges to perfect competition as the population of firms is infinitely replicated, so that each firm's limiting size is negligible relative to the size of the market. By contrast, in our framework, such an infinite replication results in the monopolistic competition outcome, as we showed in Section 2.2.

We provide two sets of approximation results: When firms have small market shares, and when industry conduct is close to monopolistic competition.

Approximation Results for Small Firms. For this set of approximations, we assume that consumers have access to an outside option. We proceed as follows. We first fix a vector of market shares $s = (s^f)_{f \in \mathcal{F}}$, and compute the welfare measures $\mathrm{CS}(s)$ and $\mathrm{AS}(s)$. Using s, we then back out the type vector $T(s) = (T^f(s))_{f \in \mathcal{F}}$ that gives rise to this profile of market shares under oligopoly. Next, using $T(s)$, we compute our welfare measure under monopolistic competition as a function of firms' market shares under oligopoly, $\mathrm{CS}^m(s)$ and $\mathrm{AS}^m(s)$. Finally, we apply Taylor's Theorem to derive a second-order approximation of the welfare distortions for small market shares:

Proposition 3. *In the neighborhood of* $s = 0$*,*

$$\begin{aligned} CS(s) - CS^m(s) &= -\alpha HHI(s) + o\left(\|s\|^2\right), \\ \text{and } \; AS(s) - AS^m(s) &= -\alpha HHI(s) + o\left(\|s\|^2\right). \end{aligned}$$

Proof. See Appendix C.1. □

The market power distortion to both consumer surplus and aggregate surplus is thus approximately proportional to the Herfindahl index, where the proportionality factor is the elasticity measure α.

To see why the distortion to consumer surplus increases with the Herfindahl index, consider a mean-preserving spread of the market share vector s under oligopoly. This raises the

[16] This is akin to the homogeneous-goods Cournot model, where consumer surplus depends only on aggregate output, and aggregate surplus is proportional to the Herfindahl index, holding aggregate output fixed.

Herfindahl index but leaves consumer surplus unchanged, as $\mathrm{CS}(s)$ depends only on the sum of market shares. The concavity of the market-share fitting-in function $S(\cdot)$, which comes from the fact that a firm with a higher type tends to charge a higher markup, implies that the mean-preserving spread of the market share vector must have been caused by a sum-increasing change in the vector of firm types.[17] As consumer surplus under monopolistic competition depends only the sum of those types, this change increases $\mathrm{CS}^m(s)$.

It may seem surprising that the distortion to consumer surplus is equal to that to aggregate surplus at the second order. In Appendix C.1, we show that those two distortions no longer coincide at the third order.[18]

Approximation Results around Monopolistic Competition Conduct. We now provide an alternative approximation of the market power distortion from oligopolistic behavior, namely one involving only small departures from monopolistic competition conduct, but without restricting the size of firms or imposing that $H^0 > 0$.

Given the conduct parameter $\theta \in [0,1]$ and the type vector $(T^f)_{f\in\mathcal{F}}$, the equilibrium aggregator level is $H^*((T^f)_{f\in\mathcal{F}}, \theta)$, equilibrium consumer surplus is

$$\mathrm{CS}((T^f)_{f\in\mathcal{F}}, \theta) = \log H^*((T^f)_{f\in\mathcal{F}}, \theta),$$

and aggregate surplus is

$$\mathrm{AS}((T^f)_{f\in\mathcal{F}}, \theta) = \log H^*((T^f)_{f\in\mathcal{F}}, \theta) + \sum_{g\in\mathcal{F}} \frac{\alpha S\left(\frac{T^g}{H^*((T^f)_{f\in\mathcal{F}},\theta)}, \theta\right)}{1 - \alpha\theta S\left(\frac{T^g}{H^*((T^f)_{f\in\mathcal{F}},\theta)}, \theta\right)}.$$

The industry-level Herfindahl index is given by

$$\mathrm{HHI}((T^f)_{f\in\mathcal{F}}, \theta) = \sum_{g\in\mathcal{F}} S\left(\frac{T^g}{H^*((T^f)_{f\in\mathcal{F}}, \theta)}, \theta\right)^2.$$

We can now provide a first-order Taylor approximation of the market power distortions to consumer surplus and aggregate surplus in the neighborhood of $\theta = 0$, i.e., close to monopolistic competition conduct. Below, we drop the argument $(T^f)_{f\in\mathcal{F}}$ from the functions H^*, CS, AS, and HHI to ease notation.

[17] The concavity of S is stated and proved in Lemma 1 in Appendix B.

[18] Specifically, at the third order in the neighborhood of $s = 0$,

$$\mathrm{CS}(s) - \mathrm{CS}^m(s) = -\alpha\left(\mathrm{HHI}(s) + \frac{1}{2}(1+2\alpha)\Gamma(s)\right) + o\left(\|s\|^3\right),$$

$$\text{and } \mathrm{AS}(s) - \mathrm{AS}^m(s) = -\alpha\left(\mathrm{HHI}(s)(1-\alpha\bar{s}) + \frac{1}{2}(1+3\alpha)\Gamma(s)\right) + o\left(\|s\|^3\right),$$

where $\bar{s} = \sum_{f\in\mathcal{F}} s^f$ and $\Gamma(s) = \sum_{f\in\mathcal{F}} (s^f)^3$.

Proposition 4. *In the neighborhood of $\theta = 0$,*

$$CS(\theta) - CS(0) = -\alpha HHI(\theta)\theta + o(\theta),$$

$$\text{and } AS(\theta) - AS(0) = -\alpha HHI(\theta)\left(1 - \alpha \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(\theta)}, \theta\right)\right)\theta + o(\theta).$$

Proof. See Appendix D.1. □

As in the approximation with small market shares, the market power distortion to consumer surplus is proportional to the Herfindahl index. In contrast, the market power distortion to aggregate surplus now contains a new term that depends on α and the aggregate market share. Holding fixed the aggregate market share, the distortion continues to be proportional to the Herfindahl index. Holding fixed the Herfindahl index, the distortion decreases with the aggregate market share. If the aggregate market share is small, the distortion is approximately the same as in Proposition 3.[19]

3.3 The Herfindahl Index as a Measure of the Market Power Effect of a Merger

In line with the literature, we define the market power effect of a merger as the impact that merger would have on consumer surplus or aggregate surplus if it involved no synergies. We now argue that the naively-computed change in the Herfindahl index induced by the merger is an adequate measure of that merger's market power effect. As in Section 3.2, we support this claim by providing approximation results around $s = 0$ and $\theta = 0$.

Approximation Results for Small Firms. As in Section 3.2, we assume that consumers have access to an outside option. We proceed as follows. We fix the pre-merger vector of market shares $s = (s^f)_{f\in\mathcal{F}}$, and use this vector to recover the pre-merger type vector $(T^f(s))_{f\in\mathcal{F}}$ and compute the pre-merger market performance measures $\text{CS}(s)$ and $\text{AS}(s)$. Assuming no synergies, the merged firm's type is $T^M(s) = \sum_{f\in\mathcal{M}} T^f(s)$. We then use the post-merger type vector $(T^f(s))_{f\in\overline{\mathcal{F}}}$ to obtain the post-merger vector of market shares $\bar{s}(s) = (\bar{s}(s))_{f\in\overline{\mathcal{F}}}$. The post-merger welfare measures are $\text{CS}(\bar{s}(s))$ and $\text{AS}(\bar{s}(s))$. Hence, the market power effect of the merger is $\text{CS}(\bar{s}(s)) - \text{CS}(s)$ or $\text{AS}(\bar{s}(s)) - \text{AS}(s)$.

The merged-induced, naively-computed variation in the Herfindahl index is:

$$\Delta^M \text{HHI}(s) = \left(\left(\sum_{f\in\mathcal{M}} s^f\right)^2 + \sum_{f\in\mathcal{O}} (s^f)^2\right) - \sum_{f\in\mathcal{F}} (s^f)^2 = \left(\sum_{f\in\mathcal{M}} s^f\right)^2 - \sum_{f\in\mathcal{M}} (s^f)^2.$$

Applying Taylor's theorem, we obtain the following approximation results:

[19] More precisely, the term $\text{HHI}(s)\sum_{f\in\mathcal{F}} s^f$ is third order in the neighborhood of $s = 0$. Note that that term does appear in the third-order Taylor approximation shown in footnote 18.

Proposition 5. *In the neighborhood of $s = 0$,*

$$CS(\bar{s}(s)) - CS(s) = -\alpha\Delta^M\, HHI(s) + o(\|s\|^2),$$
$$\text{and } \; AS(\bar{s}(s)) - AS(s) = -\alpha\Delta^M\, HHI(s) + o(\|s\|^2).$$

Proof. See Appendix C.1. □

Hence, the market power effect of a merger is proportional to the naively-computed variation in the Herfindahl index, where the proportionality coefficient is the elasticity measure α. As was the case in Proposition 3, this holds regardless of whether the market power effect is measured in terms of consumer surplus or aggregate surplus.

Approximation Results around Monopolistic Competition Conduct. Let θ be a conduct parameter and $(T^f)_{f\in\mathcal{F}}$ be the pre-merger type vector. The merged firm's type is $T^M = \sum_{f\in\mathcal{M}} T^f$, assuming no synergies. Using the notation introduced in Section 3.2, the market power effect of the merger given conduct parameter θ is

$$\mathrm{CS}((T^f)_{f\in\overline{\mathcal{F}}}, \theta) - \mathrm{CS}((T^f)_{f\in\mathcal{F}}, \theta)$$

or

$$\mathrm{AS}((T^f)_{f\in\overline{\mathcal{F}}}, \theta) - \mathrm{AS}((T^f)_{f\in\mathcal{F}}, \theta).$$

The merger-induced, naively-computed change in the Herfindahl index is:

$$\Delta^M\,\mathrm{HHI}((T^f)_{f\in\mathcal{F}}, \theta) =$$
$$\left(\sum_{g\in\mathcal{M}} S\left(\frac{T^g}{H^*((T^f)_{f\in\mathcal{F}}, \theta)}, \theta\right)\right)^2 + \sum_{g\in O} S\left(\frac{T^g}{H^*((T^f)_{f\in\mathcal{F}}, \theta)}, \theta\right)^2 - \sum_{g\in F} S\left(\frac{T^g}{H^*((T^f)_{f\in\mathcal{F}}, \theta)}, \theta\right)^2.$$

We provide a linear approximation of the market power effect of the merger around monopolistic competition conduct:

Proposition 6. *In the neighborhood of $\theta = 0$,*

$$CS((T^f)_{f\in\overline{\mathcal{F}}}, \theta) - CS((T^f)_{f\in\mathcal{F}}, \theta) = -\alpha\Delta^M\, HHI((T^f)_{f\in\mathcal{F}}, \theta)\theta + o(\theta),$$

and

$$AS((T^f)_{f\in\overline{\mathcal{F}}}, \theta) - AS((T^f)_{f\in\mathcal{F}}, \theta) =$$
$$-\alpha\Delta^M\, HHI((T^f)_{f\in\mathcal{F}}, \theta)\left(1 - \alpha\sum_{g\in\mathcal{F}} S\left(\frac{T^g}{H^*((T^f)_{f\in\mathcal{F}}, \theta)}, \theta\right)\right)\theta + o(\theta).$$

Proof. See Appendix D.2. □

As in the approximation of the aggregate surplus distortion from market power, the merger's market power on aggregate surplus when approximated around monopolistic competition conduct differs slightly from that when approximated around small market shares. That difference vanishes as market shares become small.

4 Consumer Surplus Effects of Mergers

We now turn to the consumer surplus effects of mergers—a question we already touched on in Section 3 by providing two sets of Taylor approximations. In this section, we revisit this question without approximations. We study a static framework in Section 4.1 and a dynamic one with endogenous mergers in Section 4.2.

4.1 Static Analysis

Consider a merger M between the firms in $\mathcal{M}$. Let H^* (resp., $\overline{H}^*$) denote the equilibrium value of the aggregator before (resp., after) the merger. As consumer surplus is increasing in the value of that aggregator, we say that the merger is *CS-increasing* (resp., *CS-decreasing*) if $\overline{H}^* > H^*$ (resp., $\overline{H}^* < H^*$); it is *CS-neutral* if $\overline{H}^* = H^*$.

Suppose the merger is CS-neutral. This implies that the market share of each outsider $g \in \mathcal{O}$, $S(T^g/H^*)$, and the market share of the outside option, H^0/H^*, is unaffected by the merger. Since the market shares of the firms and the outside option have to add up to one (equation (11)), this means that the post-merger market share of the merged firm is equal to the sum of the pre-merger market shares of the merger partners:

$$S\left(\frac{T^M}{H^*}\right) = \sum_{f \in \mathcal{M}} S\left(\frac{T^f}{H^*}\right),$$

where we have used the fact that $\overline{H}^* = H^*$.

As S is strictly increasing and has full range, it follows that there exists a unique cutoff type $\hat{T}^M$ such that the merger is CS-neutral if and only if $T^M = \hat{T}^M$:

$$\hat{T}^M = H^* S^{-1}\left(\sum_{f \in \mathcal{M}} S\left(\frac{T^f}{H^*}\right)\right).$$

By Proposition 2, $\overline{H}^*$ is strictly increasing in T^M, implying that the merger is CS-increasing if and only if $T^M > \hat{T}^M$, and CS-decreasing if and only if the reverse inequality holds.

As the market-share fitting-in function S is strictly concave (see Lemma 1 in Appendix B) and satisfies $S(0) = 0$, that function is sub-additive. This implies that the cutoff type satisfies

$\hat{T}^M > \sum_{f \in \mathcal{M}} T^f$. That is, for the merger to be CS-nondecreasing it has to involve synergies.[20] We summarize these insights in the following proposition:

Proposition 7. *For a merger among the firms in $\mathcal{M}$, there exists a unique $\hat{T}^M > \sum_{f \in \mathcal{M}} T^f$ such that the merger is CS-neutral if the post-merger type satisfies $T^M = \hat{T}^M$, CS-decreasing if $T^M < \hat{T}^M$, and CS-increasing if $T^M > \hat{T}^M$.*

We now turn to the comparative statics of the post-merger cutoff-type $\hat{T}^M$. First, we consider the thought experiment of changing the pre-merger aggregator level H^* while holding the characteristics of the merger fixed. Second, we compare two alternative mergers in a given industry, thus holding fixed the pre-merger aggregator level H^*.

The first comparative statics result shows that the synergies required for a merger to be CS-nondecreasing are smaller the more competitive is the market before the merger:

Proposition 8. *For a merger among the firms in $\mathcal{M}$, the post-merger cutoff type $\hat{T}^M$ is strictly decreasing in the pre-merger level of the aggregator, H^*.*

Proof. See Appendix E.1. □

To see the intuition, consider a merger between two symmetric single-product firms, producing products i and j at pre-merger marginal cost c, and charging the pre-merger price p^*. Suppose the merger-induced synergies materialize only through a symmetric marginal cost reduction. As shown by Werden (1996), for the merger to be CS-neutral, the common post-merger marginal cost $\hat{c}$ must be such that

$$\frac{c - \hat{c}}{c} = \frac{d(H^*)}{1 - d(H^*)} \frac{(p^* - c)}{c}, \tag{12}$$

where

$$d(H^*) \equiv -\frac{\partial D_j / \partial p_i}{\partial D_i / \partial p_i}$$

is the diversion ratio from good i to good j, which by symmetry is also equal to the diversion ratio from j to i.

The left-hand side of equation (12) gives the required percentage change in marginal cost whereas the right-hand side represents the increase in market power due to the post-merger internalization of competitive externalities. An increase in the pre-merger aggregator level H^* does not affect the left-hand side but reduces the right-hand side through two channels: It reduces both the pre-merger equilibrium price p^* and the diversion ratio $d(h^*)$.[21] Proposition 8 shows that this intuition generalizes to mergers between arbitrary sets of firms, involving arbitrary forms of synergies.

[20]Farrell and Shapiro (1990) obtain the same conclusion in the case of the homogeneous-goods Cournot model.

[21]In our model, the diversion ratio between two symmetric single-product firms can be shown to be equal to $\alpha s^*/(1 - \alpha s^*)$, which is increasing in the equilibrium market share s^*, and thus decreasing in H^*.

We now turn to our second comparative statics result. It shows that the synergies required for a merger to be CS-nondecreasing are larger for mergers involving larger firms, holding fixed the pre-merger aggregator level H^*.

