THE AVERAGED PERIODOGRAM FOR NONSTATIONARY VECTOR TIME SERIES*

by

P M Robinson
London School of Economics and Political Science

and

D Marinucci
Università degli Studî di Roma “La Sapienza

Contents:

The Suntory Centre
Suntory and Toyota International Centres
for Economics and Related Disciplines
London School of Economics and Political Science
Houghton Street
London WC2A 2AE
Tel.: 020-7405 7686

Discussion Paper
No. EM/00/408
December 2000

---

* Research supported by ESRC grant R000235892. The first author was also supported by a Leverhulme Trust Research Professorship and the second author by MURST.

Abstract

Frequency domain statistics are studied in the presence of fractional deterministic and stochastic trends. It is shown how the behaviour of the sample variance-covariance matrix of nonstationary processes can be dominated by components corresponding to a possibly degenerating band around zero frequency. This property is used to establish the limiting distribution of the averaged periodogram matrix, of memory estimates for nonstationary series, and for frequency domain regression estimates under nonstandard conditions.

Keywords: Averaged periodogram; nonstationary processes; fractional Brownian motion.

JEL No.: C22

1. INTRODUCTION

For a sequence of column vectors u_t, $t = 1, 2, ..., n$, with real-valued elements, define the discrete Fourier transform

$$w_u(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} (u_t - \overline{u}) e^{it\lambda},$$

where $\overline{u} = n^{-1} \sum_{t=1}^{n} u_t$ denotes the sample mean. Given also a vector sequence $v_t, t = 1, ..., n$, define the cross-periodogram matrix

$$I_{uv}(\lambda) = w_u(\lambda) w_v^*(\lambda), \tag{1.1}$$

the asterisk denoting complex conjugation combined with transposition. Writing $\lambda_j = 2\pi j/n$ for integer j, define

$$\widehat{F}_{uv}(\ell, m) = \frac{2\pi}{n} \sum_{j=\ell}^{m} I_{uv}(\lambda_j), \tag{1.2}$$

for $1 \leq \ell \leq m \leq n - 1$. We can think of (1.2) as a discrete version of the continuous average $\widetilde{F}_{uv}(\lambda_{\ell-1}, \lambda_{m+1})$, where

$$\widetilde{F}_{uv}(\nu, \omega) = \int_{\nu}^{\omega} I_{uv}(\lambda) d\lambda. \tag{1.3}$$

Indeed by the orthogonality properties of the complex exponential

$$\widehat{F}_{uv}(1, n-1) = \widetilde{F}_{uv}(0, 2\pi) = \frac{1}{n} \sum_{t=1}^{n} (u_t - \overline{u})(v_t - \overline{v})', \tag{1.4}$$

where the prime denotes transposition.

We term (1.3) the continuously averaged cross-periodogram matrix, and (1.2) the discretely averaged periodogram matrix, or more briefly the averaged periodogram. The case u_t, v_t scalar, where especially $u_t \equiv v_t$, has been stressed in the literature. Early references to the study of scalar $\widetilde{F}_{uu}(0, \lambda)$ in case of stationary short range dependent and long range dependent u_t are, respectively, Grenander and Rosenblatt (1957) and Ibragimov (1963). For stationary short range dependent vector u_t, v_t, and λ degenerating slowly to zero as $n \to \infty$, a simple function of $\widetilde{F}_{uv}(0, \lambda)$ consistently estimates the cross spectral density of u_t, v_t at frequency zero. The same role is played by $\widehat{F}_{uv}(1, m)$ when

$$\frac{1}{m} + \frac{m}{n} \to 0, \quad \text{as } n \to \infty, \tag{1.5}$$

see e.g. Brillinger (1975). In typical short range circumstances $\widehat{F}_{uv}(\ell, m)$ and $\widetilde{F}_{uv}(\lambda_{\ell-1}, \lambda_{m+1})$ have close asymptotic properties, while Robinson (1994a,b) found both similarities and differences under long range dependence.

