Valence influence in electoral competition with rank objectives*

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Abstract

In this paper we examine the effects of valence in a continuous spatial voting model between two incumbent parties and one potential entrant. All parties are rank-motivated and are driven by their place in the electoral competition. One of our main results is that a sufficiently wide valence gap between the incumbents yields an equilibrium in which no entry will occur. We also show that an increase in valence shifts the high-valence incumbent party closer to the median voter, while the low-valence incumbent selects a more extreme platform.

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1 Introduction

In political science, valence usually refers to a feature which is seen in a positive light by all voters (Mueller, 2002, Stokes, 1963). It could relate to experience, trustworthiness and other character traits of parties and candidates. Empirical studies suggest that valence is a key input in voting decisions, sometimes being even more important than policy choices by parties and candidates (Schofield and Zakharov, 2010, Clarke et.al., 2005).

Modeling political competition between candidates with a different degree of valence has been a theoretical challenge. No pure-strategy equilibrium exists if candidates are vote-motivated, since the best response of the high-valence candidate (HVC) is to mimic the policy of the low-valence candidate (LVC) which would allow HVC to attract the entire electorate. As a result, the research in this field has shifted to the investigation of mixed-strategy equilibria. The case of two vote-motivated candidates competing in one-dimensional issue space has been examined by Aragones and Xefteris (2012).[1] They found out that HVC adopts a pure strategy, while LVC mixes two pure strategies equidistant to the left and to the right from the HVC position. In fact, the positional choice of HVC is independent of the value of valence advantage, and in the case of symmetric distribution of voters, it coincides with the ideal point of the median voter. Interestingly, the strategies of the candidates are not affected in the case of sequential platform choices: if HVC moves first it locates near the center of the distribution to make sure that LVC is indifferent between entering to the right and to the left. It is also shown that the distance between the platforms of the two candidates increases with the widening of the valence gap (Berger, Munger, and Pothoff, 2000).[2]

Our analysis is related to a number of previous works, in which no incumbent has a valence advantage. Palfrey (1984) and Weber (1992) considered two incumbents and one entrant, who maximize their share of votes. It was shown (for symmetric voter density in Palfrey (1984) and for any single-peaked density in Weber (1992)) that equilibrium exists and is unique. The entrant always chooses a position between those of the two incumbents, and does not receive a vote share larger than that of any of the incumbents.

[1] Groseclose (2001) considered candidates who were motivated by both votes and policy.

[2] Similar results for a mixed-strategy equilibrium with two candidates and discrete policy space were obtained by Aragones and Palfrey (2002, 2004) and Hummel (2010). However, in a mixed-strategy equilibrium of the three-candidates setting with a discrete policy space, HVC chooses, on average, a more extreme position than the LVCs (Xefteris, 2014).

[2] In Aragones and Palfrey (2005), the policy space is discrete, the position of the median voter is uncertain, and an equilibrium always exists.

Those results were later extended in Rubincik-Weber (2007). In Greenberg and Shepsle (1987) there were several incumbents and one entrant. Instead of the vote maximization the entrant's objective is to avoid the last place by obtaining more votes than at least one of the incumbents. This may lead to an entry strategy where the entrant may to opt for a lower vote share in order to try to reduce the number of votes received by one of the incumbents. The incumbents are often unable to prevent a successful challenger entry and one can always find a distribution of voters' ideal points that the entry does occur. However, an equilibrium exists in some special cases. For example, Weber (1990) shows the existence of an equilibrium in the discrete case when the number of candidates is not too large. If the voter ideal points are symmetrically distributed around the mode, in the vote-maximizing model the two incumbents locate closer to the median voter than in the case where the entrant seeks to displace one of the incumbents. The difference between equilibria in two models depends on the degree to which the distribution of voter preferences is concentrated around its mode. Cohen (1987) and Shepsle and Cohen (1990) suggest that if the concentration increases, the difference becomes less significant.

In this paper we combine Palfrey-Weber and Cohen-Shepsle approaches and consider a setting with two incumbents, who choose their policy positions simultaneously, HVC and LVC, and one potential entrant, N. All parties are rank-motivated and are driven by their place in the electoral competition: being the outright winner is, naturally, the most preferred outcome, followed by a two-way tie for the first place, and so forth. The entrant enters the race only if it guarantees at least the sole second place, and, thus, displace one of the incumbents. Our key result is that the equilibrium exists if the valence advantage of HVC is large enough. The existence of equilibrium is enforced by a threat of entry by the third candidate. Suppose that the equilibrium position of LVC is to the right of HVC. Then two conditions must be satisfied. First, the support for LVC must be equally divided between voters to the right and to the left of the LVC position (otherwise, N can claim the larger of these two groups of voters, and relegate LVC the third place). Second, the share of vote that N can claim by entering to the left of HVC should not exceed the share of vote of LVC. It must actually be equal to that share, otherwise, HVC can select a payoff-enhancing position by moving to the right.

If the valence advantage is not large enough, the equilibrium unravels, as LVC has a rank-enhancing option of moving to the left. Then N would enter to the left of HVC and squeeze HVC into the third place. If the valence advantage is sufficiently large, then HVC is immune to such deviations; instead, LVC will be punished by the entrant, who will choose a position to his right, pushing LVC to the third place.

An increase in the valence gap between HVC and LVC always results in the high-valence candidate moving closer to the median voter, and the opposing candidate moving away from it[3]. On Figure 1, we show how the equilibrium policy positions of the two incumbent candidates depend on the magnitude of HVC's valence advantage. On the same graph, we also plot the vote share of the high-valence candidate.[4]

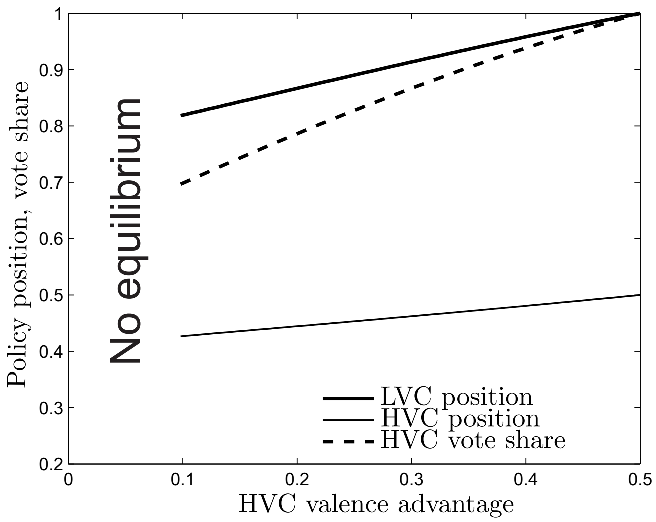

Figure 1: Positions of HVC and LVC and HVC vote share, depending on HVC valence advantage. The policy space is $[0, 1]$.

The paper is organized as follows. Section 2 describes the model. The necessary and sufficient conditions for the existence of equilibrium are presented in Section 3. Section 4 contains the comparative statics results that include an interesting example of the uniform distribution. Section 5 concludes.

[3]Note that these comparative statics are different from Groseclose (2001) in a somewhat different setting.

[4]Here, we assume that the voters prefrences over the policy parameter that are single-peaked, with ideal points distributed according to a normal distribution with mean and standard deviation of 0.5, truncated outside interval $[0, 1]$. There are no equilibria if the valence gap is less than approximately 0.095; this threshold value will be different for different distributions of voter ideal points.

2 The model

There are three candidates, H, L, and N, who compete by choosing policy positions x_H, x_L, x_N from issue space $X = [0, 1]$. Candidate H is the high-valence incumbent candidate HVC, L is the low-valence incumbent LVC. N is the entrant, choosing position after observing x_H and x_L. Candidate N can also decide not to participate in the elections. We denote that decision by $x_N = O$.

