Theil, Inequality and the Structure of Income Distribution

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Abstract

Theil's information-theoretic approach to the measurement of inequality (Theil 1967) is set in the context of subsequent developments over recent decades. It is shown that Theil's initial insight leads naturally to a very general class of inequality measures. It is thus closely related to a number of other commonly used families of inequality measures.

Keywords: Theil; inequality; independence; homotheticity; translatability.

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1 Introduction

Henri Theil's book on information theory (Theil 1967) provided a landmark in the development of the analysis of inequality measurement. The significance of the landmark was, perhaps, not fully realised for some time but his influence is now recognised in standard reference works and collections on the subject of the welfare economics of income distribution. Theil's insight provided both a method for thinking about the meaning of inequality and an introduction to an important set of functional forms for modelling and analysing inequality. The purpose of this paper is to set Theil's approach in the context of the literature that has since developed and to demonstrate that its contribution may have been more far-reaching than is usually supposed.

We will first introduce a framework for analysis (section 2) and consider Theil's approach to inequality in section 3. Section 4 introduces a general class of inequality indices foreshadowed by Theil's work and section 5 its properties. Section 6 concludes.

2 Analytical framework

2.1 Notation and terminology

Begin with some tools for the description of income distribution. The real number x denotes an individual's income: assume that issues concerning the definition of the income concept and the specification of the income receiver have been settled. Then we may speak unambiguously of an income distribution. Represent the space of all valid univariate distribution functions by $\mathfrak{F}$; income is distributed according to $F \in \mathfrak{F}$ where F has support $\mathfrak{X}$, an interval on the real line $\mathbb{R}$: for any $x \in \mathfrak{X}$, the number $F(x)$ represents the proportion of the population with incomes less than or equal to x.

Standard tools used in distributional analysis can be represented as functionals defined on $\mathfrak{F}$. The mean μ is a functional $\mathfrak{F} \mapsto \mathbb{R}$ given by $\mu(F) := \int x dF(x)$. An *inequality measure* is a functional $I : \mathfrak{F} \mapsto \mathbb{R}$ which is given meaning by axioms that incorporating criteria derived from ethics, intuition or mathematical convenience.

2.2 Properties of inequality measures

Now consider a brief list of some of the standard characteristics of inequality measures.

Definition 1 Principle of transfers. *$I(G) > I(F)$ if distribution G can be obtained from F by a* mean-preserving spread *.*

In order to characterise a number of alternative structural properties of the functional I consider a strictly monotonic continuous function $\tau : \mathbb{R} \mapsto \mathbb{R}$ and let $\mathfrak{X}^{(\tau)} := \{\tau(x) : x \in \mathfrak{X}\} \cap \mathfrak{X}$. A structural property of inequality measures then follows by determining a class of admissible transformations $\mathfrak{T}$.[1] Every $\tau \in \mathfrak{T}$ will have an inverse τ^{-1} and so, for any $F \in \mathfrak{F}$, we may define the τ-transformed distribution $F^{(\tau)} \in \mathfrak{F}$ such that

$$\forall x \in \mathfrak{X}^{(\tau)} : F^{(\tau)}(x) = F\left(\tau^{-1}(x)\right). \tag{1}$$

A general statement of the structural property is

Definition 2 $\mathfrak{T}$-Independence. *For all $\tau \in \mathfrak{T} : I\left(F^{(\tau)}\right) = I(F)$.*

Clearly, not all classes of transformations make economic sense. However two important special cases are those of *scale independence*, where $\mathfrak{T}$ consists of just proportional transformations of income by a strictly positive constant, and *translation independence* which where $\mathfrak{T}$ consists of just transformations of income by adding a constant of any sign.

The following restrictive assumption makes discussion of many issues in inequality analysis much simpler and can be justified by appeal to a number of criteria associated with decomposability of inequality comparisons (Shorrocks 1984, Yoshida 1977).

Definition 3 Additive separability. *There exist functions $\phi : \mathfrak{X} \mapsto \mathbb{R}$ and $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ such that*

$$I(F) = \psi\left(\mu(F), \int \phi(x)dF(x)\right) \tag{2}$$

Given additive separability, most other standard properties of inequality measures can be characterised in terms of the income-evaluation function ϕ and the cardinalisation function ψ. [2]

[1] See Ebert (1996) for a detailed discussion of this concept.

[2] This is well known – see, for example, Cowell (2000).