Here we shall study $\widehat{F}_{uv}(1, m)$ for nonstationary vector series u_t, v_t under the more general condition

$$m \leq \frac{n}{2}; \quad m \to \infty \text{ as } n \to \infty, \tag{1.6}$$

which also permits m to increase as fast as n. Of course eventually $m \leq n/2$ under (1.5), while on the other hand eventually (1.5) is a special case of (1.6), but in any case the requirement $m \leq n/2$ is virtually costless because, for $\pi < \lambda < 2\pi$,

$$\Re e \{I_{uv}(2\pi - \lambda)\} = \Re e \{I_{uv}(\lambda)\}, \quad \Im m \{I_{uv}(2\pi - \lambda)\} = -\Im m \{I_{uv}(\lambda)\},$$

so that, for example, (cf (1.4))

$$\widehat{F}_{uv}(1, n-1) = 2\Re e \left\{ \widehat{F}_{uv}\left(1, \frac{n-1}{2}\right) \right\}, \quad n \text{ odd} \tag{1.7}$$

$$\widehat{F}_{uv}(1, n-1) = 2\Re e \left\{ \widehat{F}_{uv}\left(1, \frac{n}{2} - 1\right) \right\} + \frac{2\pi}{n} I_{uv}(\pi), \quad n \text{ even}. \tag{1.8}$$

It is in fact only the real part of $\widehat{F}_{uv}(1, m)$ which is relevant to the particular applications considered in the present paper, but the theorem of the following section, our main result, covers both real and imaginary parts. We mentioned $\widetilde{F}_{uv}(\nu, \omega)$ for comparison's sake due to its classical interest, but it has less computational appeal than $\widehat{F}_{uv}(1, m)$ so we shall not discuss it further.

In the vector case it suffices to consider $\widehat{F}_{uu}(1, m)$, relative to $\widehat{F}_{uv}(1, m)$, because $\widehat{F}_{uu}(1, m)$ includes the averaged cross-periodogram of disjoint subvectors of u_t. Our nonstationary vector stochastic process u_t obeys the model

$$u_t = x_t + \mu_t, \quad t = 1, 2, ..., \tag{1.9}$$

where x_t is a nonstationary vector stochastic process while μ_t is a deterministic vector function of t. Detailed specification of x_t and μ_t is given in the following section, and we only mention here that the decomposition in (1.9) is uniquely defined by the property that $E(x_t) = 0$, $t \geq 1$. Many econometric models, for example, assume an additive representation covered by (1.9).

The main point of the first theorem of the following section is that $\widehat{F}_{uu}(1, n-1)$, and *a fortiori* $\widehat{F}_{uu}(1, M)$, $m < M \leq n/2$, are dominated (under (1.6)) by $\widehat{F}_{uu}(1, m)$ (c.f. (1.7), (1.8)). Note that, for such M,

$$\widehat{F}_{uu}(1, M) = \widehat{F}_{uu}(1, m) + \widehat{F}_{uu}(m+1, M), \tag{1.10}$$

and the theorem derives stochastic bounds for both components on the right side the second one's bound being the smaller. Similar results have been given by Robinson and Marinucci (1997). On the one hand, these authors do not include the deterministic component μ_t of (1.9), while on the other they obtain more refined results under otherwise milder conditions. Note that, unlike the present paper and Robinson and Marinucci (1997), the large literature on averaged periodogram statistics is restricted to stationary series, with rare exceptions such as Phillips (1991), whose proofs actually involve weighted autocovariance type spectral estimates, which can have different asymptotic properties from the averaged periodogram under nonstationarity. Section 2 also applies the theorem, along with invariance principles for vector nonstationary fractional sequences of Marinucci and Robinson (1998) (see also Akonom and Gourieroux, 1987, Silveira, 1991), to establish the limit distribution of the averaged periodogram. Section 3 provides two applications. We give limit theory for estimates of the memory parameters of nonstationary vector sequences. We also give conditions for full-band and narrow-band frequency domain least squares estimates of regression parameters to have the same normal limit distribution when both regressors and errors have form (1.9).

2. MAIN RESULT

We began by completing the specification of (1.9). Let C be a generic positive constant, and $\|\cdot\|$ denote the Euclidean norm.

Assumption A Let u_t be given by (1.9), where:

(i)

$$x_t = \sum_{s=1}^{t} \Psi_{t-s}\varepsilon_s, \tag{2.1}$$

$$E\varepsilon_t = 0,\ E\varepsilon_s\varepsilon_t' = 0,\ E\|\varepsilon_t\|^2 \leq C < \infty,\ 1 \leq s \neq t, \tag{2.2}$$

the ε_t being $p \times 1$ vectors and the Ψ_t being $p \times p$ matrices with (a,b)th elements $\psi_{ab,t}$ satisfying

$$\left|\psi_{ab,t}\right| \leq C(1+t)^{d_a-1},\ \left|\psi_{ab,t} - \psi_{ab,t+1}\right| \leq C\frac{\left|\psi_{ab,t}\right|}{t},\ a,b = 1,...,p,\ t \geq 1, \tag{2.3}$$

