Nonparametric Identification in Asymmetric
Second-Price Auctions: A New Approach

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Abstract

This paper proposes an approach to proving nonparametric identification for distributions of bidders' values in asymmetric second-price auctions. I consider the case when bidders have independent private values and the only available data pertain to the winner's identity and the transaction price. My proof of identification is constructive and is based on establishing the existence and uniqueness of a solution to the system of non-linear differential equations that describes relationships between unknown distribution functions and observable functions. The proof is conducted in two logical steps. First, I prove the existence and uniqueness of a local solution. Then I describe a method that extends this local solution to the whole support.

This paper delivers other interesting results. I show how this approach can be applied to obtain identification in more general auction settings, for instance, in auctions with stochastic number of bidders or weaker support conditions. Furthermore, I demonstrate that my results can be extended to generalized competing risks models. Moreover, contrary to results in classical competing risks (Roy model), I show that in this generalized class of models it is possible to obtain implications that can be used to check whether the risks in a model are dependent. Finally, I provide a sieve minimum distance estimator and show that it consistently estimates the underlying valuation distribution of interest.

1 Introduction

In auctions, researchers are often interested in learning models' economic primitives, particularly the joint distribution of bidders' values. Because this underlying distribution is not known a priori, it must be learned from the data. To obtain credible estimation results, a researcher must first study the identification question to determine whether the distribution of interest is identified or whether there are many distributions consistent with the data. The importance of this issue has generated many methodological papers on identification in auction models. This paper contributes to that literature.

The paper examines the nonparametric identification of the distributions of bidders' values in asymmetric second-price auctions. The identification analysis cannot be conducted without (a) imposing conditions on the joint distribution of bidders' signals and (b) specifying what data are available from the auctions' outcomes. This paper assumes that bidders have private values and that the only available data pertain to the winner's identity and the transaction price. Identification in this framework was first considered in Athey and Haile (2002).

It is well known that in second-price auctions within the private-values framework, a weakly dominant strategy for bidders entails submitting their true value.[1] I consider an equilibrium where bidders employ this strategy. In this case, even though the submitted bids directly reveal bidders' values, the joint distribution of these values cannot be identified nonparametrically because not all the bids are observed. This result is established in Athey and Haile (2002). The identification of the parameter of interest requires strengthening the model's assumptions. This paper shows that in our problem, it suffices to assume that bidders' values are independent. There are three main issues to address in obtaining this result. First, the distribution functions must be identified nonparametrically in order to avoid incorrect assumptions about their form. Second, there is the challenge posed by the asymmetry of the bidders participating in the auction. Finally, given that the transaction price is the value of the second-highest bid, the identification proof must be based on the second-order statistic.

One of the main contributions of this paper is to provide conditions on observable data sufficient to guarantee point identification. First, I present the conditions on the observables that are necessary and sufficient for the existence of a solution to the model; thus, we always know with certainty whether the model has a solution. I then show that these conditions, together with an additional condition on the observables, are sufficient to show the uniqueness of a solution and therefore to ensure the identification of distribution functions. The main sufficient identification condition can be formulated in terms of observables as well as in terms of unobservables. It is interpretable and is weaker than identification conditions usually assumed in auctions. Another contribution of the paper is to prove that

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[1]See, for example, Vickrey (1961) or Krishna (2002).

when there are only two types of bidders, identification always holds. This result is generalized for the case when there are only two types of the bidders and the joint distribution of bidders' values is given by an Archimedean copula. I obtain a condition on the generating function of a copula that is sufficient for identification. This condition is satisfied for many classes of Archimedean copulas.

A methodological contribution of this paper is to suggest a new approach to proving identification in analyzed auction models. The idea behind this method is to establish the existence and uniqueness of a solution to a system of non-linear differential equations that relate unknown underlying distribution functions to the observable data. This strategy includes two major steps. First, I show that the system has a unique solution on a subinterval of the support; this is what I call a local solution. Second, I demonstrate that this local solution can be extended to the whole support. This two-step approach is constructive and enables us to conduct a thorough qualitative analysis of the identification problem.

Furthermore, the techniques developed in the paper allow for two generalizations of the auction setting. One relaxes the support conditions and permits distributions to have different upper support points as well as holes in the support. The other considers second-price auctions in which the set of actual bidders is unknown and varies exogenously. Using the case of three bidders, I outline the specifics of proving identification in these models.

Athey and Haile (2007) formulate an identification result for second-price auctions with a reserve price. When there is a reserve price, bidders participate endogenously. The authors state that if the winner's identity, the transaction price and the set of the participants are observed, then the distributions are identified for the values above the reserve price. I explain in detail how this result can be proven by using the techniques of the basic case.

Another contribution of this paper is to uncover conditions sufficient to determine whether the model is stated correctly. These conditions do not pertain to the observable data and can be verified only after we determine the solution to the system of differential equations. To address this issue, I present an example of an incorrectly stated model that proves the possibility of finding well-defined observable functions such that the functions in the solution corresponding to them are not all monotone − that is, not all of them possess the properties of distribution functions.

Within the private-values framework, second-price auctions are equivalent to ascending auctions. For proofs of identification in these two types of auctions, when the data indicate only the winner's identity and the winning price, researchers have referred to results in the statistical literature that examines identification in generalized competing risks models. Athey and Haile (2002) were first to observe that analyzed auctions can be considered a special case of these models.

In generalized competing risks models, an object that consists of different components fails as a result of the cumulative failure of several of its elements, and the only observed data pertain to the lifetime of the object and the set of components that had failed before

the object's failure. Though the main identification result for these cases was obtained by Meilijson (1981), the Meilijson's proofs lack some essential details, most importantly, conditions on the observables or on the unknowns that guarantee identification. I show that my method, on the other hand, provides an exhaustive proof of identification in generalized models. For any of these models, I provide conditions on the observables and equivalent to them conditions on the unknowns that guarantee that the model cannot have more than one solution. I also explain why the existence of a solution cannot be proved in general and must be assumed. For a special class of generalized competing risks models (one that encompasses our auction models), I present necessary and sufficient conditions for existence.

For a thorough overview of nonparametric identification in auctions, see Athey and Haile (2002, 2006, 2007) and references therein. The authors obtain numerous nonparametric identification results for various auctions settings, and some of the point identification results rely on the work of Meilijson (1981). Brendstrup and Paarsch (2006) deal specifically with asymmetric ascending auctions within the independent-private-values framework, considering both single-unit and multi-unit settings. For nonparametric identification in single-unit auctions, they also refer to Meilijson (1981). Banerji and Meenakshi (2004) and Meenakshi and Banerji (2005) also consider asymmetric ascending auctions within the independent-private-values framework by examining wheat markets in India. Similar to Brendstrup and Paarsch (2006), they cite Meilijson (1981) to show identification.

Another thread of the literature related to this paper applies the techniques of the theory of differential equations to identification problems. In auctions, examples of such papers are Campo, Perrigne and Vuong (2003); Guerre, Perrigne and Vuong (2009) and Lebrun (1999). Campo, Perrigne and Vuong (2003) prove nonparametric identification for asymmetric first-price auctions with affiliated private values. Guerre, Perrigne and Vuong (2009) address the nonparametric identification of utility functions for bidders in first-price auctions, specifically when the bidders are risk averse and have private values. Lebrun (1999) analyzes first-price auctions with independent private values and characterizes a Bayesian equilibrium as a solution to a system of non-linear differential equations. He then refers to results in the theory of differential equations to show that an equilibrium exists and that in some special models, it is unique. In a related area of classical competing risks, Buera (2006) uses the theory of partial differential equations to prove identification in a certain class of Roy models.

Because assuming the independence of bidders' values may seem dubious in some applications, it is worthwhile to consider auctions in which private values are not independent. Though the joint distribution of bidders' values is not identified, the data are informative and allow me to derive bounds. More precisely, I obtain bounds on the joint distribution for any subset of bidders. I show that these bounds continue to hold when the equilibrium condition is replaced by two weaker assumptions on the bidders' behavior. These assump-

tions on rationality were introduced in Haile and Tamer (2003). One assumption is that the bidders do not bid more than they are willing to pay. The other assumption is that the bidders do not allow an opponent to win at a price they are willing to beat.

In addition, I analyze how the bounds change when we acquire data on other elements of the auction model. Namely, I consider the case when data on all the identities and the bids except for the highest bid become available. For simplicity, I only show bounds on the joint distribution for the set of all bidders and the marginal distributions.

Finally, the paper also proposes a sieve minimum distance estimator to estimating the underlying distribution functions in the case of independent values. This estimator is shown to be consistent in the uniform metric.

The rest of this paper is organized as follows. Section 2 reviews second-price auctions, outlines generalized competing risks models and explains their connection to auctions. Section 3 states identification results for auctions, discusses incorrect specification and considers identification in more general auction settings. Section 4 defines a sieve minimum distance estimator of the distribution functions and explores the properties of the operator that maps these functions into observable data. The continuity of the inverse of this operator allows proving the consistency of the sieve estimator in the uniform metric. Section 5 describes generalized competing risks models in detail and provides identification results for these models. Section 6 provides results for the auctions in which bidders' values are not independent. Proofs of propositions, lemmas and theorems are collected in appendices.