Proposition 9. *Consider a merger between the firms in* $\mathcal{M} = \{f, g\}$, *resp.,* $\mathcal{M}' = \{f', g'\}$, *where* $T^f \geq T^{f'}$ *and* $T^g > T^{g'}$. *Then, the "larger" merger* $\mathcal{M}$ *requires larger synergies than* $\mathcal{M}'$, *in the sense of a larger fractional increase in type:*

$$\frac{\hat{T}^M}{T^f + T^g} > \frac{\hat{T}^{M'}}{T^{f'} + T^{g'}}.$$

This in turn implies that the larger merger requires a larger absolute increase in type:

$$\hat{T}^M - (T^f + T^g) > \hat{T}^{M'} - (T^{f'} + T^{g'}).$$

Proof. See Appendix E.2. □

To see the intuition, suppose each of the two mergers involves symmetric single-product firms, and that merger-induced synergies materialize only through a symmetric reduction in the common marginal cost. The right-hand side of equation (12) is larger for merger $\mathcal{M}$ than $\mathcal{M}'$ as each merger partner in $\mathcal{M}$ has a higher pre-merger market share, implying that both its pre-merger diversion ratio $d(H^*)$ and its markup $(p^* - c)/c$ are larger. Hence, the percentage cost reduction necessary for the merger to be CS-neutral is larger for the larger merger.[22]

Propositions 8 and 9 provide theoretical support for the use of the merger-induced, naively-computed variation in the Herfindahl index to screen mergers. For merger $\mathcal{M} = \{f, g\}$, the naively-computed increase in the Herfindahl index is equal to

$$\Delta^M \text{HHI} = \left(s^f + s^g\right)^2 - ((s^f)^2 + (s^g)^2) = 2s^f s^g.$$

A merger involving larger firms will induce a larger Δ^M HHI.

Proposition 8 shows that, holding fixed the types of the merger partners, a decrease in the pre-merger equilibrium aggregator level H^*, which results in a higher Δ^M HHI, raises the required level of synergies for the merger to be CS-increasing. Proposition 9 shows that, holding fixed the pre-merger equilibrium aggregator level, a merger involving firms with higher types, and thus resulting in a higher Δ^M HHI, also raises that required level of synergies. Both propositions in conjunction suggest that the additional scrutiny received by

[22] In the case of NCES demand, there is a monotonic relationship between the percentage reduction in marginal cost and the percentage increase in type ($dT/T = (1 - \sigma)\beta dc/c$). In the case of NMNL demand, there is instead a monotonic relationship between the absolute reduction in marginal cost and the percentage increase in type ($dT/T = -(\beta/\lambda)dc$). As a larger firm charges a larger absolute markup under NMNL demand, equation (12) implies that the larger merger also requires a larger absolute reduction in marginal cost, and thus a larger percentage increase in type.

mergers resulting in a higher naively-computed increase in the Herfindahl index is indeed warranted.

4.2 Dynamic Analysis

In the previous subsection, we studied the static consumer surplus effect of a given merger. In industries in which merger opportunities are not isolated events, such a static analysis may be inappropriate: The approval decision on a currently proposed merger may affect both the consumer surplus effects of future mergers, and therefore the set of mergers that will be approved in the future, as well as the profitability of future mergers, and therefore the set of mergers that will be proposed in the future.

In the following, we show that a completely myopic merger approval policy, according to which, in every period, the antitrust authority approves only those mergers that raise consumer surplus given current market conditions, is dynamically optimal. This extends the main insight of Nocke and Whinston (2010), derived in the context of a homogeneous-goods Cournot model, to the case of differentiated-goods price competition with NMNL or NCES demands.

Framework. Following Nocke and Whinston (2010), we assume that there is a set of potential mergers, $\{\mathcal{M}_1, ..., \mathcal{M}_K\}$, and that all of these mergers are disjoint, i.e., $\mathcal{M}_k \cap \mathcal{M}_l = \emptyset$ for $k \neq l$. Disjointness means that each firm has a distinct set of natural merger partners that have the potential to create sizable synergies by merging. (Recall from the previous subsection that any merger not involving synergies is CS-decreasing.)

There are $\tau < \infty$ periods in which mergers may become feasible, and be proposed to the antitrust authority for approval. Any merger $\mathcal{M}_k$ may become feasible at the beginning of period $1 \leq t \leq \tau$ with probability $p_t^{M_k}$, where $\sum_t p_t^{M_k} \leq 1$. Once merger $\mathcal{M}_k$ has become feasible, the merger partners learn the realization of their post-merger type T^{M_k}, which is drawn from some set $\mathcal{T}_t^{M_k}$ according to some continuous probability distribution.

If merger $\mathcal{M}_k$ has become feasible in period t, or became feasible earlier but has not yet been approved, the merger partners decide whether to propose it for approval to the antitrust authority. We assume that bargaining is efficient so that the merger is proposed if and only if it is in the merger partners' joint interest to do so. When doing so, they observe the type not only of their own merger but also that of any other feasible but not yet approved merger (as well as the type of every firm).[23]

If a feasible merger is proposed, the antitrust authority observes its efficiency (i.e., the post-merger type); the authority also observes the types of all firms. Market structure (as summarized by the vector of firm types) changes according to the authority's approval decisions. Importantly, while a blocked merger cannot be consummated, it can be proposed again in the future.

[23]One of the firms in $\mathcal{M}_k$ can be thought of as acting in the role of the proposer, with the gains or losses from the merger being split in fixed proportions among its partners.

At the end of period t, firms compete in prices under complete information, as described in Section 2.1. Payoffs in each period therefore depend only on the market structure at the end of that period. Firms as well as the authority discount future payoffs with factor $\delta \leq 1$.

Results. The main result of this subsection is that a myopically CS-maximizing merger policy is dynamically optimal in that it maximizes the discounted sum of consumer surplus.

A myopically CS-maximizing merger policy is a merger approval rule that, in each period t, maximizes consumer surplus in that period, given current market structure and the set of proposed mergers. As shown in Nocke and Whinston (2010), there may be more than one set of merger approvals that maximizes consumer surplus in a given period but, if so, these sets differ only by mergers that are CS-neutral given the other mergers in those sets. However, Proposition 2 implies that any merger is generically either CS-decreasing or CS-increasing, no matter what the market structure. For simplicity of exposition, we will thus henceforth assume that the myopically CS-maximizing set of merger approvals is unique.

Our result on the dynamic optimality of a CS-maximizing merger policy comes in two parts. First, we ignore the incentive constraints for proposing mergers and show that the myopically CS-maximizing merger policy maximizes discounted consumer surplus *if all feasible but not yet approved mergers are proposed in each period*. Second, we show that there exists a subgame-perfect equilibrium in which all feasible but not yet approved mergers are indeed proposed in each period. Moreover, any subgame-perfect equilibrium induces the same optimal sequence of period-by-period consumer surpluses.

To show the first part, we begin by establishing a sign-preserving complementarity in the consumer surplus effects of mergers. Consider two disjoint mergers M_k and M_l, and suppose first that each is CS-nondecreasing given current market structure, i.e., $T^{M_k} \geq \hat{T}^{M_k}$ and $T^{M_l} \geq \hat{T}^{M_l}$. If merger M_k is implemented first, then H^* weakly increases as the merger is CS-nondecreasing. By Proposition 8, this implies that $\hat{T}^{M_l}$ weakly decreases so that the condition for merger M_l to be nondecreasing, $T^{M_l} \geq \hat{T}^{M_l}$, continues to hold. By the same argument, if both mergers are CS-decreasing given current market structure, then implementing merger M_k increases the cutoff type for the other merger M_l, implying that M_l remains CS-decreasing. This insight is summarized in the following proposition:

Proposition 10. *If merger M_l is CS-nondecreasing in isolation, it remains CS-nondecreasing if another merger M_k, $k \neq l$, that is CS-nondecreasing in isolation takes place. If merger M_l is CS-decreasing in isolation, it remains CS-decreasing if another merger M_k, $k \neq l$, that is CS-decreasing in isolation takes place.*

Proposition 8 implies that a CS-increasing merger M_k can induce an otherwise CS-decreasing merger M_l to become CS-nondecreasing. In this case, we have:

Proposition 11. *Suppose that merger M_k is CS-nondecreasing in isolation whereas merger M_l is CS-decreasing in isolation but CS-nondecreasing once merger M_k has taken place. Then, merger M_k is CS-increasing conditional on merger M_l taking place.*

Proof. As in the proof of Proposition 2 in Nocke and Whinston (2010), consider the thought experiment of reversing the order of the two mergers: Consider first implementing merger M_l (step 1) and then merger M_k (step 2). As consumer surplus must, by assumption, be weakly higher when both mergers are implemented compared to when none is, and because consumer surplus strictly falls at step 1 (again, by assumption), consumer surplus must strictly increase at step 2. □

Propositions 10 and 11 imply that if the antitrust authority approves only mergers that are CS-nondecreasing at the time of approval, then it will not have ex post regret about previously approved mergers (as these remain CS-nondecreasing) nor about previously rejected mergers (as these remain feasible and therefore can be implemented once they become CS-nondecreasing). This intuitively explains the following result:

Corollary 1. *Suppose that all feasible but not yet approved mergers are proposed in each period. Then, the myopically CS-maximizing merger policy maximizes discounted consumer surplus, no matter what the realization of feasible mergers is.*

Proof. The corollary is the analogue of Lemma 4 in Nocke and Whinston (2010), and its proof is identical to that of the lemma in the earlier paper. It suffices to make the following two observations.

First, Lemma 4 in Nocke and Whinston (2010) states the result for the "most lenient" myopically CS-maximizing merger policy. However, the result and proof also hold for the "least lenient" such policy. As noted in the text, these two policies are generically identical in our model as every merger is, generically, either CS-increasing or CS-decreasing, but not CS-neutral.

Second, the proof of Lemma 4 uses the monotonicity property of Lemma 2 in Nocke and Whinston (2010). It is straightforward to see that Lemmas 5 and 6 in Nocke and Whinston (2010) hold in our setup, implying that the monotonicity property of Lemma 2 carries over as well. □

We now turn to the second part by showing that there always exists a subgame-perfect equilibrium in which, in each period, every feasible but not yet approved merger is proposed for approval.

The first step in showing this is that a CS-nondecreasing merger is privately profitable in the sense that it raises the joint profit of the merger partners, holding fixed the market structure in the rest of the industry. We first argue that a merger that does not involve synergies is profitable, as in Deneckere and Davidson (1985). Intuitively, such a merger lowers the equilibrium aggregator level, and therefore reduces the outsiders' contribution to the aggregator. It follows that the merging parties face less competition, and therefore make strictly higher profits after the merger. By Proposition 7, a CS-nondecreasing merger must involve synergies. Hence, by Proposition 2, a merger involving synergies must be more profitable than one that does not. This explains the following result:

Proposition 12. *A CS-nondecreasing merger M_k is privately profitable in that it strictly raises the joint profit of the merger partners, holding fixed the market structure among outsiders.*

Proof. We first show that merger M_k is profitable if it is CS-neutral. Recall that the profit of a firm can be written as $\Pi = m - 1$, and its market share as $S = (m-1)/(\alpha m)$. It follows that $\Pi = \alpha m S$. Note that

$$m\left(\frac{T^{M_k}}{H^*}\right) S\left(\frac{T^{M_k}}{H^*}\right) = m\left(\frac{T^{M_k}}{H^*}\right) \sum_{f\in\mathcal{M}_k} S\left(\frac{T^f}{H^*}\right) > \sum_{f\in\mathcal{M}_k} m\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right),$$

where the equality follows because the merger is CS-neutral, and the inequality follows because $\hat{T}^{M_k} > T^f$ for every $f \in \mathcal{M}_k$ and $m'(\cdot) > 0$.

Hence, merger $\mathcal{M}_k$ is profitable if $T^{M_k} = \hat{T}^{M_k}$. Next, suppose that the merger is CS-increasing, i.e., $T^M > \hat{T}^M$. Then, by Proposition 2, the merged firm makes a strictly higher equilibrium profit than if its type were $\hat{T}^{M_k}$, i.e., if it were CS-neutral. □

The second step consists in showing that a CS-nondecreasing merger is still privately profitable even if it induces (directly or indirectly) other mergers to become CS-nondecreasing, resulting in their approval:

Proposition 13. *Suppose that merger M_k is CS-nondecreasing given current market structure whereas merger M_l is CS-decreasing but becomes CS-nondecreasing once M_k has been implemented. Then, the joint profit of the firms in M_k is strictly higher if both mergers take place than if none does.*

Proof. As in the proof of Proposition 11, think of implementing merger M_l at step one. As that merger is CS-decreasing by assumption, the equilibrium level of the aggregator, H^*, must decrease, which strictly raises the profit of each firm in $\mathcal{M}_k$. Next, implement merger M_k at step two: As that merger remains, by Proposition 11, CS-nondecreasing after M_l has taken place, it is profitable by Proposition 12. We have thus shown that the joint profit of the firms in $\mathcal{M}_k$ strictly increases at each step. □

Propositions 12 and 13 imply that if the antitrust authority adopts a myopically CS-maximizing merger policy, then—in the last period, τ—there exists an equilibrium in which all feasible but not yet approved mergers are proposed. Consider now period $\tau - 1$. As the set of mergers that the antitrust authority would want to approve can only increase over time, the set of approved mergers in period τ is independent of firms' proposal decisions in period $\tau - 1$. By the same argument as for the last period, there therefore exists an equilibrium in which all feasible but not yet approved mergers are proposed in period $\tau - 1$. Folding backward, the same holds for each of the previous periods.

The following proposition states the main result of this subsection:

Proposition 14. *Suppose that the antitrust authority adopts the myopically CS-maximizing merger policy. Then, all feasible mergers being proposed in each period after any history is a subgame-perfect equilibrium. The resulting outcome maximizes discounted consumer surplus, no matter what the realized sequence of feasible mergers. Moreover, every subgame-perfect equilibrium results in the same optimal level of consumer surplus in each period.*

Proof. The proposition is the analogue of Proposition 3, part (i) in Nocke and Whinston (2010), and its proof is identical to that of the proposition in the earlier paper. (Note that, in our model, the most and least lenient myopically CS-maximizing merger policies generically coincide.) The proof in Nocke and Whinston (2010) makes explicit use of the statement about the private profitability of CS-nondecreasing mergers in Corollary 1 as well as of Lemmas 2, 4 and 5 in that paper. The profitability statement of Corollary 1 in Nocke and Whinston (2010) corresponds to Proposition 12 in our paper whereas Lemma 4 in Nocke and Whinston (2010) corresponds to our Corollary 1. As noted in the proof of our Corollary 1, Lemmas 5 and 6 in Nocke and Whinston (2010) hold in our setup, implying that Lemma 2 in Nocke and Whinston (2010) carries over as well. □

As in Nocke and Whinston (2010)'s homogeneous-goods Cournot model, a myopically CS-maximizing merger policy is dynamically optimal in a strong sense: The antitrust authority could not improve upon the resulting outcome even if it had perfect foresight about future realizations of feasible mergers (which it does not) nor if it had the power to undo previously approved mergers (which we assume it does not).

5 Aggregate Surplus and External Effects of Mergers

Although most antitrust authorities have adopted a consumer surplus standard, it is also important to understand the impact of mergers on aggregate surplus. We prove the analogue of Proposition 7 for aggregate surplus effects in Section 5.1. That is, we establish the existence of a cutoff type above which the merger under consideration raises aggregate surplus. We then study the external effect of a merger, defined as the impact of the merger on the sum of consumer surplus and the outsiders' aggregate profit, in Section 5.2.

5.1 Aggregate Surplus Effects

Consider a merger M between the firms in $\mathcal{M}$, and let T^M be the the merged firm's type. Let AS^* (resp., $\overline{\mathrm{AS}}^*$) denote equilibrium aggregate surplus before the merger (resp., after the merger). We say that the merger is AS-increasing if $\overline{\mathrm{AS}}^* > \mathrm{AS}^*$, AS-decreasing if $\overline{\mathrm{AS}}^* < \mathrm{AS}^*$, and AS-neutral if $\overline{\mathrm{AS}}^* = \mathrm{AS}^*$. We now prove the counterpart of Proposition 7 for aggregate surplus.

If $T^M = \hat{T}^M$, where $\hat{T}^M$ is the cutoff type defined in Proposition 7, then the merger is CS-neutral. Moreover, as the merger does not affect the equilibrium value of the aggregator, it has no impact on the outsiders' equilibrium profits. Since the merger is profitable by Proposition 12, it is therefore AS-increasing.

Next, we argue that the merger is AS-decreasing if T^M is small. It is easy to show that, as T^M tends to zero, the post-merger value of aggregate surplus converges to the value that would prevail if firm M did not exist. Note that this limiting value is equal to the equilibrium value aggregate surplus would have before the merger if the firms in $\mathcal{M}$ did not exist (or, equivalently, if all their types were equal to zero). As aggregate surplus is strictly increasing in types (Proposition 2), that value is strictly lower than actual pre-merger aggregate surplus. Hence, the merger is AS-decreasing for T^M sufficiently small.

