Democracy Undone.

Systematic Minority Advantage in Competitive Vote Markets*

Alessandra Casella[†] Sébastien Turban[‡]

October 2013

Abstract

We study the competitive equilibrium of a market for votes where the choice is binary and it is known that a majority of the voters supports one of the two alternatives.Voters can trade votes for a numeraire before making a decision via majority rule.We identify a sufficient condition guaranteeing the existence of an *ex ante equilibrium* and show that the condition is satisfied with probability one in a large electorate. In equilibrium, only the most intense voter on each side demands votes, and each demands enough votes to alone control a majority. The probability of a minority victory is independent of the size of the minority and converges to one half, for *any* minority size, when the electorate is arbitrarily large. In a large electorate, the numerical advantage of the majority becomes irrelevant: democracy is undone by the market.

JEL Classification: C62, C72, D70, D72, P16

Keywords: Voting, Majority Voting, Markets for Votes, Vote Trading, Competitive Equilibrium

*We thank Aniol Llorente-Saguer, Tom Palfrey, and participants to seminars at NYU, Yale, and the university of Bologna for their comments; Antoine Arnoud for his help on Theorem 3, and Georgy Egorov for an illuminating discussion of Lemma 1. Casella thanks the National Science Foundation for its support (SES-0617934), Paris-Jourdan Sciences Economiques for its hospitality, and the Straus Institute at the NYU Law School for its support and hospitality.

[†]Columbia University, NBER and CEPR, ac186@columbia.edu

[‡]Caltech, sturban@hss.caltech.edu

1 Introduction

In a broad sense, markets function well in allocating goods. Could they function well in allocating votes? Consider a group taking a single binary decision via majority voting. Majority voting ignores the intensity of voters' preferences, allowing an intense minority to lose to a tepid majority. In markets for goods, prices induce individuals to act according to the relative strength of their preferences. If majority voting were preceded by a market for votes, could intensity of preferences be expressed appropriately ?

Markets for votes have long captured the imagination of philosophers, political scientists, and economists.[1] However, even ignoring ethical objections, finding a convincing answer to the question just raised has proved difficult. The problem is a fundamental non-convexity associated with vote trading: votes are intrinsically worthless, and their value depends on the holdings of votes by all other individuals. Thus, demands are interdependent, and payoffs discontinuous at the point at which a voter becomes pivotal. Both in a market for votes and in log-rolling games, traditional equilibrium concepts such as competitive equilibrium or the core typically fail to exist.[2]

Recently, a possible solution to the failure of equilibrium existence has been suggested. Focusing on a competitive market where voters can trade votes for a numeraire, Casella, Llorente-Saguer and Palfrey (2012) (CLP from now onward) propose the concept of *ex ante competitive equilibrium*: traders are allowed to express probabilistic demands and the market clears in expectation. At the equilibrium price, deviations from market clearing can occur, but they must be unsystematic and unexpected. Ex post, the imbalance between demand and supply is resolved by a rationing rule. CLP show that such an equilibrium exists in a symmetric model where each voter has equal probability of favoring either alternative and where without vote trading the expected outcome of the vote is a tie.[3]

[1] Among economists and political scientists, the 1960's and 1970's in particular saw a large literature on the topic, whether studying trades of votes for votes or buying and selling of votes on a market in exchange for a numeraire. See: Buchanan and Tullock (1962), Coleman (1966, 1967), Park (1967), Wilson (1969), Tullock (1970), Haefele (1971), Kadane (1972), Riker and Brams (1973), Mueller (1967, 1973), Bernholtz (1973, 1974), Ferejohn (1974), Koehler (1975), Schwartz (1977). Among later contributions, see: Piketty (1994), Philipson and Snyder (1996), Kultti and Salonen (2005). For ethical and philosophical discussions of markets for votes, see, for example, Tobin (1970), Marshall (1977), Walzer (1983), Anderson (1993), Sandel (2012), and Satz (2012).

[2] Ferejohn (1974), (Schwartz (1977, 1981), Shubik and van der Heyden (1978), Weiss (1988), Philipson and Snyder (1996).

[3] Kultti and Salonen (2005) also propose a Walrasian approach to vote markets based on probabilistic demands, but do not impose any market clearing condition.

The result addresses the existence problem that has hampered the study of vote trading, and the concept of ex ante equilibrium is found to have good predictive power in a laboratory experiment. The symmetry and the uncertainty assumptions, however, are restrictive and not ideally suited to the original question that motivated the research. What we want to know is whether minority voters can buy enough votes from the majority to overcome their numerical inferiority, *when both groups are aware of their minority and majority status.* The interest is not only theoretical. In most applications, the two opposing groups are well informed about which side is holding the majority: sides are not equal-sized and are well-established by party labels, cultural and geopolitical characteristics, or historical voting patterns. This is the environment we study in this paper.

The difficulty is that the more precise information exacerbates the non-convexity problem associated with votes. The literature has conjectured, plausibly, that any equilibrium in a market for votes requires uncertainty about the alternative preferred by a majority of the voters.[4] This paper studies and eventually contradicts this conjecture. In so doing, it establishes two general points. First, the obstacles to equilibrium existence in a competitive market for votes are logically unrelated to uncertainty about the direction of preferences. Indeed, our results hold identically under different informational assumptions, and under both complete and incomplete information, as long as voters know their own majority or minority status.[5] Second, the concept of ex ante competitive equilibrium generalizes to an asymmetric setting: the contribution in CLP is not limited to a knife-edge case. We construct an ex ante equilibrium that extends in intuitive fashion the equilibrium characterized by CLP.

We study a group of voters who take a single binary decision by majority voting. Before voting, individuals can buy and sell votes among themselves in a competitive market, in exchange for a numeraire. No individual is liquidity constrained. We obtain two main results. First, we identify a sufficient condition guaranteeing that an ex ante equilibrium with vote trading exists for arbitrary electorate size and majority/minority partition. The condition rules out the possibility that multiple members of one group all have preferences that are much more intense (in a precise sense) than any member of the opposite group. At

[4]See for example Piketty (1994).

[5]Our results hold under complete information, when each voter's direction and intensity of preferences are publicly known. But they also hold if intensities of preferences are private information, and in this case they hold under different scenarios: when each voter's individual membership in the majority or minority is publicly known; when the sizes of the two groups are known, but other voters' individual membership is not, and they hold when voters know their own minority or majority status, but cannot estimate precisely the size of the two groups.

small electorate sizes, we find such likelihood to be high for standard intensity distributions–for example, if the minority is a third of the electorate and the distribution of intensities is uniform, the equilibrium exists with probability larger than 98 percent with nine voters, and larger than 99.9 percent with 21 voters. The stronger conclusion, however, concerns large electorates, where an ex ante equilibrium with trade exists with probability arbitrarily close to 1, for any intensity distribution and any minority share.

Second, the equilibrium we characterize has strong properties that translate into a systematic bias in favor of the minority, relative to the efficient outcome: for any electorate size, any majority/minority partition, and any distribution of intensities, the minority wins more frequently than efficiency dictates. In equilibrium, only the highest intensity member of each group demands votes with positive probability; all other individuals offer their vote for sale. Of the two voters who are potential buyers, it is the voter belonging to the minority who is more aggressive: he may demand to buy with higher probability than the majority voter even when his intensity is lower. Together, these properties imply that the market works not only to weaken but to erase the advantage enjoyed by the majority. Because all other voters offer to sell their votes, the two highest-intensity individuals must each demand enough votes to single-handedly control a majority. Their distinct status as minority or majority members becomes irrelevant. Again, this is particularly clear in large electorates. In such settings, the minority is always expected to win half of the time, for any distribution of intensities and *regardless of its share of the electorate*. As we summarize in the title of this paper: democracy–the power of majority rule–is undone by the market: the numerical superiority of the majority loses all its significance.

The market for votes always falls short of the first best. How it compares to majority voting with no trade depends on the shape of the distribution of intensities. In a large electorate, however, the bias in favor of the minority is strong enough that ex ante welfare is always lower than in the absence of trade, for any distribution of intensities. Because the minority always wins with probability one half, the welfare loss is larger the smaller the minority size: the expected loss can be quantified precisely and is inversely related to the minority size.

The equilibrium we construct echoes the equilibrium in CLP: a vote market leads individuals to either demand a majority of votes or sell. The robustness of this finding to the existence of asymmetric, known groups with opposite preferences suggests that, by re-establishing existence, the concept of ex ante equilibrium allows us to tap into a deeper vein of economic intuition. Votes per se are worthless; what is traded is decision power. The market comes to resemble an auction for decision power between the two individuals who have most to gain from controlling it. The aggregate values of the two opposing groups are not internalized and the final outcome is inefficient, but the market functions as we should have expected.

In addition to supporting this interpretation of a market for votes, the asymmetric model studied here delivers a number of novel predictions. First, because in both groups most individuals are offering their vote for sale, demand for additional votes is just as likely to arise from the majority as from the minority. Second, in equilibrium, intra-group trade and super-majorities always arise with high probability, even though votes command a positive price and the majority size is known. The intuition is clear: high intensity individuals need to preempt sales to the opposite group by their own weak allies. We believe that the predictions are empirically very plausible, but intra-group trades are absent from all vote-buying models we are familiar with.[6]

Beyond its strict tie to the existing studies of vote markets, this paper is related to two other strands of literature. First, there is the important but different literature where candidates or lobbies buy voters' or legislators' votes: for example, Myerson (1993), Groseclose and Snyder (1996), Dal Bò (2007), Dekel, Jackson and Wolinsky (2008) and (2009). These papers differ from the problem we study because in our case vote trading happens *within* the committee (or the electorate). The individuals buying votes are members of the group themselves. This matters because it adds a public good aspect to vote trades: purchases of votes help all members of one's group and hurt all members of the opposite group. It is the majority's stronger temptation to free-ride that is responsible for the market's pro-minority bias.

Second, vote markets are not the only remedy advocated for majority rule's failure to recognize intensity of preferences in binary decisions. The mechanism design literature has proposed mechanisms with side payments, building on Groves-Clarke taxes (e.g., d'Apremont and Gerard-Varet 1979). However, these mechanisms have problems with bankruptcy, budget balance, and collusion (Green and Laffont 1979, Mailath and Postlewaite 1990). A recent literature suggests combining insights from mechanism design into the design of voting rules.

[6]Groseclose and Snyder's (1996) conclusion that vote-buying leads to supermajorities has the same flavor but a different origin. Their paper studies vote-buying in a legislature by two competing outside buyers, as opposed to vote trading among voters, and their result is due to the buyers taking turns in proposing a deal to the legislators, as opposed to the one-shot market studied here.

Goeree and Zhang (2012) and Weyl (2012) propose allowing voters to purchase votes from a central agency at a price equal to the square of the number of votes purchased, a scheme with strongly desirable asymptotic properties. Casella (2005, 2012), Jackson and Sonnenschein (2007), and Hortala-Vallve (2012) propose mechanisms without transfer that allow agents to express the relative intensity of their preferences by linking decisions across issues. Casella, Gelman and Palfrey (2006), Engelmann and Grimm (2012), and Hortala-Vallve and Llorente-Saguer (2010) test the performance of these mechanisms experimentally and find that efficiency levels are very close to theoretical equilibrium predictions, even in the presence of some deviations from theoretical equilibrium strategies.

The rest of the paper is organized as follows. Section 2 presents the model; section 3 characterizes the ex ante equilibrium whose properties we discuss in the rest of the paper; section 4 studies the expected frequency of minority victories and expected welfare, and compares these measures to the equivalent measures in the absence of a vote market and in the utilitarian first best. Section 5 discusses the robustness of the results to alternative assumptions about information, the rationing rule and the stochastic process generating intensities. Section 6 concludes. The Appendix collects the proofs.

2 The Model

A committee of size n (odd) must decide between two alternatives, A and B. The committee is divided into two groups with opposite preferences: M individuals prefer alternative A, and m prefer alternative B, with $m = n - M < M$. We will use M and m to indicate both the sizes of the two groups and the groups' names. Each individual knows whether he belongs to M or to m. Individuals differ not only in the direction of their preferences, but also in their intensity. The model can incorporate different informational assumptions. For concreteness, we conduct our analysis in terms of two opposing parties, whose sizes and compositions are publicly known, while individual intensities of preferences are private information. In a later section, we discuss alternative assumptions.

The remainder of the model is borrowed follows CLP. We summarize it here briefly. Intensity is indicated by a value v_i representing the utility that individual i attaches to obtaining his preferred alternative, relative to the competing one: thus the utility experienced by i as a result of the committee's decision is v_i if i's preferred alternative is chosen, and 0 if it is not. We will use *intensity* and *value* interchangeably. Individual values are independent draws from a common and commonly known distribution $F(v)$ with support $[0,1]$. We call $\mathbf{v}$ the vector of realized values.

Each individual is endowed with one indivisible vote. The group decision is taken through majority voting. Prior to voting, however, individuals can purchase or sell votes among themselves in exchange for a numeraire. The trade of a vote is an actual transfer of the vote and of all rights to its use. We normalize each voter's endowment of the numeraire to zero and allow borrowing at no cost. The important point is that no voter is budget constrained and all are treated equally. Individual utility u_i is given by:

$$u_i = v_i I + t_i \tag{1}$$

where I equals 1 if i's preferred decision is chosen and 0 otherwise, and t_i is i's net monetary transfer, positive if i is a net seller of votes, or negative if he is a net buyer.[7]

With two alternatives and a single voting decision, voting sincerely is always a weakly dominant strategy, and we restrict our attention to sincere voting equilibria: after trading, each individual casts all votes in his possession, if any, in support of the alternative he prefers. Our focus is on the vote trading mechanism, and specifically on a competitive spot market for votes.

We allow for probabilistic (mixed) demands. Let $S = \{s \in \mathbb{Z} \geq -1\}$ be the set of possible pure demands for each agent, where $\mathbb{Z}$ is the set of integers, and a negative demand corresponds to supply: agent i can offer to sell his vote, do nothing, or demand any positive integer number of votes. The set of strategies for each voter is the set of probability measures on S, ΔS, denoted by Σ. Elements of Σ are of the form $\sigma : S \to [0,1]$ where, for each voter, $\sum_{s \in S} \sigma(s) = 1$ and $\sigma(s) \geq 0$ for all $s \in S$.

If individuals adopt mixed strategies, the aggregate amounts of votes demanded and of votes offered need not coincide ex post. A rationing rule R maps the profile of voters' demands to a feasible allocation of votes. Indicating vectors by bold symbols, we denote the set of feasible vote allocations by $X = \{\mathbf{x} \in \mathbb{N}^n | \sum x_i = n\}$. The rule R is a function from realized demand profiles to the set of probability measures over vote allocations: $R : S^n \to \Delta X$. For all $\mathbf{s} \in S^n$, for any $\mathbf{x}$ in the support of $R(\mathbf{s})$, we require $x_i \in [\min(1, 1 + s_i), \max(1, 1 + s_i)]\ \forall i$, and $\mathbf{x} = \mathbf{1} + \mathbf{s}$ with probability 1 if $\sum s_i = 0$. In words, no voter with

[7] Normalizing the endowment of the numeraire to zero and allowing borrowing simplifies the notation. The analysis would be identical if each voter were granted an endowment of one unit of numeraire, with no borrowing.

positive demand can be required either to buy more votes than he demanded, or to sell his vote; no voter who offered his vote for sale can be required to buy votes, and all demands must be respected if they are all jointly feasible.

The particular mixed strategy profile, $\sigma \in \Sigma^n$, and the rationing rule, R, imply a probability distribution over the set of final vote allocations that we denote as $r_{\sigma,R}(\cdot)$. For every possible allocation $\mathbf{x} \in X$, we denote by $\varphi_{i,x}$ the probability that the committee decision coincides with voter i's favorite alternative. Thus, given some strategy profile σ, the rationing rule R, a vote price p, and equation (1), the ex ante expected utility of voter i is given by:

$$U_i(\sigma, R, p) = \sum_{\mathbf{x} \in X} r_{\sigma,R}(x) \left[\varphi_{i,x} v_i - (x_i - 1) p\right] \tag{2}$$

Each individual makes his trading and voting choices so as to maximize (2).

2.1 The Definition of Equilibrium

To allow for the existence of mixed strategies, we must depart from requiring that realized demand always clear the market at the equilibrium price. The concept of *ex ante competitive equilibrium* substitutes the traditional requirement of market balance with the weaker condition that market demand and supply coincide *in expectation*. The discipline imposed by market equilibrium is softened to the requirement that deviations from market balance be unsystematic and unpredictable.

With two opposing groups of different sizes, the notion of ex ante equilibrium proposed in CLP needs to be extended. The reason is that best response strategies will generally differ across members of the two groups. As a result, even though demands are anonymous, if the equilibrium exists, it will convey information about the direction of preferences associated to each demand, and individual strategies will take that information into account. In the spirit of rational expectations models, we call an equilibrium *fully revealing* if either: (1) the equilibrium price, together with the set of others' equilibrium demands and market equilibrium, fully convey to voter i the direction of preferences associated to each demand; or (2) the information conveyed is partial but voter i has a unique best response, identical to his best response under full information. Thus in a fully revealing equilibrium the price and individual strategies are identical to what they would be with full information. Define $\sigma_i^*(\mathbf{v})$ as individual i's equilibrium strategy when all preferences are known, where $\mathbf{v}$ stands for the vector of realized intensity values. Then:

Definition 1. *The vector of strategies σ^* and the price p^* constitute a fully revealing ex ante competitive equilibrium relative to rationing rule R if the following conditions are satisfied:*

1. *For each agent i, σ_i^* satisfies*

$$\sigma_i^* \in \arg\underset{\sigma_i \in \Sigma}{Max} U_i\left(\sigma_i, \sigma_{-i}^*, R, p^*\right)$$

2. *In expectation, the market clears, i.e.,*

$$\sum_{i=1}^{n} \sum_{\mathbf{s} \in S^n} \sigma_i^*(\mathbf{s}) \cdot \mathbf{s} = 0$$

3. *Given $\{\sigma_{-i}^*, p^*\}$ and the knowledge that the equilibrium is fully revealing,*

$$\sigma_i^* = \sigma_i^*(\mathbf{v}) \text{ for all } i.$$

In equilibrium, individuals select strategies that maximize their expected utility, given the strategies used by others and the price. Demands are interdependent and best-respond to others' demands. In a market for votes, such interdependence is inevitable because the value of a vote depends on the full profile of vote allocations.[8] In competitive equilibrium theory, it is found in analyses of contributions to public goods (for example, Arrow and Hahn. 1971, pp.132-6). In the present setting, with two opposite groups of different sizes, the interdependence of demands plays a second important role. Together with the price, it supports the information revelation that occurs in equilibrium. Surveying the literature on the existence of rational expectations equilibria, Allen and Jordan (1998) identify the "competitive message"–the price and the set of others' demands–as the smallest possible information message that generally supports a fully revealing equilibrium.[9]

In our competitive market, demands are known but anonymous. In a fully revealing equilibrium, however, the other players' demands and the price convey enough information to identify uniquely one's own best response strategy, as if all the information about the direction and intensities of preferences were known. An important corollary is that if a fully

[8]As a transparent example, all remaining votes have zero value if one voter holds a majority on his own.

[9]In general, prices alone cannot reveal all information when the dimensionality of the price set is lower than the dimensionality of the state space, as in our environment, with a single "commodity", a vote, and a large-dimension state space.

revealing equilibrium exists, then it is also an equilibrium of the complete information game. We have assumed above that individual preferences are private information. But everything we say below will apply identically if all preferences are publicly known.[10]

In general, the existence and the characterization of the equilibrium will depend on the rationing rule. Here too we follow CLP and, for most of the analysis, concentrate on a rule called $R1$ or Rationing-by-Voter. $R1$ requires that any positive demand for votes be either satisfied in full, or not at all: for any vector of realized demands $\mathbf{s}$, a final allocation $\mathbf{x}$ must satisfy $x_i \in \{1, 1+s_i\}$ $\forall i$. Under $R1$, any outstanding positive order for votes is equally likely to be selected; the order is satisfied if there exists sufficient outstanding supply to do so fully, in which case the sellers are selected with equal probability among all voters with outstanding offers to sell. If the order cannot be fully satisfied, then it remains void. A second positive order is then randomly selected from those remaining, with equal probability, and the process continues until either all orders are satisfied or the only orders left outstanding are all infeasible. $R1$ is well-suited to a market for votes because the value of a package of votes can change discontinuously with changes of a single unit.[11] In the final section of the paper, we return to the rationing rule and discuss the conditions under which our results are robust to an alternative rule that allows for partially filled orders. Up to that point, all our results are to be read as relative to rationing rule $R1$**.**

An equilibrium with no trade always exists–if no-one else is trading, an individual is rationed with probability one–and is, trivially, fully revealing–strategies are identical to what they would be with full information. Our interest is in equilibria with trade.

If an equilibrium existed in pure strategies, market balance would hold not only ex ante but ex post, and no rationing would occur. We need to allow for mixed strategies and ex ante equilibrium because in a market for votes with two opposing groups of known sizes, no fully revealing competitive equilibrium with trade exists in pure strategies. This result is well-known[12] but we reproduce it here because it is the point of departure of our analysis.

Remark 1. *For all n odd, m, F, and $\mathbf{v}$, there is no price p^* and vector of strategies $\mathbf{s}^*(\mathbf{v}, p^*)$*

[10]Note that the reverse does not hold: an equilibrium of the full information game need not be a fully revealing equilibrium of the incomplete information game, because it may be impossible for an agent to extract all relevant information.

[11]$R1$ resembles All-or-Nothing (AON) orders used in securities trading: the order is executed at the specified price only if it can be executed in full. See for example the description of AON orders by the New York Stock Exchange http://www.nyse.com/futuresoptions/nysearcaoptions/

[12]Ferejohn (1974), Philipson and Snyder (1996), Piketty (1994), Kultti and Salonen (2005), Casella, Palfrey, Turban (2012)

such that $s_i^*(\mathbf{v}, p^*) = \arg\underset{s_i \in S}{Max} U_i\left(s_i, \mathbf{s}_{-i}^*, p^*\right)$ *for all* i *and* $\sum_i s_i^*(\mathbf{v}, p^*) = 0$*, unless* $s_i^*(\mathbf{v}, p^*) = 0$ *for all* i*.*

Proof. The logic is simple. If there is trade, for all $p > 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) \in \{-m, (M - m + 1)/2\}$: if the aggregate demand of minority voters is positive, it must equal the minimum number of votes required to win; alternatively, at any positive price all losing votes must be offered for sale. But $\sum_{i \in M} s_i^*(\mathbf{v}, p) \leq 0$: in equilibrium, the aggregate demand by majority voters cannot be positive. In addition, $\sum_{i \in M} s_i^*(\mathbf{v}, p) \neq -(M - m + 1)/2$: if $(M - m + 1)/2$ votes were traded, the remaining $(M + m - 1)/2$ votes collectively held by M voters would be worthless and thus offered for sale too. Thus for all $p > 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) + \sum_{i \in M} s_i^*(\mathbf{v}, p) \neq 0$. If $p = 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) \geq (M - m + 1)/2$,[13] but $\sum_{i \in M} s_i^*(\mathbf{v}, p) \geq -(M - m - 1)/2$, because the only supply can come from M voters whose vote is not pivotal. Thus for $p = 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) + \sum_{i \in M} s_i^*(\mathbf{v}, p) > 0$. $\square$

The question this paper addresses then is whether a fully revealing *ex ante* competitive equilibrium with trade exists, given the knowledge of m and M.

