An empirical model of non-equilibrium behavior in games

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Abstract. This paper is concerned with a structural econometric model of non-equilibrium behavior in games, with the goal of identifying and estimating the solution concepts that individuals use to generate their actions in games. The model is primarily based on various notions of limited strategic reasoning, allowing multiple modes of strategic reasoning and also heterogeneity in strategic reasoning across individuals and within individuals. The paper proposes the model, and provides sufficient conditions for point identification of the unknown parameters. Then, the model is estimated on data from an experiment involving two-player guessing games. The application illustrates the empirical relevance of the main features of the model.

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1. Introduction

In game theory, there is not a definitive prediction about how players choose their actions. The Nash equilibrium solution concept (e.g., Nash (1950)) is ubiquitous, but theory also provides other solution concepts. There is considerable empirical evidence of behavior that does not conform to the predictions of Nash equilibrium (e.g., Camerer (2003)). This paper is concerned with a structural econometric model of non-equilibrium behavior in games, with the goal of determining the solution concepts that individuals use to generate their actions in games. Broadly, similar empirical questions have been a major focus in the literature on experimental game theory, although the focus in this paper is on understanding the econometrics of the newly proposed structural model. The paper proposes the model, establishes sufficient conditions for point identification of the model, and estimates the model on real data.

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The prior literature on identification in the econometrics of games[1] has tended to focus on the problem of identifying the utility function under the assumption that the econometrician knows the solution concept, whereas this paper is concerned with identifying the solution concept(s) under the assumption that the econometrician knows the utility function, as in an experiment. Prior work in the experimental economics literature, which addresses similar empirical questions, has not focused on formal identification results as much as the literature on econometrics of games. Evidently, that is because the models/"empirical strategies" used previously are "obviously" point identified, often without formal proof. However, as discussed throughout the paper, that is not the case for the model in this paper.[2]

The model allows that individuals generate their actions based on solution concepts other than Nash equilibrium, though Nash equilibrium is still included in the model. One alternative solution concept is rationalizability (e.g., Bernheim (1984) and Pearce (1984)). Rationalizability is equivalent to common knowledge of rationality and independence of actions across players (e.g., Tan and da Costa Werlang (1988)). In two player games, rationalizability is also equivalent to infinitely many steps of iterated deletion of dominated strategies.[3]

Due to limited strategic reasoning, for example bounded cognitive ability, individuals may carry out only finitely many steps of iterated deletion of dominated strategies,[4] and so the model also formalizes and includes that possibility as the "steps of unanchored strategic reasoning." In most games, a set of actions are consistent with any given number of steps of unanchored strategic reasoning. Consequently, unanchored strategic reasoning is an "incomplete model" of behavior that does not uniquely determine the action that an individual takes. Moreover, a given action can be consistent with multiple different numbers of steps of unanchored strategic reasoning. Therefore, it is not possible to simply "infer" the number of steps of unanchored strategic reasoning that an individual uses by inspecting whether an observed action is equal to that predicted by a particular number of steps of unanchored strategic reasoning. One of the contributions of the paper is to study the empirical relevance of unanchored strategic reasoning, in particular by providing a structural model in

[1]Papers in that literature (among papers studying complete information), typically based on Nash equilibrium, include Tamer (2003), Aradillas-Lopez and Tamer (2008), Bajari, Hong, and Ryan (2010), Kline and Tamer (2012), Aradillas-Lopez and Rosen (2013), Dunker, Hoderlein, and Kaido (2013), Fox and Lazzati (2013), and Kline (2015a,b). See de Paula (2013) for a review.

[2]However, important identification results in experimental economics do include Haile, Hortaçsu, and Kosenok (2008) (identification of the quantal response equilibrium model), and Gillen (2010) and An (2013) (identification of the level-k model in auctions).

[3]See for example Tan and da Costa Werlang (1988) or Fudenberg and Tirole (1991). In any given game, rationalizability might be equivalent to a certain finite number of steps of iterated deletion of dominated strategies, if additional strategies are no longer deleted in further iterations. But, in general, rationalizability requires infinitely many (or, at least, unbounded) steps of iterated deletion.

[4]Or, equivalently, individuals may use actions that are not consistent with common knowledge of rationality and independence of actions, perhaps because doing so (even in two player games) requires them to carry out infinitely many steps of iterated deletion of dominated strategies.

which it is possible to identify/estimate how many steps of unanchored strategic reasoning individuals carry out. Equivalently, resolving the identification problems mentioned above is a contribution of the paper. Unanchored strategic reasoning is found to be empirically relevant in the application.

Another solution concept related to limited strategic reasoning is the level-k model of thinking, commonly used in the experimental game theory literature.[5] In the level-k model of thinking, individuals that use 0 steps of reasoning are "anchored" to a particular distribution of actions, usually the uniform distribution over the action space. Hence, this paper uses the term "anchored strategic reasoning" to refer to this solution concept. Individuals that use more than 0 steps of anchored strategic reasoning best respond to the strategy used by an individual of the immediately lower number of steps of anchored strategic reasoning.

Rather than suppose that a single solution concept is responsible for generating all actions of all individuals, the model allows heterogeneity in the solution concept(s) that generate the actions. The model allows both across-individual and within-individual heterogeneity. Essentially, the model aims to estimate how often individuals use each of the solution concepts. Equivalently, the model aims to estimate how often individuals use each mode of strategic reasoning (anchored versus unanchored), and the number of steps of reasoning that individuals use. Across-individual heterogeneity allows that different individuals use different solution concepts, an important stylized fact from the experimental game theory literature. Similarly, within-individual heterogeneity allows that even a given individual uses multiple different solution concepts, a contribution of the structural model in this paper. Prior empirical work in the related experimental game theory literature has been based on the assumption that each individual uses just one solution concept. In particular, the prior literature based on the level-k model of thinking characterizes individuals as a "level-1" thinker or a "level-2" thinker, and so on. The model in this paper allows that a given individual is characterized by the use of multiple solution concepts, rather than just one solution concept, just as the overall population of individuals is characterized by the use of multiple solution concepts, rather than just one solution concept. For example, a given individual might sometimes carry out one step of iterated deletion of dominated strategies, but other times carry out two steps of iterated deletion of dominated strategies. Both across-individual and within-individual heterogeneity are found to be important in the empirical application.

[5]See for example Camerer (2003) or Crawford, Costa-Gomes, and Iriberri (2012) for a discussion of the related experimental literature, which includes in particular (not exhaustive): Stahl and Wilson (1994, 1995), Nagel (1995), Ho, Camerer, and Weigelt (1998), Costa-Gomes, Crawford, and Broseta (2001), Camerer, Ho, and Chong (2004), Costa-Gomes and Crawford (2006), and Crawford and Iriberri (2007a,b).

After proposing the model, the paper establishes sufficient conditions for point identification of the unknown parameters. In the absence of such sufficient conditions, it is shown by example that it can easily happen that the unknown parameters are not point identified. Many of the main sufficient conditions concern the structure of the games that subjects are observed to play, providing guidance to the design of the experiment. One of the main sufficient conditions is that the econometrician observes each subject play multiple games.

Then, the model is estimated using data that comes from the "two-person guessing game" experiment in Costa-Gomes and Crawford (2006), in order to establish the empirical relevance of the results in the context of a well-known and representative experimental design. The results suggest that both across-individual and within-individual heterogeneity, and unanchored strategic reasoning, all are important. A complete discussion of the empirical results is deferred until after formalizing the setup and identification of the model.

Beyond the difference in focus (identification versus empirical results), the model in this paper differs from prior models in experimental game theory. Those differences are the reason for the more difficult identification problem in this paper, compared to prior work in experimental game theory. In particular, allowing unanchored strategic reasoning and within-individual heterogeneity substantially complicates the identification problem, but the application shows that those features of the model are empirically relevant.

1.1. **Outline of the paper.** Section 2 sets up the model. Section 3 sets up the identification problem, and sections 4 and 5 establish sufficient conditions for point identification. Section 6 reports the empirical application. Section 7 concludes. Appendix A collects supplemental results, including additional estimation results and the proof of point identification.

2. Model

2.1. **Notation for the games.** The goal of the model is to study strategic behavior in complete information games with continuous action spaces.[6] The setup for game g is:

(1) There are M_g agents, indexed by $j = 1, 2, \ldots, M_g$. Note that the "agent indexed by j" or just "agent j" corresponds to the indexing of agents in the game, and is not the same as subject j in the dataset. (So "agent j" could alternatively be called, for example, the "row player" in the game.)

(2) The action of agent j is a_j.

(3) The utility function of agent j in game g is $u_{jg}(a_1, \ldots, a_{M_g})$.

[6] It is possible to propose a similar model for games with discrete action spaces, but games with continuous action spaces provide more scope for different solution concepts to make different predictions about the action an individual takes, which is necessary for identification.

(4) The action space for agent j in game g is the interval $[\alpha_{Lg}(j), \alpha_{Ug}(j)]$, hence continuous action space.
(5) All of these facts are common knowledge among the agents, so the game is complete information. Also, all of these facts are known by the econometrician.

As formalized in section 2.4, the econometrician has data on the behavior of subjects in these games. There is an important distinction between "agent" and "subject" in this paper. The term "subject" refers to an actual individual in the real world. The term "subject" reflects the fact that the data typically concerns "experimental subjects." The term "agent" refers to the more generic game theory concept, for example an "agent" could refer to the "row player" (i.e., characterized by preferences and action space) in a particular game.

2.2. **Setup of model.** Essentially, the model in this paper is concerned with recovering information about the solution concepts that individuals use to generate their actions in games, based on observing the actions of those individuals.

By solution concept, this paper means a possibly set-valued mapping between the structure of a game (e.g., utility functions, and action spaces) and the set of strategies for the players. Each solution concept can be viewed as making a set-valued prediction about behavior. In particular, following the literature on experimental game theory, this paper focuses on non-equilibrium solution concepts, which do not necessarily have any sort of "equilibrium" property as does, for example, Nash equilibrium. Each (non-equilibrium) solution concept therefore can be viewed as making a prediction about behavior for each player in a game, without necessarily any connection to the actual behavior of the other players in the games. Even equilibrium solution concepts like Nash equilibrium can be viewed as making "non-equilibrium" predictions, in the sense of making predictions for each individual player. Consequently, a player can be said to use (its part of) a solution concept (e.g., Nash equilibrium), without consideration of what the other players in the game actually do.

Each subject i (i.e., each individual in the real world) has a strategic decision making rule

$$\theta_i = (\lambda_i(\cdot), \delta_i, \rho_i)$$

that characterizes how it behaves in games. These are *ex ante* unknown by the econometrician. The elements of the strategic decision making rule are:

(1) $\lambda_i(\cdot)$ is a distribution over solution concepts that characterizes the probabilities that subject i uses particular solution concepts. For example, $\lambda_i(NE)$ gives the probability that subject i uses the Nash equilibrium (NE) solution concept, when it plays a game. The possibility that $\lambda_i(\cdot)$ does not place probability 1 on one solution concepts reflects the possibility of within-individual heterogeneity, one of the contributions of

this paper: a given individual might use more than one solution concept. See remark 2.1 for further discussion. The solution concepts are described in section 2.5.

(2) The parameters δ_i and ρ_i are the probability and magnitude of computational mistakes made by subject i, respectively. Roughly, this allows that a subject might "intend" to use a particular solution concept, but fail to compute the associated action correctly and actually take an action that is only approximately equal to the action predicted by the "intended" solution concept. Computational mistakes are described in section 2.6. A special case of the model rules out computational mistakes.

The behavioral implications of the model can be described in the following procedural way. The details of each step are described in subsequent subsections.

(1) Each subject is "born" and permanently assigned its strategic decision making rule $\theta_i = (\lambda_i(\cdot), \delta_i, \rho_i)$ by nature per section 2.3.
(2) Each time subject i encounters a game to play:
    (a) Subject i chooses the solution concept it intends to use in that game. The probability that subject i chooses solution concept "k" is $\lambda_i(k)$. It might choose, for example, to use the Nash equilibrium, or to use one step of unanchored strategic reasoning. The set of solution concepts is described in section 2.5.
    (b) If the intended solution concept is not subject to computational mistakes, as described in section 2.6, then the subject takes an action according to that solution concept. Otherwise, the subject attempts to calculate the action associated with the intended solution concept. The subject either correctly or incorrectly calculates the action:
        (i) The probability of correct calculation is $1 - \delta_i$. In this case, the subject actually takes the action associated with the intended solution concept.
        (ii) The probability of incorrect calculation is δ_i. In this case, the subject actually takes an action that is only approximately equal to the action associated with the intended solution concept. The details of computational mistakes are described in section 2.6.

For example, if $\lambda_i(k) = 0.2$, and $\delta_i = 0.05$, then subject i uses solution concept "k" 20 percent of the time. Supposing that "k" is subject to computational mistakes, 95 percent of the time it correctly computes the solution concept, and actually does take the associated action; but, 5 percent of the time it makes a small computational mistake, and takes an action that is only approximately equal to the "intended" action under solution concept "k".

Remark 2.1 (Within-individual heterogeneity)**.** The model does not attempt to estimate the "reason" for within-individual heterogeneity. As with other economic models, and in particular as with solution concepts in general, within-individual heterogeneity is a model of *observed behavior*, with potentially many "as if" explanations for why that observed behavior arises. One of the goals of the model is to provide a framework by which to answer empirical questions relating to individuals exhibiting (or not) within-individual heterogeneity. (Other goals include studying the empirical relevance of unanchored strategic reasoning.) Related papers are similar, providing frameworks by which to answer the empirical question of whether or not individuals conform to the predictions of some economic theory (e.g., Nash equilibrium, level-k thinking, across-individual heterogeneity, etc.), without attempting to explain the "reason why" they conform (or not) to the predictions of that theory. Nevertheless, as with other models of observed behavior, it is possible to provide some *ex ante* "as if" reasons to suspect the existence of within-individual heterogeneity. Then, the empirical application finds evidence of within-individual heterogeneity.

One possible explanation is that individuals find multiple solution concepts to make "compelling" recommendations for their strategic behavior, just as economic theorists might find many different solution concepts "compelling." As a consequence, individuals exhibit within-individual heterogeneity.

Another possible explanation is that individuals have beliefs about the type of their opponent, and each time they play a game, they resolve their uncertainty about the type of their opponent *before* they take an action. An individual that has non-degenerate beliefs about the type of the opponent can exhibit within-individual heterogeneity. For example, in the level-k model of thinking (i.e., anchored strategic reasoning in this paper, detailed in section 2.5), an individual that believes the opponent is level-0 with probability p and level-1 with probability $1-p$ will use the level-1 strategy with probability p and the level-2 strategy with probability $1-p$. This differs from the standard approach to responding to uncertainty about the type of the opponent, which would not generate within-individual heterogeneity, because it would entail individuals using the strategy that is the best response to the entire distribution of beliefs about the type of the opponent. By resolving their uncertainty about the opponent *before* taking an action, individuals can exhibit within-individual heterogeneity.

More generally, especially in the empirical experimental game theory literature concerning the level-k model of thinking (i.e., anchored strategic reasoning in this paper), issues related to but distinct from within-individual heterogeneity have been investigated as a sort of robustness check on the stability of the estimates. The details vary across papers.[7] One

[7]See Stahl and Wilson (1995) or Georganas, Healy, and Weber (2015) for some examples.

question concerns checking whether the aggregate distribution of "types" (i.e., the fraction of level-1 thinkers, the fraction of level-2 thinkers, etc.), in models that assume that each individual is exclusively one "type" of thinker, appears to be the same across multiple sets of games. Note that the example in section 3.2 shows that the fraction of individuals exhibiting any given number of steps of reasoning can be the same across games, even though particular individuals do exhibit within-individual heterogeneity. Therefore, questions concerning the aggregate distribution of "types" are distinct from questions concerning within-individual heterogeneity, and indeed within-individual heterogeneity can be obscured when investigating only the aggregate distribution of "types." Another question concerns checking whether a particular individual is "estimated" to be the same type across multiple sets of games (or, more or less equivalently, whether individuals appear to statistically conform "out of sample" to their "estimated" type). This question is more similar to, but still distinct from, questions concerning within-individual heterogeneity. In particular, note that an individual that most often uses a particular solution concept is likely to always be "estimated" to be that type (across different sets of games), since that provides the best "fit" among the types restricted to using one solution concept, regardless of underlying within-individual heterogeneity. More generally, models that are restricted to "estimating" each individual to be a "type" that exclusively uses one solution concept (e.g., level-1 or level-2) are misspecified in the presence of within-individual heterogeneity. In contrast, the model framework provided in this paper explicitly allows for within-individual heterogeneity.

2.3. **Types of strategic decision making rules.** The model is based on the assumption that there are at most R strategic decision making rules used in the population, indexed by $r = 1, 2, \dots, R$, denoted as $\Theta_r = (\Lambda_r, \Delta_r, \mathrm{P}_r)$, and known as the *strategic decision making types*. Although R is known by the econometrician, Θ_r is unknown by the econometrician. This helps make the model parsimonious, and also is required for it to be possible to point identify the model. Each subject is one of these R strategic decision making types, or equivalently uses the strategic decision making rule associated with one of these strategic decision making types. The population fraction of subjects who are type r is $\pi(r)$. This population fraction is unknown by the econometrician. So, the econometrician knows there are at most R strategic decision making types, but otherwise knows nothing about those strategic decision making types. It is allowed that $\pi(r) = 0$ for some r, so that fewer than R strategic decision making types exist. When subject i is "born" it is assigned to be strategic decision making type $\Theta_{\tau(i)}$, where $\tau(i) \in \{1, 2, \dots, R\}$, according to the distribution $\pi(\cdot)$ over $\{1, 2, \dots, R\}$. By construction, $\theta_i = \Theta_{\tau(i)}$. The identification result in this paper shows sufficient conditions for point identification of the unknown $\{\Theta_r, \pi(r)\}_{r=1}^{R}$.

2.4. **Econometrics: data and sketch of identification problem.** The data observed by the econometrician is the actions taken by each of N subjects, in games of the sort described in section 2.1. The subjects are indexed by $i = 1, 2, \ldots, N$. Each subject plays each of G games, indexed by $g = 1, 2, \ldots, G$. It is assumed (essentially without loss of generality, by re-defining the agent roles appropriately) that the subjects in the dataset are always agent 1 in the games. The observed action of subject i in game g is y_{ig}. See the empirical application in section 6 for one of many instances of such a dataset from the experimental game theory literature. As discussed in section 2.2, because of the non-equilibrium nature of the analysis, the actions of the "opponents" of a subject are not relevant to the analysis, since the analysis focuses on learning which solution concept(s) generate the behavior of individual subjects. Indeed, in principle, it would be enough for an experiment to present each subject with each of the games, without actually presenting the games to the "opponents."

The population distribution of the observed data is $P(\{y_g\}_{g=1}^G)$: the distribution of actions in the G games *across the population of subjects.* The identification problem concerns establishing sufficient conditions under which it is possible to uniquely recover the unknown model parameters $\{\Theta_r, \pi(r)\}_{r=1}^R$ from $P(\{y_g\}_{g=1}^G)$.

Identification and estimation corresponds to $N \to \infty$ while G is fixed. An alternative identification problem would be to suppose that a population of subjects is observed to play a population of games. Under that setup, identification and estimation would correspond to $N \to \infty$ and $G \to \infty$. Identification in that data setup would require making assumptions about the "population distribution of games," including games that are not actually among the finitely many observed games. It seems difficult to interpret assumptions on games that are not observed. In contrast, the identification results in the fixed G setup requires only that the econometrician verify that the observed games satisfy certain conditions. Therefore, identification results in the $G \to \infty$ setup would inevitably be less plausible than identification results in the fixed G setup.

2.5. **Solution concepts.** The model includes the following solution concepts. These solution concepts are demonstrated by example in the empirical application in section 6.

2.5.1. *Nash equilibrium.* The Nash equilibrium solution concept predicts that agents use strategies that are mutually best responses. According to Nash equilibrium, agent j in game g uses a strategy σ_{jg}, with the property that σ_{jg} is a distribution supported on the set of solutions to

$$\max_{a_j \in [\alpha_{Lg}(j), \alpha_{Ug}(j)]} E_{\sigma_{-j,g}} \left(u_{jg}(a_1, \ldots, a_{M_g}) \right)$$

where $[\alpha_{Lg}(j), \alpha_{Ug}(j)]$ is the action space for agent j in game g, and the expectation notation indicates that a_{-j} are distributed according to the Nash equilibrium strategies of the other agents in game g (i.e., according to $\sigma_{-j,g}$). The model is based on the assumption that there is a unique pure strategy Nash equilibrium that predicts that agent j in game g takes action $c_{jg}(NE)$, as is the typical case for games studied in the related experimental game theory literature. The notation for Nash (e.g., as argument in $\lambda(\cdot)$) is NE.

2.5.2. *Unanchored strategic reasoning.* The "steps of unanchored strategic reasoning" is a class of solution concepts that are iteratively-defined "steps" of increasingly sophisticated strategic reasoning closely related to iterated deletion of dominated strategies, particularly in two-player games,[8] and the rationalizability solution concept (e.g., Bernheim (1984) and Pearce (1984)). One contribution of this paper is to study the empirical relevance of unanchored strategic reasoning, in particular by providing a structural model in which it is possible to identify/estimate how many steps of unanchored strategic reasoning individuals carry out.

The following formally describes unanchored strategic reasoning. Let $\mathcal{D}_{jg}$ be the family of all strategies (i.e., distributions) supported on $[\alpha_{Lg}(j), \alpha_{Ug}(j)]$. Then, define

$$\tilde{\Sigma}_{jg}^{0} = \{\sigma_j \in \mathcal{D}_{jg}\}.$$

By definition, the strategies in $\tilde{\Sigma}_{jg}^{0}$ are exactly those strategies (i.e., all strategies) that can be used by agent j in game g that uses zero steps of unanchored strategic reasoning. Similarly, define

$$\Sigma_{jg}^{0} = [\alpha_{Lg}(j), \alpha_{Ug}(j)]$$

to be the set of actions that are consistent with the use of zero steps of unanchored strategic reasoning. Of course, by construction, Σ_{jg}^{0} is the entire action space. Then, for $s \geq 0$, define

$$\tilde{\Sigma}_{jg}^{s+1} = \{\sigma_j \in \mathcal{D}_{jg} : \exists \sigma_{-j} \in \Pi_{j' \neq j} co\left(\tilde{\Sigma}_{j'g}^{s}\right) \text{ s.t. } \sigma_j \text{ is supported on}$$
$$\text{the set of solutions to } \max_{a_j \in [\alpha_{Lg}(j), \alpha_{Ug}(j)]} E_{\sigma_{-j}}\left(u_{jg}(a_1, \ldots, a_{M_g})\right)\}.$$

By definition, the strategies in $\tilde{\Sigma}_{jg}^{s+1}$ are exactly those strategies that can be used by agent j in game g that uses $s+1$ steps of unanchored strategic reasoning. These are the strategies σ_j for which there are strategies σ_{-j} of the other agents, that can be used by other agents that use s steps of unanchored strategic reasoning[9], such that σ_j is the best response to the

[8] See for example Tan and da Costa Werlang (1988) or Fudenberg and Tirole (1991). Unanchored strategic reasoning (under the name "level-k rationality") has been *assumed* to generate the data in Aradillas-Lopez and Tamer (2008), Kline and Tamer (2012), and Kline (2015b), in order to identify the utility function.