To sum up, post-merger aggregate surplus exceeds its pre-merger value when T^M is high, and falls short of it when T^M is low. The continuity of aggregate surplus in types implies the existence of a cutoff type $\widetilde{T}^M$ that makes the merger AS-neutral. By monotonicity of aggregate surplus, that cutoff type is unique, and the merger is AS-increasing if $T^M > \widetilde{T}^M$, and AS-decreasing if $T^M < \widetilde{T}^M$. We summarize these insights in the following proposition:

Proposition 15. *For a merger among the firms in $\mathcal{M}$, there exists a unique $\widetilde{T}^M < \hat{T}^M$ such that the merger is AS-neutral if the post-merger type satisfies $T^M = \widetilde{T}^M$, AS-decreasing if $T^M < \widetilde{T}^M$, and AS-increasing if $T^M > \widetilde{T}^M$.*

Note that there is no counterpart to Proposition 15 in Farrell and Shapiro (1990)'s classical analysis. The reason is that, in the homogeneous-goods Cournot model, equilibrium aggregate surplus is not a monotonic function of firms' marginal costs (Lahiri and Ono, 1988; Zhao, 2001). By contrast, we are able to leverage the monotonicity of aggregate surplus in firms' types to obtain Proposition 15.

The proposition states that $\widetilde{T}^M < \hat{T}^M$, which follows immediately from the fact that a CS-neutral merger is AS-increasing. Whether or not $\tilde{T}^M > \sum_{f \in \mathcal{I}} T^f$, i.e., whether or not an AS-neutral merger must involve synergies, is unclear. On the one hand, a merger that does not involve synergies lowers the equilibrium aggregator level. On the other hand, it reallocates market shares toward the outsiders, which can raise social welfare if those firms are initially producing too little.

An example where a merger involving no synergies is AS-increasing can easily be constructed in the case of NMNL demand without an outside option ($H^0 = 0$). Let there be three firms, 1, 2, and 3, with pre-merger types $T^1 = 1$ and $T^2 = T^3 = 1/2$. In the aggregate-surplus-maximizing pre-merger allocation, which can be obtained by setting all markups equal to zero, firm 1 commands a market share of 1/2, whereas firms 2 and 3 each receive a market share of 1/4. The equilibrium allocation is efficient if and only if it replicates that allocation, which arises if and only if all firms charge the same markup. As firm 1's type is higher than its rivals', that firm sets an equilibrium markup that strictly exceeds that of its rivals, resulting in an inefficient equilibrium allocation. Consider now a merger M between firms 2 and 3, and, assuming no synergies, let $T^M = 1$. As firm 1 and the merged firm have the same type, they charge the same equilibrium markups, implying that the post-merger equilibrium allocation is efficient. The merger is therefore AS-increasing.[24]

5.2 External Effects

We now extend Farrell and Shapiro (1990)'s analysis of the external effects of a merger, defined as the sum of its impact on consumer surplus and outsiders' profits. To the extent that a merger is proposed by the merger partners only if it is in their joint interest to do so, a positive external effect is a sufficient ("safe harbor") condition for the merger to raise social welfare. The idea behind focusing on the external effect is that the profitability of a merger depends on the magnitude of internal cost savings, and that these are hard to assess for an antitrust authority. As we shall see below, the external-effects approach also delivers benefits in terms of tractability, by allowing us to decompose a merger into infinitesimal components.

Consider a merger M between the firms in $\mathcal{M}$, and let $\mathcal{O}$ be the set of outsiders. Let H^* and $\overline{H}^*$ denote the pre- and post-merger equilibrium values of the aggregator, respectively. The external effect of the merger is defined as

$$\mathcal{E}^M = \log \overline{H}^* - \log H^* + \sum_{f \in \mathcal{O}} \left(m\left(\frac{T^f}{\overline{H}^*}\right) - m\left(\frac{T^f}{H^*}\right)\right).$$

Defining

$$\eta(H) \equiv -1 + \sum_{f \in \mathcal{O}} \frac{T^f}{H} m'\left(\frac{T^f}{H}\right),$$

we can rewrite $\mathcal{E}^M$ as

$$\mathcal{E}^M = -\int_{H^*}^{\overline{H}^*} \frac{\eta(H)}{H} dH.$$

Hence, as in Farrell and Shapiro (1990), the merger can be thought of as a sequence of infinitesimal mergers dH, where, along the sequence, the value of the aggregator changes progressively from H^* to $\overline{H}^*$. The sign of the external effect of an infinitesimal CS-decreasing (resp. CS-increasing) merger is thus given by $\eta(H)$ (resp. $-\eta(H)$). In the following, we focus on CS-decreasing mergers to fix ideas.

An infinitesimal CS-decreasing merger $dH < 0$ reduces consumer surplus by dH/H, which corresponds to the first term in the definition of η. It also raises the profit of every outsider $f \in \mathcal{O}$ by dH/H times $(T^f/H)m'(T^f/H)$. In Appendix F.1, we show that $\eta(H)$ can be rewritten as

$$\eta(H) = -1 + \sum_{f \in \mathcal{O}} \frac{\alpha s^f(1-s^f)}{(1-\alpha s^f)(1-s^f+\alpha(s^f)^2)}, \tag{13}$$

where, for every f in $\mathcal{O}$, $s^f = S(T^f/H)$ is firm f's market share when the value of the aggregator is H. The results stated in this section are derived by exploiting the properties

[24] By the same token, with NMNL demand, no outside option, and three firms 1, 2, and 3 such that $T^1 = T^2 = T^3$, a merger between firms 2 and 3 is AS-decreasing, if it does not give rise to synergies.

of the right-hand side of equation (13).

We prove the following proposition:

Proposition 16. *Let* $\bar{\alpha} = \frac{3}{2}\left(\sqrt{57} - 7\right) \simeq 0.82$*. If* $\alpha \leq \bar{\alpha}$*, then any CS-decreasing merger has a negative external effect. If instead* $\alpha > \bar{\alpha}$*, then there exist CS-decreasing mergers that have a positive external effect, and CS-decreasing mergers that have a negative external effect.*

Proof. See Appendix F.2. □

In the non-nested CES case, the condition $\alpha \leq \bar{\alpha}$ translates into $\sigma \leq \bar{\sigma} \simeq 5.7$. More generally, in the case of NCES demand, it translates into a low value of β and/or σ. The intuition for the result is the following. After a CS-decreasing merger, the aggregate market share of the insiders falls, meaning that consumers substitute away from the insiders' products into the outsiders' products. If the insiders' and outsiders' products are poor substitutes, which is the case if σ is small and/or β is small, then such substitution gives rise to a large fall in consumer surplus, which the increase in the outsiders' profits cannot offset, implying a negative external effect.

In the following, we assume that $\alpha > \bar{\alpha}$, and derive conditions under which a CS-decreasing merger is more likely to have a positive external effect. We first argue that one such condition is that the outsiders have high market shares.

To do so, we define a partial order relation over the set of pre-merger outsider industry structures. A pre-merger outsider industry structure is a vector $(s^f)_{f\in\mathcal{O}}$ of arbitrary length, where $\mathcal{O}$ is a finite set, $s^f \in (0,1)$ for every $f \in \mathcal{O}$, and $\sum_{f\in\mathcal{O}} s^f < 1$. Let $s = (s^f)_{f\in\mathcal{O}}$ and $s' = (s'^f)_{f\in\mathcal{O}'}$ be two pre-merger industry structures. We say that the outsiders have higher market shares under s than under s', and write $s \geq_1 s'$, if there exists an injection $\iota : \mathcal{O}' \longrightarrow \mathcal{O}$ such that $s^{\iota(f)} \geq s'^f$ for every $f \in \mathcal{O}'$.[25]

We obtain the following proposition:

Proposition 17. *Let* $\alpha > \bar{\alpha}$*, and consider two infinitesimal CS-decreasing mergers,* M *and* M'*, with pre-merger outsider industry structures* $s = (s^f)_{f\in\mathcal{O}}$ *and* $s' = (s'^f)_{f\in\mathcal{O}'}$*. Suppose* $s \geq_1 s'$ *and* $s^f \leq s^* \simeq 0.68$ *for every* $f \in \mathcal{O}$*. If merger* M' *has a positive external effect, then so does merger* M*.*

Proof. See Appendix F.3. □

To understand the intuition, note that the merger-induced decrease in H has two effects on an outsider's profit. First, holding fixed outsiders' markups, it increases the profit of

[25] $\geq_1$ is clearly reflexive and transitive. It is antisymmetric under the equivalence relation

$$s = (s^f)_{f\in\mathcal{O}} \sim s' = (s'^f)_{f\in\mathcal{O}} \iff s^{b(f)} = s'^f \ \forall f \in \mathcal{O}', \text{ for some bijection } b : \mathcal{O}' \longrightarrow \mathcal{O}.$$

Hence, $\geq_1$ is a partial order relation over the set of equivalence classes of the equivalence relation defined above.

each outsider f by $\Pi^f \times |dH/H|$.[26] Hence, the "direct" effect on outsiders' joint profit is proportional to their joint profit. Second, outsiders respond by increasing their markups.

Proposition 17 is driven by the first, direct effect. If the outsiders have higher market shares before the merger, then they charge higher markups and thus make more profits. The direct effect is therefore stronger when the outsiders have higher market shares.

The reason why this intuition may fail if some of the outsiders are too large (i.e., if $s^f > s^*$ for some firm f) is the result of the second, indirect effect. Holding H fixed, the induced increase in an outsider's markup decreases its profit. This holds since oligopolistic markups are always above those of monopolistically competitive firms that perceive H as fixed, so any further increase must reduce profit for a fixed H. This second effect becomes quantitatively important when firms become too large.

Having studied the impact of the *level* of the pre-merger outsiders' market shares, we now turn our attention to the impact of their *dispersion*. We formalize the notion of dispersion by defining another partial order over the set of outsider industry structures.

To every outsider industry structure s, we associate a discrete probability measure $P_s(\cdot)$, defined as follows:

$$P_s(x) = \frac{1}{|\mathcal{O}|} \left| \left\{ f \in \mathcal{O} : \quad s^f = x \right\} \right|, \quad \forall x \in \mathbb{R}.$$

We say that outsiders' market shares are more concentrated under outsider industry structure s than under s', and write $s \geq_2 s'$, if s and s' have the same length and the same mean, and $P_{s'}$ second-order stochastically dominates P_s. Note that s and s' having the same length and the same mean implies that the aggregate market share of the outsiders is the same under s and s'.

We obtain the following proposition:

Proposition 18. *Let $\alpha > \bar{\alpha}$, and consider two infinitesimal CS-decreasing mergers, M and M', with pre-merger outsider industry structures $s = (s^f)_{f \in \mathcal{O}}$ and $s' = (s'^f)_{f \in \mathcal{O}'}$. Suppose $s \geq_2 s'$, $s^f \leq \hat{s} \simeq 0.29$ for every $f \in \mathcal{O}$, and $s'^f \leq \tilde{s}$ for every $f \in \mathcal{O}'$. If merger M' has a positive external effect, then so does merger M.*

Proof. See Appendix F.4. □

The main driving force behind Proposition 18 is the same as that behind Proposition 17. If outsiders' market shares are more concentrated under s than under s', then their aggregate profits are higher under s than under s', as argued in Section 3.1. Hence, the direct effect of the decrease in H, mentioned in the discussion after Proposition 17, is larger when outsiders' market shares are more concentrated. This intuition may fail when some of the market shares are too high, due to the indirect effect.

[26]This holds, as

$$\Pi^f = \alpha \mu^f s^f = \begin{cases} \alpha \frac{T^f}{H} \mu^f \left(1 - (1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}} & \text{under NCES,} \\ \frac{T^f}{H} \mu^f \, \mathrm{e}^{-\mu^f} & \text{under NMNL.} \end{cases}$$

Proposition 18 suggests that relying on the level of the pre-merger Herfindahl index to evaluate the social desirability of a merger can be misguided. To see this, consider two industries, and suppose that the vector of insiders' market shares is the same in both industries. Suppose also that outsiders' market shares are more concentrated in the first industry than in the second. Then, the first industry's Herfindahl index is higher than the second's. However, the merger in the first industry is more likely to have a positive external effect than the one in the second industry.

We close this section by discussing the external effect of a non-infinitesimal CS-decreasing merger. We already know from Proposition 16 that such a merger always has a negative external effect if $\alpha \leq \bar{\alpha}$. Suppose now that $\alpha > \bar{\alpha}$. By continuity, the comparative statics derived in Propositions 17 and 18 continue to obtain as long as the mergers under consideration do not have too much of an impact on the equilibrium aggregator level.

We also note that, regardless of the magnitude of the merger-induced decrease in H, a sufficient condition for the merger to have a positive external effect is that $\eta(H^*) > 0$ (i.e., at the pre-merger aggregator level, an infinitesimal CS-decreasing merger has a positive external effect). The reason is the following. The external effect of the merger is the integral of the external effects of the infinitesimal mergers along the path from H^* to $\overline{H}^* < H^*$. As the merger is CS-decreasing by assumption, outsiders' market shares increase along that sequence. Hence, if $\eta(H^*) > 0$, then, by Proposition 17, $\eta(H)$ remains positive along the sequence (provided no outsider reaches a market share larger than $\hat{s}$), and so the external effect of the merger is positive. Note that checking whether $\eta(H^*) > 0$ involves using only the outsiders' *pre*-merger market shares (see equation (13)).

6 Conclusion

We provide a merger analysis in a multiproduct-firms oligopoly model with NCES or NMNL demands. That model goes a long way towards satisfying a number of desiderata: The underlying demand system has discrete/continuous choice micro-foundations and allows for substitution patterns that go beyond those implied by the IIA property. The model allows for arbitrary product heterogeneity in terms of marginal costs and qualities, and allows firms to differ in their product portfolios. The demand system gives rise to an aggregative pricing game; the equilibrium is unique and has intuitive comparative statics. Moreover, the type aggregation property permits rich forms of merger-specific synergies through marginal cost reductions, quality improvements, or new products. Finally, consumer surplus and aggregate surplus can be expressed as functions of firm-level equilibrium market shares.

We derive three sets of results. First, we relate the Herfindahl index to market performance measures using approximation techniques. The Herfindahl index provides an adequate measure of the welfare distortions introduced by market power, relative to the monopolistic competition benchmark. Moreover, the naively-computed, merger-induced variation in the Herfindahl index approximates the market power effect of that merger.

Second, we study the consumer surplus effects of mergers, in both static and dynamic settings. For a merger to be CS-increasing requires that the merger generates efficiencies. These efficiencies need to be larger when the industry is less competitive before the merger, or when the merger partners are larger—thus providing additional justification for the use of the naively-computed change in the Herfindahl index. In a dynamic context, in which merger opportunities arise stochastically over time and merger proposals (and approvals) are endogenous, a completely myopic consumer-surplus-based merger approval policy is dynamically optimal.

Third, we study the aggregate surplus and external effects of mergers. For a merger to be AS-increasing requires fewer efficiencies than for it to be CS-increasing and may, in fact, not require any efficiencies at all. The external effect of a CS-decreasing merger is always negative when products are poor substitutes. When instead products are good substitutes, the external effect is positive if the outsiders' pre-merger market shares are sufficiently large or sufficiently concentrated.

We believe that our analysis has implications well beyond industrial organization and antitrust. The model of monopolistic competition with CES preferences is a major building block in the macroeconomics and international trade literatures. Yet, many industries are highly concentrated, with firms wielding market power. Such market power within an industry introduces several forms of misallocation, as it shifts output towards the outside good (representing other industries), the within-industry outside option, and smaller, less efficient firms that charge lower markups.[27] We show that the welfare loss associated with those misallocations is well approximated by the industry-level Herfindahl index—a measure that is often readily available in industry-level data.

Appendix

A Proof of Proposition 1: Necessity and Sufficiency of First-Order Conditions

Proof. Fix a profile of prices p^{-f} for firm f's rivals, and let $\mathcal{N}^f = \bigcup_{l \in f} l$. Define

$$H^{0\prime} = H^0 + \sum_{g \in \mathcal{F} \setminus \{f\}} \sum_{l \in g} \left(\sum_{j \in l} h_j(p_j^{-f}) \right)^{\beta} > 0,$$

and

$$G(p) = \beta \frac{\sum_{l \in f} \left(\sum_{i \in l} h_i(p_i) \right)^{\beta - 1} \sum_{j \in l} (p_j - c_j)(-h_j'(p_j))}{H^{0\prime} + \sum_{l \in f} \left(\sum_{i \in l} h_i(p_i) \right)^{\beta}},$$

[27]There is a large recent literature that attempts to measure empirically the extent of misallocation arising from market imperfections. See Restuccia and Rogerson (2017) for a survey.

for every profile of prices $p = (p_j)_{j \in \mathcal{N}^f}$. Note that $G(p)$ is the profit firm f receives when it sets the price vector p and its rivals rivals set the price vector p^{-f}. Our goal is to show that the maximization problem

$$\max_{p \in \mathbb{R}_{++}^{\mathcal{N}^f}} G(p)$$

has a unique solution, and that the price vector p solves that maximization problem if and only if it satisfies the first-order conditions.

