Linear Inequality Concepts and Social Welfare

by

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Abstract

The paper presents ab abstract definition of linear inequality concepts leading to linearly invariant measures and characterizes the class of linear concepts completely. Two general methods of deriving ethical measures are proposed. They imply an Atkinson-Kolm-Sen index and a new dual index reflecting the inequality of living standard. Then all separable social welfare orderings which generate linearly invariant measures are characterized. The measures are presented and their general properties are discussed. Dual measures prove to be additively decomposable. Linear welfare orderings defined on rank-ordered income vectors are examined. They are consistent with all linear inequality concepts and yield an inequality ordering for every concept.

Keywords: Linear inequality concepts; social welfare; ethical measures.

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1. Introduction[1]

Twenty years ago Kolm (1976a, 1976b) raised a number of problems which stimulated a lot of research on inequality measurement. He was mainly interested in the concept of inequality, the relationship between inequality and social welfare, and the characterization of inequality measures and social welfare functions by means of axioms. The present paper contributes to this discussion. First it presents a new (abstract) approach of defining inequality concepts and describes the class of linear concepts completely. Second, it investigates the relation between inequality and welfare for such concepts and proposes two general methods of deriving corresponding ethical inequality indices in this framework. One type of index is related to the familiar Atkinson-Kolm-Sen measure. The other method yields a new type of (dual) index reflecting the inequality of living standard. Third, all separable social welfare orderings which generate inequality measures compatible with linear inequality concepts are characterized; the respective measures are presented. Fourth, the paper discusses the general properties of the measures introduced. The dual measures are shown to be additively decomposable. Fifth, the particular role of linear social welfare orderings (defined on rank-ordered income vectors) is examined. They are consistent with all linear inequality concepts and therefore imply an inequality ordering for every concept.

Economists usually distinguish between the size and the distribution of income; i.e. inequality does not depend on average income. On the other hand, certain changes in income are admitted which leave inequality unchanged. For the concept of relative inequality, the ranking of incomes and relative inequality measures are invariant with respect to equal proportional changes of incomes; for absolute inequality, the addition of the same amount to all incomes does not alter inequality. These concepts of rightist and leftist inequality are well known. Most researchers seem to be interested in and adhere to relative (rightist) inequality. Nevertheless there has always been some discussion in the literature whether there exist further reasonable concepts: Kolm (1976a + b)already proposed centrist measures; Bossert and Pfingsten (1990) introduced the concept of intermediate inequality. Pfingsten/Seidl (1994) and del Rio/Ruiz-Castillo (1996) deal with other new (more complicated) concepts.

[1] I am grateful to Patrick Moyes, Oskar von dem Hagen, and John Weymark for valuable comments and discussions. The paper was finished while I was enjoying the hospitality of the LSE. The paper is partially based on an unpublished discussion paper Ebert (1996).

Furthermore, several experimental tests by questionaire have been performed in order to reveal attitudes to inequality (Amiel/Cowell (1992, 1997), Ballano/Ruiz-Castillo (1993), Harrison/ Seidl (1994a, 1994b)). The questionaires are designed to investigate the acceptance of the conventional assumptions employed in inequality measurement. In particular the support of the relative, absolute, and intermediate views of inequality is examined by these authors. They unanimously conclude that none of these concepts obtains universal consent. Some individuals accept the relative, others the absolute or the intermediate inequality concept. Even extreme rightist and leftist views are supported. Furthermore the attitude to inequality depends on the size of income: the relative inequality concept may be acceptable for low incomes, the absolute one for high incomes (or conversely). Therefore it seems necessary to investigate inequality concepts in further detail and to make an attempt at providing a theoretical model which might describe the empirical findings. This paper unifies some of the approaches by providing an abstract and general definition of an inequality concept. It makes the following contributions:

At first an inequality concept will be described by a *set of feasible transformations* leaving inequality unaltered and defining an equivalence relation at the same time. A transformation changes an income distribution in a systematic way (e.g. by scaling all incomes for relative inequality) and generates another income distribution possessing the same degree of inequality. The kind of transformations admitted is uniquely related to the view of inequality and reflects normative judgments. It is obvious that the set of transformations has to satisfy some specific properties if the definition is to make sense. Essentially, it has to be sufficiently rich and the transformations have to be consistent with one another. If an inequality measure is invariant with respect to such a set of transformations it satisfies the respective concept of inequality. Below, as a first step linear inequality concepts are considered; i.e. the corresponding set contains only *linear* transformations; the inequality measures are linearly invariant. It turns out that the class of linear concepts comprises relative, absolute, intermediate, and ‘ultra-rightist’ inequality. The latter one favours rich persons: even if the ratio of a high income to a low income increases inequality remains unchanged. Probably few people will agree with the implicit value judgement.

In the next step the relationship between social welfare orderings and inequality measures is examined. This part of the paper extends the methodology to linear concepts and to the abstract setting. Blackorby/Donaldson (1978), Ebert (1987a) considered relative inequality. Ebert (1987b) treated absolute inequality as well. Dutta/Esteban (1992) discussed some problems for relative inequality. Up to now there is no paper dealing rigorously with intermediate (and ultra-rightist) inequality (the way Bossert and Pfingsten handle this problem seems to be too ad hoc). The basic idea used below is to standardize income distributions and then to consider the correponding welfare loss due to inequality. This methodology can be adapted suitably and yields two types of measure depending on the way of standardization: If mean income is normalized to a given standard, in principle generalized Atkinson-Kolm-Sen measures (appropriately adapted to linear concepts) emerge. If distributions are normalized to a given level of social welfare, a *new type of index* is generated: a dual inequality measure, which is mainly based on the representative income instead of mean income. It has been mentioned in the literature, but has never been defined and treated rigorously.

Then both types of *normative* inequality measures are derived in this framework. They are (ordinally) unique and belong to a linear inequality concept if the underlying social welfare ordering is homogeneous with respect to the corresponding set of transformations. Therefore it is important to characterize the class of social welfare orderings which fulfill the property of homogeneity. For separable orderings these classes can be determined. Of course, one obtains the class of Atkinson and Kolm-Pollak orderings for relative and absolute inequality. For intermediate and ultra-rightist inequality a modification of Atkinson welfare orderings is required.

Both types of measure derived possess the usual properties: They are anonymous, satisfy the principle of progressive transfers, are positive and equal zero only at equality of incomes. An interesting result is that both are ordinally equivalent: Though they are derived in different ways they provide only different cardinalizations of the same inequality ordering. In so far nothing new is implied. But the dual measures prove to be *additively decomposable*. They inherit this property directly from the social welfare ordering. In this case the between-group term is defined on a smoothed income distribution consisting of the representative - not mean! - income of the respective groups. This reinforces the fact that the dual measures reflect the inequality of living standard.

Finally linear social welfare orderings defined on *rank-ordered income vectors* are considered (e.g. the welfare ordering generating the popular Gini-coefficient possesses this form). Since the analysis is restricted to linear inequality concepts a linear ordering is homogeneous with respect to all these transformations independently of the particular concept. Therefore for each inequality concept a corresponding inequality measure (or ordering) can be generated. In this case the researcher has got an additional degree of freedom. She can choose a linear welfare ordering according to her view on social welfare and independently decide in favour of a particular concept of inequality.

The results of this paper demonstrate that defining an inequality concept by means of a set of suitable transformations is worthwhile. The abstract framework necessitates a new and general approach to many problems. In particular the dual measures derived will prove helpful whenever inequality is to be decomposed.

The paper is organized as follows: Section 2 reconsiders the inequality concepts proposed in the literature. Section 3 defines linear inequality concepts; section 4 determines all linear concepts. Social welfare orderings are introduced in section 5. Then ethical inequality measures are defined (section 6) and investigated (section 7). The next section derives all ethical inequality measures which belong to a linear inequality concept if the welfare ordering is separable. The decomposition of dual measures is examined in section 9. Finally, section 10 is devoted to linear social welfare orderings and section 11 offers some conclusions.

2. Inequality concepts reconsidered

We consider a fixed population consisting of n individuals. They are supposed to be identical with respect to all attributes but possibly income. Let $\Omega \subset R$ denote the set (interval) of feasible incomes. Its definition will depend on the inequality concept chosen. Then $X_i \in \Omega$ denotes individual i's income $(i = 1, \dots, n)$. An income distribution is given by a vector $X = (X_1, \dots, X_n)$ belonging to the set of feasible income distributions $\Omega^n \subset R^n$ which will be defined below. The average income of X is abbreviated by $\mu(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$. **1** denotes a vector containing n ones.