3 The Theil Approach

3.1 Approaches to Inequality

It is useful to distinguish between the method by which a concept of inequality is derived – such as dominance criteria or the "welfare-waste" technique – and the intellectual basis on which the approach is founded. The principal intellectual bases used for founding an approach to inequality can be summarised as follows:

- An extension of welfare criteria (Atkinson 1970, Sen 1973);
- Analogy with the analysis of risk (Harsanyi 1953, 1955; Rothschild and Stiglitz 1973);
- "Fundamentalist" approaches including persuasive ad hoc criteria such as the Gini coefficient and those based on some philosophical principle of inequality such as Temkin's "complaints" (Temkin 1993).

Theil added one further intellectual basis of his own. He focused on inequality as a by-product of the information content of the structure of the income distribution. The information-theoretic idea incorporates the following main components (Kullback 1959):

1. A set of possible events each with a given probability of its occurrence.
2. An information function h for evaluating events according to their associated probabilities, similar in spirit to the income-evaluation function ("social utility"?) in welfarist approaches to inequality.
3. The entropy concept is the expected information in the distribution.

The specification of h uses three axioms:

Axiom 1 Zero-valuation of certainty*:* $h(1) = 0$.

Axiom 2 Diminishing-valuation of probability*:* $p > p' \Rightarrow h(p) < h(p')$.

Axiom 3 Additivity of independent events*:* $h(pp') = h(p) + h(p')$

The first two of these requirements appear to be reasonable: if an event were considered to be a certainty ($p = 1$) the information that it had occurred would be valueless; the greater the assumed probability of the event the lower the value of the information that it had occurred. It is then easy to establish:

Lemma 1 *Given Axioms 1-3 the information function is* $h(p) = -\log(p)$*.*

Theil's application of this to income distribution replaced the concept of event-probabilities by income shares, introduced an income-evaluation function that played the counterpart of the information function h and specified a comparison distribution, usually taken to be perfect equality. The focus on income shares imposes a requirement of homotheticity – a special case of $\mathfrak{T}$-independence – on the inequality measure and the use of the expected value induces additive separability.

Given an appropriate normalisation using the standard population principle (Dalton 1920) this approach then found expression in the following inequality index

$$I_{\text{Theil}}(F) := \int \frac{x}{\mu(F)} \log\left(\frac{x}{\mu(F)}\right) dF(x) \tag{3}$$

and also the following (which has since become more widely known as the *mean logarithmic deviation*):

$$I_{\text{MLD}}(F) := -\int \log\left(\frac{x}{\mu(F)}\right) dF(x) \tag{4}$$

The second Theil index or MLD is an example of Theil's application of the concept of conditional entropy; conditional entropy in effect introduces alternative versions of the comparison distribution and has been applied to the measurement of distributional change (Cowell 1980a).

3.2 A generalisation

However, in their original derivation, the Theil measures in 3 use an axiom (#3 in the abbreviated list above) which does not make much sense in the context of distributional shares. It has become common practice to see (3) and (4) as two important special cases of a more flexible general class; in terms of the Theil analogy this is achieved by taking a more general evaluation function for income shares.

Then the *generalised entropy* family of measures (Cowell 1977; Cowell and Kuga 1981a, 1981b; Toyoda 1975) is given by:

$$I_{\mathrm{GE}}^{\alpha}(F) := \frac{1}{\alpha^2 - \alpha} \int \left[\left[\frac{x}{\mu(F)} \right]^{\alpha} - 1 \right] dF(x) \tag{5}$$

where $\alpha \in (-\infty, +\infty)$ is a parameter that captures the distributional sensitivity: for α large and positive the index is sensitive to changes in the distribution that affect the upper tail; for α negative the index is sensitive to changes in the distribution that affect the lower tail. Measures ordinally equivalent to the class (5) include a number of pragmatic indices such as the variance and measures of industrial concentration (Gehrig 1988, Hart 1971, Herfindahl 1950).

The principal attractions of the class (5) lie not only in the generalisation of Theil's insights but also in the fact that the class embodies some of the key distributional assumptions discussed in section 2.2.

Theorem 2 *A continuous inequality measure $I : \mathfrak{F} \mapsto \mathbb{R}$ satisfies the principle of transfers, scale invariance, and decomposability if and only if it is ordinally equivalent to (5) for some α.*[3]

However it is useful to consider the class (5), and with it the Theil indices, as members of a more general and flexible class. To do this we move away from Theil's original focus on income shares, but retain the use of $\mathfrak{T}$-independence and additive separability.