The proof follows a similar development as the proof of Lemmas B–H in the Appendix of Nocke and Schutz (2018). It proceeds as follows. We first show that pricing some (or all) of the products below cost is strictly suboptimal (Step 1). We then extend the domain of G to price vectors that have infinite components (Step 2). Combining Steps 1 and 2 allows us to show that the profit maximization problem has a solution (Step 3). We then show that there exists a unique price vector satisfying the first-order conditions of profit maximization (Step 4). Combining Steps 1–4, we can conclude that the profit maximization problem has a unique solution, and that first-order conditions are necessary and sufficient for optimality.

Step 1: No product is priced below cost. We first argue that firm f's products are substitutes. Let $n, n' \in f$ and $(i, i') \in n \times n'$ such that $i \neq i'$. If $n \neq n'$, then

$$\frac{\partial D_i}{\partial p_{i'}} = \beta^2 \frac{h_i' H_n^{\beta-1} h_{i'}' H_{n'}^{\beta-1}}{H^2} > 0.$$

If instead $n = n'$, then

$$\frac{\partial D_i}{\partial p_{i'}} = \frac{\beta h_i' h_{i'}'}{H} \left((1-\beta) H_n^{\beta-2} + \beta \frac{H_n^{2(\beta-1)}}{H} \right) > 0.$$

Let p be a price vector for firm f such that $p_j < c_j$ for some product $j \in \mathcal{N}^f$. Define a new price vector $\tilde{p}$ for firm f such that for every $i \in \mathcal{N}^f$, $\tilde{p}_i = \max(c_i, p_i)$. When firm f deviates from p to $\tilde{p}$, it stops making losses on those products that were originally priced below cost, and, by substitutability, it makes more profits on those products that were priced above cost. Therefore, price vector p is not optimal for firm f. When looking for a solution to firm f's profit maximization problem, we can therefore confine our attention to price vectors in $\prod_{j \in \mathcal{N}^f} [c_j, \infty)$.

Step 2: Defining G at infinite prices. Let $\hat{p} \in \prod_{j \in \mathcal{N}^f} [c_j, \infty]$. Suppose $\hat{p}$ has at least one infinite component, and let $(p^k)_{k \geq 0}$ be a sequence over $\prod_{j \in \mathcal{N}^f} [c_j, \infty)$ such that $p^k \underset{k \to \infty}{\longrightarrow} \hat{p}$. Let

$$f' = \{l \in f : \ \exists i \in l \text{ s.t. } \hat{p}_i < \infty\}$$

and

$$\mathcal{N}^{f'} = \{j \in \mathcal{N}^f : \ \hat{p}_j < \infty\}.$$

Clearly, as k tends to infinity, the denominator of $G(p^k)$ tends to[28]

$$H^{0\prime} + \sum_{l \in f'} \left(\sum_{j \in l \cap \mathcal{N}^{f'}} h_j(\hat{p}_j) \right)^{\beta}.$$

Next, let $i \in \mathcal{N}^f \setminus \mathcal{N}^{f'}$. Let $l \in f$ be the nest that contains product i. Note that, for every $k \geq 0$,

$$(p_i^k - c_i)(-h_i'(p_i^k)) \left(\sum_{j \in l} h_j(p_j^k) \right)^{\beta - 1} \leq (p_i^k - c_i)(-h_i'(p_i^k)) \left(h_i(p_i^k) \right)^{\beta - 1}.$$

Under NCES demand,

$$(p_i^k - c_i)(-h_i'(p_i^k)) \left(h_i(p_i^k) \right)^{\beta - 1} \leq (\sigma - 1) a_i (p_i^k)^{\beta(1-\sigma)} \underset{k \to \infty}{\longrightarrow} 0.$$

Under NMNL demand,

$$(p_i^k - c_i)(-h_i'(p_i^k)) \left(h_i(p_i^k) \right)^{\beta - 1} \leq \frac{1}{\lambda} p_i^k \exp \left(\frac{\beta}{\lambda} \left(a_i - p_i^k \right) \right) \underset{k \to \infty}{\longrightarrow} 0.$$

It follows that

$$G(p^k) \underset{k \to \infty}{\longrightarrow} \beta \frac{\sum_{l \in f'} \left(\sum_{i \in l \cap \mathcal{N}^{f'}} h_i(\hat{p}_i) \right)^{\beta - 1} \sum_{j \in l \cap \mathcal{N}^{f'}} (\hat{p}_j - c_j)(-h_j'(\hat{p}_j))}{H^{0\prime} + \sum_{l \in f'} \left(\sum_{i \in l \cap \mathcal{N}^{f'}} h_i(\hat{p}_i) \right)^{\beta}} \equiv G(\hat{p}).$$

We have thus extended the domain of G to $\prod_{j \in \mathcal{N}^f} [c_j, \infty]$. Note that, at $\hat{p}$, G has smooth partial derivatives with respect to $(p_i)_{i \in \mathcal{N}^{f'}}$.

Step 3: The profit maximization problem has a solution. By continuity of G (as established in the previous step) and compactness of $\prod_{j \in \mathcal{N}^f} [c_j, \infty]$, the maximization problem

$$\max_{p \in \prod_{j \in \mathcal{N}^f} [c_j, \infty]} G(p)$$

has a solution $\hat{p}$. Clearly, $\hat{p}$ has at least one finite component, for otherwise $G(\hat{p})$ would be equal to zero, as shown above.

Assume for a contradiction that $\hat{p}$ has some infinite components, and define f' and $\mathcal{N}^{f'}$ as in the previous step. Since $\hat{p}$ maximizes G, it must be the case that $\left. \frac{\partial G}{\partial p_i} \right|_{\hat{p}} = 0$ for every $i \in \mathcal{N}^{f'}$. Manipulating the first order conditions as we did in Section 2.3, we obtain the

[28] By convention, the sum of an empty collection of reals is zero.

existence of a $\tilde{\mu}^f$ such that, for every $i \in \mathcal{N}^{f\prime}$,

$$\frac{\hat{p}_i - c_i}{\hat{p}_i}\frac{\hat{p}_i h_i''(\hat{p}_i)}{-h_i'(\hat{p}_i)} = \tilde{\mu}^f.$$

Under NCES, $(\hat{p}_i h_i''(\hat{p}_i))/(-h_i'(\hat{p}_i)) = \sigma$, so that $\tilde{\mu}^f < \sigma$. Moreover, under both NCES and NMNL demand, $\tilde{\mu}^f$ satisfies

$$\tilde{\mu}^f\left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \tilde{\alpha}\beta\tilde{\mu}^f \frac{\sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}{H^{0\prime} + \sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}, \tag{14}$$

so that $\tilde{\mu}^f > 1$.

Fix a product $i \in \mathcal{N}^f \setminus \mathcal{N}^{f\prime}$, and let $n \in f$ be the nest that contains product i. For every $x \geq c_i$, let $\tilde{G}(x)$ be the value of G when product i is priced at x and all the other products are priced according to $\hat{p}$. We showed in the previous step that $\tilde{G}(x) \underset{x\to\infty}{\longrightarrow} G(\hat{p})$. Note that, for every $x \in (c_i, \infty)$,

$$\begin{aligned}\tilde{G}'(x) = D_i \times \Bigg(& 1 - (x-c_i)\frac{h_i''(x)}{-h_i'(x)} + (1-\beta)\frac{(x-c_i)(-h_i'(x)) + \tilde{\alpha}\tilde{\mu}^f\sum_{j\in(n\cap\mathcal{N}^{f\prime})\setminus\{i\}} h_j(\hat{p}_j)}{h_i(x) + \sum_{j\in(n\cap\mathcal{N}^{f\prime})\setminus\{i\}} h_j(\hat{p}_j)} \\ &+\beta\frac{\left(h_i(x) + \sum_{j\in(n\cap\mathcal{N}^{f\prime})\setminus\{i\}} h_j(\hat{p}_j)\right)^{\beta-1}\left((x-c_i)(-h_i'(x)) + \tilde{\alpha}\tilde{\mu}^f\sum_{j\in(n\cap\mathcal{N}^{f\prime})\setminus\{i\}} h_j(\hat{p}_j)\right)}{H^{0\prime} + \left(h_i(x) + \sum_{j\in(n\cap\mathcal{N}^{f\prime})\setminus\{i\}} h_j(\hat{p}_j)\right)^\beta + \sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta} \\ &+\beta\tilde{\alpha}\tilde{\mu}^f\frac{\sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}{H^{0\prime} + \left(h_i(x) + \sum_{j\in(n\cap\mathcal{N}^{f\prime})\setminus\{i\}} h_j(\hat{p}_j)\right)^\beta + \sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}\Bigg),\end{aligned} \tag{15}$$

where we have used the simplification derived in equation (3).

We argue that $\tilde{G}'(x) < 0$ for x sufficiently high. We distinguish two cases. Assume first that $n \notin f'$, i.e., $\hat{p}_j = \infty$ for every $j \in n$. Then, $\tilde{G}'(x)$ simplifies to

$$\begin{aligned}\tilde{G}'(x) = D_i \Bigg(& 1 - (x-c_i)\frac{h_i''(x)}{-h_i'(x)} + (1-\beta)(x-c_i)\frac{-h_i'(x)}{h_i(x)} \\ &+\beta\frac{h_i(x)^{\beta-1}(x-c_i)(-h_i'(x)) + \tilde{\alpha}\tilde{\mu}^f\sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}{H^{0\prime} + h_i(x)^\beta + \sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}\Bigg).\end{aligned} \tag{16}$$

Under NCES demand, $(x-c_i)\frac{h_i''(x)}{-h_i'(x)}$ and $(x-c_i)\frac{-h_i'(x)}{h_i(x)}$ tend to σ and $\sigma - 1$, respectively, as

x goes to infinity, whereas

$$h_i(x)^{\beta-1}(x-c_i)(-h_i'(x)) = (\sigma-1)a_i x^{\beta(1-\sigma)}\frac{x-c_i}{x}$$

tends to zero. It follows that the term in parenthesis in equation (16) tends to

$$1-\sigma+(1-\beta)(\sigma-1)+\beta\tilde{\alpha}\tilde{\mu}^f\frac{\sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}}h_j(\hat{p}_j)\right)^\beta}{H^{0\prime}+\sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}}h_j(\hat{p}_j)\right)^\beta},$$

which, using equation (14), simplifies to

$$\begin{aligned}-\beta(\sigma-1)+\tilde{\mu}^f(1-\tilde{\alpha}(1-\beta))-1 &< -\beta(\sigma-1)+\sigma(1-\tilde{\alpha}(1-\beta))-1,\\ &=\frac{1}{1-\tilde{\alpha}}\Big(-\beta\tilde{\alpha}+(1-\tilde{\alpha}(1-\beta))-(1-\tilde{\alpha})\Big),\\ &=0.\end{aligned}$$

Hence, $\tilde{G}'(x)<0$ for high enough x.

Under NMNL demand,

$$h_i(x)^{\beta-1}(x-c_i)(-h_i'(x)) = \frac{x-c_i}{\lambda}\exp\left(\frac{\beta}{\lambda}(a_i-x)\right)\underset{x\to\infty}{\longrightarrow}0,$$

and

$$1-(x-c_i)\frac{h_i''(x)}{-h_i'(x)}+(1-\beta)(x-c_i)\frac{-h_i'(x)}{h_i(x)} = 1-\frac{\beta}{\lambda}(x-c_i)\underset{x\to\infty}{\longrightarrow}-\infty.$$

Hence, we also have that $\tilde{G}'(x)<0$ for high enough x.

Next, assume instead that $n\in f'$. Under NCES demand, the term in parenthesis in equation (15) tends to

$$1-\sigma+(1-\beta)\tilde{\alpha}\tilde{\mu}^f+\beta\tilde{\alpha}\tilde{\mu}^f\frac{\sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f'}}h_j(\hat{p}_j)\right)^\beta}{H^{0\prime}+\sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f'}}h_j(\hat{p}_j)\right)^\beta},$$

which, using equation (14), simplifies to

$$1-\sigma+(1-\beta)\tilde{\alpha}\tilde{\mu}^f+\tilde{\mu}^f(1-\tilde{\alpha}(1-\beta))-1=-\sigma+\tilde{\mu}^f<0,$$

implying that $\tilde{G}'(x)<0$ for x high enough.

Under NMNL demand, the term in parenthesis in equation (15) tends again to $-\infty$, so that $\tilde{G}'(x)<0$ for x high enough.

It follows that $\tilde{G}$ is strictly decreasing over some interval (x^0,∞). Therefore, $\tilde{G}(x^0)>$

$\lim_{x\to\infty} \tilde{G}(x) = G(\hat{p})$, and $\hat{p}$ does not maximize G, a contradiction. Hence, $\hat{p} \in \prod_{j\in\mathcal{N}^f}[c_j,\infty)$ maximizes G, which concludes Step 3.

Step 4: There exists a unique price vector satisfying the first-order optimality conditions. The analysis in Section 2.3 implies that the price vector $\hat{p} \in \prod_{j\in\mathcal{N}^f}[c_j,\infty)$ satisfies the first-order conditions if and only if there exists a $\tilde{\mu}^f$ that is such that for every $i \in \mathcal{N}^f$, $\hat{p}_i = r_i(\tilde{\mu}^f)$, where

$$r_i(x) \equiv \begin{cases} \frac{\sigma}{\sigma - x} c_i & \text{in the case of NCES,} \\ \lambda x + c_i & \text{in the case of NMNL,} \end{cases}$$

and that satisfies

$$\tilde{\mu}^f \left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \tilde{\alpha}\beta\tilde{\mu}^f \frac{\sum_{l\in f}\left(\sum_{j\in f} h_j(r_j(\tilde{\mu}^f))\right)^\beta}{H^{0\prime} + \sum_{l\in f}\left(\sum_{j\in f} h_j(r_j(\tilde{\mu}^f))\right)^\beta},$$

or, equivalently,

$$\tilde{\mu}^f(1-\tilde{\alpha}) = 1 - \tilde{\alpha}\beta\tilde{\mu}^f \frac{H^{0\prime}}{H^{0\prime} + \sum_{l\in f}\left(\sum_{j\in f} h_j(r_j(\tilde{\mu}^f))\right)^\beta}. \tag{17}$$

As the left-hand side of equation (17) is strictly increasing, whereas the right-hand side is strictly decreasing, that equation has at most one solution. By Step 3, that equation has a solution. Hence, there exists a unique price vector satisfying the first-order conditions. □

B Technical Results on Fitting-In Functions

The following results are proved in Nocke and Schutz (2018):

Lemma 1. *The following holds for every $\alpha \in (0,1]$:*

(a) For every $x > 0$,

$$S'(x) = \frac{1}{x}\frac{S(x)(1-S(x))(1-\alpha S(x))}{1 - S(x) + \alpha S(x)^2}. \tag{18}$$

(b) The elasticity of S, $\varepsilon(x) = xS'(x)/S(x)$, is strictly decreasing in x.

(c) S is strictly concave.

Proof. See Section XIII.3 in the Online Appendix to Nocke and Schutz (2018). □

We also require the following lemma:

Lemma 2. *The continuous extension of S to $\mathbb{R}_+$ is $\mathcal{C}^3$. Moreover, $S(0) = 0$,*

$$S'(0) = \begin{cases} \alpha^{\frac{\alpha}{1-\alpha}} & \text{under NCES demand,} \\ e^{-1} & \text{under NMNL demand,} \end{cases}$$

$S''(0) = -2\alpha S'(0)^2$, and $S'''(0) = -3\alpha(1-2\alpha)S'(0)^3$.

The inverse function $\Theta \equiv S^{-1}$ is $\mathcal{C}^3$ on $[0,1)$. Moreover, $\Theta(0) = 0$, $\Theta'(0) = 1/S'(0)$, $\Theta''(0) = 2\alpha/S'(0)$, and $\Theta'''(0) = 3\alpha(1+2\alpha)/S'(0)$.