[9] Or, technically, are mixtures of such strategies, in cases of non-convexity.

other agents using those strategies. Similarly, define

$$\Sigma_{jg}^{s+1} = \{a_j \in [\alpha_{Lg}(j), \alpha_{Ug}(j)] : \exists \sigma_{-j} \in \Pi_{j' \neq j} co\left(\tilde{\Sigma}_{j'g}^{s}\right) \text{ s.t.}$$
$$a_j \in \arg \max_{a_j \in [\alpha_{Lg}(j), \alpha_{Ug}(j)]} E_{\sigma_{-j}}\left(u_{jg}(a_1, \ldots, a_{M_g})\right)\}.$$

to be the set of actions that are consistent with the use of $s+1$ steps of unanchored strategic reasoning. Σ_{jg}^{s+1} can be viewed as the set of *pure strategies* consistent with $s+1$ steps of unanchored strategic reasoning, noting that the mixed strategies consistent with $s+1$ steps of unanchored strategic reasoning are mixtures over Σ_{jg}^{s+1}.

Note the intuitive appeal of s steps of unanchored strategic reasoning, in terms of iterated deletion of dominated strategies, especially in the case of two-player games. Intuitively, strategies in $\tilde{\Sigma}_{jg}^{1}$ are best responses to some strategies of the opponents, and therefore survive 1 round of deletion of dominated strategies; $\tilde{\Sigma}_{jg}^{2}$ are best responses to some strategies of the opponents that survive 1 round of deletion of dominated strategies, and therefore survive 2 rounds of deletion of dominated strategies; and so forth. See Tan and da Costa Werlang (1988) or Fudenberg and Tirole (1991) for further details.

By the principle of indifference, agent j in game g that uses $s+1$ steps of unanchored strategic reasoning takes an action drawn from the uniform distribution on Σ_{jg}^{s+1}.[10] See remark 4.1 for further discussion.

By construction, $\Sigma_{jg}^{s'} \subseteq \Sigma_{jg}^{s}$ for $0 \leq s \leq s'$, so any action consistent with s' steps of unanchored strategic reasoning is also consistent with s steps of unanchored strategic reasoning. This complicates the association between the action than an individual takes and the number of steps of unanchored strategic reasoning that individual used to generate that action. A given action can be consistent with many different numbers of steps of unanchored strategic reasoning, making it difficult to infer the number of steps of unanchored strategic reasoning used to generate that action. The notation for s steps of unanchored strategic reasoning (e.g., as argument in $\lambda(\cdot)$) is s_{unanch}.

Remark 2.2 (Epistemic interpretation)**.** The results of Tan and da Costa Werlang (1988) can be used to provide an epistemic interpretation of s steps of unanchored strategic reasoning. Using $s = 1$ step of unanchored strategic reasoning is "equivalent" to being rational (at least), and for $s \geq 2$, using s steps of unanchored strategic reasoning is "equivalent"

[10] The principle of indifference applies, since all actions in Σ_{jg}^{s+1} are "equally" consistent with $s+1$ steps of unanchored strategic reasoning. In contrast, if some actions in Σ_{jg}^{s+1} were used less often than other actions, that suggests the use of a solution concept other than $s+1$ steps of unanchored strategic reasoning. For example, if some actions in Σ_{jg}^{s+1} were used with 0 probability, that would suggest the use of some refinement of $s+1$ steps of unanchored strategic reasoning. This implicitly requires, as a technical regularity condition, that such a uniform distribution is well-defined. Consequently, it is implicitly assumed that Σ_{jg}^{s} either: (a) is a finite set, or (b) is Lebesgue measurable with non-zero measure.

to being rational and also knowing everyone (knows everyone)$^{s-2}$ is rational (at least), in addition to some other conditions (including those related to players acting independently of each other). Rationalizability, or $s = \infty$ steps of unanchored strategic reasoning, is roughly equivalent to common knowledge of rationality.

Remark 2.3 (Example)**.** For example, in a two-player game, agent 1 who uses two steps of unanchored strategic reasoning can be interpreted "as if" to use the following strategic reasoning: I think my opponent will use strategy σ_2. I think my opponent will use σ_2 because σ_2 would be a best response from the perspective of my opponent, if I were to use strategy σ_1. And given that I think my opponent will use σ_2, I should use the strategy σ_1', which is a best response to σ_2.

Remark 2.4 (Consistency with unanchored strategic reasoning)**.** The experimental game theory literature has sometimes checked whether observed actions are *consistent with* some number of steps of unanchored strategic reasoning (under names like "iterated deletion of dominated strategies"), as a standalone exercise separate from, for example, estimating a structural "level-k model." In contrast, in this paper, "unanchored strategic reasoning" is included as a solution concept in a model alongside other solution concepts (e.g., alongside the "level-k model"), making it possible to answer the question of how often (and/or whether) a subject uses a given number of steps of unanchored strategic reasoning (versus, for example, a different number of steps that also is consistent with an observed action), similar to the same question about other solution concepts. Note the fundamental distinction between these approaches, which is exacerbated by the fact that generally a given action will be consistent with multiple steps of unanchored strategic reasoning, so that the fact that an action is *consistent* with a given number of steps of unanchored strategic reasoning is not necessarily evidence that the subject taking that action actually used exactly that number of steps of unanchored strategic reasoning.

2.5.3. *Anchored strategic reasoning.* It is possible to add to the above iterated definitions the condition that, for all agents j and games g, $\tilde{\Sigma}_{jg}^{0}$ consists of only one strategy: the uniform distribution over the action space. This results in "anchored strategic reasoning," because the steps of strategic reasoning become "anchored" to the uniform distribution being used by agents that use zero steps of strategic reasoning. In the experimental game theory literature, this is known as the "level-k model," but "anchored" and "unanchored" are used in this paper to emphasize the relationship between the two classes of solution concept.[11]

[11] See citations on the level-k model in the introduction.

Zero steps of unanchored strategic reasoning is observationally equivalent to zero steps of anchored strategic reasoning, but "anchoring" does revise the implications of using more than zero steps of strategic reasoning, by working through the iterated definition of steps of strategic reasoning described in section 2.5.2. For example, an agent that uses one step of *anchored* strategic reasoning would use a strategy that is a best response to the other agents using the strategy that is the uniform distribution over the action space, and an agent that uses two steps of *anchored* strategic reasoning would use a strategy that is the best response to the other agents using a strategy consistent with one step of *anchored* strategic reasoning.

Generically, in the sorts of games studied in experimental game theory, there is a unique action (or equivalently, a unique pure strategy) consistent with s steps of *anchored* strategic reasoning (for each $s \geq 1$), whereas there is a range of actions consistent with s steps of *unanchored* strategic reasoning. The results are derived based on the assumption that there is a unique action consistent with anchored strategic reasoning, as is typically the case for games studied in the related experimental game theory literature: agent j in game g that uses s steps of anchored strategic reasoning takes action $c_{jg}(s_{anch})$. The notation for s steps of anchored strategic reasoning (e.g., as argument in $\lambda(\cdot)$) is s_{anch}.

It is possible to distinguish between an individual that uses *unanchored* strategic reasoning and an individual that uses the special case of *anchored* strategic reasoning, because the latter will *always* take the action associated with anchored strategic reasoning, whereas the former will not.

2.5.4. *Assumptions on strategic reasoning.* Assumption 2.1 supposes that the set of steps of strategic reasoning that subjects possibly use is known by the econometrician to be a finite set. Note that this is consistent with prior experimental results, which indicate individuals use a very small number of steps of reasoning. It would be extremely difficult to distinguish between infinitely many solution concepts, especially given finite data. The consequence of assumption 2.1 is that $\lambda_i(\cdot)$ is a distribution over a finite set of solution concepts, rather than an infinite set of solution concepts.

Assumption 2.1 (Steps of strategic reasoning)**.** *The numbers of steps of unanchored strategic reasoning that subjects might use is the known finite set $\mathcal{U}$. The numbers of steps of anchored strategic reasoning that subjects might use is the known finite set $\mathcal{A}$.*

2.6. **Computational mistakes.** Roughly following the literature on experimental game theory, computational mistakes arise when a subject "intends" to use a certain solution concept, but fails to correctly take the associated action. The solution concepts subject to computational mistakes are the solution concepts that are associated with a *unique* action, collected in the set $\mathcal{M}$: the steps of anchored strategic reasoning and Nash equilibrium.[12] The econometrician can assume *ex ante* that subjects do not make computational mistakes, in which case the sufficient conditions for point identification are weaker.

Let $\xi(\cdot)$ be a known bounded and continuous density defined on support $[-1, 1]$ that is bounded away from zero, in the sense that $\xi(x) \geq \kappa > 0$ for all $x \in [-1, 1]$ for some κ.[13] Suppose that subject i "intends" to use a particular solution concept in $\mathcal{M}$ that predicts the action c, and that subject i is playing in the role of agent j in game g. There is δ_i probability that the subject makes a computational mistake. If there is a computational mistake, then the subject actually takes an action according to the $\xi(\cdot)$ density, translated to an interval of radius $\rho_i\left(\alpha_{Ug}(j) - \alpha_{Lg}(j)\right)$ that is centered at the "intended" action c, intersected with the action space $[\alpha_{Lg}(j), \alpha_{Ug}(j)]$:

$$[\alpha_{Lg}(j), \alpha_{Ug}(j)] \cap [c - \rho_i\left(\alpha_{Ug}(j) - \alpha_{Lg}(j)\right), c + \rho_i\left(\alpha_{Ug}(j) - \alpha_{Lg}(j)\right)].$$

The intersection with $[\alpha_{Lg}(j), \alpha_{Ug}(j)]$ guarantees that the action is within the action space. Consequently, the subject takes an action a according to the density

$$\begin{aligned}\omega_{jg,c,\rho_i}(a) \equiv & \frac{2}{\min\{\alpha_{Ug}(j), c + \rho_i\left(\alpha_{Ug}(j) - \alpha_{Lg}(j)\right)\} - \max\{\alpha_{Lg}(j), c - \rho_i\left(\alpha_{Ug}(j) - \alpha_{Lg}(j)\right)\}} \times \\ & \xi\left(\frac{a - \frac{\min\{\alpha_{Ug}(j), c+\rho_i\left(\alpha_{Ug}(j)-\alpha_{Lg}(j)\right)\}+\max\{\alpha_{Lg}(j), c-\rho_i\left(\alpha_{Ug}(j)-\alpha_{Lg}(j)\right)\}}{2}}{\frac{\min\{\alpha_{Ug}(j), c+\rho_i\left(\alpha_{Ug}(j)-\alpha_{Lg}(j)\right)\}-\max\{\alpha_{Lg}(j), c-\rho_i\left(\alpha_{Ug}(j)-\alpha_{Lg}(j)\right)\}}{2}}\right).\end{aligned}$$

The parameter ρ_i characterizes the magnitude of computational mistakes: larger ρ_i imply the possibility of larger computational mistakes. The range of computational mistakes is ρ_i multiplied by the width of the action space (i.e., $(\alpha_{Ug}(j) - \alpha_{Lg}(j))$) to reflect the fact that games with larger action spaces are more subject to relatively larger computational mistakes. The model of computational mistakes is formalized in assumption 2.2. Similar identification strategies could be used for similar models of computational mistakes.

Assumption 2.2 (Computational mistakes)**.** *Either:*

(1) The econometrician allows the possibility of computational mistakes. The probability that subject i makes a computational mistake is $0 \leq \delta_i < 1$. The magnitude of the mistakes made by subject i is $\rho_i > 0$. If subject i makes a computational mistake in game g as role j, and intended to use a solution concept that would result in taking action c, then subject i takes an action according to the $\xi(\cdot)$ density, translated to

[12] Computational mistakes arise only with solution concepts that are associated with a unique action (which is where computational mistakes have been allowed in the prior literature), avoiding the ambiguity about what it would mean to "incorrectly compute the action" associated with a solution concept that is consistent with a range of actions, as in unanchored strategic reasoning.

[13] Conversely, $\xi(\cdot)$ is zero off the support $[-1, 1]$, by definition of support. The continuity at the endpoints -1 and 1 is implicitly understood to be right- and left- continuity.

$[\alpha_{Lg}(j), \alpha_{Ug}(j)] \cap [c - \rho_i (\alpha_{Ug}(j) - \alpha_{Lg}(j)), c + \rho_i (\alpha_{Ug}(j) - \alpha_{Lg}(j))]$. *The econometrician knows $\overline{\rho}$ such that $\rho_i < \overline{\rho}$ for all subjects i.*

(2) *The econometrician does not allow the possibility of computational mistakes, and therefore knows that $\delta_i \equiv 0$ and $\rho_i \equiv 0$ for all subjects i. For the purposes of future assumptions, the econometrician sets $\overline{\rho} = 0$.*

If the econometrician allows the possibility of computational mistakes, it is assumed that $\rho_i > 0$ for all subjects i. If it were allowed that $\rho_i = 0$, then there would be a complication relating to the fact that players that do not make computational mistakes ($\delta_i = 0$) are observationally equivalent to players that do make computational mistakes with zero magnitude ($\delta_i > 0$ but $\rho_i = 0$).

3. Setup of the identification problem

The identification problem in this model concerns the question of whether it is possible to recover the parameters of the model (i.e., $\{\Theta_r, \pi(r)\}_{r=1}^{R}$) from the population distribution of the data. The model will fail to be point identified if it happens that more than one specification of the parameters generate the same distribution of the data, because then the "true" specification of the parameters cannot be distinguished from a "false" specification of the parameters. Therefore, point identification is a necessary logical prerequisite for estimating the parameters of the model.[14] The parameters of the model are not point identified without non-trivial sufficient conditions, as section 3.2 provides a counterexample to point identification in the absence of the sufficient conditions for point identification and section 3.3 provides a discussion of further threats to point identification.

3.1. **Definition of point identification.** In order to define point identification, it is necessary to define observational equivalence of strategic decision making types. If there are two strategic decision making types that *are not* observationally equivalent, then at least in principle (for some logically possible game) those two strategic decision making types could generate different observed behavior, and therefore be distinguished from each other. Conversely, if there are two strategic decision making types that *are* observationally equivalent, then there are no games in which those two strategic decision making types would generate different observed behavior. Therefore, it is impossible to distinguish between observationally equivalent strategic decision making types.

[14] If the model were not point identified, inference following Chernozhukov, Hong, and Tamer (2007), Beresteanu and Molinari (2008), Rosen (2008), Andrews and Soares (2010), Bugni (2010), Canay (2010), Romano and Shaikh (2010), Kline (2011), or Kline and Tamer (2015), among others, would be necessary.

It follows that any point identification result can *at most* be expected to achieve point identification up to observational equivalence of strategic decision making types. But, by definition, point identification up to observational equivalence is enough to answer any interesting question about behavior, precisely because point identification up to observational equivalence exhausts the relevant information needed to understand the behavior generated from the strategic decision making types.

Definition 1 (Observational equivalence of strategic decision making types)**.** $\Theta_1 = (\Lambda_1, \Delta_1, \mathrm{P}_1)$ and $\Theta_2 = (\Lambda_2, \Delta_2, \mathrm{P}_2)$ are observationally equivalent if:

(1.1) $\Lambda_1 = \Lambda_2$

(1.2) $\Delta_1 1[\sum_{k \in \mathcal{M}} \Lambda_1(k) > 0] = \Delta_2 1[\sum_{k \in \mathcal{M}} \Lambda_2(k) > 0]$

(1.3) $\mathrm{P}_1 1[\Delta_1 > 0] 1[\sum_{k \in \mathcal{M}} \Lambda_1(k) > 0] = \mathrm{P}_2 1[\Delta_2 > 0] 1[\sum_{k \in \mathcal{M}} \Lambda_2(k) > 0]$

Therefore, two strategic decision making types are observationally equivalent if: they use the solution concepts with the same probability (i.e., condition 1.1), make computational mistakes with the same probability provided that the types actually use solution concepts subject to computational mistakes (i.e., condition 1.2), and make computational mistakes with the same magnitude provided that the types actually use solution concepts subject to computational mistakes and make computational mistakes with positive probability (i.e., condition 1.3). It is not possible to require that "observationally equivalent" types have the same probability of making computational mistakes if those types never use solution concepts subject to computational mistakes, because in that case the probability of making a computational mistake has no observable implications *in any game.*[15] Similarly, it is not possible to require that "observationally equivalent" types have the same magnitude of computational mistakes if those types never use solution concepts subject to computational mistakes, or never make computational mistakes, because in that case the magnitude of computational mistakes has no observable implications *in any game.*

Then, the following is the definition of point identification.

Definition 2 (Point identification of model parameters)**.** The model parameters are point identified if: for any specifications $\{\Theta_{0r}, \pi_0(r)\}_{r=1}^{\tilde{R}_0}$ and $\{\Theta_{1r}, \pi_1(r)\}_{r=1}^{\tilde{R}_1}$ of the model parameters that satisfy the assumptions and also are such that

(1) $\{\Theta_{0r}, \pi_0(r)\}_{r=1}^{\tilde{R}_0}$ and $\{\Theta_{1r}, \pi_1(r)\}_{r=1}^{\tilde{R}_1}$ both generate the observable data

[15] In other words, if "observationally equivalent" types were required to have the same probability of making computational mistakes even if the types never use solution concepts subject to computational mistakes, then two strategic decision making types that generate the same behavior (i.e., two types that use the solution concepts with the same probabiltiies, never use solution concepts subject to computational mistakes, and have different probabilities of making a computational mistake) would be defined as *not* "observationally equivalent."

(2) $\pi_0(\cdot) > 0$ and $\pi_1(\cdot) > 0$

(3) Θ_{0r} and $\Theta_{0r'}$ are not observationally equivalent for all $r \neq r'$, and Θ_{1r} and $\Theta_{1r'}$ are not observationally equivalent for all $r \neq r'$.

then $\tilde{R}_0 = \tilde{R} = \tilde{R}_1$ and there is a permutation ϕ of $\{1, 2, \ldots, \tilde{R}\}$ such that for each $r = 1, 2, \ldots, \tilde{R}$ it holds that $\pi_0(r) = \pi_1(\phi(r))$ and Θ_{0r} is observationally equivalent to $\Theta_{1\phi(r)}$.

This is the standard definition of point identification, adjusted for two issues. First, point identification can only be up to "observationally equivalent" strategic decision making types, as discussed above. (Note that this only concerns parameters relating to computational mistakes, which are assumed known by the econometrician when the model is specified to have no computational mistakes, and otherwise might be viewed as "nuisance parameters.") And second, point identification can only be up to permutations of the labeling of the strategic decision making types, because the labeling has no observable implications.[16]

The condition that $\pi(\cdot) > 0$ is required because it is not possible to point identify the strategic decision making types that are "used" with zero probability. (Types that are "used" by zero percent of the population have no observable implications.) So, in a specification that has $\tilde{R}$ strategic decision making types, it is assumed that indeed all $\tilde{R}$ types are used with positive probability. This can be taken as the definition of a specification "using" $\tilde{R}$ strategic decision making types, ruling out "using" a type with zero probability. Moreover, the condition that the strategic decision making types in a specification are not observationally equivalent is required because it is always possible to "split" a strategic decision making type into two identical copies of that type, and generate the same observable data, as long as the sum of the probabilities of the use of those two types equals the probability of the use of the original type. By requiring that the types are not observationally equivalent, this uninteresting source of non-identification is ruled out.

3.2. **Counterexample to point identification.** It is possible to give a counterexample to point identification in the absence of the sufficient conditions established in this paper. This counterexample illustrates the difficulty in distinguishing between across-individual heterogeneity and within-individual heterogeneity.

The counterexample involves two specifications of the parameters. In the first specification, $R = 1$, and $(\lambda_1(NE), \lambda_1(1_{anch})) = (\frac{1}{2}, \frac{1}{2})$, and $\delta_1 = 0$. In the second specification, $R = 2$, with $\pi_r = \frac{1}{2}$ and $(\lambda_r(NE), \lambda_r(1_{anch})) = (1[r = 1], 1[r = 2])$, and $\delta_r = 0$, for $1 \leq r \leq 2$.

[16] It is not possible to identify which strategic decision making type is "truly" type r since being type r rather than type r' has no observable implications. This arises in any model with "types," and has no substantive consequence.

There are a total of three types across these two specifications, and no pair of types are observationally equivalent according to definition 1.

In the first specification, all subjects use the same strategic decision making rule, and that rule uses the Nash equilibrium and one step of anchored strategic reasoning with equal probability. In the second specification, there are two equally probable strategic decision making rules, and each rule uses just one of the solution concepts.

These two specifications generate the same distribution of the data in any one game: the distribution that is an equally weighted mixture of point masses at the actions associated with Nash equilibrium, and one step of anchored strategic reasoning. Consequently, these two specifications cannot be distinguished on the basis of observing subjects play just one game, and therefore the parameters of the model are not point identified if the econometrician observes subjects play just one game. Note that in particular this shows that within-individual heterogeneity can be obscured by across-individual heterogeneity, in the data from just one game: fundamentally, this is because within-individual heterogeneity is a property of individuals, and therefore individuals must be observed to play multiple games in order to identify within-individual heterogeneity. This counterexample is unrelated to the additional complications introduced by computational mistakes, or unanchored strategic reasoning, which are discussed in section 3.3.

Similar counterexamples can be shown in the context of data on more than one game, but less than the number of games established as sufficient for point identification. These counterexamples become quite notationally cumbersome, when the number of games is large but not large enough for point identification, but it is possible to provide another relatively simple counterexample when there are two games. Consider the parameterized specification that $R = 2$, with parameters π_r and $(\lambda_r(NE), \lambda_r(1_{anch})) = (\lambda_r, 1 - \lambda_r)$ (by some abuse of notation), and $\delta_r = 0$, for $1 \leq r \leq 2$. Note that $\pi_2 = 1 - \pi_1$. The free parameters are π_1, λ_1, and λ_2. The data when $G = 2$ can be summarized by the following four observed probabilities concerning the distribution of individuals' behavior across $G = 2$ games:

(1) probability that a subject uses Nash in both games: $P(NE, NE) = \lambda_1^2\pi_1 + \lambda_2^2(1-\pi_1)$
(2) probability that a subject uses Nash and then 1 step of anchored strategic reasoning: $P(NE, 1_{anch}) = \lambda_1(1 - \lambda_1)\pi_1 + \lambda_2(1 - \lambda_2)(1 - \pi_1)$
(3) equally, due to the assumption that behavior is independent across games, so the order of games doesn't matter, the probability that a subject uses 1 step of anchored strategic reasoning and then Nash: $P(1_{anch}, NE) = \lambda_1(1-\lambda_1)\pi_1 + \lambda_2(1-\lambda_2)(1-\pi_1)$
(4) probability that a subject uses 1 step of anchored strategic reasoning in both games: $P(1_{anch}, 1_{anch}) = (1 - \lambda_1)^2\pi_1 + (1 - \lambda_2)^2(1 - \pi_1)$

Consequently, if there are two distinct specifications of π_1, λ_1, and λ_2 that give rise to the same numerical values for these four probabilities (i.e., $P(NE, NE), P(NE, 1_{anch}), P(1_{anch}, NE), P(1_{anch}, 1_{anch})$), then the model is not point identified. It is a fairly straightforward computational exercise to establish. For just one example, the specification ($\pi_1 = 0.16$, $\lambda_1 = 0.65$, and $\lambda_2 = 0.4$) generates the same values for these four probabilities as does ($\pi_1 = 0.3$, $\lambda_1 = 0.3$, and $\lambda_2 = 0.5$).