3 Equilibrium Existence and Characterization

In this section we derive two theorems. Theorem 1 identifies a sufficient condition guaranteeing that an ex ante equilibrium with trade exists. Theorem 2 shows that with large electorates the sufficient condition must be satisfied with probability arbitrarily close to 1.

Given realized values $\mathbf{v}$, we denote by $v_{(1)}$ the highest realized value; by $G \in \{m, M\}$ the group such that $v_{(1)} \in G$–the group to which the highest intensity individual belongs–, and by g the opposite group. We call $\overline{v}_G$ ($\overline{v}_g$) the highest realized value in G (g) (thus by definition $\overline{v}_G = v_{(1)}$).[14] Finally, we denote by $v_{(2)G}$ the second highest value in G: $v_{(2)G} = \max(v_i \in \{G \setminus \overline{v}_G\})$.

Theorem 1. *For all* n *odd,* m *and* F*, there exists a threshold* $\mu(n) \in (0, 1)$ *such that if* $\overline{v}_g \geq \mu(n) v_{(2)G}$*, there exists a fully revealing ex ante equilibrium with trade where* $\overline{v}_G$ *and* $\overline{v}_g$

[13] We are assuming that at $p = 0$, voters on the losing side demand rather than sell votes. This is equivalent to the standard assumption that goods are in excess demand at 0 price.

[14] Throughout the paper, we use v_i to denote the value of i but also occasionally, with abuse of notation, the name of voter i. We use the notation $v_{(1)}$ to indicate the highest draw, as opposed to the more standard $v_{(n)}$, for consistency with $v_{(2)G}$.

randomize between demanding $\frac{n-1}{2}$ votes (with probabilities $q_{\overline{G}}$ and $q_{\overline{g}}$ respectively) and selling their vote, and all other individuals sell. The randomization probabilities $q_{\overline{G}}$ and $q_{\overline{g}}$ and the price p depend on n, G, $\overline{v}_g$, and $\overline{v}_G$, but for all G, $\overline{v}_G$ and $\overline{v}_g \geq \mu(n)v_{(2)G}$, $q_{\overline{G}} \in \left[\frac{n-1}{n+1}, 1\right]$ and $q_{\overline{g}} \in \left[\frac{n-1}{n+1}, 1\right]$. The threshold $\mu(n)$ is given by:

$$\mu(n) = \begin{cases} \frac{2}{3} & \text{if } n = 3 \\ \max\left\{\frac{(n-2)(n-1)}{2(n^2+n-5)}, \frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)}\right\} & \text{if } n > 3 \end{cases} \tag{3}$$

The theorem is proved in the Appendix, where we also report the explicit solutions for $q_{\overline{G}}$, $q_{\overline{g}}$ and p.

An important observation is that $\mu(n) < 1$ for all n, and $\mu(n) < 1/2$ for all $n > 3$.[15] The condition $\overline{v}_g \geq \mu(n)v_{(2)G}$ is necessary and sufficient for the existence of the equilibrium characterized in the theorem, and is thus sufficient for the existence of a fully revealing ex ante equilibrium with trade.[16]

The need to account for all possible rankings in the value realizations of the two groups– $v_{(1)}$ may belong to M or to m; and so may $v_{(2)}$, $v_{(3)}$, ..–explains the notation, but an example will help make the theorem more transparent. Suppose $\overline{v}_m \geq \overline{v}_M \geq v_{(2)m}$: the (weakly) highest intensity voter belongs to the minority, and the second (weakly) highest to the majority. In this case, $G = m$, $g = M$, and $\overline{v}_g \geq v_{(2)G}$, implying that the condition in the theorem is satisfied. Then there exists an ex ante fully revealing equilibrium where all individuals with the exception of $\overline{v}_m$ and $\overline{v}_M$ offer their vote for sale; $\overline{v}_m$ and $\overline{v}_M$ randomize between offering their vote for sale and demanding $(n-1)/2$ votes, enough to yield the buyer a strict majority of votes.[17]

The theorem says that if the sufficient condition is satisfied, an equilibrium exists that always assumes this form, *regardless of the realized rankings in the values of the two groups*: the highest-value individual belonging to M and the highest-value individual belonging to m compete for dictatorship, while all others sell their votes. If the sufficient condition is satisfied, the equilibrium exists whether $G = m$, as in the example, or $G = M$, and because $\mu(n) < 1$ for all n, the equilibrium exists whether the two highest value voters are opposite

[15]For all $n > 3$, $\mu(n)$ is increasing in n, and approaches $1/2$ asymptotically for n arbitrarily large.

[16]Theorem 1 does not state that no fully revealing equilibrium with trade exists if $\overline{v}_g < \mu(n)v_{(2)G}$. In a specific example ($M = 3$, $m = 2$), we have constructed such an equilibrium for some value realizations that violate the condition (Casella, Palfrey and Turban, 2012).

[17]In the specific case $\overline{v}_m \geq \overline{v}_M \geq v_{(2)m}$, $q_{\overline{m}} = 1$, $q_{\overline{M}} = (n-1)/(n+1)$, and $p = (2\overline{v}_M)/(n+1)$.

sides, as in the example, or on the same side, as long as $\overline{v}_g \geq \mu(n)v_{(2)G}$. The price p and the mixing probabilities, $q_{\overline{G}}$ and $q_{\overline{g}}$, depend on $\overline{v}_G$, $\overline{v}_g$, and on whether $G = m$, or $G = M$, but the structure of the equilibrium is unchanged. For clarity, recall that individual preferences are private information: the group membership of the two highest-value voters and a voter's own position in the values' ranking are revealed in equilibrium.

When it exists, the equilibrium recalls the equilibrium in CLP. In that paper's symmetric environment, the competition for dictatorship is between the two highest-value individuals overall; here it is between the two individuals with highest value *and* opposite preferences. The conclusion that the vote market does not allocate votes smoothly among higher value individuals seems counter-intuitive, but the robustness of the result to the different assumptions in the two models suggests a central aspect of markets for votes. Votes have no value in themselves, and in this equilibrium a well-functioning market for votes approximates a market for decision power. The market allocates decision power to one of the two individuals with the highest incentive to compete for it.

In the scenario studied here, with two opposing groups of different sizes, the equilibrium has a number of additional features. The first and most striking is that both the existence and the properties of the equilibrium do not depend directly on the size of the minority m. The value of m affects the probabilities of the inter-party ranking in the realizations of values $\mathbf{v}$, but, given n and $\mathbf{v}$, if the equilibrium exists, the strategies and the price are identical whether $m = 1$ or $m = M - 1$. The intuition is clear: since all individuals but $\overline{v}_m$ and $\overline{v}_M$ always offer their vote for sale, the precise numerical advantage of the majority is irrelevant in equilibrium: either $\overline{v}_m$ too offers his vote for sale, and the majority wins, for any m; or $\overline{v}_m$ demands $(n-1)/2$ votes, and any demand by $\overline{v}_M$ lower then $(n-1)/2$ results in defeat with probability 1, for any m.

Second, there is a positive probability that the only realized purchases are made by $\overline{v}_M$, that is, by the majority. The result is less paradoxical than it seems: all other majority members are offering their votes for sale, and $\overline{v}_M$ buys to prevent the transfer of votes to the minority. Preemptive purchases by the majority are very plausible: any sponsor of a bill needs to worry about the support of his weakest allies. But to our knowledge they have no role in usual formalizations of vote trading. For the same reason, the equilibrium predicts intra-group trading with high probability for all m and M. Again, most voters are offering their vote for sale, and high value individuals need to preempt sales to the opposite group by their own weak allies.

Finally, unless all of one's group votes are purchased, the winning majority will be larger than the minimal winning coalition. Thus in general the equilibrium predicts super-majority, a counter-intuitive result in a market for votes where votes command a positive price and the number of additional votes the minority needs to win is common knowledge.

The explicit solutions for $q_{\overline{G}}$, $q_{\overline{g}}$ and p are in the Appendix because, although quite simple, they are not very enlightening: we need to consider different cases, depending on the realizations of $\overline{v}_g$ and $\overline{v}_G$. One property of the randomization probabilities, however, deserves notice, and we report it in the following remark:

Remark 2. *In the equilibrium in Theorem 1, there exist realizations of values v such that $q_{\overline{g}} > q_{\overline{G}}$ if and only if $g = m$.*

Thus not only do $\overline{v}_M$ and $\overline{v}_m$ demand the same number of votes, if they demand votes at all, but the minority's strategy is weakly more aggressive: $q_{\overline{m}} > q_{\overline{M}}$ whenever $\overline{v}_m > \overline{v}_M$ *and* over a range of values such that $\overline{v}_m \leq \overline{v}_M$. It is not difficult to see why: if no trade is concluded, $\overline{v}_M$ is sure to win, while $\overline{v}_m$ is sure to lose. The less desirable outside option predisposes $\overline{v}_m$ towards buying.

Figure 1 represents graphically the regions of values over which the equilibrium described in Theorem 1 exists and uses different colors to describe the equilibrium mixing probabilities. In all panels, the vertical axis measures $\overline{v}_g/\overline{v}_G$ and the horizontal axis $v_{(2)G}/\overline{v}_G$, and thus both axes range between 0 and 1. The panels on the left are drawn for the case $G = m$ and the panels on the right for $G = M$. The upper panels correspond to $n = 9$, and the lower panels to $n = 21$. Because the existence and characterization of the equilibrium do not depend on the size of the minority, the figure applies for any $m < M$, as long as $v_{(2)m}$ exists and thus $m \geq 2$.[18]

In all panels, the equilibrium exists above the line $\overline{v}_g = \mu(n)v_{(2)G}$. Blue areas correspond to an equilibrium where $\overline{v}_m$, demands $(n-1)/2$ votes with probability 1; $\overline{v}_M$ demands $(n-1)/2$ votes with probability $(n-1)/(n+1)$ and sells his vote otherwise, and all other voters sell. In line with Remark 2 above, such an equilibrium exists not only when the highest value belongs to the minority (the panels on the left) but also when the highest value belongs to the majority (the panels on the right) as long as $\overline{v}_m$ is high enough, relative to $\overline{v}_M$–higher than a value $\overline{\rho}(n)\overline{v}_M < \overline{v}_M$ that appears as the upper horizontal line in the panels on the

[18] If $m = 1$, the panel on the right ($G = M$) is unchanged; the panel on the left ($G = m$) has no white area in the lower right corner because the condition $\overline{v}_M > \mu(n)v_{(2)m}$ is trivially satisfied.

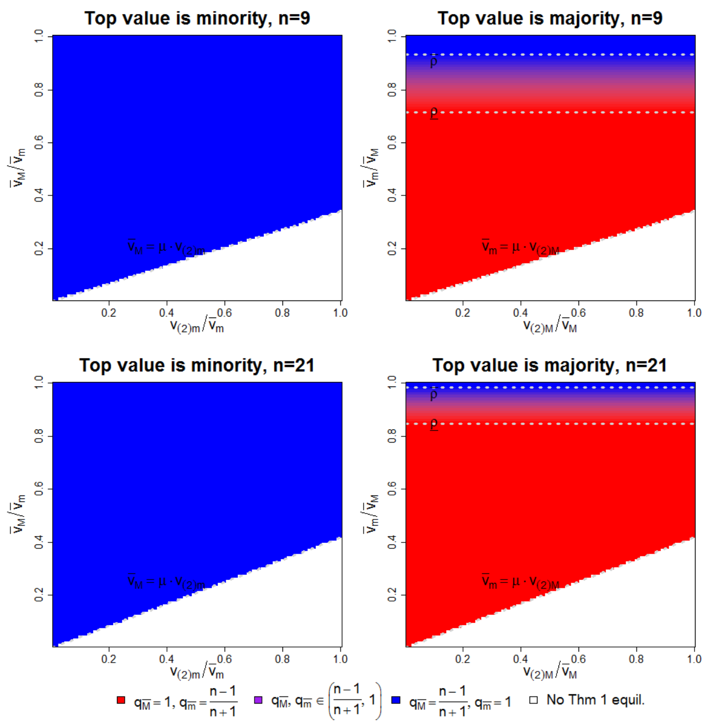

Figure 1: Equilibrium strategies in Theorem 1, as function of $\overline{v}_G$, $\overline{v}_g$, and $\overline{v}_{(2)G}$.

right. The red area corresponds to an equilibrium where $\overline{v}_M$ demands $(n-1)/2$ votes with probability 1, $\overline{v}_m$ demands $(n-1)/2$ votes with probability $(n-1)/(n+1)$ and sells his vote otherwise, and all other voters sell. Such an equilibrium exists when the highest value belongs to the majority and $\overline{v}_m$ is low enough, relative to $\overline{v}_M$–lower than a value $\underline{\rho}(n)\overline{v}_M < \overline{\rho}(n)\overline{v}_M$ that appears as the lower horizontal line in the panels on the right. Both $\underline{\rho}(n)$ and $\overline{\rho}(n)$ are defined precisely in the Appendix; for all n they satisfy $1/2 \leq \underline{\rho}(n) < \overline{\rho}(n) < 1$, and both converge to 1 at large n. Finally, in the purple area, for $\overline{v}_m \in (\underline{\rho}(n)\overline{v}_M < \overline{\rho}(n)\overline{v}_M)$, both $\overline{v}_m$ and $\overline{v}_M$ randomize between demanding $(n-1)/2$ votes, with probabilities $q_{\overline{m}}$ and $q_{\overline{M}}$ strictly between $(n-1)/(n+1)$ and 1, and selling their vote, and all others sell. The values of $\mu(n)$, $\underline{\rho}(n)$, and $\overline{\rho}(n)$, and thus the exact borders between the different areas, depend on n, but qualitatively the figure is unchanged for all n.

Figure 1 represents the equilibrium strategies in Theorem 1 sharply, but could be misleading. It is important to note that the relative size of an area in the figure is *not* informative about the probability with which values in the area are realized. The figure's axes correspond to ratios of order statistics whose realizations depend on F, n, and the size of the two

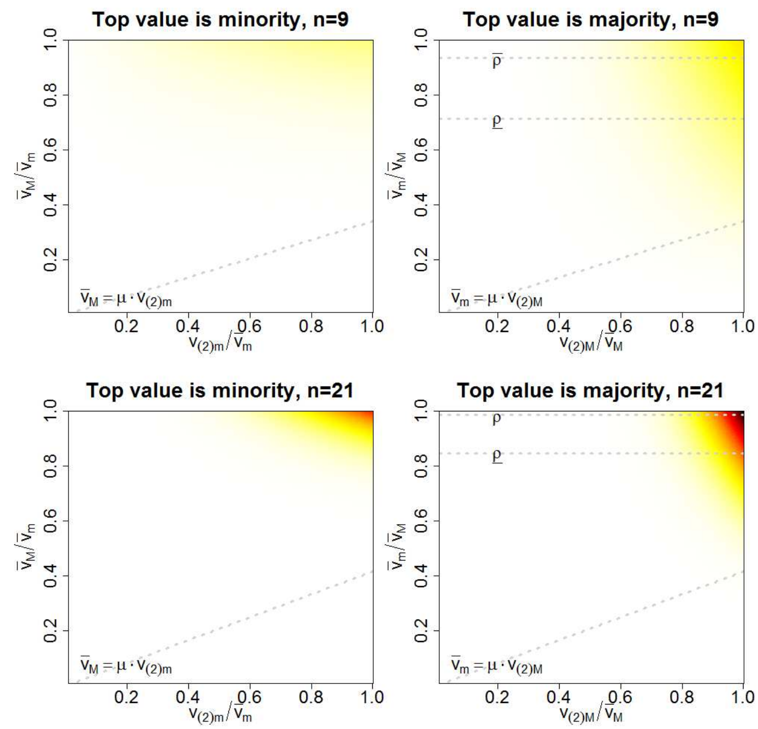

Figure 2: Probability of ordered value realizations; $F(v)$ uniform. A darker shade indicates higher probability.

groups, m and M. Figure 2 reports the same panels drawn in Figure 1, now using shading to represents probability mass: darker areas correspond to value realizations with higher probability. The probabilities were obtained from one hundred million random independent draws from a uniform distribution, fixing $m = (1/3)n$. As in Figure 1, the upper panels report results for $n = 9$, and the lower panels for $n = 21$; the left panels are drawn for the case $G = m$ and the right panels for $G = M$. Because the minority is by definition small, realizations in the right panels always have higher probability than realizations in the left panels, as reflected in the slightly darker shades. Given Remark 2, this does not imply that a majority victory is necessarily more probable than a minority victory.

Figure 2 shows two main regularities: first, in each panel, the probability mass is concentrated in the upper right corner; second, the concentration is stronger at higher n.[19]

[19] The different patterns in the left and right panels reflect the different sizes of the two groups. Because $M > m$, $\overline{v}_M$ is likely to be higher than $v_{(2)m}$ (and thus the probability mass in the left panels concentrates around the upper horizontal boundary), and because $M > m + 1$, $v_{(2)M}$ is likely to be higher than $\overline{v}_m$ (and thus the probability mass in the right panels concentrates around the upper vertical boundary).

The figure gives a clear visual representation, but both results can be obtained analytically. As shown in Figure 1, the realizations of $\overline{v}_g$, $\overline{v}_G$, and $v_{(2)G}$ that support the equilibrium of Theorem 1 can be divided into three areas, corresponding to the different mixing probabilities: blue (B), where $q_{\overline{m}} = 1$, red (R), where $q_{\overline{M}} = 1$, and purple (P), where both $q_{\overline{m}}$ and $q_{\overline{M}} \in (\frac{n-1}{n+1}, 1)$. Call $\Pr(B)$ the probability of value realizations in B, and similarly for R and P. Thus:

$$
\begin{aligned}
\Pr(B) &= \Pr(\overline{v}_m \geq \overline{\rho}\overline{v}_M, \overline{v}_M \geq \mu v_{(2)m}) \\
\Pr(P) &= \Pr(\underline{\rho}\overline{v}_M < \overline{v}_m < \overline{\rho}\overline{v}_M) \\
\Pr(R) &= \Pr(\overline{v}_m \leq \underline{\rho}\overline{v}_M, \overline{v}_m \geq \mu v_{(2)M})
\end{aligned}
$$

Given F, the different probabilities can be calculated. Suppose, for example, that F is uniform. Then:[20]

$$
\begin{aligned}
\Pr(B) &= 1 - \frac{m(m-1)}{n(n-1)}\mu^M - \frac{M}{n}\overline{\rho}^m \\
\Pr(P) &= \frac{M}{n}(\overline{\rho}^m - \underline{\rho}^m) \\
\Pr(R) &= \underline{\rho}^m \frac{M}{n} - \frac{M(M-1)}{n(n-1)}\mu^m
\end{aligned}
\tag{4}
$$

Specific values of n and m will then yield precise numerical values. For example, if $n = 9$ and $m = 3$, as in the upper panels of Figure 2, the probability of falling in the blue area is 47.8 percent, in the red area is 22.6 percent, and in the purple area 29.9 percent. Thus the probability of value realizations for which the equilibrium of Theorem 1 does not exist is 1.7 percent. At $n = 21$ and $m = 7$, as in the lower panels of Figure 2, the numbers become: $\Pr(B) = 0.401$, $\Pr(P) = 0.392$, and $\Pr(R) = 0.206$; the probability of value realizations that do not support the equilibrium of Theorem 1 is less than 1 in $1,000$.

As n increases, both the concentration of probability mass in the upper right corner of each panel and the sharply decreased likelihood of realizations outside the equilibrium area are clear both from the figure and from the numbers. These features arise from the increase in n and are independent, qualitatively, from the uniform distribution assumption used in these examples. If the minority is a non-vanishing fraction of the electorate,[21] then with

[20] See the details in the Appendix.

[21] I.e. $\frac{m}{n}$ is bounded away from 0 as $n \longrightarrow \infty$.

independent draws from any common distribution F, at large n, both $\overline{v}_g/\overline{v}_G$ and $v_{(2)G}/\overline{v}_G$ must approach the upper boundary of the distribution's support. It then follows that when the electorate is large, the restriction on realized values required for the existence of the equilibrium described in Theorem 1 is almost certainly satisfied. Indeed this is our second result. Suppose $m = \lfloor \alpha n \rfloor$, for all n, where $\lfloor \alpha n \rfloor$ is the largest integer not greater than αn, and α is a constant in $(0, 1/2)$. Adding a subscript n to indicate explicitly the dependence on the size of the market, we can state:

Theorem 2. *Consider a sequence of vote markets. For any* $\alpha \in \left(0, \frac{1}{2}\right)$ *and* F, $\lim\limits_{n\to\infty} \Pr{}_n(\overline{v}_{g,n} \geq \mu(n) v_{(2)G,n}) = 1.$

Proof. The proof of the theorem is immediate. Given $\mu(n) < 1/2$, the theorem follows if $\lim_{n\to\infty} \Pr_n(\overline{v}_{g,n} > 1/2) = 1$. But $\lim_{n\to\infty} \Pr_n(\overline{v}_{g,n} > 1/2) = \lim_{n\to\infty} 1 - [F(1/2)]^{\lfloor \alpha n \rfloor} = 1$, and the result is established. $\square$

The uniform distribution provides a transparent example. From (4):

$$\begin{aligned}
&\lim_{n\to\infty} \Pr{}_n(B) = \alpha \\
&\lim_{n\to\infty} \Pr{}_n(P) = (1-\alpha)(1-e^{-4\alpha}) \\
&\lim_{n\to\infty} \Pr{}_n(R) = (1-\alpha)e^{-4\alpha}.
\end{aligned} \tag{5}$$

As predicted, $\lim_{n\to\infty}(\Pr(B) + \Pr(P) + \Pr(R)) = 1$.[22]

The uniform distribution provides a clean example, but Theorem 2 holds generally. It implies that for large n the equilibrium described in Theorem 1 exists with probability that approaches 1. In addition, because in such an equilibrium the probabilities with which $\overline{v}_G$ *and* $\overline{v}_g$ demand $(n-1)/2$ votes are bounded below by $(n-1)/(n+1)$, at large n both probabilities must also approach 1. Theorem 2 thus leads to the following Corollary:[23]

[22] As expected, the probability of $\overline{v}_m/\overline{v}_M$ realizations high enough to support $q_{\overline{m}} = 1$ (the Blue area) increases monotonically with α; conversely, the probability of low enough $\overline{v}_m/\overline{v}_M$ realizations to support $q_{\overline{M}} = 1$ (the Red area) falls monotonically with α; the intermediate case where both $q_{\overline{m}}$ and $q_{\overline{M}} \in (\frac{n-1}{n+1}, 1)$ (the Purple area) is not monotonic in α.