Proof. We start by computing $\lim_{x\downarrow 0} \frac{S(x)}{x}$. In the NMNL case,

$$\frac{S(x)}{x} = \mathrm{e}^{-m(x)} = \exp\left(\frac{-1}{1-S(x)}\right) \underset{x\downarrow 0}{\longrightarrow} \mathrm{e}^{-1}.$$

In the NCES case,

$$\frac{S(x)}{x} = (1-(1-\alpha)m(x))^{\frac{\alpha}{1-\alpha}} = \left(1 - \frac{1-\alpha}{1-\alpha S(x)}\right)^{\frac{\alpha}{1-\alpha}} \underset{x\downarrow 0}{\longrightarrow} \alpha^{\frac{\alpha}{1-\alpha}}.$$

Differentiating equation (18), we obtain

$$S''(x) = -\left(\frac{S(x)}{x}\right)^2 \frac{\alpha(2-S(x))(1-S(x))(1-\alpha S(x))}{(1-S(x)+\alpha S(x)^2)^3}. \tag{19}$$

Differentiating once more gives

$$S'''(x) = -\left(\frac{S(x)}{x}\right)^3 \frac{\alpha(1-S(x))(1-\alpha S(x))}{(1-S(x)+\alpha S(x)^2)^5}\Big(3(1-2\alpha) - 4(1+\alpha)S(x) + (1+13\alpha+6\alpha^2)S(x)^2 - 2\alpha(2+5\alpha)S(x)^3 + 3\alpha^2 S(x)^4\Big). \tag{20}$$

Taking limits in equations (19) and (20) gives us the values of $S''(0)$ and $S'''(0)$.

Since S is $\mathcal{C}^3$ with strictly positive derivative on $\mathbb{R}_+$, that function establishes a $\mathcal{C}^3$-diffeomorphism from $\mathbb{R}_+$ to

$$\Big[S(0), \lim_{x\to\infty} S(x)\Big) = [0,1).$$

It follows that Θ is $\mathcal{C}^3$. Moreover,

$$\Theta'(s) = \frac{1}{S' \circ S^{-1}(s)},$$

$$\Theta''(s) = -\frac{S'' \circ S^{-1}(s)}{(S' \circ S^{-1}(s))^3},$$

$$\Theta'''(s) = -\frac{\frac{S'''\circ S^{-1}(s)}{S'\circ S^{-1}(s)}\left(S'\circ S^{-1}(s)\right)^3 - S''\circ S^{-1}(s) \times 3\left(S'\circ S^{-1}(s)\right)^2 \frac{S''\circ S^{-1}(s)}{S'\circ S^{-1}(s)}}{\left(S'\circ S^{-1}(s)\right)^6},$$

$$= \frac{1}{S' \circ S^{-1}(s)} \left(-\frac{S''' \circ S^{-1}(s)}{(S' \circ S^{-1}(s))^3} + 3 \left(\frac{S'' \circ S^{-1}(s)}{(S' \circ S^{-1}(s))^2} \right)^2 \right).$$

Hence,

$$\begin{aligned}
\Theta'(0) &= \frac{1}{S'(0)}, \\
\Theta''(0) &= -\frac{1}{S'(0)} \frac{S''(0)}{S'(0)^2} = \frac{2\alpha}{S'(0)}, \\
\Theta'''(0) &= \frac{1}{S'(0)} \left(-\frac{S'''(0)}{S'(0)^3} + 3 \left(\frac{S''(0)}{S'(0)^2} \right)^2 \right), \\
&= \frac{1}{S'(0)} \left(3\alpha(1-2\alpha) + 3(2\alpha)^2 \right), \\
&= \frac{3\alpha(1+2\alpha)}{S'(0)}.
\end{aligned}$$

□

C Approximation Results Around Small Market Shares

C.1 Proof of Proposition 3

We prove a series of lemmas that jointly imply Proposition 3 as well as the third-order approximation stated in footnote 18.

We first approximate consumer surplus under oligopoly:

Lemma 3. *$H^*(s) = \frac{H^0}{1-\sum_{g \in \mathcal{F}} s^g}$. Moreover, in the neighborhood of $s = 0$,*

$$CS(s) = \log H^0 + \sum_{f \in \mathcal{F}} s^f + \frac{1}{2} \left(\sum_{f \in \mathcal{F}} s^f \right)^2 + \frac{1}{3} \left(\sum_{f \in \mathcal{F}} s^f \right)^3 + o\left(\|s\|^3 \right).$$

Proof. The first part of the lemma follows immediately from the equilibrium condition

$$\frac{H^0}{H^*} + \sum_{g \in \mathcal{F}} s^g = 1.$$

The second part of the lemma follows from the fact that, in the neighborhood of $x = 0$,

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3).$$

□

Next, we compute the first, second, and third (cross-)partial derivatives of the type vector $T(s)$:

Lemma 4. *For every* $(f, f') \in \mathcal{F}^2$,

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{s=0} = \begin{cases} \frac{H^0}{S'(0)} & \text{if } f = f', \\ 0 & \text{otherwise.} \end{cases}$$

For every $(f, f', f'') \in \mathcal{F}^3$,

$$\left.\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = \begin{cases} \frac{H^0}{S'(0)} 2(1+\alpha) & \text{if } f = f' = f'', \\ 0 & \text{if } f' \neq f \text{ and } f'' \neq f, \\ \frac{H^0}{S'(0)} & \text{otherwise.} \end{cases}$$

Finally, for every $(f, f', f'', f''') \in \mathcal{F}^4$,

$$\left.\frac{\partial^3 T^f}{\partial s^{f'} \partial s^{f''} \partial s^{f'''}}\right|_{s=0} = \frac{H^0}{S'(0)} \begin{cases} 6 + 9\alpha + 6\alpha^2 & \text{if } f = f' = f'' = f''', \\ 2\alpha + 4 & \text{if } (f', f'', f''') \in \mathcal{P}^2(f), \\ 2 & \text{if } (f', f'', f''') \in \mathcal{P}^1(f), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{P}^1(f) = \left\{(f^1, f^2, f^3) \in \mathcal{F}^3 : \ f = f^i \neq f^j, f^k, \text{ for some permutation } (i, j, k) \text{ of } (1, 2, 3)\right\},$$

and

$$\mathcal{P}^2(f) = \left\{(f^1, f^2, f^3) \in \mathcal{F}^3 : \ f = f^i = f^j \neq f^k, \text{ for some permutation } (i, j, k) \text{ of } (1, 2, 3)\right\}.$$

Proof. Let $f \in \mathcal{F}$. Since $s^f = S\left(\frac{T^f}{H^*(s)}\right)$, we have that

$$T^f = H^* S^{-1}(s^f) = H^0 \frac{\Theta(s^f)}{1 - \sum_{g \in \mathcal{F}} s^g} \equiv H^0 \Theta(s^f) \Psi(s),$$

where we have used the inverse function Θ that was defined in Lemma 2.

Note that, for every $(f, f', f'') \in \mathcal{F}^3$,

$$\begin{aligned} \Psi(0) = \left.\frac{\partial \Psi}{\partial s^f}\right|_{s=0} &= 1, \\ \left.\frac{\partial^2 \Psi}{\partial s^f \partial s^{f'}}\right|_{s=0} &= 2, \\ \left.\frac{\partial^3 \Psi}{\partial s^f \partial s^{f'} \partial s^{f''}}\right|_{s=0} &= 6. \end{aligned}$$

Therefore, for every $(f, f') \in \mathcal{F}^2$,

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{s=0} = H^0 \left.\left(\frac{\partial \Theta(s^f)}{\partial s^{f'}}\Psi(s) + \Theta(s^f)\frac{\partial \Psi}{\partial s^{f'}}\right)\right|_{s=0} = \begin{cases} \frac{H^0}{S'(0)} & \text{if } f = f', \\ 0 & \text{if } f \neq f'. \end{cases}$$

For every $(f, f', f'') \in \mathcal{F}^3$,

$$\begin{aligned} \left.\frac{\partial^2 T^f}{\partial s^{f'}\partial s^{f''}}\right|_{s=0} &= H^0 \left.\left(\frac{\partial^2 \Theta(s^f)}{\partial s^{f'}\partial s^{f''}}\Psi(s) + \Theta(s^f)\frac{\partial^2 \Psi}{\partial s^{f'}\partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}}\frac{\partial \Psi}{\partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}}\frac{\partial \Psi}{\partial s^{f'}}\right)\right|_{s=0}, \\ &= H^0 \left.\left(\frac{\partial^2 \Theta(s^f)}{\partial s^{f'}\partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}}\right)\right|_{s=0}, \\ &= H^0 \times \begin{cases} \Theta''(0) + 2\Theta'(0) & \text{if } f = f' = f'', \\ 0 & \text{if } f', f'' \neq f, \\ \Theta'(0) & \text{otherwise}, \end{cases} \\ &= \frac{H^0}{S'(0)} \times \begin{cases} 2(\alpha+1) & \text{if } f = f' = f'', \\ 0 & \text{if } f', f'' \neq f, \\ 1 & \text{otherwise}. \end{cases} \end{aligned}$$

Finally, for every $(f, f', f'', f''') \in \mathcal{F}^3$,

$$\begin{aligned} \left.\frac{\partial^3 T^f}{\partial s^{f'}\partial s^{f''}\partial s^{f'''}}\right|_{s=0} &= H^0 \left(\frac{\partial^3 \Theta(s^f)}{\partial s^{f'}\partial s^{f''}\partial s^{f'''}}\Psi(s) + \Theta(s^f)\frac{\partial^3 \Psi}{\partial s^{f'}\partial s^{f''}\partial s^{f'''}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f'}\partial s^{f''}}\frac{\partial \Psi}{\partial s^{f'''}}\right. \\ &\quad + \frac{\partial \Theta(s^f)}{\partial s^{f'''}}\frac{\partial^2 \Psi}{\partial s^{f'}\partial s^{f''}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f'}\partial s^{f'''}}\frac{\partial \Psi}{\partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}}\frac{\partial^2 \Psi}{\partial s^{f''}\partial s^{f'''}} \\ &\quad \left.\left. + \frac{\partial^2 \Theta(s^f)}{\partial s^{f''}\partial s^{f'''}}\frac{\partial \Psi}{\partial s^{f'}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}}\frac{\partial^2 \Psi}{\partial s^{f'}\partial s^{f'''}}\right)\right|_{s=0}, \\ &= H^0 \left(\frac{\partial^3 \Theta(s^f)}{\partial s^{f'}\partial s^{f''}\partial s^{f'''}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f'}\partial s^{f''}} + 2\frac{\partial \Theta(s^f)}{\partial s^{f'''}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f'}\partial s^{f'''}}\right. \\ &\quad \left.\left. + 2\frac{\partial \Theta(s^f)}{\partial s^{f'}} + \frac{\partial^2 \Theta(s^f)}{\partial s^{f''}\partial s^{f'''}} + 2\frac{\partial \Theta(s^f)}{\partial s^{f''}}\right)\right|_{s=0}, \\ &= \frac{H^0}{S'(0)} \begin{cases} 3\alpha(1+2\alpha) + 3(2\alpha+2) & \text{if } f = f' = f'' = f''', \\ 2\alpha + 2 + 2 & \text{if } (f', f'', f''') \in \mathcal{P}^2(f), \\ 2 & \text{if } (f', f'', f''') \in \mathcal{P}^1(f), \\ 0 & \text{otherwise}, \end{cases} \end{aligned}$$

$$= \frac{H^0}{S'(0)} \begin{cases} 6 + 9\alpha + 6\alpha^2 & \text{if } f = f' = f'' = f''', \\ 2\alpha + 4 & \text{if } (f', f'', f''') \in \mathcal{P}^2(f), \\ 2 & \text{if } (f', f'', f''') \in \mathcal{P}^1(f), \\ 0 & \text{otherwise.} \end{cases}$$

□

To ease notation, let $\bar{s} = \sum_{g \in \mathcal{F}} s^g$. We now use Lemma 4 to obtain a third-order Taylor approximation of $T^f(s)$ in the neighborhood of $s = 0$:

Lemma 5. *In the neighborhood of $s = 0$,*

$$T^f(s) = \frac{H^0}{S'(0)} \left(s^f + \left(\alpha (s^f)^2 + s^f \bar{s}\right) + \left(\frac{\alpha(1+2\alpha)}{2} (s^f)^3 + \alpha (s^f)^2 \bar{s} + s^f \bar{s}^2 \right) \right) + o(\|s\|^3).$$

Proof. By Lemma 4, first-order terms are simply given by $\frac{H^0}{S'(0)} s^f$. Second-order terms are given by

$$\frac{H^0}{S'(0)} \frac{1}{2} \left(2(1+\alpha)(s^f)^2 + 2 s^f \sum_{g \neq f} s^g \right) = \frac{H^0}{S'(0)} \left(\alpha (s^f)^2 + s^f \bar{s} \right).$$

Finally, third-order terms are:

$$\frac{H^0}{S'(0)} \frac{1}{6} \left((6 + 9\alpha + 6\alpha^2)(s^f)^3 + (2\alpha + 4) \sum_{(f', f'', f''') \in \mathcal{P}^2(f)} s^{f'} s^{f''} s^{f'''} + 2 \sum_{(f', f'', f''') \in \mathcal{P}^1(f)} s^{f'} s^{f''} s^{f'''} \right),$$
$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((6 + 9\alpha + 6\alpha^2)(s^f)^3 + 3(2\alpha + 4)(s^f)^2 \sum_{g \neq f} s^g + 6 s^f \sum_{g, g' \neq f} s^g s^{g'} \right),$$
$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((6 + 9\alpha + 6\alpha^2)(s^f)^3 + 3(2\alpha + 4)(s^f)^2 (\bar{s} - s^f) + 6 s^f (\bar{s} - s^f)^2 \right),$$
$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((-6 + 3\alpha + 6\alpha^2)(s^f)^3 + 3(2\alpha + 4)(s^f)^2 \bar{s} + 6 s^f (\bar{s}^2 - 2\bar{s} s^f + (s^f)^2) \right),$$
$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((3\alpha + 6\alpha^2)(s^f)^3 + 6\alpha (s^f)^2 \bar{s} + 6 s^f \bar{s}^2 \right),$$

The lemma follows by Taylor's theorem. □

We recall the definition of the dispersion measure $\Gamma(s)$:

$$\Gamma(s) = \sum_{f \in \mathcal{F}} (s^f)^3.$$

The following lemma gives a third-order Taylor approximation of the sum of the types:

Lemma 6. *In the neighborhood of $s = 0$,*

$$\sum_{f\in\mathcal{F}} \frac{S'(0)}{H^0} T^f(s) = \bar{s} + \left(\alpha\, HHI(s) + \bar{s}^2\right) + \left(\frac{\alpha(1+2\alpha)}{2}\Gamma(s) + \alpha\, HHI(s)\bar{s} + \bar{s}^3\right) + o(\|s\|^3).$$

Proof. Immediate. □

Let $\mathrm{CS}^m(s)$ be consumer surplus under monopolistic competition. Recall that all the firms set their normalized markups equal to 1 under monopolistic competition. Hence, in the case of NMNL demand,

$$\mathrm{CS}^m(s) = \log\left(H^0 + \sum_{f\in\mathcal{F}} T^f(s)\,\mathrm{e}^{-1}\right) = \log H^0 + \log\left(1 + \sum_{f\in\mathcal{F}} T^f(s)\frac{S'(0)}{H^0}\right).$$

Similarly, in the case of NCES demand,

$$\mathrm{CS}^m(s) = \log\left(H^0 + \sum_{f\in\mathcal{F}} T^f(s)\alpha^{\frac{\alpha}{1-\alpha}}\right) = \log H^0 + \log\left(1 + \sum_{f\in\mathcal{F}} T^f(s)\frac{S'(0)}{H^0}\right).$$

We now provide a third-order Taylor expansion of $\mathrm{CS}^m(s)$:

Lemma 7. *In the neighborhood of $s = 0$,*

$$CS^m(s) = \log H^0 + \bar{s} + \frac{1}{2}\bar{s}^2 + \frac{1}{3}\bar{s}^3 + \alpha\, HHI(s) + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + o(\|s\|^3).$$

Proof. Recall that, at the third order in the neighborhood of $x = 0$,

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3).$$

Combining this with Lemma 6, and eliminating higher-order terms, we obtain

$$\begin{aligned}
\mathrm{CS}^m(s) = {} & \log H^0 + \bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2 + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + \alpha\,\mathrm{HHI}(s)\bar{s} + \bar{s}^3 \\
& - \frac{1}{2}\left(\bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2\right)^2 + \frac{1}{3}\bar{s}^3 + o(\|s\|^3), \\
= {} & \log H^0 + \bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2 + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + \alpha\,\mathrm{HHI}(s)\bar{s} + \bar{s}^3 \\
& - \frac{1}{2}\left(\bar{s}^2 + 2\alpha\,\mathrm{HHI}(s)\bar{s} + 2\bar{s}^3\right) + \frac{1}{3}\bar{s}^3 + o(\|s\|^3), \\
= {} & \log H^0 + \bar{s} + \frac{1}{2}\bar{s}^2 + \frac{1}{3}\bar{s}^3 + \alpha\,\mathrm{HHI}(s) + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + o(\|s\|^3).
\end{aligned}$$

□

Combining Lemmas 3 and 7, we obtain the approximation results for the distortion to consumer surplus that were announced in Proposition 3 and footnote 18:

Lemma 8. *In the neighborhood of $s = 0$,*

$$CS(s) - CS^m(s) = -\alpha\, HHI(s) - \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + o\left(\|s\|^3\right).$$

Next, we turn our attention to profits. Let

$$\Pi(s) = \sum_{f\in\mathcal{F}} \left(\frac{1}{1-\alpha s^f} - 1 \right)$$

be aggregate profit.