3.3. **Further threats to point identification.** Section 3.2 is an example of one issue that threatens point identification, but many other issues also threaten point identification.

First, even if a subject does not use an action associated with a particular solution concept, it may still be that that subject used that solution concept, because of computational mistakes. So, it is not enough to check whether a subject uses the associated action in order to check whether that subject used that solution concept.

Second, when multiple solution concepts predict the same action in a given game, then based on observing a subject take that action it is impossible to uniquely determine the solution concept. In particular, any action that is predicted by s' steps of unanchored strategic reasoning is also predicted by s steps of unanchored strategic reasoning for $0 \leq s \leq s'$, as discussed in section 2.5.

Third, the distribution of observed actions is not necessarily identical across games. For example, in one game, it might be that a particular range of actions is consistent with both zero and one steps of unanchored strategic reasoning, but in another game, that "same" range of actions is consistent with only zero steps of unanchored strategic reasoning. Consequently, the probability of observing actions in that range would be different across the two games, even holding fixed the probabilities that subjects use the various solution concepts. This implies that observed actions across games are not necessarily identically distributed, despite the fact that the use of solution concepts is identically distributed across games per $\lambda_i(\cdot)$.

4. Sufficient conditions for point identification of all model parameters

This section provides the main sufficient conditions for point identification of all unknown model parameters, in the sense of definition 2. Because the main sufficient conditions for point identification concern the properties of the games that subjects are observed to play, the identification result can be interpreted as a result on *experimental design*: an econometrician with the goal of identifying the solution concepts should run an experiment that has subjects play games that satisfy the conditions of the identification result. Mechanically, estimation is straightforward, under the sufficient conditions for point identification, and proceeds by maximizing the likelihood derived in appendix A.1.

Obviously, the sufficient conditions for point identification must be at least as strong as any necessary condition for point identification. And, it is a necessary condition for point identification that each pair of solution concepts in the model makes distinct predictions about the action in the games that subjects are observed to play. Otherwise, if two solution concepts in the model make the same predictions about the action, in all of the games that subjects are observed to play, then obviously those two solution concepts are observationally equivalent relative to the observed games. Also, per the counterexample in section 3.2, a necessary condition for point identification is that each subject is observed to play multiple games. The sufficient conditions for point identification are, therefore, necessarily related to these two necessary conditions.

In particular, in order to distinguish between the use of different numbers of steps of unanchored strategic reasoning, it is necessary that the different numbers of steps of unanchored strategic reasoning make distinct predictions about the action in the games that subjects are observed to play. However, section 2.5.2 discussed the fact that, in every game, some actions are consistent with multiple different numbers of steps of unanchored strategic reasoning. Nevertheless, it is possible to distinguish between the use of different numbers of steps of unanchored strategic reasoning, because some actions are *inconsistent* with certain numbers of steps of unanchored strategic reasoning.

So, define the set $U_{jg}(s, \epsilon)$ to be a (possibly empty) set of actions for agent j in game g that: are consistent with s steps of unanchored strategic reasoning, are not consistent with $s' \in \mathcal{U}$ with $s' > s$ steps of unanchored strategic reasoning, and collectively will be taken with zero probability by subjects that use any solution concept $k \in \mathcal{M}$ and possibly make a computational mistake of magnitude at most ϵ. The set $U_{jg}(s, \epsilon)$ can be written as:

$$U^1_{jg}(s,\epsilon) = \Sigma^s_{jg} \cap \bigcap_{k \in \mathcal{M}} [c_{jg}(k) - \epsilon(\alpha_{Ug}(j) - \alpha_{Lg}(j)), c_{jg}(k) + \epsilon(\alpha_{Ug}(j) - \alpha_{Lg}(j))]^C \cap \bigcap_{s' > s, s' \in \mathcal{U}} (\Sigma^{s'}_{jg})^C$$

$$U^0_{jg}(s) = \Sigma^s_{jg} \cap \bigcap_{k \in \mathcal{M}} \{c_{jg}(k)\}^C \cap \bigcap_{s' > s, s' \in \mathcal{U}} (\Sigma^s_{jg})^C$$

$$U_{jg}(s,\epsilon) = \begin{cases} U^1_{jg}(s,\epsilon) & \text{if } \Sigma^s_{jg} \text{ is not a finite set} \\ U^0_{jg}(s) & \text{if } \Sigma^s_{jg} \text{ is a finite set} \end{cases}$$

Let $R_{jg}(s, s', \epsilon)$ be the probability of $U_{jg}(s, \epsilon)$ under the uniform distribution on $\Sigma^{s'}_{jg}$.[17] By construction, $R_{jg}(s, s', \epsilon) = 0$ if $s' > s$ and $s' \in \mathcal{U}$. Let $U_{jg}(s) = U_{jg}(s, \overline{\rho})$, where $\overline{\rho}$ comes from assumption 2.2. Also, let $\Omega_{jg} = \alpha_{Ug}(j) - \alpha_{Lg}(j)$.

[17] For example, if $s' \leq s$ and $\Sigma^{s'}_{jg} = [c_{Ljg}(s'), c_{Ujg}(s')]$ is a non-degenerate interval, then $R_{jg}(s, s', \epsilon)$ is the ratio of the Lebesgue measure of $U_{jg}(s, \epsilon)$ to $c_{Ujg}(s') - c_{Ljg}(s')$.

The addition of assumption 4.1 is sufficient for point identification. A stylized depiction of the assumption is provided in figure 1, showing the arrangement of various quantities in the action space, in the case that $\Sigma^s_{1g} = [c_{L1g}(s), c_{U1g}(s)]$. Recall from section 2.4 that without loss of generality the subjects in the dataset are always agent 1 in the games. Assumption 4.1 is discussed in more detail in the context of the empirical application in section 6.

Assumption 4.1 (Conditions on the games)**.** *The dataset includes at least $2R-1$ games, such that each game g of those $2R-1$ games satisfies all of the following conditions:*

(4.1.1) $\Omega_{1g} > 0$

(4.1.2) For each $k \in \mathcal{M}$ and $k' \in \mathcal{M}$ such that $k \neq k'$, $|c_{1g}(k) - c_{1g}(k')| > 2\overline{\rho}\Omega_{1g}$

(4.1.3) For each $k \in \mathcal{M}$ and $s \in \mathcal{U}$ such that Σ^s_{1g} is a finite set, $c_{1g}(k) \notin \Sigma^s_{1g}$

(4.1.4) For each $k \in \mathcal{M}$, $\overline{\rho}\Omega_{1g} < \max\{\alpha_{Ug}(1) - c_{1g}(k), c_{1g}(k) - \alpha_{Lg}(1)\}$

(4.1.5) For each $s \in \mathcal{U}$, $R_{1g}(s, s, \overline{\rho}) > 0$

Condition 4.1.1 requires that the game has non-degenerate action space. If the game had a degenerate action space, then all solution concepts would make the same prediction, and therefore would be observationally equivalent.

Condition 4.1.2 requires that the game be such that the actions predicted by solution concepts subject to computational mistakes are far enough apart from each other (relative to the largest possible computational mistakes), so that a subject that uses solution concept $k \in \mathcal{M}$ will take a different action than a subject that uses solution concept $k' \in \mathcal{M}$ for $k' \neq k$, even if the subjects make computational mistakes. Note that if the econometrician specifies the model to have no computational mistakes (i.e., $\overline{\rho} = 0$), this requires simply that $c_{1g}(k) \neq c_{1g}(k')$. Despite this condition, note that it is not necessarily possible to determine the intended solution concept of a subject even if a subject is observed to take an action "close" to an action predicted by a particular solution concept $k^* \in \mathcal{M}$, because it is still possible that the subject used some number of steps of unanchored strategic reasoning that resulted in taking an action "close" to the action predicted by solution concept k^*. Moreover, it is not possible to determine the probability that a subject intends to use a solution concept $k^* \in \mathcal{M}$ by checking how often the subject takes the action exactly predicted by solution concept k^*, because with unknown probability the subject will make a computational mistake. In figure 1, this condition is reflected by the fact that $[c_{1g}(NE) - \overline{\rho}\Omega_{1g}, c_{1g}(NE) + \overline{\rho}\Omega_{1g}]$ is disjoint from $[c_{1g}(1_{anch}) - \overline{\rho}\Omega_{1g}, c_{1g}(1_{anch}) + \overline{\rho}\Omega_{1g}]$.

Condition 4.1.3 requires that the game be such that, if it happens that s steps of unanchored strategic reasoning predicts a finite set of actions, then the actions predicted by solution concepts subject to computational mistakes are not equal to one of the finitely many

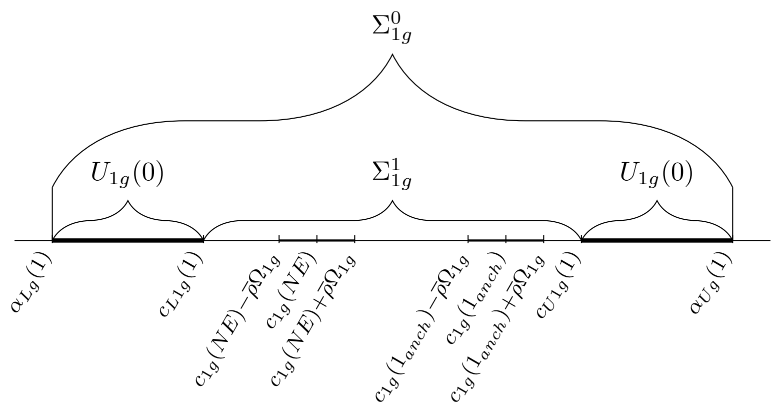

FIGURE 1. Stylized graphical depiction of assumption 4.1: This figure complements the discussion of assumption 4.1, showing a stylized depiction of the arrangement of various quantities in the action space. In this depiction, subjects might use 0 or 1 steps of unanchored strategic reasoning, or 1 step of anchored strategic reasoning, or Nash equilibrium. (Recall 0 steps of anchored strategic reasoning is the same as 0 steps of unanchored strategic reasoning.)

actions predicted by s steps of unanchored strategic reasoning. This helps identification, in particular it helps distinguish between anchored and unanchored strategic reasoning, because it implies that the actions predicted by solution concepts subject to computational mistakes will not arise with positive probability due to the use of unanchored strategic reasoning. In figure 1, this condition is not relevant as it is assumed that Σ^s_{1g} is a non-degenerate interval.

Condition 4.1.4 requires that the game be such that the actions predicted by solution concepts subject to computational mistakes are sufficiently far from at least one of the boundaries of the action space so that there will be some actions between the largest (or, respectively, smallest) action that arises due to computational mistakes and the upper bound (or, respectively, lower bound) of the action space. Otherwise, it would not be possible to determine the "true" magnitude of computational mistakes. It allows that the action predicted by a solution concept subject to computational mistakes equals one of the boundaries of the action space. In figure 1, this condition is reflected by the fact that $[c_{1g}(NE)-\overline{\rho}\Omega_{1g}, c_{1g}(NE)+\overline{\rho}\Omega_{1g}]$ and $[c_{1g}(1_{anch}) - \overline{\rho}\Omega_{1g}, c_{1g}(1_{anch}) + \overline{\rho}\Omega_{1g}]$ are strictly contained in the action space.

Condition 4.1.5 requires that the game be such that for each number of steps of unanchored strategic reasoning $s \in \mathcal{U}$ that there is a set of actions that can only arise from s or fewer steps of unanchored strategic reasoning. This helps to identify the probability of using $s+1$ steps of unanchored strategic reasoning, by the difference between the probabilities of using s or fewer and using $s+1$ or fewer steps of unanchored strategic reasoning. In figure 1, this condition is reflected by $U_{1g}(0)$, which can arise from the use of 0 but not 1 step of unanchored strategic reasoning (and also not the use of other solution concepts).

Assumption 4.1 requires that the econometrician observes each of the subjects play at least $2R-1$ games satisfying these conditions. This is necessary to avoid the threat to point identification that was described in section 3.2.

The econometrician must also observe subjects play at least one game that satisfies some of the above conditions, and a condition described in the following assumption.

Assumption 4.2 (Conditions on at least one game)**.** *The econometrician observes in the dataset at least one game g satisfying conditions 4.1.1, 4.1.2, and 4.1.4 in assumption 4.1, and the condition that:*

(4.2.1) For each $k \in \mathcal{M}$ and $s \in \mathcal{U} \cup \{0_{unanch}\}$, one of the following holds:

- *(a) $[c_{1g}(k) - \overline{\rho}\Omega_{1g}, c_{1g}(k) + \overline{\rho}\Omega_{1g}]$ is a subset of Σ^s_{1g}*
- *(b) $[c_{1g}(k) - \overline{\rho}\Omega_{1g}, c_{1g}(k) + \overline{\rho}\Omega_{1g}]$ is disjoint from Σ^s_{1g}*
- *(c) $[c_{1g}(k), c_{1g}(k) + \overline{\rho}\Omega_{1g}]$ is a subset of Σ^s_{1g} and $c_{1g}(k) = \alpha_{Lg}(1)$*
- *(d) $[c_{1g}(k) - \overline{\rho}\Omega_{1g}, c_{1g}(k)]$ is a subset of Σ^s_{1g} and $c_{1g}(k) = \alpha_{Ug}(1)$*

Condition 4.2.1 requires that the range of possible "computational mistakes" from any solution concept $k \in \mathcal{M}$ cannot overlap the boundary of the range of predictions from any number of steps of unanchored strategic reasoning. This assumption is used to identify the magnitude of computational mistakes, by inspecting whether actions slightly closer to the actions predicted by solution concepts subject to computational mistakes are more likely than those slightly further. This assumption guarantees that over the relevant range of possible computational mistakes, the use of unanchored strategic reasoning cannot either "mimic" or alternatively "mask" computational mistakes. Parts 4.2.1a and 4.2.1b of assumption 4.2 can be viewed, roughly, as meaning that assumption 4.2 is satisfied whenever the actions associated with the strategies in $\mathcal{M}$ are suitably distinct from the boundaries of the sets of actions associated with unanchored strategic reasoning. Parts 4.2.1c and 4.2.1d of assumption 4.2 allows that an action associated with a strategy in $\mathcal{M}$ is on the boundary of the action space. Recall from above that $\overline{\rho} = 0$ whenever computational mistakes are ruled out: in that case, note that logically either 4.2.1a or 4.2.1b must be true, since the singleton $c_{1g}(k)$ must either be a subset or disjoint from any given set. In figure 1, this is reflected by the fact that $c_{1g}(NE)$ and $c_{1g}(1_{anch})$ are distinct from the boundaries of the sets of actions associated with unanchored strategic reasoning, hence $[c_{1g}(NE) - \overline{\rho}\Omega_{1g}, c_{1g}(NE) + \overline{\rho}\Omega_{1g}]$ and $[c_{1g}(1_{anch}) - \overline{\rho}\Omega_{1g}, c_{1g}(1_{anch}) + \overline{\rho}\Omega_{1g}]$ are contained in both Σ^0_{1g} and Σ^1_{1g}. As with the other assumptions, this assumption is further discussed in the context of the empirical application in section 6.3.1.

The following theorem establishes that the model is point identified under the above assumptions. The lengthy proof of this theorem is collected in appendix A.2. A stylized sketch of the proof is provided in section 4.1.

Theorem 4.1. *Under assumptions 2.1, 2.2, 4.1, and 4.2, the parameters of the model are point identified in the sense of definition 2.*

This theorem does not imply that only the games that satisfy the conditions in assumptions 4.1 or 4.2 are informative about model parameters, or that only such games should be used in estimation. All games should be used in estimation for the purposes of maximizing the efficiency of the estimator relative to the available data.

Remark 4.1 (The role of distributional assumptions)**.** In section 2.5.2, it was assumed that individuals that use s steps of unanchored strategic reasoning choose an action uniformly at random from the set of actions consistent with s steps of unanchored strategic reasoning. This assumption was motivated on theoretical grounds (e.g., appealing to the principle of indifference), and is discussed further in the context of the empirical application, where it is shown to be reasonable based on inspecting the overall empirical distributions of actions in each game, and finding that they exhibit features compatible with this distributional assumption (see in particular sections 6.2).

Moreover, in addition to being theoretically and empirically justified, it is fundamental for point identification of the model for the econometrician to maintain these sorts of distributional assumptions, though potentially the uniform distribution could be replaced by some other distribution that is *known by the econometrician*. Essentially, this requirement to maintain a distributional assumption is an implication of the more general fact that arbitrary mixtures of densities are not point identified. For example, suppose that there is only one strategic decision making type (i.e., $R = 1$), and suppose that the econometrician is willing to assume that strategic decision making type uses either 0 or 1 steps of unanchored strategic reasoning (i.e., $\mathcal{U} = \{0_{unanch}, 1_{unanch}\}$ and $\mathcal{A} = \mathcal{M} = \emptyset$). Then a specification of the model would entail, for that one type: the specification of $\Lambda_1(0_{unanch})$ and $\Lambda_1(1_{unanch})$, and also, for each game, a distribution H_{g0} that is supported on the actions associated with 0 steps of unanchored strategic reasoning in game g, and a distribution H_{1g} that is supported on the actions associated with 1 step of unanchored strategic reasoning in game g. With the uniform distributional assumption, H_{0g} and H_{1g} are assumed to be uniform distributions on Σ^0_{1g} and Σ^1_{1g} respectively, in all games g. But, without a distributional assumption, H_{0g} and H_{1g} could be any distribution with the appropriate support. The specification $(\Lambda_1(0_{unach}), \Lambda_1(1_{unanch}), H_{0g}, H_{1g})$ implies the observed distribution of actions in game g that is given by the mixture $\Lambda_1(0_{unanch})H_{0g}+\Lambda_1(1_{unanch})H_{1g}$. Now, consider the strategic decision making type that uses 0 steps of unanchored strategic reasoning with probability 1, and uses the $\Lambda_1(0_{unanch})H_0 + \Lambda_1(1_{unanch})H_1$ distribution on Σ^0_{1g}. By construction, that results in the same observed distribution of actions. Intuitively, this can happen if individuals that use 0 steps of unanchored strategic reasoning are "biased" toward using the actions that are also consistent with using 1 step of unanchored strategic reasoning, but in that case, evidently the individual is using a refinement of 0 steps of unanchored strategic reasoning leading to that "bias" toward actually using 1 step of unanchored strategic reasoning. Consequently, these two specifications of the model are observationally equivalent, establishing the sense in which the distributional assumptions are important for point identification.

4.1. **Sketch of proof.** The proof is lengthy and technical, but it is possible to provide a sketch. The discussion of assumptions 4.1 and 4.2 already describes the sources of identification, and this sketch describes how that is formalized in the proof. This sketch states without justification main claims that are non-trivial to prove, and proving those claims comprises a significant fraction of the proof.

It can be shown that a vector of probabilities of events related to the observed actions (e.g., "the probability of an observed action within a certain range") in game g due to a subject that uses strategic decision making rule θ, $P_{g,\theta}$, can be written as a matrix Q_g (that depends on the structure of game g) times a vector that is a known function $\eta^*(\cdot)$ (defined in the appendix) of strategic decision making rule θ. So, $P_{g,\theta} = Q_g\eta^*(\theta)$. $P_{g,\theta}$ is not observable, since the population uses more than one strategic decision making rule. Critically, Q_g is non-singular under the identification assumptions, although that is not obvious and requires a lengthy proof. That implies that if it *were* possible to observe $P_{g,\theta}$, then it would be possible to recover $\eta^*(\theta)$. Let $\mathcal{G}$ be a subset of games of $\{1, 2, \ldots, G\}$. Let $\mathcal{G}(p)$ be the p-th smallest element of $\mathcal{G}$, and let $\mathcal{G}_p = \{\mathcal{G}(1), \ldots, \mathcal{G}(p)\}$.

Then, by the algebra of the Kronecker product, the joint distribution of those events across games in the first p games out of $\mathcal{G}$ is $P_{\mathcal{G},\theta,p} \equiv \bigotimes_{g\in\mathcal{G}_p} P_{g,\theta} = \bigotimes_{g\in\mathcal{G}_p}(Q_g\eta^*(\theta)) = (\bigotimes_{g\in\mathcal{G}_p} Q_g)(\bigotimes^p \eta^*(\theta)) = Q^{(p)}_{\mathcal{G}}\eta^*(\theta)^{(p)}$. Again by the algebra of the Kronecker product, $Q^{(p)}_{\mathcal{G}} \equiv \bigotimes_{g\in\mathcal{G}_p} Q_g$ is non-singular since each Q_g is non-singular. Let $P_{\mathcal{G},\theta} = (1, P_{\mathcal{G},\theta,1}, \ldots, P_{\mathcal{G},\theta,|\mathcal{G}|})$. Let $\eta^*(\theta)^{(0)} = 1$ and $\eta^*(\theta)^{(p)} = \eta^*(\theta) \otimes \cdots \otimes \eta^*(\theta)$ be the p-times Kronecker product. Let $\overline{\eta}^*(\theta) = (1, \eta^*(\theta)^{(1)}, \ldots, \eta^*(\theta)^{(|\mathcal{G}|)})$. Let $Q_{\mathcal{G}}$ be the block diagonal matrix with blocks along the diagonal equal to $Q_{\mathcal{G}}^{(0)}, \ldots, Q_{\mathcal{G}}^{(|\mathcal{G}|)}$, which is non-singular as each term is non-singular. And, $P_{\mathcal{G},\theta} = Q_{\mathcal{G}} \overline{\eta}^*(\theta)$.

Suppose that the true parameters of the data generating process are rules $\Theta_{0,1}, \ldots, \Theta_{0,R}$, that are used by $\pi_0(1), \ldots, \pi_0(R)$ percent of the population. Let Υ_0^* be a matrix that stacks $(\overline{\eta}^*(\Theta_{0,r}))$ for $r = 1, 2, \ldots, R$ as its columns. So, then, the observable joint distribution of those events across games is $P_{\mathcal{G}} = Q_{\mathcal{G}} \Upsilon_0^* \pi_0$. Suppose that another specification of the parameters with rules $\Theta_{1,1}, \ldots, \Theta_{1,R}$, that are used by $\pi_1(1), \ldots, \pi_1(R)$ percent of the population is observationally equivalent, so that there is an Υ_1^* derived from those parameters so that $P_{\mathcal{G}} = Q_{\mathcal{G}} \Upsilon_1^* \pi_1$. Then, it would hold that $0 = Q_{\mathcal{G}} \overline{\Upsilon}^* \overline{\pi}$, where $\overline{\Upsilon}^*$ collect the unique columns of Υ_0^* and Υ_1^*. Correspondingly, $\overline{\pi}$ collects the difference between π_0 and π_1. The value of π_0 (or π_1) for a strategic decision making rule that does not appear in specification 0 (or 1) is by convention zero, reflecting the fact that that strategic decision making rule is used by zero percent of the population under specification 0 (or 1).