2 Second-price auctions and generalized competing risks models

In this section, I first review second-price auctions. Next, I describe generalized competing risks models and show their connection to these auctions.

2.1 Second-price auctions within the private-values framework

A single object is up for sale, and d buyers are bidding on it. The set of all bidders is known. Bids are submitted in sealed envelopes. The highest bidder wins and pays the value of the second-highest bid; thus, in these auctions, the second-highest bid is the winning price. Suppose that the bidders have private values and that they are aware of their value. It is known that in this setting, a weakly dominant strategy for bidders is to submit their true value − and this is an equilibrium that I consider later. In this paper, only the winner's identity and the winning price are observed in auction's outcomes.

It is worth mentioning that within the private-values framework, second-price auctions are equivalent to open ascending auctions. One form of ascending auctions is a "button

auction," in which bidders hold down a button as the auctioneer raises the price. When the price gets too high for a bidder, she drops out by releasing the button. The auction ends when only one bidder remains. This person wins the object and pays the price at which the auction stopped.

2.2 Generalized competing risks models

Now I turn to a brief description of generalized competing risks models. Consider a machine that consists of several elements. A special case of these models are classical competing risks models. The classical models correspond to a situation in which a machine breaks down as soon as one of its components fails; the data available after the breakdown are the machine's lifetime and the element that caused the failure. One example of these models in economics is duration models. Also, the Roy model is isomorphic to classical competing risks. In the Roy model, a person chooses from a finite set of occupational alternatives to obtain the highest income, and the outcomes of the choice (occupation and income) are observed. In biometrics, the death of an individual because of a particular disease when that person is also facing several other diseases presents a classical competing risks model, based on a fundamental assumption that a single cause is behind every death.

Generalized competing risks models relax this assumption and consider cases in which a machine fails because of the cumulative failure of some of its elements rather than a single one. A *fatal set* for the machine is a subset of parts such that the failure of all the parts in the subset causes the failure of the machine; in other words, it is a set of the elements that failed before the machine broke down. In this paper, the machine's failure provides information only about the fatal set and the machine's lifetime. More details about generalized competing risks are given in section 5.

2.3 Second-price auctions as a special case of generalized competing risks models

Athey and Haile (2002) were among the first ones to notice the connection between second-price auctions and generalized competing risks models. To clarify the connection, I use the equivalence of second-price and ascending auctions within the private-values paradigm.

Consider a button auction, as described above, with d bidders. Notice that observing the identity of the winner is equivalent to observing the identities of the bidders who dropped out. Compare this auction framework to the following generalized competing risks model. Assume that a machine consists of d elements and works as long as at least two of its elements are functioning; in other words, the machine breaks once $d - 1$ of its elements are dead. The set of these $d - 1$ elements is fatal. Clearly, the breakdown of other $d - 1$ components would also be fatal. A fatal set in this model is an analog of the set of bidders

that dropped out, and the machine's lifetime is an analog of the winning price.

3   Identification in second-price auctions

In this section, I formulate identification results and present a mathematical description of the identification problem. Also, I discuss generalizations of the identification results and model's misspecification. The proofs of the theorems, propositions and lemmas of this section are collected in Appendix A.

3.1   Statement of identification problem

Denote bidders' private values as X_i, $i = 1, \ldots, d$. Assume that these values are independent and have absolutely continuous distributions on a common support $[t_0, T]$. Also assume that bidders' values at each auction are independent draws from the same joint distribution. We aim to learn this distribution from the available data. Note that in the equilibrium, the bids' joint distribution coincides with the distribution of the bidders' private values. Therefore, if all the bids are observed, then the distribution of values can be clearly identified. If some of them are not observed, however, then neither the joint nor the marginal value distributions can be identified, as shown in Athey and Haile (2002). Given that our knowledge is often limited to the second-highest bid, I show that when the only available data pertain to the bid and the winner's identity, the marginal distributions of bidders' values can be identified if these values are independent.

Notation

Throughout this paper, I use the following notations. A bid submitted by player i is denoted as b_i. Symbol M^{tr} represents the transpose of matrix M. The distribution function of X_i is denoted as F_i, $i = 1, \ldots, d$. Function F_i is called positive (negative) if $F_i(t) > 0$ ($F_i(t) < 0$) for $t > t_0$. A vector-valued function $F = (F_1, \ldots, F_d)^{tr}$ on $[t_0, T]$ is called positive (negative) if each of its components F_i is a positive (negative) function. F is referred to as strictly increasing if each F_i is strictly increasing on $[t_0, T]$.

For simplicity, I first consider the case of three bidders, then generalize the results to any number of bidders. Because the winner's identity and the winning price are observed in an auction's outcome, then the probability of an event {price $\leq t$, i wins} is known for any $t \in [t_0, T]$ and any $i = 1, 2, 3$. So, for each bidder i, we observe the following function G_i on $[t_0, T]$:

$$G_i(t) = Pr(\text{price} \leq t, i \text{ wins}), \quad i = 1, 2, 3.$$

The identification problem is to determine whether there is only one collection of private values distribution functions F_1, F_2 and F_3 that rationalize observable functions G_1, G_2 and G_3.

3.2 Necessary conditions on observables

I start by describing the properties of observable functions G_i that follow from the model.

I will say that *the model is not stated correctly* if at least one of the following conditions fails to hold: 1) bidders submit their true values; 2) bidders have independent private values; 3) bidders' values have absolutely continuous distributions; 4) bidders' values are distributed on $[t_0, T]$.

The next proposition indicates **necessary conditions on observable functions** G_i implied by the model.

Proposition 3.1. *If the model is stated correctly, then the following conditions hold:*

Necessary conditions (I)

1. $G_i(t_0) = 0$, $i = 1, 2, 3$
2. G_i *are absolutely continuous on* $[t_0, T]$, $i = 1, 2, 3$
3. G_i *are strictly increasing on* $[t_0, T]$, $i = 1, 2, 3$

Proof. By assumption, the distributions of private values X_i are absolutely continuous. This implies, in particular, that players submit bids equal to t_0 with probability 0. Also, t_0 is the lower support point for all distributions. These two facts give Condition 1. Condition 2 follows from the absolute continuity of the distributions of X_i. Condition 3 is true because the support of each X_i is the connected interval $[t_0, T]$, without any holes in it. □

Even though these conditions are simple, it is worth indicating them because they are useful in the proof of identification. As we can see, all the properties of the private values distributions, except for the assumption of independence and the boundary conditions $F_i(T) = 1$, $i = 1, 2, 3$, are used in establishing Proposition 3.1. The independence assumption, combined with necessary conditions (I), allows me to obtain the following result.

Proposition 3.2. *Suppose that the model is stated correctly. Let F be a solution to the model. Then*

$$\lim_{t \downarrow t_0} \frac{F_1}{\sqrt{\frac{G_2 G_3}{G_1}}}(t) = 1, \quad \lim_{t \downarrow t_0} \frac{F_2}{\sqrt{\frac{G_1 G_3}{G_2}}}(t) = 1, \quad \lim_{t \downarrow t_0} \frac{F_3}{\sqrt{\frac{G_1 G_2}{G_3}}}(t) = 1. \tag{3.1}$$

Conditions (3.1) are formulated in terms of both observable and unobservable functions. They characterize a solution F to the model only in a neighborhood of t_0. To be more precise, they find the rate of convergence of unknown distribution functions F_i at t_0 in terms of observable functions G_i. These conditions are essential for proving identification.

The properties of G_i formulated in the next corollary also play an important role in identification.

Corollary 3.3. *Suppose that the model is stated correctly. Then the following conditions hold:*

Necessary conditions (II)

$$\lim_{t \downarrow t_0} \frac{G_2 G_3}{G_1}(t) = 0, \quad \lim_{t \downarrow t_0} \frac{G_1 G_3}{G_2}(t) = 0, \quad \lim_{t \downarrow t_0} \frac{G_1 G_2}{G_3}(t) = 0. \tag{3.2}$$

The reasoning behind conditions (II) is that, no matter how different the underlying distributions are, bidders' probabilities of winning do not have considerably different rates of convergence at t_0.

Now that I have presented necessary conditions on observables, I turn to describing the mathematical model of identification and explain how necessary conditions (I) and (II) are employed in the identification proof.

3.3   Mathematical model of the identification problem

Assuming the independence of bidders' values, functions G_i can be expressed through F_i as follows. Let b_i, $i = 1, 2, 3$, indicate the submitted bids. Then

$$G_1(t) = Pr(\max\{b_2, b_3\} < b_1, \max\{b_2, b_3\} \le t)$$

$$= Pr(\max\{X_2, X_3\} < X_1, \max\{X_2, X_3\} \le t) = \int_{t_0}^{t} (F_2 F_3)'(1 - F_1) ds,$$

where integration is to be understood in the sense of Lebesgue. Functions G_2 and G_3 have similar expressions. Therefore, unknown distribution functions F_i are related to observable functions G_i by means of this system of integral-differential equations:

$$G_1(t) = \int_{t_0}^{t} (F_2 F_3)'(1 - F_1) ds$$

$$G_2(t) = \int_{t_0}^{t} (F_1 F_3)'(1 - F_2) ds \tag{3.3}$$

$$G_3(t) = \int_{t_0}^{t} (F_1 F_2)'(1 - F_3) ds.$$

Notice that the left-hand and right-hand sides of the equations in (3.3) are absolutely continuous functions, allowing us to differentiate them and obtain the following system of differential equations almost everywhere (a.e.) on $[t_0, T]$:

$$g_1 = (F_2 F_3)'(1 - F_1) \tag{DE}$$

$$\text{Main system} \qquad g_2 = (F_1 F_3)'(1 - F_2)$$

$$g_3 = (F_1 F_2)'(1 - F_3),$$

where g_i stands for the a.e. derivative of G_i. I will refer to system (DE) as the main system. Distribution functions F_i in this system must satisfy the following initial conditions:

$$\textit{Initial conditions} \qquad F_i(t_0) = 0, \quad i = 1, 2, 3. \qquad (IC)$$

I will refer to problem (DE)-(IC) as the main problem. The definition below explains the meaning of a solution to (DE)-(IC).