[23] We provide a rigorous proof in the Appendix. In Theorem 1, the mixing probabilities are written for a given $\mathbf{v}$. For any n, we can define random variables $q_{\overline{G},n}(\mathbf{v})$ and $q_{\overline{g},n}(\mathbf{v})$ which take the values given by Theorem 1 if the condition on $\mathbf{v}$ is satisfied, and 0 otherwise.

Corollary 1. *For any* $\alpha \in \left(0, \frac{1}{2}\right)$ *and* F, $\Pr[\lim_{n\to\infty} q_{\overline{G},n}(\mathbf{v}) = 1] = 1$, *and* $\Pr[\lim_{n\to\infty} q_{\overline{G},n}(v) = 1] = 1$

4 Market Outcomes

4.1 Frequency of minority victories

The most unexpected feature of Theorem 1 is that when the equilibrium exists the market outcome depends on the size of the minority only indirectly. As we remarked, if the equilibrium exists, given realized values the expected outcome is the same whether there is a single minority voter or the minority comprises almost half of the electorate. This result suggests a systematic vote market bias in favor of the minority group: a higher frequency of minority victories than efficiency dictates.

To evaluate this conjecture, we need to construct an equilibrium that exists for all value draws, and define an efficiency benchmark. Since an equilibrium with no trade exists trivially for all value realizations, we can construct an equilibrium such that if $\overline{v}_g \geq \mu(n)v_{(2)G}$, then trade occurs and the equilibrium of Theorem 1 is selected; if $\overline{v}_g < \mu(n)v_{(2)G}$, then no vote-trading takes place and the majority wins with probability 1. Our equilibrium construction thus minimizes the frequency of minority victories when the condition is not met.[24] We call θ_m the ex ante expected frequency of minority victories in such an equilibrium, before values are drawn. Recall that $x(\mathbf{v})$ is a random variable denoting a final allocation of votes for a given value profile. Hence: $\theta_m \equiv \Pr_F(\sum_{i\in m} x_i(\mathbf{v}) > \sum_{j\in M} x_j(\mathbf{v}))$.

In line with the anonymity of the competitive market and of majority voting, we measure efficiency by ex ante efficiency, treating each voter identically–expected utility before the voter knows the group he belongs to and before values are drawn. Ex ante efficiency is equivalent to the utilitarian criterion: it is maximized when, for each realization of values, the group with higher aggregate value prevails. We call θ_m^* the expected frequency of minority victories under this efficiency benchmark: $\theta_m^* \equiv \Pr_F(\sum_{i\in m} v_i > \sum_{j\in M} v_i)$. To evaluate whether a systematic pro-minority bias is indeed realized, in this section we compare θ_m to θ_m^*.

We begin by establishing a preliminary result. Because it can be of some general interest,

[24] As noted earlier, equilibria with trade may exist when $\overline{v}_g < \mu(n)v_{(2)G}$, and thus the expected fraction of minority victories must be weakly higher than in our equilibrium construction.

we report it here as a separate lemma.

Lemma 1. *If all v_i, $i \in m$ and $i \in M$, are i.i.d. according to some $F(v)$, then for all F, n, and m ,* $\theta_m^* \leq \frac{1}{1+\binom{M}{m}} \leq \frac{m}{n}$.

The lemma is proved in the Appendix. It states that if values are i.i.d., then for any distribution F the expected share of value configurations such that the aggregate minority value is larger than the aggregate majority value, and thus a minority victory is efficient, cannot be larger than the share of the minority in the electorate. The statement is intuitive and is useful here because it establishes an upper bound for θ_m^* that holds for all F, n, and m and can be compared to θ_m, the equilibrium fraction of expected minority victories.

Conditional on value realizations, $\theta_m(\mathbf{v})$ is either characterized precisely by the strategies in Theorem 1, or equals 0, by our equilibrium construction, if the condition in Theorem 1 is not satisfied. In particular, because under Theorem 1 the final votes' allocation depends only on the probability with which $v_{\overline{m}}$ and $v_{\overline{M}}$ demand votes, we can write:

$$\theta_m(\mathbf{v}) = \begin{cases} q_{\overline{m}}(\mathbf{v})(1 - q_{\overline{M}}(\mathbf{v})) + \frac{1}{2} \cdot q_{\overline{m}}(\mathbf{v}) q_{\overline{M}}(\mathbf{v}) & \text{if } \overline{v}_g \geq \mu(n) v_{(2)G} \\ 0 & \text{if } \overline{v}_g < \mu(n) v_{(2)G} \end{cases}$$

where the equilibrium values of $q_{\overline{m}}$ and $q_{\overline{M}}$ depend on the realized values. It is convenient to refer to the regions of the value space according to their color in Figure 1: recall that Blue (B) corresponds to value realizations such that $\overline{v}_M \in \left[\mu(n) v_{(2)m}, \overline{v}_m / \overline{\rho}(n)\right]$; Red ($R$) corresponds to $\overline{v}_m \in \left[\mu(n) v_{(2)M}, \underline{\rho}(n) \overline{v}_M\right]$, and Purple ($P$) to $\overline{v}_m \in [\underline{\rho}(n) \overline{v}_M, \overline{\rho}(n) \overline{v}_M]$. Then:

$$q_{\overline{m}}(\mathbf{v}) = \begin{cases} 1 \text{ if } \mathbf{v} \in B \\ \frac{n-1}{n+1} \text{ if } \mathbf{v} \in R \\ q'_{\overline{m}} \in (\frac{n-1}{n+1}, 1) > \frac{n-1}{n+1} \text{ if } \mathbf{v} \in P \end{cases} \qquad q_{\overline{M}}(\mathbf{v}) = \begin{cases} \frac{n-1}{n+1} \text{ if } \mathbf{v} \in B \\ 1 \text{ if } \mathbf{v} \in R \\ q'_{\overline{M}} \in (\frac{n-1}{n+1}, 1) < 1 \text{ if } \mathbf{v} \in P \end{cases}$$

Hence:

$$\theta_m \geq \left[\left(1 - \frac{n-1}{n+1}\right) + \frac{1}{2} \left(\frac{n-1}{n+1}\right) \right] \Pr(B) + \left[\frac{1}{2} \left(\frac{n-1}{n+1}\right) \right] \Pr(R) + \left[\frac{1}{2} \left(\frac{n-1}{n+1}\right) \right] \Pr(P)$$

with strong inequality if $\Pr(P) > 0$. Or:

$$\theta_m \geq \left(\frac{n+3}{2(n+1)}\right) \Pr(B) + \left(\frac{n-1}{2(n+1)}\right) [\Pr(R) + \Pr(P)] \equiv \underline{\theta_m} \tag{6}$$

The probability of realizations in the different regions of the value space depends on F, and thus so does θ_m. Yet, as we prove in the Appendix:

Proposition 1. *For all n, m , and F, $\theta_m > \theta_m^*$.*

Relatively to utilitarian efficiency, the market, at least in the equilibrium we have characterized, *always* leads to excessive minority victories. Remarkably, the conclusion holds for all electorate sizes, regardless of the size of the minority and of the shape of the values distribution. An example can help in making the proposition concrete. Suppose that F is uniform. Substituting (4) in (6), we obtain an explicit expression for $\underline{\theta_m}$, as function of n and m. Figure 3 plots $\underline{\theta_m}$, on the vertical axis, against $m/n \equiv \alpha$ on the horizontal axis, with $m = 1, .., (n-1)/2$. The different panels correspond to different values of n: $n = 9$, 15, and 21. In each panel, the 45^0 line thus equals $m/n = \alpha$, and by Lemma 1, since $\theta_m^* \leq m/n$, if $\underline{\theta_m} > m/n$, it follows that $\theta_m > \theta_m^*$. The figure shows that $\underline{\theta_m}$ can be surprisingly large, especially at low m/n. For example, if $m = 1$, $\underline{\theta_m}$ is 33 percent at $n = 9$ (when m is 11 percent of the voters) and remains almost 29 percent at $n = 21$ (when m is just below 5 percent of the voters).

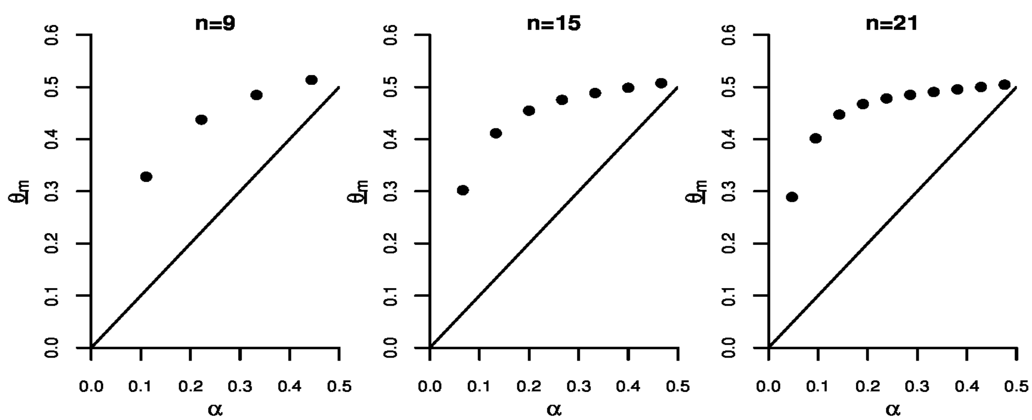

Figure 3: Lower bound on the probability of minority victories, as function of $\alpha = \frac{m}{n}$. $F(v)$ uniform.

In a large electorate, the expected fraction of equilibrium minority victories can be made precise. The result confirms the magnitude of the pro-minority bias at low m/n highlighted by Figure 3. The points of departure are Theorem 2 and its Corollary in the previous section: if n is large, with probability approaching 1, realized values satisfy the condition in Theorem

1, and again with probability approaching 1, voters $\overline{v}_m$ and $\overline{v}_M$ both demand $(n-1)/2$ votes, while all other voters offer their votes for sale. An immediate and unexpected result then follows: the final outcome depends exclusively on which one of $\overline{v}_m$ and $\overline{v}_M$ has his order filled, and since both have identical chances, both win with equal probability. Theorem 2 and its Corollary directly imply:[25]

Proposition 2. *Consider a sequence of vote markets, such that for all n, $m = \lfloor \alpha n \rfloor$, with $\alpha \in \left(0, \frac{1}{2}\right)$. Then for any α and F, $\lim\limits_{n\to\infty} \theta_{m,n} = \frac{1}{2}$. Moreover, $\Pr[\lim\limits_{n\to\infty} \theta_{m,n}(\mathbf{v}) = \frac{1}{2}] = 1$.*

At sufficiently large market size, the minority is expected to win with probability arbitrarily close to 1/2, for *any* minority share and for *any* distribution from which values are drawn. Note that the proposition is very strong; it states not only that the ex ante expected frequency of minority victories (θ_m) converges to 1/2, but that the expected frequency of minority victories converges to 1/2 for all value realizations, except on a set with zero probability ($\theta_m(\mathbf{v}) \underset{a.s}{\longrightarrow} 1/2$).

Given the previous results, the intuition is straightforward, but the result remains surprising. Whether the minority is 40 percent of the total electorate, 25 percent, or 10 percent, as long as it is not negligible, in a sufficiently large vote market there is an equilibrium such that the minority wins with probability 1/2 for any shape of the value distribution. After trade, the minority and the majority group are *equally likely* to control a majority of the votes. The market nullifies majority voting: following the will of the electorate becomes identical to flipping a coin.

4.2 Welfare

Beyond the existence of a bias, we are finally interested in the welfare properties of the market. Since $\theta_m > \theta_m^*$, we know that the market falls short of efficiency. But how does the market compare to majority voting in the absence of vote trading? To address this question we need a direct comparison of ex ante utilities. We call W the ex ante expected utility in the equilibrium we have constructed, and W_0 the ex ante expected utility in the absence of vote trading (i.e. with simple majority voting):

[25] For $\mathbf{v}$ satisfying the condition in Theorem 1, $\theta_{m,n}(\mathbf{v})$ is a continuous function of $q_{\bar{G},n}(\mathbf{v})$ and $q_{\bar{g},n}(\mathbf{v})$. By Theorem 2, its Corollary, and the continuous mapping theorem, $\theta_{m,n}(\mathbf{v}) \underset{a.s}{\longrightarrow} 1/2$.

$$
\begin{aligned}
nW &= \int_{\mathbf{v} \in R \cup B \cup P} \left[(1-\theta_m(\mathbf{v})) \sum_{i \in M} v_i + \theta_m(\mathbf{v}) \sum_{j \in m} v_j \right] dF^n(\mathbf{v}) + \\
&+ \int_{\mathbf{v} \notin R \cup B \cup P} \left[\sum_{i \in M} v_i \right] dF^n(\mathbf{v})
\end{aligned}
\tag{7}
$$

$$
nW_0 = \int_{\mathbf{v}} \left[\sum_{i \in M} v_i \right] dF(\mathbf{v}) \tag{8}
$$

If n is small, the welfare comparison between the vote market and no-trade depends on the shape of the value distribution. We find:

Proposition 3. *For all n and m, there exist distributions $\mathbf{F}'$ such that if $F \in \mathbf{F}'$ then $W < W_0$ for all n and m.*

The Appendix shows that $F = v^b$, with $b \geq 1$, belongs to $\mathbf{F}'$. Allowing for arbitrary $b > 0$ provides a simple intuition for the role played by F. The higher is b, the larger the probability mass at high value realizations, the smaller the ratio $Ev_{(1)}/Ev$–the ratio of the expected highest order statistics to the mean–and the smaller the probability that some unusually high value realization can compensate for the minority's smaller size. Hence the higher is b the lower is the probability that the aggregate minority value is higher than the aggregate majority value. Conversely, the lower is b, the larger the probability mass at low value realizations, the larger the ratio $Ev_{(1)}/Ev$, and the less important the relative size of the two groups in determining which group has higher aggregate value. Hence the lower is b, the less costly is the high frequency of minority victories built into the vote market. Thus, as stated, if $b \geq 1$, $F = v^b \in \mathbf{F}'$, but there exists a $\underline{b} \in (0,1)$ such that for $b \leq \underline{b}$, $F = v^b \notin \mathbf{F}'$, and $W > W_0$ is possible.[26]

The complications tied to the specific shape of F disappear when the market is large. By Theorem 2, we can ignore the second integral in (7); by Proposition 2, for all value realizations but a set of probability zero, $\theta_{m,n}(\mathbf{v})$ converges to 1/2; finally, with i.i.d. value draws, by the strong law of large numbers, both $\sum_{i \in M} v_i/M$ and $\sum_{i \in m} v_i/m$ converge to $Ev = \int_0^1 v dF(v)$. Thus:

[26]An example for which the market is welfare superior to no trading is $F = v^b$ with $b = 0.1$, $n = 7$, and $m = 3$. As discussed below, however, the market can be welfare improving only for small n.

$$\lim_{n\longrightarrow\infty} W_n = \left(\frac{1}{2}\right) Ev$$
$$\lim_{n\longrightarrow\infty} W_{0n} = (1-\alpha)Ev.$$

It then follows that:

$$\lim_{n\longrightarrow\infty}\left(\frac{W_n}{W_{0n}}\right) = \frac{1}{2(1-\alpha)} < 1$$

Note that the limit is independent of the distribution of valuations. As in the previous asymptotic results, the convergence can be stated in stronger terms: not only in terms of expected welfare but as almost sure convergence; that is, in the limit, for all realizations of values, except a zero probability set:[27]

Proposition 4. *Consider a sequence of vote markets. For any* $\alpha \in (0, \frac{1}{2})$ *and* F, $\lim_{n\longrightarrow\infty}(W_n/W_{0n}) = \frac{1}{2(1-\alpha)} < 1$. *Moreover:* $\Pr[\lim\limits_{n\longrightarrow\infty} \frac{W_n(\mathbf{v})}{W_{0n}(\mathbf{v})} = \frac{1}{2(1-\alpha)}] = 1$.

For any minority size and for any distribution of values, with a sufficiently large electorate vote-trading lowers welfare. Note the contribution of the proposition. The assumption of i.i.d. value draws implies that, in terms of ex ante expected utility, majority voting without trade must be asymptotically efficient. But there is no a priori reason why a market for votes should not be. If the price becomes negligible (as the probability that a single vote be pivotal becomes negligible), a market for votes could in principle support an equilibrium with negligible minority victories, and negligible efficiency losses. By Proposition 2, however, we know that this is not the case: the minority is always expected to win as frequently as the

[27]For all $\mathbf{v}$ that satisfy the condition in Theorem 1:

$$nW_n(\mathbf{v}) = \left[(1-\theta_{m,n}(\mathbf{v}))\sum_{i\in M} v_i + \theta_{m,n}(\mathbf{v})\sum_{j\in m} v_j\right]$$

In addition for any such $\mathbf{v}$, by Theorem 1, $\theta_{m,n}(\mathbf{v}) \in \left[\frac{n-1}{2(n+1)}, \frac{n+3}{2(n+1)}\right]$. Thus, for such values:

$$nW_n(\mathbf{v}) \in \left[\frac{n-1}{2(n+1)}\sum_{i=1}^{n} v_i, \frac{n+3}{2(n+1)}\sum_{i=1}^{n} v_i\right]$$

Theorem 2, the continuous mapping theorem, and the strong law of large numbers then give us immediately $W_n(\mathbf{v}) \underset{a.s}{\longrightarrow} Ev/2$. But by (8) and the strong law of large numbers, $W_{0,n}(\mathbf{v}) \underset{a.s}{\longrightarrow} (1-\alpha)Ev$. Using the continuous mapping theorem a final time, we then obtain $(W_n(\mathbf{v})/W_{0,n}(\mathbf{v})) \underset{a.s}{\longrightarrow} \frac{1}{2(1-\alpha)}$.

majority. wins. As a result, the efficiency loss is both precisely quantifiable and significant. If the minority is a third of the electorate, for example, the loss in ex ante utility is 25 percent; if it is 15 percent, the loss is more than 40 percent.

5 Robustness of the equilibrium

5.1 Alternative information assumptions

We described the model in Section 2 by stating that both the precise values of m and M and the compositions of the two group are commonly known. As mentioned earlier, however, our results extend to a range of different informational scenarios.

Knowing the exact composition of each group–which voter belongs to which group–plays no role in the analysis because demands in the competitive market are anonymous. The assumption that group membership is known seems preferable in the case of small committees and inappropriate in the case of a large electorate. Both scenarios are consistent with the equilibrium we have characterized.

Exact knowledge of the sizes of the two groups is not required either. Theorem 1 relies on one central assumption: each voter knows that a majority and a minority exist and knows which group he belongs to. Given this, the proof does not depend on precise information on the values of m and M. In particular, equilibrium strategies do not require individuals to form expectations of the two group sizes. Intuitively, the exact sizes are irrelevant because in equilibrium, for *any* m and M, the only two demands with positive probability correspond to $(n-1)/2$ votes, while everyone else sells. The results on the expected frequency of minority victories and on ex ante expected utility also hold unchanged if there is uncertainty about group sizes: because they hold for any m and M, they hold when the sizes are uncertain[28].

[28] In the case of large electorate, suppose $m = \lfloor \alpha n \rfloor$, where α is a random variable distributed according to some CDF H over $[a, b]$ with $a > 0$, $b < \frac{1}{2}$. For the proof of Theorem 2, note that $P(\bar{v}_g \geq \mu(n)v_{(2)G}) \geq P(\bar{v}_g \geq \frac{1}{2}) \geq 1 - [F(1/2)]^{\lfloor an \rfloor}$. For $a > 0, lim_{n\to\infty} 1 - [F(1/2)]^{\lfloor an \rfloor} = 1$. The result follows. For Propositions 2 and 4, we have not verified whether almost sure convergence holds when α is uncertain, but the results on expectations extend immediately. For Proposition 2, denote $\theta_{m,n}(\alpha)$ the expected fraction of minority victories, given α. Hence $\theta_{m,n} = \int_a^b \theta_{m,n}(\alpha) dH(\alpha)$. For all α, $\theta m,_n(\alpha) \to \frac{1}{2}$. In addition, for all n, α, $|\theta_{m,n}(\alpha)| < 1$. Hence by the bounded convergence theorem, $\theta_{m,n} \to \int_a^b \frac{1}{2} dH(\alpha) = \frac{1}{2}$. Identical reasoning can be used for Proposition 4. For any given α, denote $W_n(\alpha)$ the equilibrium welfare. Thus $W_n = \int_a^b W_n(\alpha) dH(\alpha)$. For all α, $W_n(\alpha) \to \frac{Ev}{2}$, and for all n, α, $|W_n(\alpha)| < 1$. By the bounded convergence theorem, $W_n \to \int_a^b \frac{Ev}{2} dH(\alpha) = \frac{Ev}{2}$. We can proceed likewise for W_0.

The results are robust not only to introducing more uncertainty, but also to the opposite change in assumptions, to introducing more information. Because of the focus on a fully revealing equilibrium, the analysis remains identical if we assume that not only m and M and the groups' composition are publicly known, but so are all mixed demands, with identifiers attaching them to each individual voter. In fact, as noted in passing, full revelation in equilibrium means that the results remain identical when all voters' preferences are public information: not only the direction of preferences of each voter, but also the full profile of values.