Therefore, $\overline{\pi}$ is in the null space of $Q_{\mathcal{G}} \overline{\Upsilon}^*$. It can be shown (as a non-trivial claim under the conditions of the identification results) that there is a non-singular matrix T such that $0 = Q_{\mathcal{G}} T^{-1} T \overline{\Upsilon}^*$ where $T \overline{\Upsilon}^*$ has full column rank. This step critically uses the fact that the econometrician observes at least $2R - 1$ games satisfying the conditions of assumption 4.1. Since $Q_{\mathcal{G}}$ has full column rank and therefore $Q_{\mathcal{G}} T^{-1}$ has full column rank, it follows that $Q_{\mathcal{G}} \overline{\Upsilon}^*$ has full column rank, so it must be that $\overline{\pi} = 0$, so that the columns of Υ_0^* and Υ_1^* are the same (up to permutations of the order of the columns). That implies that (up to permutations of the labels), that $\eta^*(\cdot)$ applied to the strategic decision making rules in specification 0 is the same as $\eta^*(\cdot)$ applied to the strategic decision making rules in specification 1. It can be shown that $\eta^*(\cdot)$ is "injective" (up to the issues relating to possible lack of observable implications of parameters relating to computational mistakes accounted for in definition 1). So, the parameters are point identified in the sense of definition 2.

5. Sufficient conditions for point identification except for the magnitude of computational mistakes

This section establishes sufficient conditions for point identification of all unknown parameters except for those related to the *magnitude* of computational mistakes, under weaker conditions than used by theorem 4.1. The result does still allow that individuals might make computational mistakes. This can be interpreted as a *partial identification* result, showing that some but not necessarily all of the parameters are point identified. Alternatively, this can be interpreted as a *point identification* result, showing that a model without computational mistakes (or even a model with computational mistakes with *known* magnitudes of computational mistakes) is point identified.

The identification result in this section uses a different definition of observational equivalence of strategic decision making types. Essentially, the alternative definition treats the magnitude of computational mistakes as irrelevant and is similar to definition 1, except the last condition involving P is dropped. There is a corresponding definition of point identification, ignoring the magnitude of computational mistakes.

Definition 3 (Observational equivalence of strategic decision making types, ignoring the magnitude of computational mistakes)**.** $\Theta_1 = (\Lambda_1, \Delta_1, \mathrm{P}_1)$ and $\Theta_2 = (\Lambda_2, \Delta_2, \mathrm{P}_2)$ are observationally equivalent ignoring the magnitude of computational mistakes if:

(3.1) $\Lambda_1 = \Lambda_2$

(3.2) $\Delta_1 1[\sum_{k\in\mathcal{M}} \Lambda_1(k) > 0] = \Delta_2 1[\sum_{k\in\mathcal{M}} \Lambda_2(k) > 0]$

Definition 4 (Point identification of model parameters, ignoring the magnitude of computational mistakes)**.** The model parameters are point identified ignoring the magnitude of computational mistakes if: for any specifications $\{\Theta_{0r}, \pi_0(r)\}_{r=1}^{\tilde{R}_0}$ and $\{\Theta_{1r}, \pi_1(r)\}_{r=1}^{\tilde{R}_1}$ of the model parameters that satisfy the assumptions and also are such that

1. $\{\Theta_{0r}, \pi_0(r)\}_{r=1}^{\tilde{R}_0}$ and $\{\Theta_{1r}, \pi_1(r)\}_{r=1}^{\tilde{R}_1}$ both generate the observable data
2. $\pi_0(\cdot) > 0$ and $\pi_1(\cdot) > 0$
3. Θ_{0r} and $\Theta_{0r'}$ are not observationally equivalent ignoring the magnitude of computational mistakes for all $r \neq r'$, and Θ_{1r} and $\Theta_{1r'}$ are not observationally equivalent ignoring the magnitude of computational mistakes for all $r \neq r'$.

then $\tilde{R}_0 = \tilde{R} = \tilde{R}_1$ and there is a permutation ϕ of $\{1, 2, \ldots, \tilde{R}\}$ such that for each $r = 1, 2, \ldots, \tilde{R}$ it holds that $\pi_0(r) = \pi_1(\phi(r))$ and Θ_{0r} is observationally equivalent ignoring the magnitude of computational mistakes to $\Theta_{1\phi(r)}$.

The main difference between the sufficient conditions of this section, and the sufficient conditions of section 4, is that assumption 4.1 is dropped in favor of the weaker assumption 5.1. Moreover, assumption 4.2 is dropped entirely.

Assumption 5.1 (Conditions on the games)**.** *The dataset includes at least* $2R - 1$ *games, such that each game* g *of those* $2R - 1$ *games satisfies all of the following three conditions:*

(5.1.1) $\Omega_{1g} > 0$

(5.1.2) For each $k \in \mathcal{M}$ *and* $k' \in \mathcal{M}$ *such that* $k \neq k'$, $c_{1g}(k) \neq c_{1g}(k')$

(5.1.3) For each $k \in \mathcal{M}$ *and* $s \in \mathcal{U}$ *such that* Σ_{1g}^s *is a finite set,* $c_{1g}(k) \notin \Sigma_{1g}^s$

The dataset includes at least $2R-1$ games, such that each game g of those $2R-1$ games satisfies the following condition:

(5.1.4) For each $s \in \mathcal{U}$, $R_{1g}(s, s, \overline{\rho}) > 0$

Assumption 4.1 requires that the same games satisfy all of the conditions stated in assumption 4.1, whereas assumption 5.1 allows that some games satisfy conditions 5.1.1, 5.1.2, and 5.1.3, and other games satisfy condition 5.1.4. However, it is allowed that the set of games satisfying conditions 5.1.1, 5.1.2, and 5.1.3 arbitrarily overlaps with the set of games satisfying condition 5.1.4.

The next assumption disallows certain "knife-edge" cases and requires additional notation. Use the notation that $\mathcal{M}(r)$ is the r-th smallest element of $\mathcal{M}$ (with Nash equilibrium the largest element by convention) and $\mathcal{U}(r)$ is the r-th smallest element of $\mathcal{U}$.

Assumption 5.2 (No knife-edge strategic decision making rules)**.** *There are $\tilde{R}$ strategic decision making rules used in the population, with $\pi(r) > 0$ for $r = 1, 2, \ldots, \tilde{R}$. For each $r' \neq r$, it holds that:*

(5.2.1) $((1-\Delta_r)\Lambda_r(\mathcal{M}(1)), \ldots, (1-\Delta_r)\Lambda_r(\mathcal{M}(|\mathcal{M}|))) \neq ((1-\Delta_{r'})\Lambda_{r'}(\mathcal{M}(1)), \ldots, (1-\Delta_{r'})\Lambda_{r'}(\mathcal{M}(|\mathcal{M}|)))$ *and* $(\Lambda_r(\mathcal{U}(1)), \ldots, \Lambda_r(\mathcal{U}(|\mathcal{U}|))) \neq (\Lambda_{r'}(\mathcal{U}(1)), \ldots, \Lambda_{r'}(\mathcal{U}(|\mathcal{U}|)))$

(5.2.2) $\pi(r) \neq \pi(r')$

Condition 5.2.1 rules out the knife-edge case that strategic decision making rules r and r' used in the population, despite being distinct, are such that $((1-\Delta_r)\Lambda_r(\mathcal{M}(1)), \ldots, (1-\Delta_r)\Lambda_r(\mathcal{M}(|\mathcal{M}|))) = ((1-\Delta_{r'})\Lambda_{r'}(\mathcal{M}(1)), \ldots, (1-\Delta_{r'})\Lambda_{r'}(\mathcal{M}(|\mathcal{M}|)))$ or $(\Lambda_r(\mathcal{U}(1)), \ldots, \Lambda_r(\mathcal{U}(|\mathcal{U}|))) = (\Lambda_{r'}(\mathcal{U}(1)), \ldots, \Lambda_{r'}(\mathcal{U}(|\mathcal{U}|)))$. Condition 5.2.2 rules out the knife-edge case that two strategic decision making rules are used with the same probability.

Theorem 5.1. *Under assumptions 2.1, 2.2, 5.1, and 5.2, the parameters of the model are point identified in the sense of definition 4.*

6. Empirical application

The application shows that the features of the model are empirically relevant, in the context of a well-known and representative experimental design, motivating the main contributions of the paper: proposing and understanding identification of the model. Specifically, the empirical application establishes evidence for within-individual heterogeneity, and also unanchored strategic reasoning.

6.1. **Data.** The data for the empirical application comes from the "two-player guessing game" experiment conducted in Costa-Gomes and Crawford (2006). The following briefly describes the data. The data concerns $N = 88$ subjects, each of whom play $G = 16$ games. An important feature of the experimental design is that the subjects face new opponents in each game and do not learn the actions of their opponents until after the conclusion of the experiment. This eliminates basically any role for "learning," or "specializing" their play against their "perception" of their current opponent. This is consistent with the broader view that non-equilibrium models are best studied in a setting without learning.[18] The empirical analysis of this data is quite different in Costa-Gomes and Crawford (2006), because of the difference in models. In Costa-Gomes and Crawford (2006), as representative of the literature, each subject is assumed to have no within-individual heterogeneity, and the model does not include "unanchored strategic reasoning," which means that the "main result" of estimating the model is essentially assigning each subject to its level out of the "level-k model."[19] The analysis in this current paper does not use the novel "information search data" that is also studied in Costa-Gomes and Crawford (2006), simply because the dataset without the "information search data" is more representative of the literature, since most studies do not (yet) use such data. Because of these fundamental differences, the analysis in this current paper is not in any sense an attempt to "replicate" the results of Costa-Gomes and Crawford (2006), though section 6.4 does show how the results are related. Rather, the analysis is intended to show the empirical relevance of the theoretical results of this current paper (proposing and point identifying the model), in the context of a well-known and representative experimental design.[20]

All of the games are "two-player guessing games," which are related to the beauty contests studied by Nagel (1995), Ho, Camerer, and Weigelt (1998), and Bosch-Domenech, Montalvo, Nagel, and Satorra (2002), among others. In a two-player guessing game, two agents simultaneously make a "guess." The utility function for an agent j in game g is a decreasing function of the difference between its own guess (a_j), and that agent's "target" (p_{jg}) times the guess of the other agent (a_{-j}). In game g, the action space for agent j is $[\alpha_{Lg}(j), \alpha_{Ug}(j)]$. The utility function for agent j in game g is:

[18] Another important feature of the experimental design is that the experiment involves only 8 different two-player games in the traditional sense of the definition of "game." However, each subject plays each game once in each of the player roles (i.e., row player and column player), so that each subject plays 16 times. Each such game × player role pair is denoted a separate "game." Essentially the same convention is maintained in Costa-Gomes and Crawford (2006).

[19] The model in Costa-Gomes and Crawford (2006) also allows Nash equilibrium, and certain "dominance" or "sophisticated" strategies (which are rare). Note that the "dominance" type is distinct from "unanchored strategic reasoning" despite the fact that "unanchored strategic reasoning" relates to iterated dominance. Specifically, all of the "dominance" or "sophisticated" types make a unique prediction, fundamentally unlike unanchored strategic reasoning. Costa-Gomes and Crawford (2006) also check for consistency with iterated deletion of dominated strategies, in the sense discussed in section 2.5.2.

[20] Using prior experimental data also avoids the time and financial cost of running an experiment that would, in any case, attempt to be representative of other experiments. So since the point is not to innovate the experimental design, it seems to make most sense to use prior experimental data.

	Game specification						Predictions of solution concepts			
	Agent 1		Agent 2		Targets		Anchored reasoning		Unanchored reasoning	
g	$\alpha_L(1)$	$\alpha_U(1)$	$\alpha_L(2)$	$\alpha_U(2)$	p_1	p_2	$c_1(1_{anch})$	$c_1(2_{anch})$	Δ_1^1	$c_1(NE)$
1	100	500	100	900	0.70	0.50	350	105	[100, 500]	100
2	100	900	100	500	0.50	0.70	150	175	[100, 250]	100
3	100	900	300	500	0.50	0.70	200	175	[150, 250]	150
4	300	500	100	900	0.70	0.50	350	300	[300, 500]	300
5	300	500	300	900	1.50	1.30	500	500	[450, 500]	500
6	300	900	300	500	1.30	1.50	520	650	[390, 650]	650
7	300	900	300	900	1.30	1.30	780	900	[390, 900]	900
8	300	900	300	900	1.30	1.30	780	900	[390, 900]	900
9	100	900	100	500	0.50	1.50	150	250	[100, 250]	100
10	100	500	100	900	1.50	0.50	500	225	[150, 500]	150
11	300	900	100	900	0.70	1.30	350	546	[300, 630]	300
12	100	900	300	900	1.30	0.70	780	455	[390, 900]	390
13	300	500	100	900	0.70	1.50	350	420	[300, 500]	500
14	100	900	300	500	1.50	0.70	600	525	[450, 750]	750
15	100	500	100	500	0.70	1.50	210	315	[100, 350]	350
16	100	500	100	500	1.50	0.70	450	315	[150, 500]	500

Some numbers are rounded to the nearest integer in this table, in order to avoid clutter. However, in the econometric analysis, the un-rounded numbers are used. These numerical values for these strategies are derived using the method described in the text.

TABLE 1. Experimental design

$$u_{jg}(a_1, a_2) = \max\{0, 200 - (a_j - p_{jg}a_{-j})\} + \max\left\{0, 100 - \frac{(a_j - p_{jg}a_{-j})}{10}\right\}.$$

For example, if an agent's target is $\frac{2}{3}$, then that agent's utility is maximized, holding fixed the other agent's guess, by guessing two-thirds of the other agent's guess. As displayed in table 1, the 16 games differ along two dimensions: the action spaces and the targets. The experimental design (and arrangement of the dataset) is such that when a subject is observed to play some game g, that subject is agent 1 in the game.

The strategies corresponding to the various solution concepts described in section 2 for agent 1 are in the last columns of the table.[21] In these games, the Nash equilibrium solution

[21]Strategies for agent 2 are not explicitly shown, but the experimental design described in footnote 18 implies that the strategies of agent 2 in even (odd) numbered games are the strategies of agent 1 in the previous (next) game in the table.

concept is indistinguishable from the rationalizability solution concept, since they imply the same guess (i.e., same pure strategy).

As detailed in Costa-Gomes and Crawford (2006), the derivation of the guesses predicted by anchored strategic reasoning (the "level-k model") in these games is straightforward. Similarly, the derivation of the ranges of guesses predicted by unanchored strategic reasoning is also straightforward. Let

$$\chi_{jg}(a) = \begin{cases} \alpha_{Lg}(j) \text{ if } a < \alpha_{Lg}(j) \\ a \text{ if } \alpha_{Lg}(j) \leq a \leq \alpha_{Ug}(j) \\ \alpha_{Ug}(j) \text{ if } a > \alpha_{Ug}(j) \end{cases}$$

The result is that $\Sigma_{jg}^{s} = [c_{Ljg}(s), c_{Ujg}(s)]$ is an interval. The biggest guess that agent j in game g that uses one step of unanchored strategic reasoning can make is $c_{Ujg}(1) = \chi_{jg}(p_{jg}\alpha_{Ug}(-j))$. That is because the biggest justifiable guess is the biggest possible guess of the opponent times the target. (And if that would be outside the action space, then the boundary of the action space is the biggest guess.) Similarly, the smallest guess that agent j in game g that uses one step of unanchored strategic reasoning can make is $c_{Ljg}(1) = \chi_{jg}(p_{jg}\alpha_{Lg}(-j))$. More generally, the biggest (respectively, smallest) guess that agent j in game g that uses s steps of unanchored strategic reasoning can make is $c_{Ujg}(s) = \chi_{jg}(p_{jg}c_{U,-j,g}(s-1))$ (respectively, $c_{Ljg}(s) = \chi_{jg}(p_{jg}c_{L,-j,g}(s-1))$).

6.2. **Non-parametric estimates.** It is useful to plot the empirical cumulative distribution functions of the observed actions in each of the games. Figure 2 shows this for game 1. The rest of the figures are displayed in appendix A.3, to save space.[22]

Along the bottom of the figure, along the horizontal axis, is displayed the action predicted by 1, 2, and 3 steps of anchored strategic reasoning, and the Nash equilibrium. Along the top of the figure is displayed via red endpoints the (interval of) actions predicted by 1, 2, and 3 steps of unanchored strategic reasoning. The actions predicted by 0 steps of unanchored strategic reasoning is necessarily the entire action space, so is not specifically noted.

Figure 2, and the other estimates in appendix A.3, shows clear evidence of mass points corresponding to a small number of actions, and otherwise a roughly continuous distribution of actions. In this game, it appears that there are mass points corresponding to using one and two steps of anchored strategic reasoning, and the Nash equilibrium, and otherwise a uniform distribution over the action space. The uniform distribution of actions is exactly consistent with the model, which supposes in section 2.5.2 the uniform distribution from

[22]See Appendix D of Costa-Gomes and Crawford (2006) for a different way of displaying the actions.

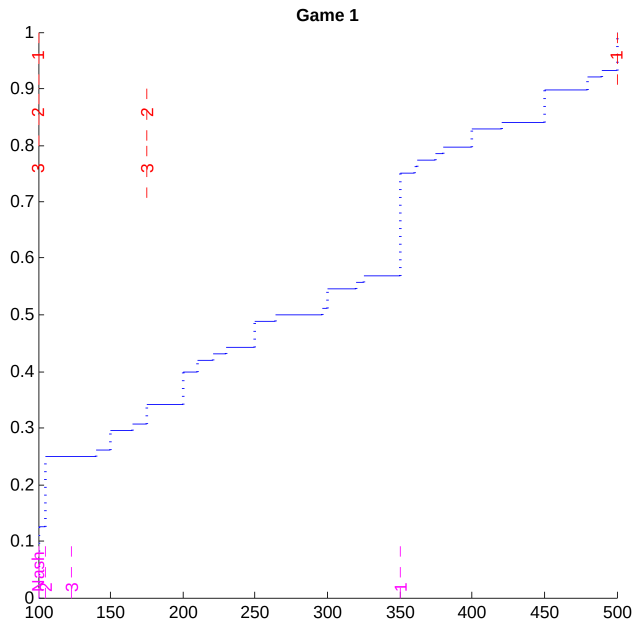

FIGURE 2. Distribution of actions of subjects, game 1

using unanchored strategic reasoning (thus, helping to motivate that part of the model).[23] In this game, zero and one steps of unanchored strategic reasoning make the same predictions about actions, but in other games displayed in appendix A.3 the predictions are different.

6.3. **Model specification and estimation results.** This subsection discusses the final details of model specification and the estimation results. Section 6.3.1 establishes that the sufficient conditions for identification hold in this application. Section 6.3.2 discusses estimation of R, based on model selection. Sections 6.3.3 and 6.4 discuss the estimation results.

The estimated model does not allow computational mistakes. As a robustness check, the estimation results that do allow computational mistakes are almost identical, as displayed in appendix A.4. It is not surprising that the results allowing computational mistakes are almost identical, based on the following argument involving the figures in section 6.2 and appendix A.3. Note that computational mistakes would imply a higher density of actions in the neighborhoods around the actions associated with the strategies subject to computational mistakes (i.e., the steps of anchored strategic reasoning or Nash equilibrium), compared to the density of actions slightly further away from the actions associated with those same strategies. In the figures, which display the empirical cumulative distribution functions, that would translate to a "greater slope" of the empirical cumulative distribution functions in those

[23]The defining characteristic of a uniformly distributed random variable is a cumulative distribution function with constant slope, which seems to essentially be the case here, after accounting for the mass points. That is, the displayed empirical cumulative distribution function is essentially that of a mixture of point masses and a uniform distribution over the action space. Uniform distributions over the actions consistent with various numbers of steps of unanchored strategic reasoning also appear in the other figures in appendix A.3, consistent with section 2.5.2.

same neighborhoods, compared to the slope just outside of those neighborhoods. However, there appears to be no such feature in the figures. Note that this argument is agnostic about the exact model of computational mistakes: although this paper has specified a particular model of computational mistakes, it seems that any reasonable model of computational mistakes would have similar implications for the density of actions. The actions that do not correspond to anchored strategic reasoning nor Nash equilibrium appear better explained by unanchored strategic reasoning, not computational mistakes, as the estimation formalizes.

6.3.1. *Model assumptions.* This section establishes that the sufficient conditions for identification are satisfied in this empirical application. The same basic approach would be taken in any empirical application.

First, it is necessary to specify the sets $\mathcal{A}$ and $\mathcal{U}$ from assumption 2.1. Overall, based on visually inspecting the figures from section 6.2 and appendix A.3, it appears that there is essentially no subject that uses three or more steps of anchored strategic reasoning, basically the standard finding in experimental game theory. Therefore, assumption 2.1 is maintained with $\mathcal{A} = \{1_{anch}, 2_{anch}\}$. Further, assumption 2.1 is maintained with $\mathcal{U} = \{0_{unanch}, 1_{unanch}\}$, largely because there are not enough games in this dataset such that the predictions of 1 and 2 steps of unanchored strategic reasoning differ sufficiently to guarantee point identification of the model with a larger set for $\mathcal{U}$, given the conditions in assumptions 4.1 or 5.1. (See below for further discussion of assumptions 4.1 or 5.1.)

Second, assumption 2.2 is simply the assumption that the model of computational mistakes is correct, and therefore is directly assumed by the econometrician. Specifically, the empirical exercise rules out computational mistakes. (Note that, particularly when verifying latter assumptions, because computational mistakes are ruled out, $\overline{\rho} = 0$.)

Third, verifying assumption 4.1 (or, by similar steps, assumption 5.1) requires inspecting table 1 and checking which games satisfy the conditions in assumption 4.1 (or, the weaker conditions in assumption 5.1):

(1) Condition 4.1.1: requires that the game has a non-degenerate action space. Obviously, all games in this dataset satisfy this.
(2) Condition 4.1.2: requires that the game is such that the actions associated with the strategies in $\mathcal{M}$ (in this application: 1 and 2 steps of anchored strategic reasoning, and Nash equilibrium) are all distinct. It is easy to directly verify by inspecting table 1 that games 1, 2, 3, 9, 10, 11, 12, 13, 14, 15, and 16 satisfy this condition. (More generally, the condition requires that if computational mistakes were to be allowed, then those actions would need to separated from each other by a sufficient magnitude.)

(3) Condition 4.1.3: requires that if a certain number of steps of unanchored strategic reasoning in $\mathcal{U}$ (in this application: 0 or 1 steps) predicts a finite set of actions, then those actions are distinct from the predictions of the steps of anchored strategic reasoning in $\mathcal{A}$ and Nash equilibrium. Since no game is such that 0 or 1 steps of unanchored strategic reasoning predicts a finite set of actions, this condition is satisfied in all games in the dataset.