There are essentially three concepts of inequality dealt with in the literature. Each concept is characterized by an invariance property of the respective measures and orderings. For the relative inequality concept inequality is unchanged if all incomes are changed in the same proportion; i.e. a relative inequality measure $I(X)$ satisfies

$$I(X) = I\big(T_\lambda(X)\big) \quad \text{for all} \quad X \in \Omega^n_{rel} := R^n_{++} \quad \text{and} \quad \lambda > 0,$$

where T_λ belongs to

$$\mathrm{T}^*_{rel} = \left\{ T_\lambda : \Omega^n_{rel} \to \Omega^n_{rel} \middle| T_\lambda(X) = \lambda \cdot X, \quad \lambda \in R_{++} \right\}.$$

Similarly an absolute inequality measure is characterized by

$$I(X) = I(T_\alpha(X)) \quad \text{for all} \quad X \in \Omega^n_{abs} := R^n \quad \text{and} \quad \alpha \in R$$

where

$$T_\alpha \in \mathrm{T}^*_{abs} = \left\{ T_\beta : \Omega^n_{abs} \to \Omega^n_{abs} \middle| T_\beta(X) = X + \beta \cdot \mathbf{1}, \quad \beta \in R \right\}.$$

Here inequality is not altered if all individuals receive the same amount of income in addition to their income X_i. These inequality concepts are well-known and seem to present two extreme views of inequality. Kolm (1976 a) calls the respective measures rightist and leftist. For relative inequality the ratio of two individuals' incomes remains fixed, for absolute inequality the difference. Bossert/Pfingsten (1990) filled the gap between both extremes by introducing the concept of intermediate inequality. An inequality measure defined on Ω^n_θ satisfies the concept of θ-inequality for $\theta \in (0,1)$ if it is invariant with respect to the set of transformations

$$\mathrm{T}^*_\theta = \left\{ T_\kappa : \Omega^n_\theta \to \Omega^n_\theta \middle| T_\kappa(X) = X + \kappa \cdot (\theta \cdot X + (1-\theta) \cdot \mathbf{1}) \quad \text{for} \quad \kappa > -1/\theta \right\}$$

where $\Omega_\theta := (-1/s, \infty)$ for $s := (1-\theta)/\theta$.

A transformation T_κ is a combination of scaling and translating incomes. In this sense the θ-inequality concept is an intermediate one: setting $\theta = 0$ we obtain $\mathrm{T}^*_\theta = \mathrm{T}^*_{abs}$, for $\theta = 1$ we get $\mathrm{T}^*_\theta = \mathrm{T}^*_{rel}$. Furthermore one can prove that for an increase in income by means of a transformation $\mathrm{T}_\kappa (\kappa > 0)$ θ-inequality is unchanged, relative inequality measures register a decrease and absolute ones an increase of inequality.

Obviously the inequality measures are invariant with respect to the transformations admitted. Conversely, the respective set of transformations defines an equivalence relation. Consider e.g. the case of relative inequality and define

$$X \sim_{\mathrm{T}^*_{rel}} Y :\Leftrightarrow \quad \text{there exists} \quad T_\lambda \in \mathrm{T}^*_{rel} \quad \text{such that} \quad Y = T_\lambda(X).$$

$X \sim_{\mathrm{T}^*_{rel}} Y$ is an equivalence relation since it satisfies reflexivity, transitivity, and symmetry:

$$X \sim_{\mathrm{T}^*_{rel}} X \quad \text{since} \quad X = T_1(X),$$

$$X \sim_{T^*_{rel}} Y \quad \text{and} \quad Y \sim_{T^*_{rel}} Z \quad \text{implies} \quad X \sim_{T^*_{rel}} Z$$

since, if $Y = T_\lambda(X)$ and $Z = T_\nu(Y)$, then $Z = T_\nu\big(T_\lambda(X)\big) = T_{\nu\cdot\lambda}(X)$.

Finally, $X \sim_{T^*_{rel}} Y$ implies $Y \sim_{T^*_{rel}} X$ since $X = T_\lambda(Y)$ yields $Y = T_{1/\lambda}(X)$.

Therefore, by definition X and Y possess the same degree of inequality whenever $X \sim_{T^*_{rel}} Y$.

One can argue analogously for absolute and intermediate inequality. The set of transformations admitted forms a basic characteristic of the respective inequality concept.

3. Definition of linear inequality concepts

The above inequality concepts were characterized by a corresponding invariance property of inequality measures. In the following we will *define* inequality concepts by a corresponding set of transformations. We postulate that inequality is not affected if an income distribution is changed by a feasible transformation. It is obvious that the set of transformations has to fulfill some properties if the definition is to be reasonable. These properties will be discussed in this section . At first we introduce the set of transformations

$$T^* = \left\{ T_\tau : \Omega^n \to \Omega^n \,\middle|\, \tau \in D \right\},$$

where the set of parameters D must be a nondegenerate interval of R. Therefore the set of transformations can be characterized by one parameter. The parametrization is not unique, but this aspect is not important in the following.

In this paper we confine ourselves to *linear* transformations satisfying

Property LIN (linear transformation)

There are two continuous functions, at least one strictly increasing, $a: D \to R_{++}$ *and* $b: D \to R$ *such that*

$$T_\tau(X) = a(\tau)\cdot X + b(\tau)\cdot \mathbf{1} \quad \textit{for} \quad X \in \Omega^n \quad \textit{and} \quad \tau \in D. \tag{1}$$

A transformation is defined for income vectors X. The same notation will be employed for a transformation of its components since there will be no ambiguity:

$$\left[T_\tau(X)\right]_i := T_\tau(X_i) \qquad i = 1, \dots, n.$$

$T_\tau(X)$ is continuous and strictly increasing in *X* and continuous in the parameter τ. Since *D* is connected the set $\{T_\tau(X) | \tau \in D\}$ is also a connected subset of Ω^n for each *X*. It contains all income distributions possessing the same inequality as distribution *X* according to the inequality concept defined by T^*. Transformations satisfying LIN are linear in *X*. One easily recognizes that the concepts introduced in section 2 fulfill LIN. We observe that

$$a(\lambda) = \lambda \quad \text{and} \quad b(\lambda) = 0 \quad \text{for} \quad \mathrm{T}^*_{rel},$$
$$a(\alpha) = 1 \quad \text{and} \quad b(\alpha) = \alpha \quad \text{for} \quad \mathrm{T}^*_{abs}, \quad \text{and}$$
$$a(\kappa) = (1 + \kappa\theta) \quad \text{and} \quad b(\kappa) = \kappa(1 - \theta) \quad \text{for} \quad \mathrm{T}^*_\theta \quad \text{and} \quad \theta \in (0,1).$$

By definition $T_\tau(X)$ possesses the same inequality as *X*; but of course the reverse should be true as well. In other words, ‘possessing the same inequality’ should be an equivalence relation. Then three more properties are required:

Property REF (reflexivity)

The identity T_e*, defined by* $T_e(X) = X$ *for all* $X \in \Omega^n$*, belongs to* T^*.

Property TRANS (transitivity)

If $T_\sigma, T_\tau \in \mathrm{T}^*$*, then the transformation* $T_{\sigma \circ \tau}(X) := T_\sigma\left(T_\tau(X)\right)$ *belongs to* T^*.

Property SYM (symmetry)

If $T_\tau \in \mathrm{T}^*$*, then its inverse* T_τ^{-1} *belongs to* T^*.

They guarantee that T^* defines an equivalence relation on Ω^n if one defines

$$X \sim_{\mathrm{T}^*} Y \quad :\Leftrightarrow \quad \text{there is } T_\tau \in \mathrm{T}^* \text{ such that } Y = T_\tau(X). \tag{2}$$

REF implies reflexivity, TRANS transitivity and SYM the symmetry of the relation. As a consequence $\{T_\tau(X) | \tau \in D\}$ is really an equivalence class of $\sim_{\mathrm{T}^*}$. Furthermore, the set T^*

forms a group (in the mathematical sense) if the composition of transformations is taken as operation.