There is a continuum of voters. Each voter is characterized by the most preferred alternative, or ideal point in X. Let F be the cumulative distribution of voter's ideal points, with $F(0) = 0$ and $F(1) = 1$. We introduce three assumptions on the distribution of voter ideal points. Most of the results in our work require either all three assumptions to be satisfied, or only the first two.

Assumption 1. The probability density $f(x) = F'(x)$ is continuous, differentiable, and strictly positive on $X = [0, 1]$.

The second assumption is known as the *gradually escalating median* (see Haimanko, Le Breton, and Weber, 2005). It is satisfied if the density function of voter ideal points does not have sharp peaks where voter ideal points are concentrated. Formally, take a point $x \in X$ and consider the medians $m_l(x)$ and $m_r(x)$ of the distribution F over intervals $[0, x]$ and $[x, 1]$, respectively. Then if we slightly move x to the right, i.e., consider $x + \epsilon$, where ϵ is a small positive number, the corresponding medians $m_l(x + \epsilon)$ and $m_r(x + \epsilon)$ shift to the right by the distance which is smaller than ϵ.

Assumption 2. Let $m_l(x) = F^{-1}\big(F(x)/2\big) : [0, x] \to [0, x]$ and $m_r(x) = F^{-1}\big((1 + F(x))/2\big) : [x, 1] \to [x, 1]$ be the left F-median and the right F-median respectively. Then $m_l'(x) < 1$, $m_r'(x) < 1$.

Denote by

$$\lambda_F = \max_{x,y \in [0,1]} f(x)/f(y), \tag{1}$$

the maximal ratio of values of the density function in over all possible pairs $x, y \in [0, 1]$. Since $m_l'(x) = \frac{f(x)}{2f(F(x/2))}$, it is easy to see that the Assumption 2 is satisfied if $\lambda_F < 2$. For our purposes we will also use a slightly tighter bound that is needed in Theorem 1 to prove that if an equilibrium exists for some level of valence advantage, it will also exist if the level of valence advantage is higher. This assumption is also used to establish some comparative statics.

Assumption 2′. $\lambda_F \leq \frac{4}{3}$.

After the policy positions of candidates are chosen, each voter votes for one of the three candidates. No abstentions are allowed. Voters evaluate the candidates on the basis of their platforms and valence. Define

$$u_j(y) = \delta_j - |x_j - y| \tag{2}$$

to be the utility that voter with ideal point y would derive from the position chosen by candidate $j = H, L, N$. Let $u_N(y) = -\infty$ if $x_N = O$. The value $\delta_j \geq 0$ is the valence of candidate j. We assume that $\delta_H = \delta \in (0, \frac{1}{2})$ and $\delta_L = \delta_N = 0$. Hence, candidate H is the high-valence candidate, and candidates N and L are low-valence candidates. We assume that the valence of candidate H is not too high; if $\delta \geq \frac{1}{2}$, then that candidate can choose position $x_H = \frac{1}{2}$ and win the support of all voters.

A voter with ideal point y is assumed to vote for candidate j for which $u_j(y)$ is the highest. If two candidates deliver the same maximal utility to the voter, then the voter chooses each candidate with probability $\frac{1}{2}$. If all three candidates give the same utility to the voter, he chooses each candidate with probability $\frac{1}{3}$.

All three candidates have preferences that are lexicographical in rank and vote share. Define the rank $r_j(x_H, x_L, x_N)$ of candidate j as follows:

1. The sole first place,
2. Two-way tie for first place,
3. The sole second place,
4. Two-way tie for the second place,
5. Three-way tie,
6. Two-way tie for the last place,
7. The sole last place.

In addition we assume that candidate N prefers to rank second after candidate L to ranking second after candidate H, given that the vote shares associated with the two choices are equal.

In other words, candidate N enters only if he at least captures the sole second place.

For every pair of incumbents' positions (x_H, x_L) let

$$W_1(x_H, x_L) = \{x_N \in [0,1] |\, V_N > \max\{V_H, V_L\}\} \tag{3}$$

be the set of policy positions that guarantee candidate N the sole possession of the first place. Similarly, let

$$W_2(x_H, x_L) = \{x_N \in [0,1] |\, V_N = \max\{V_H, V_L\} \text{ and } V_N > \min\{V_H, V_L\}\} \tag{4}$$

be the set of positions that result in a tie for the first place between the entrant and one of the incumbents. Also let

$$W_3(x_H, x_L) = \{x_N \in [0,1] |\, V_N < \max\{V_H, V_L\} \text{ and } V_N > \min\{V_H, V_L\}\} \tag{5}$$

be the set of positions that result in the entrant placing second.[5]

If the entrant decides to enter, it chooses a policy position that guarantees the first place, if such policy position exists; otherwise, it chooses a policy that results in a tie for the first place or (absent such policy) one that places him second. The set of possible entry decisions is described as follows:

$$E(x_H, x_L) = \begin{cases} W_1(x_H, x_L), & \text{if} \quad W_1(x_H, x_L) \neq \emptyset \\ W_2(x_H, x_L), & \text{if} \quad W_1(x_H, x_L) = \emptyset, W_2(x_H, x_L) \neq \emptyset \\ W_3(x_H, x_L), & \text{if} \quad W_1(x_H, x_L) = W_2(x_H, x_L) = \emptyset, \\ & \qquad W_3(x_H, x_L) \neq \emptyset. \end{cases} \tag{6}$$

We need to identify the payoffs of the players when the entrant attempts to determine his best response over the set E. However, a common feature of electoral competition models with two incumbents and one entrant is the lack of a well-defined best response function for the entrant (Palfrey (1984)). Let $V_j(x_H, x_L, x_N)$ be the vote share of candidate j. The best-response is defined by

$$b(x_H, x_L) = \begin{cases} \{x_N | r_N(x_H, x_L, x_N) \leq 3 \text{ and } \nexists x'_N \in [0,1] \text{ such that} \\ \qquad r_N(x_H, x_L, x'_N) > r_N(x_H, x_L, x_N) \text{ or} \\ \qquad r_N(x_H, x_L, x'_N) = r_N(x_H, x_L, x_N), \\ \qquad V_N(x_H, x_L, x'_N) > V_N(x_H, x_L, x_N)\} \\ \{O\}, r_N(x_H, x_L, x_N) \geq 4 \text{ for all } x_N. \end{cases} \tag{7}$$

The best response function for candidate N can have empty values because his payoffs are discontinuous in x_N at points $x_H - \delta$, $x_H + \delta$, and (possibly) at x_L.

Throughout the paper, we will assume, without loss of generality, that $x_H < x_L$. Then we have the following result:

[5] In principle, we could have defined W_4, W_5, W_6 and W_7, but we will not utilize this notation in the paper.

Lemma 1. If the set $b(x_H, x_L)$ is nonempty then either $b(x_H, x_L) = \{O\}$ or $b(x_H, x_L) = \{x_L\}$.

To define the payoffs of candidates when the $b(x_H, x_L)$ is empty but $E(x_H, x_L)$ is not, following Palfrey (1984), we assume that for a given positive ϵ, candidate N randomly chooses a point from a set of ϵ-best responses and then takes a limit of his expected payoffs when ϵ approaches zero. Formally, for each $\epsilon > 0$, define the set of ϵ-best responses of candidate N to policy positions of candidates H and L. Let

$$b_\epsilon(x_H, x_L) = \{x_N \in E(x_H, x_L) | V_N(x_H, x_L, x_N) \geq \sup_{y \in E(x_H, x_L)} V_N(x_H, x_L, y) - \epsilon\}. \tag{8}$$

Define the following subset of b_ϵ:

$$\tilde{b}_\epsilon(x_H, x_L) = \{x_N \in b_\epsilon(x_H, x_L) | r_L(x_H, x_L, x_N) = \min_{y \in b_\epsilon(x_H, x_L)} r_L(x_H, x_L, y)\}. \tag{9}$$

This is the adjusted set of ϵ-best responses for candidate N over which the rank of candidate L is minimized; it reflects our assumption that, other things being equal, candidate N would prefer candidate L to rank as low as possible.