Definition 3.1. *Function $F = (F_1, F_2, F_3)^{tr}$ is a solution to problem (DE)-(IC) on an interval $[t_0, t_0 + a]$, $t_0 + a \leq T$, if F_i, $i = 1, 2, 3$, are absolutely continuous on $[t_0, t_0 + a]$, satisfy equations (DE) a.e. on $[t_0, t_0 + a]$ and satisfy (IC).*

The system of differential equations (DE) is a convenient tool because identifying functions F_i is equivalent to proving that problem (DE)-(IC) has a unique positive solution F on $[t_0, T]$.

Notice that I did not mention anything about the monotonicity of the solution. There are two reasons for that. First, as I will show in section 3.7, functions F_i in a solution to (DE)-(IC) may be non-monotone. Second, it is not clear whether one can find conditions on G_i that guarantee the monotonicity of all F_i. Therefore, the monotonicity of a solution to problem (DE)-(IC) will be assumed.

3.4 Main results

The theorem below formulate the existence result for problem (DE)-(IC).

Theorem 3.4. (*Existence of a solution*) *Let observable functions G_i satisfy conditions (I) and (II). Then problem (DE)-(IC) has a positive solution on $[t_0, T]$.*

Remember that all conditions on G_i required in this theorem are necessary conditions implied by the model. Therefore, conditions (I) and (II) are both necessary and sufficient conditions for the existence of a solution to the model. In particular, if even one of the conditions in (I) and (II) fails to hold, we can immediately conclude that the model is not stated correctly.

The next theorem describes conditions on G_i sufficient to guarantee the identification of F_i.

Theorem 3.5. (*Uniqueness of a solution*) *Let observable functions G_i satisfy conditions (I), (II) and*
Sufficient condition (III): The function

$$\left(\frac{g_1}{G_1} + \frac{g_2}{G_2} + \frac{g_3}{G_3} \right) \left(\sqrt{\frac{G_2 G_3}{G_1}} + \sqrt{\frac{G_1 G_3}{G_2}} + \sqrt{\frac{G_1 G_2}{G_3}} \right) \qquad (3.4)$$

is Lebesgue integrable − that is, belongs to the class L^1 − in a neighborhood of t_0.

    Then problem (DE)-(IC) has a unique positive solution on $[t_0, T]$.

Because the function in condition (III) is non-negative, Lebesgue integrability means that the integral of this function is finite.
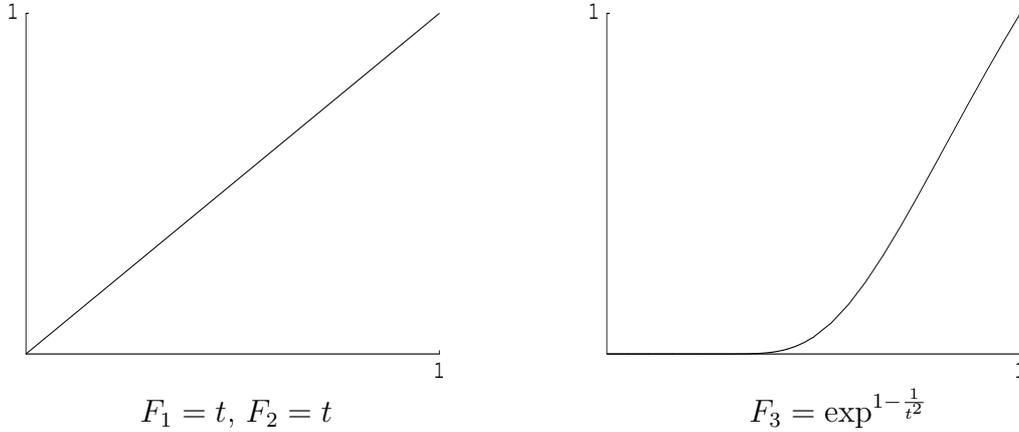

$$F_1 = t, F_2 = t \qquad\qquad\qquad F_3 = \exp^{1-\frac{1}{t^2}}$$

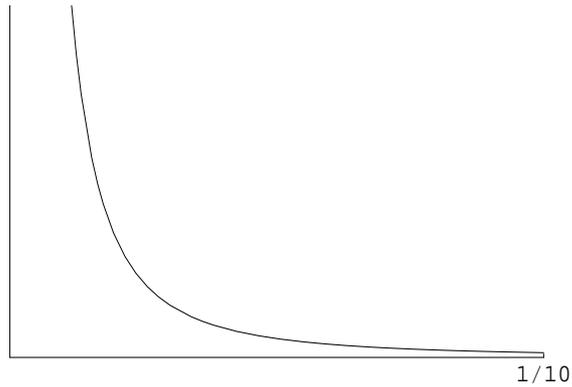Figure 1. Underlying distribution functions.

$$1/10$$

Figure 2. Function in (3.5).

The most important element in obtaining sufficient condition (III) is the result of Proposition 3.2. To acquire a better understanding of this condition, I write it in terms of distribution functions F_i.

Remark. *Condition (III) is equivalent to the following condition:*

The function

$$\left(\frac{F_1'}{F_1} + \frac{F_2'}{F_2} + \frac{F_3'}{F_3} \right) (F_1 + F_2 + F_3) \tag{3.5}$$

is Lebesgue integrable in a neighborhood of t_0.

Now it is intuitive that the reasoning behind this condition is that the underlying distribution functions F_1, F_2 and F_3 are not too different around t_0 in a certain sense. For instance, if the underlying distribution functions are $F_1 = t$, $F_2 = t$ and $F_3 = \exp(1 - \frac{1}{t^2})$

on $[0, 1]$, then the corresponding observable functions G_i do not satisfy condition (III). Figure 1 depicts these distribution functions F_i. As we can see, the value distribution for the third bidder has a very small mass around point 0, whereas values for the first and second bidders are distributed uniformly on $[0, 1]$. This means that bidder 3 wins very rarely when all the bidders submit bids close to t_0. Figure 2 shows the function in (3.5). This function has the same behavior in a neighborhood of point 0 as the observable function in (3.4). It is non-integrable because in a neighborhood of 0 it behaves as function $\frac{1}{t^2}$.

Condition (3.5) is satisfied if all F_i behave as power functions around t_0:

$$F_i(t) = O((t - t_0)^{\alpha_i}) \quad \text{as} \quad t \downarrow t_0,$$

where $\alpha_i > 0$, $i = 1, 2, 3$.

In identification results for the first-price auctions, it is usually assumed that the densities of all the distributions of the bidders' values are bounded from zero and are finite on the support. For example, these conditions are imposed in Guerre, Perrigne and Vuong (2009). Condition (III) is much weaker than these restrictions. Indeed, if the densities are bounded from zero and are bounded, then all the ratios $\frac{F_i'}{F_j'}$ are bounded from above, which implies that all the rations $\frac{F_i}{F_j}$ are bounded from above, and hence condition (III) is obviously satisfied.

3.5 Local and global identification

My identification proof comprises two major steps: establishing the local identification result and the global identification result. Namely, I first prove that problem (DE)-(IC) has only one positive solution F in a small neighborhood of t_0; this solution is what I call a local solution. Establishing the existence and the uniqueness of a local solution is the most challenging part of the identification result because problem (DE)-(IC) has a singularity at t_0. Notice that conditions (II) and (III) describe the behavior of observable functions G_i only in a neighborhood of t_0.

After I prove the existence and the uniqueness of a local solution to (DE)-(IC), I show that it can be extended to a positive solution on the entire interval $[t_0, T]$, and that such extension is unique.