5.2 An alternative rationing rule

The equilibrium strategies have an extreme flavor: individuals either demand a majority of votes or sell. As in CLP, we want to verify that this is not an artefact of the all-or-nothing rationing rule (either an order is fully filled or it is passed over). CLP consider the following alternative rule, which they call $R2$, or rationing-by-vote: If voters' orders result in excess supply, the votes to be sold are chosen randomly from each seller, with equal probability. If instead there is excess demand, any vote supplied is randomly allocated to one of the individuals with outstanding purchasing orders, with equal probability. An order remains outstanding until it has been completely filled. When all supply is allocated, each individual who put in an order must purchase all units that have been directed to him, even if the order is only partially filled. Formally, we require $x_i \in \{0, 1, 2, .., 1+s_i\}$ for any x in the support of $R2\,(s)$. Like $R1$, $R2$ is anonymous. Contrary to $R1$, it guarantees that only one side of the market is ever rationed, but its requirement that partially filled orders be accepted seems ill-suited to a market for votes, where the value of votes hinges on pivotality, and thus on the exact number of votes transacted.

At $n = 3$, $R2$ and $R1$ are identical and Theorem 1 applies. Suppose then $n > 3$:

Theorem 3. *Suppose $R2$ is the rationing rule. For all $n > 3$ odd, m, and F, there exists a threshold $\mu_{R2}(n) > 0$ such that if $\overline{v}_g \geq \mu_{R2}(n)Max[v_{(2)G}, v_{(2)g}]$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $(n-1)/2$ votes (with probabilities $q'_{\overline{G}}$ and $q'_{\overline{g}}$ respectively) and selling their vote, and all other individuals sell. The randomization probabilities $q'_{\overline{G}}$ and $q'_{\overline{g}}$ and the price p' depend on the realized values $\overline{v}_g$ and $\overline{v}_G$, but for all $\overline{v}_G$ and $\overline{v}_g \geq \mu_{R2}(n)Max[v_{(2)G}, v_{(2)g}]$, $q'_{\overline{G}} \in [\frac{n-1}{n+1}, 1]$ and $q'_{\overline{g}} \in [\frac{n-1}{n+1}, 1]$. The* *threshold $\mu_{R2}(n)$ is given by:*

$$\mu_{R2}(n) = \frac{(n-1)^2}{2^{n-2}n}\binom{n-3}{\frac{n-3}{2}}$$

The theorem is proved in Appendix B. Its similarity to Theorem 1 is apparent. There are two main differences: first, the thresholds in the two theorems differ, and $\mu_{R2}(n) > \mu(n)$, implying that the equilibrium exists under $R2$ under more restrictive conditions than under $R1$. In particular, $\lim_{n\longrightarrow\infty}\mu_{R2}(n) = \infty$: whereas under $R1$ the probability that the equilibrium exists in a very large market converges to 1, the probability converges to 0 under $R2$. Second, as can be verified in the Appendix, when the equilibrium exists, the equilibrium price p' is consistently lower than p, the equilibrium price under $R1$. The intuition is clear: when both $\overline{v}_G$ and $\overline{v}_g$ submit demands for $(n-1)/2$ votes, one of the two will receive and be charged for $(n-3)/2$ votes, useless votes, since the opponent will hold a majority. To compensate for this risk, the equilibrium price must be lower.[29]

The choice of rationing rule poses a number of interesting but challenging questions. We know that in general the equilibrium must depend on the exact rule, and we can debate whether the rationing rule is better thought of as part of the institution, controlled by the market designer, or as part of the equilibrium, and interpreted as reduced form for the complex, decentralized system of search that underlies the trades.[30] Our goal here is not to address these broad questions but to make a narrower point: Theorem 3 shows that the equilibrium discussed in this paper is not the artefact of one specific rationing rule, and in particular of the all-or-nothing nature of $R1$.

5.3 Correlated and not identically distributed values

We have assumed so far that values are independent both across groups and within groups, and identically distributed according to some distribution F. The assumption allowed us to provide simple closed form solutions, but the logic of the arguments shows that neither independence nor a common distribution are necessary for our more substantive results. Theorem 1 states a sufficient condition for a trading equilibrium that depends only on the

[29] There is a third difference as well. As the proof in Appendix B makes clear, the condition $\overline{v}_g \geq \mu_{R2}(n)Max[v_{(2)G}, v_{(2)g}]$ is sufficient for the existence of the equilibrium in Theorem 3–there are value realizations for which weaker conditions are necessary–whereas under $R1$ the condition in Theorem 1 is necessary and sufficient for the equilibrium characterized there.

[30] See for example Green (1980) for a compelling exposition of the second interpretation.

existence of a sufficient wedge between $\overline{v}_g$ and $\overline{v}_{(2)G}$, the *realized* highest values in the two groups. Nor does the equilibrium depend on F: given m, M, R, p, and others' strategies, a voter's best response is fully identified. The probability that the condition in Theorem 1 is satisfied does depend on F, but the asymptotic result in Theorem 2 is robust to significant generalization.

Particularly relevant to our voting environment is the possibility of correlation in values. Consider then the following standard model, where the assumption of independence is weakened to conditional independence:

$$\begin{aligned} v_i &= v_m + \varepsilon_i \text{ for all } i \in m \\ v_j &= v_M + u_j \text{ for all } j \in M \end{aligned}$$

where v_m (v_M) is a common value shared by all m (M) voters, and ε_i and u_j are idiosyncratic components, independently drawn from distribution $G_m(\varepsilon)$, with full support $[0, \overline{\varepsilon}]$, and $G_M(u)$, with full support $[0, \overline{u}]$. For all fixed $\alpha \in (0, 1/2)$, as $n \longrightarrow \infty$, $\overline{v}_m \longrightarrow v_m + \overline{\varepsilon}$, and $\overline{v}_M \longrightarrow v_M + \overline{u}$. Thus for all $2(v_M + \overline{u}) \geq (v_m + \overline{\varepsilon}) \geq \frac{v_M + \overline{u}}{2}$ the equilibrium of Theorem 1 exists with probability approaching 1 asymptotically.[31] And if the equilibrium exists, Proposition 3 follows: asymptotically, the minority is expected to win with probability 1/2.

Relative to our previous results, there are then two qualifications. First, to ensure that the equilibrium always exists asymptotically, we need additional conditions on the distributions of values, here on v_m, $\overline{v}_M$, $\overline{\varepsilon}$, and $\overline{u}$. Second, the welfare results need to be re-evaluated and again in general will depend on the distributions. In this example, if $v_m + E_{G_m}(\varepsilon)$ is sufficiently larger than $v_M + E_{G_M}(u)$, then, depending on α, the vote market could be asymptotically superior to simple majority voting. If the distributions differ between the two groups, predictably the conclusions will depend on *how* they differ. Note however that neither qualification stems from relaxing independence. Our asymptotic results require that the extremum statistic of the value draws in each group should converge to the upper bound of the support. The condition is violated if all values are perfectly correlated, but can accommodate high degrees of dependence[32].

[31] We are using $\lim_{n \longrightarrow \infty} \mu(n) = 1/2$.

[32] For example, statisticians working on limit distributions for maxima have proposed the concept of *m-dependence*. When values are drawn in a natural sequence (think of floods over time), m-dependence applies when there exists a finite m such that draws that are more than m steps apart are independent (Hoeffding and Robbins, 1948). In our application, the concept could be relevant for geographically or ideologically concentrated subgroups of voters. Theorem 2 and Proposition 3 continue to hold in this case, under minor

6 Conclusions

How does a vote market function when voters are aware of their minority and majority status? In this paper, we have borrowed the concept of ex ante competitive equilibrium from Casella, Llorente-Saguer and Palfrey (2012) (CLP) and extended their model to an asymmetric setting where each voter knows that a majority and a minority exist, and knows which of the two groups he belongs to. We have characterized a sufficient condition for the existence of an ex ante equilibrium with trade for any electorate size, any majority advantage, and any distribution of intensities. In equilibrium, only two voters, the highest intensity voters on each side, demand votes with positive probabilities; all others offer their votes for sale. The two voters who randomize assign positive probability to only two actions: either selling, or demanding enough votes to alone control a majority of all votes. The equilibrium exists unless multiple members of one group, whether the majority or the minority, have intensities that are much higher, in a precise sense, than all members of the opposite group. We show that in a large electorate, the probability of such realizations of intensities is arbitrarily close to zero: the equilibrium exists with probability one.

The similarity to the equilibrium in CLP, where individuals are symmetric and equally likely to favor either alternative, suggests to us that, by re-establishing existence, the concept of ex ante equilibrium sheds light on a fundamental aspect of vote markets: votes per se are worthless; what is traded is decision power. The market becomes an auction between the two individuals who value the ownership of such power most. In the presence of a clear majority, the equilibrium has a number of additional properties. First, because all but the two highest-value voters offer their vote for sale, there is a large volume of intra-group trades. Second, for the same reason, the majority can win only of its highest value individual actively purchases votes. Finally, even if the numerical advantage of the majority is known precisely, the equilibrium results in a supermajority.

The probability of either group's victory depends only on the action of its most intense member and gives no direct weight to the size of the group. Because in addition the most intense minority member demands votes with probability that, at equal value, is always weakly higher than for the most intense majority member, the equilibrium yields a systematic minority bias. For any number of voters, any minority size, and any distribution of intensities, the market results in more frequent minority victories than efficiency dictates. In a large

regularity assumptions.

electorate, strikingly, the minority always wins with probability one half, regardless of its relative size.

The systematic bias in favor of the minority exacts welfare costs, and the market can be welfare inferior to simple majority voting with no vote trading. In a small electorate, whether this conclusion holds depends on the distribution of intensities. In a large electorate, however, the conclusion always holds. The welfare loss is precisely quantifiable and does not approach zero asymptotically.

The results we have obtained are surprisingly clear-cut for such a long-debated problem. They depend, however, on the specific equilibrium we have studied. It would be good to know to what extent the minority bias we uncovered is a general property of competitive markets for votes. The experimental results in Casella, Palfrey and Turban (2012) support the conjecture: in every experimental session, in fact in every committee of voters, the frequency of minority victories is higher than efficiency dictates. The experiment, however, concerns a specific case: a committee of five voters, with a minority of size two. Can the theory tell us more?

This is difficult question because it addresses the possible multiplicity of equilibria, an issue we are unable to resolve satisfactorily. We have not identified any other equilibrium with trade under the assumptiosn of our model. We know however that other equilibria can be supported in special cases. Casella and Turban (2012) discuss an example, again with five voters and a minority of two, in which the distribution of intensities is degenerate: all voters in the same group share the same value. If the majority and the minority values are sufficiently similar, an ex ante equilibrium exists where all voters randomize between demanding one vote, staying out of the market, and offering their vote for sale. The equilibrium is interesting because its strategies are such that voters do not demand bundles of votes, contrary to the equilibrium characterized in Theorem 1.[33] However, it remains true in this example that equilibrium strategies induce a bias in favor of the minority: the minority wins with higher probability than efficiency dictates. The bias arises because the minority consistently adopts more aggressive strategies than the majority: the minority is smaller and suffers from a weaker free-rider problem. We are lead to conjecture that a similar factor may be present more generally, whenever an equilibrium exists in a market for votes. A more solid evaluation of this claim, however, will have to wait for further research.

[33]The equilibrium of Theorem 1 continues to exist in this example, and exists over a larger range of value realizations.

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A Proofs

Proof of Theorem 1.

Theorem 1. *For all n odd, m, and F, there exists a threshold $\mu(n) \in (0,1)$ such that if $\overline{v}_g \geq \mu(n) v_{(2)G}$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $(n-1)/2$ votes (with probabilities $q_{\overline{G}}$ and $q_{\overline{g}}$ respectively) and selling their vote, and all other individuals sell. The randomization probabilities $q_{\overline{G}}$ and $q_{\overline{g}}$ and the price p depend on $\overline{v}_g$ and $\overline{v}_G$, but for all $\overline{v}_G$ and $\overline{v}_g \geq \mu(n)v_{(2)G}$, $q_{\overline{G}} \in [\frac{n-1}{n+1}, 1]$ and $q_{\overline{g}} \in [\frac{n-1}{n+1}, 1]$. The threshold $\mu(n)$ is given by:*

$$\mu(n) = \begin{cases} \frac{2}{3} & \text{if } n = 3 \\ \max\left\{\frac{(n-2)(n-1)}{2(n^2+n-5)}, \frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)}\right\} & \text{if } n > 3 \end{cases} \tag{9}$$

Proof. The theorem is implied by the following two lemmas. Lemma 2 characterizes the case $G = M$ and Lemma 3 the case $G = m$.

Lemma 2. *Suppose $G = M$ (or $\overline{v}_G = \overline{v}_M$ $\overline{v}_g = \overline{v}_{m,}$). Then if $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \overline{v}_M\right]$, the strategies described in the theorem are a fully revealing ex ante competitive equilibrium for all n odd, m, and F. The mixing probabilities $q_{\overline{M}}$ and $q_{\overline{m}}$ and the price p depend on the realizations of $\overline{v}_m$ and $\overline{v}_M$. There exist two thresholds $\frac{1}{2} \leq \underline{\rho}(n) < \overline{\rho}(n) < 1$ such that:*

(a) *Case $n > 3$*

1. *If $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M\right]$, $q_{\overline{M}}$, $q_{\overline{m}}$, and p satisfy:*

$$\begin{aligned} q_{\overline{M}} &= 1 \\ q_{\overline{m}} &= \frac{n-1}{n+1} \\ p &= 2\frac{\overline{v}_m}{n+1} \end{aligned} \tag{10}$$

2. *If* $\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \overline{\rho}(n)\overline{v}_M]$, $q_{\overline{M}}$, $q_{\overline{m}}$, *and* p *satisfy:*

$$\begin{aligned} q_{\overline{m}} + q_{\overline{M}} &= \frac{2n}{n+1} \\ p &= \frac{2q_{\overline{m}}\overline{v}_M}{2(n-1)-(n-3)q_{\overline{m}}} \\ p &= \frac{2(2-q_{\overline{M}})\overline{v}_m}{2(n-1)-(n-3)q_{\overline{M}}}. \end{aligned} \tag{11}$$

3. *If* $\overline{v}_m \in [\overline{\rho}(n)\overline{v}_M, \overline{v}_M]$, $q_{\overline{M}}$, $q_{\overline{m}}$, *and* p *satisfy:*

$$\begin{aligned} q_{\overline{m}} &= 1 \\ q_{\overline{M}} &= \frac{n-1}{(n+1)} \\ p &= 2\frac{\overline{v}_M}{n+1} \end{aligned} \tag{12}$$

The two thresholds $\underline{\rho}(n)$ *and* $\overline{\rho}(n)$ *are given by:*

$$\begin{aligned} \underline{\rho}(n) &= \frac{n+1}{n+5} \\ \overline{\rho}(n) &= \frac{(n-1)(n+5)}{(n+3)(n+1)} \end{aligned} \tag{13}$$

(b) *Case* $n = 3$

1. *If* $v_{(2)M} \leq \frac{3}{4}\overline{v}_M$, *then* $\mu(3)v_{(2)M} \leq \underline{\rho}(3)\overline{v}_M$, *and the characterization in part (a) above applies unchanged. If* $v_{(2)M} > \frac{3}{4}\overline{v}_M$, *then:*

2. *If* $\overline{v}_m \in [\mu(3)v_{(2)M}, \overline{\rho}(3)\overline{v}_M]$, $q_{\overline{M}}$, $q_{\overline{m}}$, *and* p *satisfy system 11; if* $\overline{v}_m \in [\overline{\rho}(3)\overline{v}_M, \overline{v}_M]$, $q_{\overline{M}}$, $q_{\overline{m}}$, *andp satisfy system 12.*

Lemma 3. *Suppose* $G = m$ *(or* $\overline{v}_G = \overline{v}_m$ *and* $\overline{v}_g = \overline{v}_M$*). Then if* $\overline{v}_M \in \left[\mu(n)v_{(2)m}, \overline{v}_m\right]$, *where* $\mu(n)$ *is given by relation 3 above, the strategies described in the theorem, together with the price and mixing probabilities given by system 12 are a fully revealing ex ante competitive equilibrium for all* n *odd,* m*, and* F.

The proof is organized in two stages. First, we show that if the direction of preferences associated with each demand is commonly known, the strategies and price described above are an equilibrium. Second, we show that when preferences are private information the equilibrium is fully revealing: given others' strategies and the market price, each individual's best response is identical to what it would be under full information. Others' strategies and the market price, together with the notion that the market is in equilibrium, fully reveal others' direction of preferences.

Ex ante equilibrium with full information

Suppose first that preferences are publicly known. We show here that the three systems 10, 11, and 12 characterize an ex ante equilibrium for each corresponding range of realized valuations.

1. Consider a candidate equilibrium with $q_{\overline{M}} \in (0,1)$, $q_{\overline{m}} \in (0,1)$. Expected market balance requires $(q_{\overline{M}} + q_{\overline{m}})(n-1)/2 = (n-2) + (1-q_{\overline{M}}) + (1-q_{\overline{m}})$, or:

$$q_{\overline{M}} + q_{\overline{m}} = \frac{2n}{n+1} \tag{14}$$

Denote by $U_M(s)$ the expected utility to voter $\overline{v}_M$ from demand s. Then:

$$\begin{aligned} U_M\left(\frac{n-1}{2}\right) &= q_{\overline{m}}\left(\frac{\overline{v}_M}{2} - \frac{n-1}{4}p\right) + (1-q_{\overline{m}})\left(\overline{v}_M - \frac{n-1}{2}p\right) \\ U_M(-1) &= q_{\overline{m}}\left(\frac{p}{2}\right) + (1-q_{\overline{m}})\left(\overline{v}_M\right) \end{aligned}$$

where we are assuming that voter $\overline{v}_M$ is informed that the other voter randomizing with probability $q_{\overline{m}}$ belongs to the minority. Voter $\overline{v}_M$ is indifferent between the two pure demands if and only if:

$$p = \frac{2q_{\overline{m}}\overline{v}_M}{n+1+(n-3)(1-q_{\overline{m}})} \tag{15}$$

Similarly, denoting by $U_m(s)$ the expected utility from demand s to voter $\overline{v}_m$:

$$\begin{aligned} U_m\left(\frac{n-1}{2}\right) &= q_{\overline{M}}\left(\frac{\overline{v}_m}{2} - \frac{n-1}{4}p\right) + (1-q_{\overline{M}})\left(\overline{v}_m - \frac{n-1}{2}p\right) \\ U_m(-1) &= q_{\overline{M}}\left(\frac{p}{2}\right) + (1-q_{\overline{M}})(0), \end{aligned} \tag{16}$$

again assuming full information. Indifference requires:

$$p = \frac{2(2 - q_{\overline{M}})\overline{v}_m}{n + 1 + (n - 3)q_{\overline{M}}} \tag{17}$$

Equations 14, 15 and 17 corresponds to system 10 in Lemma 2. The existence of a solution is not guaranteed. There is a solution if and only if there exists $q_{\overline{M}} \in [0, 1]$ and $q_{\overline{m}} \in [0, 1]$ with $q_{\overline{M}} + q_{\overline{m}} = \frac{2n}{n+1}$ such that 15=17. Such conditions are satisfied if and only if:

$$\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \bar{\rho}(n)\overline{v}_M]$$

where:

$$\begin{aligned} \underline{\rho}(n) &= \frac{n+1}{n+5} \\ \bar{\rho}(n) &= \frac{(n-1)(n+5)}{(n+3)(n+1)}, \end{aligned}$$

conditions 13 in Lemma 3. Note that $\frac{1}{2} \leq \underline{\rho}(n) < \bar{\rho}(n) < 1$ for all $n \geq 3$. To verify that this is indeed an equilibrium, we need to rule out profitable deviations. Note that for any voter any demand $s_i > n - 1$ is always fully rationed, and thus is equivalent to $s_i = 0$.

(i) Consider first voter $\overline{v}_M$. For any $s_M \in (\frac{n-1}{2}, n - 1]$, $U_M(s_M) < U_M(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority does not affect the probability of rationing and is strictly costly. For any $s_M \in [0, \frac{n-1}{2})$, $U_M(s_M) < U_M(-1)$: demanding less than $\frac{n-1}{2}$ votes is dominated by selling. To see this, note that when $s_m = \frac{n-1}{2}$, any $s_M < \frac{n-1}{2}$ guarantees that $\overline{v}_m$ will not be rationed and will win (because all other voters are selling). Thus, whether $s_M \in (0, \frac{n-1}{2})$ and the action is strictly costly, or $s_M = 0$ and voter $\overline{v}_M$ stays out of the market, when $s_m = \frac{n-1}{2}$, any $s_M \in [0, \frac{n-1}{2})$ is strictly dominated by selling. When $s_m = -1$, any $s_M \in (0, \frac{n-1}{2}]$ is dominated by $s_M \in \{-1, 0\}$ and these two actions are equivalent because both $s_M = -1$ and $s_M = 0$ induce no trade and guarantee a majority victory. Therefore, when facing the strategy profile defined in the candidate equilibrium, $\overline{v}_M$'s best response can only be either $s_M = -1$ or $s_M = \frac{n-1}{2}$. System 10 guarantees that $\overline{v}_M$ is indifferent between the two demands.

(ii) Consider now voter $\overline{v}_m$. As above, for any $s_m \in \left(\frac{n-1}{2}, n - 1\right]$, $U_m(s_m) < U_m(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority does not affect the probability of rationing and is strictly costly. It is also clear that $U_m(0) < U_m(-1)$: the two demands are equivalent if $s_M = -1$ and selling is strictly superior to staying out of the market if $s_M = \frac{n-1}{2}$. The question is whether $\overline{v}_m$ could gain by demanding less than $\frac{n-1}{2}$ votes: although such a strategy is dominated by selling when $s_M = \frac{n-1}{2}$, it could in principle be superior when $s_M = -1$. Consider the relevant expected utilities:

$$\begin{aligned} U_m\left(\frac{n-1}{2}\right) &= (1-q_{\bar{M}})\left(\bar{v}_m - \frac{n-1}{2}p\right) + q_{\bar{M}}\left(\frac{\bar{v}_m}{2} - \frac{n-1}{4}p\right) \\ U_m(-1) &= (1-q_{\bar{M}})\cdot 0 + q_{\bar{M}}\left(\frac{p}{2}\right) \\ U_m(x) &= (1-q_{\bar{M}})\left(P(x)\bar{v}_m - xp\right) + q_{\bar{M}}\left(-xp\right) \end{aligned} \tag{18}$$

where $P(x)$ is the probability of a minority victory when $\overline{v}_m$ demands $x \in (0, \frac{n-1}{2})$ votes and $\overline{v}_M$ offers his vote for sale. Since $P(x) < 1$ for all $x \in (0, \frac{n-1}{2})$, and $U_m(x)$ is increasing in $P(x)$ and decreasing in x, it follows that $U_m(x) < (1 - q_{\bar{M}})(\overline{v}_m - p) + q_{\bar{M}}(-p)$. Hence $U_m\left(\frac{n-1}{2}\right) > (1 - q_{\bar{M}})(\overline{v}_m - p) + q_{\bar{M}}(-p)$ is sufficient to rule out a profitable deviation to $x \in (0, \frac{n-1}{2})$. The condition is equivalent to:

$$\frac{q_{\bar{M}}}{2}\bar{v}_m \geq \frac{2(1-q_{\bar{M}})(n-1) + q_{\bar{M}}(n-1) - 4}{4}p$$

Substituting p from (17) and simplifying, the condition amounts to:

$$(2-n)q_{\bar{M}}^2 + (3n-5)q_{\bar{M}} - 2n + 6 \geq 0$$

This function is increasing in $q_{\bar{M}}$ for all $n \geq 3$. By equation 14, $q_{\bar{M}} \geq \frac{n-1}{n+1}$. Hence, we can evaluate the condition at $q_{\bar{M}} = \frac{n-1}{n+1}$. If it is positive, the deviation is not profitable. Substituting, we obtain:

$$n^2 + 2n + 13 \geq 0$$

which is trivially satisfied for all n. Hence for any $s_m \in [1, \frac{n-1}{2})$, $U_m(s_m) < U_m(\frac{n-1}{2})$. We can conclude that when facing the strategy profile defined in the candidate equilibrium, $\overline{v}_m$'s best response can only be either $s_M = -1$ or $s_M = \frac{n-1}{2}$. System 10 guarantees that $\overline{v}_m$ is indifferent between them.