for

$$d_a > \frac{1}{2},\quad a = 1,...,p. \tag{2.4}$$

(ii) μ_t has $a-$th element μ_{at} satisfying

$$|\mu_{at}| \leq C(1+t)^{\delta_a-\frac{1}{2}},\ \left|\mu_{at} - \mu_{a,t+1}\right| \leq C\frac{|\mu_{at}|}{t},\ a,b = 1,...,p,\ t \geq 1, \tag{2.5}$$

for

$$\delta_a > 0,\quad a = 1,...,p. \tag{2.6}$$

In (2.2), the innovations ε_t need not be identically distributed or independent or martingale differences, while some stable heterogeneity is allowed for. Moreover, finiteness of only second moments of x_t, and thus of u_t, is implied, by contrast with Robinson and Marinucci's (1997) fourth order stationarity assumption on ε_t. The latter, however, assists these authors in deriving results for averaged cross-periodograms involving a nonstationary and (asymptotically) stationary series, whereas no element of x_t can be asymptotically stationary in view of (2.4). Robinson and Marinucci (1997) also relaxed the uncorrelatedness assumption on ε_t to a much more general one of short range dependence.

The different roles played by the d_a and δ_a in the exponents in (2.3) and (2.5) will be of subsequent notational convenience. Clearly the conditions (2.3) and (2.5) on the Ψ_t and μ_t are of the same character, though in case of μ_t one thinks of polynomials as leading examples, whereas in case of Ψ_t it follows, in view also of (2.2), that x_t can be the nonstationary fractionally integrated vector autoregressive moving average

$$x_t = \Delta^{-1}(L)\Phi^{-1}(L)\Theta(L)\left\{\varepsilon_t 1_{t\geq 0}\right\}, \tag{2.7}$$

where L is the lag operator, 1_A is the indicator function of the set A, $\Phi(L)$ and $\Theta(L)$ are finite order matrix lag polynomials with zeros outside the unit circle, and

$$\Delta^{-1}(L) = diag\left\{(1-L)^{-d_1}, ..., (1-L)^{-d_p}\right\}, \tag{2.8}$$

$$(1-L)^{-d} = \sum_{k=0}^{\infty}\frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)}L^k, \quad \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx. \tag{2.9}$$

Define

$$\theta_a = \max(d_a, \delta_a), \quad 1 \leq a \leq p.$$

Let ε be an arbitrarily small positive number, and write I_{uv}^{ab} and $\widehat{F}_{uv}^{ab}$ for the (a,b)th elements of I_{uv} and $\widehat{F}_{uv}$ in (1.1), (1.2).

Theorem 1 Under Assumption A and (1.6), for $1 \leq m < M \leq n/2$, as $n \to \infty$, and $a, b = 1, ..., p$,

$$\begin{aligned}\widehat{F}_{uu}^{ab}(1,m) &= O_p\left(n^{\theta_a+\theta_b-1}\right), \\ \widehat{F}_{uu}^{ab}(m+1,M) &= o_p\left(n^{\theta_a+\theta_b-1}\right),\end{aligned} \tag{2.10}$$

when $d_a \neq 1$, $\delta_a \neq \frac{1}{2}$, $d_b \neq 1$, $\delta_b \neq \frac{1}{2}$, and thus if the bound in (2.10) is an exact rate

$$\widehat{F}_{uu}(1,m) = \frac{1}{2n}\sum_{t=1}^{n}(u_t - \overline{u})(u_t - \overline{u})'(1 + o_p(1)), \tag{2.11}$$

whereas when $d_a = 1$ or $\delta_a = \frac{1}{2}$ or $d_b = 1$ or $\delta_b = \frac{1}{2}$

$$\begin{aligned}\widehat{F}_{uu}^{ab}(1,m) &= O_p\left(n^{\theta_a+\theta_b-1+\varepsilon}\right), \\ \widehat{F}_{uu}^{ab}(m+1,M) &= o_p\left(n^{\theta_a+\theta_b-1+\varepsilon}\right),\end{aligned}$$

for any $\varepsilon > 0$.