Limit points of the adjusted set of ϵ-best responses are called *limit best responses*:

Definition 1. Let x be a limit point of set $\tilde{b}_\epsilon(x_H, x_L)$ as $\epsilon \to 0$. If for every ϵ there exists x' such that the interval $(x', x) \in \tilde{b}_\epsilon(x_H, x_L)$, then we say that x is a *left limit best response* (or *LLBR*) of candidate N. If the interval $(x, x') \in \tilde{b}_\epsilon(x_H, x_L)$ for some x', then x is a *right limit best response* (or *RLBR*).

A limit point of $\tilde{b}_\epsilon(x_H, x_L)$ can be either a right limit best response, or a left limit best response, or both. Let

$$\mu_\epsilon(x_H, x_L) = \int_{\tilde{b}_\epsilon(x_H, x_L)} dx \tag{10}$$

be the measure of the adjusted set of ϵ-best responses. The set $\tilde{b}_\epsilon(x_H, x_L)$ is nonempty for every $\epsilon > 0$. Given ϵ, we can define the expected share of votes for each candidate H, L if the entering candidate N randomly chooses a position that is uniformly distributed on the set of ϵ-best responses. According to Palfrey (1984), the limits of such expected vote shares are well defined for each $j = H, L$:

$$v_j(x_H, x_L) = \lim_{\epsilon \to 0} \frac{1}{\mu_\epsilon(x_H, x_L)} \int_{\tilde{b}_\epsilon(x_H, x_L)} V_i(x_H, x_L, x_N) dx_N. \tag{11}$$

We can now define the vote share for candidates H and L. They are equal to the limits v_H, v_L if the set of best responses b is empty, but limit best responses exist. Otherwise, it is equal to the vote share of the candidates if candidate N makes the optimal decision $x_N(x_H, x_L)$ — which is to choose the unique element of $b(x_H, x_L)$ if the best response exists, or choose not to enter if no best response or limit best response can guarantee the entrant a second or higher place.

$$\pi_j(x_H, x_L) = \begin{cases} v_j(x_H, x_L), & \text{if } E(x_H, x_L) \neq \emptyset \text{ and } b(x_H, x_L) = \emptyset, \\ V_j(x_H, x_L, \tilde{x}_N(x_H, x_L)), & \text{otherwise.} \end{cases} \tag{12}$$

We shall now consider the electoral competition game between candidates H and L with lexicographical preferences over r_j and π_j, who anticipate the best response of candidate N. Our objective is to examine pure strategies Nash equilibria of this game.

3 Equilibrium conditions

Our main result shows that, under Assumptions 1 and 2, the equilibrium does not exist if the level of valence δ is sufficiently small, namely, there exists $\delta_0 \in (0, \frac{1}{2})$ such that for any $\delta \in (0, \delta_0]$, there are no equilibria in our electoral game.

This however leaves open the question what happens for the levels of δ between δ_0 and $\frac{1}{2}$. It turns out that if we impose the stronger Assumption $2'$, then we are able to fill this gap and to demonstrate that for any δ in this interval the equilibrium exists and is unique. Moreover, in such equilibrium the third candidate chooses the strategy O and does not enter.

Theorem 1. Suppose that Assumptions 1 and 2 hold. Then there exists $\delta_0 \in (0, \frac{1}{2}]$ such that for any $\delta \in (0, \delta_0)$, there are no equilibria in our electoral game.

If, in addition, Assumption $2'$ is imposed, then for any $\delta \in (\delta_0, \frac{1}{2})$ there exists a unique equilibrium. In this equilibrium the candidate N does not enter.

In order to prove the last assertion of our theorem we shall show that necessary and sufficient condition for a pair (x_H^*, x_L^*) to be an equilibrium are given by the following conditions:

$$\phi = \gamma = \frac{\alpha}{2}, \tag{13}$$

$$x_H^* - x_L^* > \delta, \tag{14}$$

and

$$F\left(\frac{3x_H^* + x_L^* + 3\delta}{4}\right) > 2\alpha, \tag{15}$$

where $\alpha = F(x_H^* - \delta)$, $\gamma = F(x_L^*) - F(\frac{x_H + x_L + \delta}{2})$, and $\phi = 1 - F(x_L^*)$.

We prove this theorem by establishing a series of intermediate results. Here we only sketch some of the proofs. The formal proofs are relegated to Appendix B.

First, we show that in an equilibrium, candidate N is not able to enter and win a second place.

Lemma 2. Suppose that (x_H^*, x_L^*) is an equilibrium. Then candidate N does not enter.

If there is entry in equilibrium, then one of the two incumbent candidates will eventually be ranked below second place. This cannot be HVC as it can guarantee itself a rank at least that high by matching the position of LVC and leaving that candidate with zero votes. Therefore, in any entry equilibrium candidate L cannot help being pushed below second place by the entrant. But it can be shown that, for any x_H, candidate L can always select a position that will prevent the entrant from achieving a higher rank than L.

Our next step is to outline the conditions under which a pair of strategies is an equilibrium. For candidate N, the conditions are as follows.

Lemma 3. Let (x_H, x_L) be a pair of candidate strategies. If Assumptions 1 and 2 are satisfied, then no entry of candidate N can occur if and only if

$$\phi = \gamma, \tag{16}$$

$$\alpha \leq \min\{\phi + \gamma, \theta + \beta\}, \tag{17}$$

and

$$\beta \leq \gamma + \phi, \tag{18}$$

where $\beta = F(\frac{x_H^* + x_L^* + \delta}{2}) - F(x_H^* + \delta)$ and $\theta = F(x_H^* + \delta) - F(x_H^* - \delta)$.

Potentially, candidate N can respond in six ways. It can either choose a position approaching $x_H - \delta$ from the left, approaching $x_H + \delta$ from the right, approaching x_L from left or right, x_L itself, or somewhere in the interval between x_H and x_L. Conditions (16), (17), and (18) ensure that the first five responses do not give candidate N more than a tie for the last place. Assumption 2, together with (16), rules out the sixth response.

Additional constraints on (x_H, x_L) are imposed by the requirement that candidates H and L cannot deviate from their strategies.

Lemma 4. Suppose that conditions of Lemma 3 are satisfied. Then there does not exist x'_H that gives a higher payoff to candidate H if and only if

$$\alpha = \gamma + \phi. \tag{19}$$

There does not exist x'_L that gives a higher payoff to candidate L if and only if

$$F\left(\frac{3x_H + x_L + 3\delta}{4}\right) > 2\alpha. \tag{20}$$

Lemmas 3 and 4 state that conditions (13), (15), $\alpha \leq \theta + \beta$, and $\beta \leq \alpha$, are necessary and sufficient for an equilibrium to exist. We now need to prove that a pair of candidate strategies that satisfies those conditions exists only if and only if δ is large enough, and is unique if it exists. We proceed with the following statement:

Lemma 5. Under Assumptions 1 and 2, for any $\delta \in (0, \frac{1}{2})$ there exists a unique solution to equations (13), satisfying $x_L - x_H > \delta$.

Our goal is to prove that the equilibrium exists for all $\delta \in (\delta_0, \frac{1}{2})$ whenever it exists for some $\delta_0 \in (0, \frac{1}{2})$. As $\frac{3x_H+x_L+3\delta}{4} < \frac{x_H+x_L+\delta}{2}$, the condition (15) is stronger than $\alpha \leq \beta + \theta$. We investigate how do the inequality condition (15) and $\alpha \geq \beta$ change depending on δ.