Lemma 9. *In the neighborhood of $s = 0$,*

$$\Pi(s) = \alpha\bar{s} + \alpha^2\, HHI(s) + \alpha^3\Gamma(s) + o\left(\|s\|^3\right).$$

Proof. This follows immediately from the fact that, in the neighborhood of $x = 0$,

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + o(x^3). \qquad \square$$

Let $\Pi^m(s)$ be aggregate profit under monopolistic competition. Under NMNL demand,

$$\Pi^m(s) = \sum_{f\in\mathcal{F}} \frac{T^f(s)\,\mathrm{e}^{-1}}{H^0 + \sum_{g\in\mathcal{F}} T^g(s)\,\mathrm{e}^{-1}} = 1 - \frac{1}{1 + \sum_{g\in\mathcal{F}} T^g(s)\frac{S'(0)}{H^0}}.$$

Under NCES demand,

$$\Pi^m(s) = \sum_{f\in\mathcal{F}} \alpha \frac{T^f(s)\alpha^{\frac{\alpha}{1-\alpha}}}{H^0 + \sum_{g\in\mathcal{F}} T^g(s)\alpha^{\frac{\alpha}{1-\alpha}}} = \alpha\left(1 - \frac{1}{1 + \sum_{g\in\mathcal{F}} T^g(s)\frac{S'(0)}{H^0}}\right).$$

Lemma 10. *In the neighborhood of $s = 0$,*

$$\Pi^m(s) = \alpha\left(\bar{s} + \alpha\, HHI(s) + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) - \alpha\, HHI(s)\bar{s}\right) + o(\|s\|^3).$$

Proof. Note that, at the third order in the neighborhood of $x = 0$,

$$1 - \frac{1}{1+x} = x - x^2 + x^3 + o(x^3).$$

Combining this with the definition of Π^m and Lemma 6, and eliminating higher-order terms, we obtain:

$$\Pi^m(s) = \alpha\left(\bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2 + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + \alpha\,\mathrm{HHI}(s)\bar{s} + \bar{s}^3\right.$$

$$- \left(\bar{s} + \alpha \,\mathrm{HHI}(s) + \bar{s}^2\right)^2 + \bar{s}^3\Big) + o(\|s\|^3),$$

$$= \alpha \left(\bar{s} + \alpha \,\mathrm{HHI}(s) + \bar{s}^2 + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + \alpha \,\mathrm{HHI}(s)\bar{s} + \bar{s}^3 \right.$$

$$\left. - \left(\bar{s}^2 + 2\alpha \,\mathrm{HHI}(s)\bar{s} + 2\bar{s}^3\right) + \bar{s}^3\right) + o(\|s\|^3),$$

$$= \alpha \left(\bar{s} + \alpha \,\mathrm{HHI}(s) + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) - \alpha \,\mathrm{HHI}(s)\bar{s} \right) + o(\|s\|^3). \qquad \square$$

Combining Lemmas 8, 9, and 10 delivers the approximation of the aggregate surplus distortion announced in Proposition 3 and footnote 18:

Lemma 11. *In the neighborhood of $s = 0$,*

$$AS(s) - AS^m(s) = -\alpha \left(HHI(s)(1 - \alpha\bar{s}) + \frac{1}{2}(1+3\alpha)\Gamma(s) \right) + o\left(\|s\|^3\right).$$

C.2 Proof of Proposition 5

We prove a series of lemmas that jointly imply Proposition 5.

Recall from Appendix C.1 that $H^*(s)$ is the equilibrium value of the aggregator given the vector of market shares s. To ease notation, let $\overline{H}(s) \equiv H^*(\bar{s}(s))$ be the post-merger equilibrium value of the aggregator. We first provide an approximation of the market power effect of the merger, measured in terms of consumer surplus—the first part of Proposition 5:

Lemma 12. *In the neighborhood of $s = 0$,*

$$CS(\bar{s}(s)) - CS(s) = -\alpha\Delta^M \, HHI(s) + o(\|s\|^2).$$

Proof. By definition of $\overline{H}$, we have that

$$\frac{H^0}{\overline{H}} + \sum_{g \in \overline{\mathcal{F}}} S\left(\frac{T^g}{\overline{H}}\right) = 1.$$

Totally differentiating this expression, we obtain:

$$-\frac{d\overline{H}}{\overline{H}} \left(\frac{H^0}{\overline{H}} + \sum_{g \in \overline{\mathcal{F}}} \frac{T^g}{\overline{H}} S'\left(\frac{T^g}{\overline{H}}\right) \right) + \frac{1}{\overline{H}} \sum_{g \in \overline{\mathcal{F}}} S'\left(\frac{T^g}{\overline{H}}\right) \sum_{f \in \mathcal{F}} \frac{\partial T^g}{\partial s^f} ds^f = 0.$$

Hence,

$$\frac{\partial \overline{H}}{\partial s^f} = \overline{H} \frac{\sum_{g \in \overline{\mathcal{F}}} S'\left(\frac{T^g}{\overline{H}}\right) \frac{\partial T^g}{\partial s^f}}{H^0 + \sum_{g \in \overline{\mathcal{F}}} T^g S'\left(\frac{T^g}{\overline{H}}\right)}.$$

Hence, by Lemma 5 and since $T^M = \sum_{g \in \mathcal{M}} T^g$,

$$\left.\frac{\partial \overline{H}}{\partial s^f}\right|_{s=0} = H^0.$$

Next, we compute the Hessian of $\overline{H}$. Note that, for every $f, f' \in \mathcal{F}$

$$\begin{aligned}
\left.\frac{\partial^2 \overline{H}}{\partial s^f \partial s^{f'}}\right|_{s=0} &= \frac{\partial \overline{H}}{\partial s^{f'}} \times 1 + H^0 \times \frac{1}{(H^0)^2} \left(\left(\sum_{g \in \overline{\mathcal{F}}} \left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0} \frac{\partial T^g}{\partial s^f} \frac{\partial T^g}{\partial s^{f'}} S''(0) \right) \right) H^0 \right. \\
&\quad \left. - H^0 \left(\sum_{g \in \overline{\mathcal{F}}} \frac{\partial T^g}{\partial s^f} S'(0) \right) \right), \\
&= H^0 + \sum_{g \in \overline{\mathcal{F}}} \left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0} \frac{\partial T^g}{\partial s^f} \frac{\partial T^g}{\partial s^{f'}} S''(0) \right) - \sum_{g \in \overline{\mathcal{F}}} \frac{\partial T^g}{\partial s^f} S'(0), \\
&= \sum_{g \in \overline{\mathcal{F}}} \left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0} \frac{\partial T^g}{\partial s^f} \frac{\partial T^g}{\partial s^{f'}} S''(0) \right), \\
&= \left(\frac{\partial^2 T^M}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0} \frac{\partial T^M}{\partial s^f} \frac{\partial T^M}{\partial s^{f'}} S''(0) \right) \\
&\quad + \sum_{g \in \mathcal{O}} \left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0} \frac{\partial T^g}{\partial s^f} \frac{\partial T^g}{\partial s^{f'}} S''(0) \right).
\end{aligned}$$

Assume first that $f \in \mathcal{O}$ and/or $f' \in \mathcal{O}$. Then, by Lemma 5 and since $T^M = \sum_{g \in \mathcal{M}} T^g$,

$$\begin{aligned}
\left.\frac{\partial^2 \overline{H}}{\partial s^f \partial s^{f'}}\right|_{s=0} &= \begin{cases} 2H^0 & \text{if } f \neq f', \\ \frac{H^0}{S'(0)} 2(1+\alpha) S'(0) + \frac{1}{H^0} \left(\frac{H^0}{S'(0)} \right)^2 S''(0) & \text{if } f = f', \end{cases} \\
&= 2H^0.
\end{aligned}$$

Next, assume instead that $f, f' \in \mathcal{M}$. Then,

$$\begin{aligned}
\left.\frac{\partial^2 \overline{H}}{\partial s^f \partial s^{f'}}\right|_{s=0} &= \begin{cases} 2H^0 + \frac{1}{H^0} \left(\frac{H^0}{S'(0)} \right)^2 S''(0) & \text{if } f \neq f', \\ \frac{H^0}{S'(0)} 2(1+\alpha) S'(0) + \frac{1}{H^0} \left(\frac{H^0}{S'(0)} \right)^2 S''(0) & \text{if } f = f', \end{cases} \\
&= \begin{cases} 2H^0(1-\alpha) & \text{if } f \neq f', \\ 2H^0 & \text{if } f = f'. \end{cases}
\end{aligned}$$

By Taylor's theorem,

$$\begin{aligned}
\overline{H}(s) &= H^0 + H^0 \sum_{f\in\mathcal{F}} s^f + \frac{H^0}{2}\left(2\sum_{f,g\in\mathcal{F}} s^f s^g - 2\alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g\right) + o(\|s\|^2),\\
&= H^0\left(1 + \sum_{f\in\mathcal{F}} s^f + \left(\sum_{f\in\mathcal{F}} s^f\right)^2 - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g\right) + o(\|s\|^2).
\end{aligned}$$

Using the fact that $\log(1+x) = x - \frac{1}{2}x^2 + o(x^2)$ in the neighborhood of $x = 0$, this implies that

$$\begin{aligned}
\log \overline{H}(s) &= \log H^0 + \sum_{f\in\mathcal{F}} s^f + \left(\sum_{f\in\mathcal{F}} s^f\right)^2 - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g - \frac{1}{2}\left(\sum_{f\in\mathcal{F}} s^f\right)^2 + o(\|s\|^2),\\
&= \log H^*(s) - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g + o(\|s\|^2), \text{ by Lemma 3,}\\
&= \log H^*(s) - \alpha \Delta^M \, \mathrm{HHI}(s) + o(\|s\|^2).
\end{aligned}$$

□

Next, we approximate post-merger market shares:

Lemma 13. *In the neighborhood of $s = 0$, for every $f \in \mathcal{O}$*

$$\bar{s}^f = s^f + o(\|s\|^2),$$

and

$$\bar{s}^M = \sum_{f\in\mathcal{M}} s^f - \alpha\Delta^M \, HHI(s) + o(\|s\|^2).$$

Proof. By definition, for every $f \in \overline{\mathcal{F}}$,

$$\bar{s}^f = S\left(\frac{T^f}{\overline{H}}\right).$$

For every $f \in \overline{\mathcal{F}}$ and $f' \in \mathcal{F}$,

$$\frac{\partial \bar{s}^f}{\partial s^{f'}} = \frac{1}{\overline{H}}\left(\frac{\partial T^f}{\partial s^{f'}} - \frac{T^f}{\overline{H}}\frac{\partial \overline{H}}{\partial s^{f'}}\right) S'\left(\frac{T^f}{\overline{H}}\right).$$

It follows that

$$\left.\frac{\partial \bar{s}^f}{\partial s^{f'}}\right|_{s=0} = \begin{cases} 0 & \text{if } f \neq f' \text{ and } (f \neq M \text{ or } f' \notin \mathcal{M}),\\ 1 & \text{otherwise.}\end{cases}$$

For every $f \in \overline{\mathcal{F}}$ and $f', f'' \in \mathcal{F}$,

$$\begin{aligned}
\left.\frac{\partial^2 \bar{s}^f}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} &= -\frac{\partial \overline{H}}{\partial s^{f''}} \frac{1}{\overline{H}^2} \frac{\partial T^f}{\partial s^{f'}} S'(0) + \frac{1}{\overline{H}} \left(\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}} - \frac{1}{\overline{H}} \frac{\partial T^f}{\partial s^{f''}} \frac{\partial \overline{H}}{\partial s^{f'}} \right) S'(0) \\
&\quad + \frac{1}{\overline{H}^2} \frac{\partial T^f}{\partial s^{f'}} \frac{\partial T^f}{\partial s^{f''}} S''(0), \\
&= -\frac{1}{H^0} \frac{\partial T^f}{\partial s^{f'}} S'(0) + \frac{1}{H^0} \left(\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^f}{\partial s^{f''}} \right) S'(0) + \frac{1}{(H^0)^2} \frac{\partial T^f}{\partial s^{f'}} \frac{\partial T^f}{\partial s^{f''}} S''(0), \\
&= \frac{S'(0)}{H^0} \left(\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^f}{\partial s^{f'}} - \frac{\partial T^f}{\partial s^{f''}} \right) + \frac{S''(0)}{(H^0)^2} \frac{\partial T^f}{\partial s^{f'}} \frac{\partial T^f}{\partial s^{f''}}.
\end{aligned}$$

Suppose first that $f \neq M$, so that $f \in \mathcal{F}$. Clearly, if $f' \neq f$ and $f'' \neq f$, then,

$$\left.\frac{\partial^2 \bar{s}^f}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = 0.$$

If $f'' \neq f$, then

$$\left.\frac{\partial^2 \bar{s}^f}{\partial s^{f} \partial s^{f''}}\right|_{s=0} = \frac{S'(0)}{H^0} \left(\frac{\partial^2 T^f}{\partial s^{f} \partial s^{f''}} - \frac{\partial T^f}{\partial s^{f}} \right) = 0.$$

Finally,

$$\begin{aligned}
\left.\frac{\partial^2 \bar{s}^f}{\partial (s^{f})^2}\right|_{s=0} &= \frac{S'(0)}{H^0} \left(\frac{\partial^2 T^f}{\partial (s^{f})^2} - 2\frac{\partial T^f}{\partial s^{f}} \right) + \frac{S''(0)}{(H^0)^2} \left(\frac{\partial T^f}{\partial s^f} \right)^2, \\
&= \frac{S'(0)}{H^0} \left(\frac{H^0}{S'(0)} 2(1+\alpha) - 2\frac{H^0}{S'(0)} \right) + \frac{S''(0)}{(H^0)^2} \left(\frac{H^0}{S'(0)} \right)^2, \\
&= 0.
\end{aligned}$$

Next, assume that $f = M$. Clearly, if $f', f'' \notin \mathcal{M}$, then

$$\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = 0.$$

Next assume that $f'' \notin \mathcal{M}$ and $f' \in \mathcal{M}$. Then,

$$\begin{aligned}
\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} &= \frac{S'(0)}{H^0} \left(\frac{\partial^2 T^M}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^M}{\partial s^{f'}} \right), \\
&= \frac{S'(0)}{H^0} \left(\frac{\partial^2 T^{f'}}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^{f'}}{\partial s^{f'}} \right), \\
&= 0.
\end{aligned}$$

Next, assume that $f', f'' \in \mathcal{M}$. Then,

$$\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^M}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^{f'}}{\partial s^{f'}} - \frac{\partial T^{f''}}{\partial s^{f''}}\right) + \frac{S''(0)}{(H^0)^2}\frac{\partial T^{f'}}{\partial s^{f'}}\frac{\partial T^{f''}}{\partial s^{f''}},$$

$$= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^M}{\partial s^{f'} \partial s^{f''}} - 2\frac{H^0}{S'(0)}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2.$$

Hence, if $f' = f''$, then

$$\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^{f'}}{\partial (s^{f'})^2} - 2\frac{H^0}{S'(0)}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2,$$

$$= 0.$$

If instead $f' \neq f''$, then

$$\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^{f'}}{\partial s^{f'} \partial s^{f''}} + \frac{\partial^2 T^{f''}}{\partial s^{f'} \partial s^{f''}} - 2\frac{H^0}{S'(0)}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2,$$

$$= -2\alpha.$$

The lemma follows by Taylor's theorem. □

Let

$$\Pi(s) = \sum_{f \in \mathcal{F}}\left(\frac{1}{1 - \alpha s^f} - 1\right),$$

$$\text{and } \overline{\Pi}(s) = \sum_{f \in \overline{\mathcal{F}}}\left(\frac{1}{1 - \alpha \bar{s}^f} - 1\right),$$

be aggregate profits, pre- and post-merger, respectively.