To gain intuition, consider Figure 3. The picture on the left shows the local solution F found on some interval $[t_0, t_0 + c]$. The idea of constructing a global solution is to extend this solution F to the right at least to a small interval $(t_0 + c, t_0 + c_1]$, $c_1 > c$, in such a way that the extended solution solves (DE)-(IC) on $[t_0, t_0 + c_1]$. The picture on the right in Figure 3 shows this extended solution. Then this solution is extended even farther to the right and so on. I show that if we continue this process in a certain way, then we will reach the upper support point T, and thus, find the solution on the whole support.
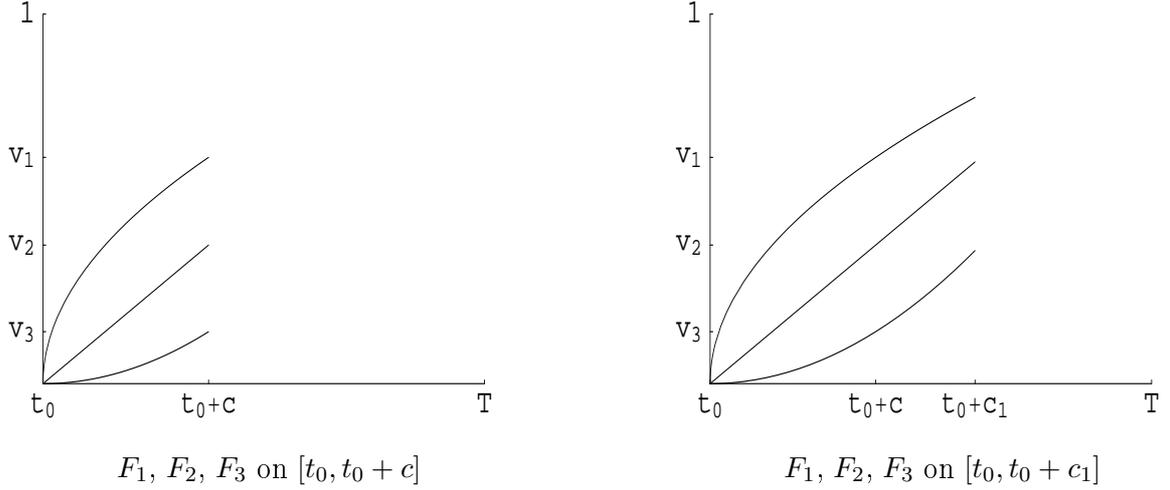
F_1, F_2, F_3 on $[t_0, t_0 + c]$ F_1, F_2, F_3 on $[t_0, t_0 + c_1]$

Figure 3. Solution to the main problem on $[t_0, t_0 + c]$ (left) and extended solution to the main problem on $[t_0, t_0 + c_1]$ (right).

3.6   Generalizations

3.6.1   Any number of bidders

In this section, I show how the identification result for auctions with three bidders can be generalized to auctions with any number of bidders. I state main results and outline their proofs in Appendix A. The interpretations and intuitiveness of these results are similar to those in the case of three bidders.

The observable functions are

$$G_i(t) = Pr(\text{price} \leq t, i \text{ wins}) = Pr(\max_{j \neq i} b_j \leq t, \max_{j \neq i} b_j < b_i), \quad i = 1, \ldots, d.$$

Propositions 3.6 and 3.7 below are the analogs of propositions 3.1 and 3.2. Corollary 3.8 is analogous to Corollary 3.3.

Proposition 3.6. *If the model is stated correctly, then the following conditions hold:*

Necessary conditions (Id)

    1. $G_i(t_0) = 0$, $i = 1, \ldots, d$

    2. G_i are absolutely continuous on $[t_0, T]$, $i = 1, \ldots, d$

    3. G_i are strictly increasing on $[t_0, T]$, $i = 1, \ldots, d$

Proposition 3.7. *Suppose that the model is stated correctly. Let F be a solution to the model. Then*

$$\lim_{t \downarrow t_0} \frac{F_i}{\left(\frac{G_1 G_2 \ldots G_{i-1} G_{i+1} \ldots G_d}{G_i^{d-1}} \right)^{\frac{1}{d-2}}}(t) = 1, \quad i = 1, \ldots, d.$$

Corollary 3.8. *Suppose that the model is stated correctly. Then the following conditions hold:*

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Necessary conditions (IId)

$$\lim_{t \downarrow t_0} \frac{G_1 G_2 \ldots G_{i-1} G_{i+1} \ldots G_d}{G_i^{d-1}}(t) = 0, \quad i = 1, \ldots, d.$$

The mathematical model of the identification problem is obtained in the following way. The definition of G_i and the independence of private values yield the following system of integral-differential equations that describes relationships between observable functions G_i and unknown distribution functions F_i:

$$G_i(t) = \int_{t_0}^{t} (F_1 \ldots F_{i-1} F_{i+1} \ldots F_d)'(1 - F_i) ds, \quad i = 1, \ldots, d.$$

The differentiation of both sides of these equations gives us a system of differential equations

$$g_i = (F_1 \ldots F_{i-1} F_{i+1} \ldots F_d)'(1 - F_i), \quad i = 1, \ldots, d. \tag{3.6}$$

Functions F_i in this system must satisfy initial conditions

$$F_i(t_0) = 0, \quad i = 1, \ldots, d. \tag{3.7}$$

Theorem 3.9 below gives necessary and sufficient conditions for the existence of a solution to the model. Theorem 3.10 presents an identification result.

Theorem 3.9. *(**Existence of a solution**) Let observable functions G_i satisfy conditions (Id) and (IId). Then problem (3.6)-(3.7) has a positive solution on $[t_0, T]$.*

Theorem 3.10. *(**Uniqueness of a solution**) Let observable functions G_i satisfy conditions (Id), (IId) and*

Sufficient condition (IIId): The function

$$\sum_{i=1}^{d} \frac{g_i}{G_i} \cdot \sum_{i=1}^{d} \left(\frac{G_1 G_2 \ldots G_{i-1} G_{i+1} \ldots G_d}{G_i^{d-1}} \right)^{\frac{1}{d-2}}$$

is Lebesgue integrable in a neighborhood of t_0. Then problem (3.6)-(3.7) has a unique positive solution on $[t_0, T]$.

The main identification condition (IIId) has an equivalent form in terms of the primitives of the model:

The function

$$\sum_{i=1}^{d} \frac{F_i'}{F_i} \cdot \sum_{i=1}^{d} F_i$$

is Lebesgue integrable in a neighborhood of t_0.

3.6.2 Only two types of bidders

Suppose that there are only two types of bidders. Without a loss of generality, bidders 1, ..., k have type I and bidders $k + 1$, ..., d are of type II. In this case, there are two observable functions:

$$G_I(t) = Pr(\text{price} \leq t, \text{ bidder of type I wins}),$$

$$G_{II}(t) = Pr(\text{price} \leq t, \text{ bidder of type II wins}).$$

Clearly, $G_i = G_I$, $i = 1, \ldots, k$, and $G_i = G_{II}$, $i = k + 1, \ldots, d$. The following theorem gives the conditions on observables that are both necessary and sufficient for identification. It shows that in the situation with only two types, condition (IIId) is not required for identification.

Theorem 3.11. *If observable functions G_i satisfy conditions (Id), then the distributions of bidders' values are identified. In other words, conditions (Id) are necessary and sufficient for identification.*

The result in theorem 3.11 can be extended to the case when bidders' private values are dependent and their joint distribution is described by an Archimedean copula:

$$C(u_1, u_2, \ldots, u_d) = \psi^{-1}\left(\psi(u_1) + \psi(u_2) + \ldots + \psi(u_d)\right),$$

where function ψ is defined on $(0, 1]$ and

$$\psi(1) = 0, \quad \lim_{x \to 0} \psi(x) = \infty, \quad \psi'(x) < 0, \quad \psi''(x) > 0.$$

Theorem 3.12. *If observable functions G_i satisfy conditions (Id) and the function $\frac{\psi''(x)}{(\psi'(x))^2}$ is increasing, then the distributions of bidders' values are identified.*

This theorem can be applied, for instance, to the following copulas:

Clayton copulas: $\psi(x) = \frac{1}{\theta}(x^{-\theta} - 1)$, $\theta \in (0, \infty)$.

Gumbel copulas: $\psi(x) = (-\ln x)^\theta$, $\theta \in [1, \infty)$.

Frank copulas:

$$\psi(x) = -\ln \frac{e^{-\theta x} - 1}{e^{-\theta} - 1}, \quad \theta > 0,$$

$$\psi(x) = -\ln x, \quad \theta = 0.$$

Joe copulas: $\psi(x) = -\ln(1 - (1 - x)^\theta)$, $\theta \in [1, \infty)$.

Ali-Mikhail-Haq (AMH) copulas: $\psi(x) = \ln \frac{1 + \theta(1 - x)^{-\theta}}{x}$, $\theta \in (-1, 0]$.

Identification within the Archimedean family of copulas in a different auction framework is considered in Brendstrup and Paarsch (2007).

3.6.3 Weaker support conditions

The assumption that bidders' values have the same support can be relaxed. For instance, bidders can have different upper support point or holes in their supports, but the assumption that has to be maintained is that bidders' values have the same lower support point. This result is explained by the fact that in the identification proof, I first established the existence and uniqueness of a local solution and then extended it to the whole support.

I start by considering a case of three bidders and analyzing identification when their distributions' upper support points differ from each other. I then generalize this analysis for auctions with any number of bidders. Finally, I briefly discuss what happens when distributions have holes in their supports.

Auctions with three bidders

Let τ_1, τ_2, τ_3 be the upper support points of the distributions of bidders' valuations. Without a loss of generality, assume that $\tau_1 \leq \tau_2 \leq \tau_3$. Given these inequalities, there are four possibilities for the locations of τ_1, τ_2, τ_3 with respect to each other; they are illustrated in figures 4 and 5.