Lemma 6. Suppose that (x_H, x_L) satisfy conditions (13). Then

$$D_1 = F\left(\frac{3x_H + x_L + 3\delta}{4}\right) - 2\alpha \tag{21}$$

increases in δ. Assumption $2'$ yields that the value

$$D_2 = \alpha - \beta \tag{22}$$

decreases in δ.

We evaluate the constraints (21) and (22) as $\delta \to \frac{1}{2}$. As δ increases, candidate H moves closer to the middle of the interval $[0, 1]$, while candidate L moves right, toward 1, and α and β both tend to zero. From this and Lemma 6 it follows that D_1 is strictly positive if δ is large enough, while $\lim_{\delta \to \frac{1}{2}} D_2 = 0$, so condition $\beta \leq \alpha$ holds for any (x_H, x_L) that satisfy conditions (13). It remains to show that condition (15) is violated if δ is small enough.

Lemma 7. Let Assumption 1 be satisfied. If $\alpha \geq \beta$ and (13) hold, then there exists $\delta_0 \in (0, \frac{1}{2})$ such that (15) is violated for all $\delta \in (0, \delta_0]$.

As $\alpha \geq \beta$, the equilibrium satisfying conditions (13), (14), and (15) would exist for all $\delta \in (\delta_0, \frac{1}{2})$, and would fail to exist if $\delta \in (0, \delta_0]$. This completes the proof of Theorem 1.

4 Comparative statics

We first see how do the equilibrium positions change as the valence difference δ changes.

Theorem 2. Suppose that the pair (x_H^*, x_L^*) is an equilibrium. Let Assumptions 1 and 2 be satisfied. Then we have

$$\frac{\partial x_L^*}{\partial \delta} > 0. \tag{23}$$

Suppose that Assumption 3 is satisfied as well. Then the following is true:

$$\frac{\partial x_H^*}{\partial \delta} > 0. \tag{24}$$

As the valence gap increases, the low-valence incumbent L will lose votes to his left. He will have to shift his position to the right, in order to make sure that an equal number of voters supporting L are located to the left and to the right of that candidate (otherwise, candidate N will be able to enter to one side of L or another, and push him into third place).

Increase in δ will have two countervailing effects on the position of the high-valence candidate H. First, as the vote share of candidate L is reduced, candidate N will require a smaller vote share for a successful entry to the left of H, so the latter candidate is forced to shift to the left in order to prevent entry. On the other hand, increase in H's valence reduces the set of available policy positions to the left of H with which candidate N can enter; thus, H can gain votes from L by shifting his policy position to the right. If the distribution of voter ideal points is uniform enough, the second effect dominates, and increase in valence results in H moving right, toward the median voter.

Let us illustrate the statement of Theorem 2 by means of the uniform distribution.

Example. Uniform distribution of ideal points. Suppose that the ideal points of the voters are distributed uniformly on $[0, 1]$. We then have $\theta = 2\delta$ and $\alpha = x_H - \delta$. Conditions (13) give us $x_L = 1 - \frac{\alpha}{2} = 1 + \frac{\delta}{2} - \frac{x_H}{2}$ and $x_m = \frac{x_H+\delta+x_L}{2} = 1 - x_H + \delta = 1 - \alpha$. Solving the two equations, we get

$$x_H = \frac{2 + \delta}{5} \tag{25}$$

and

$$x_L = \frac{4+2\delta}{5}. \tag{26}$$

In an equilibrium, if one exists, the positions of both candidates will shift to the right as δ increases. The vote share of candidatre H will also increase in δ:

$$v_H = \frac{6+4\delta}{5}. \tag{27}$$

We now derive the conditions on δ which guarantee the existance of equilibrium. Consider candidate N entering with some $x_N \in [0,1]$. The following cases are possible:

1. $x_N \in [0, x_H - \delta]$. If $x_N < x_H - \delta$, the share of vote for the entrant will be $V_N = \frac{x_N + x_H - \delta}{2}$, which will always be less than V_L, which is equal to α by condition (13). In the limit case $x_N = x_H - \delta$, the vote share of candidate N will be $V_N = \alpha$. The share of vote of candidate H will be $V_H = x_m - \frac{x_H - \delta + x_N}{2} = \frac{x_L - x_N}{2} + \delta$. If $\delta \geq \frac{1}{12}$, which corresponds to condition $\alpha \leq \theta + \beta$, then candidate N will not be able to enter to the left of candidate H, pushing him into the third place. The best that candidate N can do is to share the last place with candidate L, which is not acceptable. Hence there will be no entry at any $x_N \in [0, x_H - \delta)$ as long as $\delta \geq \frac{1}{12}$.

2. $x_N \in (x_H - \delta, x_H + \delta)$. The share of vote for the entrant will be zero, as all voters will prefer candidate H to candidate N.

3. $x_N \in (x_H + \delta, x_L)$. The share of vote for the entrant will be

$$V_N = \frac{x_L - x_H - \delta}{2} = \frac{1-2\delta}{5}.$$

   If candidate N enters at $x_N = x_L - e$, the share of vote for candidate L will be

$$V_L = 1 - x_L + \frac{e}{2} = \frac{1-2\delta}{2} + \frac{e}{2},$$

   which for all $e > 0$ will be greater than V_N. Hence, no entry with $x_N \in (x_H + \delta, x_L)$ is possible.

4. $x_N = x_L$. Candidates L and N will share second and third place, hence there will be no entry.

5. $x_N \in (x_L, 1]$. Candidate N's share of votes will be $V_N = \frac{1+x_L}{2}$, and candidate L's share of votes will be $V_L = \frac{x_N - x_m}{2}$. We will have $V_L > V_N$ by (13).

So, candidate N will not be able to do better than share second and third place with candidate L. Hence, no entry will occur for (x_H, x_L) defined by (26), (25) if $\delta \geq \frac{1}{12}$.

Candidate H will not deviate from policy position x_H. If he shifts left to $x'_H > x_H$, he risks entry at $x_N = x_H - \delta - e$. Given that, candidate H will not benefit from such a shift even if $x'_H = x_L$ and candidate L is eliminated. If candidate H shifts right to $x'_H < x_H$, he is bound to lose votes to candidate N, who will now find it profitable to enter at $x_N = x_L - e$ for some small e.

Likewise, candidate L will not change her position: if $x'_L > x_L$, then candidate N will enter at the left of x'_L and gain second place. Candidate L may deviate to $x'_L = x_m$, prompting candidate N to enter at the left of $x_H - \delta$. This deviation, as well as any other deviation $x'_L \in [x_m, x_L)$ is prevented if

$$\frac{3x_H + x_L + 3\delta}{4} \geq 2(x_H - \delta)$$

or

$$\delta \geq \frac{3}{26}.$$

In that case, any deviation $x'_L \in [x_m, x_L)$ will instead prompt candidate N to enter to the right of x'_L, pushing candidate L into third place. Equilibrium conditions for the uniform distribution of voter ideal points are summarized in Table 1.

Condition	What happens when it is violated	Condition for uniform distribution
$\alpha \leq \theta + \beta$	Candidate N has a LLBR at $x_N = x_H - \delta$. As Candidate N enters, Candidate H is pushed to third place.	$\delta \geq \frac{1}{12}$
(15)	Candidate L can deviate to $x'_L = x_m$. Candidate N subsequently has a LLBR at $x_N = x_H - \delta$ (instead of a RLBR at $x_N = x'_L$). As candidate N enters, Candidate H is pushed to third place.	$\delta \geq \frac{3}{26}$

Table 1: Equilibrium conditions for uniform distribution of voter ideal points and $x_H = \frac{2+\delta}{5}$, $x_L = \frac{4+2\delta}{5}$.