(iii) Consider $v_i \in M$, $v_i \neq \overline{v}_M$. We show here that, given others' specified strategies, v_i's best response is selling: $s_i = -1$. First notice that, as argued above and for the same reasons, $U_i(s_i) < U_i(\frac{n-1}{2})$ for any $s_i \in (\frac{n-1}{2}, n-1]$. We need to treat the cases $n \geq 5$ and $n = 3$ separately.

(iii.a) Suppose first $n > 3$. In this case, for the same reasons described above $U_i(0) < U_i(-1)$. If a deviation from $s_i = -1$ is profitable, it must be to some $s_i \in (0, \frac{n-1}{2}]$. Suppose first $s_M = -1$. Then in the candidate equilibrium the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_M$. In particular, $U_i(-1) = U_M(-1) = U_M(\frac{n-1}{2}) > U_M(s)$ for all $s \in [0, \frac{n-1}{2})$. But $U_i(s)$ is increasing in v_i for all $s \in (0, \frac{n-1}{2}]$; hence for all s in this interval $U_i(s) < U_M(s)$, and thus $U_i(-1) > U_i(s)$ for all $s \in (0, \frac{n-1}{2}]$. Thus if $s_M = -1$, $s_i = -1$ is v_i's best response. Suppose then $s_M = \frac{n-1}{2}$. For all $s_i \in [0, \frac{n-3}{2})$, v_i is never rationed, but there is always another voter, either $\overline{v}_M$ or $\overline{v}_m$, who exits the market holding a majority of the votes. Hence the strategy is costly for v_i and never increases the probability of his side winning. It is dominated by $s_i = -1$. Consider then the two remaining strategies $s_i = \frac{n-1}{2}$, and $s_i = \frac{n-3}{2}$. Conditional on $s_M = \frac{n-1}{2}$, the relevant expected utilities are:

$$\begin{aligned}
U_{i\in M}\left(\frac{n-1}{2}\right)\Bigg|_{s_M = \frac{n-1}{2}} &= (1-q_{\overline{m}})\left(v_i - \frac{n-1}{4}p\right) + q_{\overline{m}}\left(\frac{2v_i}{3} - \frac{n-1}{6}p\right) \\
U_{i\in M}\left(\frac{n-3}{2}\right)\Bigg|_{s_M = \frac{n-1}{2}} &= (1-q_{\overline{m}})\left(v_i - \frac{n-3}{2}p\right) + q_{\overline{m}}\left(\frac{2v_i}{3} - \frac{n-3}{6}p\right) \qquad (19) \\
U_{i\in M}(-1)\Bigg|_{s_M = \frac{n-1}{2}} &= (1-q_{\overline{m}})\left(v_i + \frac{p}{2}\right) + q_{\overline{m}}\left(\frac{v_i}{2} + \frac{n-1}{2(n-2)}p\right)
\end{aligned}$$

Taking into account $q_{\overline{m}} \in \left[\frac{n-1}{n+1}, 1\right]$, equation 15, and $v_i \leq \overline{v}_M$, it is then straightforward to show that, conditional on $s_M = \frac{n-1}{2}$, $U_{i\in M}(-1) > U_{i\in M}\left(\frac{n-1}{2}\right)$, and $U_{i\in M}(-1) > U_{i\in M}\left(\frac{n-3}{2}\right)$. But if $s_i = -1$ is v_i's best response both when $s_M = -1$ and when $s_M = \frac{n-1}{2}$, than it is v_i's best response when $\overline{v}_M$ randomizes between $s_M = -1$ and $s_M = \frac{n-1}{2}$. No profitable deviation exists.

(iii.b) Suppose now $n = 3$. There are two M voters; hence $v_i \in M$, $v_i \leq \overline{v}_M$, is $v_{(2)M}$, the M voter with second highest value. This case must be considered separately because if $n = 3$, and only if $n = 3$, $v_{(2)M}$ can induce no trade with probability $q_{\overline{m}}q_{\overline{M}}$ by unilaterally deviating and staying out of the market.

Conditional on $s_M = \frac{n-1}{2} = 1$, the relevant expected utilities are:

$$\begin{aligned} U_{(2)M}(1)\Big|_{s_M=1} &= (1-q_{\overline{m}})\left(v_i - \frac{n-1}{4}p\right) + q_{\overline{m}}v_i \\ U_{(2)M}(0)\Big|_{s_M=1} &= v_i \\ U_{(2)M}(-1)\Big|_{s_M=1} &= (1-q_{\overline{m}})\left(v_i + \frac{p}{2}\right) + q_{\overline{m}}\left(\frac{v_i}{2}+p\right) \end{aligned} \tag{20}$$

It is immediately clear that $U_{(2)M}(0) > U_{(2)M}(1)$. Given equations 17 and 14, $U_{(2)M}(-1) > U_{(2)M}(0)$ for all $\overline{v}_m \in [\underline{\rho}(3)\overline{v}_M, \bar{\rho}(3)\overline{v}_M] \iff \overline{v}_m > (2/3)v_{(2)M}$. Thus $s_i = -1$ is indeed a best response for $v_{(2)M}$ as long as

$$\overline{v}_m \in \left[\max\{(2/3)v_{(2)M}, \underline{\rho}(3)\overline{v}_M\}, \bar{\rho}(3)\overline{v}_M\right]$$

.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. Note that such a voter only exists for $n > 3$. Again, we show here that, given others' specified strategies, v_i's best response is selling: $s_i = -1$. The proof proceeds as above. First notice that, as above, $U_i(s_i) < U_i(\frac{n-1}{2})$ for any $s_i \in (\frac{n-1}{2}, n-1]$, and $U_i(0) < U_i(-1)$. If a deviation from $s_i = -1$ is profitable, it must be to some $s_i \in (0, \frac{n-1}{2}]$. Suppose first $s_m = -1$. Then in the candidate equilibrium the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_m$. In particular, $U_i(-1) = U_m(-1) = U_m(\frac{n-1}{2}) > U_m(s)$ for all $s \in [0, \frac{n-1}{2})$. But $U_i(s)$ is increasing in v_i for all $s \in (0, \frac{n-1}{2}]$; hence for all s in this interval $U_i(s) < U_m(s)$, and thus $U_i(-1) > U_i(s)$ for all $s \in (0, \frac{n-1}{2}]$. Thus if $s_m = -1$, $s_i = -1$ is v_i's best response. Suppose then $s_m = \frac{n-1}{2}$. Exactly as argued above, if $s_i \in [0, \frac{n-3}{2})$, v_i is never rationed, but there is always another voter, either $\overline{v}_M$ or $\overline{v}_m$, who exits the market holding a majority of the votes. Hence the strategy is costly for v_i and never increases the probability of his side winning. It is dominated by $s_i = -1$. Consider then the two remaining strategies $s_i = \frac{n-1}{2}$, and $s_i = \frac{n-3}{2}$. Conditional on $s_m = \frac{n-1}{2}$, the relevant expected utilities are:

$$
\begin{aligned}
U_{i\in m}\left(\frac{n-1}{2}\right)\Bigg|_{s_m\,=\,\frac{n-1}{2}} &= (1-q_{\bar{M}})\left(v_i-\frac{n-1}{4}p\right)+q_{\bar{M}}\left(\frac{2v_i}{3}-\frac{n-1}{6}p\right)\\
U_{i\in m}\left(\frac{n-3}{2}\right)\Bigg|_{s_m\,=\,\frac{n-1}{2}} &= (1-q_{\bar{M}})\left(v_i-\frac{n-3}{2}p\right)+q_{\bar{M}}\left(\frac{2v_i}{3}-\frac{n-3}{6}p\right)\\
U_{i\in m}\left(-1\right)\Bigg|_{s_m\,=\,\frac{n-1}{2}} &= (1-q_{\bar{M}})\left(v_i+\frac{p}{2}\right)+q_{\bar{M}}\left(\frac{v_i}{2}+\frac{n-1}{2(n-2)}p\right)
\end{aligned}
$$

Taking into account $q_{\overline{M}} \in \left[\frac{n-1}{n+1}, 1\right]$, equation 17, and $v_i \leq \overline{v}_m$, it is then straightforward to show that, conditional on $s_m = \frac{n-1}{2}$, $U_{i\in m}(-1) > U_{i\in m}\left(\frac{n-1}{2}\right)$, and $U_{i\in m}(-1) > U_{i\in m}\left(\frac{n-3}{2}\right)$. But if $s_i = -1$ is v_i's best response both when $s_m = -1$ and when $s_m = \frac{n-1}{2}$, than it is v_i's best response when $\overline{v}_m$ randomizes between $s_m = -1$ and $s_m = \frac{n-1}{2}$. No profitable deviation exists. We can conclude that if $\overline{v}_m \in [\max\{\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M\}, \bar{\rho}(n)\overline{v}_M]$, where $\mu(n)$ is given by equation 3, and $\underline{\rho}(n)$ and $\bar{\rho}(n)$ are given by system 13, the strategies described in the theorem, together with the price and the mixing probabilities characterized in system 11, are indeed an ex ante equilibrium of the full information game. Note that $\underline{\rho}(n)\overline{v}_M > \mu(n)v_{(2)M}$ for all $n > 3$; if $n = 3$, $\underline{\rho}(3)\overline{v}_M > (2/3)v_{(2)M} \iff v_{(2)M} < (3/4)\overline{v}_M$.

2. Consider now $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M]$, where $\mu(n)$ is given by relation 3. Note that this case is relevant if $\underline{\rho}(n)\overline{v}_M > \mu(n)v_{(2)M}$, and thus for all $n > 3$, or for $v_{(2)M} < (3/4)\overline{v}_M$ if $n = 3$. Suppose all voters adopt the strategies described in the theorem, and $q_{\overline{M}} = 1$. Expected market clearing (equation 14) implies $q_{\overline{m}} = \frac{n-1}{n+1}$, and $U_m(-1) = U_m(\frac{n-1}{2})$ (or equation 17) implies $p = \frac{2\overline{v}_m}{n+1}$. Thus suppose system 11 holds. We show here that such strategies and price are an ex ante equilibrium of the full information game. As above, we rule out any profitable deviation for each voter in turn. Again, note that for any voter any demand $s_i > n-1$ is always fully rationed, and thus is equivalent to $s_i = 0$.

   (i) Consider first voter $\overline{v}_M$. In the candidate equilibrium, $s_M = \frac{n-1}{2}$. As argued earlier, it remains true that for any $s_M \in (\frac{n-1}{2}, n-1]$, $U_M(s_M) < U_M(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority does not affect the probability of rationing and is strictly costly. Similarly, it remains true that for any $s_M \in [0, \frac{n-1}{2})$, $U_M(s_M) < U_M(-1)$: demanding less than $\frac{n-1}{2}$ votes is dominated by selling. The argument is identical to what described earlier. Thus the only deviation we need to consider is to $s_M = -1$. The relevant expected utilities are:

$$\begin{aligned} U_M\left(\frac{n-1}{2}\right) &= q_{\bar{m}}\left(\frac{\overline{v}_M}{2} - \frac{n-1}{4}p\right) + (1-q_{\bar{m}})\left(\overline{v}_M - \frac{n-1}{2}\right) \\ U_M(-1) &= q_{\bar{m}}\left(\frac{p}{2}\right) + (1-q_{\bar{m}})(\overline{v}_M) \end{aligned}$$

Substituting $q_{\overline{m}} = \frac{n-1}{n+1}$, we obtain:

$$U_M\left(\frac{n-1}{2}\right) \geq U_M(-1) \Leftrightarrow \frac{\overline{v}_M}{p} > \frac{n+5}{2}$$

Given $p = \frac{2\overline{v}_m}{n+1}$, the condition amounts to:

$$U_M\left(\frac{n-1}{2}\right) \geq U_M(-1) \Leftrightarrow \overline{v}_M \geq \frac{n+5}{n+1}\overline{v}_m = \frac{1}{\underline{\rho}(n)}\overline{v}_m$$

The requirement established the upper bound of the range of $\overline{v}_m$ values considered here: $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M]$.

(ii) Consider voter $\overline{v}_m$. The arguments discussed under point 1.(ii) apply. With $s_M = \frac{n-1}{2}$ and all other voters selling, $s_m = \frac{n-1}{2}$ and $s_m = -1$ dominate all other $v_{\overline{m}}$'s strategies. With $p = \frac{2\overline{v}_m}{n+1}$, $\overline{v}_m$ is indifferent between them and has no profitable deviation.

(iii) Consider now $v_i \in M$, $v_i \neq \overline{v}_M$. We show here that, given others' specified strategies, v_i's best response is selling: $s_i = -1$. By the arguments under point 1.(iii) above, the only deviations we need to consider are $s_i = \frac{n-1}{2}$ and $s_i = \frac{n-3}{2}$. The relevant expected utilities are given by system 19 for $n > 3$, and system 20 for $n = 3$. Substituting $p = \frac{2\overline{v}_m}{n+1}$, and $q_{\overline{m}} = \frac{n-1}{n+1}$, we derive the following conditions. If $n > 3$:

$$U_{i\in M}\left(\frac{n-1}{2}\right) \leq U_{i\in M}(-1) \Leftrightarrow v_i\frac{(n-2)(n-1)}{2(n^2+n-5)} \leq \overline{v}_m$$

and

$$U_{i\in M}\left(\frac{n-3}{2}\right) \leq U_{i\in M}(-1) \Leftrightarrow v_i\frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)} \leq \overline{v}_m$$

The two conditions are satisfied if and only if $\mu(n)v_i \leq \overline{v}_m$. Thus they are satisfied for all $v_i \in M$, $v_i \leq \overline{v}_M$ if they are satisfied for $v_i = v_{(2)M}$. If $n = 3$:

$$U_{(2)M}(1) \leq U_{(2)M}(-1) \Leftrightarrow \frac{v_{(2)M}}{2} \leq \overline{v}_m$$

and:

$$U_{(2)M}(0) \leq U_{(2)M}(-1) \Leftrightarrow \frac{2}{3}v_{(2)M} \leq \overline{v}_m$$

This latter condition is stricter and again is satisfied if and only if $\mu(3)v_{(2)M} \leq \overline{v}_m$. For all n, we have established the lower bound of the range of $\overline{v}_m$ values considered here: $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_m]$. Recall that$\underline{\rho}(n)\overline{v}_M > \mu(n)v_{(2)M}$ for all $n > 3$, but if $n = 3$, $\underline{\rho}(3)\overline{v}_M > \mu(3)v_{(2)M} \iff v_{(2)M} < (3/4)\overline{v}_M$ if $n = 3$.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. Again, this voter only exists if $n > 3$. The arguments in 1.(iv) above can be applied identically here and establish that $s_i = -1$ is v_i's unique best response. In particular, if $s_m = -1$, the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_m$. Given others' specified strategies, the differential utility from selling, relative to any other action, is decreasing in v_i; hence if $s_m = -1$ is $\overline{v}_m$'s best response, then it must be a best response for $v_i \leq \overline{v}_m$. If $s_m = \frac{n-1}{2}$, the identical proof detailed in 1.(iv) is relevant. The proof made use of the constraint $q_{\bar{M}} \in \left[\frac{n-1}{n+1}, 1\right]$, which is still satisfied here.

We conclude that for all $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M\right]$, where $\mu(n)$ is given by relation 3, the strategies described in the theorem, together with the price and the mixing probabilities characterized in system 10, are indeed an ex ante equilibrium of the full information game. If $n = 3$, this case is only relevant if $v_{(2)M} < \frac{3}{4}\overline{v}_M$.

3. Consider now $\overline{v}_m > \bar{\rho}(n)\overline{v}_M$, where $\bar{\rho}(n)$ is defined in system 13. Suppose all voters adopt the strategies described in the theorem, and $q_{\overline{m}} = 1$. Expected market clearing (equation 14) implies $q_{\overline{M}} = \frac{n-1}{n+1}$, and $U_M(-1) = U_M(\frac{n-1}{2})$ (or equation 15) implies $p = \frac{2\overline{v}_M}{n+1}$. Thus suppose system 12 holds. We show here that such strategies and price are an ex ante equilibrium of the full information game. As above, we rule out any profitable deviation for each voter in turn. The proofs follow immediately from the arguments used earlier. In particular:

   (i) Consider first voter $\overline{v}_M$. The arguments discussed under point 1.(i) apply. With

$s_m = \frac{n-1}{2}$ and all other voters selling, $s_M = \frac{n-1}{2}$ and $s_M = -1$ dominate all other $\overline{v}_M$'s strategies. With $p = \frac{2\overline{v}_M}{n+1}$, $\overline{v}_M$ is indifferent between them and has no profitable deviation.

(ii) Consider then voter $\overline{v}_m$. Recall that when $\overline{v}_M$ randomizes between $s_M = \frac{n-1}{2}$ and $s_M = -1$ and all others sell, $s_m = \frac{n-1}{2}$ and $s_m = -1$ dominate all other $\overline{v}_m$'s strategies. The relevant expected utilities are given by equation 16. Hence, substituting $q_{\overline{M}} = \frac{n-1}{n+1}$:

$$U_m\left(\frac{n-1}{2}\right) \geq U_m\left(-1\right) \Leftrightarrow \frac{\overline{v}_m}{p} \geq \frac{(n-1)(n+5)}{2(n+3)}$$

With $p = \frac{2\overline{v}_M}{n+1}$, therefore:

$$U_m\left(\frac{n-1}{2}\right) \geq U_m\left(-1\right) \Leftrightarrow \overline{v}_m \geq \frac{(n-1)(n+5)}{(n+1)(n+3)}\overline{v}_M = \bar{\rho}(n)\overline{v}_M$$

The condition establishes the lower bound of the range of $\overline{v}_m$ values considered under this case.

(iii) Consider $v_i \in M$, $v_i \neq \overline{v}_M$. If $n > 3$, the arguments in 1.(iii.a) above can be applied identically here and establish that $s_i = -1$ is v_i's unique best response. In particular, if $s_M = -1$, the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_M$. Hence if $s_M = -1$ is $\overline{v}_M$'s best response, then it must be a best response for $v_i \leq \overline{v}_M$. If $s_M = \frac{n-1}{2}$, the identical proof detailed in 1.(iii) is relevant. The proof made use of the constraint $q_m \in \left[\frac{n-1}{n+1}, 1\right]$, which is still satisfied here. If $n = 3$, $v_i \equiv v_{(2)M}$ and:

$$\begin{aligned}
U_{(2)M}\left(1\right)\Big|_{s_m=1} &= q_{\overline{M}}v_{(2)M} + (1-q_{\overline{M}})\left(\frac{v_{(2)M}}{2} + \frac{p}{2}\right) \\
U_{(2)M}\left(0\right)\Big|_{s_m=1} &= q_{\overline{M}}v_{(2)M} \\
U_{(2)M}\left(-1\right)\Big|_{s_m=1} &= q_{\overline{M}}\left(\frac{v_{(2)M}}{2} + p\right) + (1-q_{\overline{M}})\left(\frac{p}{2}\right)
\end{aligned}$$

With $p = \frac{2\overline{v}_M}{n+1}$ and $q_{\overline{M}} = \frac{1}{2}$ by equation 14, it is trivial to verify that $U_{(2)M}\left(-1\right) > U_{(2)M}\left(1\right)$ and $U_{(2)M}\left(-1\right) > U_{(2)M}\left(0\right)$.

(iv) Finally, when $n > 3$, consider $v_i \in m$, $v_i \neq \overline{v}_m$. The problem faced here by $v_i \in m$ is identical to the problem faced by $v_i \in M$, $v_i \neq \overline{v}_M$ in case 2.(iii) above, when

$q_{\overline{M}} = 1$, $q_{\overline{m}} = \frac{n-1}{n+1}$. Taking into account $p = \frac{2\overline{v}_M}{n+1}$, all profitable deviations can be ruled out if and only if $v_i \max\left\{\frac{(n-2)(n-1)}{2(n^2+n-5)}, \frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)}\right\} \leq \overline{v}_M$, or $v_i\mu(n) \leq \overline{v}_M$.

Because $\mu(n) < 1$, two observations follow immediately. First, if $\overline{v}_M \geq \overline{v}_m$, the condition $v_i\mu(n) \leq \overline{v}_M$ for all $v_i \in m$, $v_i \neq \overline{v}_m$ is always satisfied. Thus the strategies described in the theorem, together with the price and mixing probabilities characterized in system 12 are indeed an ex ante equilibrium of the full information game for all $\overline{v}_m \in (\bar{\rho}(n)\overline{v}_M, \overline{v}_M]$. Second, the condition $\overline{v}_M \geq \overline{v}_m$ has not been imposed anywhere in the proof of the equilibrium of case 3. The equilibrium requires $\overline{v}_m > \bar{\rho}(n)\overline{v}_M$, where $\bar{\rho}(n) < 1$, and, for $n > 3$, $v_i\mu(n) \leq \overline{v}_M$ $\forall v_i \in m$, $v_i \neq \overline{v}_m$. Thus it is compatible with $\overline{v}_m > \overline{v}_M$, as long as $\overline{v}_M \geq \mu(n)v_{(2)m}$ if $n > 5$, and with no additional constraint if $n = 3$. Hence Lemma 3 follows immediately.