Case 1: $\tau_1 = \tau_2 = \tau_3$

This is the case analyzed in this paper. Clearly, functions G_i are defined on $[t_0, \tau_1]$ and satisfy the boundary condition

$$G_1(\tau_1) + G_2(\tau_1) + G_3(\tau_1) = 1$$

because $G_1(\tau_1) + G_2(\tau_1) + G_3(\tau_1) = Pr(\text{price} \leq \tau_1) = 1$. As I have already established, given G_1, G_2 and G_3, distribution functions F_1, F_2 and F_3 are identified on $[t_0, \tau_1]$. This case is illustrated in figure 4.

Case 2: $\tau_1 < \tau_2 = \tau_3$

Because the first player never submits bids higher than τ_1, function G_1 is defined on $[t_0, \tau_1]$. The second and the third players have positive probability of submitting bids in $[\tau_1, \tau_2]$, so G_2 and G_3 are defined on $[t_0, \tau_2]$. The definitions of G_1, G_2 and G_3 imply the boundary condition

$$G_1(\tau_1) + G_2(\tau_2) + G_3(\tau_2) = 1.$$

Identification is obtained in the following way. On $[t_0, \tau_1]$, functions F_i must solve (DE)-(IC). First, I find the unique solution to (DE)-(IC) in a small neighborhood of t_0. I then use methods from section 7.4.2 in Appendix A to extend it farther to the right until one of the functions F_1, F_2, F_3 reaches value 1. Because by assumption $\tau_1 < \tau_2 = \tau_3$, function F_1 will be the first one to reach value 1, and that will happen at τ_1. Thus, all F_i can be

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identified on $[t_0, \tau_1]$, and I can find values $x_2 = F_2(\tau_1) > 0$ and $x_3 = F_3(\tau_1) > 0$. The next step is to identify functions F_2 and F_3 on $(\tau_1, \tau_2]$. On this interval, functions F_2 and F_3 must solve the system

$$g_2 = F_3'(1 - F_2) \qquad\qquad (3.8)$$
$$g_3 = F_2'(1 - F_3)$$

and satisfy initial conditions

$$F_2(\tau_1) = x_2, \quad F_3(\tau_1) = x_3.$$

Applying techniques from section 7.4.1 in Appendix A, I can show that this problem has the unique solution in a right-hand neighborhood of τ_1. Employing extension methods, I can demonstrate that this local solution can be extended to the interval $[\tau_1, \tau_2]$ and that such extension is unique.[2] This case is illustrated in figure 5.

Case 3: $\tau_1 = \tau_2 < \tau_3$

In this case, there are no observable data on $(\tau_1, \tau_3]$. So, functions G_i are defined on $[t_0, \tau_1]$ and satisfy the boundary condition

$$G_1(\tau_1) + G_2(\tau_1) + G_3(\tau_1) = 1.$$

Given G_1, G_2 and G_3, I find the unique solution F to (DE)-(IC) on $[t_0, \tau_1]$. Hence, F_1 and F_2 are identified. As for F_3, nothing can be learned about this function for $t > \tau_1$ because there are no observations corresponding to those t, so this function is only partially identified. This case is illustrated in figure 4.

Case 4: $\tau_1 < \tau_2 < \tau_3$

This situation is similar to case 3. Function G_1 is defined on $[t_0, \tau_1]$, G_2 is defined on $[t_0, \tau_2]$, and because a transaction price never exceeds τ_2, G_3 is defined only on $[t_0, \tau_2]$. Functions G_1, G_2 and G_3 satisfy the boundary condition

$$G_1(\tau_1) + G_2(\tau_2) + G_3(\tau_2) = 1.$$

To identify F_1 and F_2 as well as partially identify F_3, I proceed as follows. On $[t_0, \tau_1]$, functions F_1, F_2 and F_3 must solve problem (DE)-(IC). On $[\tau_1, \tau_2]$, functions F_2 and F_3 must solve system (3.8) and satisfy certain initial conditions at τ_1. Therefore, F_1 and F_2 are identified. Because there are no observed data for $t > \tau_2$, function F_3 cannot be learned

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[2]The methodology presented here can be used for auctions with any number of bidders. In this particular case, however, system (3.8) can be handled much more easily once it is rewritten as a system of two linear equations: $F_2' = \frac{g_3(1-F_2)}{1-G_2-G_3}$ and $F_3' = \frac{g_2(1-F_3)}{1-G_2-G_3}$. This system has a closed-form solution: $F_2(t) = 1 - (1-x_2)\exp(-\int_{\tau_1}^t \frac{g_3(s)}{1-G_2(s)-G_3(s)}ds)$, $F_3(t) = 1 - (1-x_3)\exp(-\int_{\tau_1}^t \frac{g_2(s)}{1-G_2(s)-G_3(s)}ds)$.

for $t > \tau_2$; that is, F_3 is only partially identified. This case is illustrated in figure 5.

Thus, in auctions with three bidders, two distribution functions F_1 and F_2 are always identified. In two cases, the third distribution function F_3 is identified too. In two other cases, it is identified only on a subinterval of the corresponding support.

Auctions with any number of bidders

Now I consider auctions with any number of bidders and briefly discuss identification in this case. Suppose I found the unique local solution to (DE)-(IC). I extend it farther and farther to the right until I reach a point where one of the functions F_i hits the value of 1. Beyond this point, system (DE) does not contain the equation corresponding to bidder i. Thus, $d - 1$ equations (or possibly even fewer equations, if several functions F_i hit value 1 at the same point) remain in system (DE). Next, I find the unique local solution to the reduced system and extend this solution to the right until I reach a point where one of the remaining functions F_i hits value 1. Beyond this point, the number of equations in the system decreases again. Proceeding in this way, I eventually come to a final system such that all functions F_i in its solution, with the possible exception of one function, reach value 1 at the same point.

The result of the next proposition is intuitive from case of the three bidders. I do not prove this result because it follows from the local existence and local uniqueness results, as well as the extension techniques.

Proposition 3.13. *Suppose that d bidders are participating in the auction, and the distributions of their values are locally identified. Let $[t_0, \tau_i]$ stand for the support of bidder i, $i = 1, \ldots, d$. Without a loss of generality, assume that*

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_{d-1} \leq \tau_d.$$

If $\tau_{d-1} = \tau_d$, then all distribution functions F_1, $\ldots$, F_d are identified. If $\tau_{d-1} < \tau_d$, then $d - 1$ functions $F_1, \ldots$, F_{d-1} are identified, and F_d is identified only on $[t_0, \tau_{d-1}]$.

Holes in the support

Finally, I want to informally discuss identification in a situation where the distributions of bidders' values can have holes in the supports. In this case, distribution functions F_i can have intervals on which they are constant. If all distributions have the same lower support point t_0, then $F_i(t) > 0$ for $t > t_0$ and therefore local identification can be shown by using techniques from section 7.3. Extension to the global solution depends on distributions' upper support points, as explained in Proposition 3.13.

3.6.4   Reserve price

The case of a reserve price is considered in Athey and Haile (2006) . The authors state that the distributions of bidders' values are identified for $t \geq r$ when the transaction price,

Case 1: $\tau_1 = \tau_2 = \tau_3$
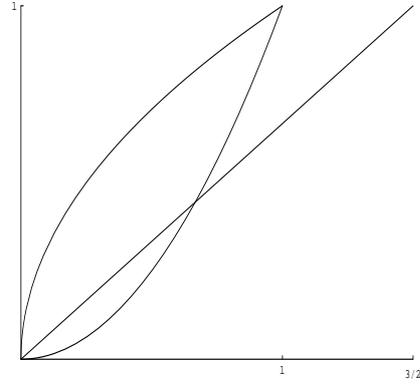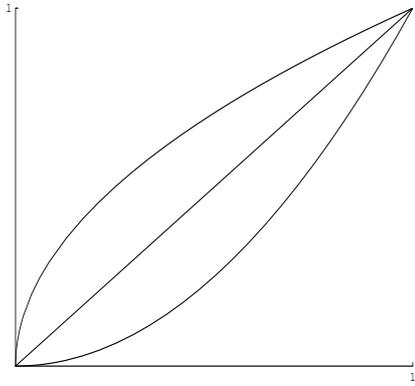
Underlying distribution functions
$F_1 = \sqrt{t}, \quad F_2 = t^2, \quad F_3 = t$
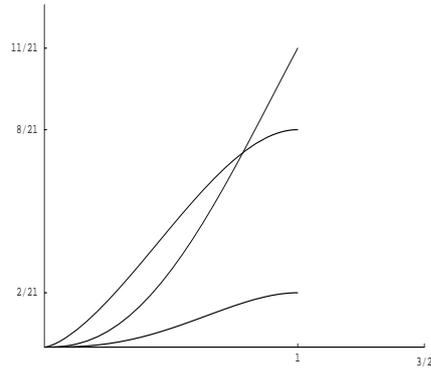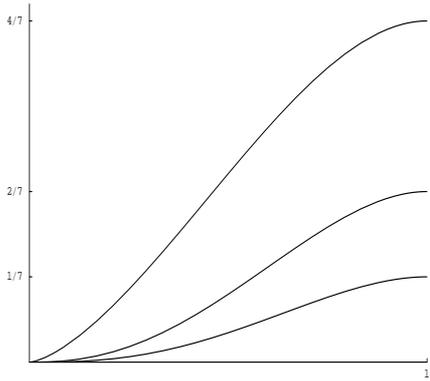
Observed functions G_1, G_2, G_3