It follows that, for $\delta \in [\frac{3}{26}, \frac{1}{2})$, the pair $x^*_H = \frac{2+\delta}{5}$, $x^*_L = \frac{4+2\delta}{5}$ will give us the unique equilibrium at which no entry by candidate N occurs.

We next look at the effects on equilibrium of changes in the distribution function F. Let x^*_L and x^*_H be equilibrium, and suppose that a mass a

of voters migrates from interval $(x_H^* - \delta, x_m^*)$ to interval $[0, x_H^* - \delta)$, with no movement of voters outside those intervals. Formally, let (x_L^*, x_H^*) be an equilibrium given the distribution of voter ideal points $F(x)$, and let $\mathbf{F}(x, a) : [0,1] \times [0, \infty) \to [0,1]$ be any twice differentiable function such that

1. $\mathbf{F}(x, a)$ satisfies Assumptions 1, 2 for all a.
2. $\mathbf{F}(x, 0) = F(x)$.
3. For all a, $\mathbf{F}(x_L^*, a) = F(x_L^*)$, $\mathbf{F}(x_m^*, a) = F(x_m^*)$, $\mathbf{f}(x_L^*, a) = f(x_L^*)$, and $\mathbf{f}(x_m^*, a) = f(x_m^*)$, where $\mathbf{f}$ is the derivative of $\mathbf{F}$ with respect to x.
4. $\frac{\partial \mathbf{F}(x_H^* - \delta, a)}{\partial a} = 1$.

Similarly, define function $\mathbf{F}(x, b)$ such that a mass b of voters migrates from (x_M^*, x_L^*) to $(x_L^*, 1]$, with voters outside those intervals remaining stationary. Then the following statement is true.

Theorem 3. Suppose that (x_H^*, x_L^*) is an equilibrium. Then we have

$$\frac{\partial x_H^*}{\partial a} < 0, \qquad \frac{\partial x_H^*}{\partial b} < 0, \tag{28}$$

$$\frac{\partial x_L^*}{\partial a} < 0, \qquad \frac{\partial x_L^*}{\partial b} > 0 \tag{29}$$

at $a = 0$ and $b = 0$.

As the mass of voters to the right of $x_H^* - \delta$ increases, candidate N can make a successful entry with the limit best response $x_N = x_H^* - \delta$. Candidates H and L both shift their positions to the left to prevent this entry.

A migration of voters to the interval $(x_L^*, 1]$ from (x_m^*, x_L^*) has an opposing effect on candidate policy positions. Candidate L is now threatened with candidate N's right limit best response $x_N = x_L^*$, and is forced to shift his position to the right. This move decreases the vote share of candidate L, so candidate N now has a left limit best response at $x_N = x_H^* - \delta$. To prevent that, candidate H must shift his position to the left.

For (15), we can evaluate how will x_H^*, x_L^*, and δ change with a and b if we force δ to be such that the condition is satisfied as an equality. We can thus see how the set of δ for which equilibrium exists changes with the distribution of voter ideal points.

Theorem 4. Let Assumption $2'$ be satisfied. Put

$$\delta_0 = \inf\{\delta | \text{ There exists an equilibrium } (x_H^*, x_L^*)\}. \tag{30}$$

Then

$$\frac{\partial \delta_0}{\partial a} > 0, \ \frac{\partial \delta_0}{\partial b} < 0 \tag{31}$$

at $a = 0$ and $b = 0$.

It turns out that the set of δ for which equilibrium exists shrinks as a increases, and expends as b increases.

The effect of changes in the distribution function $F(\cdot)$ on equilibrium existence can be studied in greater detail using numeric methods. Figure 2 shows the results of a numeric experiment where equilibrium existence was evaluated for various values of δ and various parameters of the distribution function.

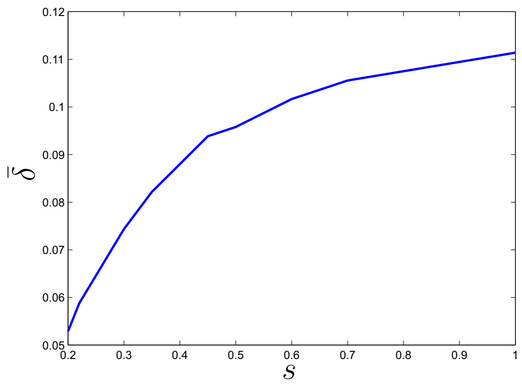

Figure 2: Equilibrium exists if and only if $\delta \geq \bar{\delta}$.

In particular, we assume that voter ideal points are distributed according to a normal distribution with mean 0.5 and standard deviation s, truncated at $[0, 1]$.[6] The minimum valence advantage for which equilibrium exists increases with s. It turns out that in this example the condition $\alpha \geq \beta$ of Lemma 6 always holds in equilibrium, even though the sufficient Assumption $2'$ was not satisfied for $s < .66$.

5 Conclusion

We study a model of electoral competition in which there are two incumbents and one entrant, candidates are office-motivated, and one of the incumbents has a valence advantage over the other two candidates. We show

[6]The experiment was carried out using Matlab 7.11. Equilibrium existence was evaluated for values of δ between 0 and 0.2, in increments of 0.001.

that a unique equilibrium exists if the valence advantage is large enough, and that entry is prevented in equilibrium. The candidate positions change as the valence advantage increases: the high-valence candidate moves closer to the median voter, and the low-valence candidate chooses a more extreme position. The minimum value of valence advantage that is required for equilibrium to exist depends on the distribution of voter ideal points. Numeric experiments suggest that as the variance of voter ideal points decreases, equilibrium becomes more likely, as it exists for a broader range of δ.

Appendix A. Supplementary result.

The following result will be used in the proof of Lemmas 3 and 6.

Lemma 8. Let $y > z$ be arbitrary points from the interval $[0, 1]$. Then inequalities in the following pairs are equivalent.

$$m_l(y) < z \quad \text{and} \quad F(y) - F(z) < F(z) \tag{32}$$

$$m_r(z) > y \quad \text{and} \quad 1 - F(y) > F(y) - F(z). \tag{33}$$

Proof of Lemma 8. Since the function F is monotonous, the first inequality (32) leads to inequality

$$F\big(m_l(y)\big) < F(z). \tag{34}$$

From the definition of the median $2F\big(m_l(y)\big) = F(y)$ and inequality (34) it follows that $F(y) < 2F(z)$ that coincides with the second inequality (32). These reasoning are invertible so that the inequalities in (32) are equivalent. The equivalence of the inequalities in (33) is established in the same way. □

Appendix B. Proofs of statements.

Proof of Lemma 1. Assume that $x_H \leq x_L$. Consider the following cases. *Case 1.* $x_H + \delta \geq x_L$. Then for any $x_N < x_H - \delta$ we have $r_N(x_H, x_L, x_N) \geq 3$ and there exists $x'_N \in (x_N, x_H - \delta)$ such that $r_N(x_H, x_L, x_N) \geq r_N(x_H, x_L, x'_N)$ and $V_N(x_H, x_L, x_N) > V_N(x_H, x_L, x'_N)$. So, x_N is not a best response. Similarly, $x_N > x_H + \delta$ is not a best response.

Case 2. $x_H + \delta < x_L$. Take $x_N \in [0, 1]$.

Case 2A. $x_N \in [0, x_H - \delta)$. If $x'_N \in (x_N, x_H - \delta)$, then $V'_N > V_N$ and $r'_N \geq r_N$.

Case 2B. $x_N = x_H - \delta$. Then there exists $x'_N = x_N - e$ such that $V'_N > V_N$ and $r'_N \geq r_N$ if e is small enough.