Now we are able to introduce the

Definition

A linear inequality concept is defined by a set of transformations T^* *satisfying LIN, REF, TRANS, and SYM.*

Relative, absolute and θ-intermediate inequality are linear inequality concepts since the properties REF, TRANS, and SYM are fulfilled:

$$T_1(X) = X, \quad T_{\lambda\cdot\mu} = T_\lambda \circ T_\mu \quad \text{and} \quad T_\lambda^{-1} = T_{1/\lambda}, \text{ (see the discussion above)}$$

$$T_0(X) = X, \quad T_{\alpha+\beta} = T_\alpha \circ T_\beta \quad \text{and} \quad T_\alpha^{-1} = T_{-\alpha},$$

$$T_0(X) = X, \quad T_\kappa \circ T_\nu = T_\rho \quad \text{with} \quad \rho = \kappa + \nu + \kappa \cdot \nu \cdot \theta$$

$$\text{and} \quad T_\kappa^{-1} = T_\sigma \quad \text{with} \quad \sigma = -\kappa / (1+\kappa\theta).$$

The next section determines *all* linear inequality concepts.

4. The class of linear inequality concepts

The properties defining a linear inequality concept seem to be weak at first sight but have strong implications. They impose some structure on the set of transformations. We establish

Proposition 1

A set of transformations T^* *satisfies LIN, REF, TRANS, and SYM if and only if there is a parameterization of* T^* *such that either*

(i) $\quad a(\tau) = 1 \quad and \quad b(\tau) = \tau \quad for \quad \tau \in D = R \quad and \quad \Omega^n = R^n$ *(type (i))* (3)

or

(ii) $\quad a(\tau) = \tau \quad and \quad b(\tau) = d(1-\tau) \quad for \quad \tau \in D = R_{++},$

an arbitrary $d \in R$, *and* $\Omega^n = \Omega_d^n$ *where* $\Omega_d := (d, \infty)$ *(type (ii))* (4)

Obviously there are not many possibilities of choosing $a(\tau)$ and $b(\tau)$. Essentially there are only two types of linear inequality concepts. Type (i) corresponds to absolute inequality. Type (ii) depends on the additional parameter *d*. For $d = 0$ we obtain the concept of relative inequality. If *d* is negative θ-inequality is implied since

$$T_{\kappa}(X) = X + \kappa \cdot (\theta \cdot X + (1-\theta) \cdot \mathbf{1})$$

can be rearranged to

$$T_{\tau}(X) = \tau \cdot X + d \cdot (1-\tau) \cdot \mathbf{1} \quad \text{for} \quad \tau = 1 + \kappa \cdot \theta \quad \text{and} \quad d = -(1-\theta)/\theta .$$

Then T^* is identical with T^*_{θ} for $\theta \in (0,1)$ and Ω^n_d with Ω^n_{θ}.

New concepts are given for positive parameters *d*. Their respective set of transformations is denoted by T^*_d and their domain by Ω_d ; i.e. the set of feasible income distributions Ω^n_d forms a subset of the positive orthant. (The notation T^*_d will also be employed if $d < 0$.) A close inspection of these transformations demonstrates that these concepts are even 'more rightist' than that of relative inequality. Suppose e.g. that *d* equals unity: $d = 1$ and choose the income distribution $X = (2000, 4000)$. Application of T_{τ} for $\tau = 2$ implies $T_{\tau}(X) = (3000, 7000)$. According to this concept both distributions possess the same degree of inequality. The lower income is increased by 50 %, the higher one by 75 %. Therefore these concepts are called d-ultra-rightist. There will probably be few adherents of these concepts.

Summarizing this discussion we obtain

Proposition 2

The class of linear inequality concepts comprises (only) the concepts of relative, absolute, θ-intermediate, and d-ultra-rightist inequality.

Thus the class can be completely described. Now we prove the above result:

Proof of Proposition 1

Choose any transformations $T_{\sigma}, T_{\tau} \in \mathrm{T}^*$. LIN implies

$$T_{\sigma}(T_{\tau}(X)) = a(\sigma) \cdot a(\tau) \cdot X + (a(\sigma) \cdot b(\tau) + b(\sigma)) \cdot \mathbf{1}.$$

Now define T_ρ by $T_\rho := T_\sigma \circ T_\tau$ and κ by $\rho = \kappa(\sigma, \tau)$. By LIN T_ρ must possess the form (1). Because of TRANS it belongs to T^*. Therefore the functions a and b have to satisfy the functional equations

$$a(\rho) = a\big(\kappa(\sigma, \tau)\big) = a(\sigma) \cdot a(\tau) \tag{5}$$

$$b(\rho) = b\big(\kappa(\sigma, \tau)\big) = a(\sigma) \cdot a(\tau) + b(\sigma) \tag{6}$$

Since D is a nondegenerate interval by assumption and since REF and SYM are satisfied, Theorem 3 in Aczel (1966) can be applied to (6).

The theorem yields that κ is a continuous group operation for the parameter σ and τ. Thus there is a monotone and continuous function $g: D \to R$ such that $\kappa(\sigma, \tau) = g^{-1}\big(g(\sigma) + g(\tau)\big)$ and therefore

$$a\big(g^{-1}(g(\sigma) + g(\tau))\big) = a\big(g^{-1}(g(\sigma))\big) \cdot a\big(g^{-1}(g(\tau))\big)$$

and

$$b\big(g^{-1}(g(\sigma) + g(\tau))\big) = a\big(g^{-1}(g(\sigma))\big) \cdot b\big(g^{-1}(g(\tau))\big) + b\big(g^{-1}(g(\sigma))\big).$$

Defining $\hat{a}(v) = a\big(g^{-1}(v)\big)$, $\hat{b}(v) = b\big(g^{-1}(v)\big)$, $v = g(\sigma)$, and $w = g(\tau)$ we obtain

$$\hat{a}(v + w) = \hat{a}(v) \cdot \hat{a}(w)$$
$$\hat{b}(v + w) = \hat{a}(v) \cdot \hat{b}(w) + \hat{b}(v).$$

Theorem 1, p. 150 in Aczel (1966) and its Corollary provide the general solution of the second functional equation (which also satisfies the first one): There are constants c and d such that

either (i) $\hat{a}(t) = 1$ and $\hat{b}(t) = d \cdot t$ for $d \neq 0$ and $t \in R$

or (ii) $\hat{a}(t) = e^{ct}$ and $\hat{b}(t) = d(1 - e^{ct})$ for $t \in R$.

(The trivial solution is impossible since the transformation T_τ has to be increasing.)

Inserting the solution and returning to a and b yields

$$T_\tau(X) = X + d \cdot g(\tau)\mathbf{1}$$

or $$T_\tau(X) = e^{cg(\tau)} \cdot X + d(1 - e^{cg(\tau)})\mathbf{1}.$$

Conversely, it is obvious that these transformations satisfy REF, TRANS, and SYM. □

Finally we consider some particular properties of the set of transformations characterized by Proposition 1. Usually they change the average income of the distribution transformed; i.e. an equivalence class of $\sim_{\mathrm{T}^*}$ contains distributions having different average incomes. Later on, when ethical inequality measures are defined, income distributions will be normalized for income. Thus the question arises whether for each degree of inequality (or in each equivalence class) there is an income distribution for any possible per-capita income $\varepsilon \in \Omega$. The answer is affirmative. Let S_ε^X denote a transformation in T^* which maps a given distribution *X* into a distribution with the average income ε.

Then we obtain

Proposition 3

Assume that T^* defines a linear inequality concept.

a) *There is a transformation $S_\varepsilon^X \in \mathrm{T}^*$ for every $X \in \Omega^n$ and $\varepsilon \in \Omega$.*

b) *S_ε^X is unique.*

c) *T^* is equal to $\{S_\varepsilon^X | \varepsilon \in \Omega\}$ for every $X \in \Omega^n$.*

d) *$S_\varepsilon^{T(X)}(T(X)) = S_\varepsilon^X(X)$ for $X \in \Omega^n$, $T \in \mathrm{T}^*$, and $\varepsilon \in \Omega$.*

e) *$S_\varepsilon^X(Y)$ is continuous in ε for all $X, Y \in \Omega^n$.*

For a linear inequality concept a transformation S_ε^X exists and is unique. Therefore there is exactly one income distribution with average income ε in each equivalence class. Part c) of Proposition 3 demonstrates again that the set T^* depends only on one parameter. Part d) seems to be a bit technical, but it makes perfect sense: *X* and $T(X)$ possess the same degree of inequality and belong to the same equivalence class of $\sim_{\mathrm{T}^*}$. Since there is exactly *one* (normalized) distribution *X* with mean income ε in each equivalence class (by part b), every income distribution belonging to the equivalence class of *X* is mapped into the *same* normalized distribution.