Identified functions

Case 3: $\tau_1 = \tau_2 < \tau_3$

Underlying distribution functions
$F_1 = \sqrt{t}, \quad F_2 = t^2, \quad F_3 = \frac{2}{3}t$

Observed functions G_1, G_2, G_3
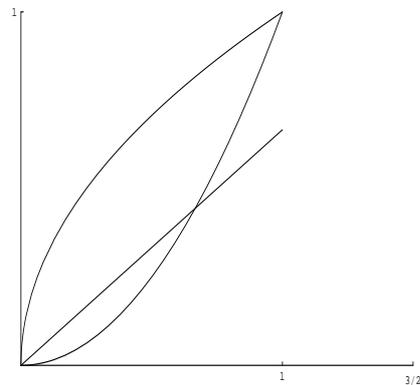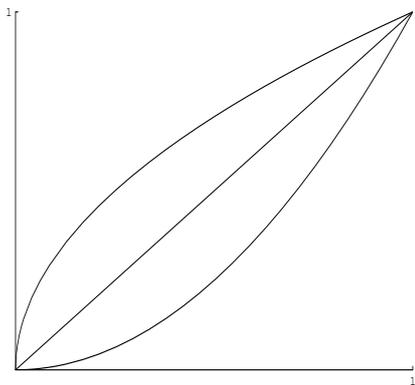
Identified functions

Figure 4. Supports for cases 1 and 3

Case 2: $\tau_1 < \tau_2 = \tau_3$
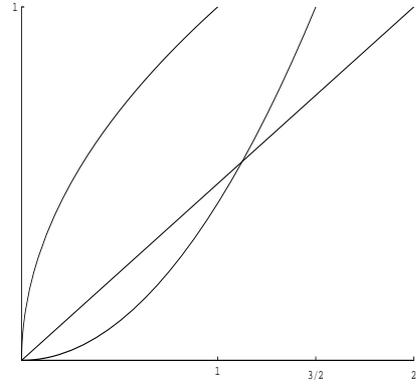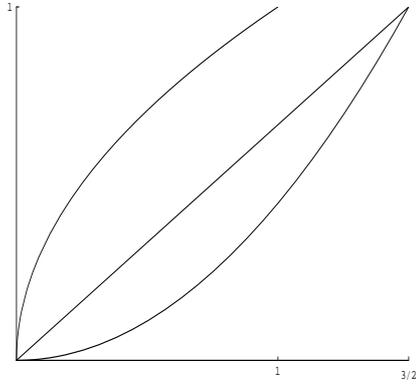
Underlying distribution functions
$F_1 = \sqrt{t}, \quad F_2 = \frac{4}{9}t^2, \quad F_3 = \frac{2}{3}t$

Observed functions G_1, G_2, G_3

Identified functions

Case 4: $\tau_1 < \tau_2 < \tau_3$

Underlying distribution functions
$F_1 = \sqrt{t}, \quad F_2 = \frac{4}{9}t^2, \quad F_3 = \frac{1}{2}t$
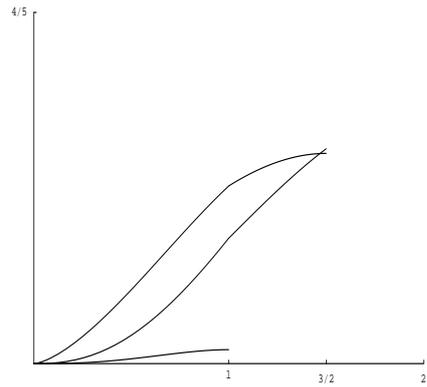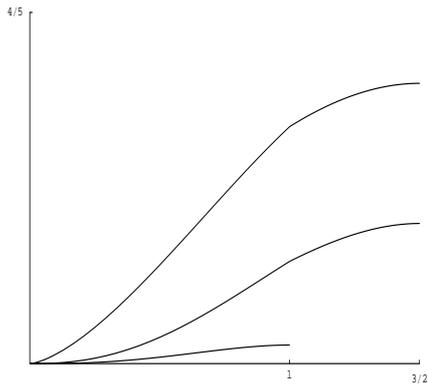
Observed functions G_1, G_2, G_3
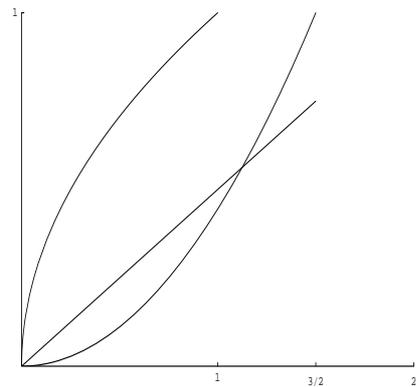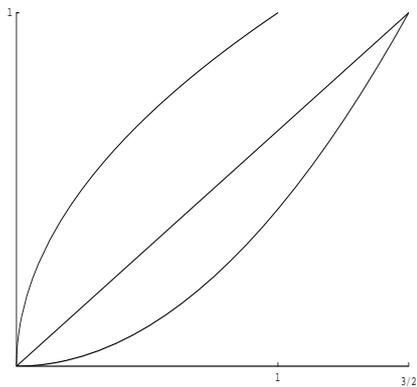
Identified functions

Figure 5. Supports for cases 2 and 4

the identity of the winning bidder, and the set of actual bidders are observable. Below I briefly explain why this is so.

Suppose that the seller sets a reserve price of r. In second-price auctions with private values, a reserve price does not change bidders' behavior because it is still a weakly dominant strategy to bid one's value.

Assume that the reserve price is known to the bidders, and that a bidder does not submit a bid if her value lies below the reserve price. Also suppose that r lies in the intersection of the supports of all bidders[3] and in any right-hand side neighborhood of r, densities F_i' are positive on sets that have positive Lebesgue measure.

Define the following truncated distribution functions:

$$\bar{F}_i(t) = \frac{F_i(t) - F_i(r)}{1 - F_i(r)}, \quad t \geq r.$$

Because for bidder i the probability of the event {all bidders participate, i wins} is positive, then all the truncated distribution functions can be identified from the system

$$g_i = (\bar{F}_1 \ldots \bar{F}_{i-1} \bar{F}_{i+1} \ldots \bar{F}_d)'(1 - \bar{F}_i), \quad i = 1, \ldots, d,$$

with the initial conditions

$$\bar{F}_i(r) = 0, \quad i = 1, \ldots, d.$$

Here g_i is the derivative of the following function G_i:

$$G_i(t) = Pr(r \leq \text{price} \leq t, \, i \text{ wins}),$$

and it is assumed that sufficient conditions for identification are satisfied.

Note that the values $F_i(r)$ are identified for each i because

$$F_i(r) = P(i \text{ does not participate in the auction}),$$

and the probability of this event is known due to the assumption that the set of actual bidders is observed.

The identification of the truncated distributions functions for $t \geq r$ and the identification of the values $F_i(r)$ imply that distributions functions F_i are identified for $t \geq r$.

3.6.5   Auctions with stochastic number of bidders

In this section, I assume that the number of potential buyers is known and does not change, but the number of actual bidders is unknown and varies exogenously. For instance, this may happen because of entrance fees or the different costs of acquiring information. In this

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[3]We can allow bidders to have different lower support points.

setting, I do not aim to present a complete analysis of identification in general. Rather, I want to illustrate how the methods developed in this paper allow us to approach the identification problem. To gain some insight while keeping the problem simple, I consider the case of three buyers.

Suppose that the number of bidders and their identities are determined by chance and the process through which bidders are selected is taken to be exogenous, embodied in probabilities p_A, $A \subset \{1, 2, 3\}$. I assume that at least two bidders are participating in the auction:

$$p_{12} + p_{13} + p_{23} + p_{123} = 1,$$

and each buyer has a positive probability of participation:

$$\sum_{j \neq i} p_{ij} + p_{123} > 0, \quad i = 1, 2, 3.$$

Suppose that buyers' private values are absolutely continuous and distributed on a common support $[t_0, T]$. As before, assume that the available data tells us the winner's identity and the transaction price, and therefore we observe functions $G_i(t) = Pr(\text{price} \leq t, i \text{ wins})$, $i = 1, 2, 3$. Using the law of total probability, it can be found that

$$G_1(t) = P(\text{ price} \leq t, 1 \text{ wins } |\{1, 2\})p_{12} + P(\text{ price} \leq t, 1 \text{ wins } |\{1, 3\})p_{13}$$

$$+ P(\text{ price} \leq t, 1 \text{ wins } |\{1, 2, 3\})p_{123} = \int_{t_0}^{t} (p_{12}F_2' + p_{13}F_3' + p_{123}(F_2F_3)')(1 - F_1)ds.$$

Likewise,

$$G_2(t) = \int_{t_0}^{t} (p_{12}F_1' + p_{23}F_3' + p_{123}(F_1F_3)')(1 - F_2)ds$$

$$G_3(t) = \int_{t_0}^{t} (p_{13}F_1' + p_{23}F_2' + p_{123}(F_1F_2)')(1 - F_3)ds.$$

The differentiation of these equations yields that a.e. on $[t_0, T]$

$$
\begin{aligned}
g_1 &= (p_{12}F_2' + p_{13}F_3' + p_{123}(F_2F_3)')(1 - F_1) \\
g_2 &= (p_{12}F_1' + p_{23}F_3' + p_{123}(F_1F_3)')(1 - F_2) \\
g_3 &= (p_{13}F_1' + p_{23}F_2' + p_{123}(F_1F_2)')(1 - F_3).
\end{aligned}
\tag{3.9}
$$