Case 2C. $x_N \in (x_H - \delta, x_H + \delta)$. In this case $V_N = 0$.

Case 2D. $x_N = x_H + \delta$. Then there exists $x'_N = x_N + e$ such that $V'_N > V_N$ and $r'_N \geq r_N$ if e is small enough.

Case 2E. $x_N \in (x_H + \delta, x_L)$. By definition of right F-median,

$$m_r\left(\frac{x_H + \delta + x_L}{2}\right) = x_L. \tag{35}$$

Integrating inequality $m'_r(x) < 1$ over the interval $[(x_H + \delta + x_N)/2, (x_H + \delta + x_L)/2]$, we get

$$m_r\left(\frac{x_H + \delta + x_L}{2}\right) - m_r\left(\frac{x_H + \delta + x_N}{2}\right) < \frac{x_L - x_N}{2}.$$

Combining the last inequality and (35), we obtain

$$m_r\left(\frac{x_H + \delta + x_N}{2}\right) > \frac{x_N + x_L}{2}. \tag{36}$$

Be Lemma 8, taking $z = (x_H + \delta + x_N)/2$ and $y = (x_N + x_L)/2$ inequality (36) gives us

$$F\left(\frac{x_N + x_L}{2}\right) - F\left(\frac{x_H + \delta + x_N}{2}\right) < 1 - F\left(\frac{x_N + x_L}{2}\right), \tag{37}$$

or $V_N < V_L$. By a symmetric argument we show that $V_H > V_N$, and we cannot have $b(x_H, x_L) = x_N$.

Case 2E. $x_N \in (x_L, 1]$. If $x'_N \in (x_L, x_N)$, then $V'_N > V_N$ and $r'_L \geq r_L$.

If follows that if $x \in b(x_H, x_L)$, then $x = x_L$ or $x = O$.

□

Proof of Lemma 2. Suppose that (x_H, x_L) is an equilibrium that does not prevent the entry of Candidate N. Then candidate N receives strictly more votes than at least one of the other candidates. Candidate H cannot rank third in equilibrium, because deviation $x'_H = x_L$ will guarantee it at least a second place, with candidate L ranking last with zero vote share. Therefore candidate L finishes third. Consider the following cases.

Case 1. $\theta < \min\{\alpha, \alpha'\}$, where $\alpha = F(x_H - \delta)$ and $\alpha' = 1 - \alpha - \theta$.

Case 1A: $\alpha \leq \alpha'$. Take $x'_L = x_H - \delta - e$, where e is small enough. Then the limit best response of N will be $x_N = x_H + \delta$. Candidate N will rank first, candidate L second, and candidate H third, which is a contradiction.

Case 1B: $\alpha > \alpha'$ This case is symmetric to Case 1A.

Case 2: $\alpha < \alpha'$ and $\theta \geq \min\{\alpha, \alpha'\}$. Define x^*_L be such that $1 - F(x^*_L) = F(x_L) - F(\frac{x_H + x_L + \delta}{2}) \equiv \gamma$.

Case 2A: $\gamma \geq \frac{\alpha}{2}$.

Case 2A1: $x_L \neq x_L^*$. Take $x'_L = x_L^*$. If $x_N \in [x_L, 1]$, then $V_N \leq V_L$ because $\gamma = \varphi$. If $x_N \in (x_H + \delta, x_L)$, we have $V_L > V_N$ by Lemma 1.

Finally, for any $x_N \in [0, x_H - \delta)$, we will have $V_N < \alpha \leq V_L$. Thus, candidate L improves its rank with $x'_L = x_L^*$, and (x_H, x_L) is not an equilibrium.

Case 2A2: $x_L = x_L^*$. By the above argument, $V_N < V_L$ for any x_N. So, (x_H, x_L) prevent the entry of Candidate N.

Case 2B: $\gamma < \alpha/2$. Consider the following cases.

Case 2B1: $x_L < x_H - \delta$. Then the RLBR will be $x_N = x_H + \delta$. It follows that for any $x'_L \in (x_L, x_H - \delta)$, we will have $V_L < V'_L < V_N$, with rank of candidate L remaining the same. So, (x_H, x_L) is not an equilibrium.

Case 2B2: $x_L \in (\bar{x}_L, 1]$, where $\bar{x}_L$ is the solution to $F(\frac{x_H + x_L + \delta}{2}) = 1 - \alpha$. Then the LLBR will be $x_N = x_H - \delta$ and $V_H > \alpha > V_L$. If $x'_L = x_H - \delta - e$, then the RLBR will be $x_N = x_H + \delta$, and we will have $V'_L > V_L$ if e is small enough, with rank of candidate L remaining the same. So, (x_H, x_L) is not an equilibrium.

Case 2B3: $x_L \in (x_H + \delta, \bar{x}_L]$. Because $\alpha/2 > \gamma$, we have $x_L^* > \bar{x}_L$ and $1 - F(x_L) > F(x_L) - F(\frac{x_H + x_L + \delta}{2})$. Take $x'_H = x_H + e$. The RLBR will be $x_N = x_L$ for both x_H and x'_H, if e is small enough. Hence $V'_H > V_H$, with rank of candidate H remaining the same. So, (x_H, x_L) is not an equilibrium.

Case 3: $\alpha' \leq \alpha$ and $\theta \geq \min\{\alpha, \alpha'\}$ This case is symmetric to Case 2.

□

Proof of Lemma 3. Take some $x_N \in [0, 1]$.

Case 1: $x_N \in [0, x_H - \delta)$ is weakly dominated by $x'_N \in (x_N, x_H - \delta)$.

Case 2: $x_N = x_H - \delta$. The following conditions are necessary and sufficient for $x_N = x_H - \delta$ not to be a left limit best response:

$$\alpha \leq \phi + \gamma, \tag{38}$$

or $\lim_{x_N \to x_H - \delta} V_N > V_L$, and

$$\alpha \leq \theta + \beta, \tag{39}$$

or $\lim_{x_N \to x_H - \delta^-} V_N > \lim_{x_N \to x_H - \delta^-} V_H$.

Case 3: $x_N \in (x_H - \delta, x_H + \delta)$. Candidate N gets zero votes.

Case 4: $x_N = x_H + \delta$. The following condition is necessary and sufficient for $x_N = x_H + \delta$ not to be a right limit best response:

$$\gamma + \phi \geq \beta, \tag{40}$$

or $\lim_{x_N \to x_H + \delta^+} V_N > V_L$.

Case 5: $x_N = x_L$. If $\phi > \gamma$, then $\lim_{x_N \to x_L^+} V_N > \lim_{x_N \to x_L^-} V_N$. In that case $x_N = x_L$ is a right limit best response that guarantees a second place. If $\phi < \gamma$, then $x_N = x_L$ is a left limit best response because $\lim_{x_N \to x_L^-} V_N > \lim_{x_N \to x_L^+} V_N$. If

$$\psi = \gamma, \tag{41}$$

then $\lim_{x_N \to x_L^-} V_N = \lim_{x_N \to x_L^+} V_N = V_N$, but $x_N = x_L$ is not a best response because candidates L and N will share second and third place. Hence condition (41) must hold if there is no entry at $x_N = x_L$.

Case 6: $x_N \in (x_H + \delta, x_L)$. For x_N in this interval, we should have $V_N < V_L$ and $V_N < V_H$ (this statement is proven in Lemma 1).

Case 7: $x_N \in (x_L, 1]$ is weakly dominated by any $x'_N \in (x_L, x_N)$. $\square$

Proof of Lemma 4. Consider deviation $x'_L \neq x_L$:

Case 1: $x'_L \in [0, x_H - \delta)$. Due to (38), candidate L's new vote share will not be greater than his previous vote share $\phi + \gamma$. Because of (39) and (38), candidate L will not be able to improve his rank above second.