4 A class of inequality measures

4.1 Intermediate measures

Consider now the "centrist" concept of inequality introduced by Kolm (1969, 1976a, 1976b). This concept has been developed further in a number of papers, and the basic notion has re-emerged under the label "intermediate

[3] See Bourguignon (1979), Cowell (1980b), Shorrocks (1980, 1984).

inequality" (Bossert 1988, 1990). Unsurprisingly, as the labels suggest, centrist concepts have been shown to be related in limiting cases to measures described as "leftist" or "rightist" in Kolm's terminology; intermediate inequality measures in their limiting forms are related to "relative" and "absolute" measures. However a general treatment of these types of measures runs into a number of difficulties:

- In some cases the inequality measures are well-defined only with domain restrictions. The nature of these restrictions is familiar from the well-known relative inequality measures which are defined only for positive incomes.
- In the literature results on the limiting cases are available for only a subset of the potentially interesting ordinal inequality indices.

In what follows I will introduce a general structure that allows one to address these difficulties, that will be found to subsume many of the standard families of inequality measures, and that shows the inter-relationships between these families and Theil's fundamental contributions.

4.2 Definitions

We consider first a convenient cardinalisation of the principal type of inequality index:

Definition 4 *For any* $\alpha \in (-\infty, 1)$ *and any finite* $k \in \mathbb{R}_+$ *an* intermediate inequality measure *is a function* $I_{\text{int}}^{\alpha,k} : \mathfrak{F} \mapsto \mathbb{R}$ *such that*

$$I_{\text{int}}^{\alpha,k}(F) = \frac{1}{\alpha^2 - \alpha}\left[\int \left[\frac{x+k}{\mu(F)+k}\right]^{\alpha} dF(x) - 1\right] \tag{6}$$

Intermediate inequality measures have usually appeared in other cardinalisations, for example

$$[1+k]\left[1 - \left[1 + \left[\alpha^2 - \alpha\right] I_{\text{int}}^{\alpha,k}(F)\right]^{1/\alpha}\right]$$

(Bossert and Pfingsten 1990, Eichhorn 1988).[4] From (6) we may characterise a class of measures that are of particular interest.

Definition 5 *The* intermediate inequality class *is the set of functions*

$$\Im := \left\{ I_{\text{int}}^{\alpha,k} : \alpha \in (-\infty, 1), 0 < k < \infty \right\} \tag{7}$$

where $I_{\text{int}}^{\alpha,k}$ *is given by definition 4.*

The set $\Im$ can be generalised in a number of ways. Obviously one could consider relaxing the domain restrictions upon the sensitivity parameter α and the location parameter k. Further insights can be obtained if we introduce the possibility of a functional dependence of α upon k. Let $\mathfrak{T}' \subset \mathfrak{T}$ be the subset of affine transformations and consider $\alpha \in \mathfrak{T}'$ such that

$$\alpha(k) := \gamma + \beta k \tag{8}$$

where $\gamma \in \mathbb{R}, \beta \in \mathbb{R}_+$. Distributional sensitivity depends upon the location parameter k. Then the class $\Im$ (7) is equivalent to a subset of the following related class of functions

Definition 6 *The* extended intermediate inequality class *is the set of functions.*[5]

$$\bar{\Im} := \left\{ I_{\text{ext}}^{\alpha,k}(F) : \alpha \in \mathfrak{T}', k \in \mathbb{R} \right\} \tag{9}$$

where

$$I_{\text{ext}}^{\alpha,k}(F) := A(k) \int \left[\left[\frac{x+k}{\mu(F)+k} \right]^{\alpha(k)} - 1 \right] dF(x) \tag{10}$$

[4] See for example Bossert and Pfingsten (1990) page 129 where the definition (in the present notation) is given as $[1+k]\left[1 - \int \left[\frac{x+k}{\mu(F)+k}\right]^{\alpha} dF(x)\right]^{1/\alpha}$. Kolm's standard formulation (Kolm 1976a, page 435) is found by multiplying this by a factor $\frac{\mu(F)+k}{1+k}$; Kolm has suggested a number of other cardinalisations (Kolm 1996, page 17) .