Proof From the Cauchy inequality, for $\ell \leq m$,

$$\left|\widehat{F}_{uu}^{ab}(\ell,m)\right| \leq \left\{\widehat{F}_{uu}^{aa}(\ell,m)F_{uu}^{bb}(\ell,m)\right\}^{\frac{1}{2}}. \tag{2.12}$$

We can thus deduce results for $a \neq b$ from those for $a = b$. Further, we can study stochastic and deterministic components separately because

$$\widehat{F}_{uu}^{aa}(\ell, m) \leq 2\widehat{F}_{xx}^{aa}(\ell, m) + 2\widehat{F}_{\mu\mu}^{aa}(\ell, m). \tag{2.13}$$

With ψ'_{at} the $a-$th row of Ψ_t,

$$\begin{aligned} I_{xx}^{aa}(\lambda) &= \frac{1}{2\pi n}\sum_{t=1}^{n}\sum_{s=1}^{n} e^{i(s-t)\lambda}\sum_{j=1}^{s}\sum_{k=1}^{t}\psi'_{a,s-j}\varepsilon_j\varepsilon'_k\psi_{a,t-k} \\ &= \frac{1}{2\pi n}\sum_{s=1}^{n}\sum_{t=1}^{n} e^{i(s-t)\lambda}\sum_{k=1}^{\min(s,t)}\psi'_{a,s-k}\varepsilon_k\varepsilon'_k\psi_{a,t-k} \qquad (2.14) \\ &\quad + \frac{1}{2\pi n}\sum_{s=1}^{n}\sum_{t=1}^{n} e^{i(s-t)\lambda}\sum_{j=1}^{s}\sum_{k=1,k\neq j}^{t}\psi'_{a,s-j}\varepsilon_j\varepsilon'_k\psi_{a,t-k}. \end{aligned}$$

The expectation of this, non-negative scalar random variable, equals the expectation of (2.14) alone, and is thus, writing $\psi_{at} = 0$, $t < 0$,

$$\begin{aligned} &\frac{1}{2\pi n}\sum_{s=1}^{n}\sum_{t=1}^{n} e^{i(s-t)\lambda}\sum_{k=1}^{\infty}\psi'_{a,s-k}E(\varepsilon_k\varepsilon'_k)\psi_{a,t-k} \\ &= \frac{1}{2\pi n}\sum_{k=1}^{\infty}\left\{\sum_{s=1}^{n} e^{is\lambda}\psi_{a,s-k}\right\}' E(\varepsilon_k\varepsilon'_k)\left\{\sum_{t=1}^{n} e^{-it\lambda}\psi_{a,t-n}\right\}. \end{aligned} \tag{2.15}$$

Since the kth summand is a non-negative quadratic form, it follows from (2.2) that (2.15) is bounded by a constant times

$$\begin{aligned} \frac{1}{2\pi n}\sum_{k=1}^{\infty}\left\|\sum_{s=1}^{n} e^{is\lambda}\psi_{a,s-k}\right\|^2 &= \frac{1}{2\pi n}\sum_{k=1}^{n}\left\|\sum_{s=1}^{n} e^{is\lambda}\psi_{a,s-k}\right\|^2 \\ &= \frac{1}{2\pi n}\sum_{k=1}^{n}\left\|\sum_{t=0}^{n-k} e^{it\lambda}\psi_{at}\right\|^2. \end{aligned} \tag{2.16}$$

On the other hand, for the deterministic μ_t,

$$I_{\mu\mu}^{aa}(\lambda) = \frac{1}{2\pi n}\left|\sum_{t=1}^{n} e^{it\lambda}\mu_{at}\right|^2. \tag{2.17}$$

Now from Robinson and Marinucci (1997), we deduce that for a real valued sequence κ_t such that

$$|\kappa_t| \leq Ct^{\rho-1}, \ \ |\kappa_t - \kappa_{t+1}| \leq \frac{C\,|\kappa_t|}{t}, \ t \geq 0, \ \rho > \frac{1}{2}, \tag{2.18}$$

we have, for $0 < |\lambda| \le \pi$, $v \ge 1$,

$$\left|\sum_{t=1}^{v} e^{it\lambda}\kappa_t\right| \le \frac{C}{\lambda^{\rho}},\ \frac{1}{2} < \rho < 1, \qquad (2.19)$$

$$\le \frac{Cv^{\rho-1}}{|\lambda|},\ \rho > 1, \qquad (2.20)$$

cf. Kokoszka and Taqqu (1996, Lemma 3.1). For $\rho = 1$ the definition of κ_t in Robinson and Marinucci (1997) is more special than that in (2.18), so their result does not necessarily apply, but by proceeding in a similar way as they did for $\rho \neq 1$, applying summation-by-parts and the usual bound for the Dirichlet kernel,

$$\left|\sum_{t-1}^{v} e^{it\lambda}\kappa_t\right| \le \frac{C}{|\lambda|}\sum_{t=1}^{v}(t^{-1}+1) \le \frac{C\log v}{|\lambda|},\ \rho = 1. \qquad (2.21)$$