To prove identification, I have to show that system (3.9) with initial conditions

$$F(t_0) = 0, \quad i = 1, 2, 3,\tag{3.10}$$

has a unique positive solution on $[t_0, T]$. My approach is to construct an auxiliary system

by introducing new functions

$$H_1 = p_{12}F_2 + p_{13}F_3 + p_{123}F_2F_3,$$
$$H_2 = p_{12}F_1 + p_{23}F_3 + p_{123}F_1F_3,$$
$$H_3 = p_{13}F_1 + p_{23}F_2 + p_{123}F_1F_2.$$

Below I demonstrate that, in general, functions F_i have a unique representation in terms of H_i. Let $q_i(H)$ denote the representation of F_i in terms of H. Then (3.9) can be written as the system of differential equations

$$H'_i = \frac{g_i}{1 - q_i(H)}, \quad i = 1, 2, 3.$$

The initial conditions on H_i are

$$\lim_{t \downarrow t_0} H_i(t) = 0, \quad i = 1, 2, 3.$$

The existence of a local solution to the auxiliary problem can be proved by applying techniques from section 7.3. First, I would find necessary conditions on G_i. Assuming these conditions I would use the Tonelli approximations method to prove the local existence of a solution H to the auxiliary problem. Then, I would find a solution F to (3.9)-(3.10) from H by using formulas $F_i = q_i(H)$, $i = 1, 2, 3$. The extension techniques in section 7.4 would be used to show global identification.

Now I demonstrate that F_i have unique representations through H_i. I consider two cases: one with $p_{12} > 0$, $p_{13} > 0$, $p_{23} > 0$, $p_{123} > 0$, and the other where $p_{12} > 0$, $p_{13} > 0$, $p_{23} > 0$, $p_{123} = 0$. In both cases, the only conditions required for uniqueness are $g_i \in L^1$ in a small neighborhood of t_0; these conditions are obviously satisfied. Note that the example in which $p_{123} = 1$ constitutes the paper's original problem.

Case $p_{12} > 0$, $p_{13} > 0$, $p_{23} > 0$, $p_{123} > 0$
Observe that

$$H_1 = p_{123}\left(F_2 + \frac{p_{13}}{p_{123}}\right)\left(F_3 + \frac{p_{12}}{p_{123}}\right) - \frac{p_{12}p_{13}}{p_{123}}$$
$$H_2 = p_{123}\left(F_1 + \frac{p_{23}}{p_{123}}\right)\left(F_3 + \frac{p_{12}}{p_{123}}\right) - \frac{p_{12}p_{23}}{p_{123}}$$
$$H_3 = p_{123}\left(F_1 + \frac{p_{23}}{p_{123}}\right)\left(F_2 + \frac{p_{13}}{p_{123}}\right) - \frac{p_{13}p_{23}}{p_{123}}.$$

Taking into account that F_i are positive for $t > t_0$, I derive the following formulas:

$$F_1 = -\frac{p_{23}}{p_{123}} + \frac{1}{p_{123}}\sqrt{\frac{(p_{123}H_2 + p_{12}p_{23})(p_{123}H_3 + p_{13}p_{23})}{p_{123}H_1 + p_{12}p_{13}}}$$

$$F_2 = -\frac{p_{13}}{p_{123}} + \frac{1}{p_{123}}\sqrt{\frac{(p_{123}H_1 + p_{12}p_{13})(p_{123}H_3 + p_{13}p_{23})}{p_{123}H_2 + p_{12}p_{23}}}$$

$$F_3 = -\frac{p_{12}}{p_{123}} + \frac{1}{p_{123}}\sqrt{\frac{(p_{123}H_1 + p_{12}p_{13})(p_{123}H_2 + p_{12}p_{23})}{p_{123}H_3 + p_{13}p_{23}}}.$$

The expressions on the right-hand sides of these equations are $q_1(H)$, $q_2(H)$ and $q_3(H)$, respectively.

Case $p_{12} > 0$, $p_{13} > 0$, $p_{23} > 0$, $p_{123} = 0$
Because

$$H_1 = p_{12}F_2 + p_{13}F_3$$

$$H_2 = p_{12}F_1 + p_{23}F_3$$

$$H_3 = p_{13}F_1 + p_{23}F_2,$$

F_i can be expressed through H_i by inverting matrix

$$P = \begin{pmatrix} 0 & p_{12} & p_{13} \\ p_{12} & 0 & p_{23} \\ p_{13} & p_{23} & 0 \end{pmatrix}.$$

Because

$$P^{-1} = \begin{pmatrix} -\frac{p_{23}}{2p_{13}p_{12}} & \frac{1}{2p_{12}} & \frac{1}{2p_{13}} \\ \frac{1}{2p_{12}} & -\frac{p_{13}}{2p_{12}p_{23}} & \frac{1}{2p_{23}} \\ \frac{1}{2p_{13}} & \frac{1}{2p_{23}} & -\frac{p_{12}}{2p_{13}p_{23}} \end{pmatrix},$$

then

$$F_1 = -\frac{p_{23}}{2p_{13}p_{12}}H_1 + \frac{1}{2p_{12}}H_2 + \frac{1}{2p_{13}}H_3$$

$$F_2 = \frac{1}{2p_{12}}H_1 - \frac{p_{13}}{2p_{12}p_{23}}H_2 + \frac{1}{2p_{23}}H_3$$

$$F_3 = \frac{1}{2p_{13}}H_1 + \frac{1}{2p_{23}}H_2 - \frac{p_{12}}{2p_{13}p_{23}}H_3.$$

The expressions on the right-hand sides of these equations are $q_1(H)$, $q_2(H)$ and $q_3(H)$, respectively. As we can see, in both cases F_i are uniquely expressed in terms of H_i.

Several papers have explored other types of auctions with exogenous variation in the number of bidders. For instance, McAfee and McMillan (1987) allow the number of actual

bidders to be stochastic in first-price, sealed-bid auctions with independent private values. They investigate how bidders' uncertainty about the number of actual rivals affects their equilibrium behavior, not to mention the seller's expected revenue and other issues. In another study, Harstad, Kagel and Levin (1990) consider symmetric first-price and second-price auctions with an uncertain number of actual bidders. They show that first-price and second-price auctions, each with the number of bidders known or uncertain, and English auctions are revenue-equivalent.

3.7   Non-monotonicity of the solution

Conditions (I), (II) and (III) guarantee that problem (DE)-(IC) has a unique positive solution. However, it is possible that functions in this solution are not all increasing. In this situation, one can conclude that the auction model is not stated correctly.

The example below describes a system of well-defined functions G_i such that one of functions F_i in the corresponding unique solution is not monotone.

Example 3.1. This example is illustrated in Figure 6. On $[0,1]$ consider $F_2(t) = t$, $F_3(t) = t$, and define function F_1 in the following way:

$$
F_1(t) = \begin{cases} -2t + 2^{-2m+1}, & t \in [2^{-2m-2}, 2^{-2m-1}], \quad m \geq 1, \\ 10t - 2^{-2m+2}, & t \in [2^{-2m-1}, 2^{-2m}], \quad m \geq 2, \\ (6t+1)/7, & t \in [1/8, 1]. \end{cases}
$$

Function F_1 has the Lipschitz property and therefore it is absolutely continuous. It is strictly decreasing on intervals $[2^{-2m-2}, 2^{-2m-1}]$, $m \geq 1$, and strictly increasing on other intervals. In particular, F_1 *is not increasing in any small neighborhood of* t_0.

I now demonstrate that functions $F_2 F_3$, $F_1 F_3$ and $F_1 F_2$ are strictly increasing. Clearly, $F_2 F_3 = t^2$ is strictly increasing on $[0,1]$. Consider $F_1 F_3$ on an interval $[2^{-2m-2}, 2^{-2m-1}]$, $m \geq 1$:

$$
F_1 F_3(t) = t\left(-2t + 2^{-2m+1}\right).
$$

Function $t\left(-2t + 2^{-2m+1}\right)$ is quadratic; it strictly increases until point 2^{-2m-1}, then it strictly decreases. So, $F_1 F_3$ is strictly increasing on $[2^{-2m-2}, 2^{-2m-1}]$.

Now consider $F_1 F_3$ on an interval $[2^{-2m-1}, 2^{-2m}]$, $m \geq 2$:

$$
F_1 F_3(t) = t\left(10t - 2^{-2m+2}\right).
$$

Quadratic function $t\left(10t - 2^{-2m+2}\right)$ strictly decreases until point $2^{-2m}/5$ and strictly increases afterward. Because $2^{-2m-1} > 2^{-2m}/5$, then $F_1 F_3$ strictly increases on $[2^{-2m-1}, 2^{-2m}]$. Obviously, on $[1/8, 1]$ function

$$
F_1 F_3(t) = t\left(6t + 1\right)/7
$$

strictly increases. Thus, F_1F_3 and, consequently, F_1F_2 are strictly increasing on $[0,1]$.