We now show that when preferences are private information, the strategies and price identified above constitute a fully revealing ex ante equilibrium.

Fully revealing equilibrium

We conjecture an equilibrium identical to the full information equilibrium characterized above and show that given others' strategies, the equilibrium price and the knowledge that the market is in a fully revealing equilibrium, each voter's best response when preferences are private information is uniquely identified and equals the voter's best response with full information. Thus the equilibrium exists when preferences are private information and is indeed fully revealing.

1. Consider first the perspective of voter $\overline{v}_M$, in equilibrium. In any of the scenarios identified above, expected market equilibrium requires $\overline{v}_M$ to demand a positive number of votes with positive probability. It then follows that the other voter who demands a positive number of votes with positive probability must belong to the minority. If not, $\overline{v}_M$'s best response would be to sell, violating expected market equilibrium. Thus $\overline{v}_M$ also knows that $M-1$ majority members and $m-1$ minority members are offering their vote for sale; he cannot identify them individually, but that is irrelevant. Given that the other net demand for votes comes from a minority voter, $\overline{v}_M$'s best response is identified uniquely and is identical to his best response under full information.

2. Consider then the perspective of voter $\overline{v}_m$. If $n = 3$, he is the only minority voter and the problem is trivial. Suppose $n > 3$. Suppose first that $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \underline{\rho_M}(n)\overline{v}_M\right]$,

and hence $s_M = \frac{n-1}{2}$ with probability 1. Expected market balance requires $\overline{v}_m$ to demand a positive number of votes with positive probability. But that can only be a best response if the voter who demands $\frac{n-1}{2}$ votes belongs to the majority; if not, $\overline{v}_m$'s best response would be to sell. Again, $\overline{v}_m$ also knows that $M-1$ majority members and $m-1$ minority members are offering their vote for sale; he cannot identify them individually, but that is irrelevant. Suppose now $\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \bar{\rho}(n)\overline{v}_M]$. Expected market balance rules out that $\overline{v}_m$ could sell with probability 1 (because over this range of valuations the minimal expected demand of votes by $\overline{v}_m$ required for expected market balance is $\min(q_{\overline{m}})(\frac{n-1}{2})+(1-\min(q_{\overline{m}})(-1) = \left(\frac{n-1}{n+1}\right)\left(\frac{n-1}{2}\right)+\left(1-\frac{n-1}{n+1}\right)(-1) = \frac{n-5}{2(n+1)} > -1$ for all $n \geq 3$). Given the profile of strategies faced by $\overline{v}_m$, staying out of the market ($s_m = 0$) is always dominated by selling. Thus $\overline{v}_m$'s best response in equilibrium must include demanding a positive number of votes with positive probability. As in all previous cases, demanding more than $\frac{n-1}{2}$ votes is always dominated by demanding $\frac{n-1}{2}$ votes. Thus the actions over which $\overline{v}_m$ can randomize with positive probability are $s_m = \frac{n-1}{2}$, $s_m = x$, with $0 \leq x < \frac{n-1}{2}$, and $s_m = -1$. Suppose that the voter demanding $\frac{n-1}{2}$ with probability $q_{\overline{M}}$ (with $q_{\overline{M}}$ identified in system 10), and selling otherwise, belonged to the minority. Then:

$$\begin{aligned}
U_m\left(\frac{n-1}{2}\right)\Big|_{(\overline{v}_M \in m)_e} &= (1-q_{\bar{M}})\left(\overline{v}_m - \frac{n-1}{2}p\right) + q_{\bar{M}}\left(\overline{v}_m - \frac{n-1}{4}p\right) \\
U_m(-1)\Big|_{(\overline{v}_M \in m)_e} &= (1-q_{\bar{M}})\cdot 0 + q_{\bar{M}}\left(\overline{v}_m + \frac{p}{2}\right) \\
U_m(x)\Big|_{(\overline{v}_M \in m)_e} &= (1-q_{\bar{M}})\left(P(x)\overline{v}_m - xp\right) + q_{\bar{M}}\left(\overline{v}_m - xp\right)
\end{aligned} \tag{21}$$

where the index $(\overline{v}_m \in m)_e$ indicates the belief that the other voter with positive expected demand belongs to the minority. System 21 is similar to system 18. In particular: (1) The differential utility from selling relative to demanding $x \in [0, \frac{n-1}{2})$ votes, $U_m(-1) - U_m(x)$, is identical. We saw earlier that such term must be positive for all $q_{\bar{M}} \in \left[\frac{n-1}{n+1}, 1\right]$, a result that thus applies immediately here. (2) For all $\overline{v}_m > 0$, the differential utility from selling relative to demanding $\frac{n-1}{2}$ votes, $U_m(-1) - U_m\left(\frac{n-1}{2}\right)$, is strictly higher than in system 18, where, at equilibrium $q_{\bar{M}}$, it equalled 0. Hence at equilibrium $q_{\bar{M}}$ it must be positive here. It follows that if the voter demanding $\frac{n-1}{2}$ with probability $q_{\overline{M}}$ belonged to the minority, $\overline{v}_m$'s best response would be to sell. But that would violate expected market balance. Hence the voter demanding

$\frac{n-1}{2}$ with probability $q_{\overline{M}}$ must belong to the majority. Of all remaining voters offering their votes for sale, $M-1$ belongs to the majority, and $m-1$ to the minority. They cannot be distinguished but that has no impact on $\overline{v}_m$'s unique best response. Finally, suppose either $\overline{v}_m \in (\bar{\rho}(n)\overline{v}_M, \overline{v}_M]$, or $\overline{v}_M \in \left[\mu(n)v_{(2)m}, \overline{v}_m\right]$. Expected market balance requires $s_m = \frac{n-1}{2}$ with probability 1. But then the other voter demanding $\frac{n-1}{2}$ votes with positive probability cannot belong to the minority (because in a fully revealing equilibrium, if $s_m = \frac{n-1}{2}$ with probability 1, all other minority voters would prefer to sell). Hence again the other voter with positive demand for votes must be a majority voter. All remaining voters are sellers; identifying the group each of them belongs to is not possible but has no impact on $\overline{v}_m$'s unique best response.

3. Consider now the perspective of all voters who in the full information equilibrium offer their vote for sale with probability 1: $v_i \in M$, $v_i \neq \overline{v}_M$, or $v_i \in m$, $v_i \neq \overline{v}_m$. By the arguments above, each of them knows that in a fully revealing equilibrium the two voters with positive expected demand must belong to the two different parties. Which one belongs to the majority and which one to the minority cannot be distinguished, but is irrelevant: since in the full information case v_i's best response is $s_i = -1$ with probability 1 whether $v_i \in M$, or $v_i \in m$, it follows that identifying which of the two voters with positive expected demand belongs to which group is irrelevant to v_i's best response. Equally irrelevant is identifying which of the sellers belongs to which group. Although the direction of preferences associated with each individual voter cannot be identified, v_i's best response is unique and identical to his best response with full information.

We can conclude that the equilibrium strategies and price identified by Lemmas 2 and 3 are indeed a fully revealing ex ante equilibrium with private information. □

A.1 Derivation of system 4

Using our notation, call $x_{(1)}$ and $x_{(2)}$ the two highest order statistics out of n independent draws, where each variable is distributed according to the cumulative distribution function G_x, with density g_x. Then the joint density of $x_{(1)}$ and $x_{(2)}$, $g_{x_{(1)},x_{(2)}}$ is given by:

$$g_{x_{(1)},x_{(2)}}(y,x) = n(n-1)[G_x(x)]^{n-2}g_{x_{(1)}}(y)g_{x_{(2)}}(x)$$

where, calling $x_{(r)}$ the r^{th} highest order statistics:

$$g_{x_{(r)}}(x) = \frac{n!}{(n-r)!(r-1)!}[G_x(x)]^{n-r}[1-G_x(x)]^{r-1}g_x(x)$$

See Gibbons (2003). The expressions in system 4 are obtained from solving the integrals in:

$$\begin{aligned}
\Pr(B) &= \int_{\overline{v}_M=0}^{1}\int_{v_{(2)m=0}}^{\min\left(\frac{\overline{v}_M}{\mu},1\right)}\int_{\overline{v}_m=\max\left(v_{(2)m},\overline{\rho}\overline{v}_M\right)}^{1} m(m-1)(v_{(2)m})^{m-2}M(\overline{v}_M)^{M-1}d\overline{v}_m dv_{(2)m}d\overline{v}_M \\
\Pr(P) &= \frac{M}{M+m}(\overline{\rho}^m - \underline{\rho}^m) \\
\Pr(R) &= \int_{\overline{v}_m=0}^{\underline{\rho}}\int_{v_{(2)M=0}}^{\min\left(\frac{\overline{v}_m}{\mu},1\right)}\int_{\overline{v}_M=\max\left(v_{(2)M},\frac{\overline{v}_m}{\underline{\rho}}\right)}^{1} M(M-1)(v_{(2)M})^{M-2}m(\overline{v}_m)^{m-1}d\overline{v}_M dv_{(2)M}d\overline{v}_m
\end{aligned}$$

A.2 Proof of the Corollary to Theorem 2

Corollary 1. *For any* $\alpha \in \left(0, \frac{1}{2}\right)$ *and* F*,* $\Pr[\lim_{n\to\infty} q_{\overline{G},n}(\mathbf{v}) = 1] = 1$*, and* $\Pr[\lim_{n\to\infty} q_{\overline{G},n}(v) = 1] = 1$

Proof. For $h = g, G$, define $q_{\bar{h},n}(\mathbf{v})$ as a sequence of random variables that take the values specified in Theorem 1 if the condition in the theorem is satisfied, and 0 otherwise. We will use the Borel Cantelli lemma. In the context of almost sure convergence, it implies that a sufficient condition for a sequence of random variable X_n to converge almost surely to X is that $\forall\epsilon > 0, \sum_{k=1}^{\infty}\Pr(|X_k - X| > \epsilon) < \infty$. In the specific case of the corollary to Theorem 2, we want to show that for $h = g, G$, $\forall\epsilon > 0, \sum_{k=1}^{\infty}\Pr(|q_{\bar{h},k} - 1| > \epsilon) < \infty$. Fix $\epsilon > 0$. Choose n_0 a positive integer such that $\frac{n_0-1}{n_0+1} \geq 1-\epsilon$ and $\alpha\cdot n_0 > 1$ so that $\frac{\alpha k}{2} \leq \lfloor\alpha k\rfloor$ for $k > n_0$. Then, for all $k \geq n_0$, $\Pr(\{|q_{\bar{h},k} - 1| \geq \epsilon\}) \leq \Pr(G = m \cap \bar{v}_M \leq \mu(n)v_{(2)m}) + \Pr(G = M \cap \bar{v}_m \leq \mu(n)v_{(2)M})$. For $k \geq n_0$, $m = \lfloor\alpha k\rfloor$, $M = k - m$, we know that $P(G = m \cap \bar{v}_M \leq \mu(n)v_{(2)m}) \leq F\left(\frac{1}{2}\right)^M$ and $P(G = M \cap \bar{v}_m \leq \mu(n)v_{(2)M}) \leq F\left(\frac{1}{2}\right)^m$. We can then write for all $k \geq n_0$ that $P(\{|q_{\bar{h},k} - 1| \geq \epsilon\}) \leq 2F\left(\frac{1}{2}\right)^{\frac{\alpha}{2}k}$. Hence,

$$\sum_{k=n_0}^{\infty} P(|q_{\bar{h},k} - 1| > \epsilon) \leq \sum_{k=n_0}^{\infty} 2F\left(\frac{1}{2}\right)^{\frac{\alpha}{2}k}$$

The latter is the partial sum of a geometric sum with a multiplicative term strictly between 0 and 1. This sum is finite. By the Borel Cantelli lemma, the result is proven. □

A.3 Proof of Lemma 1

Lemma 1. *If all v_i, $i \in m$ and $i \in M$, are i.i.d. according to some $F(v)$, then for all F, n, and m , $\theta^*_m \leq \frac{1}{1+\binom{M}{m}} \leq \frac{m}{n}$.*

Proof. Call a realization of n values a *profile* Π, and call a *partition* $\mathcal{P}(\Pi)$ a corresponding minority profile $\mathfrak{m}$ and majority profile $\mathfrak{M}$: $\mathcal{P}(\Pi)$=$\{\mathfrak{m}$,$\mathfrak{M}\}$[34]. The probability of a profile Π depends on the distribution F, but note that because values are i.i.d., given Π any partition $\mathcal{P}(\Pi)$ is equally likely. Call V_m the sum of realized minority values ($V_m = \sum_{i \in m} v_i$), and similarly for V_M ($V_M = \sum_{j \in M} v_j$). Consider any $\mathcal{P}(\Pi)$=$\{\mathfrak{m}, \mathfrak{M}\}$ such that $V_m > V_M$, supposing that at least one such profile Π and partition $\mathcal{P}(\Pi)$ exist. Now, keeping Π fixed, consider an alternative partition $\mathcal{P}'(\Pi)$ such that the values in the minority profile $\mathfrak{m}$ are reassigned to majority voters. By construction, $V_M > V_m$. The values assigned to the remaining $M - m$ majority voters are chosen freely among all realized values in the original majority profile $\mathfrak{M}$. Thus for any $\mathfrak{m}$, there are $\binom{n-m}{M-m} = \binom{M}{M-m} = \binom{M}{m}$ equally likely partitions $\mathcal{P}'(\Pi)$ such that $V_M > V_m$. But then:

$$\Pr(V_M > V_m|\Pi) \geq \binom{M}{m} \Pr(V_m > V_M|\Pi),$$

with inequality because for given Π we are ignoring partitions $\mathcal{P}''(\Pi)$ such that some of $\mathfrak{m}$ values are associated with minority and some with majority voters and $V_M > V_m$.[35]. Now:

$$\Pr(V_m > V_M) = \int_\Pi \Pr(V_m > V_M|\Pi)dG$$

$$\Pr(V_M > V_m) = \int_\Pi \Pr(V_M > V_m|\Pi)dG \geq \binom{M}{m} \int_\Pi \Pr(V_m > V_M|\Pi)dG = \binom{M}{m} \Pr(V_m > V_M)$$

where $G = F^n$ is the joint density of a profile Π. But $\Pr(V_m > V_M) = 1 - \Pr(V_M > V_m)$. Hence:

$$\Pr(V_m > V_M) \leq \frac{1}{1 + \binom{M}{m}}.$$

[34]For clarity: for any Π, there are $\binom{n}{m}$ possible partitions $\mathcal{P}(\Pi)$, and for any partition $\mathcal{P}(\Pi)$ there are $m!M!$ possible permutations of values among the different voters, all keeping $\mathcal{P}(\Pi) = \{\mathfrak{m}, \mathfrak{M}\}$ constant.

[35]We are not ignoring those such that $V_m > V_M$ because they are taken into account as different initial partitions $\widetilde{\mathcal{P}}(\Pi)$.

To establish that:

$$\frac{1}{1+\binom{M}{m}} \leq \frac{m}{m+M}, \tag{22}$$

note that condition 22 is equivalent to:

$$\frac{m!(M-m)!}{m!(M-m)!+M!} \leq \frac{m}{m+M}$$

or, after some manipulations:

$$(m-1)!(M-m)! \leq (M-1)!$$

which is equivalent to:

$$\binom{M-1}{m-1} \geq 1,$$

an inequality that holds for all $m \geq 1$. □

A.4 Proof of Proposition 1

Proposition 1. *For all n, m, and F, $\theta_m > \theta^*_m$.*

Proof. We know that if $\overline{v}_g > v_{(2)G}$, the equilibrium in Theorem 1 always applies. If $G = m$ (i.e. $v_n \in m$), m wins with probability $\frac{n+3}{2(n+1)}$; if $G = M$ (i.e. $v_n \in M$), m wins with probability $\frac{n-1}{2(n+1)}$ if $\overline{v}_m < \underline{\rho}\overline{v}_M$, and with some probability $\in (\frac{n-1}{2(n+1)}, \frac{n+3}{2(n+1)})$ otherwise. Hence:

$$\theta_m > \frac{n+3}{2(n+1)}\Pr(G = m \cap \overline{v}_M > v_{(2)m}) + \frac{n-1}{2(n+1)}\Pr(G = M \cap \overline{v}_m > v_{(2)M}) \tag{23}$$

The inequality is strict both because equation 23 sets to $\frac{n-1}{2(n+1)}$ the probability of minority victories whenever $\overline{v}_g > \overline{v}_{(2)G}$ and $G = M$, and because it ignores value realizations such that $\overline{v}_g \in (\mu(n)v_{(2)G}, v_{(2)G})$–the condition in Theorem 1 is satisfied, and the minority wins with positive probability.[36] With i.i.d. value draws:

$$\Pr(G = m \cap \overline{v}_M > \overline{v}_{(2)m}) = \Pr(G = M \cap \overline{v}_m > \overline{v}_{(2)M}) = \frac{mM}{n(n-1)}$$

[36]Note that such realizations have positive probability for all F with full support.

Thus:

$$\theta_m > \frac{n+3}{2(n+1)}\frac{mM}{n(n-1)} + \frac{n-1}{2(n+1)}\frac{mM}{n(n-1)} = \frac{mM}{n(n-1)}$$

Given Lemma 1, the proposition follows if for all m, n, $\frac{m(n-m)}{n(n-1)} \geq \frac{1}{1+\binom{n-m}{m}}$. Define $f_n(m) = m(n-m)\binom{n-m}{m}$. The inequality then amounts to $f_n(m) \geq n(n-1)$. We first show that for given n, $\forall m \in \{1, \ldots, \frac{n-1}{2}\}$, $f_n(m) \geq min\left(f(2), f_n\left(\frac{n-1}{2}\right)\right)$. For $m \in \{2, \ldots, \frac{n-1}{2}\}$:

$$\frac{f_n(m)}{f_n(m-1)} = \frac{m}{m-1}\frac{n-m}{n-m+1}\frac{(n-2m+1)(n-2m+2)}{(n-m+1)m}$$

Define $g(x) = ln\left(\frac{f_n(x)}{f_n(x-1)}\right)$ for $x > 1$. Then $\forall x > 1$, $g'(x) = -\frac{1}{n-x} - \frac{1}{x-1} + \frac{2}{n-x+1} - \frac{2}{n-2x+1} - \frac{2}{n-2x+2}$. Because $\frac{2}{n-x+1} < \frac{2}{n-2x+1}$ for any positive x, $g'(x) < 0$ for all $x > 1$. Consequently, $\frac{f_n(m)}{f_n(m-1)}$ decreases in m on $\{2, \ldots, \frac{n-1}{2}\}$. Moreover, $\frac{f_n(2)}{f_n(1)} = \frac{(n-2)^2(n-3)}{(n-1)^2} \geq 1$ and $\frac{f_n\left(\frac{n-1}{2}\right)}{f_n\left(\frac{n-3}{2}\right)} = \frac{8(n+1)}{(n-3)(n+3)^2} \leq 1$. Therefore, $f_n(m) \geq min\left(f_n(,2), f_n\left(\frac{n-1}{2}\right)\right)$ for all $m \in \{1, \ldots, \frac{n-1}{2}\}$. Substituting $m = \frac{n-1}{2}$ in $f_n(m)$, we find that $f_n\left(\frac{n-1}{2}\right) \geq n(n-1) \Leftrightarrow n^3 - 7n^2 + 7n - 1 \geq 0$, which holds for all $n > 5$. Substituting $m = 2$ in $f_n(m)$, we find that $f_n(2) \geq n(n-1) \Leftrightarrow n^3 - 8n^2 + 17n - 12 \geq 0$, which holds for all $n > 8$. Therefore, if $n \geq 9$, for all m, F, $\theta_m > \theta^*_m$. For $n \in \{3, 5, 7\}$ we can compute directly the lower bound for θ_m, $\frac{m(n-m)}{n(n-1)}$, and the upper bound for θ^*_m, $\frac{1}{1+\binom{n-m}{m}}$, for $m \in \{1, \ldots, \frac{n-1}{2}\}$ and verify that the result continues to hold. □

A.5 Proof of Proposition 3

Proposition 3. *For all n and m, there exist distributions $\mathbf{F}'$ such that if $F \in \mathbf{F}'$ then $W < W_0$ for all n and m.*

Proof. Recall that V_m denotes the sum of realized minority values ($V_m = \sum_{i \in m} v_i$), and V_M the sum of realized majority values ($V_M = \sum_{j \in M} v_j$). Suppose $F(v) = v^b$, $b > 0$. We show here that $W < W_0$ if $b \geq 1$, for all n, m. For value realizations such that the condition in Theorem 1 is not satisfied, the equilibrium construction selects the majority voting outcome, and thus $(W|\overline{v}_g < \mu v_{(2)G}) = (W_0|\overline{v}_g < \mu v_{(2)G})$. When the value realizations are in areas R ($\underline{\rho}\overline{v}_M > \overline{v}_m > \mu v_{(2)M}$) and P ($\overline{\rho}\overline{v}_M > \overline{v}_m > \underline{\rho}\overline{v}_M$), $\overline{v}_M > \overline{v}_m$, and given $m < M$ and i.i.d. values, it follows that $E[V_M|R, P] > E[V_m|R, P]$. Thus $(W|\overline{\rho}\overline{v}_M > \overline{v}_m > \mu v_{(2)M}) < (W_0|\overline{\rho}\overline{v}_M > \overline{v}_m > \mu v_{(2)M})$. Hence, for all n and m, a sufficient condition for $W < W_0$ is $E[V_M|B] > E[V_m|B]$, where B is the area of value realizations such that $\overline{v}_m > \overline{\rho v}_M, \overline{v}_M > \mu v_{(2)m}$. The proposition is an immediate result of the following Lemma.