[5] If $\alpha(k) \to 0$ applying L'Hôpital's rule shows that the limiting form (10) is

$$[1+k^2] \int \left[\log \left(\frac{\mu(F)+k}{x+k} \right) \right] dF(x)$$

Likewise if $\alpha(k) = 1$ (10) becomes

$$[1+k^2] \int \left[\left[\frac{x+k}{\mu(F)+k} \right] \log \left(\frac{x+k}{\mu(F)+k} \right) \right] dF(x)$$

and $A(k)$ is a normalisation term given by

$$A(k) := \frac{1+k^2}{\alpha(k)^2 - \alpha(k)} \ , \tag{11}$$

This class will be the principal focus of attention in the rest of this paper.

5 Properties of the class

The class of extended intermediate inequality measures possesses several interesting properties and contains a number of important special cases.

First, it has the property that it is $\mathfrak{T}'$-independent. Second, notice that the measure (10) can be written in the form (2) thus

$$A(k)\left[\int \frac{\phi(x)}{\phi(\mu(F))} dF(x) - 1\right] \tag{12}$$

where the income-evaluation function ϕ is given by

$$\phi(x) = \frac{1}{\alpha(k)} \left[x+k\right]^{\alpha(k)} \ , \tag{13}$$

and $\alpha(k), A(k)$ are as defined in (8) and (11): the income-evaluation function interpretation is useful in examining the behaviour of the class of inequality measures in limiting cases of the location parameter k. The important special cases of $I_{\text{ext}}^{\alpha,k}(F)$ correspond to commonly-used families of inequality measures:

- The generalised entropy indices are given by $\left\{I_{\text{ext}}^{\alpha,0}\right\}$ (Cowell 1977)
- The Theil indices (Theil 1967) are a subset of these given by the cases $\alpha(0) = 0$ and $\alpha(0) = 1$ (see equations 4 and 3 respectively).
- The Atkinson indices (Atkinson 1970) are ordinally equivalent to a subset of $\left\{I_{\text{ext}}^{\alpha,0}\right\}$: $1 - \left[I_{\text{ext}}^{\alpha,0}/A(0) + 1\right]^{1/\alpha(0)}$, $\alpha(0) < 1$.

There are other measures that can be shown to belong to this class for certain values of the location parameter k. However, here we encounter a problem of domain for the income-evaluation function ϕ. This problem routinely arises except for the special case where $\alpha(k)$ is an even positive integer;[6] otherwise one has to be sure that the argument of the power function used in (13) is never negative. Because of this it is convenient to discuss two important subcases.

5.1 Restricted domain: x bounded below

We first consider the case that corresponds to many standard treatments of the problem of inequality measurement: $-k \leq \inf(\mathfrak{X})$. This restriction enables us to consider what happens as the location parameter goes to (positive) infinity.

Theorem 3 *As $k \rightarrow \infty$ the extended intermediate inequality class (9) becomes the class of Kolm indices*

$$\left\{I_{\mathrm{K}}^{\beta}(F) : \beta \in \mathbb{R}_{+}\right\}$$

where:

$$I_{\mathrm{K}}^{\beta}(F) := \frac{1}{\beta}\left[\int e^{\beta[x-\mu(F)]} dF(x) - 1\right] \tag{14}$$

Proof. To examine the limiting form of (10) note that the parameter restriction ensures that, for finite $x \in \mathfrak{X}$ and k sufficiently large, we have $\frac{x}{k} \in (-1, 1)$. So, consider the function

$$\chi(x, y, \alpha, k) := \log\left(\frac{\phi(x)}{\phi(y)}\right) = \alpha(k)\left[\log\left(1+\frac{x}{k}\right) - \log\left(1+\frac{y}{k}\right)\right]. \tag{15}$$

Using the standard expansion

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \ldots \tag{16}$$

and (8) we find

$$\chi(x, y, \alpha, k) = \left[\beta + \frac{\gamma}{k}\right]\left[x - y - \frac{x^2}{2k} + \frac{y^2}{2k} + \frac{x^3}{2k^2} - \frac{y^3}{2k^2} - \ldots\right]. \tag{17}$$

[6] This condition is very restrictive. Indices with values of $\alpha(k) \geq 4$ are likely to be impractical and may also be regarded as ethically unattractive, in that they are very sensitive to income transfers amongst the rich and the super-rich.