Proof of Proposition 3

a) S_{ε}^{X} must be equal to a $T_{\tau} \in \mathrm{T}^{*}$. Choose $\tau = \varepsilon - \mu(X)$ and $\tau = (\varepsilon - d)/(\mu(X) - d)$ for type (i) and type (ii), respectively.

b) Obvious.

c) $\{S_{\varepsilon}^{X} | \varepsilon \in \Omega\} \subset \mathrm{T}^{*}$ by definition. Conversely, consider any $T_{\tau} \in \mathrm{T}^{*}$ and define $\varepsilon := \mu(T_{\tau}(X))$. Then $T_{\tau} = S_{\varepsilon}^{X}$.

d) T must be equal to a transformation S_{η}^{X}. The definition of these transformations implies

$$S_{\varepsilon}^{S_{\eta}^{X}(X)}\left(S_{\eta}^{X}(X)\right) = S_{\varepsilon}^{X}(X)$$

e) Obvious. □

5. Inequality concepts and social welfare

Since it is our objective to derive *ethical* inequality measures we have to introduce social welfare orderings $\geq_W$ in a first step. A welfare ordering $\geq_W$ is defined on Ω^n and has to satisfy

Property WELF (welfare)

$\geq_W$ *is continuous, strictly increasing and S-concave.*

Monotonicity is related to the Pareto principle: Any increase in income improves welfare as well. S-concavity implies anonymity and the Pigou-Dalton principle: Progressive transfers from a richer to a poorer individual which do not change the ranking of their incomes increases social welfare. These are basic attributes of a social welfare ordering.

In order to obtain normatively significant inequality measures we need two more properties:

Property EDEI

For every $X \in \Omega^n$ *an equally distributed equivalent income (EDEI)* $\xi(X)$ *exists such that*

$$X \sim_W \xi(X)\mathbf{1}$$

Property T^***-HOM (homogeneity)**

$$X \sim_W Y \Rightarrow T(X) \sim_W T(Y) \quad \textit{for every} \quad X, Y \in \Omega^n \quad \textit{and} \quad T \in \mathrm{T}^*$$

The existence of an equally distributed equivalent income is in general not guaranteed. Therefore it has to be postulated explicitly. It is well-known that $\xi(X)$ is a representation of the welfare ordering $\geq_W$. It can be interpreted as the representative income of the population considered. T^*-homogeneity is the generalization of (linear) homogeneity of the welfare function in the framework of relative inequality, or the analogue to translatability for the concept of absolute inequality. These properties have some further implications:

Proposition 4

a) *EDEI implies that* $\xi(\varepsilon\mathbf{1}) = \varepsilon$ *for all* $\varepsilon \in \Omega$.

b) *If EDEI and* T^**-HOM are satisfied, then*

$$\xi\big(T(X)\big) = T\big(\xi(X)\big) \textit{ for } X \in \Omega^n,\ T \in \mathrm{T}^*. \tag{7}$$

c) *EDEI and* T^**-HOM imply:*

$$X \geq_W Y \Rightarrow T(X) \geq_W T(Y) \quad \textit{for} \quad X, Y \in \Omega^n, T \in \mathrm{T}^*.$$

The equally distributed equivalent income is a particular representation of $\geq_W$. Its normalization proves to be attractive if the welfare ordering is in addition T^*-homogeneous. The EDEI of a transformed income distribution $T(X)$ can be computed by the transformation of the EDEI of X. This property allows to extend T^*-homogeneity from the symmetric part of $\geq_W$ to the entire ordering.

For the rest of the paper we assume that WELF, EDEI, and T^*-HOM are satisfied.

The proofs of these assertions are simple:

Proof of Proposition 4

a) Suppose that $X = \varepsilon \mathbf{1}$. Then

$$\xi(\varepsilon \mathbf{1}) \sim_W \varepsilon \mathbf{1}$$

which yields the result because of the monotonicity of $\geq_W$.

b) By EDEI $X \sim_W \xi(X)\mathbf{1}$ and $T(X) \sim_W \xi\big(T(X)\big)\mathbf{1}$. T^*-HOM implies

$$T(X) \sim_W T\big(\xi(X)\big)\mathbf{1},$$

and therefore $\xi\big(T(X)\big) = T\big(\xi(X)\big)$.

c) $X \geq_W Y \Leftrightarrow \xi(X) \geq \xi(Y)$

$\Leftrightarrow T\big(\xi(X)\big) \geq T\big(\xi(Y)\big)$ since T is increasing

$\Leftrightarrow \xi\big(T(X)\big) \geq \xi\big(T(Y)\big)$ because of b)

$\Leftrightarrow T(X) \geq_W T(Y)$ by definition □

Above it was demonstrated that each equivalence class of $\sim_{\mathrm{T}^*}$ contains exactly one distribution with mean income ε (for every $\varepsilon \in \Omega$). If a social welfare ordering is T^*-homogeneous a similar result obtains: The intersection of a T^*-equivalence class with any social indifference curve is always nonempty and consists of exactly one income distribution. Given a certain level of social welfare we can find a distribution for any degree of inequality and, conversely, for any degree of inequality we are able to determine an income distribution which yields any level of welfare. This property will be useful in the next section when a new type of ethical inequality measure is defined.

But at first we have to prove the above claim. It is easy to see that the range of $\xi(X)$ coincides with Ω. Therefore we introduce the notation R_u^X in analogy to S_ε^X for any $X \in \Omega^n$ and $u \in \Omega$. While S_ε^X maps X into a distribution with the *average income* ε, R_u^X denotes a transformation in T^* which maps the distribution X into a distribution yielding the *level of social welfare* u when measured by the EDEI: $\xi\big(R_u^X(X)\big) = u$. Thus there is an analogy between the definition of S_ε^X and that of R_u^X. But there is also one important difference: S_ε^X is completely determined by the underlying inequality concept T^*. Unlike S_ε^X the transformation R_u^X depends also on the (social welfare function and the) EDEI considered. Since it is always clear which EDEI ξ is under conderation the shorthand $R_u^X(Y)$ is used instead of the more precise $R_u^X(Y,\xi)$. Then we obtain:

Proposition 5

Assume that T^* *defines a linear inequality concept and that* $\geq_W$ *satisfies WELF, EDEI, and* T^**-HOM.*

a) *There is a transformation* $R_u^X \in \mathrm{T}^*$ *for every* $X \in \Omega^n$ *and* $u \in \Omega$.

b) R_u^X *is unique.*

c) T^* *is equal to* $\left\{R_u^X \middle| u \in \Omega\right\}$ *for every* $X \in \Omega^n$.

d) $R_u^{T(X)}\left(T(X)\right) = R_u^X(X)$ *for* $X \in \Omega^n$, $T \in \mathrm{T}^*$, *and* $u \in \Omega$.

e) $R_u^X(Y)$ *is continuous in u for all* $X, Y \in \Omega^n$.

The proof runs along the same lines as that of Proposition 3:

Proof of Proposition 5

a) R_u^X must be equal to a transformation $\mathrm{T}_\tau \in \mathrm{T}^*$.

Choose $\tau := u - \mu(X)$ and $\tau = (u-d)/\left(\xi(X) - d\right)$ for type (i) and type (ii), respectively.

b) Obvious.

c) $\left\{R_u^X \middle| u \in \Omega\right\} \subset \mathrm{T}^*$ by definition. Conversely, consider any $T_\tau \in \mathrm{T}^*$ and define $u := \xi\left(T_\tau(X)\right)$. Then $T_\tau = R_u^X$.

d) T must be equal to a transformation R_v^X . The definition of these transformations implies

$$R_u^{R_v^X(X)}\left(R_v^X(X)\right) = R_u^X(X).$$

e) Obvious. □

Thus the properties of R_u^X are in principle the same as those of S_ε^X, only the *normalization* is different. Therefore the interpretation provided for Proposition 3 applies analogously.

6. Ethical inequality measures: definition

For a derivation of normatively significant inequality measures (or orderings) a decomposition of social welfare functions (or orderings) is necessary: One has to distinguish between the size and the distribution of income as determinants of social welfare (cf. Ebert (1987a)). Informally, we have to split up[2] the social welfare function $\xi(X)$ into two components, average income $\mu(X)$ and inequality $K(X)$:

$$\xi(X) = F\big(\mu(X), K(X)\big)$$

where F is increasing in mean income and decreasing in inequality. One kind of decomposition can be accomplished easily:

$$\xi(X) = \mu(X) - \big(\mu(X) - \xi(X)\big). \tag{8}$$

For such a decomposition, inequality measures can be defined by standardizing the income distribution X and using the measure K. We will present two different methods of standardization. The first one normalizes with respect to average income. Choose an arbitrary average income ε and keep it fixed. Then transform the income distribution X to $S_\varepsilon^X(X)$. By definition the average income is changed to ε. We obtain

$$\xi\big(S_\varepsilon^X(X)\big) = F\big(\varepsilon, K(S_\varepsilon^X(X))\big) \tag{9}$$

and therefore $K\big(S_\varepsilon^X(X)\big) = F^{-1}\big(\varepsilon, \xi(S_\varepsilon^X(X)\big)$.