Lemma 4. *If $F(v) = v^b$, then:*

$$\Pr(B)E(V_m|B) = \frac{bm}{b+1} - \frac{b^2Mm}{(bn+1)(b+1)}\rho^{bm+1} - \\ - \mu^{bM}\frac{b^2m(m-1)}{bn+1}\left[\frac{1}{b(n-1)} + \frac{(b(m-1)+1)}{(b+1)(b(n-1)+1)}\right]$$

$$\Pr(B)E(V_M|B) = \frac{bM}{b+1} - \frac{bM(bM+1)}{(b+1)(bn+1)}\bar{\rho}^{bm} - \frac{b^3Mm(m-1)\mu^{bM+1}}{(b+1)(b(n-1)+1)(bn+1)}$$

Proof. Recall that $\bar{v}_m > \overline{\rho v}_M, \bar{v}_M > \mu v_{(2)m}$. If we call $x = \bar{v}_m, y = v_{(2)m}$, and $z = \bar{v}_M$, then:

$$\Pr(B)E[V_M|B] = \int_{x=0}^{1}\int_{y=0}^{x}\int_{z=\mu y}^{min\left(\frac{x}{\rho},1\right)} \left[\frac{bM+1}{b+1}z\right] b^2m(m-1)y^{b(m-1)-1}x^{b-1}Mbz^{bM-1}$$
$$= \frac{bM}{b+1} - \frac{bM(bM+1)}{(b+1)(bn+1)}\bar{\rho}^{bm} - \frac{b^3Mm(m-1)\mu^{bM+1}}{(b+1)(b(n-1)+1)(bn+1)}$$

$$\Pr(B)E[V_m|B] = \int_{x=0}^{1}\int_{y=0}^{x}\int_{z=\mu y}^{min\left(\frac{x}{\rho},1\right)} \left[x + \frac{b(m-1)+1}{b+1}y\right] b^2m(m-1)y^{b(m-1)-1}x^{b-1}Mbz^{bM-1}$$
$$= \frac{bm}{b+1} - \frac{b^2Mm}{(bn+1)(b+1)}\rho^{bm+1} - \mu^{bM}\frac{b^2m(m-1)}{bn+1}\left[\frac{1}{b(n-1)} + \frac{(b(m-1)+1)}{(b+1)(b(n-1)+1)}\right]$$

□

The proof of Proposition 3 proceeds in two stages. First, we show that if $W < W_0$ for $b = 1$, the uniform case, then $W < W_0$ for $b > 1$. Second, we show that $W < W_0$ for $b = 1$. Given Lemma 4, for any b, a sufficient condition for $W < W_0$ is:

$$
\begin{aligned}
W < W_0 \;\Leftarrow\;& 2\frac{b}{b+1}\left[\frac{M-m}{2} - \frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm}\right] \\
-\;& \frac{b^2 m(m-1)}{(b+1)(b(n-1)+1)(bn+1)}\left[bM\mu - \frac{b^2 m(n-1)+b(n-1)+bn+1}{b(n-1)}\right]\mu^{bM} > 0 \\
\Leftrightarrow\;& 2\frac{b}{b+1}\left[\frac{M-m}{2} - \frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm}\right] \\
-\;& 2\frac{b}{b+1}\left[\frac{bm(m-1)}{b(n-1)}\frac{1}{2(b(n-1)+1)(bn+1)}\left[b^2(n-1)(M\mu-m-1)-bn-1\right]\mu^{bM}\right] > 0
\end{aligned}
$$

Note first that:

$$
\begin{aligned}
b^2(n-1)(M\mu-m)-bn-1 &< b^2(n-1)(n\mu-1)-b(n-1) \\
&< b^2(n-1)(n-1)-b(n-1)
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\frac{b^2(n-1)(M\mu-m-1)-bn-1}{2(b(n-1)+1)(bn+1)} &< \frac{b(n-1)}{2(bn+1)} \\
&\leq \frac{1}{2}
\end{aligned}
$$

Thus $W < W_0$ if:

$$
\frac{M-m}{2} - \frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm} - \frac{m(m-1)}{(n-1)}\frac{\mu^{bM}}{2} \geq 0 \tag{24}
$$

Straightforward manipulations show that $\frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm}$ is decreasing in b. The term in μ^{bM} is obviously decreasing in b. Hence, if condition (24) is satisfied at $b=1$, then it is satisfied for all $b>1$. We can thus focus on the case $b=1$. First, consider the case $n=3$. If $n=3$, then $m=1$, $M=2$, $\bar{\rho}=2/3$, and $\mu=2/3$. Condition (24) becomes: $\frac{1}{2}-7/18>0$ and is satisfied. Suppose then $n>3$. Substituting $b=1$ and $M=n-m$, condition (24) becomes:

$$
\frac{n-2m}{2} - \frac{(n-m)(n-m+1-m\bar{\rho})}{2(n+1)}\bar{\rho}^{m} - \frac{m(m-1)}{2(n-1)}\mu^{n-m} \geq 0
$$

Suppose first $n \geq (2m+3)$, or $m \leq (n-3)/2$. Given $n>3$ and $\bar{\rho} = 1-\frac{8}{(n+1)(n+3)}$, if

$n \geq (2m+3)$**:**

$$\frac{n-2m}{2} - \frac{(n-m)(n-m+1-m\bar{\rho})}{2(n+1)}\bar{\rho}^m \geq \frac{37m}{2(n+1)^2(n+3)}$$

Hence the condition becomes:

$$\frac{37m}{(n+1)^2(n+3)} - \frac{m(m-1)}{(n-1)}\left(\frac{1}{2}\right)^{n-m} \geq 0$$

But $37m(n-1) - m(m-1)(n+1)^2(n+3)\left(\frac{1}{2}\right)^{n-m} \geq 37m(n-1) - m(m-1)(n+1)^2(n+3)\left(\frac{1}{2}\right)^{\frac{n+1}{2}} \geq 37(n-1) - (m-1)(n+1)^2(n+3)\left(\frac{1}{2}\right)^{\frac{n+1}{2}}$. Finally, notice that $37(n-1) - (m-1)(n+1)^2(n+3)\left(\frac{1}{2}\right)^{\frac{n+1}{2}}$ evaluated at $m = \frac{n-1}{2}$ is always positive for any $n > 3$, and thus must be positive for all $m \leq (n-3)/2$. Therefore, condition (24) is always satisfied for $n > 3$ and $n \geq (2m+3)$ The condition $n \geq (2m+3)$ excludes the only case $m = \frac{n-1}{2}$. Suppose then $m = \frac{n-1}{2}$. In this case, $M\mu - m < 0$ and the term in μ^M in condition 24 is positive. A sufficient condition for $W < W_0$ is then:

$$2\frac{b}{b+1}\left[\frac{M-m}{2} - \frac{M(bM+1-bm\bar{\rho})}{2(bn+1)}\bar{\rho}^{bm}\right] > 0$$

or, with $m = \frac{n-1}{2}$:

$$\frac{1}{2} - \frac{\frac{n+3}{2} - \frac{n-1}{2}\bar{\rho}}{4}\bar{\rho}^m = \frac{1}{2} - \frac{2 + \frac{8}{(n+1)(n+3)}}{4}\bar{\rho}^m > 0$$

Or:

$$\left[1 + \frac{4}{(n+1)(n+3)}\right] exp\left(\frac{n-1}{2} ln\left(1 - \frac{8}{(n+1)(n+3)}\right)\right) < 1$$

Denote $x = \frac{4}{(n+1)(n+3)}$. Note that:

$$\begin{aligned} exp\left(\frac{n-1}{2} ln\left(1 - \frac{8}{(n+1)(n+3)}\right)\right) &= exp\left(\frac{n-1}{2} ln\left(1-2x\right)\right) \\ &< exp\left(-(n-1)x\right) \end{aligned}$$

But $f(x) = (1+x)exp(-(n-1)x)$ is decreasing in x and is equal to 1 at $x = 0$. Hence, the inequality is satisfied, for any n. This concludes the proof. $\square$

Supplementary material not for publication

B Proof of Theorem 3

Theorem 3. *Suppose R2 is the rationing rule. For all $n > 3$ odd, m, and F, there exists a threshold $\mu_{R2}(n) > 0$ such that if $\overline{v}_g \geq \mu_{R2}(n)Max[v_{(2)G}, v_{(2)g}]$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $(n-1)/2$ votes (with probabilities $q'_{\overline{G}}$ and $q'_{\overline{g}}$ respectively) and selling their vote, and all other individuals sell. The randomization probabilities $q'_{\overline{G}}$ and $q'_{\overline{g}}$ and the price p' depend on the realized values $\overline{v}_g$ and $\overline{v}_G$, but for all $\overline{v}_G$ and $\overline{v}_g \geq \mu_{R2}(n)Max[v_{(2)G}, v_{(2)g}]$, $q'_{\overline{G}} \in [\frac{n-1}{n+1}, 1]$ and $q'_{\overline{g}} \in [\frac{n-1}{n+1}, 1]$. The threshold $\mu_{R2}(n)$ is given by:*

$$\mu_{R2}(n) = \frac{(n-1)^2}{2^{n-2}n}\binom{n-3}{\frac{n-3}{2}}$$

Proof. The theorem is implied by the following three lemmas.

Lemma 5. *Suppose $\frac{\overline{v}_M}{\overline{v}_m} \geq \frac{n+1}{n-1}$. Then for all $n > 3$ odd, m, and F, if $\overline{v}_m \geq \mu_{R2}(n)Max[v_{(2)M}, v_{(2)m}]$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_M$ demands $\frac{n-1}{2}$ votes with probability 1, $\overline{v}_m$ randomizes between demanding $\frac{n-1}{2}$ votes (with probability $q'_{\overline{m}} = \frac{n-1}{n+1}$ and selling, and all others sell. The equilibrium price p' equals $\frac{\overline{v}_m}{n-1}$.*

Lemma 6. *Suppose $\frac{\overline{v}_M}{\overline{v}_m} \leq \frac{n+3}{n+1}$. Then for all $n > 3$ odd, m, and F, if $\overline{v}_M \geq \mu_{R2}(n)Max[v_{(2)M}, v_{(2)m}]$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_m$ demands $\frac{n-1}{2}$ votes with probability 1, $\overline{v}_M$ randomizes between demanding $\frac{n-1}{2}$ votes (with probability $q'_{\overline{M}} = \frac{n-1}{n+1}$ and selling, and all others sell. The equilibrium price p' equals $\frac{\overline{v}_M}{n-1}$.*

Lemma 7. *Suppose $\frac{\overline{v}_M}{\overline{v}_m} \in \left(\frac{n+3}{n+1}, \frac{n-1}{n+1}\right)$. Then for all $n > 3$ odd, m, and F, if :*

$$\overline{v}_m \geq \mu_{R2}(n)\frac{2(nx-n-1)}{(n-1)(x-1)x}Max[v_{(2)M}, v_{(2)m}]$$

where $x \equiv \frac{\overline{v}_M}{\overline{v}_m}$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_M$ and $\overline{v}_m$ randomize between demanding $\frac{n-1}{2}$ votes (with probabilities $q'_{\overline{M}}$ and $q'_{\overline{m}}$ respectively) and *selling their vote, and all other individuals sell. The randomization probabilities* $q'_{\overline{M}}$ *and* $q'_{\overline{m}}$ *and the price* p' *solve:*

$$\begin{aligned} q'_{\overline{M}} + q'_{\overline{m}} &= \frac{2n}{n+1} \\ p' &= \left(\frac{2 - q'_{\overline{M}}}{n-1}\right) \overline{v}_m \\ p' &= \left(\frac{q'_{\overline{m}}}{n-1}\right) \overline{v}_M. \end{aligned}$$

Note that in lemmas 5 and 7, $\overline{v}_m < \overline{v}_M$, or $\overline{v}_m \equiv \overline{v}_g$, and the condition thus applies to $\overline{v}_g$, as stated in the theorem. In Lemma 6 the condition is stated in terms of $\overline{v}_M$, and $\overline{v}_M \lesseqgtr \overline{v}_m$, but if the condition is satisfied for $\overline{v}_g = \min[\overline{v}_M, \overline{v}_m]$, then it is always satisfied for $\overline{v}_M$ (i.e. the condition stated in the theorem is sufficient for the condition stated in the lemma). Finally, in Lemma 7, the condition depends on $x \equiv \frac{\overline{v}_M}{\overline{v}_m}$. Over the interval $x \in (\frac{n+3}{n+1}, \frac{n+1}{n-1})$, the expression $\frac{2(nx-n-1)}{(n-1)(x-1)x}$ is increasing in x, and maximal at $x = \frac{n+1}{n-1}$ where $\frac{2(nx-n-1)}{(n-1)(x-1)x} = 1$ for all n. Hence again the condition stated in the theorem is sufficient for the condition stated in the lemma. As in the case of Theorem 1, the proof is organized in two stages. First, we show that the strategies and price described in the lemmas are an equilibrium if the direction of preferences associated with each demand is commonly known. Second, we show that when preferences are private information the equilibrium is fully revealing.

Ex ante equilibrium with full information

Suppose first that the direction of preferences associated with each demand is commonly known. Expected market balance requires $(q'_{\overline{M}} + q'_{\overline{m}})(n-1)/2 = (n-2) + (1 - q'_{\overline{M}}) + (1 - q'_{\overline{m}})$, or:

$$q'_{\overline{M}} + q'_{\overline{m}} = \frac{2n}{n+1}$$

We begin by proving Lemma 5.

Proof of Lemma 5.

Recall that we denote by $U_m(s)$ the expected utility to voter $\overline{v}_m$ from demand s (and similarly for $U_M(s)$). Then, in the candidate equilibrium:

$$\begin{aligned} U_m(-1) &= \frac{p'}{2} \\ U_m\left(\frac{n-1}{2}\right) &= \frac{\overline{v}_m}{2} - \frac{n-2}{2} p' \end{aligned}$$

Indifference between the two actions requires:

$$p' = \frac{\overline{v}_m}{n-1}$$

By expected market balance, if $q'_{\overline{M}} = 1$, then:

$$q'_{\overline{m}} = \frac{n-1}{n+1}.$$

To verify that this is indeed an equilibrium, we need to rule out profitable deviations.

(i) Consider first voter $\overline{v}_M$. For any $s_M \in (\frac{n-1}{2}, n-1]$, $U_M(s_M) < U_M(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority is strictly costly and does not affect the probability of rationing $\overline{v}_m$ (because $s_M > \frac{n-1}{2}$ becomes relevant only once $s_M = \frac{n-1}{2}$ is satisfied, at which point $\overline{v}_m$ is already rationed and $\overline{v}_M$ holds a majority of votes). For any $s_M \in [0, \frac{n-1}{2})$, $U_M(s_M) < U_M(-1)$: demanding less than $\frac{n-1}{2}$ votes is dominated by selling because demanding any positive number of votes less than $\frac{n-1}{2}$ would be costly and not affect the outcome, whether $\overline{v}_m$ is selling or demanding $\frac{n-1}{2}$. Therefore, the majority leader is optimizing if and only if the deviation to selling is not profitable. In the candidate equilibrium:

$$\begin{aligned} U_M(-1) &= q'_{\overline{m}}(\frac{1}{2}p') + (1 - q'_{\overline{m}})(\overline{v}_M) \\ U_M\left(\frac{n-1}{2}\right) &= q'_{\overline{M}}(\frac{\overline{v}_m}{2} - \frac{n-2}{2}p') + (1 - q'_{\overline{M}})(\overline{v}_m - \frac{n-1}{2}p) \end{aligned}$$

The deviation is not desirable if and only if $\frac{\overline{v}_M}{\overline{v}\ m} \geq \frac{n+1}{n-1}$.

(ii) Consider voter $\overline{v}_m$. Given $s_M = \frac{n-1}{2}$, $U_m(s_m) < U_m(\frac{n-1}{2})$ for all $s_m > 0 \neq \frac{n-1}{2}$, and $U_m(0) < U_m(-1)$. Hence no deviation dominates randomizing over selling or demanding $\frac{n-1}{2}$.

(iii) Consider now $v_i \in M$, $v_i \neq \overline{v}_M$. Here the rationing rule makes an important difference. With $R2$, any incremental demand has a positive incremental impact on the probability that $\overline{v}_m$ and/or $\overline{v}_M$ will be rationed. We need to consider and exclude deviation to any $s_i \in \left[0, \frac{n-1}{2}\right]$. We show here, however, that for all $v_i \in M$, $v_i \neq \overline{v}_M$, $U_i(-1) \geq U_i(0)$ is sufficient to guarantee $U_i(-1) \geq U_i(s_i)$ for all $s_i \in \left[0, \frac{n-1}{2}\right]$. Hence only one possible deviation, to $s_i = 0$, needs to be ruled out. It is this step that makes the proof possible. Consider the utilities from demanding $s+1$ votes and demanding s. The probability of receiving 0 to $s-1$ votes is identical when demanding s or $s+1$ votes. The probability of receiving s votes when demanding s votes is equal to the probability of receiving s or $s+1$ votes when demanding $s+1$ votes. Therefore, calling x the number of votes received after rationing, for all $s \in \left[0, \frac{n-5}{2}\right]$:

$$\begin{aligned} U_i(s+1) - U_i(s) &= (1-q'_{\overline{m}})(-p') + q'_{\overline{m}}[P(x_i = s+1|s+1)] \cdot \\ &\cdot \left[(P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s) - P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s+1))v - p'\right] \end{aligned}$$

Calling $[(P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s) - P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s+1))v - p'] \equiv \Delta(s)$, we can rewrite the expression more concisely as:

$$U_i(s+1) - U_i(s) = q'_{\overline{m}}[P(x_i = s+1|s+1)\Delta(s)] - (1-q'_{\overline{m}})p' \tag{25}$$

and thus, for $s \in [0, \frac{n-5}{2}]$:

$$U_i(s) - U_i(s-1) = q'_{\overline{m}}(P(x_i = s|s)\Delta(s-1) - (1-q'_{\overline{m}})p' \tag{26}$$

where, as argued above, $P(x_i = s|s) > P(x_i = s+1|s+1)$. Given:

$$P(x_{\overline{m}} = \frac{n-1}{2}|s_i = s) = \sum_{z=\frac{n-1}{2}}^{n-3-s} \binom{n-3-s}{z} \left(\frac{1}{2}\right)^{n-3-s} \quad \forall s \in \left[0, \frac{n-5}{2}\right],$$

and hence:

$$\Delta(s) = \sum_{z=\frac{n-1}{2}}^{n-3-s} \left(\frac{1}{2}\right)^{n-4-s} \left[\binom{n-4-s}{z-1}(1 - \frac{n-3-s}{2z})\right] v - p',$$

it is possible to show that $\Delta(s) \leq 0$ implies $\Delta(s+1) \leq 0$ for all $s \in \left[0, \frac{n-5}{2}\right]$[37]. It follows that if 0 is preferred to 1, then 0 dominates all strategies up to buying $\frac{n-3}{2}$

[37] The proof requires some work. Details are posted at: columbia.edu/~st2511/demundone/theorem3_supp.pdf.

votes. From equation 25:

$$U_i(1) - U_i(0) = q'_{\overline{m}}(P(x_i = 1|1)\Delta(0) - (1 - q'_{\overline{m}})p'$$

and since

$$\Delta(0) = \left[\frac{\binom{n-4}{\frac{n-7}{2}}}{2^{n-4}} - \frac{\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}}\right] v_i - p'$$

it follows that $U_i(1) < U_i(0)$ if $\Delta(0) \leq 0$ or, given $p' = \frac{\overline{v}_m}{n-1}$, $U_i(1) < U_i(0)$ for all $v_i \in M$, $v_i \neq \overline{v}_M$, if $\overline{v}_m \geq \frac{(n-1)\left[4\binom{n-4}{\frac{n-7}{2}} - \binom{n-3}{\frac{n-3}{2}}\right]}{2^{n-2}} v_{(2)M}$. But:

$$\begin{aligned} U_i(0) &= q'_{\overline{m}}\left[\frac{1}{2} + \frac{\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}}\right] v_i + (1 - q'_{\overline{m}})v_i \\ U_i(-1) &= q'_{\overline{m}}\left(\frac{v_i}{2} + p'\right) + (1 - q'_{\overline{m}})\left(v_i + \frac{p'}{2}\right) \end{aligned}$$

and thus:

$$U_i(0) < U_i(-1) \text{ for all } v_i \in M, v_i \neq \overline{v}_M \text{ if } \frac{\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} v_{(2)M} \leq \frac{v_{\bar{m}}}{n-1}$$

or:

$$\overline{v}_m \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} v_{(2)M}. \tag{27}$$

Note that $4\binom{n-4}{\frac{n-7}{2}} - \binom{n-3}{\frac{n-3}{2}} \leq \binom{n-3}{\frac{n-3}{2}}$. Hence, the last condition is sufficient for $\Delta(0) \leq 0$. It is the condition in the lemma, and it is sufficient to establish both that $s_i = -1$ dominates $s_i = 0$, and that $s_i = 0$, and hence $s_i = -1$, dominate all $s_i \in [1, \frac{n-3}{2}]$. The last step in proof is verifying that a deviation to $\frac{n-1}{2}$is not profitable. Note that $P(x_{\overline{m}} = \frac{n-1}{2}|s_i = \frac{n-1}{2}) = P(x_{\overline{m}} = \frac{n-1}{2}|s_i = \frac{n-3}{2})$: demanding $\frac{n-1}{2}$ does not change the probability that $\overline{v}_m$ receive $\frac{n-1}{2}$ votes, relative to demanding $\frac{n-3}{2}$. It may however lead to a higher number of votes paid. Thus $s_i = \frac{n-1}{2}$ is dominated by $s_i = \frac{n-3}{2}$ which, as we have seen, is dominated by $s_i = 0$. Ruling out a profitable deviation to 0 is thus sufficient to rule out all other deviations. It follows that no deviation is profitable if equation 27 is satisfied.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. With probability $q'_{\overline{m}}$, $\overline{v}_m$ demands $\frac{n-1}{2}$ votes, as does $\overline{v}_M$. In this case, a demand of votes by v_i is justified if it increases the probability that $\overline{v}_M$ is rationed. This is exactly the reasoning we considered in point (iii) above, for $v_i \in M$. We established there that if $s_i = -1$ dominates $s_i = 0$, then it dominates all $s_i \in [0, \frac{n-3}{2}]$. With probability $(1 - q'_{\overline{m}})$, however, $\overline{v}_m$ sells his vote. Since $\overline{v}_M$ demands $\frac{n-1}{2}$ votes with probability 1, in this case $s_i = 0$ is dominated by $s_i = -1$ and any $s_i \in [1, \frac{n-3}{2}]$ is dominated by $s_i = \frac{n-1}{2}$ (because for any $s_i \in [1, \frac{n-3}{2}]$, neither $\overline{v}_M$ nor v_i are rationed, $\overline{v}_M$ wins, and v_i pays $s_i p'$). We conclude the only deviations from $s_i = -1$ that cannot be excluded are to $s_i = 0$, and $s_i = \frac{n-1}{2}$. The condition $U_i(0) < U_i(-1)$ leads to a condition parallel to equation 27):

$$\overline{v}_m \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} v_{(2)m}. \tag{28}$$

Consider now $U_i\left(\frac{n-1}{2}\right)$. If $\overline{v}_m$ demands $\frac{n-1}{2}$ votes, v_i can expect to receive $\dfrac{n-3}{3}$ votes. The minority wins unless $\overline{v}_M$ receives $\dfrac{n-1}{2}$ votes. If $\overline{v}_m$ sells his vote, v_i receives $\dfrac{n-1}{2}$ votes with probability $\frac{1}{2}$ (and wins), and $\dfrac{n-3}{2}$ votes with probability $\frac{1}{2}$ (and loses). Hence:

$$\begin{aligned} U_i\left(\frac{n-1}{2}\right) &= q'_{\overline{m}}[(1 - P(x_{\overline{M}} = \frac{n-1}{2} | s_m = \frac{n-1}{2}, s_i = \frac{n-1}{2}))v_i - \frac{n-3}{3}p'] + \\ &\quad +(1-q'_{\overline{m}})(\frac{1}{2}v_i - \frac{n-2}{2}p') \end{aligned}$$

Call $P(x_{\overline{M}} = \frac{n-1}{2} | s_m = \frac{n-1}{2}, s_i = \frac{n-1}{2}) = \sum_{z=\frac{n-1}{2}}^{n-3} \sum_{y=0}^{n-3-z} \binom{n-3}{n-3-z-y, z, y} \left(\frac{1}{3}\right)^{n-3} \equiv \delta$. For all $v_i \in m$, $v_i \neq \overline{v}_m$, the deviation to buying $\frac{n-1}{2}$ is not desirable if:

$$\overline{v}_m \geq \frac{n(3-6\delta) + 3 + 6\delta}{2n+6} v_{(2)m}.$$

This constraint is not binding if $n = 5$ (when the ratio equals $\frac{24}{23} > 1$) and when $n = 7$ (when the ratio equals 1), and it is less stringent than equation 28 for all $n \geq 9$.