For finite γ, β, x, y we have:

$$\lim_{k\to\infty} \chi(x, y, \alpha, k) = \beta [x - y] \tag{18}$$

and

$$\lim_{k\to\infty} A(k) = \lim_{k\to\infty} \frac{1 + \frac{1}{k^2}}{\left[\beta + \frac{\gamma}{k}\right]^2 - \frac{1}{k}\left[\beta + \frac{\gamma}{k}\right]} = \frac{1}{\beta^2}. \tag{19}$$

So we obtain

$$\lim_{k\to\infty} I_{\text{ext}}^{\alpha,k}(F) = \frac{1}{\beta^2} \int [\exp(\beta [x - \mu(F)]) - 1] \, dF(x). \tag{20}$$

■

This family of Kolm indices form the translation-invariant counterparts of the family (5) (Eichhorn and Gehrig 1982, Toyoda 1980).[7]

Theorem 4 *As $k \to \infty$ and $\beta \to 0$ (10) converges to the variance*

Proof. An expansion of (20) gives

$$\int \left[\frac{1}{2!}[x - \mu(F)]^2 + \frac{1}{3!}\beta[x - \mu(F)]^3 + \frac{1}{4!}\beta^2[x - \mu(F)]^4 + \ldots \right] dF(x)$$

As $\beta \to 0$ this becomes the variance. ■

5.2 Restricted domain: x bounded above

A number of papers in the mainstream literature make the assumption that there is a finite maximum income.[8] If we adopt this assumption then it makes sense to consider parameter values such that $-k \geq \sup(\mathfrak{X})$. However, it is immediate that the new parameter restriction again ensures that, for finite $x \in \mathfrak{X}$ and $(-k)$ sufficiently large, we have $\frac{x}{k} \in (-1, 1)$. Therefore the same argument can be applied as in equations (15) to (20) above: again the evaluation function converges to that of the Kolm class of leftist inequality measures.

[7] See Foster and Shneyerov (1999).

[8] See for example Atkinson (1970).

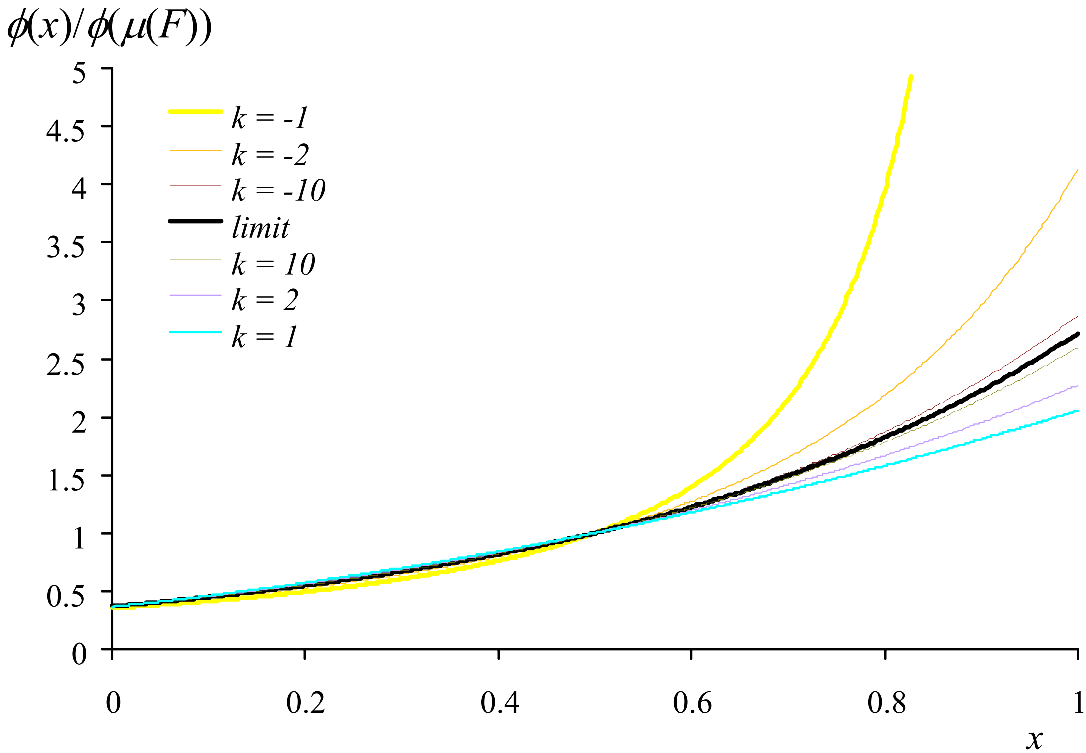

Figure 1: Values of $\phi(x)/\phi(\mu(F))$ as k varies: $\mathfrak{X} = [0,1], \gamma = 0.5, \beta = 2, \mu(F) = 0.5$