Now *define the inequality of X by* $K\big(S_\varepsilon^X(X)\big)$. The measure is normatively significant because

$$K\big(S_\varepsilon^X(X)\big) \geq K\big(S_\varepsilon^Y(Y)\big) \Leftrightarrow \xi\big(S_\varepsilon^X(X)\big) \leq \xi\big(S_\varepsilon^Y(Y)\big) \quad \text{for} \quad X, Y \in \Omega^n.$$

Since average income is standardized to ε, inequality and welfare are negatively correlated.

Using (8) the inequality measure is defined by

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[2] The first part of this section generalizes the method proposed in Ebert (1997) for homothetic social welfare functions to arbitrary welfare orderings.

$$I(X,\varepsilon) := \mu\left(S_\varepsilon^X(X)\right) - \xi\left(S_\varepsilon^X(X)\right) \quad \text{for all} \quad X \in \Omega^n \quad \text{and} \quad \varepsilon \in \Omega. \tag{10}$$

It can be interpreted in the usual way: $I(X,\varepsilon)$ reflects the welfare loss due to inequality if a distribution is normalized to $\mu\left(S_\varepsilon^X(X)\right) = \varepsilon$ (cf. Figure 1). Indeed, the definition is an extension of the Atkinson-Kolm-Sen concept to the more general framework. Since $\geq_W$ is T^*-homogeneous and S_ε^X can be determined explicitly (cf. the proof of Proposition 3 a) we obtain

$$I(X,\varepsilon) = \varepsilon - S_\varepsilon^X\left(\xi(X)\right)$$

$$= \begin{cases} \mu(X) - \xi(X) & \text{for type (i)} \\ (\varepsilon - d)\left(1 - \dfrac{\xi(X) - d}{\mu(X) - d}\right) & \text{for type (ii)} \end{cases} \tag{11}$$

In other words, $I(X,\varepsilon)$ is related[3] to the ethical measures known: For absolute inequality (type (i)) it possesses the usual form. The same is true for relative inequality if $d = 0$ and ε is set to unity. The properties of $I(X,\varepsilon)$ will be discussed in the next section.

The decomposition of social welfare (8) can be exploited in another way. Instead of standardizing with respect to average income ε (as $I(X,\varepsilon)$ does) we can normalize with respect to the level of social welfare *u*: Fix $u \in \Omega$ and transform an income distribution *X* to $R_u^X(X)$. By definition of R_u^X we get $\xi\left(R_u^X(X)\right) = u$ and obtain

$$u = \xi\left(R_u^X(X)\right) = F\left(\mu(R_u^X(X)), K(R_u^X(X))\right). \tag{12}$$

Correspondingly we *define the inequality of X by* $K\left(R_u^X(X)\right)$. It can also be measured by the average income $\mu\left(R_u^X(X)\right)$ because

$$K\left(R_u^X(X)\right) \geq K\left(R_u^Y(Y)\right) \Leftrightarrow \mu\left(R_u^X(X)\right) \geq \mu\left(R_u^Y(Y)\right) \quad \text{for} \quad X, Y \in \Omega^n.$$

Since the level of social welfare is fixed to *u*, greater inequality has to be compensated by greater average income. Therefore inequality and average income are positively correlated. Employing the decomposition (8) we define

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[3] The indexes $I(X,\varepsilon)$ depend on the (type of) inequality concept. We abstain from using an additional subscript.

$$I(X,u) := \mu\left(R_u^X(X)\right) - \xi\left(R_u^X(X)\right) \quad \text{for all} \quad X \in \Omega^n \quad \text{and} \quad u \in \Omega\,. \tag{13}$$

It can be interpreted as welfare loss due to inequality as well. Figure 2 illustrates the definition. Since welfare is standardized to *u*, the inequality ordering represented by $I(X,u) = \mu\left(R_u^X(X)\right) - u$ is the same as that implied by $\mu\left(R_u^X(X)\right)$.

The linearity of the inequality concept again yields a simple representation of $I(X,u)$ (cf. the proof of Proposition 5 a):

$$I(X,u) = R_u^X\left(\mu(X)\right) - u$$
$$= \begin{cases} \mu(X) - \xi(X) & \text{for type (i)} \\ (u-d)\left(\dfrac{\mu(X)-d}{\xi(X)-d} - 1\right) & \text{for type (ii)} \end{cases} \tag{14}$$

For a type (i) inequality concept $I(X,u)$ coincides with $I(X,\varepsilon)$ for all $\varepsilon, u \in \Omega$. Thus nothing new is implied. But if the transformations of T^* belong to type (ii) the measures $I(X,u)$ are new. Both types of measure are examined in greater detail in the next section.

Equation (8) presents one possible way of decomposing social welfare. Obviously other forms of decomposition could have been used instead, e.g.

$$\xi(X) = \mu(X)\left(1 - \left(1 - \frac{\xi(X)}{\mu(X)}\right)\right).$$

Therefore the question arises how the inequality measures developed depend on the split-up. Fortunately, it turns out that the measures $I(X,\varepsilon)$ and $I(X,u)$, respectively, always represent the same inequality ordering: Suppose that $\xi(X) = \overline{F}\left(\mu(X), \overline{K}(X)\right)$ where $\overline{F}$ is increasing in μ and decreasing in $\overline{K}$. Employing (9) and $\xi\left(S_\varepsilon^X(X)\right) = \overline{F}\left(\varepsilon, \overline{K}(S_\varepsilon^X(X))\right)$ we obtain

$$\begin{aligned} I(X,\varepsilon) &= K(S_\varepsilon^X(X)) = F^{-1}\left(\varepsilon, \overline{F}(\mu(S_\varepsilon^X(X)), \overline{K}(S_\varepsilon^X(X)))\right) \\ &= F^{-1}\left(\varepsilon, \overline{F}(\varepsilon, \overline{K}(S_\varepsilon^X(X))\right) = F^{-1}\left(\varepsilon, \overline{F}(\varepsilon, \overline{I}(X,\varepsilon))\right) \end{aligned};$$

i.e. both inequality measures are ordinally equivalent, since the function $F^{-1}\left(\varepsilon,\overline{F}(\varepsilon,\cdot)\right)$ is strictly increasing. Similarly, equation (12) yields that $K(R_u^X(X)) = G\left(\mu(R_u^X(X))\right)$ and $\overline{K}(R_u^X(X)) = \overline{G}\left(\mu(R_u^X(X))\right)$, where G and $\overline{G}$ are strictly increasing. That implies

$$I(X,u) = K\left(R_u^X(X)\right) = G\left(\overline{G}^{-1}(\overline{K}(R_u^X(X)))\right) = G\left(\overline{G}^{-1}(\overline{I}(X,u))\right).$$

$I(X,u)$ and $\overline{I}(X,u)$ are possibly different cardinalizations of the same inequality ordering. Thus the way of splitting-up does not matter: One can choose any function F and index $K(X)$, respectively; i.e. we have established

Proposition 6

The measures

$$I(X,\varepsilon) = \mu\left(S_\varepsilon^X(X)\right) - \xi\left(S_\varepsilon^X(X)\right)$$

and $$I(X,u) = \mu\left(R_u^X(X)\right) - \xi\left(R_u^X(X)\right)$$

define a unique inequality ordering for every $\varepsilon, u \in \Omega$.

The measures $I(X,u)$ and $I(X,\varepsilon)$ are called dual measures: Deriving $I(X,\varepsilon)$ the average income of X is changed to ε, the measure is ordinally equivalent to the resulting equally distributed income and level of social welfare. For $I(X,u)$ the level of social welfare implied by X is altered to u. Then the measure is essentially defined by the corresponding average income.