Case 2: $x'_L \in [x_H - \delta, x_H + \delta]$. Candidate L's new vote share will be zero.

Case 3: $x'_L \in (x_H + \delta, x_m)$: The right limit best response of candidate N will be $x_N = x'_L$, moving candidate L to third place.

Case 4: $x'_L \in (x_m, x_L)$. Put

$$x'_m = \frac{x_H + x'_L + \delta}{2}.$$

Candidate N will have the following strategy:

1. If $\alpha > F(x'_m) - \alpha$, the limit best response of candidate N will be $x_N = x_H - \delta$, giving him a vote share of α and a second place.
2. If $\alpha \leq F(x'_m) - \alpha$ and $1 - F(x'_L) > F(x'_L) - F(x'_m)$, the limit best response of candidate N will be $x_N = x'_L$, giving him a vote share of $1 - F(x'_L) < \alpha$ and a second place.
3. Otherwise $x_N = O$.

There will be no entry at $x_N \in (x_H + \delta, x'_L)$ because $V_N(x_H, x_N, x'_L) < V_N(x_H, x_N, x_L) < V_L(x_H, x_N, x_L) < V_L(x_H, x_N, x'_L)$. The first and third inequalities are because $x_N < x'_L < x_L$. The second inequality has been proven in Lemma 1. It follows that a deviation $x'_L \in (x_m, x_L)$ will not be feasible if $x_N = x_H - \delta$ is not a left limit best response of candidate N, but $x_N = x'_L$ *is* a right limit best response. This translates into the following two conditions: $2\alpha < F(x'_m)$ and $1 + F(x'_m) > 2F(x'_L)$. As x'_m is monotonic in x'_L, the first of these two conditions is guaranteed to be satisfied for all $x'_L \in (x_m, x_L)$ if it is satisfied for $x'_L = x_m$:

$$F\left(\frac{3x_H + x_L + 3\delta}{4}\right) - 2\alpha > 0. \tag{42}$$

The condition $1+F(x'_m) > 2F(x'_L)$ is always satisfied because (see Lemma 1) we know that for any $x_N \in (x_H + \delta, x_L)$ we have $V_N(x_H, x_N, x_L) < V_N(x_H, x_N, x_L)$. It follows that for any $x'_L \in (x_N, x_L)$ we must also have $V_N(x_H, x_N, x'_L) < V_N(x_H, x_N, x'_L)$.

Case 5:$x'_L \in (x_L, 1]$. Not feasible. As $\gamma = \phi$, candidate N's limit best response will be $x_N = x'_L$, moving candidate L to third place.

It follows that, as long as conditions of (3) are satisfied, candidate L will not be able to improve its payoff if and only if (15) holds. Now consider a deviation $x'_H \neq x_H$.

Case 1: $x'_H \in [0, x_H)$. This will never increase candidate H's vote share $F(x_m)$, because x_m will decrease with x_H.

Case 2: $x'_H \in (x_H, x_L - \delta)$. If

$$\alpha < \phi + \gamma,$$

then there exists $x'_H > x_H$ such that $x_N = x'_H - \delta$ will not be the limit best response of candidate N. That will allow candidate H to increase his vote share. Hence, together with (38), we impose the condition

$$\alpha = \phi + \gamma. \tag{43}$$

If (43) holds, the limit best response of candidate N will be $x_N = x'_H - \delta$. We know that $\phi = \gamma = \alpha/2$. The vote share of candidate H will decrease by at least α (as voters $[0, x_N)$ will go to candidate N), but will not increase by more than $\frac{\alpha}{2}$. Hence, this will never increase candidate H's vote share.

Case 3: $x'_H = x_L - \delta$. Candidate H will gain at most α (the voters $[x_m, 1]$ who previously voted for candidate L). At the same time candidate H will lose at least α votes to candidate N, whose limit best response will be $x_N = x_H - \delta$, as $x_L - x_H > \delta$.

Case 4: $x'_H \in (x_L - \delta, 1]$ is dominated by $x'_H = x_L - \delta$.

It follows that if conditions of Lemma 3 are satisfied, then (19) and (20) are necessary and sufficient for candidates H and L not to change their positions. □

Proof of Lemma 5. Denote $f_L = f(x_L)$, $f_{H-} = f(x_H - \delta)$, and $f_m = f(x_m)$.

Rewrite conditions (13) as

$$H_1 = 1 + F\left(\frac{x_H + x_L + \delta}{2}\right) - 2F(x_L) = 0 \tag{44}$$

and

$$H_2 = F(x_H - \delta) + 2F(x_L) - 2 = 0. \tag{45}$$

We have the following derivatives:

$$\begin{array}{ll} \frac{\partial H_1}{\partial \delta} = \frac{1}{2} f_m & \frac{\partial H_2}{\partial \delta} = -f_{H-} \\ \frac{\partial H_1}{\partial x_H} = \frac{1}{2} f_m & \frac{\partial H_2}{\partial x_H} = f_{H-} \\ \frac{\partial H_1}{\partial x_L} = \frac{1}{2} f_m - 2f_L & \frac{\partial H_2}{\partial x_L} = 2f_L \end{array} \tag{46}$$

Let $x_L^1(x_H)$ be a solution to $H_1 = 0$. Since $f(x) > 0$ for all $x \in [0,1]$, by implicit function theorem we have

$$\frac{\partial x_L^1(x_H)}{\partial x_H} = -\frac{\frac{1}{2} f_m}{\frac{1}{2} f_m - 2f_L} > 0, \tag{47}$$

since from Assumption 2 it follows that $f_L > \frac{1}{2} f_m$. As $\frac{\partial H_1}{\partial x_L} < 0$, this solution is unique. Moreover, $H_1(x_H, x_H + \delta) = 1 - F(x_H + \delta) > 0$ and $H_1(x_H + \delta, 1) = F(\frac{1}{2} + \frac{x_H + \delta}{2}) - 1 < 0$, so from $\frac{\partial H_1}{\partial x_L} < 0$ it follows that $x_L^1(x_H) \in (x_H + \delta, 1)$.

Similarly, let $x_L^2(x_H)$ be a solution to $H_2 = 0$. From (46) it follows that

$$\frac{\partial x_L^2(x_H)}{\partial x_H} = -\frac{f_{H-}}{2f_L} < 0. \tag{48}$$

We have $x_L^2(\delta) = 1$ and $x_L^1(\delta) < 1$. We also have $x_L^1(1-\delta) = 1$ and $H_2(1 - \delta, 1) > 0$. Since $\frac{\partial H_2}{\partial x_L} > 0$, we must have $x_L^2(1-\delta) < 1$. It follows that the system $H_1 = 0$, $H_2 = 0$ has a unique solution.