We then deduce from (2.16), (2.17), (2.19) - (2.21) that, for $0 < |\lambda| \le \pi$,

$$\begin{aligned} EI_{xx}^{aa}(\lambda) &\le \frac{C}{|\lambda|^{2d_a}},\ \frac{1}{2} < d_a < 1, \\ &\le C\frac{(\log n)^2}{\lambda^2},\ d_a = 1, \\ &\le C\frac{n^{2d_a-2}}{\lambda^2},\ d_a > 1, \\ I_{\mu\mu}^{aa}(\lambda) &\le \frac{C}{n\,|\lambda|^{2\delta_a+1}},\ 0 < \delta_a < \frac{1}{2}, \\ &\le \frac{C}{n}\frac{(\log n)^2}{\lambda^2},\ \delta_a = \frac{1}{2}, \\ &\le \frac{Cn^{2\delta_a-2}}{\lambda^2},\ \delta_a > \frac{1}{2}. \end{aligned}$$

It follows that from (1.2)

$$\begin{aligned} E\widehat{F}_{xx}^{aa}(\ell,m) &\le C\left(\frac{n}{\ell}\right)^{2d_a-1},\ \frac{1}{2} < d_a < 1, \\ &\le Cn\frac{(\log n)^2}{\ell},\ d_a = 1, \\ &\le C\frac{n^{2d_a-1}}{\ell},\ d_a > 1, \end{aligned}$$

and

$$
\begin{aligned}
\widehat{F}^{aa}_{\mu\mu}(\ell, m) &\leq C\frac{n^{2\delta_a - 1}}{\ell^{2\delta_a}}, \; 0 < \delta_a < \frac{1}{2}, \\
&\leq C\frac{(\log n)^2}{\ell}, \; \delta_a = \frac{1}{2}, \\
&\leq C\frac{n^{2\delta_a - 1}}{\ell}, \; \delta_a > \frac{1}{2}.
\end{aligned}
$$

Altogether we can write, for some $\varepsilon > 0$,

$$
\begin{aligned}
E\widehat{F}^{aa}_{xx}(\ell, m) &\leq C\frac{n^{2d_a - 1}}{\ell^{\varepsilon}} \left\{1 + 1_{d_a = 1}(\log n)^2\right\}, \\
\widehat{F}^{aa}_{\mu\mu}(\ell, m) &\leq C\frac{n^{2\delta_a - 1}}{\ell^{\varepsilon}} \left\{1 + 1_{\delta_a = \frac{1}{2}}(\log n)^2\right\},
\end{aligned}
$$

so that from (2.13)

$$
E\widehat{F}^{aa}_{uu}(\ell, m) \leq C\frac{n^{2\theta_a - 1}}{\ell^{\varepsilon}} \left[1 + \left\{1_{d_a = 1} + 1_{\delta_a = \frac{1}{2}}\right\} n^{\varepsilon}\right],
$$

whence the results follow quickly from (2.12), (1.4), (1.6) - (1.8) and (1.10). □

It follows that the contribution from all but an arbitrarily slowly increasing number of Fourier frequencies to half the sample covariance matrix $\widehat{F}_{uu}(1, n-1)$ (see (1.4), (2.11)) is negligible when the bound in (2.10) represents an exact rate, which may be shown to be true under stronger assumptions. Such assumptions can also lead, indeed, to a representation for the limit distribution of $\widehat{F}_{uu}(1, n-1)$, and thence of $\widehat{F}_{uu}(1, m)$ and of various functions of these statistics that are of interest. We introduce

Assumption B: Let Assumption A hold, let $\min_{1 \leq a \leq p} \delta_a \geq 0$ and let ε_t be an independent identically distributed zero mean sequence such that, for $d_* = \min_{1 \leq a \leq p} d_a$,

$$
E\varepsilon_t \varepsilon_t' = \Omega, \quad E\left\|\varepsilon_t\right\|^{\gamma} \leq C, \; \gamma > \max\left(2, \frac{1}{2d_* - 1}\right);
$$

also, for $a, b = 1, ..., p$, as $t \to \infty$ let

$$
\psi_{an,t} \sim g_{ab}(1+t)^{d_a - 1}, \quad \mu_{at} \sim h_a(1+t)^{\delta_a},
$$

for finite nonzero constants g_{ab}, h_a where "$\sim$" indicates that the ratio of left and right hand sides tends to 1; "$\sim$" is replaced by "$=$" for $d_a = 1, \delta_a = 0$.