Lemma 14. *In the neighborhood of $s = 0$,*

$$\overline{\Pi}(s) - \Pi(s) = o(\|s\|^2).$$

Proof. By Lemma 13, and since $\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + o(\|x\|^2)$ in the neighborhood of $x = 0$, we have that

$$\Pi(s) = \alpha \sum_{f \in \mathcal{F}} s^f + \alpha^2 \sum_{f \in \mathcal{F}} (s^f)^2 + o(\|s\|^2),$$

and

$$\overline{\Pi}(s) = \frac{1}{1 - \alpha \bar{s}^M} - 1 + \sum_{f \in \mathcal{O}}\left(\frac{1}{1 - \alpha \bar{s}^f} - 1\right),$$

$$= \alpha \left(\sum_{f\in\mathcal{M}} s^f - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g \right) + \alpha^2 \left(\sum_{f\in\mathcal{M}} s^f \right)^2 + \alpha \sum_{f\in\mathcal{O}} s^f + \alpha^2 \sum_{f\in\mathcal{O}} (s^f)^2 + o(\|s\|^2),$$
$$= \alpha \sum_{f\in\mathcal{F}} s^f + \alpha^2 \sum_{f\in\mathcal{F}} (s^f)^2 + o(\|s\|^2),$$
$$= \Pi(s) + o(\|s\|^2).$$
□

Combining Lemmas 12 and 14 proves the second part of Proposition 5:

Lemma 15. *In the neighborhood of $s = 0$,*

$$AS(\bar{s}(s)) - AS(s) = -\alpha \Delta^M \, HHI(s) + o(\|s\|^2).$$

D Approximation Results Around Monopolistic Competition Conduct

D.1 Proof of Proposition 4

We prove a series of lemmas that jointly imply Proposition 4.

Recall that the markup and market-share fitting-in functions, $m(x,\theta)$ and $S(x,\theta)$, jointly solve the system

$$\mu^f = \frac{1}{1-\theta\alpha s^f},$$
$$s^f = \begin{cases} x\left(1-(1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}} & \text{in the case of NCES,} \\ x\,\mathrm{e}^{-\mu^f} & \text{in the case of NMNL.} \end{cases}$$

We compute the derivatives of S with respect to x and θ at $\theta = 0$:

Lemma 16. *For every $\alpha \in (0,1]$ and $x > 0$,*

$$\left.\frac{\partial S}{\partial x}\right|_{(x,0)} = \frac{S(x,0)}{x},$$
$$\textit{and} \quad \left.\frac{\partial S}{\partial \theta}\right|_{(x,0)} = -\alpha S(x,0)^2.$$

Proof. Under NMNL demand,

$$S = x\,\mathrm{e}^{-m} = x \exp\left(-\frac{1}{1-\theta S}\right).$$

Hence, at $\theta = 0$,

$$dS = \frac{S}{x}dx - S^2 d\theta,$$

which proves the lemma for the case $\alpha = 1$.

Under NCES demand,

$$S = x\left(1-(1-\alpha)m\right)^{\frac{\alpha}{1-\alpha}} = x\left(1 - \frac{1-\alpha}{1-\theta\alpha S}\right)^{\frac{\alpha}{1-\alpha}} = x\left(\alpha\frac{1-\theta S}{1-\theta\alpha S}\right)^{\frac{\alpha}{1-\alpha}}.$$

Hence, at $\theta = 0$,

$$\begin{aligned} dS &= \frac{S}{x}dx + \frac{\alpha}{1-\alpha}S\frac{1-\alpha\theta S}{1-\theta S}\frac{1}{(1-\alpha\theta S)^2}\Big(\left(-\theta(1-\alpha\theta S) + \alpha\theta(1-\theta S)\right) dS \\ &\quad + \left(-S(1-\alpha\theta S) + \alpha S(1-\theta S)\right) d\theta\Big), \\ &= \frac{S}{x}dx - \alpha S^2, \end{aligned}$$

which proves the lemma for the case $\alpha < 1$. □

Fix a profile of types $(T^f)_{f\in\mathcal{F}}$ and a value of the outside option $H^0 \geq 0$, and let $H^*(\theta)$ be the equilibrium value of the aggregator when the conduct parameter is θ. We compute $H^{*\prime}(0)$, and use this derivative to obtain the first part of Proposition 4:

Lemma 17. *The following holds:*

$$\left.\frac{d\log H^*}{d\theta}\right|_{\theta=0} = -\alpha\sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*}, 0\right)^2.$$

This implies that, in the neighborhood of $\theta = 0$,

$$CS(\theta) - CS(0) = -\alpha\, HHI(\theta)\theta + o(\theta).$$

Proof. Recall that $H^*(\theta)$ is pinned down by the equilibrium condition

$$\frac{H^0}{H^*} + \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*}, \theta\right) = 1.$$

Totally differentiating the equilibrium condition, we obtain:

$$-\frac{dH^*}{H^*}\left(\frac{H^0}{H^*} + \sum_{f\in\mathcal{F}}\frac{T^f}{H^*}\frac{\partial S}{\partial(T^f/H^*)}\left(\frac{T^f}{H^*},\theta\right)\right) + d\theta\sum_{f\in\mathcal{F}}\frac{\partial S}{\partial\theta}\left(\frac{T^f}{H^*},\theta\right) = 0.$$

Evaluating the above expression at $\theta = 0$, and using Lemma 16 and the equilibrium condition, we obtain:

$$-\frac{dH^*}{H^*(0)} - d\theta \sum_{f\in\mathcal{F}} \alpha S\left(\frac{T^f}{H^*(0)}, 0\right)^2 = 0,$$

which proves the first part of the lemma.

The second part of the lemma follows by Taylor's theorem:

$$\begin{aligned}\mathrm{CS}(\theta) - \mathrm{CS}(0) &= -\alpha\,\mathrm{HHI}(0)\theta + o(\theta),\\ &= -\alpha\,\mathrm{HHI}(\theta)\theta + o(\theta),\end{aligned}$$

where the second line follows from the fact that $\mathrm{HHI}(\theta) - \mathrm{HHI}(0)$ is at most first order. □

Let $\Pi(\theta)$ denote aggregate equilibrium profits when the conduct parameter is θ. We compute $\Pi'(0)$:

Lemma 18. $\Pi'(0) = \alpha^2\, HHI(0) \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(0)}, 0\right)$.

Proof. Let $s^f(\theta) = S(T^f/H^*(\theta), \theta)$ and

$$\pi^f(\theta) = \alpha \frac{s^f(\theta)}{1 - \alpha\theta s^f(\theta)}$$

be firm f's equilibrium market share and profit, respectively. Then,

$$\begin{aligned}s^{f\prime}(0) &= \left(-\frac{T^f}{H^*}\frac{d\log H^*}{d\theta}\frac{\partial S}{\partial(T^f/H^*)} + \frac{\partial S}{\partial\theta}\right)\bigg|_{\theta=0},\\ &= \alpha\,\mathrm{HHI}(0)s^f(0) - \alpha(s^f(0))^2.\end{aligned}$$

Hence,

$$\pi^{f\prime}(0) = \alpha\left(s^{f\prime}(0) - s^f(0)\left(-\alpha s^f(0)\right)\right) = \alpha^2\,\mathrm{HHI}(0)s^f(0).$$

Adding up over all firms proves the lemma. □

Combining Lemmas 17 and 18, we obtain the second part of Proposition 4:

Lemma 19. *In the neighborhood of* $\theta = 0$,

$$AS(\theta) - AS(0) = -\alpha\, HHI(\theta)\left(1 - \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(\theta)}, \theta\right)\right)\theta + o(\theta).$$

Proof. Lemmas 17 and 18 and Taylor's theorem imply that

$$\mathrm{AS}(\theta) - \mathrm{AS}(0) = -\alpha\,\mathrm{HHI}(0)\left(1 - \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(0)}, 0\right)\right)\theta + o(\theta).$$

The lemma follows from the fact that

$$\mathrm{HHI}(0)\left(1-\sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(0)},0\right)\right)-\mathrm{HHI}(\theta)\left(1-\sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(\theta)},\theta\right)\right)$$

is at most first order. □

D.2 Proof of Proposition 6

Proof. Let $\mathrm{CS}(\theta)$ and $\mathrm{AS}(\theta)$ be pre-merger equilibrium consumer surplus and aggregate surplus, respectively. Let $\mathrm{HHI}(\theta)$ (resp., $H^*(\theta)$) be the pre-merger equilibrium value of the Herfindahl index (resp., aggregator), and

$$\Sigma(\theta)\equiv\sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(\theta)},\theta\right)$$

be the firms' aggregate market share. The post-merger values of those quantities are $\overline{\mathrm{CS}}(\theta)$, $\overline{\mathrm{AS}}(\theta)$, $\overline{\mathrm{HHI}}(\theta)$, $\overline{H}^*(\theta)$, and $\overline{\Sigma}(\theta)$, respectively.

Note that $\mathrm{CS}(0)=\overline{\mathrm{CS}}(0)$, $\mathrm{AS}(0)=\overline{\mathrm{AS}}(0)$, $H^*(0)=\overline{H}^*(0)$, $\Sigma(0)=\overline{\Sigma}(0)$, and

$$\overline{\mathrm{HHI}}(0)-\mathrm{HHI}(0)=\Delta^M\,\mathrm{HHI}(0),$$

where $\Delta^M\,\mathrm{HHI}(\theta)$ is the merged-induced, naively computed variation in the Herfindahl index.

Using these facts and Proposition 4, we obtain:

$$\begin{aligned}
\overline{\mathrm{CS}}(\theta)-\mathrm{CS}(\theta) &= -\alpha\left(\overline{\mathrm{HHI}}(\theta)-\mathrm{HHI}(\theta)\right)\theta+o(\theta),\\
&= -\alpha\left(\overline{\mathrm{HHI}}(0)-\mathrm{HHI}(0)+o(1)\right)\theta+o(\theta),\\
&= -\alpha\Delta^M\,\mathrm{HHI}(0)\theta+o(\theta),\\
&= -\alpha\left(\Delta^M\,\mathrm{HHI}(\theta)+o(1)\right)\theta+o(\theta),\\
&= -\alpha\Delta^M\,\mathrm{HHI}(\theta)\theta+o(\theta),
\end{aligned}$$

which proves the first part of the proposition.

Similarly,

$$\begin{aligned}
\overline{\mathrm{AS}}(\theta)-\mathrm{AS}(\theta) &= -\alpha\Big(\overline{\mathrm{HHI}}(\theta)\left(1-\alpha\overline{\Sigma}(\theta)\right)-\mathrm{HHI}(\theta)\left(1-\alpha\Sigma(\theta)\right)\Big)\theta+o(\theta),\\
&= -\alpha\Big(\overline{\mathrm{HHI}}(0)\left(1-\alpha\Sigma(0)\right)-\mathrm{HHI}(0)\left(1-\alpha\Sigma(0)\right)+o(1)\Big)\theta+o(\theta),\\
&= -\alpha\left(1-\alpha\Sigma(0)\right)\left(\overline{\mathrm{HHI}}(0)-\mathrm{HHI}(0)\right)\theta+o(\theta),\\
&= -\alpha\left(1-\alpha\Sigma(\theta)+o(1)\right)\left(\Delta^M\,\mathrm{HHI}(\theta)+o(1)\right)\theta+o(\theta),\\
&= -\alpha\left(1-\alpha\Sigma(\theta)\right)\Delta^M\,\mathrm{HHI}(\theta)\theta+o(\theta),
\end{aligned}$$

which proves the second part of the proposition. □

E Consumer Surplus Effects: Static Analysis

E.1 Proof of Proposition 8

Proof. Recall that $\varepsilon(\cdot)$ is the elasticity of S (see Lemma 1) and that the cutoff type solves the equation:

$$S\left(\frac{\hat{T}^M}{H^*}\right) = \sum_{f\in\mathcal{M}} S\left(\frac{T^f}{H^*}\right).$$

Totally differentiating this equation, we obtain:

$$\begin{aligned}
S'\left(\frac{\hat{T}^M}{H^*}\right)\frac{d\hat{T}^M}{dH^*} &= \frac{\hat{T}^M}{H^*}S'\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f\in\mathcal{M}}\frac{T^f}{H^*}S'\left(\frac{T^f}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right)S\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f\in\mathcal{M}}\varepsilon\left(\frac{T^f}{H^*}\right)S\left(\frac{T^f}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right)\sum_{f\in\mathcal{M}}S\left(\frac{T^f}{H^*}\right) - \sum_{f\in\mathcal{M}}\varepsilon\left(\frac{T^f}{H^*}\right)S\left(\frac{T^f}{H^*}\right), \\
&= \sum_{f\in\mathcal{M}}\left(\varepsilon\left(\frac{\hat{T}^M}{H^*}\right) - \varepsilon\left(\frac{T^f}{H^*}\right)\right)S\left(\frac{T^f}{H^*}\right), \\
&< 0,
\end{aligned}$$

where the third line follows by definition of $\hat{T}^M$ and the last line follows from Lemma 1 and from the fact that $\hat{T}^M > T^f$ for every $f \in \mathcal{M}$. □

E.2 Proof of Proposition 9

Proof. Note that

$$\frac{\hat{T}^M}{T^f + T^g} = \frac{S^{-1}\left(S\left(\frac{T^f}{H^*}\right) + S\left(\frac{T^g}{H^*}\right)\right)}{\frac{T^f}{H^*} + \frac{T^g}{H^*}} = \xi\left(\frac{T^f}{H^*}, \frac{T^g}{H^*}\right),$$

where

$$\xi(x,y) \equiv \frac{S^{-1}\left(S(x) + S(y)\right)}{x+y}, \quad \forall x, y > 0.$$

Proving the first part of the lemma therefore boils down to showing that $\partial\xi/\partial x > 0$ and $\partial\xi/\partial y > 0$. By symmetry, this is equivalent to proving that $\partial\xi/\partial x > 0$, which we undertake next.

Differentiating ξ with respect to x, we obtain:

$$\frac{\partial \xi}{\partial x} = \frac{S^{-1}(S(x)+S(y))}{(x+y)^2}\left(\underbrace{\frac{(x+y)\times S'(x)}{S^{-1}(S(x)+S(y))\times S'\circ S^{-1}(S(x)+S(y))}}_{\equiv \psi(x,y)} - 1\right).$$

Let $z = S^{-1}(S(x)+S(y))$. By definition, $S(z) = S(x)+S(y)$. Moreover, by subadditivity of S, $z > x+y$. Assume first that $x \leq y$. Note that

$$\begin{aligned}
\psi(x,y) &= \frac{(x+y)S'(x)}{zS'(z)},\\
&= \frac{(x+y)S'(x)/(S(x)+S(y))}{zS'(z)/S(z)},\\
&= \frac{\frac{xS'(x)}{S(x)}\frac{S(x)}{S(x)+S(y)} + \frac{yS'(x)}{S(y)}\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)},\\
&\geq \frac{\frac{xS'(x)}{S(x)}\frac{S(x)}{S(x)+S(y)} + \frac{yS'(y)}{S(y)}\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \text{ by concavity of } S \text{ (see Lemma 1)},\\
&= \frac{\varepsilon(x)\frac{S(x)}{S(x)+S(y)} + \varepsilon(y)\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)},\\
&> \frac{\varepsilon(z)\frac{S(x)}{S(x)+S(y)} + \varepsilon(z)\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \text{ since } \varepsilon \text{ is decreasing (see Lemma 1)},\\
&= 1.
\end{aligned}$$

Therefore, $\partial\xi/\partial x > 0$ whenever $x \leq y$.

Next, assume for a contradiction that $\psi(x,y) \leq 1$ for some $x > y$. Take the smallest such x. By continuity, this x exists, and satisfies $x > y$ (as shown in the first step of the proof) and $\psi(x,y) = 1$. Note that

$$\begin{aligned}
\frac{\partial \psi}{\partial x} &= \frac{1}{(zS'(z))^2}\left(\left(S'(x)+(x+y)S''(x)\right)zS'(z) - (x+y)S'(x)\left(S'(x)+S'(x)\frac{zS''(z)}{S'(z)}\right)\right),\\
&= \frac{1}{(zS'(z))^2}\left((x+y)S''(x)zS'(z) - (x+y)(S'(x))^2\frac{zS''(z)}{S'(z)}\right), \text{ since } \psi(x,y)=1,\\
&= \frac{(x+y)z}{(zS'(z))^2}\left(S''(x)S'(z) - (S'(x))^2\frac{S''(z)}{S'(z)}\right),\\
&= \frac{(x+y)z(S'(x))^2S'(z)}{(zS'(z))^2}\left(\frac{S''(x)}{(S'(x))^2} - \frac{S''(z)}{(S'(z))^2}\right).
\end{aligned}$$

Next, we argue that $S''(\cdot)/(S'(\cdot))^2$ is decreasing. Recall from Lemma 1 that

$$S'(x) = \frac{1}{x}\frac{S(x)(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S(x)^2}.$$

It follows that

$$S''(x) = -\frac{\alpha(2-S(x))(1-S(x))(1-\alpha S(x))S(x)^2}{x^2\left(1-S(x)1+\alpha S(x)^2\right)^3}.$$

Hence,

$$\frac{S''(x)}{(S'(x))^2} = -\frac{\alpha(2-S(x))}{(1-S(x))(1-\alpha S(x))\left(1-S(x)1+\alpha S(x)^2\right)}.$$

Since $S(\cdot)$ is strictly increasing, the above expression is strictly decreasing in x if and only if

$$\varphi(s) = \frac{\alpha(2-s)}{(1-s)(1-\alpha s)\left(1-s1+\alpha s^2\right)}$$

is strictly increasing in s. Routine calculations show that $\varphi'(s) > 0$ for every $s \in (0,1)$ and $\alpha \in (0,1]$. Therefore, $\partial\psi(x,y)/\partial x > 0$. It follows that $\psi(x',y) < 1$ in a small neighborhood to the left of x. This contradicts the definition of x. We can conclude that ξ is increasing in both of its arguments, which proves the first part of the lemma.