(4) Condition 4.1.4: requires that the game be such that the actions associated with the strategies in $\mathcal{M}$ (in this application: 1 and 2 steps of anchored strategic reasoning, and Nash equilibrium), are not on *both* end points of the action space. Since the action spaces are all intervals, it is not possible for any given action to be on both end points, so all games in this dataset satisfy this. (More generally, the condition requires that if computational mistakes were to be allowed, then those actions would be required to be separated from *at least one of* the end points of the action space by a sufficient magnitude.)

(5) Condition 4.1.5: requires that the game be such that, for each $s \in \mathcal{U}$, there are actions used by s steps of unanchored strategic reasoning that are *not used* by s' steps of unanchored strategic reasoning (for each $s' \in \mathcal{U}$ with $s' > s$), *nor* used by the strategies in $\mathcal{M}$. In this application, that means there must be actions used by 0 step of unanchored strategic reasoning, but not used by 1 step of unanchored strategic reasoning, nor used by 1 or 2 steps of anchored strategic reasoning, nor used by Nash equilibrium. And also this means there must be actions used by 1 step of unanchored strategic reasoning but not used by 1 or 2 steps of anchored strategic reasoning, nor used by Nash equilibrium. It is easy to directly verify by inspecting table 1 that games 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, and 16 satisfy this condition. (More generally, the condition requires that if computational mistakes were to be allowed, it would be necessary that these actions are not just different from the actions used by the strategies in $\mathcal{M}$, but also separated from the actions used by the strategies in $\mathcal{M}$ by a sufficient magnitude).

Therefore, games 2, 3, 9, 10, 11, 12, 14, 15, and 16 satisfy all these conditions, a total of 9 games. And therefore assumption 4.1 is satisfied for any $R \leq 5$.

Finally, verifying assumption 4.2 requires establishing at least one game satisfies condition 4.2.1, among the games satisfying conditions 4.1.1, 4.1.2, and 4.1.4, or in other words in this application among games 1, 2, 3, 9, 10, 11, 12, 13, 14, 15, and 16. But recall from above that $\overline{\rho} = 0$ since computational mistakes are ruled out in the empirical application. In that case, note that logically either 4.2.1a or 4.2.1b must be true, since the singleton $c_{1g}(k)$ must either be a subset or disjoint from any given set. Therefore, assumption 4.2 is clearly satisfied, for all games satisfying conditions 4.1.1, 4.1.2, and 4.1.4.

Note that even if computational mistakes were to be allowed, this assumption can be easily verified as true for sufficiently small $\overline{\rho}$ (maximum magnitude of computational mistakes). For example, consider game $g = 2$. Verifying assumption 4.2 holds for game $g = 2$ and sufficiently small $\overline{\rho}$ requires simply verifying the following based on inspecting table 1:

(1) For $k = 1_{anch}$ and $s = 0_{unanch}$: notice that $c_{1g}(1_{anch}) = 150$ is in the interior of $\Sigma^0_{1g} = [100, 900]$ so clearly $[c_{1g}(1_{anch}) - \overline{\rho}\Omega_{1g}, c_{1g}(1_{anch}) + \overline{\rho}\Omega_{1g}]$ is a subset of Σ^0_{1g} for small enough $\overline{\rho}$.

(2) For $k = 1_{anch}$ and $s = 1_{unanch}$: notice that $c_{1g}(1_{anch}) = 150$ is in the interior of $\Sigma^1_{1g} = [100, 250]$ so clearly again $[c_{1g}(1_{anch}) - \overline{\rho}\Omega_{1g}, c_{1g}(1_{anch}) + \overline{\rho}\Omega_{1g}]$ is a subset of Σ^1_{1g} for small enough $\overline{\rho}$.

(3) For $k = 2_{anch}$ and $s = 0_{unanch}$: notice that $c_{1g}(2_{anch}) = 175$ is in the interior of $\Sigma^0_{1g} = [100, 900]$ so clearly $[c_{1g}(2_{anch}) - \overline{\rho}\Omega_{1g}, c_{1g}(2_{anch}) + \overline{\rho}\Omega_{1g}]$ is a subset of Σ^0_{1g} for small enough $\overline{\rho}$.

(4) For $k = 2_{anch}$ and $s = 1_{unanch}$: notice that $c_{1g}(2_{anch}) = 175$ is in the interior of $\Sigma^1_{1g} = [100, 250]$ so clearly again $[c_{1g}(2_{anch}) - \overline{\rho}\Omega_{1g}, c_{1g}(2_{anch}) + \overline{\rho}\Omega_{1g}]$ is a subset of Σ^1_{1g} for small enough $\overline{\rho}$.

(5) For $k = NE$ and $s = 0_{unanch}$: notice that $c_{1g}(NE) = 100 = \alpha_{Lg}(1)$ is on the lower bound of $\Sigma^0_{1g} = [100, 900]$ so clearly $[c_{1g}(NE), c_{1g}(NE) + \overline{\rho}\Omega_{1g}]$ is a subset of Σ^0_{1g} for small enough $\overline{\rho}$.

(6) For $k = NE$ and $s = 1_{unanch}$: notice that $c_{1g}(NE) = 100 = \alpha_{Lg}(1)$ is on the lower bound of $\Sigma^1_{1g} = [100, 250]$ so clearly again $[c_{1g}(NE), c_{1g}(NE) + \overline{\rho}\Omega_{1g}]$ is a subset of Σ^1_{1g} for small enough $\overline{\rho}$.

More generally, establishing assumptions 4.1 and 4.2 can be accomplished by a computerized algorithm that takes as inputs the information in table 1, and replicates the steps of verifying the assumptions just described.

Finally, note that verifying assumption 5.1 follows similar steps to verifying assumption 4.1, since the assumptions are similar. Assumption 5.2 is simply the assumption that rules out the described "knife-edge" situations, and is directly assumed by the econometrician.

6.3.2. *Model selection.* Economic theory does not predict the number of strategic decision making types (i.e., R). Therefore, R is part of the estimation problem. The selection of R is based on comparing the likelihood of the models with different R adjusted by a measure of model complexity, penalizing models that have more types (and therefore more parameters).

A generic information criterion is $-2\log L_R(\hat{\theta}_R)+h(R,N)$, where L_R is the likelihood function of the data for the model with R types, $\hat{\theta}_R$ is the estimate of the parameters of the model with R types, and h penalizes model complexity as a function of the number of types and sample size. Models with low values of the information criterion are preferred models.

R	Bayesian	Δ_{Bayesian}	Akaike	Δ_{Akaike}
1	12016.81	631.74	12007.38	662.72
2	11686.71	301.64	11666.72	322.06
3	11484.37	99.30	11455.44	110.78
4	11404.61	19.54	11368.71	24.06
5	11385.07	0.00	11344.66	0.00
6	11386.54	1.47	11344.69	0.04
7	11395.22	10.16	11355.90	11.24

TABLE 2. Model selection

There is not a uniquely "correct" information criterion, so this paper uses two specifications of h that are commonly used in the general statistical literature. Suppose that S is the total number of solution concepts potentially used by the subjects, per assumption 2.1. Then, there are $g_S(R) = R(S) - 1$ free parameters.[24]

The specification $h(R,N) = g_S(R)\log(N)$ results in the Bayesian information criterion (e.g., Schwarz (1978)). The specification $h(R,N) = 2(g_S(R)) + \frac{2g_S(R)(g_S(R)+1)}{N-g_S(R)-1}$ results in the corrected Akaike information criterion (e.g. Akaike (1974), Sugiura (1978), Hurvich and Tsai (1989)). See Konishi and Kitagawa (2008) for details on information criteria. Since the information criteria depend on the unknown parameters only through the likelihood, identifiability of the model parameters is irrelevant. (Per theorems 4.1 and 5.1 the model will not necessarily be point identified with R too large.)

The results of model selection are displayed in table 2, showing for each specification of R: the values of the Bayesian and Akaike information criteria, and also the Δ difference between the information criterion for that R and the information criterion for the specification of R with the smallest value of the information criterion. The R with a "Δ" of zero is preferred by the associated information criterion, since that corresponds to the specification of R with smallest information criterion. The results suggest $R = 5$ and both criteria show overwhelming support for more than one type, since Δ_{Bayesian} and Δ_{Akaike} for the model with $R = 1$ are extremely large.

[24] There are $R-1$ free parameters in $\pi(\cdot)$, and $S-1$ free parameters per type from $\Lambda_r(\cdot)$. If computational mistakes were allowed, there would be two more free parameters per type.

	Λ					Probability
	Anchored reasoning		Unanchored reasoning			of type
	1	2	0	1	Nash	
r	$\Lambda_r(1_{anch})$	$\Lambda_r(2_{anch})$	$\Lambda_r(0_{unanch})$	$\Lambda_r(1_{unanch})$	$\Lambda_r(NE)$	$\pi(r)$
1	0.10 (0.07, 0.12)	0.04 (0.02, 0.06)	0.49 (0.39, 0.57)	0.31 (0.23, 0.41)	0.07 (0.03, 0.10)	0.44 (0.38, 0.56)
2	0.70 (0.52, 0.77)	0.00 (0.00, 0.00)	0.15 (0.10, 0.27)	0.11 (0.06, 0.20)	0.04 (0.02, 0.06)	0.20 (0.14, 0.31)
3	0.19 (0.00, 0.35)	0.42 (0.36, 0.77)	0.11 (0.00, 0.20)	0.24 (0.00, 0.43)	0.04 (0.00, 0.06)	0.15 (0.09, 0.25)
4	0.06 (0.03, 0.09)	0.04 (0.00, 0.06)	0.04 (0.00, 0.08)	0.40 (0.31, 0.51)	0.45 (0.40, 0.57)	0.15 (0.04, 0.24)
5	0.08 (0.00, 0.15)	0.90 (0.87, 1.00)	0.00 (0.00, 0.00)	0.02 (0.00, 0.03)	0.00 (0.00, 0.00)	0.06 (0.00, 0.10)

95% confidence intervals reported in parentheses, estimated according to the standard subsampling algorithm for maximum likelihood (e.g., Politis, Romano, and Wolf (1999)) by re-sampling $N_s = floor(\frac{2}{3}88) = 58$ people from the dataset, without replacement. Conventional asymptotic approximations and bootstraps are likely invalid in this model (with this data), because many of the estimated probabilities are 0, which suggests a "parameter on the boundary" problem.

TABLE 3. Estimates

6.3.3. *Parameter estimates.* The results of estimating the model are displayed in table 3. Each row of table 3 corresponds to one of the estimated types. The first five columns (not counting the "r" column) show the probabilities that type uses the various solution concepts described in section 2.5. The sixth column shows the fraction of the population of that type. Also displayed are 95% confidence intervals. Types are listed in decreasing order of the fraction of the population that are that type.

The most common type, 44% of the population, primarily uses zero steps of unanchored strategic reasoning (49%), and also uses one step of unanchored strategic reasoning (31%).

The second most common type, 20% of the population, primarily uses one step of anchored strategic reasoning (70%), and also uses zero steps of unanchored strategic reasoning (15%) and one step of unanchored strategic reasoning (11%).

The third most common type, 15% of the population, primarily uses two steps of anchored strategic reasoning (42%), and also uses one step of anchored strategic reasoning (19%) and one step of unanchored strategic reasoning (24%).

The fourth most common type, 15% of the population, primarily uses the Nash equilibrium (45%), and also uses one step of unanchored strategic reasoning (40%).

Finally, the least common type, 6% of the population, primarily uses two steps of anchored strategic reasoning (90%), and also uses one step of anchored strategic reasoning (8%).

So, all types involve within-individual heterogeneity, since no type involves the exclusive use of just one solution concept. (The least common type does have perhaps "little" within-individual heterogeneity.) This shows that allowing within-individual heterogeneity is important. The estimated strategic decision making types generally have the sensible feature that they emphasize the use of just one "mode" of strategic reasoning (anchored or unanchored). Rules 1 and 4 predominantly use unanchored strategic reasoning, while rules 2 and 5 predominantly use anchored strategic reasoning. Rule 3 shows a slightly more even "mix" of modes of strategic reasoning. This shows that allowing both modes of strategic reasoning is important, and that different subjects use different modes of strategic reasoning. The fact that the estimates are sensible in this way was not imposed by the model or the estimation method.

6.4. **Relationship to prior estimates.** There is a logical relationship between these estimates (allowing within-individual heterogeneity) and the estimates of Costa-Gomes and Crawford (2006) (not allowing within-individual heterogeneity). Costa-Gomes and Crawford (2006) observe that roughly half of the subjects can be assigned their "type" (out of the standard level-k model, and not allowing within-individual heterogeneity) based on type being "apparent from guesses," which means using the action associated with the type in at least 7 out of the 16 games. (Recall that another contribution of the model in this paper is including unanchored strategic reasoning, which is not included in this discussion, since Costa-Gomes and Crawford (2006) focus on the level-k model.)

Costa-Gomes and Crawford (2006) find that 20 subjects (22.7%) are the type to use one step of anchored strategic reasoning. This is consistent with the current estimates, because type 2 (20% of the population) uses one step of anchored strategic reasoning with probability 70% (but not 100%, reflecting within-individual heterogeneity). Using the Binomial distribution, such subjects will almost surely use one step of anchored strategic reasoning in at least 7 out of 16 games, and therefore will appear to be the type that uses one step of anchored strategic reasoning, explaining the concordance between the estimates of 22.7% and 20%. (Other types use one step of anchored strategic reasoning so rarely that such subjects are extremely unlikely to use it in 7 out of 16 games, and thus will not appear to be that type.) Costa-Gomes and Crawford (2006) also find that 12 subjects (13.6%) are the type to use two steps of anchored strategic reasoning. This is also consistent with the current estimates. Type 5 (6% of the population) uses two steps of anchored strategic reasoning with probability 90%. Such subjects will almost certainly use two steps of anchored strategic reasoning in at least 7 out of 16 games. Moreover, using the Binomial distribution, roughly 54% of type 3 subjects (a type comprising 15% of the population) will use two steps of anchored strategic reasoning in at least 7 out of 16 games. Thus, roughly, based on these estimates there will be $6\% + 54\% \times 15\% \approx 14.1\%$ of subjects that will use two steps of anchored strategic reasoning in at least 7 out of 16 games, hence the concordance between the estimates of 13.6% and 14.1%. Finally, Costa-Gomes and Crawford (2006) find that 8 subjects (9.1%) are the type to use Nash equilibrium. This is also consistent with these current estimates, because type 4 (15% of the population) uses Nash with probability 45%. Using the Binomial distribution, approximately 63% of such subjects will use Nash in at least 7 out of 16 games, hence the concordance between the estimates of 9.1% and $9.5\% = 63\% \times 15\%$. (Other types use Nash so rarely that such subjects are extremely unlikely to use it in 7 out of 16 games.)

Costa-Gomes and Crawford (2006) also estimate a formal structural model that assigns the subjects to a type, the one that "fits" (in a maximum likelihood sense) as a best approximation to their underlying within-individual heterogeneity.

7. Conclusion

This paper proposes a structural model of non-equilibrium behavior in games with continuous action spaces, in order to learn about the solution concepts that individuals use to generate their actions. The model allows two different modes of strategic reasoning (anchored and unanchored), and computational mistakes. And, the model allows both across-individual and within-individual heterogeneity. The paper proposes the model, and provides sufficient conditions for point identification. The point identification result can be interpreted as a result on experimental design, informing the sorts of experiments that should be run to learn about the solution concepts that individuals use. Then, the model is estimated on data from an experiment involving two-player guessing games. The empirical application both illustrates the empirical relevance of the features of the model, and provides empirical results of independent interest. The estimation results indicate both across-individual heterogeneity and within-individual heterogeneity, and that both modes of strategic reasoning are used.

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Appendix A. Supplemental results

A.1. **Appendix: model likelihood.** Use the notation that y is the entire dataset, y_i is the data of subject i, and y_{ig} is the data of subject i in game g. Also, $\tau(i)$ is the strategic decision making rule used by subject i. Suppose that γ_{ig} is the intended solution concept for subject i in game g. Neither $\tau(i)$ nor γ_{ig} are observed by the econometrician. Then the likelihood is as follows, for observing subjects $i = 1, 2 \ldots, N$ take actions in games $g = 1, 2, \ldots, G$:

$$\begin{aligned}
\log L(y|\theta) &= \sum_{i=1}^{N} \log L(y_i|\theta) \\
&= \sum_{i=1}^{N} \log\left(\sum_{r=1}^{R}\left(P(y_i|\tau(i)=r,\theta)P(\tau(i)=r|\theta)\right)\right) \\
&= \sum_{i=1}^{N} \log\left(\sum_{r=1}^{R}\left(\left(\prod_{g=1}^{G} P(y_{ig}|\tau(i)=r,\theta)\right)\pi(r)\right)\right) \\
&= \sum_{i=1}^{N} \log\left(\sum_{r=1}^{R}\left(\left(\prod_{g=1}^{G}\left(\sum_{k}\left(P(y_{ig}|\tau(i)=r,\gamma_{ig}=k,\theta)P(\gamma_{ig}=k|\tau(i)=r,\theta)\right)\right)\right)\pi(r)\right)\right) \\
&= \sum_{i=1}^{N} \log\left(\sum_{r=1}^{R}\left(\left(\prod_{g=1}^{G}\left(\sum_{k}\left(P(y_{ig}|\tau(i)=r,\gamma_{ig}=k,\theta)\Lambda_r(k)\right)\right)\right)\pi(r)\right)\right)
\end{aligned}$$

where θ collects all of the parameters of the model. The sum over k corresponds to the sum over the solution concepts that subjects might use, per assumption 2.1. It remains to derive the form of $P(y_{ig}|\tau(i)=r,\gamma_{ig}=k,\theta)$ from the model specification.

For $k = s_{unanch}$, for some s:

$$P(y_{ig} \leq t|\tau(i)=r,\gamma_{ig}=s_{unanch},\theta) = F_{gs_{unanch}}(t),$$

where $F_{gs_{unanch}}(\cdot)$ is the cumulative distribution function of a uniformly distributed random variable on Σ_{1g}^{s}.

For $k \in \mathcal{M}$ corresponding to any solution concept subject to computational mistakes (i.e., for $k = s_{anch}$ for some s or $k = NE$), and letting m_{ig} be a binary variable to indicate whether subject i incorrectly computes the action associated with solution concept k in game g, which is not observed by the econometrician:

$$\begin{aligned}
P(y_{ig} \leq t|\tau(i)=r,\gamma_{ig}=k,\theta) &= P(y_{ig} \leq t|\tau(i)=r,\gamma_{ig}=k,m_{ig}=1,\theta) \times P(m_{ig}=1|\tau(i)=r,\gamma_{ig}=k,\theta) \\
&\quad + P(y_{ig} \leq t|\tau(i)=r,\gamma_{ig}=k,m_{ig}=0,\theta) \times P(m_{ig}=0|\tau(i)=c,\gamma_{ig}=k,\theta) \\
&= F_{rgk}(t)\Delta_r + 1[t \geq c_{1g}(k)](1-\Delta_r),
\end{aligned}$$

where $F_{rgk}(\cdot)$ is the cumulative distribution function of computational mistakes on $[\alpha_{Lg}(1),\alpha_{Ug}(1)] \cap [c_{1g}(k) - \mathrm{P}_r\left(\alpha_{Ug}(1)-\alpha_{Lg}(1)\right), c_{1g}(k) + \mathrm{P}_r\left(\alpha_{Ug}(1)-\alpha_{Lg}(1)\right)]$, per section 2.6.

A.2. **Appendix: proof of point identification.** Use the notation that $\mathcal{M}(r)$ is the r-th smallest element of $\mathcal{M}$ (with Nash equilibrium the largest element by convention), $\mathcal{U}(r)$ is the r-th smallest element of $\mathcal{U}$, $U_g(s) = U_{1g}(s)$, $R_g(s,s',\epsilon) = R_{1g}(s,s',\epsilon)$ and $\Omega_g =$

$\alpha_{Ug}(1) - \alpha_{Lg}(1)$. And,

$$M_g(k, \epsilon, \mathrm{P}_r) = \begin{cases} \int_{c_{1g}(k)-\epsilon\Omega_g}^{c_{1g}(k)+\epsilon\Omega_g} \omega_{1g,c_{1g}(k),\mathrm{P}_r}(a)\mathrm{da} & \text{if } \mathrm{P}_r > 0 \\ 1 & \text{if } \mathrm{P}_r = 0. \end{cases}$$

Let the set of non-zero unique values of $\{\mathrm{P}_r 1[\Delta_r > 0]1[\sum_{k\in\mathcal{M}} \Lambda_r(k) > 0]1[\pi(r) > 0]\}_{r=1}^R$ together with $\bar{\rho}$ be $\{\tilde{\mathrm{P}}_w\}_{w=1}^W$, and without loss of generality assume that $0 \leq \tilde{\mathrm{P}}_1 < \tilde{\mathrm{P}}_2 < \cdots < \tilde{\mathrm{P}}_W$, and that $1 \leq W \leq R+1$. By assumption 2.2, $\tilde{\mathrm{P}}_W = \bar{\rho}$.

For any solution concept $k \in \mathcal{M}$, let $C_g(k, \epsilon)$ be the event that a subject takes an action weakly within $\epsilon\Omega_g$ of the action predicted by solution concept k in game g, but excluding the action exactly predicted by solution concept k in game g. For any solution concept $k \in \mathcal{M}$, let the event that a subject takes the action exactly predicted by solution concept k in game g be $C_g(k)$. (Note that $C_g(k) \neq C_g(k, 0)$.)

Use the generic notation that P_θ refers to the distribution of observables based on strategic decision making rule θ, and that $P_{g,\theta}$ refers to the distribution of observables based on strategic decision making rule θ in game g. By some abuse of notation, let $P_{g,\theta}$ be the $(|\mathcal{M}| + |\mathcal{U}| + W|\mathcal{M}|) \times 1$ vector:

(1) the first $|\mathcal{M}|$ rows are $(P_{g,\theta}(C_g(\mathcal{M}(1))), \ldots, P_{g,\theta}(C_g(\mathcal{M}(|\mathcal{M}|))))$;
(2) the next $|\mathcal{U}|$ rows are $(P_{g,\theta}(U_g(\mathcal{U}(1))), \ldots, P_{g,\theta}(U_g(\mathcal{U}(|\mathcal{U}|))))$;
(3) the final $W|\mathcal{M}|$ rows are $\Big(P_{g,\theta}(C_g(\mathcal{M}(1), \tilde{\mathrm{P}}_1)), \ldots, P_{g,\theta}(C_g(\mathcal{M}(1), \tilde{\mathrm{P}}_W)), P_{g,\theta}(C_g(\mathcal{M}(2), \tilde{\mathrm{P}}_1)), \ldots\Big)$.

Use the notation that $\bigotimes^n b = \underbrace{b \otimes b \otimes \cdots \otimes b}_{n \text{ times}}$, for $n \in \mathbb{N}$.

Lemma A.1. *The following claims are true:*

A.1.1 For a game g that satisfies condition 4.1.1, and for $\rho_i > 0$, the density $\omega_{jg,c,\rho_i}(a)$ has discontinuities at, and only at, $\min\{\alpha_{Ug}(j), c+\rho_i\,(\alpha_{Ug}(j) - \alpha_{Lg}(j))\}$ and $\max\{\alpha_{Lg}(j), c-\rho_i\,(\alpha_{Ug}(j) - \alpha_{Lg}(j))\}$.