7. Ethical inequality measures: properties

Whenever the transformations characterizing a linear inequality concept T^* are of type (ii), i.e. if $\mathrm{T}^* = \mathrm{T}_d^*$, the measures $I(X,\varepsilon)$ and $I(X,u)$ possess different forms:

$$I(X,\varepsilon) = (\varepsilon - d)\frac{\mu(X) - \xi(X)}{\mu(X) - d} \qquad \text{and}$$

$$I(X,u) = (u - d)\frac{\mu(X) - \xi(X)}{\xi(X) - d}.$$

According to the way of standardization they normalize the welfare loss due to inequality differently: $I(X,\varepsilon)$ is essentially normalized with resepct to the average income (neglecting for the moment the other constants!). Therefore it equals the per-capita welfare loss due to inequality expressed in money per $ income and is bounded above by 1. On the other hand, $I(X,u)$ is normalized by the equally distributed equivalent income $\xi(X)$. The latter can be interpreted as representative income, being equivalent to the average living standard of the population whereas the average income $\mu(X)$ is related to the possibilities individuals face (cf. Blackorby/Donaldson/Auersperg (1981)). The dual measure is unbounded above since the welfare loss might be arbitrarily high in comparison to $\xi(X)$.

The attributes of $I(X,\varepsilon)$ and $I(X,u)$ depend on the properties of the underlying welfare ordering and of the set of transformation T^*. It turns out that the measures are attractive. We establish:

Proposition 7

Assume that T^* *defines a linear inequality concept and that* $\geq_W$ *satisfies WELF, EDEI, and* T^**-HOM.*

a) $I(X,\varepsilon)$ *and* $I(X,u)$ *are* T^**-invariant:*

$$I\big(T(X),\varepsilon\big) = I(X,\varepsilon) \quad \text{and} \quad I\big(T(X),u\big) = I(X,u)$$
$$\text{for all} \quad X \in \Omega^n, T \in \mathrm{T}^*, u, \varepsilon \in \Omega.$$

b) $I(X,\varepsilon)$ *and* $I(X,u)$ *are symmetric.*

c) $I(X,\varepsilon)$ *and* $I(X,u)$ *satisfy the principle of progressive transfers (inequality is decreased).*

d) $I(X,\varepsilon)$ *and* $I(X,u)$ *are nonnegative. They equal zero if and only if* $X = \alpha\mathbf{1}$ *for* $\alpha \in \Omega$.

e) $I(X,\varepsilon)$ *and* $I(X,u)$ *represent the same inequality ordering for all* $\varepsilon, u \in \Omega$.

T^*-homogeneity and the procedure proposed above guarantee (in connection with the other assumptions) that the inequality measures are T^*-invariant. This is obviously a necessary condition for any sensible definition of measures. On the other hand it implies that all income distributions belonging to an equivalence class of $\sim_{\mathrm{T}^*}$ possess the same degree of inequality or are contained in the same equivalence class of the *inequality* ordering. The converse is not true since the inequality ordering is not only T^*-invariant, but also symmetric. This property is inherited from the social welfare ordering. Therefore several equivalence classes of the equivalence relation $\sim_{\mathrm{T}^*}$ yield the same degree of inequality. Progressive transfers diminish inequality. Thus the Pigou-Dalton principle is fulfilled. The range of $I(X,\varepsilon)$ and $I(X,u)$ is appropriate. Furthermore the measures indicate inequality whenever incomes are not equal. The last result e) is surprising: Looking at the functional forms (11) and (14) of $I(X,\varepsilon)$ and $I(X,u)$, respectively, one immediately recognizes that the *ordering* represented by $I(X,\varepsilon)\;[I(X,u)]$ does not depend on $\varepsilon \in \Omega\;[u \in \Omega]$: A change in ε and u implies only a scaling of the respective measure. But the measures $I(X,\varepsilon)$ and $I(X,u)$ seem to be different, at first sight, since they normalize the welfare loss due to inequality differently: Nevertheless, the orderings implied coincide. Therefore, for a given linear inequality concept only *one* inequality ordering is derivable from $\geq_W$.

Proof of Proposition 7

a) Consider $I(X,\varepsilon)$ and any $T \in \mathrm{T}^*$. Then Proposition 3 d) and the definition of $I(X,\varepsilon)$ yield

$$I(T(X),\varepsilon) = \mu\left(S_\varepsilon^{T(X)}(T(X))\right) - \xi\left(S_\varepsilon^{T(X)}(T(X))\right)$$
$$= \mu\left(S_\varepsilon^{X}(X)\right) - \xi\left(S_\varepsilon^{X}(X)\right) = I(X,\varepsilon)\,.$$

For $I(X,u)$ apply Proposition 5 d).

b) Obvious because of WELF.

c) A progressive transfer increases $\xi(X)$ and therefore decreases $I(X,\varepsilon)$ and $I(X,u)$.

d) Obvious.

e) We obtain for an inequality concept characterized by T_d^*.

$$I(X,\varepsilon) = \frac{(\varepsilon - d)}{(\eta - d)}\, I(X,\eta) \quad \text{and}$$

$$I(X,u) = \frac{(u-d)}{(v-d)}\, I(X,v) \quad \text{for} \quad \varepsilon,\eta,u,v \in \Omega\,.$$

Furthermore set $\varepsilon = u = d + 1$. Then

$$I(X,\varepsilon) = g\big(I(X,u)\big)$$

for $\quad g(t) = 1 - 1/(t+1)$

which is a strictly increasing function. □

8. Determination of T^*-invariant inequality measures

A survey of the literature on inequality measurement demonstrates that most ethical measures considered are based on separable welfare orderings. Therefore we now focus the analysis on this type of ordering and introduce

Property SEP (separability)

The social welfare ordering $\geq_W$ can be represented by a separable welfare function; i.e. there is a continuous monotone function f such that

$$\xi(X) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right).$$

f is called a characteristic function of $\geq_W$.

The characteristic function is not unique. Any increasing affine transformation of *f* can be used as well. Furthermore, the formulation of SEP excludes rank-dependent orderings a priori. Some of them will be investigated in the next section. Property SEP implies Property EDEI. Thus it is stronger.

Imposing SEP we are able to characterize and derive the class of T^*-homogeneous welfare functions for any linear inequality concept T^*. At first we obtain a technical result which is interesting in itself:

Lemma 8

A social welfare ordering $\geq_W$ satisfying the property SEP is T^*-homogeneous if and only if

$$f\big(T_\tau(f^{-1}(t))\big) = c(\tau)\cdot t + e(\tau) \quad \text{for} \quad T_\tau \in \mathrm{T}^* \quad \text{and} \quad \tau \in D.$$

Separability has strong implications for the interplay of the characteristic function *f* and the transformations $T_\tau \in \mathrm{T}^*$.

Proof of Lemma 8

Suppose that $\geq_W$ is T^*-homogeneous. Then Proposition 4 yields

$$T\big(\xi(X)\big) = \xi\big(T(X)\big) \quad \text{for} \quad X \in \Omega^n \quad \text{and} \quad T \in \mathrm{T}^*.$$

Because of SEP we obtain

$$T\left(f^{-1}\left(\sum \frac{1}{n} f(X_i)\right)\right) = f^{-1}\left(\sum \frac{1}{n} f(T(X_i))\right)$$

which is equivalent to

$$f\left(T\left(f^{-1}(\sum \frac{1}{n} f(X_i))\right)\right) = \sum \frac{1}{n} f\big(T(X_i)\big).$$

Define $t_i := \frac{1}{n} f(X_i)$. Then X_i is equal to $f^{-1}(nt_i)$. Substitution of X_i leads to

$$f\Big(T\Big(f^{-1}(\sum t_i)\Big)\Big) = \sum \frac{1}{n} f\Big(T(f^{-1}(nt_i))\Big).$$

Theorem 1 and its Corollary in Aczel (1966), p. 142 imply that there are constants *c* and *e* such that

$$f\Big(T(f^{-1}(t))\Big) = c \cdot t + e$$

(and $\quad \frac{1}{n} f\Big(T(f^{-1}(nt))\Big) = c \cdot t + \frac{e}{n} \quad$ which is equivalent).