Assumption B can be checked for the model given by (2.7) - (2.9).

For $d = (d_1, ..., d_p)$, $\delta = (\delta_1, ..., \delta_p)$, define the matrices

$$D(n; d, \delta) = diag\left\{n^{\theta_1 - \frac{1}{2}}, ..., n^{\theta_p - \frac{1}{2}}\right\}, \quad G(r; d) = \left\{g_{ab} r^{d_a - 1}\right\},$$

and the variates

$$\begin{aligned} W(r; d) &= \int_0^r G(r - s; d) dB(s; \Omega), \quad W(d) = \int_0^1 W(r; d) dr, \\ V(d) &= \int_0^1 \left\{W(r; d)W(r; d)' - W(d)W(d)'\right\} dr, \end{aligned}$$

where $B(r; \Omega)$ is multivariate scaled Brownian motion, defined as a zero mean continuous Gaussian process with independent increments and variance matrix $EB(r; \Omega)B(r; \Omega)' = \Omega r$, $0 < r < 1$; $W(r; d)$ was referred to as Type II multivariate fractional Brownian motion by Marinucci and Robinson (1999). Also define

$$\begin{aligned} \mu(r) &= \left(h_1 r^{\delta_1 - \frac{1}{2}}, ..., h_p r^{\delta_p - \frac{1}{2}}\right), \quad \overline{\mu} = \int_0^1 \mu(r) dr, \\ H(\delta) &= \int_0^1 \left\{\mu(r)\mu(r)' - \overline{\mu}\overline{\mu}'\right\} dr, \quad T(d, \delta) = \int_0^1 \mu(r) W(r; d)' dr, \end{aligned}$$

and denote by $V^{ab}(d)$, $H^{ab}(\delta)$, $T^{ab}(d, \delta)$ the (a, b)th elements of $V(d)$, $H(\delta)$ and $T(d, \delta)$.

Theorem 2 Under Assumptions A and B and (1.6), as $n \to \infty$

$$2D^{-1}(n; d, \delta)\widehat{F}_{uu}(1, m)D^{-1}(n; d, \delta) \to_d L(d, \delta),$$

where the variate $L(d, \delta)$ has (a, b)th element

$$L^{ab} = V^{ab}(d)1_{d_a + d_b = \theta_a + \theta_b} + T^{ab}(d, \delta)1_{d_b + \delta_a = \theta_a + \theta_b} + T^{ba}(d, \delta)1_{d_a + \delta_b = \theta_a + \theta_b} + H^{ab}(\delta)1_{\delta_a + \delta_b = \theta_a + \theta_b}.$$

Proof We have

$$\begin{aligned} \widehat{F}_{uu}(1, m) &= \left\{\widehat{F}_{uu}(1, m) - \frac{1}{2}\widehat{F}_{uu}(1, n - 1)\right\} \\ &\quad + \frac{1}{2}\left\{\widehat{F}_{xx}(1, n - 1) + \widehat{F}_{x\mu}(1, n - 1) + \widehat{F}_{\mu x}(1, n - 1) + \widehat{F}_{\mu\mu}(1, n - 1)\right\}. \end{aligned}$$

From Theorem 1 (see (2.11), (1.4), (1.7), (1.8))

$$D^{-1}\left\{\widehat{F}_{uu}(1,m)-\frac{1}{2}\widehat{F}_{uu}(1,n-1)\right\}D^{-1}=o_p(1), \text{ as } n\to\infty,$$

where D abbreviates $D(n;d,\delta)$. Writing $D_1 = diag\left\{n^{d_1-\frac{1}{2}},...,n^{d_p-\frac{1}{2}}\right\}$, we have from Theorem 1 of Marinucci and Robinson (1997) that as $n\to\infty$

$$D_1^{-1}\left(x_{\lfloor nr\rfloor}-\overline{x}_n\right)\Rightarrow W(r;d)-W(d), \quad 0<r<1,$$

the convergence holding in the sense of Billingsley (1968) in the Skorokhod space $D[0,1]^p$. Now consider the space of mappings $G\in\mathcal{G}$ from $R\to R^{p\times p}$ such that $\left\|\int_0^1 Gdr\right\|<\infty$: because $W(r;d)'$ is continuous with probability one we have $\mu(r)W(r;d)'\in\mathcal{G}$ a.s., and hence from the continuous mapping theorem

$$D_1^{-1}\widehat{F}_{xx}(1,n-1)D_1^{-1}\rightarrow_d V(d), \quad D_1^{-1}\widehat{F}_{uu}(1,n-1)D_2^{-1}\rightarrow_d T(d,\delta),$$

as $n\to\infty$, where D_2 is $diag\left\{n^{\delta_1-\frac{1}{2}},...,n^{\delta_p-\frac{1}{2}}\right\}$, while by dominated convergence

$$D_2^{-1}\widehat{F}_{\mu\mu}(1,n-1)D_2^{-1}\rightarrow H(\delta), \quad \text{as } n\to\infty.$$