A.1.2 For a game g that satisfies the conditions of assumption 4.2, and for $\mathrm{P}_r > 0$, for any $k \in \mathcal{M}$, and $0 < \epsilon < \bar{\rho}$, $M_g(k, \epsilon, \mathrm{P}_r)$ has a kink at, and only at, $\epsilon = \mathrm{P}_r$.

A.1.3 For a game g that satisfies conditions 4.1.1 and 4.1.4, for any $k \in \mathcal{M}$, $M_g(k, \epsilon, \mathrm{P}_r) = M_g(k, \mathrm{P}_r, \mathrm{P}_r)$ if $\epsilon \geq \mathrm{P}_r$ and $M_g(k, \epsilon_1, \mathrm{P}_r) < M_g(k, \epsilon_2, \mathrm{P}_r)$ if $0 \leq \epsilon_1 < \epsilon_2 \leq \mathrm{P}_r$.

Proof of Lemma A.1. Because the game g satisfies condition 4.1.1, and $\rho_i > 0$, the density $\omega_{jg,c,\rho_i}(a)$ does not involve dividing by zero, and therefore is well-defined.

For A.1.1: Because $\xi(\cdot)$ is continuous on $[-1, 1]$, discontinuities in $\omega_{jg,c,\rho_i}(a)$ can occur only at a such that the argument of $\xi(\cdot)$ in the definition of $\omega_{jg,c,\rho_i}(a)$ is either -1 or 1. Therefore, discontinuities can occur only at $a = \min\{\alpha_{Ug}(j), c + \rho_i\,(\alpha_{Ug}(j) - \alpha_{Lg}(j))\}$ and $a = \max\{\alpha_{Lg}(j), c - \rho_i\,(\alpha_{Ug}(j) - \alpha_{Lg}(j))\}$. Moreover, by assumption, ξ is bounded away from zero on $[-1,1]$, but equals zero off $[-1,1]$, and therefore indeed $\omega_{jg,c,\rho_i}(a)$ does have discontinuities at the claimed points.

For A.1.2: By part A.1.1, the integrand in $M_g(k,\epsilon,\mathrm{P}_r) = \int_{c_{1g}(k)-\epsilon\Omega_g}^{c_{1g}(k)+\epsilon\Omega_g} \omega_{1g,c_{1g}(k),\mathrm{P}_r}(a)\mathrm{da}$ has discontinuities at, and only at, $\min\{\alpha_{Ug}(1), c_{1g}(k)+\mathrm{P}_r(\alpha_{Ug}(1)-\alpha_{Lg}(1))\}$ and $\max\{\alpha_{Lg}(1), c_{1g}(k)-\mathrm{P}_r(\alpha_{Ug}(1)-\alpha_{Lg}(1))\}$. Because the game g satisfies the conditions of assumption 4.2, $0 < \mathrm{P}_r\Omega_g < \overline{\rho}\Omega_g < \alpha_{Ug}(1) - c_{1g}(k)$ or $0 < \mathrm{P}_r\Omega_g < \overline{\rho}\Omega_g < c_{1g}(k) - \alpha_{Lg}(1)$ by condition 4.1.4. Therefore, either $\min\{\alpha_{Ug}(1), c_{1g}(k)+\mathrm{P}_r(\alpha_{Ug}(1)-\alpha_{Lg}(1))\} = c_{1g}(k)+\mathrm{P}_r\Omega_g$ or $\max\{\alpha_{Lg}(1), c_{1g}(k)-\mathrm{P}_r(\alpha_{Ug}(1)-\alpha_{Lg}(1))\} = c_{1g}(k)-\mathrm{P}_r\Omega_g$. Therefore, $M_g(k,\epsilon,\mathrm{P}_r)$ has a kink at $\epsilon = \mathrm{P}_r$. Moreover, there can be no other kinks in $M_g(k,\epsilon,\mathrm{P}_r)$ for any $k \in \mathcal{M}$ and $0 < \epsilon < \overline{\rho}$, by condition 4.2.1. That follows because any other kink would be located at $\epsilon = \frac{c_{1g}(k)-\alpha_{Lg}(1)}{\Omega_g}$ or $\epsilon = \frac{\alpha_{Ug}(1)-c_{1g}(k)}{\Omega_g}$. But by condition 4.2.1 evaluated at $s=0$, such ϵ would either equal 0 or 1 under conditions 4.2.1c or 4.2.1d, or would be weakly greater than $\overline{\rho}$ under condition 4.2.1a. However, $0 < \epsilon < \overline{\rho}$ and by assumption 4.1.4, $\overline{\rho} < 1$.

For A.1.3: Note that $M_g(k,\epsilon,\mathrm{P}_r) = \int_{c_{1g}(k)-\epsilon\Omega_g}^{c_{1g}(k)+\epsilon\Omega_g} \omega_{1g,c_{1g}(k),\mathrm{P}_r}(a)\mathrm{da}$, where the integrand is 0 for $a > \min\{\alpha_{Ug}(1), c_{1g}(k)+\mathrm{P}_r\Omega_g\}$ and $a < \max\{\alpha_{Lg}(1), c_{1g}(k)-\mathrm{P}_r\Omega_g\}$. Therefore, $M_g(k,\epsilon,\mathrm{P}_r) = \int_{\max\{c_{1g}(k)-\epsilon\Omega_g,\max\{\alpha_{Lg}(1),c_{1g}(k)-\mathrm{P}_r\Omega_g\}\}}^{\min\{c_{1g}(k)+\epsilon\Omega_g,\min\{\alpha_{Ug}(1),c_{1g}(k)+\mathrm{P}_r\Omega_g\}\}} \omega_{1g,c_{1g}(k),\mathrm{P}_r}(a)\mathrm{da}$. Therefore, since the bounds of integration are $[\max\{\alpha_{Lg}(1), c_{1g}(k)-\mathrm{P}_r\Omega_g\}, \min\{\alpha_{Ug}(1), c_{1g}(k)+\mathrm{P}_r\Omega_g\}]$ for $\epsilon \geq \mathrm{P}_r$, it follows that $M_g(k,\epsilon,\mathrm{P}_r) = M_g(k,\mathrm{P}_r,\mathrm{P}_r)$ if $\epsilon \geq \mathrm{P}_r$. Since the bounds of integration are $[\max\{\alpha_{Lg}(1), c_{1g}(k)-\epsilon\Omega_g\}, \min\{\alpha_{Ug}(1), c_{1g}(k)+\epsilon\Omega_g\}]$ for $\epsilon \leq \mathrm{P}_r$, and the integrand is positive over that range for all $\epsilon \leq \mathrm{P}_r$, and by assumption 4.1.4, either the lower bound equals $c_{1g}(k)-\epsilon\Omega_g$ or the upper bound equals $c_{1g}(k)+\epsilon\Omega_g$, which both depend non-trivially on ϵ by assumption 4.1.1, it follows that $M_g(k,\epsilon_1,\mathrm{P}_r) < M_g(k,\epsilon_2,\mathrm{P}_r)$ if $0 \leq \epsilon_1 < \epsilon_2 \leq \mathrm{P}_r$. □

Lemma A.2. *Let $R \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfy $m \geq R-1$. Let $C(m,n) = \sum_{p=0}^{m} n^p$. Let $\gamma_{p,n}(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n^p}$ be defined by $\gamma_{p,n}(z) = \bigotimes^p z$. Let $\Gamma_{m,n}(\cdot) : \mathbb{R}^n \to \mathbb{R}^{C(m,n)}$ be defined by $\Gamma_{m,n}(z) = (1, \gamma_{1,n}(z), \ldots, \gamma_{m,n}(z))$. Thus, $\Gamma_{m,n}(z)$ gives all monomials of the argument vector z, of order between 0 and m, in ascending order (i.e., the order 0 monomial in the first row, then order 1 monomials in the next rows, etc.). Suppose $b_1, \ldots, b_R \in \mathbb{R}^n$ are distinct. Let $B^* = (\Gamma_{m,n}(b_1)\ \Gamma_{m,n}(b_2) \cdots \Gamma_{m,n}(b_R)) \in \mathbb{R}^{C(m,n)\times R}$. Then, there is a $C(m,n) \times C(m,n)$ non-singular matrix T such that TB^* has full column rank.*

Proof of Lemma A.2. The following argument establishes that since $b_k \neq b_l$ for $k \neq l$, there exists a $t \in \mathbb{R}^n_{++}$ such that $t'b_k \neq t'b_l$ for all $k \neq l$. Let $\mathcal{D}(t) = \{(k,l) : t'b_k = t'b_l, k \neq l\}$. Let $t_0 \in \mathbb{R}^n_{++}$. If $|\mathcal{D}(t_0)| = 0$, then the claim is established. Otherwise, for some k^* and l^* such that $k^* \neq l^*$, $t_0'b_{k^*} = t_0'b_{l^*}$. By slightly perturbing t_0 in the element of t_0 corresponding to the element where b_{k^*} and b_{l^*} are not equal (which must exist since $b_{k^*} \neq b_{l^*}$), there exists $t_1 \in \mathbb{R}^n_{++}$ such that $t_1' b_{k^*} \neq t_1' b_{l^*}$. If the perturbation is sufficiently small, then $t_1'(b_k - b_l) \approx t_0'(b_k - b_l)$ uniformly for all k and l. Therefore, for any (k, l) such that $t_0 b_k \neq t_0 b_l$, also $t_1 b_k \neq t_1 b_l$. Therefore, $|\mathcal{D}(t_1)| < |\mathcal{D}(t_0)|$. Similarly, it is possible to perturb t_1 to construct t_2 such that $|\mathcal{D}(t_2)| < |\mathcal{D}(t_1)|$ if $|\mathcal{D}(t_1)| > 0$. Necessarily, this process terminates at $t \in \mathbb{R}^n_{++}$ such that $t' b_k \neq t' b_l$ for all $k \neq l$.

The matrix T is defined constructively, using the notation that $z \in \mathbb{R}^n$ is a free variable. For each integer $p \in \{0, 1, \ldots, m\}$, row $C(p, n)$ of T has $C(p-1, n)$ leading zeros, then is equal to $\bigotimes^p t'$, and then has trailing zeros. Therefore, row $C(p, n)$ of $T\Gamma_{m,n}(z)$ is $(t'z)^p$, since $(t'z)^p = \bigotimes^p (t'z) = \bigotimes^p t' \bigotimes^p z$. In particular, for $p = 0$, use the convention that $(t'z)^p = 1$. So, since the first element of $\Gamma_{m,n}(z)$ is 1, the first row of T has a 1 along the diagonal and is equal to zero everywhere else. Since $t'z$ is the sum of n terms, there are n^p terms in the series expansion of $(t'z)^p$. Therefore, the last non-zero term in row $C(p, n)$ is in column $C(p-1, n) + n^p = C(p, n)$. All other rows are zeros except for a 1 along the diagonal. In particular, note that T is a lower triangular matrix, with non-zero entries along the diagonal (since $t \in \mathbb{R}^n_{++}$). Therefore, T is non-singular.

By construction of T, the element of TB^* in row $C(p, n)$ and column c is $(t' b_c)^p$. Therefore, one submatrix of TB^* is a Vandermonde matrix of dimension $(m+1) \times R$, in terms of the powers of $(t' b_c)$ for $c = 1, \ldots, R$. Since $m + 1 \geq R$, in particular one submatrix of T is the Vandermonde matrix of dimension $R \times R$. Since $t' b_c \neq t' b_{c'}$ for $c \neq c'$ by choice of t, these Vandermonde matrices are based on distinct "parameters," which implies that the square Vandermonde matrix is non-singular. So, TB^* contains an $R \times R$ non-singular submatrix. Since TB^* is $C(m, n) \times R$, this implies that TB^* has full column rank. □

Lemma A.3. *Let* $\tilde{\mathrm{P}} = \{\tilde{\mathrm{P}}_w\}_{w=1}^W$ *be a set of possible magnitudes of computational mistakes with* $\tilde{\mathrm{P}}_1 < \tilde{\mathrm{P}}_2 < \cdots < \tilde{\mathrm{P}}_W$. *Based on* $\tilde{\mathrm{P}}$, *define vector-valued mappings* η_1, η_2 *and* η_3 *of the strategic decision making rules* $\Theta = (\Lambda, \Delta, \mathrm{P})$:

(1) $\eta_1(\Theta) = ((1-\Delta)\Lambda(\mathcal{M}(1)), \ldots, (1-\Delta)\Lambda(\mathcal{M}(|\mathcal{M}|)))$. *So,* η_1 *gives the vector of* $(1-\Delta)\Lambda(k)$ *for solution concepts* $k \in \mathcal{M}$.

(2) $\eta_2(\Theta) = (\Lambda(\mathcal{U}(1)), \ldots, \Lambda(\mathcal{U}(|\mathcal{U}|)))$. *So,* η_2 *gives the vector of* $\Lambda(k)$ *for solution concepts* $k \in \mathcal{U}$.

(3) $\eta_3(\Theta) = \Big(\Delta\Lambda(\mathcal{M}(1))1[\mathrm{P} = \tilde{\mathrm{P}}_1], \ldots, \Delta\Lambda(\mathcal{M}(1))1[\mathrm{P} = \tilde{\mathrm{P}}_W], \Delta\Lambda(\mathcal{M}(2))1[\mathrm{P} = \tilde{\mathrm{P}}_1], \ldots\Big)$. *So,* η_3 *gives* $\Delta\Lambda(k)1[\mathrm{P} = \tilde{\mathrm{P}}_w]$ *for solution concepts* $k \in \mathcal{M}$ *and* $w = 1, 2, \ldots, W$.

Let $\eta^*(\Theta) = (\eta_1(\Theta), \eta_2(\Theta), \eta_3(\Theta))$ *and* $\eta^{**}(\Theta) = (\eta_1(\Theta), \eta_2(\Theta))$.

Suppose Θ_1 and Θ_2 are two strategic decision making rules such that $\mathrm{P}_1 \in \tilde{\mathrm{P}}$ and $\mathrm{P}_2 \in \tilde{\mathrm{P}}$. If $\eta^*(\Theta_1) = \eta^*(\Theta_2)$, then $\Lambda_1 = \Lambda_2$, $\Delta_1 1[\sum_{k\in\mathcal{M}} \Lambda_1(k) > 0] = \Delta_2 1[\sum_{k\in\mathcal{M}} \Lambda_2(k) > 0]$, and $\mathrm{P}_1 1[\Delta_1 > 0] 1[\sum_{k\in\mathcal{M}} \Lambda_1(k) > 0] = \mathrm{P}_2 1[\Delta_2 > 0] 1[\sum_{k\in\mathcal{M}} \Lambda_2(k) > 0]$.

Suppose Θ_1 and Θ_2 are two strategic decision making rules. If $\eta^{**}(\Theta_1) = \eta^{**}(\Theta_2)$, then $\Lambda_1 = \Lambda_2$, and $\Delta_1 1[\sum_{k\in\mathcal{M}} \Lambda_1(k) > 0] = \Delta_2 1[\sum_{k\in\mathcal{M}} \Lambda_2(k) > 0]$.

Proof of Lemma A.3. Suppose that $\eta^{**}(\Theta_1) = \eta^{**}(\Theta_2)$. It is immediate from the definition of η_1 that $(1-\Delta_1)\Lambda_1(k) = (1-\Delta_2)\Lambda_2(k)$ for any solution concept $k \in \mathcal{M}$. Also, it is immediate from the definition of η_2 that $\Lambda_1(k) = \Lambda_2(k)$ for any solution concept $k \in \mathcal{U}$. Necessarily, $1 = \sum_k \Lambda(k) = \sum_{k\in\mathcal{M}} \Lambda(k) + \sum_{k\in\mathcal{U}} \Lambda(k)$. Therefore, it must be that $\sum_{k\in\mathcal{M}} \Lambda_1(k) = \sum_{k\in\mathcal{M}} \Lambda_2(k)$ since $\sum_{k\in\mathcal{U}} \Lambda_1(k) = \sum_{k\in\mathcal{U}} \Lambda_2(k)$. Therefore, since $\sum_{k\in\mathcal{M}}(1-\Delta_1)\Lambda_1(k) = \sum_{k\in\mathcal{M}}(1-\Delta_2)\Lambda_2(k)$, it must be that $\Delta_1 1[\sum_{k\in\mathcal{M}} \Lambda_1(k) > 0] = \Delta_2 1[\sum_{k\in\mathcal{M}} \Lambda_2(k) > 0]$. Suppose that for all $k \in \mathcal{M}$ it holds that $\Lambda_1(k) = 0$. Then, since $\Delta_2 < 1$, it must be that $\Lambda_2(k) = 0$ for all $k \in \mathcal{M}$ by definition of η_1. So, in that case, $\Lambda_1(k) = \Lambda_2(k)$ for all $k \in \mathcal{M}$. If there is $k^* \in \mathcal{M}$ such that $\Lambda_1(k^*) > 0$, then since $\Delta_1 < 0$ it must be that $\Lambda_2(k^*) > 0$ by definition of η_1. In that case, it must indeed be that $\Delta_1 = \Delta_2$ since $1[\sum_{k\in\mathcal{M}} \Lambda_1(k) > 0] = 1[\sum_{k\in\mathcal{M}} \Lambda_2(k) > 0] = 1$. So, then, by definition of η_1 it must be that $\Lambda_1(k) = \Lambda_2(k)$ for all $k \in \mathcal{M}$. So, again, in that case, $\Lambda_1(k) = \Lambda_2(k)$ for all $k \in \mathcal{M}$.

Now suppose in addition that $\eta^*(\Theta_1) = \eta^*(\Theta_2)$. If $\Delta_1 = \Delta_2 > 0$ and $\sum_{k\in\mathcal{M}} \Lambda_1(k) = \sum_{k\in\mathcal{M}} \Lambda_2(k) > 0$, note that $\Delta_1 \sum_{k\in\mathcal{M}} \Lambda_1(k) 1[\mathrm{P}_1 = \tilde{\mathrm{P}}_w]$ (or, respectively, $\Delta_2 \sum_{k\in\mathcal{M}} \Lambda_2(k) 1[\mathrm{P}_2 = \tilde{\mathrm{P}}_w]$) is non-zero if and only if $\mathrm{P}_1 = \tilde{\mathrm{P}}_w$ (or $\mathrm{P}_2 = \tilde{\mathrm{P}}_w$). Therefore, by definition of η_3, it must be that $\mathrm{P}_1 1[\Delta_1 > 0] 1[\sum_{k\in\mathcal{M}} \Lambda_1(k) > 0] = \mathrm{P}_2 1[\Delta_2 > 0] 1[\sum_{k\in\mathcal{M}} \Lambda_2(k) > 0]$. □

Lemma A.4. *The following claims are true:*

A.4.1 Suppose that $k \in \mathcal{M}$. In a game g that satisfies conditions 4.1.2 and 4.1.3, or a game g that satisfies conditions 5.1.2 and 5.1.3, it holds that

$$P_{rg}(C_g(k)) = (1 - \Delta_r)\Lambda_r(k).$$

A.4.2 Suppose that $k \in \mathcal{M}$. Suppose that $0 < \epsilon$. In a game g that satisfies condition 4.1.1, or equivalently a game g that satisfies condition 5.1.1, it holds that

$$P_{rg}(C_g(k,\epsilon)) = \sum_{k' \neq k} P_{rg}(C_g(k,\epsilon) | \gamma_g = k')\Lambda_r(k') + \Delta_r M_g(k, \epsilon, \mathrm{P}_r)\Lambda_r(k)$$

A.4.3 Suppose that $k \in \mathcal{M}$. Suppose that $0 < \epsilon \leq \overline{\rho}$, where $\overline{\rho}$ arises from assumption 4.1. In a game g that satisfies conditions 4.1.1 and 4.1.2, it holds that

$$P_{rg}(C_g(k,\epsilon)) = \sum_{s\in\mathcal{U}} P_{rg}(C_g(k,\epsilon) | \gamma_g = s_{unanch})\Lambda_r(s_{unanch}) + \Delta_r M_g(k, \epsilon, \mathrm{P}_r)\Lambda_r(k)$$

A.4.4 Suppose that $s \in \mathcal{U}$. It holds that

$$P_{rg}(U_g(s)) = \sum_{0 \leq s' \leq s, s' \in \mathcal{U}} R_g(s, s', \overline{\rho})\Lambda_r(s'_{unanch})$$

Proof of Lemma A.4. For A.4.1: By the law of total probability,

$$P_{rg}(C_g(k)) = \sum_{k'} P_{rg}(C_g(k)|\gamma_g = k')P_{rg}(\gamma_g = k').$$

Under conditions 4.1.2 and 4.1.3, or conditions 5.1.2 and 5.1.3, there are no solution concepts $k' \neq k$ that take the action associated with solution concept k with positive probability, so $P_{rg}(C_g(k)) = P_{rg}(C_g(k)|\gamma_g = k)\Lambda_r(k)$. And, a subject that uses rule r and solution concept k will actually take the action predicted by solution concept k with probability $1 - \Delta_r$, since with probability Δ_r it makes a computational mistake and takes an action according to the density on a non-degenerate interval since $\mathrm{P}_r > 0$ by assumption when $\Delta_r > 0$. So, $P_{rg}(C_g(k)|\gamma_g = k) = 1 - \Delta_r$.

For A.4.2: A subject that uses rule r and intends to use solution concept k in game g and that makes a computational mistake will take an action that is distributed according to $\xi(\cdot)$ translated to the interval with radius $\mathrm{P}_r(\alpha_{Ug}(1) - \alpha_{Lg}(1))$ centered at the action predicted by solution concept k, and intersected with the action space. Therefore:

$$\begin{aligned} P_{rg}(C_g(k, \epsilon)) &= \sum_{k'} P_{rg}(C_g(k, \epsilon)|\gamma_g = k')P_{rg}(\gamma_g = k') \\ &= \sum_{k' \neq k} P_{rg}(C_g(k, \epsilon)|\gamma_g = k')\Lambda_r(k') + \Delta_r M_g(k, \epsilon, \mathrm{P}_r)\Lambda_r(k) \end{aligned}$$

By condition 4.1.1, $\alpha_{Lg}(1) < \alpha_{Ug}(1)$, so as long as $\mathrm{P}_r > 0$, this last expression does not involve dividing by zero in the definition of $\omega_{1g, c_{1g}(k), \mathrm{P}_r}(\cdot)$ that appears as the integrand in $M_g(k, \epsilon, \mathrm{P}_r)$. The condition that $\mathrm{P}_r > 0$ is assumed in section 2.6 when $\Delta_r > 0$. Otherwise, if $\mathrm{P}_r = 0$ then $\Delta_r = 0$ and the expression is still correct.

For A.4.3: Since g is a game that additionally satisfies condition 4.1.2 and $\epsilon \leq \overline{\rho}$, $P_{rg}(C_g(k, \epsilon)|\gamma_g = k') = 0$ for any solution concept $k' \in \mathcal{M}$.