Therefore, if $\geq_W$ is T^*-homogeneous, the characteristic function *f* has to satisfy

$$f\Big(T_\tau(f^{-1}(t))\Big) = c(\tau) \cdot t + e(\tau) \quad \text{for} \quad T_\tau \in \mathrm{T}^* \quad \text{and} \quad \tau \in D.$$

The converse is obvious. □

Using the Lemma we are able to establish

Proposition 9

A T^**-homogeneous welfare ordering* $\geq_W$ *satisfying the Properties WELF and SEP is represented either by (i)*

$$W^\gamma(X) = -\frac{1}{\gamma}\ln\frac{1}{n}\sum_{i=1}^{n} e^{-\gamma X_i} \qquad \textit{for} \quad \gamma > 0 \quad \textit{if} \quad \mathrm{T}^* = \mathrm{T}^*_{abs}$$

or by (ii)

$$W_d^\delta(X) = \begin{cases} \left(\frac{1}{n}\sum\limits_{i=1}^{n}(X_i - d)^\delta\right)^{1/\delta} + d & \textit{for} \quad \delta < 1, \delta \neq 0 \\ \prod\limits_{i=1}^{n}(X_i - d)^{1/n} + d & \textit{for} \quad \delta = 0 \quad \textit{if} \quad \mathrm{T}^* = \mathrm{T}^*_d \end{cases}.$$

The class (i) contains all Kolm-Pollak social welfare functions. Here the elements of T^* are of type (i). If the transformation belonging to T^* belongs to type (ii), $\geq_W$ is represented by a function W_d^δ which coincides with an Atkinson welfare function for $d = 0$. The characterization result is not really surprising. It is well-known that intermediate inequality measures are somehow related to some extended form of the Atkinson social functions (cf. Bossert/Pfingsten (1990)). Therefore the same should be true for ultra-rightist measures, but the class W_d^δ for $d \neq 0$ has not been characterized up to now.

Proof of Proposition 9

If $T_\tau(t)$ is of type (i), it is well known that the social welfare ordering is represented by a Kolm-Pollak welfare function (cf. e.g. Blackorby/Donaldson (1980), Ebert (1988)).

Employ Lemma 7 and suppose that T_τ possesses the form

$$T_\tau(X) = \tau \cdot X + d(1-\tau)\mathbf{1} \qquad \text{(type (ii))}.$$

Then we obtain

$$f\big(\tau(t-d)+d\big) = c(\tau)\cdot f(t) + e(\tau).$$

Define $t := \hat{t} + d$ and $\hat{f}(t) := f(t+d)$.

We get

$$f(\tau\hat{t} + d) = c(\tau)\cdot f(\hat{t}+d) + e(\tau)$$

and $\hat{f}(\tau\hat{t}) = c(\tau)\cdot\hat{f}(\hat{t}) + e(\tau)$.

The solution to this equation is given by Theorem 2.7.3 in Eichhorn (1978):

$$\hat{f}(\hat{t}) = h\cdot\log\hat{t} + k$$

or $\hat{f}(\hat{t}) = l\cdot\hat{t}^{\delta} + k$

where $h \neq 0$, $l \neq 0$, $\delta \neq 0$ and k are arbitrary real constants. This implies

$$\hat{f}^{-1}\left(\frac{1}{n}\sum\hat{f}(X_i)\right) = \begin{cases}\left(\frac{1}{n}\sum X_i^{\delta}\right)^{1/\delta} & \text{for} \quad \delta \neq 0 \\ \prod X_i^{1/n} & \text{otherwise}\end{cases}.$$

The rest of part (ii) follows from the definition of f and S-concavity. □

Using Proposition 9 one can directly determine the corresponding inequality measures based on both methods proposed above. We obtain

Proposition 10

(i) *The ordering $\geq_W$ represented by $W^{\gamma}(X)$ generates*

$$I(X,\varepsilon) = I(X,u) = J^{\gamma}(X) := -\frac{1}{\gamma}\ln \quad \frac{1}{n}\sum_{i=1}^{n} e^{\gamma(X_i-\mu(X))} \quad \textit{for all} \quad X \in \Omega^n$$

$$\textit{and} \quad \varepsilon, u \in \Omega.$$

(ii) *The ordering $\geq_W$ represented by $W_d^{\delta}(X)$ generates*

$$I(X,\varepsilon) = J_d^{\delta}(X,\varepsilon) := \begin{cases}(\varepsilon-d)\left(1-\frac{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-d)^{\delta}\right)^{1/\delta}}{\mu(X-d\mathbf{1})}\right) & \textit{for} \quad \delta<1, \delta\neq 0 \\ (\varepsilon-d)\left(\frac{\prod_{i=1}^{n}(X_i-d)^{1/n}}{\mu(X-d\mathbf{1})}\right) & \textit{for} \quad \delta = 0\end{cases}$$

and

$$I(X,u) = J_d^{\delta}(X,u) := \begin{cases} (u-d)\left(\dfrac{\mu(X-d\mathbf{1})}{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-d)^{\delta}\right)^{1/\delta}} - 1\right) & \textit{for} \quad \delta < 1, \delta \neq 0 \\ (u-d)\left(\dfrac{\mu(X-d\mathbf{1})}{\prod_{i=1}^{n}(X_i-d)^{1/n}} - 1\right) & \textit{for} \quad \delta = 0 \end{cases}$$

A Kolm-Pollak social welfare function implies an absolute inequality measure. The normalization does not play a role: $I(X,\varepsilon)$ and $I(X,u)$ are independent of ε and u, respectively and coincide. $W_d^{\delta}(X)$ is related to a T_d^*-invariant inequality measure. For $d = 0$ we obtain measures of relative inequality[4] (implied by an Atkinson-welfare function). For negative d, measures of intermediate inequality[5] are generated, for positive d-ultra-rightist measures. As proved in Proposition 7 $I(X,\varepsilon)$ and $I(X,u)$ represent the same inequality ordering. But the dual measures $I(X,u)$ presented in Proposition 10 are nevertheless interesting in themselves: they possess an attractive *decomposition* property which will be discussed now.

9. Decomposition of dual measures

Suppose in this section that the population size is variable and that there is a social welfare ordering $\geq_{W_n}$ on Ω^n for each $n \geq 2$, represented by

$$\xi_n(X) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right).$$

Then the welfare ordering fulfills a decomposition property

$$X \sim_{W_n} \left(\xi_k(X^k)\mathbf{1}_k, \xi_l(X^l)\mathbf{1}_l\right)$$

whenever $X = (X^k, X^l) = \left((X_1,\ldots,X_k),(X_{k+1},\ldots,X_n)\right)$ and $n = k + l$ (cf. e.g. Ebert (1988) or proof by direct computation). It means that the level of social welfare is unchanged if an

[4] The measures $J_0^d(X,u)$ are already discussed in Ebert (1997).

income distribution is smoothed, i.e. if each individual receives its group's representative income $\xi_i(X_i)\ (i=1,2)$ instead of its actual income. This property can be exploited to establish

Proposition 11

If $I^n(X,u)$ *is implied by* $\geq_{W_n}$ *represented by* $W^\gamma(X)$ *or* $W_d^\delta(X)$*, then it satisfies*

$$I^n(X,u) = \left[w_k \cdot I^k(X^k,u) + w_l \cdot I^l(X^l,u)\right] + I^n\left(\left(\xi_k(X^k)\mathbf{1}_k, \xi_l(X^l)\mathbf{1}_l\right), u\right)$$

for all $k,l>1, \quad k+l=n \quad$ and $\quad X=(X^k,X^l)\in\Omega^n$,

where (i) $w_k = k/n$ *and* $w_l = l/n$ *if* $W^\gamma(X)$ *represents the generating welfare ordering*

and (ii) $w_k = \dfrac{k}{n}\dfrac{\xi_k(X^k)-d}{\xi_n(X)-d}$, $w_l = \dfrac{l}{n}\dfrac{\xi_l(X^l)-d}{\xi_n(X)-d}$ *and* $w_k + w_l \geq 1$ *otherwise.*

Obviously the dual measures are additively decomposable into the sum of a within-group term and a between-group term. In contrast to the usual form of additively decomposable measures (cf. e.g. Shorrocks (1980, 1984)) between-group inequality does not depend on mean incomes, but on the groups' *equally distributed equivalent income*. Since the indices $I(X,u)$ compare the welfare loss due to inequality with the representative income, the (different) smoothing procedure makes perfect sense: The dual measures focus on the living standard. Within-group inequality corresponds to a weighted sum of the inequality within subgroups. For absolute inequality the weights correspond to the population shares and sum up to unity. For relative, intermediate and *d*-ultra-rightist inequality the sum of the weighting factors generally exceeds one, a phenomenon also known for most generalized entropy measures.

At this point one should recall that the indices $I(X,\varepsilon)$ and $I(X,u)$ represent the same inequality ordering and that - if the concept of relative inequality is considered - they must be equivalent to a generalized entropy measure. Therefore the inequality orderings implied by

[5] Eichhorn (1988) characterizes the family $J_d^\delta(X,\varepsilon)$ directly (for $d<0$).