To complete the proof note that as $n\to\infty$

$$D^{-1}D_1\rightarrow diag\left\{1_{\theta_1=d_1},...,1_{\theta_p=d_p}\right\}, \; D^{-1}D_2\rightarrow diag\left\{1_{\theta_1=\delta_1},...,1_{\theta_p=\delta_p}\right\}. \qquad \square$$

3. APPLICATIONS

We apply Theorems 1 and 2 to two examples, namely memory parameter estimation and regression estimation.

Consider the following estimate of the "maximal memory" θ_a,

$$\widehat{\theta}_a=\frac{1}{2}\left\{1+\frac{\log\widehat{F}_{uu}^{aa}(1,m)}{\log n}\right\}, \qquad a=1,...,p.$$

Theorem 3 Under Assumptions A, B and (1.6), as $n\to\infty$

$$(\log n)(\widehat{\theta}_a-\theta_a)\rightarrow_d\frac{1}{2}\log L^{aa}(d,\delta), \quad a=1,...,p. \tag{3.1}$$

Proof We can write $(\log n)(\widehat{\theta}_a - \theta_a)$ as

$$\frac{1}{2}\log\left\{n^{1-2\theta_a}\widehat{F}_{uu}^{aa}(1,m)\right\},$$

whence (3.1) follows from Theorem 2 and the continuous mapping theorem. □

The estimate $\widehat{\theta}_a$ automatically adjusts to whether the stochastic or the deterministic component dominates. Its structure is somewhat similar to that of the averaged periodogram memory parameter estimate of Robinson (1994a,b), but it does not require his tuning parameter and does not require the bandwidth m to increase slower than n. On the other hand, the Theorem implies $\widehat{\theta}_a$ is only $(\log n)$-consistent, though note the generality of the conditions.

For the other application, partition the variables in (1.9) as

$$u_t = \begin{pmatrix} v_t \\ w_t \end{pmatrix}, \qquad x_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}, \qquad \mu_t = \begin{bmatrix} \zeta_t \\ \xi_t \end{bmatrix},$$

where v_t, y_t and ζ_t are $(p-1)\times 1$ vectors and w_t, z_t and ξ_t are scalars, and suppose that, of these, only v_t is observable, along with the scalar s_t such that

$$s_t = \beta' v_t + w_t, \qquad t = 1, 2, ..., \tag{3.2}$$

where β is an unknown $(p-1)\times 1$ vector. To estimate β consider

$$\widehat{\beta}_m = \left[\Re e\left\{\widehat{F}_{vv}(1,m)\right\}\right]^{-1}\Re e\left\{\widehat{F}_{vs}(1,m)\right\},$$

as in Robinson (1994a), Robinson and Marinucci (1997). Note from (1.4) that $\widehat{\beta}_{n-1}$ is just the least squares estimate with intercept correction, while under (1.5) $\widehat{\beta}_m$ is a least squares estimate using only the discrete Fourier transforms closest to the origin.

Denote by $\zeta(r)$ the leading $(p-1)\times 1$ sub-vector of $\mu(r)$ and $H_{vv}(\delta)$ the leading $(p-1)\times(p-1)$ submatrix of $H(\delta)$, and let $D_3 = diag\left\{n^{\delta_1 - d_p}, ..., n^{\delta_{p-1}-d_p}\right\}$, $g_p = (g_{p1}, ..., g_{pp})'$.