For A.4.4: By construction, the only time $U_g(s)$ happens (with positive probability) is from subjects that use s' steps of unanchored strategic reasoning for some $0 \leq s' \leq s$ with $s' \in \mathcal{U}$, so it follows that:

$$\begin{aligned} P_{rg}(U_g(s)) &= \sum_{k'} P_{rg}(U_g(s)|\gamma_g = k')P_{rg}(\gamma_g = k') \\ &= \sum_{0 \leq s' \leq s, s' \in \mathcal{U}} R_g(s, s', \overline{\rho})\Lambda_r(s'_{unanch}) \end{aligned}$$

□

Lemma A.5. *Suppose assumptions 2.2 and 4.1. Suppose that the econometrician allows the possibility of computational mistakes. Suppose that g is a game that satisfies conditions 4.1.1, 4.1.2, 4.1.4, and 4.2.1. Then, $\{\tilde{\mathrm{P}}_w\}_{w=1}^{W-1}$ is identified by the locations of the kinks in $\{P_g(C_g(k,\epsilon))\}_{k\in\mathcal{M}}$ as a function of ϵ, for $0<\epsilon<\overline{\rho}$.*

Proof of Lemma A.5. Suppose that $k\in\mathcal{M}$. For $0<\epsilon\leq\overline{\rho}$, the probability of the event $C_g(k,\epsilon)$ in game g is, using the result of lemma A.4.2 and condition 4.1.1,

$$P_g(C_g(k,\epsilon))=\sum_{r=1}^{R}P_{rg}(C_g(k,\epsilon))\pi(r)=\sum_{r=1}^{R}\left(\sum_{k'\neq k}P_{rg}(C_g(k,\epsilon)|\gamma_g=k')\Lambda_r(k')\right)\pi(r)$$
$$+\sum_{r=1}^{R}\left(\Delta_r M_g(k,\epsilon,\mathrm{P}_r)\Lambda_r(k)\right)\pi(r)$$

Since g is a game that satisfies condition 4.1.2, it follows that $P_{rg}(C_g(k,\epsilon)|\gamma_g=k')=0$ for all such solution concepts $k'\in\mathcal{M}$ with $k'\neq k$, since $\epsilon\leq\overline{\rho}$. So:

$$P_g(C_g(k,\epsilon))=\sum_{r=1}^{R}\left(\sum_{s\in\mathcal{U}}P_{rg}(C_g(k,\epsilon)|\gamma_g=s_{unanch})\Lambda_r(s_{unanch})\right)\pi(r)$$
$$+\sum_{r=1}^{R}\left(\Delta_r M_g(k,\epsilon,\mathrm{P}_r)\Lambda_r(k)\right)\pi(r)$$

Since g satisfies condition 4.2.1, for any $s\in\mathcal{U}$, $P_{rg}(C_g(k,\epsilon)|\gamma_g=s_{unanch})$ is a differentiable function of ϵ, for all $0<\epsilon<\overline{\rho}$. Under conditions 4.2.1a, 4.2.1c, or 4.2.1d, $P_{rg}(C_g(k,\epsilon)|\gamma_g=s_{unanch})$ is linear in ϵ. Under condition 4.2.1b, $P_{rg}(C_g(k,\epsilon)|\gamma_g=s_{unanch})=0$ for all $0<\epsilon<\overline{\rho}$.

Suppose that r is such that $\pi(r)>0$ and $\Delta_r>0$. Suppose that r uses at least one $k_r^*\in\mathcal{M}$ with positive probability. So, it holds that $\Delta_r\Lambda_r(k_r^*)\pi(r)>0$. Therefore, there is a kink in $P_g(C_g(k_r^*,\epsilon))$ at $\epsilon=\mathrm{P}_r$ since there is a kink in $M_g(k_r^*,\epsilon,\mathrm{P}_r)$ at $\epsilon=\mathrm{P}_r$ by lemma A.1.2. This uses the fact that $\mathrm{P}_r<\overline{\rho}$ for all r by assumption 2.2, whereas the above expression for $P_g(C_g(k,\epsilon))$ is valid for all $\epsilon\leq\overline{\rho}$, so that the location of all relevant kinks are indeed identified. Moreover, there can be no other kinks in $M_g(k,\epsilon,\mathrm{P}_r)$ for any $k\in\mathcal{M}$ and $0<\epsilon<\overline{\rho}$, by lemma A.1.2. Consequently, the list of non-zero unique values corresponding to $\{\mathrm{P}_r 1[\Delta_r>0]1[\sum_{k\in\mathcal{M}}\Lambda_r(k)>0]1[\pi(r)>0]\}_r$ is identified by the list of the locations of the kinks in $\{P_g(C_g(k,\epsilon))\}_{k\in\mathcal{M}}$ as a function of ϵ, for $0<\epsilon<\overline{\rho}$. □

Lemma A.6. *For each game g, define the following:*

(1) Let Q_{2g} be the $|\mathcal{U}|\times|\mathcal{U}|$ matrix that has element in row r and column c that equals $R_g(\mathcal{U}(r),\mathcal{U}(c),\overline{\rho})$.

(2) *Let Q_{3g} be the $(W|\mathcal{M}|)\times|\mathcal{U}|$ matrix that has element in row r and column c that equals the probability in game g of the event $C_g\left(\mathcal{M}\left(\left\lceil \frac{r}{W}\right\rceil\right), \tilde{\mathrm{P}}_{mod(r-1,W)+1}\right)$ according to the distribution of actions used by subjects that use $\mathcal{U}(c)$ steps of unanchored strategic reasoning.*

(3) *For each $k \in \mathcal{M}$, let Q_{4gk} be the $W\times W$ matrix that has element in row r and column c that equals $M_g\left(k, \tilde{\mathrm{P}}_r, \tilde{\mathrm{P}}_c\right)$. Then, let Q_{4g} be the $(W|\mathcal{M}|)\times(W|\mathcal{M}|)$ matrix that has $(Q_{4g\mathcal{M}(1)}, \ldots, Q_{4g\mathcal{M}(|\mathcal{M}|)})$ along the diagonal.*

Then, let

$$Q_g = \begin{pmatrix} I_{|\mathcal{M}|\times|\mathcal{M}|} & 0 & 0 \\ 0 & Q_{2g} & 0 \\ 0 & Q_{3g} & Q_{4g} \end{pmatrix}.$$

For any game g satisfying conditions 4.1.1, 4.1.2, 4.1.3, 4.1.4, and 4.1.5, $P_{g,\theta} = Q_g\eta^*(\theta)$ and Q_g is non-singular.

For any game g satisfying condition 4.1.5, or equivalently any game g satisfying condition 5.1.4, Q_{2g} is non-singular.

For any game g satisfying conditions 4.1.2 and 4.1.3, or any game g satisfying conditions 5.1.2 and 5.1.3, the first $|\mathcal{M}|$ rows of $P_{g,\theta}$ are equal to the first $|\mathcal{M}|$ rows of $Q_g\eta^*(\theta)$.

For any game g, rows $|\mathcal{M}|+1$ through $|\mathcal{M}|+|\mathcal{U}|$ of $P_{g,\theta}$ are equal to rows $|\mathcal{M}|+1$ through $|\mathcal{M}|+|\mathcal{U}|$ of $Q_g\eta^*(\theta)$.

For any game g satisfying conditions 4.1.1 and 4.1.2, the last block of $|W|\mathcal{M}$ rows of $P_{g,\theta}$ are equal to the last block of $W|\mathcal{M}|$ rows of $Q_g\eta^*(\theta)$.

Proof of Lemma A.6. Since $R_g(s, s', \overline{\rho}) = 0$ for $s' > s$ by construction, it follows that Q_{2g} is lower triangular. Since g is a game that satisfies condition 4.1.5, the diagonal elements are non-zero, implying that Q_{2g} is non-singular.

By the following arguments, for a game g that satisfies conditions 4.1.1 and 4.1.4, Q_{4gk} is non-singular for each $k \in \mathcal{M}$. First, consider the case that the econometrician allows the possibility of computational mistakes. Apply repeated elementary row operations: for rows $r \geq 2$ (if indeed $W \geq 2$), starting with row W and then moving to the next higher row, subtract row $r-1$ from row r and substitute the result into row r. The resulting matrix $\tilde{\tilde{Q}}_{4gk}$ has element in row $r \geq 2$ and column c that equals $M_g(k, \tilde{\mathrm{P}}_r, \tilde{\mathrm{P}}_c) - M_g(k, \tilde{\mathrm{P}}_{r-1}, \tilde{\mathrm{P}}_c)$. For a game g that satisfies conditions 4.1.1 and 4.1.4, by lemma A.1.3, this difference is 0 if $r-1 \geq c$ and is strictly positive if $r \leq c$. Therefore, row $r \geq 2$ has $r-1$ leading zeros and then positive elements. In row 1 and column c, the element is $M_g(k, \tilde{\mathrm{P}}_1, \tilde{\mathrm{P}}_c) > 0$. Therefore, for a game g that satisfies conditions 4.1.1 and 4.1.4, $\tilde{\tilde{Q}}_{4gk}$ is an upper-diagonal matrix with non-zero elements along the diagonal, so is non-singular. Therefore, Q_{4gk} is non-singular for a game g that satisfies conditions 4.1.1 and 4.1.4. And therefore the matrix Q_{4g} has full rank if g satisfies conditions 4.1.1 and 4.1.4. Second, consider the case that the econometrician does not allow computational mistakes. In that case, $W = 1$, and $\tilde{\mathrm{P}}_1 = 0$, so $Q_{4gk} = 1$ has full rank.

Then, Q_g is non-singular since all of the diagonal matrices are non-singular.

The first block of $|\mathcal{M}|$ rows of $\eta^*(\theta)$ gives the vector of $((1-\Delta)\Lambda(\mathcal{M}(1)), \ldots, (1-\Delta)\Lambda(\mathcal{M}(|\mathcal{M}|)))$. Therefore, since g is a game that satisfies condition 4.1.2 and 4.1.3, the first block of $\mathcal{M}$ rows of $Q_g\eta^*(\theta)$ is indeed the first block of $\mathcal{M}$ rows of $P_{g,\theta}$ by lemma A.4.1. (And similarly the same would be true if g were a game satisfying conditions 5.1.2 and 5.1.3.) The second block of $|\mathcal{U}|$ rows of $\eta^*(\theta)$ gives the vector of $(\Lambda(\mathcal{U}(1)), \ldots, \Lambda(\mathcal{U}(|\mathcal{U}|)))$. Therefore, by lemma A.4.4, by definition, it follows that the second block of $|\mathcal{U}|$ rows of $Q_g\eta^*(\theta)$ is indeed the second block of $|\mathcal{U}|$ rows of $P_{g,\theta}$. Finally, the last block of $W|\mathcal{M}|$ rows of $\eta^*(\theta)$ gives the vector of $(\Delta\Lambda(\mathcal{M}(1))1[\mathrm{P} = \tilde{\mathrm{P}}_1], \ldots, \Delta\Lambda(\mathcal{M}(1))1[\mathrm{P} = \tilde{\mathrm{P}}_W], \Delta\Lambda(\mathcal{M}(2))1[\mathrm{P} = \tilde{\mathrm{P}}_1], \ldots)$. Also, the last block of $W|\mathcal{M}|$ rows of $P_{g,\theta}$ is $(P_{g,\theta}(C_g(\mathcal{M}(1), \tilde{\mathrm{P}}_1)), \ldots, P_{g,\theta}(C_g(\mathcal{M}(1), \tilde{\mathrm{P}}_W)), P_{g,\theta}(C_g(\mathcal{M}(2), \tilde{\mathrm{P}}_1)), \ldots)$. Therefore, it follows from lemma A.4.3, and the fact that g is a game that satisfies conditions 4.1.1 and 4.1.2 and the definition of Q_{3g}, that indeed the last block of $W|\mathcal{M}|$ rows of $Q_g\eta^*(\theta)$ is indeed the last block of $|W|\mathcal{M}$ rows of $P_{g,\theta}$. □

Proof of Theorem 4.1. Using the game g that satisfies the conditions of assumption 4.2, and lemma A.5, it is possible to identify $\{\tilde{\mathrm{P}}_w\}_{w=1}^{W}$.

Let $\mathcal{G}$ be a subset of $\{1, 2, \ldots, G\}$ with $|\mathcal{G}| \geq 2R - 1$ games that satisfy the conditions of assumption 4.1. Let $\mathcal{G}(p)$ be the p-th smallest element of $\mathcal{G}$. Let $\mathcal{G}_p = \{\mathcal{G}(1), \ldots, \mathcal{G}(p)\}$. Let $Q_{\mathcal{G}}^{(0)} = 1$, and $Q_{\mathcal{G}}^{(p)} = Q_{\mathcal{G}(1)} \otimes \cdots \otimes Q_{\mathcal{G}(p)}$. Let $Q_{\mathcal{G}}$ be the block diagonal matrix with the blocks along the diagonal equal to $Q_{\mathcal{G}}^{(0)}, \ldots, Q_{\mathcal{G}}^{(|\mathcal{G}|)}$. $Q_{\mathcal{G}}$ is non-singular as long as each diagonal block is non-singular. So, since $Q_{\mathcal{G}(p)}$ is non-singular for all p by lemma A.6, which implies that $Q_{\mathcal{G}}^{(p)}$ is non-singular by the algebra of the Kronecker product, $Q_{\mathcal{G}}$ is non-singular.

Let $P_{\mathcal{G},\theta,p} \equiv P_{\mathcal{G}(1),\theta} \otimes \cdots \otimes P_{\mathcal{G}(p),\theta}$. Since actions are independent across games, $P_{\mathcal{G},\theta,p}$ gives the joint distribution of the events $C(\cdot)$, $U(\cdot)$, and $C(\cdot,\cdot)$ across games $\mathcal{G}_p$. Let $P_{\mathcal{G},\theta} = (1, P_{\mathcal{G},\theta,1}, \ldots, P_{\mathcal{G},\theta,|\mathcal{G}|})$. Let $\eta^*(\theta)^{(0)} = 1$ and $\eta^*(\theta)^{(p)} = \eta^*(\theta) \otimes \cdots \otimes \eta^*(\theta)$ be the p-times Kronecker product. Let $\overline{\eta}^*(\theta) = (1, \eta^*(\theta)^{(1)}, \ldots, \eta^*(\theta)^{(|\mathcal{G}|)})$.

Then, using the results of lemma A.6, it follows from the algebra of the Kronecker product that $P_{\mathcal{G},\theta,p} \equiv P_{\mathcal{G}(1),\theta} \otimes \cdots \otimes P_{\mathcal{G}(p),\theta} = (Q_{\mathcal{G}(1)}\eta^*(\theta)) \otimes \cdots \otimes (Q_{\mathcal{G}(p)}\eta^*(\theta)) = (Q_{\mathcal{G}(1)} \otimes \cdots \otimes Q_{\mathcal{G}(p)})(\eta^*(\theta) \otimes \cdots \otimes \eta^*(\theta)) = Q_{\mathcal{G}}^{(p)}\eta^*(\theta)^{(p)}$. Also, $P_{\mathcal{G},\theta} = Q_{\mathcal{G}}\overline{\eta}^*(\theta)$.

Let the true parameters of the data generating process be $\Theta_{01}, \ldots, \Theta_{0\tilde{R}_0}$ and $\pi_0(1), \ldots, \pi_0(\tilde{R}_0)$, where $\tilde{R}_0 \leq R$ is the number of strategic decision making rules that are used in the population and Θ_{0r} is not observationally equivalent to $\Theta_{0r'}$ for all $r \neq r'$ per definition 1. So, by construction, $\pi_0(\cdot) > 0$. Then, by the above, it follows that $P_{\mathcal{G},\Theta_{0r}} = Q_{\mathcal{G}}\overline{\eta}^*(\Theta_{0r})$ for each r. Let $\Upsilon_0^* = \left(\overline{\eta}^*(\Theta_{01}) \ \cdots \ \overline{\eta}^*(\Theta_{0\tilde{R}_0})\right)$. Since no pair of strategic decision making rules are observationally equivalent, by lemma A.3 the columns of Υ_0^* are distinct. Then, $P_{\mathcal{G},0} = Q_{\mathcal{G}}\Upsilon_0^*\pi_0$, where $P_{\mathcal{G},0}$ is the observed joint distribution of actions in games $\mathcal{G}$.

Suppose that there were an observationally equivalent specification of the parameters $\Theta_{1\cdot}$ and $\pi_1(\cdot)$, with corresponding Υ_1^*, such that $P_{\mathcal{G},0} = Q_{\mathcal{G}}\Upsilon_1^*\pi_1$, where again by construction no columns of Υ_1^* correspond to a rule r such that $\pi_1(r) = 0$ and no pair of strategic decision making rules are observationally equivalent. Let $\overline{\Upsilon}^*$ collect the unique columns of $(\Upsilon_0^* \ \Upsilon_1^*)$. Similarly, let $\overline{\pi}$ be the corresponding differences between π_0 and π_1. If column c of $\overline{\Upsilon}^*$ exists in both Υ_0^* and Υ_1^*, as columns c_0 and c_1 respectively, then set $\overline{\pi}_c = \pi_0(c_0) - \pi_1(c_1)$. If column c of $\overline{\Upsilon}^*$ exists only in Υ_0^* as column c_0, then set $\overline{\pi}_c = \pi_0(c_0)$. And if column c of $\overline{\Upsilon}^*$ exists only in Υ_1^* as column c_1, then set $\overline{\pi}_c = -\pi_1(c_1)$. Then, $0 = Q_{\mathcal{G}}\overline{\Upsilon}^*\overline{\pi}$. By lemma A.2, since the number of columns of $\overline{\Upsilon}^*$ is at most $2R$, and $|\mathcal{G}| \geq 2R - 1$, $0 = Q_{\mathcal{G}}T^{-1}T\overline{\Upsilon}^*\overline{\pi}$ where T is non-singular and $T\overline{\Upsilon}^*$ has full column rank. Therefore, $Q_{\mathcal{G}}T^{-1}T\overline{\Upsilon}^*$ has full column rank, so $\overline{\pi} = 0$. Therefore, any strategic decision making rules that appear in specifications 0 and 1 are used with equal probability, and there are no strategic decision making rules used only in specifications 0 and 1, since no elements of π_0 and π_1 are equal to zero by construction.

Therefore, Υ_0^* and Υ_1^* contain exactly the same columns, up to permuting the order of the columns. And, the probabilities of the corresponding strategic decision making rules are also equal across specifications. Note, in particular, this implies that the set of $\eta^*(\Theta_{0r})$ for $r = 1, 2, \ldots, \tilde{R}$ and the set of $\eta^*(\Theta_{1r})$ for $r = 1, 2, \ldots, \tilde{R}$ are equal up to permutations of the labels. Since η^* is "injective" in the sense of lemma A.3, the two specifications of the parameters are the same up to observational equivalence in definition 1 (up to permutations of the labels), so the parameters are point identified in the sense of definition 2. □

Proof of Theorem 5.1. Let $\mathcal{G}_{\mathcal{M}}$ be a subset of $\{1, 2, \ldots, G\}$ with at least $|\mathcal{G}_{\mathcal{M}}| \geq 2R - 1$ games that satisfy the first set of conditions of assumption 5.1. Let $\mathcal{G}_{\mathcal{M}}(p)$ be the p-th smallest element of $\mathcal{G}_{\mathcal{M}}$. Let $\mathcal{G}_{p,\mathcal{M}} = \{\mathcal{G}_{\mathcal{M}}(1), \ldots, \mathcal{G}_{\mathcal{M}}(p)\}$. Let $Q_{\mathcal{G}_{\mathcal{M}}}^{(0)} = 1$, and $Q_{\mathcal{G}_{\mathcal{M}}}^{(p)} = I_{|\mathcal{M}|\times|\mathcal{M}|} \otimes \cdots \otimes I_{|\mathcal{M}|\times|\mathcal{M}|}$ be the p-times Kronecker product of $I_{|\mathcal{M}|\times|\mathcal{M}|}$. Let $Q_{\mathcal{G}_{\mathcal{M}}}$ be the block diagonal matrix with the blocks along the diagonal equal to $Q_{\mathcal{G}_{\mathcal{M}}}^{(0)}, \ldots, Q_{\mathcal{G}_{\mathcal{M}}}^{(|\mathcal{G}_{\mathcal{M}}|)}$. $Q_{\mathcal{G}_{\mathcal{M}}}$ is non-singular as long as each diagonal block is non-singular. So, since $Q_{\mathcal{G}_{\mathcal{M}}}^{(p)}$ is non-singular by the algebra of the Kronecker product, $Q_{\mathcal{G}_{\mathcal{M}}}$ is non-singular.

Let $\mathcal{G}_{\mathcal{U}}$ be a subset of $\{1, 2, \ldots, G\}$ with at least $|\mathcal{G}_{\mathcal{U}}| \geq 2R - 1$ games that satisfy the second set of conditions of assumption 5.1. Let $\mathcal{G}_{\mathcal{U}}(p)$ be the p-th smallest element of $\mathcal{G}_{\mathcal{U}}$. Let $\mathcal{G}_{p,\mathcal{U}} = \{\mathcal{G}_{\mathcal{U}}(1), \ldots, \mathcal{G}_{\mathcal{U}}(p)\}$. Let $Q_{\mathcal{G}_{\mathcal{U}}}^{(0)} = 1$, and $Q_{\mathcal{G}_{\mathcal{U}}}^{(p)} = Q_{2\mathcal{G}_{\mathcal{U}}(1)} \otimes \cdots \otimes Q_{2\mathcal{G}_{\mathcal{U}}(p)}$. Let $Q_{\mathcal{G}_{\mathcal{U}}}$ be the block diagonal matrix with the blocks along the diagonal equal to $Q_{\mathcal{G}_{\mathcal{U}}}^{(0)}, \ldots, Q_{\mathcal{G}_{\mathcal{U}}}^{(|\mathcal{G}_{\mathcal{U}}|)}$.

$Q_{\mathcal{G}_{\mathcal{U}}}$ is non-singular as long as each diagonal block is non-singular. So, since $Q_{2\mathcal{G}_{\mathcal{U}}(p)}$ is non-singular for all p by lemma A.6, which implies that $Q^{(p)}_{\mathcal{G}_{\mathcal{U}}}$ is non-singular by the algebra of the Kronecker product, $Q_{\mathcal{G}_{\mathcal{U}}}$ is non-singular.

Let $P_{\mathcal{G}_{\mathcal{M}}(p),\theta,\mathcal{M}}$ be the first $|\mathcal{M}|$ rows of $P_{\mathcal{G}_{\mathcal{M}}(p),\theta}$. Let $P_{\mathcal{G}_{\mathcal{M}},\theta,p,\mathcal{M}} \equiv P_{\mathcal{G}_{\mathcal{M}}(1),\theta,\mathcal{M}} \otimes \cdots \otimes P_{\mathcal{G}_{\mathcal{M}}(p),\theta,\mathcal{M}}$. Since the actions in the games are independent across games, $P_{\mathcal{G}_{\mathcal{M}},\theta,p,\mathcal{M}}$ gives the joint distribution of the events $C(\cdot)$ across games $\mathcal{G}_{p,\mathcal{M}}$. Let $P_{\mathcal{G}_{\mathcal{M}},\theta,\mathcal{M}} = (1, P_{\mathcal{G}_{\mathcal{M}},\theta,1,\mathcal{M}}, \ldots, P_{\mathcal{G}_{\mathcal{M}},\theta,|\mathcal{G}_{\mathcal{M}}|,\mathcal{M}})$. Let $\eta^*_{\mathcal{M}}(\theta)$ be the first $|\mathcal{M}|$ rows of $\eta^*(\theta)$. Let $\eta^*(\theta)^{(0)} = 1$ and $\eta^*_{\mathcal{M}}(\theta)^{(p)} = \eta^*_{\mathcal{M}}(\theta)\otimes\cdots\otimes\eta^*_{\mathcal{M}}(\theta)$ be the p-times Kronecker product. Let $\overline{\eta}^*_{\mathcal{M}}(\theta) = (1, \eta^*_{\mathcal{M}}(\theta)^{(1)}, \ldots, \eta^*_{\mathcal{M}}(\theta)^{(|\mathcal{G}_{\mathcal{M}}|)})$.