To prove the second part of the lemma, note that

$$\frac{\hat{T}^M - (T^f + T^g)}{T^{f'} + T^{g'}} > \frac{\hat{T}^M - (T^f + T^g)}{T^f + T^g} > \frac{\hat{T}^{M'} - (T^{f'} + T^{g'})}{T^{f'} + T^{g'}},$$

where the first inequality follows from the fact that $T^f + T^g > T^{f'} + T^{g'}$ and the second inequality follows from the first part of the lemma. □

F External Effects

F.1 Preliminaries

We first derive the formula for $\eta(H)$:

Lemma 20. *$\eta(H)$ is given by:*

$$\eta(H) = -1 + \sum_{f\in\mathcal{O}} \phi(s^f, \alpha),$$

where $s^f = S(T^f/H)$, and

$$\phi(s,\alpha) = \frac{\alpha s(1-s)}{(1-\alpha s)(1-s+\alpha s^2)}, \quad \forall s \in (0,1),\ \forall \alpha \in (0,1].$$

Proof. This follows from the definition of η and from the fact that

$$\begin{aligned} xm'(x) &= x\alpha\frac{S'(x)}{(1-\alpha S(x))^2}, \text{ since } m(x) = \frac{1}{1-\alpha S(x)}, \\ &= \frac{\alpha}{(1-\alpha S(x))^2}\frac{S(x)(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S(x)^2}, \text{ by Lemma 1,} \\ &= \frac{\alpha S(x)(1-S(x))}{(1-\alpha S(x))(1-S(x)+\alpha S(x)^2)}, \\ &= \phi(S(x),\alpha). \end{aligned}$$

□

Next, we put on record the following facts about the function ϕ:

Lemma 21. *Let $\hat{\alpha} = \frac{1}{2} + \frac{\sqrt{33}}{18} \simeq 0.82$. The function ϕ has the following properties:*

(a) For every $s \in (0,1)$, $\phi(s,\cdot)$ is strictly increasing.

(b) If $\alpha \leq \hat{\alpha}$, then $\phi(s,\alpha) \leq s$ for every $s \in (0,1)$.

Moreover, if $\alpha > \hat{\alpha}$, then there exist thresholds $s^*(\alpha) \in (0,1]$ and $\hat{s}(\alpha) \in (1/4,1)$ such that:

(c) $\phi(\cdot,\alpha)$ is strictly increasing on $(0,s^*(\alpha))$ and strictly decreasing on $(s^*(\alpha),1)$.

(d) $\phi(\cdot,\alpha)$ is strictly convex on $(0,\hat{s}(\alpha))$ and strictly concave on $(\hat{s}(\alpha),1)$.

Proof. We prove the lemma (analytically) using Mathematica. Mathematica files are available upon request. □

F.2 Proof of Proposition 16

Proof. If $\alpha \leq \hat{\alpha}$, then, by Lemma 21, $\phi(x,\alpha) \leq x$ for every $x \in (0,1)$. As outsiders' market shares add up to strictly less than 1, Lemma 20 immediately implies that any infinitesimal CS-decreasing merger has a negative external effect. Hence, any (not necessarily infinitesimal) CS-decreasing merger has a negative external effect.

Next, suppose $\alpha > \hat{\alpha}$, and define

$$\mathcal{S} = \bigcup_{n\geq 1} \mathcal{S}^n, \text{ where } \mathcal{S}^n = \{s \in [0,1]^n : \sum_{i=1}^n s_i \leq 1\} \ \forall n \geq 1,$$

$$\bar{\mathcal{S}} = \bigcup_{n\geq 1} \mathcal{S}^n, \text{ where } \bar{\mathcal{S}^n} = \{s \in [0,1]^n : \sum_{i=1}^n s_i = 1\} \ \forall n \geq 1,$$

and

$$\Psi(\alpha) = \sup_{s\in\mathcal{S}} \sum_s \phi(\cdot,\alpha), \ \forall \alpha \in (\hat{\alpha},1],$$

where

$$\sum_s \phi(\cdot,\alpha) \equiv \sum_{i=1}^{n} \phi(s_i,\alpha),\ \forall s=(s_i)_{1\le i\le n}\in\mathcal{S},\ \forall \alpha\in(0,1].$$

Clearly, since $\phi(x,\alpha)\ge 0$ for all x, we have that $\Psi(\alpha)=\sup_{s\in\bar{\mathcal{S}}}\sum_s \phi(\cdot,\alpha)$. Next, we claim that $\Psi(\alpha)=\sup_{s\in\bar{\mathcal{S}}^4}\sum_s \phi(\cdot,\alpha)$. To prove this, we show that, for every $s\in\bar{\mathcal{S}}$, there exists $s'\in\bar{\mathcal{S}}^4$ such that

$$\sum_s \phi(\cdot,\alpha)\le \sum_{s'}\phi(\cdot,\alpha).$$

If s belongs to $\mathcal{S}^n$ for some $n\le 4$, or, more generally, if s has at most four components different from zero, then this is obvious. Assume instead that s has five or more components different from zero. Assume without loss of generality that $s\in\bar{\mathcal{S}}^n$ for some $n\ge 5$, that $s_i>0$ for every i, and that the components of s_i have been sorted in increasing order. We construct s' by induction.

Let us first define a function ξ, which takes as argument a profile of market shares $\tilde{s}\in\bar{\mathcal{S}}^n$ sorted in increasing order and with strictly positive components, and returns a profile of market shares $\xi(\tilde{s})$ sorted in increasing order and with strictly positive components, such that either $\xi(\tilde{s})\in\bar{\mathcal{S}}^n$, or $\xi(\tilde{s})\in\bar{\mathcal{S}}^{n-1}$. $\xi(\tilde{s})$ is defined as follows:

- If $\tilde{s}_2\ge\hat{s}(\alpha)$ (or if $\tilde{s}\in\mathcal{S}^1$), then $\xi(\tilde{s})=\tilde{s}$.
- If $\tilde{s}_2<\hat{s}(\alpha)$, then do the following:
  - If $\tilde{s}_1+\tilde{s}_2\le\hat{s}(\alpha)$, then form the $(n-1)$-dimensional vector with first component $\tilde{s}_1+\tilde{s}_2$ and remaining components $(\tilde{s}_i)_{3\le i\le n}$, and sort that vector in increasing order to obtain $\xi(\tilde{s})$.
  - If instead $\tilde{s}_1+\tilde{s}_2>\hat{s}(\alpha)$, then form the n-dimensional with first component $\tilde{s}_1+\tilde{s}_2-\hat{s}(\alpha)$, second component $\hat{s}(\alpha)$, and remaining components $(\tilde{s}_i)_{3\le i\le n}$, and sort that vector in increasing order to obtain $\xi(\tilde{s})$.

Note that, since $\phi_\alpha(\cdot)$ is convex on $[0,\hat{s}(\alpha)]$, we have that, for every $\tilde{s}\in\bar{\mathcal{S}}$

$$\sum_{\tilde{s}}\phi(\cdot,\alpha)\le\sum_{\xi(\tilde{s})}\phi(\cdot,\alpha).$$

We can now define the sequence $(s^k)_{k\ge 0}$ by induction: $s^0=s$; $s^{k+1}=\xi(s^k)$ for every $k\ge 0$. Let m^k denote the number of components of s^k greater or equal to $\hat{s}(\alpha)$, and n^k denote the dimensionality of the vector s^k. By definition of ξ and of the sequence $(s^k)_{k\ge0}$, the sequence of integers $(m^k)_{k\ge0}$ (resp. $(n^k)_{k\ge0}$) is non-decreasing (resp. non-increasing) and bounded above by n (resp. bounded below by 1). Therefore, those sequences of integers are eventually stationary: There exists $K\ge 0$ such that $m^k=m^{k+1}$ and $n^k=n^{k+1}$ for every $k\ge K$. It follows that $(s^k)_{k\ge0}$ is also stationary after K. Let s' be the stationary value of the sequence $(s^k)_{k\geq 0}$. Then, by induction on k,

$$\sum_s \phi(\cdot,\alpha) \leq \sum_{s'} \phi_\alpha(\cdot,\alpha).$$

Moreover, s' has at most one component in $[0,\hat{s}(\alpha))$ (for otherwise, $\xi(s')$ would not be equal to s'). Let n' be the dimensionality of the vector s'. We claim that $n' \leq 4$. Suppose $n' > 1$. Then,

$$1 = \sum_{i=1}^{n'} s'_i \geq (n'-1)\hat{s}(\alpha) > \frac{1}{4} \times (n'-1),$$

where the last inequality follows by Lemma 21. Hence, $n' \leq 4$. Having constructed s', we can conclude that

$$\Psi(\alpha) = \sup_{s\in\bar{\mathcal{S}}^4} \sum_s \phi_\alpha(\cdot,\alpha). \tag{21}$$

By continuity of $\phi(\cdot,\alpha)$ (or, rather, of $\phi(\cdot,\alpha)$'s continuous extension to $[0,1]$) and compactness of $\bar{\mathcal{S}}^4$, the maximization problem defined in equation (21) has a solution. Let s be such a solution. Then, by the convexity argument used in the construction of s', s has a most one component in $(0,\hat{s}(\alpha))$. Moreover, since $\phi(\cdot,\alpha)$ is strictly concave on $[\hat{s}(\alpha),1]$, the components of s that are greater or equal to $\hat{s}(\alpha)$ must be equal to each other. It follows that

$$\Psi(\alpha) = \max_{x\in[0,1]} \max\left(\phi(x,\alpha)+\phi(1-x,\alpha), \phi(x,\alpha)+2\phi\left(\frac{1-x}{2},\alpha\right), \phi(x,\alpha)+3\phi\left(\frac{1-x}{3},\alpha\right)\right).$$

We (analytically) solve the above maximization problem using Mathematica. We obtain:

$$\Psi(\alpha) = \begin{cases} \frac{18\alpha}{18-3\alpha-\alpha^2} & \text{if } \alpha \leq \frac{6}{7}, \\ \frac{4\alpha}{4-\alpha^2} & \text{otherwise.} \end{cases}$$

It is straightforward to check that Ψ is strictly increasing, and that $\Psi(\hat{\alpha}) < 1 < \Psi(1)$. The unique solution of equation $\Psi(\alpha) = 1$ on the interval $(\hat{\alpha},1]$ is $\bar{\alpha} = \frac{3}{2}(\sqrt{57}-7)$.

We can conclude. Assume first that $\alpha \in (\hat{\alpha},\bar{\alpha}]$. Then, for every profile of outsiders' market shares $(s^f)_{f\in\mathcal{O}}$,

$$\sum_{f\in\mathcal{O}} \phi(s^f,\alpha) < \phi\left(1-\sum_{f\in\mathcal{O}} s^f,\alpha\right) + \sum_{f\in\mathcal{O}} \phi(s^f,\alpha) \leq \Psi(\alpha) \leq \Psi(\bar{\alpha}) = 1.$$

Therefore, any CS-decreasing merger must have a negative external effect.

Assume instead that $\alpha > \bar{\alpha}$. We first show that there exists an infinitesimal CS-decreasing merger that has a negative external effect. Let $\mathcal{O} = \{1\}$ and $\mathcal{I} = \{2,3\}$. Since $\phi(\cdot,\alpha)$ is continuous and $\phi(0,\alpha) = 0$, there exists $s \in (0,1)$ such that $\phi(s,\alpha) < 1$. Let $T^1 = S^{-1}(s)$,

$T^2 = T^3 = S^{-1}((1-s)/2)$, and $H^0 = 0$. Then, by construction, the pre-merger equilibrium aggregator level is $H = 1$, and market shares are as follows: $s^1 = s$, $s^2 = s^3 = (1-s)/2$. The external effect of an infinitesimal CS-decreasing merger between firms 2 and 3 is given by $\phi(s,\alpha) - 1$, which is strictly negative by construction.

Next, we claim that there exists an infinitesimal CS-decreasing merger that has a positive external effect. Since $\Psi(\alpha) > 1$, there exists $(s_i)_{1\leq i\leq n} \in (0,1]^n$ such that $\sum_{i=1}^n s_i \leq 1$ and $\sum_{i=1}^n \phi(s_i,\alpha) > 1$. By continuity, for $\varepsilon > 0$ small enough, $\sum_{i=1}^n \phi(s_i - \varepsilon, \alpha) > 1$. Let $\mathcal{O} = \{1,\dots,n\}$, $\mathcal{I} = \{n+1, n+2\}$, $s'^i = s_i - \varepsilon$ for every $i \in \mathcal{O}$, $s'^i = \frac{1}{2}\left(1 - \sum_{j=1}^n s'^j\right)$ for $i \in \mathcal{I}$, $T^i = S^{-1}(s'^i)$ for every $i \in \mathcal{I}\cup\mathcal{O}$, and $H^0 = 0$. Then, by construction, an infinitesimal CS-decreasing merger between the insiders has a positive external effect.

Since any CS-decreasing merger can be decomposed into the integral of infinitesimal CS-decreasing mergers, and since a CS-decreasing merger can be made infinitesimal by tweaking the post-merger type of the merged entity, the above existence results extend immediately to non-infinitesimal mergers: If $\alpha > \bar{\alpha}$, then there exist CS-decreasing mergers that have a positive external effect, and CS-decreasing mergers that have a negative external effect. □

F.3 Proof of Proposition 17

Proof. It is easy to show that $s^* \equiv \inf_{\alpha\in[\bar{\alpha},1]} s^*(\alpha) \simeq 0.68$, where $s^*(\alpha)$ was defined in Lemma 21. Let $s = (s^f)_{f\in\mathcal{O}}$ and $s' = (s'^f)_{f\in\mathcal{O}'}$ such that $s \geq_1 s'$, and $s^f \leq s^*$ for every $f \in \mathcal{O}$. There exists an injection $\iota : \mathcal{O}' \longrightarrow \mathcal{O}$ such that $s^{\iota(f)} \geq s'^f$ for every $f \in \mathcal{O}'$. Note that

$$-1 + \sum_{f\in\mathcal{O}'} \phi(s'^f) \leq -1 + \sum_{f\in\mathcal{O}'} \phi(s^{\iota(f)}) \leq -1 + \sum_{f\in\mathcal{F}} \phi(s^f,\alpha),$$

where the first inequality follows by Lemma 21, and the second inequality follows by injectivity of ι and non-negativity of ϕ. This proves the proposition. □

F.4 Proof of Proposition 18

Proof. It is easy to show that $\hat{s} \equiv \inf_{\alpha\in[\bar{\alpha},1]} \hat{s}(\alpha) \simeq 0.29$, where $\hat{s}(\alpha)$ was defined in Lemma 21. Let $s = (s^f)_{f\in\mathcal{O}}$ and $s' = (s'^f)_{f\in\mathcal{O}'}$ such that $s \geq_2 s'$, $s^f \leq \hat{s}$ for every $f \in \mathcal{O}$, and $s'^f \leq \hat{s}$ for every $f \in \mathcal{O}'$. Since $s \geq_2 s'$, those vectors have the same length, and we can assume that $\mathcal{O} = \mathcal{O}' = \{1,\dots,n\}$ without loss of generality. Note that

$$\begin{aligned} -1 + \sum_{f=1}^n \phi(s^f,\alpha) &= -1 + n\int_0^{\hat{s}} \phi(x,\alpha)dP_s(x), \\ &\geq -1 + n\int_0^{\hat{s}} \phi(x,\alpha)dP_{s'}(x), \\ &= -1 + \sum_{f=1}^n \phi(s'^f,\alpha), \end{aligned}$$

where the inequality follows from the convexity of $\phi(\cdot, \alpha)$ on $[0, \hat{s}]$ (see Lemma 21), and the fact that $\int_0^{\hat{s}} x dP_s(x) = \int_0^{\hat{s}} x dP_{s'}(x)$ and $P_{s'}$ second-order stochastically dominates P_s. This proves the proposition. □

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