Theorem 4 Let Assumptions A, B and (1.6) hold, with

$$\delta_a > d_a, \quad \delta_a > d_p, \quad i = 1, ..., p-1, \tag{3.3}$$

$$d_p > \delta_p, \tag{3.4}$$

and let

$$r\left\{H_{vv}(\delta)\right\} = p - 1. \tag{3.5}$$

Then as $n \to \infty$

$$D_3(\widehat{\beta}_m - \beta) \to_d N(0, \sum),$$

where

$$\sum = \frac{g_p' \Omega g_p}{2d_p - 1} H_{vv}^{-1}(\delta) \int_0^1 r^{2d_p - 1} \zeta(r)\zeta(r)' dr H_{vv}^{-1}(\delta).$$

<u>Proof</u> From (3.3), (3.4) the leading $(p-1) \times (p-1)$ submatrix of D is $D_v = diag\left\{n^{\delta_1 - \frac{1}{2}}, ..., n^{\delta_{p-1} - \frac{1}{2}}\right\}$, while its remaining nonzero element is $n^{d_p - \frac{1}{2}}$. It follows that

$$D_3(\widehat{\beta}_m - \beta) = \left\{D_v^{-1} \Re_e \left\{\widehat{F}_{vv}(1,m)\right\} D_v^{-1}\right\}^{-1} D_v^{-1} \Re_e \left\{\widehat{F}_{vw}(1,m)\right\} n^{\frac{1}{2} - d_p}.$$

From Theorem 2 and (3.3), (3.4), as $n \to \infty$

$$D_v^{-1} \widehat{F}_{vv}(1,m) D_v^{-1} \underset{p}{\to} H_{vv}(\delta),$$
$$D_v^{-1} \widehat{F}_{vw}(1,m) n^{\frac{1}{2} - d_p} \to_d T_{vw}(d, \delta),$$

where $T_{vw}(d, \delta)$ is the leading $(p-1) \times 1$ subvector of $T(d, \delta)$. However it is straightforward to show that

$$T_{vw}(d, \delta) \sim N\left(0, \frac{g_p' \Omega g_p}{2d_p - 1} \int_0^1 r^{2d_p - 1} \zeta(v)\zeta(v)' dv\right). \qquad \square$$

Our specification of regressors and errors in (3.2) is nonstandard in view of the presence of stochastic nonstationarity and possible deterministic trends in the errors, that is a nonstationary ζ_t and a non-constant ξ_t, and the possible correlation between regressors and errors, that is between the components y_t of v_t and z_t of w_t. The latter feature, though not the former ones, was present in the treatments of Robinson (1994a), Robinson and Marinucci (1997), indeed these authors included no deterministic components and the first reference assumed stationarity throughout. In the situations covered by these authors nonstandard limit distributions result, and the reason for the normal limit in the present circumstances is the domination of the deterministic components of the regressors over the stochastic components of both regressors and errors, and the domination of the stochastic component of the errors over their deterministic component, as provided by (3.3), (3.4).

References

Akonom, J. and Gourieroux, C. (1987), "A functional central limit theorem for fractional processes", preprint, CEPREMAP.

Billingsley, P. (1968) *Convergence of Probability Measures*, Wiley, New York.

Brillinger, D.R. (1975) *Time Series Data Analysis and Theory.* Holt, Rinehart and Winston, New York.

Grenander, U. and Rosenblatt, M. (1957) *Statistical Analysis of Stationary Time Series*, Wiley, New York.

Ibragimov, I.A. (1963) "On estimation of the spectral function of a stationary Gaussian process", *Theory of Probability and its Applications* 8, 366-401.

Kokoszka, P.S. and Taqqu, M.S. (1996), "Parameter estimation for infinite variance fractional ARIMA", *Annals of Statistics*, 24, 1880-1913.

Marinucci, D. and Robinson, P.M. (1998) "Weak convergence to fractional Brownian motion", *Stochastic Processes and their Applications*, forthcoming.

Marinucci, D. and Robinson, P.M. (1999) "Alternative forms of fractional Brownian motion", *Journal of Statistical Planning and Inference*, 80, 111-122.

Phillips, P.C.B. (1991) "Spectral regression for cointegrated time series", *Nonparametric and Semiparametric Methods in Econometrics and Statistics* (W.A. Barnett, J. Powell and G. Tauchen, eds.), Cambridge: Cambridge University Press, 413-435.

Robinson, P.M. (1994a) "Semiparametric analysis of long-memory time series", *Annals of Statistics*, 22, 515-539.

Robinson, P.M. (1994b) "Rates of convergence and optimal spectral bandwidth for long range dependence", *Probability Theory and Related Fields*, 99, 443-473.

Robinson, P.M. and Marinucci, D. (1997) "Semiparametric frequency-domain analysis of fractional cointegration", preprint.

Silveira, G. (1991) *Contributions to Strong Approximations in Time Series with Applications in Nonparametric Statistics and Functional Central Limit Theorems*, Ph.D. Thesis, University of London.