$W_d^{\delta}(X)$ possess different cardinalizations which are additively decomposable: The dual measure $J_d^{\delta}(X,u)$ is ordinally equivalent to the generalized entropy measure[6]

$$E_d^{\delta}(X)=\begin{cases}\dfrac{1}{\delta^2-\delta}\dfrac{1}{n}\sum_{i=1}^{n}\left[\left(\dfrac{X_i-d}{\mu(X)-d}\right)^{\delta}-1\right] & \text{for} \quad \delta<1,\delta\neq 0\\ \dfrac{1}{n}\sum_{i=1}^{n}\log\left(\dfrac{\mu(X)-d}{X_i-d}\right) & \text{for} \quad \delta=0\\ \dfrac{1}{n}\sum_{i=1}^{n}\dfrac{X_i-d}{\mu(X)-d}\log\left(\dfrac{X_i-d}{\mu(X)-d}\right) & \text{for} \quad \delta=1.\end{cases}$$

Of course, the weights and the smoothing procedure used in a decomposition of a generalized entropy class differ from those presented above. They are based on the respective mean incomes.

Proof of Proposition 11

We consider case (ii). Using the definition of measures and weights we obtain

$$w_k I^k(X^k,u)+w_l I^l(X^l,u)+I^n\left((\xi_k(X^k)\mathbf{1}_k,\xi_l(X^l)\mathbf{1}_l),u\right)$$

$$=\frac{k}{n}\frac{\xi_k(X^k)-d}{\xi_n(X)-d}(u-d)\left(1-\frac{\mu(X^k-d\mathbf{1}_k)}{\xi_k(X^k)-d}\right)$$

$$+\frac{l}{n}\frac{\xi_l(X^l)-d}{\xi_n(X)-d}(u-d)\left(1-\frac{\mu(X^l-d\mathbf{1}_l)}{\xi_l(X^l)-d}\right)$$

$$+(u-d)\cdot\left(1-\frac{\mu\left((\xi_k(X^k)\mathbf{1}_k,\xi_l(X^l)\mathbf{1}_l)-d\mathbf{1}\right)}{\xi_n\left(\xi_k(X^k)\mathbf{1}_k,\xi_l(X^l)\mathbf{1}_l\right)-d}\right)$$

for the right-hand side.

It reduces to

$$(u-d)\left(1-\frac{\mu(X-d\mathbf{1}_n)}{\xi_n(X)-d}\right)$$

because of the decomposition property of $\geq_{W_n}$.

[6] This ‘generalization’ of the generalized entropy class is proposed by Cowell (1997).

Finally

$$w_k + w_l = \frac{\frac{k}{n}\xi_k(X^k) + \frac{l}{n}\xi_l(X^l) - d}{\xi_n(X) - d}.$$

Now observe that

$$f\left(\frac{k}{n}\xi_k(X^k) + \frac{l}{n}\xi_l(X^l)\right) \geq \frac{k}{n} f\left(\xi_k(X^k)\right) + \frac{l}{n} f\left(\xi_l(X^l)\right)$$

by Jensen's inequality.

Since f^{-1} is strictly increasing we obtain

$$\begin{aligned}\frac{k}{n}\xi_k(X^k) + \frac{l}{n}\xi_l(X^l) &\geq f^{-1}\left(\frac{k}{n} f(\xi_k(X^k)) + \frac{l}{n} f(\xi_l(X^l))\right) \\ &= \xi_n\left(\xi_k(X^k)\mathbf{1}_k, \xi_l(X^l)\mathbf{1}_l\right) = \xi_n(X)\end{aligned}$$

which implies $w_k + w_l \geq 1$.

The proof for case (i) runs along the same lines. □

10. Linear social welfare orderings

The Pigou-Dalton principle forms one of the basic welfare judgements on a social welfare ordering. As a consequence any representation is strictly Schur-concave, a property a linear welfare function normally does not satisfy. Nevertheless there is a large class of linear social welfare functions (and of underlying welfare orderings), namely those defined on rank-ordered income vectors. Probably the most popular one is the welfare ordering generating the Gini-coefficient. Donaldson/Weymark (1980) and Weymark (1981) were the first who introduced and investigated some generalizations. The entire class is characterized, e.g. in Ebert (1988). We briefly discuss these orderings since each of them is T^*-homogeneous for *all* linear inequality concepts T^*.

Let $X_{[\,]} = (X_{[1]}, \ldots, X_{[n]}) \in \Omega^n$ be the ordered vector *X*; i.e. $X_{[\,]}$ is a permutation of *X* such that $X_{[i]} \geq X_{[i+1]}$ for $i = 1, \ldots n-1$. Incomes are decreasing in *i*. Then a linear social welfare ordering $\geq_W$ is defined by any social welfare function having the form

$$\xi(X) = \sum_{i=1}^{n} \alpha_i X_{[i]} \qquad (15)$$

where $\sum \alpha_i = 1$ and $\alpha_i > 0$ for $i = 1, \ldots, n$.

$\xi(X)$ is obviously symmetric and strictly Schur-concave if $\alpha_i < \alpha_{i+1}$ for $i = 1, \ldots, n-1$. Then $\geq_W$ satisfies the properties WELF and EDEI. Furthermore we obtain

Proposition 12

If $\geq_W$ is represented by $\xi(X)$ having form (15), then $\geq_W$ is T^*-homogeneous for all linear inequality concepts.

Proof

Suppose that $X \sim_W Y$, then

$$\sum \alpha_i X_{[i]} = \sum \alpha_i Y_{[i]} .$$

Applying any linear *T* preserves the equality. Therefore

$$T(X) \sim_W T(Y) \quad \text{for all} \quad X, Y \in \Omega^n, T \in \mathrm{T}^* . \qquad \square$$

This result is relevant since it allows one to choose a social welfare ordering without commitment to an inequality concept. These welfare orderings are compatible with any linear concept. Therefore, one is able to derive several *different* inequality orderings (depending on T^*). They possess the form presented in section 8 if $\xi(X)$ is replaced by the corresponding linear welfare function. Furthermore, one can try to perform a kind of sensitivity analysis: If *X* is more unequal than Y for the T_d^*-concept, one can examine for which $d' \geq d$ or $d' \leq d$ that remains true (given that $X, Y \in \Omega_{d'}^n$).

Consider the Gini-welfare ordering as an example. It is represented by

$$\xi(X) = \frac{1}{n^2} \sum_{i=1}^{n} (2i-1) X_{[i]} .$$

The corresponding inequality measures for absolute inequality are then given by

$$I(X,\varepsilon)=I(X,u)=\frac{1}{n^2}\sum_{i=1}^{n}(n+1-2i)X_{[i]}.$$

For the other concepts we obtain

$$I(X,\varepsilon)=(\varepsilon-d)\frac{\frac{1}{n^2}\sum_{i=1}^{n}(n+1-2i)X_{[i]}}{\frac{1}{n}\sum_{i=1}^{n}(X_{[i]}-d)}$$

and

$$I(X,u)=(u-d)\frac{\frac{1}{n^2}\sum_{i=1}^{n}(n+1-2i)X_{[i]}}{\frac{1}{n^2}\sum_{i=1}^{n}(2i-1)(X_{[i]}-d)} \quad \text{for} \quad d\in R.$$

Unfortunately the decomposition property discussed above is not owned by linear welfare functions. It requires that $\alpha_i=\alpha_j(=1/n)$ for $i,j=1,\ldots,n$ (cf. Ebert (1988)). But then $\xi(X)$ does no longer satisfy the principle of progressive transfers.

11. Conclusion

The abstract definition of a linear inequality concept seems to be appropriate: The definition comprises relative, absolute, and intermediate inequality (the concepts discussed in the literature). The characterization of all linear concepts has revealed the ultra-rightist ones. Furthermore the class of all separable social welfare orderings compatible with these concepts and the inequality orderings implied could be derived. For relative and absolute inequality these results were well-known, for intermediate and ultra-rightist inequality they had not been proven before. In so far, the relevant questions concerning linear inequality concepts and the corresponding ethical inequality orderings have been answered completely. Dual inequality measures have been presented. Their definition is appealing and they are an attractive alternative to the generalized entropy measures in any analysis requiring the decomposition of measures.

One question implicitly raised in the introduction has not been answered yet, namely whether the empirical evidence on the view of inequality can be described by a linear inequality concept. Here the answer, probably, has to be negative. These concepts are still too simple to explain the empirical findings; i.e. exactly the concepts which are treated in the literature – the concepts of relative, absolute, and intermediate inequality – can be viewed only as approximation to the views people hold on inequality. Therefore the next step of tackling the problem is to consider nonlinear inequality concepts.

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