□

Proof of Lemma 6. Denote $f_{H+} = f(x_H + \delta)$, $f_m = f(x_m)$, and $f_t = f((3x_H + x_L + 3\delta)/4)$

Let

$$M = \begin{pmatrix} \frac{\partial H_1}{\partial x_H} & \frac{\partial H_1}{\partial x_L} \\ \frac{\partial H_2}{\partial x_H} & \frac{\partial H_2}{\partial x_L} \end{pmatrix} \tag{49}$$

be the Jacobian matrix of $(H_1 H_2)$. We have

$$|M| = f_m f_L - f(x_H - \delta)\left(\frac{1}{2} f_m - 2f_L\right), \tag{50}$$

and a sufficient condition for $|M| > 0$ is $4f_L \geq f_m$, which is satisfied because of Assumption 2. The derivatives of x_H and x_L with respect to δ will be given by the implicit function theorem:

$$\begin{pmatrix} \frac{\partial x_H}{\partial \delta} \\ \frac{\partial x_L}{\partial \delta} \end{pmatrix} = -\frac{1}{|M|} \begin{pmatrix} 2f_L & 2f_L - \frac{1}{2}f_m \\ -f_{H-} & \frac{1}{2}f_m \end{pmatrix} \begin{pmatrix} \frac{1}{2}f_m \\ -f_{H-} \end{pmatrix} \tag{51}$$

It follows that

$$\frac{\partial x_H}{\partial \delta} = -\frac{1}{|M|}(f_L(f_m - 2f_{H-}) + \frac{1}{2}f_m f_{H-}) \tag{52}$$

$$\frac{\partial x_L}{\partial \delta} = \frac{1}{|M|} f_m f_{H-}. \tag{53}$$

We have

$$\begin{aligned} \frac{\partial D_1}{\partial \delta} &= \frac{1}{4}\left(3\frac{\partial x_H}{\partial \delta} + \frac{\partial x_L}{\partial \delta} + 3\right) f_t - 2\left(\frac{\partial x_H}{\partial \delta} + 1\right) f_{H-} = \\ &= f_{H-}(f_L(3f_L - \frac{f_m}{2}) + 4f_L f_m) > 0, \end{aligned} \tag{54}$$

since from Assumption 2 we have $f_m < 2f_L$. Finally,

$$\frac{\partial D_2}{\partial \delta} = \left(\frac{\partial x_H}{\partial \delta} - 1\right) f_{H-} - \frac{1}{2}\left(\frac{\partial x_H}{\partial \delta} + \frac{\partial x_L}{\partial \delta} + 1\right) f_m + \left(\frac{\partial x_H}{\partial \delta} + 1\right) f_{H-}. \tag{55}$$

Substituting (53) and (50) into (55) and simplifying, we get

$$\frac{\partial D_2}{\partial \delta} = \frac{1}{|M|} f_{H-}(4f_L f_{H-} - 4f_L f_m - f_m f_{H+}). \tag{56}$$

Now let $f_{H+} = af_L$ and $f_m = bf_L$. The derivative of D_2 with respect to δ will have the same sign as $4a - 4b - ab$. We now find the minimum $c < 1$ such that for all $a, b \in [c, 1/c]$, $a/b \in [c, 1/c]$, we have $4a - 4b - ab \leq 0$. This function, subject to the above constraints, obtains a maximum at $a = \frac{1}{c}, b = 1$, and is equal to zero at this point if $c = \frac{3}{4}$. Hence, if Assumption 2′ is satisfied, then $\frac{\partial D_2}{\partial \delta}$ will always be negative. □

Proof of Lemma 7. Condition (15) is satisfied for $\delta = \frac{1}{2}$. Violation of (15) means inequality

$$F\left(\frac{3x_H + x_L + 3\delta}{4}\right) - \alpha \leq \alpha.$$

It is equivalent to

$$\theta + \beta \leq F(x_m) - F\left(\frac{3x_H + x_L + 3\delta}{4}\right) + \alpha.$$

Since $\beta \le \alpha$, it is enough to prove that

$$\theta \le F(x_m) - F\left(\frac{3x_H + x_L + 3\delta}{4}\right). \tag{57}$$

The right hand side of (57) tends to $F(x_m) - F((x_H + x_m)/2)$ whereas the left hand side tends to zero. Then (57) is valid for sufficiently small δ because x_L does not go to x_H when δ is small. Indeed, if it holds, candidate N can win the second place with a left limit best response $x_N = x_H - \delta$ or a right limit best response $x_N = x_L$, so (x_H, x_L) is not a no-entry equilibrium. $\square$

Proof of Theorem 2. The second of the derivatives (53) is positive. Now suppose that $f_L = af_m$ and $f_{H-} = bf_m$. We have $\frac{\partial x_H^*}{\partial \delta} = -f_m^2/|M| \cdot (a(1-2b) + \frac{b}{2})$. We must find the minimum $c < 1$ for which inequality $(a(1-2b) + \frac{b}{2}) \ge 0$ will hold for all a, b such that $a \in [c, 1/c]$, $b \in [c, 1/c]$, and $a/b \in [c, 1/c]$. The solution to this problem is $c = \frac{3}{4}$ Hence, the derivative is positive as long as Assumption $2'$ is true. $\square$

Proof of Theorem 3. Differentiating (44) and (45) with respect to a and B, we get

$$\begin{array}{ll} \frac{\partial H_1}{\partial a} = 0 & \frac{\partial H_2}{\partial a} = 1 \\ \frac{\partial H_1}{\partial b} = 2 & \frac{\partial H_2}{\partial b} = -2 \end{array} \tag{58}$$

By implicit function theorem, (46) and (49) we have

$$\begin{pmatrix} \frac{\partial x_H^*}{\partial a} & \frac{\partial x_L^*}{\partial a} \\ \frac{\partial x_H^*}{\partial b} & \frac{\partial x_L^*}{\partial b} \end{pmatrix} = M^{-1} \cdot \begin{pmatrix} \frac{\partial H_1}{\partial a} & \frac{\partial H_1}{\partial b} \\ \frac{\partial H_2}{\partial a} & \frac{\partial H_2}{\partial b} \end{pmatrix} = \begin{pmatrix} f_m - 4f_L & -2f_m \\ -f_m & 4f_{H-} + 2f_m \end{pmatrix} \cdot \frac{1}{2|M|}, \tag{59}$$

where $|M|$ is given by (50). The required signs of partial derivatives follow from Assumption 2. $\square$

Proof of Theorem 4. We differentiate condition (21) with respect to a and b:

$$\frac{\partial D_1}{\partial a} = -2, \ \frac{\partial D_1}{\partial b} = 0 \tag{60}$$

Taking $D_1 = 0$, we have

$$\begin{pmatrix} \frac{\partial x_H^*}{\partial a} & \frac{\partial x_H^*}{\partial b} \\ \frac{\partial x_L^*}{\partial a} & \frac{\partial x_L^*}{\partial b} \\ \frac{\partial \delta}{\partial a} & \frac{\partial \delta}{\partial b} \end{pmatrix} = - \begin{pmatrix} \frac{\partial H_1}{\partial x_H^*} & \frac{\partial H_1}{\partial x_L^*} & \frac{\partial H_1}{\partial \delta} \\ \frac{\partial H_2}{\partial x_H^*} & \frac{\partial H_2}{\partial x_L^*} & \frac{\partial H_2}{\partial \delta} \\ \frac{\partial D_1}{\partial x_H^*} & \frac{\partial D_1}{\partial x_L^*} & \frac{\partial D_1}{\partial \delta} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial H_1}{\partial a} & \frac{\partial H_1}{\partial b} \\ \frac{\partial H_2}{\partial a} & \frac{\partial H_2}{\partial b} \\ \frac{\partial D_1}{\partial a} & \frac{\partial D_1}{\partial b} \end{pmatrix} \tag{61}$$

where $\frac{\partial D_1}{\partial \delta}$ is evaluated keeping x_H^*, x_L^* constant. Evaluating this expression, we get

$$\frac{\partial \delta}{\partial a} = (16 f_m f_L - 2 f_m f_t + 12 f_L f_t) \frac{1}{K} \qquad (62)$$

$$\frac{\partial \delta}{\partial b} = (-16 f_m f_{H-} + 4 f_m f_t - 4 f_{H-} f_t) \frac{1}{K}, \qquad (63)$$

where

$$K = 32 f_{H-} f_m f_L - f_{H-} f_m f_t + 12 f_{H-} f_L f_t - 6 f_m f_L f_t. \qquad (64)$$

If Assumption $2'$ is satisfied, then all derivatives have the required signs. □

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