We conclude that the equilibrium exists if $\frac{\overline{v}_M}{\overline{v}\ _m} \geq \frac{n+1}{n-1}$, and

$$\overline{v}_m \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} Max[v_{(2)M}, v_{(2)m}],$$

as stated in the lemma. □

Proof of Lemma 6 .

In the candidate equilibrium:

$$\begin{aligned} U_M(-1) &= \frac{p'}{2} \\ U_M\left(\frac{n-1}{2}\right) &= \frac{\overline{v}_M}{2} - \frac{n-2}{2}p' \end{aligned}$$

Indifference between the two actions requires:

$$p' = \frac{\overline{v}_M}{n-1}$$

By expected market balance, if $q'_{\overline{m}} = 1$, then:

$$q'_{\overline{M}} = \frac{n-1}{n+1}.$$

To verify that this is indeed an equilibrium, we need to rule out profitable deviations.

(i) Consider first voter $\overline{v}_M$. Given $s_m = \frac{n-1}{2}$, $U_M(s_M) < U_M(\frac{n-1}{2})$ for all $s_M > 0 \neq \frac{n-1}{2}$, and $U_M(0) < U_M(-1)$. Hence no deviation dominates randomizing over selling or demanding $\frac{n-1}{2}$.

(ii) Consider voter $\overline{v}_m$. Call $P(k)$ the probability of a minority victory when $\overline{v}_m$ demands k votes, for $k < \frac{n-1}{2}$. Then:

$$\begin{aligned} U_m(-1) &= q'_{\overline{M}}(\frac{1}{2}p') \\ U_m\left(\frac{n-1}{2}\right) &= q'_{\overline{M}}(\frac{\overline{v}_m}{2} - \frac{n-2}{2}p') + (1-q'_{\overline{M}})(\overline{v}_m - \frac{n-1}{2}p) \\ U_m(k) &= q'_{\overline{M}}(-kp') + (1-q'_{\overline{M}})(P(k)\overline{v}_m - kp') \end{aligned}$$

where

$$P(k|n,m)=\frac{\sum_{i=\frac{n+1}{2}-m}^{k}\binom{n-m}{i}\binom{m-1}{k-i}}{\binom{n-1}{k}}$$

Note that $P(k)=0$ if $k<\frac{n+1}{2}-m$. Moreover, with n fixed, $P(k)$ is increasing in m for $k\in\left[\frac{n+1}{2}-m,\frac{n-3}{2}\right]$[38] Thus if $U_m\left(\frac{n-1}{2}\right)>U_m(k)$ when $m=M-1$, then $U_m\left(\frac{n-1}{2}\right)>U_m(k)$ for all $m<M$. Suppose then $m=M-1$. In this case:

$$P(k)=1-\frac{\binom{m-1}{k}}{\binom{n-1}{k}}$$

and, for $0\le k\le\frac{n-5}{2}$:

$$\begin{aligned}U(k+1)-U(k) &= -p+(1-q'_M)(P(k+1)-P(k))\overline{v}_m\\ &= -p+\frac{\binom{m-1}{k}}{\binom{n-1}{k}}\frac{n-m}{n-1-k}\overline{v}_m\end{aligned}$$

The difference is decreasing in k: if one defines $h(k)=\frac{\binom{m-1}{k}}{\binom{n-1}{k}}\frac{n-m}{n-1-k}$,

$$\frac{h(k-1)}{h(k)}=\frac{n-1-k}{m-k}>1$$

It follows that $U_m\left(\frac{n-1}{2}\right)\ge U_m(k)$ for all $k\in[0,\frac{n-3}{2}]$ if $U_m\left(\frac{n-1}{2}\right)\ge U_m(0)$. But note that $U_m(-1)\ge U_m(0)=0$. Hence $s_m=-1$ is the only possibly profitable deviation for $\overline{v}_m$. $U_m\left(\frac{n-1}{2}\right)\ge U_m(-1)$ yields the condition: $\frac{\overline{v}_M}{\overline{v}}{}_m\le\frac{n+3}{n+1}$.

(iii) Consider now $v_i\in M$, $v_i\neq\overline{v}_M$. The incentives are identical to (iv) in the proof of Lemma 5 with $\overline{v}_m$ demanding $\frac{n-1}{2}$ with probability 1, the only possibly profitable deviation for $v_i\in M$ are either $s_i=0$ or $s_i=\frac{n-1}{2}$. For all $v_i\in M$, $v_i\le v_{(2)M}$, $U_i(-1)\ge U_i(0)$ if:

$$\overline{v}_m\ge\frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}}v_{(2)M},\tag{29}$$

[38] See columbia.edu/~st2511/demundone/theorem3_supp.pdf.

and $U_i(-1) \geq U_i(\frac{n-1}{2})$ if:

$$\overline{v}_m \geq \frac{n(3-6\delta)+3+6\delta}{2n+6} v_{(2)M}.$$

where $\delta \equiv P(x_{\overline{m}} = \frac{n-1}{2} | s_M = \frac{n-1}{2}, s_i = \frac{n-1}{2}) = .\sum_{z=\frac{n-1}{2}}^{n-3} \sum_{y=0}^{n-3-z} \binom{n-3}{n-3-z-y,z,y} \left(\frac{1}{3}\right)^{n-3}$. This latter condition is not binding for $n = \{5, 7\}$ and is less stringent than equation29 for all $n \geq 9$. Thus equation 29 is sufficient to guarantee that no $v_i \in M$, $v_i \neq \overline{v}_M$ has an incentive to deviate.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. The incentives are identical to (iii) in the proof of Lemma 5 with $\overline{v}_m$ demanding $\frac{n-1}{2}$ with probability 1, the only possibly profitable deviation for $v_i \in m$ is to stay out of the market. For all $v_i \in m$, $v_i \leq v_{(2)m}$, $U_i(-1) \geq U_i(0)$ if:

$$\overline{v}_M \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} v_{(2)m}.$$

We conclude that the equilibrium exists if $\frac{\overline{v}_M}{\overline{v}\ _m} \leq \frac{n+3}{n+1}$, and

$$\overline{v}_M \geq \frac{(n-1)\binom{n-3}{\frac{n-3}{2}}}{2^{n-2}} Max[v_{(2)M}, v_{(2)m}],$$

as stated in the lemma. □

Proof of Lemma 7. In the candidate equilibrium:

$$\begin{aligned} U_m(-1) &= q'_{\overline{M}}(\frac{1}{2}p') \\ U_m\left(\frac{n-1}{2}\right) &= q'_{\overline{M}}(\frac{\overline{v}_m}{2} - \frac{n-2}{2}p') + (1-q'_{\overline{M}})(\overline{v}_m - \frac{n-1}{2}p') \end{aligned}$$

and

$$\begin{aligned} U_M(-1) &= q'_{\overline{m}}(\frac{1}{2}p') + (1-q'_{\overline{m}})(\overline{v}_M) \\ U_M\left(\frac{n-1}{2}\right) &= q'_{\overline{m}}(\frac{\overline{v}_M}{2} - \frac{n-2}{2}p') + (1-q'_{\overline{m}})(\overline{v}_M - \frac{n-1}{2}p') \end{aligned}$$

The two equivalence conditions yield:

$$\begin{aligned} p' &= \left(\frac{2 - q'_{\overline{M}}}{n-1}\right) \overline{v}_m \\ p' &= \left(\frac{q'_{\overline{m}}}{n-1}\right) \overline{v}_M \end{aligned}$$

This system has a solution if and only if:

$$\frac{n+3}{n+1} \leq \frac{\overline{v}_M}{\overline{v}_m} \leq \frac{n+1}{n-1}$$

Given the expected market clearing constraint $q'_{\overline{M}} + q'_{\overline{m}} = \frac{2n}{n+1}$, we obtain:

$$\begin{aligned} q'_{\overline{M}} &= 2\frac{nx - n - 1}{(n+1)(x-1)} \\ q'_{\overline{m}} &= \frac{2}{(n+1)(x-1)} \end{aligned}$$

with $x = \frac{\overline{v}_M}{\overline{v}_m}$. Consider now the scope for deviations.

(i) Consider first voter $\overline{v}_M$. As in the proof of Lemma 6, $U_M(s_M) < U_M(\frac{n-1}{2})$ for all $s_M > 0 \neq \frac{n-1}{2}$, and $U_M(0) < U_M(-1)$. Hence no deviation dominates randomizing over $s_M = -1$ and $s_M = \frac{n-1}{2}$.

(ii) Consider voter $\overline{v}_m$. Again, exactly as in the proof of Lemma 6, the only two possible best responses are $s_m = -1$ and $s_m = \frac{n-1}{2}$. Hence no profitable deviation exists when $\overline{v}_m$ randomizes over the two actions.

(iii) Consider now $v_i \in M$, $v_i \neq \overline{v}_M$. We have established above that if $s_M = \frac{n-1}{2}$ and $s_m = \frac{n-1}{2}$, v_i's best response can put positive probability on only two actions, either $s_i = -1$ or $s_i = 0$. If $s_M = \frac{n-1}{2}$ and $s_m = -1$, v_i's best response is $s_i = -1$. If $s_M = -1$ and $s_m = -1$, v_i's best response is either $s_i = -1$ or $s_i = 0$, which in this case are equivalent. Finally, if $s_M = -1$ and $s_m = \frac{n-1}{2}$, v_i's best response can put positive probability on only two actions, either $s_i = -1$ or $s_i = \frac{n-1}{2}$. It follows that all demands $s_i \in [1, \frac{n-3}{2}]$ are strictly dominated. Only $s_i = 0$ and $s_i = \frac{n-1}{2}$ are possible alternatives to $s_i = -1$: no profitable deviation exists if $U_i(-1) \geq U_i(0)$, and

$U_i(-1) \geq U_i(\frac{n-1}{2})$. We have:

$$U_i(-1) = q'_{\overline{M}} q'_{\overline{m}} (p' + \frac{1}{2} v_i) + q'_{\overline{M}} (1 - q'_{\overline{m}})(v_i + \frac{1}{2} p') + (1 - q'_{\overline{M}}) q'_{\overline{m}} \frac{1}{2} p' + (1 - q'_{\overline{M}})(1 - q'_{\overline{m}}) v_i;$$

$$U_i(0) = q'_{\overline{M}} q'_{\overline{m}} \left(\frac{1 + \binom{n-3}{(n-3)/2} (\frac{1}{2})^{n-3}}{2} v_i \right) + q'_{\overline{M}} (1 - q'_{\overline{m}}) v_i + (1 - q'_{\overline{M}})(1 - q'_{\overline{m}}) v_i,$$

where $\left(\frac{1 + \binom{n-3}{(n-3)/2} (\frac{1}{2})^{n-3}}{2} \right) = P(x_{\overline{M}} = \frac{n-1}{2} | s_m = \frac{n-1}{2}, s_i = 0)$, and:

$$\begin{aligned} U\left(\frac{n-1}{2}\right) &= q'_{\overline{m}} \left[q'_{\overline{M}} \left((1-\delta) v_i - \frac{n-3}{3} p' \right) + (1 - q'_{\overline{M}}) \left(\frac{v_i}{2} - \frac{n-2}{2} p' \right) \right] + \\ &\quad + (1 - q'_{\overline{m}}) \left[v_i - \left(q'_{\overline{M}} \cdot \frac{n-2}{2} + (1 - q'_{\overline{M}}) \cdot \frac{n-1}{2} \right) p' \right]. \end{aligned}$$

Hence, for all $v_i \in M$, $v_i \leq v_{(2)M}$, $U_i(-1) \geq U_i(0)$ if:

$$\overline{v}_M \geq \frac{\binom{n-3}{\frac{n-3}{2}}(n-1)}{2^{n-2} n} \frac{2(nx - n - 1)}{(x-1)} v_{(2)M}. \tag{30}$$

and $U_i(-1) \geq U_i(\frac{n-1}{2})$ if:

$$\overline{v}_M \geq \frac{3(n^2-1)(x-1)[(1+n)(1-x-4\delta) + 4\delta n x)]}{15 + 11n - 7n^2 - 3n^3 - 6x - 18nx + 10n^2 x + 6n^3 x + 3x^2 + 3nx^2 - 3n^2 x^2 - 3n^3 x^2} v_{(2)M}. \tag{31}$$

where $x \equiv \frac{\overline{v}_M}{\overline{v}_m}$. For $n = 5, 7$, the right-hand side of (31) is above 1 for any $x \in [\frac{n+3}{n+1}, \frac{n+1}{n-1}]$ and therefore the constraint is not binding. For $n \geq 9$, (31) is less stringent than (30)[39]. Hence (30) is sufficient to guarantee that all $v_i \in M$, $v_i \leq v_{(2)M}$, have no profitable deviation. By dividing both sides of (30) by x, we obtain the condition in the lemma.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. Exactly as described in point (iii) above, the analysis so far has established that only $s_i = 0$ and $s_i = \frac{n-1}{2}$ are possible alternatives to $s_i = -1$: no profitable deviation exists if $U_i(-1) \geq U_i(0)$, and $U_i(-1) \geq U_i(\frac{n-1}{2})$.

[39]The details are available from the authors.

We have:

$$U_i(-1) = q'_{\overline{M}} q'_{\overline{m}}(p' + \frac{1}{2}v_i) + q'_{\overline{M}}(1 - q'_{\overline{m}})\frac{1}{2}p' + (1 - q'_{\overline{M}})q'_{\overline{m}}(v_i + \frac{1}{2}p');$$

$$U_i(0) = q'_{\overline{M}} q'_{\overline{m}}(\frac{1 + \binom{n-3}{(n-3)/2}(\frac{1}{2})^{n-3}}{2}v_i) + (1 - q'_{\overline{M}})q'_{\overline{m}}v_i,$$

and:

$$\begin{aligned} U_i\left(\frac{n-1}{2}\right) &= q'_{\overline{M}} q'_{\overline{m}}\left[(1-\delta)v_i - \frac{n-3}{3}p'\right] + q'_{\overline{M}}(1 - q'_{\overline{m}})(\frac{1}{2}v_i - \frac{n-2}{2}p') + \\ &\quad +(1 - q'_{\overline{M}})q'_{\overline{m}}(v_i - \frac{n-2}{2}p') + (1 - q'_{\overline{M}})(1 - q'_{\overline{m}})(v_i - \frac{n-1}{2}p'). \end{aligned}$$

Thus, for all $v_i \in m$, $v_i \leq v_{(2)m}$, $U_i(-1) \geq U_i(0)$ if:

$$\overline{v}_m \geq \frac{\binom{n-3}{\frac{n-3}{2}}(n-1)}{2^{n-2}n}\frac{2(nx-n-1)}{(x-1)x}v_{(2)m} \tag{32}$$

and $U_i(-1) \geq U_i(\frac{n-1}{2})$ if:

$$\overline{v}_m \geq \frac{3(n^2-1)(x-1)[(1+n)(1+3x+x^2-4\delta)+4\delta nx)]}{x(15+11n-7n^2-3n^3-6x-18nx+10n^2x+6n^3x+3x^2+3nx^2-3n^2x^2-3n^3x^2)}v_{(2)m}. \tag{33}$$

As under point (iii) above, it is possible to show that (32) is a more stringent condition than (33)[40]. It is then the sufficient condition, guaranteeing that no profitable deviation exists for all $v_i \in m$, $v_i \neq \overline{v}_m$. $\square$

We now show that when preferences are private information, the strategies and price identified above constitute a fully revealing ex ante equilibrium.

Fully revealing equilibrium

We proceed as for Theorem 1. We conjecture an equilibrium identical to the full information equilibrium characterized above and show that given others' strategies, the equilibrium price and the knowledge that the market is in a fully revealing equilibrium, each voter's best response when preferences are private information is uniquely identified and equals the voter's best response with full information. Thus the equilibrium exists when preferences are private information and is indeed fully revealing.

[40]See columbia.edu/˜st2511/demundone/theorem3_supp.pdf

(i) Consider first the perspective of voter $\overline{v}_M$, in equilibrium. When the equilibrium exists, expected market balance requires $\overline{v}_M$ to demand a positive number of votes with positive probability. It then follows that the other voter who demands a positive number of votes with positive probability must belong to the minority. If not, $\overline{v}_M$'s best response would be to sell, violating expected market equilibrium. Thus $\overline{v}_M$ also knows that $M-1$ majority members and $m-1$ minority members are offering their vote for sale; he cannot identify them individually, but that is irrelevant. Given that the other net demand for votes comes from a minority voter, $\overline{v}_M$'s best response is identified uniquely and is identical to his best response under full information.

(ii) Consider then the perspective of voter $\overline{v}_m$. Suppose first that $\frac{\overline{v}_M}{\overline{v}\ m} \geq \frac{n+1}{n-1}$, and hence $s_M = \frac{n-1}{2}$ with probability 1. Expected market balance requires $\overline{v}_m$ to demand a positive number of votes with positive probability. But that can only be a best response if the voter who demands $\frac{n-1}{2}$ votes belongs to the majority. Again, $\overline{v}_m$ also knows that $M-1$ majority members and $m-1$ minority members are offering their vote for sale; he cannot identify them individually, but that is irrelevant. Suppose now $\frac{\overline{v}_M}{\overline{v}\ m} \in (\frac{n+3}{n+1}, \frac{n+1}{n-1})$. By market balance, the minimal demand on which $\overline{v}_m$ must put positive probability is $\frac{n-3}{2}$ (because $\frac{n-3}{2} = \left(\frac{n-1}{n+1}\right)\left(\frac{n-1}{2}\right) - (1 - .\frac{n-1}{n+1})$). Suppose that the voter demanding $\frac{n-1}{2}$ votes with probability $q'_{\overline{M}}$ were in fact a member of group m. Then, given that all others offer to sell:

$$\begin{aligned} U_{m,m}(-1) &= q'_{\overline{M}}(\overline{v}_m + \frac{p'}{2}) \\ U_{m,m}(\frac{n-3}{2}) &= q'_{\overline{M}}(\overline{v}_m - \frac{n-3}{2}p') + (1 - q'_{\overline{M}})(P(\frac{n-3}{2})\overline{v}_m - \frac{n-3}{2}p') \\ &\leq \overline{v}_m - \frac{n-3}{2}p' \end{aligned}$$

where $P(\frac{n-3}{2}) < 1$ is, as earlier, the probability that the minority wins when $\overline{v}_m$ is the only buyer in the market and purchases $\frac{n-3}{2}$ votes. The index m, m indicates $\overline{v}_m$'s expected utility if the voter demanding $\frac{n-1}{2}$ votes with probability $q'_{\overline{M}}$ is a member of group m. Given $p' = \overline{v}_m(2 - q'_{\overline{M}})/(n-1)$, it is easy to verify that $U_{m,m}(-1) > U_{m,m}(\frac{n-3}{2})$ for all $q'_{\overline{M}} \in (\frac{n-1}{n+1}, 1)$ if $U_{m,m}(-1) > U_{m,m}(\frac{n-3}{2})$ at $q'_{\overline{M}} = \frac{n-1}{n+1}$, a condition satisfied for all $n \geq 5$. Thus, any strategy for $\overline{v}_m$ that satisfies expected market balance cannot be his best response, if the voter demanding $\frac{n-1}{2}$ votes with probability $q'_{\overline{M}}$ belongs to group m. Hence such a voter must belong to group M. Of all remaining voters offering their votes for sale, $M-1$ belongs to the majority, and $m-1$ to the minority. They cannot be distinguished but that has no impact on $\overline{v}_m$'s unique best response. Finally, suppose either $\frac{\overline{v}_M}{\overline{v}}_m \leq \frac{n+3}{n+1}$. Expected market balance requires $s_m = \frac{n-1}{2}$ with probability 1. But then the other voter demanding $\frac{n-1}{2}$ votes with positive probability cannot belong to the minority (because in a fully revealing equilibrium, if $s_m = \frac{n-1}{2}$ with probability 1, all other minority voters would prefer to sell). Hence again the other voter with positive demand for votes must be a majority voter. All remaining voters are sellers; identifying the group each of them belongs to is not possible but has no impact on $\overline{v}_m$'s unique best response.

(iii) Consider now the perspective of all voters who in the full information equilibrium offer their vote for sale with probability 1: $v_i \in M$, $v_i \neq \overline{v}_M$, or $v_i \in m$, $v_i \neq \overline{v}_m$. By the arguments above, each of them knows that in a fully revealing equilibrium the two voters with positive expected demand must belong to the two different parties. Which one belongs to the majority and which one to the minority cannot be distinguished, but is irrelevant: since in the full information case v_i's best response is $s_i = -1$ with probability 1 whether $v_i \in M$, or $v_i \in m$, it follows that identifying which of the two voters with positive expected demand belongs to which group is irrelevant to v_i's best response. Equally irrelevant is identifying which of the sellers belongs to which group. Although the direction of preferences associated with each individual voter cannot be identified, v_i's best response is unique and identical to his best response with full information.

We can conclude that the equilibrium strategies and price identified by Lemmas 5, 6, and 7 are indeed a fully revealing ex ante equilibrium with private information. $\square$