Let $P_{\mathcal{G}_{\mathcal{U}}(p),\theta,\mathcal{U}}$ be rows $|\mathcal{M}|+1$ through $|\mathcal{M}|+|\mathcal{U}|$ of $P_{\mathcal{G}_{\mathcal{U}}(p),\theta}$. Let $P_{\mathcal{G}_{\mathcal{U}},\theta,p,\mathcal{U}} \equiv P_{\mathcal{G}_{\mathcal{U}}(1),\theta,\mathcal{U}}\otimes\cdots\otimes P_{\mathcal{G}_{\mathcal{U}}(p),\theta,\mathcal{U}}$. Since the actions in the games are independent across games, $P_{\mathcal{G}_{\mathcal{U}},\theta,p,\mathcal{U}}$ gives the joint distribution of the events $U(\cdot)$ across games $\mathcal{G}_{p,\mathcal{U}}$. Let $P_{\mathcal{G}_{\mathcal{U}},\theta,\mathcal{U}} = (1, P_{\mathcal{G}_{\mathcal{U}},\theta,1,\mathcal{U}}, \ldots, P_{\mathcal{G}_{\mathcal{U}},\theta,|\mathcal{G}_{\mathcal{U}}|,\mathcal{U}})$. Let $\eta^*_{\mathcal{U}}(\theta)$ be rows $|\mathcal{M}|+1$ through $|\mathcal{M}|+|\mathcal{U}|$ of $\eta^*(\theta)$. Let $\eta^*(\theta)^{(0)} = 1$ and $\eta^*_{\mathcal{U}}(\theta)^{(p)} = \eta^*_{\mathcal{U}}(\theta)\otimes\cdots\otimes\eta^*_{\mathcal{U}}(\theta)$ be the p-times Kronecker product. Let $\overline{\eta}^*_{\mathcal{U}}(\theta) = (1, \eta^*_{\mathcal{U}}(\theta)^{(1)}, \ldots, \eta^*_{\mathcal{U}}(\theta)^{(|\mathcal{G}_{\mathcal{U}}|)})$.

Then, using the results of lemma A.6, it follows from the algebra of the Kronecker product that $P_{\mathcal{G}_{\mathcal{M}},\theta,p,\mathcal{M}} \equiv P_{\mathcal{G}_{\mathcal{M}}(1),\theta,\mathcal{M}} \otimes\cdots\otimes P_{\mathcal{G}_{\mathcal{M}}(p),\theta,\mathcal{M}} = (I_{|\mathcal{M}|\times|\mathcal{M}|}\eta^*_{\mathcal{M}}(\theta)) \otimes\cdots\otimes (I_{|\mathcal{M}|\times|\mathcal{M}|}\eta^*_{\mathcal{M}}(\theta)) = (I_{|\mathcal{M}|\times|\mathcal{M}|}\otimes\cdots\otimes I_{|\mathcal{M}|\times|\mathcal{M}|})(\eta^*_{\mathcal{M}}(\theta)\otimes\cdots\otimes\eta^*_{\mathcal{M}}(\theta)) = Q^{(p)}_{\mathcal{G}_{\mathcal{M}}}\eta^*_{\mathcal{M}}(\theta)^{(p)}$. Also, $P_{\mathcal{G}_{\mathcal{M}},\theta,\mathcal{M}} = Q_{\mathcal{G}_{\mathcal{M}}}\overline{\eta}^*_{\mathcal{M}}(\theta)$.

Similarly, using the results of lemma A.6, it follows from the algebra of the Kronecker product that $P_{\mathcal{G}_{\mathcal{U}},\theta,p,\mathcal{U}} \equiv P_{\mathcal{G}_{\mathcal{U}}(1),\theta,\mathcal{U}} \otimes\cdots\otimes P_{\mathcal{G}_{\mathcal{U}}(p),\theta,\mathcal{U}} = (Q_{2\mathcal{G}_{\mathcal{U}}(1)}\eta^*_{\mathcal{U}}(\theta)) \otimes\cdots\otimes (Q_{2\mathcal{G}_{\mathcal{U}}(p)}\eta^*_{\mathcal{U}}(\theta)) = (Q_{2\mathcal{G}_{\mathcal{U}}(1)} \otimes\cdots\otimes Q_{2\mathcal{G}_{\mathcal{U}}(p)})(\eta^*_{\mathcal{U}}(\theta)\otimes\cdots\otimes\eta^*_{\mathcal{U}}(\theta)) = Q^{(p)}_{\mathcal{G}_{\mathcal{U}}}\eta^*_{\mathcal{U}}(\theta)^{(p)}$. Also, $P_{\mathcal{G}_{\mathcal{U}},\theta,\mathcal{U}} = Q_{\mathcal{G}_{\mathcal{U}}}\overline{\eta}^*_{\mathcal{U}}(\theta)$.

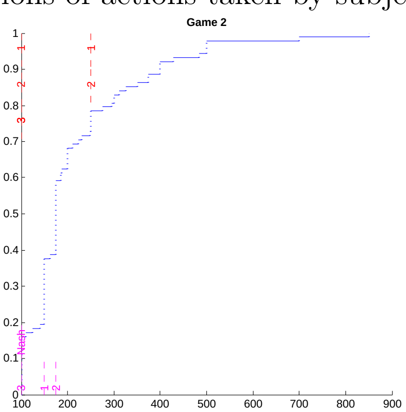
Then, let $\tilde{P}_{\mathcal{G}_{\mathcal{M}},\mathcal{G}_{\mathcal{U}},\theta} = (P_{\mathcal{G}_{\mathcal{M}},\theta,\mathcal{M}}, P_{\mathcal{G}_{\mathcal{U}},\theta,\mathcal{U}})$. Let $\overline{\eta}^*_{\mathcal{M},\mathcal{U}}(\theta) = (\overline{\eta}^*_{\mathcal{M}}(\theta), \overline{\eta}^*_{\mathcal{U}}(\theta))$. And let $Q_{\mathcal{G}_{\mathcal{M}},\mathcal{G}_{\mathcal{U}}}$ be the partioned matrix with $(Q_{\mathcal{G}_{\mathcal{M}}}, Q_{\mathcal{G}_{\mathcal{U}}})$ along the diagonal.

Let the true parameters of the data generating process be $\Theta_{01}, \ldots, \Theta_{0\tilde{R}_0}$ and $\pi_0(1), \ldots, \pi_0(\tilde{R}_0)$, where $\tilde{R}_0 \leq R$ is the number of strategic decision making rules that are used in the population and Θ_{0r} is not observationally equivalent ignoring the magnitude of computational mistakes to $\Theta_{0r'}$ for all $r \neq r'$ per definition 3. So, by construction, $\pi_0(\cdot) > 0$. Then, by the above, it follows that $P_{\mathcal{G}_{\mathcal{M}},\Theta_{0r},\mathcal{M}} = Q_{\mathcal{G}_{\mathcal{M}}}\overline{\eta}^*_{\mathcal{M}}(\Theta_{0r})$ and $P_{\mathcal{G}_{\mathcal{U}},\Theta_{0r},\mathcal{U}} = Q_{\mathcal{G}_{\mathcal{U}}}\overline{\eta}^*_{\mathcal{U}}(\Theta_{0r})$. Let $\Upsilon^*_{0,\mathcal{M}} = \left(\overline{\eta}^*_{\mathcal{M}}(\Theta_{01}) \ \cdots \ \overline{\eta}^*_{\mathcal{M}}(\Theta_{0\tilde{R}_0})\right)$ and $\Upsilon^*_{0,\mathcal{U}} = \left(\overline{\eta}^*_{\mathcal{U}}(\Theta_{01}) \ \cdots \ \overline{\eta}^*_{\mathcal{U}}(\Theta_{0\tilde{R}_0})\right)$. By assumption 5.2, the columns of $\Upsilon^*_{0,\mathcal{M}}$ are distinct and the columns of $\Upsilon^*_{0,\mathcal{U}}$ are distinct. Then, $P_{\mathcal{G}_{\mathcal{M}},0,\mathcal{M}} = Q_{\mathcal{G}_{\mathcal{M}}}\Upsilon^*_{0,\mathcal{M}}\pi_0$, where $P_{\mathcal{G}_{\mathcal{M}},0,\mathcal{M}}$ is the observed joint distribution of actions in games $\mathcal{G}_{\mathcal{M}}$. And, $P_{\mathcal{G}_{\mathcal{U}},0,\mathcal{U}} = Q_{\mathcal{G}_{\mathcal{U}}}\Upsilon^*_{0,\mathcal{U}}\pi_0$, where $P_{\mathcal{G}_{\mathcal{U}},0,\mathcal{U}}$ is the observed joint distribution of actions in games $\mathcal{G}_{\mathcal{U}}$.

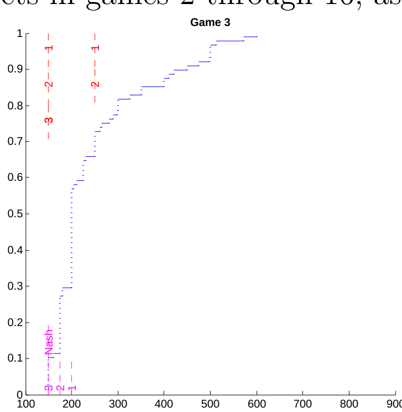
Suppose that there were an observationally equivalent specification of the parameters $\Theta_{1\cdot}$ and $\pi_1(\cdot)$, with corresponding $\Upsilon^*_{1,\mathcal{M}}$ and $\Upsilon^*_{1,\mathcal{U}}$, such that $P_{\mathcal{G}_{\mathcal{M}},0,\mathcal{M}} = Q_{\mathcal{G}_{\mathcal{M}}}\Upsilon^*_{1,\mathcal{M}}\pi_1$ and $P_{\mathcal{G}_{\mathcal{U}},0,\mathcal{U}} = Q_{\mathcal{G}_{\mathcal{U}}}\Upsilon^*_{1,\mathcal{U}}\pi_1$, where again by construction the columns of $\Upsilon^*_{1,\mathcal{M}}$ and the columns of $\Upsilon^*_{1,\mathcal{U}}$ are distinct, and no columns correspond to a rule r such that $\pi_1(r) = 0$. By the same arguments as finishes the proof of theorem 4.1, since $|\mathcal{G}_{\mathcal{M}}| \geq 2R - 1$ and $|\mathcal{G}_{\mathcal{U}}| \geq 2R - 1$, $(\pi(r), (1 - \Delta_r)\Lambda_r(\mathcal{M}(1)), \ldots, (1 - \Delta_r)\Lambda_r(\mathcal{M}(|\mathcal{M}|)))$ and $(\pi(r), \Lambda_r(\mathcal{M}(1)), \ldots, \Lambda_r(\mathcal{M}(|\mathcal{M}|)))$ are point identified up to permutations of the labels in the sense that the values of those two quantities must be equal across specifications of the parameters, up to permutations of the labels. And then, since $\pi(r)$ and $\pi(r')$ are distinct for $r' \neq r$ by assumption 5.2, it is possible to point identify $(\pi(r), (1 - \Delta_r)\Lambda_r(\mathcal{M}(1)), \ldots, (1 - \Delta_r)\Lambda_r(\mathcal{M}(|\mathcal{M}|)), \Lambda_r(\mathcal{M}(1)), \ldots, \Lambda_r(\mathcal{M}(|\mathcal{M}|)))$, in the sense that that quantity must be equal across specifications of the parameters, up to permutations of the labels, by "piecing together" the two point identification results on $(\pi(r), (1-\Delta_r)\Lambda_r(\mathcal{M}(1)), \ldots, (1-\Delta_r)\Lambda_r(\mathcal{M}(|\mathcal{M}|)))$ and $(\pi(r), \Lambda_r(\mathcal{M}(1)), \ldots, \Lambda_r(\mathcal{M}(|\mathcal{M}|)))$.

Note, in particular, this implies that the set of $\eta^{**}(\Theta_{0r})$ for $r = 1, 2, \ldots, \tilde{R}$ and the set of $\eta^{**}(\Theta_{1r})$ for $r = 1, 2, \ldots, \tilde{R}$ are equal up to permutations of the labels. Since η^{**} is "injective" in the sense of lemma A.3, the two specifications of the parameters are the same up to observational equivalence in definition 3 (up to permutations of the labels), so the parameters are point identified in the sense of definition 4. □

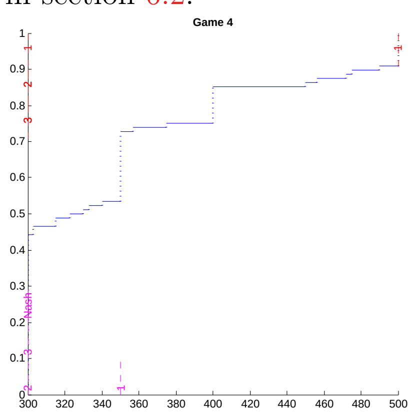
A.3. **Appendix: other games.** The following are empirical cumulative distribution functions of actions taken by subjects in games 2 through 16, as in section 6.2.



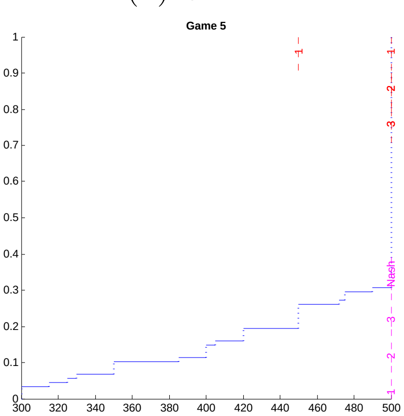
(A) Game 2



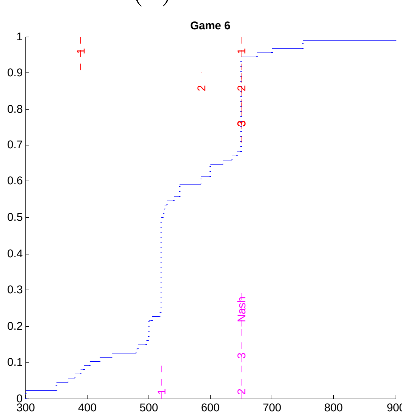
(B) Game 3



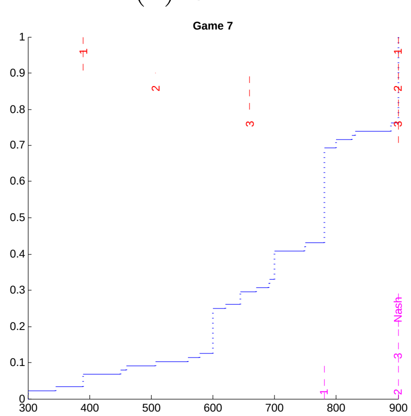
(C) Game 4



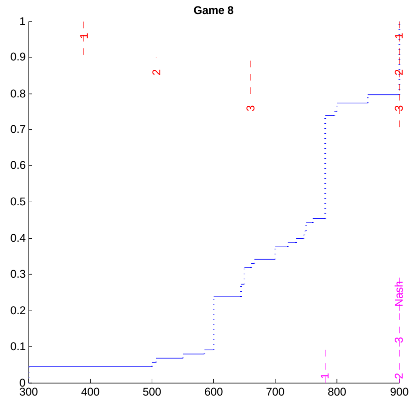
(A) Game 5



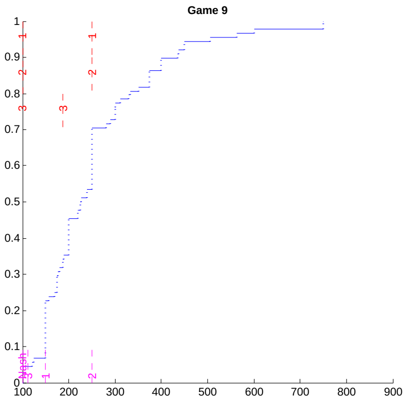
(B) Game 6



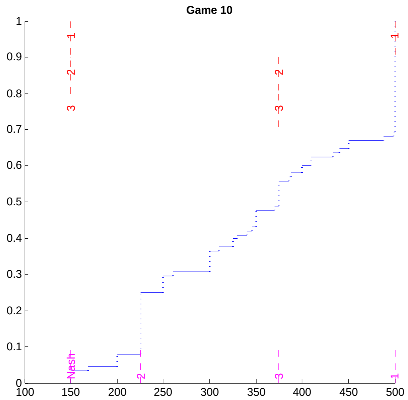
(C) Game 7



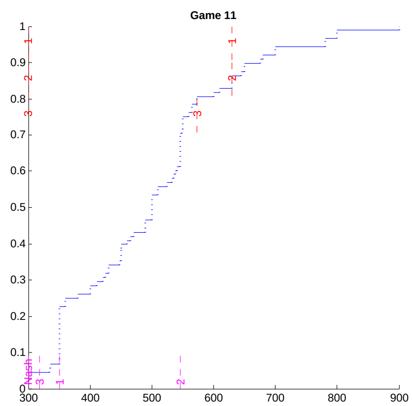
(A) Game 8



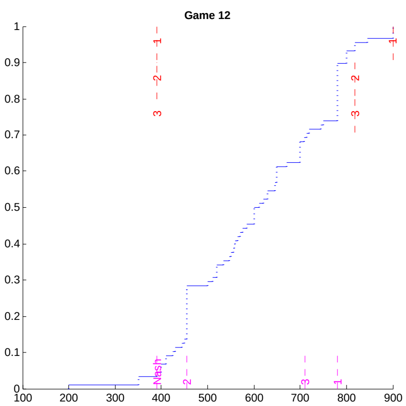
(B) Game 9



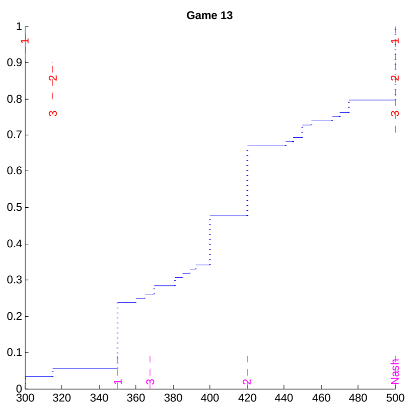
(C) Game 10



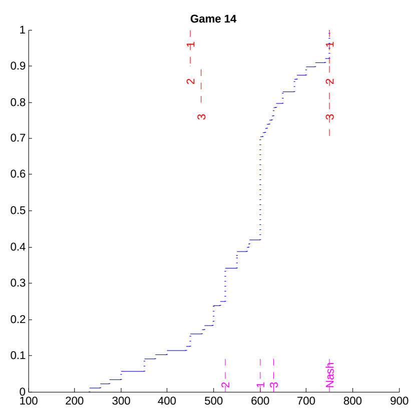
(A) Game 11



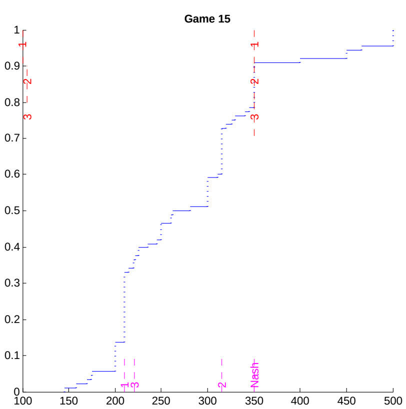
(B) Game 12



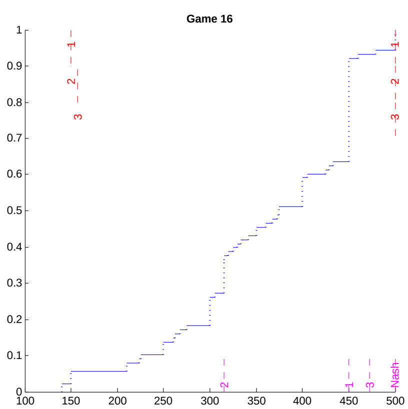
(C) Game 13



(A) Game 14



(B) Game 15



(C) Game 16

A.4. **Appendix: results of the model allowing computational mistakes.** Table 4 reports estimates of the model allowing uniformly distributed computational mistakes. It is assumed that $\mathrm{P}_r = \frac{2.5}{200}$ for all rules r.[25] The results are almost identical to the model not allowing computational mistakes, and estimates of Δ_r are close to zero for all r.

	Λ					Probability of...	
	Anchored reasoning		Unanchored reasoning			type	mistake
	1	2	0	1	Nash		
r	$\Lambda_r(1_{anch})$	$\Lambda_r(2_{anch})$	$\Lambda_r(0_{unanch})$	$\Lambda_r(1_{unanch})$	$\Lambda_r(NE)$	$\pi(r)$	Δ_r
1	0.10 (0.08, 0.12)	0.04 (0.03, 0.06)	0.49 (0.40, 0.54)	0.31 (0.25, 0.40)	0.07 (0.04, 0.10)	0.44 (0.39, 0.55)	0.00 (0.00, 0.00)
2	0.70 (0.59, 0.78)	0.00 (0.00, 0.00)	0.15 (0.09, 0.25)	0.11 (0.06, 0.18)	0.04 (0.02, 0.06)	0.20 (0.14, 0.28)	0.00 (0.00, 0.00)
3	0.21 (0.00, 0.31)	0.44 (0.40, 0.79)	0.10 (0.00, 0.19)	0.20 (0.00, 0.32)	0.05 (0.00, 0.09)	0.15 (0.10, 0.24)	0.07 (0.00, 0.12)
4	0.05 (0.01, 0.08)	0.04 (0.00, 0.07)	0.05 (0.00, 0.08)	0.40 (0.32, 0.50)	0.46 (0.42, 0.60)	0.14 (0.08, 0.20)	0.00 (0.00, 0.00)
5	0.09 (0.00, 0.16)	0.89 (0.86, 1.00)	0.00 (0.00, 0.00)	0.02 (0.00, 0.04)	0.00 (0.00, 0.00)	0.06 (0.00, 0.08)	0.00 (0.00, 0.00)

See notes to table 3.

TABLE 4. Estimates

[25]Theorem 4.1 establishes that the magnitude of the computational mistakes P_r is identified if $\Delta_r > 0$. That is required because if $\Delta_r = 0$ for some rule r, then that rule does not make computational mistakes, so P_r has no observable implications. This is not a concern based on theorem 5.1, which applies when P_r are known by the econometrician. The estimates of Δ_r's are very close to 0, which makes identification and estimation of the corresponding P_r's very tenuous. Indeed, precisely because of the small values of the Δ_r's, the P_r's are essentially irrelevant.