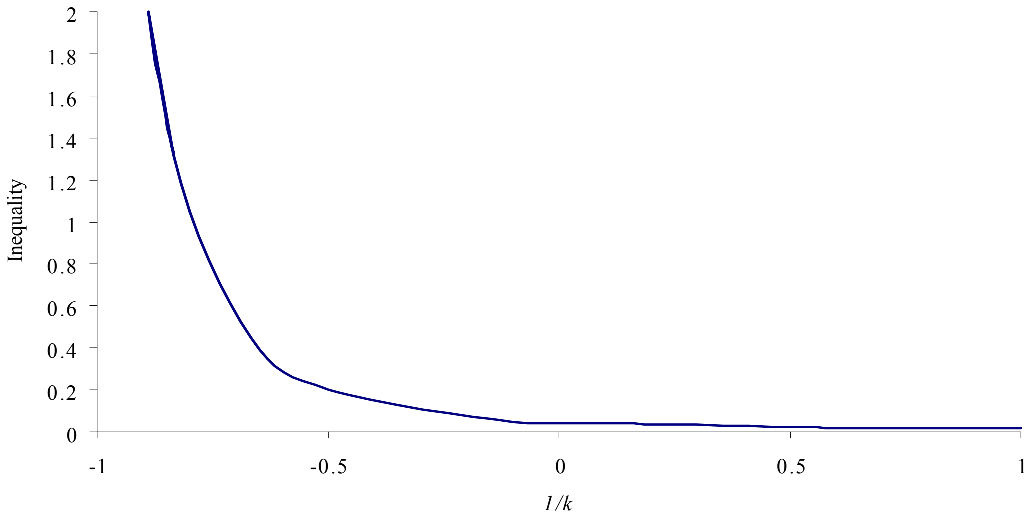

Figure 2: Inequality and k for a rectangular distribution on $[0, 1]$.

The behaviour of the evaluation function ϕ as the location parameter changes is illustrated in Figure 1: the limiting form is the heavy line in the middle of the figure. As $k \to +\infty$ the evaluation functions of the $\bar{\Im}$ class approach this from the bottom right; as $k \to -\infty$ the evaluation functions approach it from the top left. Figure 2 shows the relationship of overall inequality to the parameter k when income is distributed uniformly on the unit interval: note that the limiting case (where the inequality measure is ordinally equivalent to the "leftist" Kolm index) is given by the point $1/k = 0$.

5.3 Interpretations

The reformulation (10) is equivalent to (6) in that, given any arbitrary values of the location parameter k and the exponent in the evaluation function (13), one can always find values of γ, β such that $\alpha(k) = \gamma + \beta k$. Clearly there is a redundancy in parameters (for finite positive k one can always arbitrarily fix either γ or β), but that does not matter because the important special cases drop out naturally as we let k go to 0 (Generalised Entropy) or to ∞ (Kolm). Of course the normalisation constant $A(k)$ could be specified in some other way for convenience, but this does not matter either.

The general formulation allows one to set up a correspondence between the Generalised Entropy class of measures, including the Theil indices and the Kolm leftist class of measures ($k = \infty$). Consider, for example the subclass that is defined by the restriction $\beta = \gamma$

$$I_{\text{ext}}^{\beta,k}(F) := A(k) \int \left[\left[\frac{x+k}{\mu(F)+k} \right]^{\beta[1+k]} - 1 \right] dF(x) \tag{21}$$

Putting $k = 0$ one immediately recovers the Generalised Entropy class with parameter β. However, letting $k \to \infty$ Theorem 3 gives the Kolm index with parameter β.

6 Conclusion

In the decades since Theil's book a number of families of inequality measures have become standard tools in the analysis of income distribution. A relatively small number of key properties characterise each family and the sets of characteristic properties bear a notable resemblance to each other. This paper has shown that many of these standard families of inequality measures are interrelated in a way that is somewhat deeper than that which has been noted in previous consideration of Theil's contribution.

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