Now find functions G_i according to integral-differential equations (3.3). Because F_1F_2, F_1F_3 and F_2F_3 are strictly increasing, functions G_i are strictly increasing as well. These G_i clearly satisfy other conditions in (I) and also condition (II). It can be shown that F_i satisfy condition (3.5) and, therefore, G_i satisfy the main identification condition (III). Hence, we can find well-defined, observable G_i whose corresponding F_i are not all monotone.
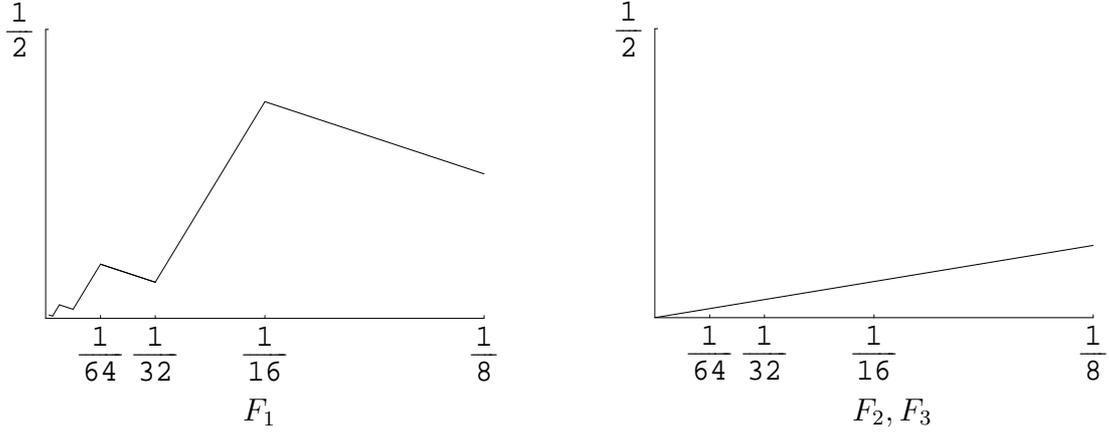
Figure 6. F_1, F_2, F_3 on $[0, 1/8]$ (Example 3.1).

The last thing that has to be to shown is that functions G_i in this example can indeed be observed in auction's outcomes. To do this, I present an example of a joint absolutely continuous distribution that rationalizes G_i. Find functions Φ_i, Ψ_i, Ξ_i, $i = 1,2,3$, that satisfy the following conditions:

1. They are defined on $[0,1]$, positive, increasing and absolutely continuous on $[0,1]$.
2. $\Phi_i(0) = 0$, $\Psi_i(0) = 0$ and $\Xi_i(0) = 0$, $i = 1,2,3$.
3.

$$G_1(t) = \int_{t_0}^{t} (\Phi_2\Phi_3)'(1 - \Phi_1)ds$$

$$G_2(t) = \int_{t_0}^{t} (\Psi_1\Psi_3)'(1 - \Psi_2)ds$$

$$G_3(t) = \int_{t_0}^{t} (\Xi_1\Xi_2)'(1 - \Xi_3)ds.$$

Denote $\phi_i = \Phi_i'$, $\psi_i = \Psi_i'$ and $\xi_i = \Xi_i'$. Consider the following function defined on $[0,1]^3$:

$$f(x_1, x_2, x_3) = \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)1(x_1 \geq x_2)1(x_1 \geq x_3)$$
$$+ \psi_1(x_1)\psi_2(x_2)\psi_3(x_3)1(x_2 > x_1)1(x_2 \geq x_3)$$
$$+ \xi_1(x_1)\xi_2(x_2)\xi_3(x_3)1(x_3 > x_1)1(x_3 > x_2).$$

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Function f is an example of the joint density of the distribution of bidders' values that rationalizes observable G_i. Note that because we can choose functions Φ_i, Ψ_i, Ξ_i in many different ways, there are many observationally equivalent densities of this type.

This non-monotonicity result is possible because of the complexity of auction data. Indeed, in auctions with d bidders, data provide direct knowledge regarding groups of $d-1$ bidders. To learn each bidder's value separately, this information has to be disentangled further.

It is important to mention that non-monotonicity is sufficient but not necessary for detecting an incorrectly stated model. That is, considering such models under the assumption of independence, it is possible to obtain a solution in which all functions are distribution functions.

The non-monotonicity result carries important implications for generalized competing risks models and contrasts sharply with results for classical competing risks models. In classical models (Roy model), every dependent risks model has a unique, observationally equivalent independent risks model, as shown in Tsiatis (1975). Example 3.1 shows that in generalized competing risks models this is not necessarily so because the non-monotonicity of at least one function F_i means that the risks are dependent. In other words, in generalized competing risks models, in some situations dependent and independent risks can be distinguished nonparametrically.

In auctions, the non-monotonicity of F_i can occur for a variety of reasons. For instance, it can happen if bidders' private values are not independent, or if bidders do not behave rationally, or if bidders' values are not private.

4   Sieve estimation of distribution functions

This section presents an approach to estimating the distribution functions of private values from a random sample. First, I define an operator A that maps unknown distribution functions F_1, F_2, F_3 to observable functions G_1, G_2, G_3. I show that this operator is Lipschitz and that under weak conditions on the set of $F = (F_1, F_2, F_3)$, its inverse operator A^{-1} is continuous. I then derive sieve estimators of F_i and use the properties of A to show their consistency.

4.1   Operator A

For an absolutely continuous function $F = (F_1, F_2, F_3)^{tr}$ define $A(F) = (A(F)_1, A(F)_2, A(F)_3)^{tr}$ as follows:

$$A(F)_1(t) = \int_{t_0}^{t} (F_2 F_3)^{'}(1 - F_1)ds$$

$$A(F)_2(t) = \int_{t_0}^{t} (F_1 F_3)^{'}(1 - F_2)ds$$

$$A(F)_3(t) = \int_{t_0}^{t} (F_1 F_2)^{'}(1 - F_3)ds.$$

Let Λ be the set of functions $F = (F_1, F_2, F_3)^{tr}$ defined on $[t_0, T]$ and satisfying the following conditions:

Conditions CI.

1. *F_i are absolutely continuous on $[t_0, T]$.*

2. *F_i are strictly increasing on $[t_0, T]$.*

3. *$F_i(t_0) = 0$, $F_i(T) = 1$, $i = 1, 2, 3$.*

4. *The function*
$$\left(\frac{F_1'}{F_1} + \frac{F_2'}{F_2} + \frac{F_3'}{F_3} \right) (F_1 + F_2 + F_3)$$
    is Lebesgue integrable in a neighborhood of t_0.

Let A be defined on Λ. Properties of the image $A(\Lambda)$ are described in Proposition 3.1: G_i are absolutely continuous on $[t_0, T]$, strictly increasing on $[t_0, T]$ and $G_i(t_0) = 0$. Also, $G_1(T) + G_2(T) + G_3(T) = 1$. As shown in this paper, there exists the inverse operator $A^{-1} : A(\Lambda) \to \Lambda$.

Endow both domain Λ and its image $A(\Lambda)$ with the following uniform metric:

$$d(F, \tilde{F}) = \sup_{t \in [0,1]} \sqrt{(F(t) - \tilde{F}(t))^{tr}(F(t) - \tilde{F}(t))}$$

$$d(G, \tilde{G}) = \sup_{t \in [0,1]} \sqrt{(G(t) - \tilde{G}(t))^{tr}(G(t) - \tilde{G}(t))}.$$

The proposition below implies that A is continuous in this metric.

Proposition 4.1. *For any $F, \tilde{F} \in \Lambda$,*

$$d(A(F), A(\tilde{F})) \leq 9\sqrt{3}d(F, \tilde{F});$$

that is, operator A is Lipschitz on Λ.

The properties of A are important for proving the consistency of the estimators of F_i. Usually, it is easier to establish consistency when the space of functions is compact. I compactify Λ by bounding the densities of F_i by the same function in L^1:

Condition CII.

$$F_i'(t) \le \phi'(t) \quad a.e. \quad [t_0, T], \quad i = 1, 2, 3,$$

where ϕ is some absolutely continuous function on $[t_0, T]$.

Let Λ_ϕ be a subset of Λ such that all functions F from Λ_ϕ satisfy condition *CII*. This condition guarantees that Λ_ϕ is relatively compact under the uniform metric. Indeed, for any $F \in \Lambda_\phi$ and $t, \tau \in [t_0, T]$,

$$|F_i(t) - F_i(\tau)| = \left| \int_\tau^t F_i'(s) ds \right| \le |\phi(t) - \phi(\tau)|, \quad i = 1, 2, 3.$$

Because ϕ is absolutely continuous, the last inequality implies that the set Λ_ϕ is equicontinuous. It is also uniformly bounded because the values of F_i do not exceed 1. According to the Arzela-Ascoli theorem, Λ_ϕ is relatively compact in metric $d(\cdot, \cdot)$. Note that if $F \in \Lambda_\phi$, then function $G = A(F)$ satisfies this condition too:

$$g_i(t) \le \phi'(t) \quad a.e. \quad [t_0, T], \quad i = 1, 2, 3.$$

Let $\overline{\Lambda}_\phi$ stand for the closure of Λ_ϕ under metric $d(\cdot, \cdot)$. Because Λ_ϕ is relatively compact, $\overline{\Lambda}_\phi$ is a compact set. To consider operator A on $\overline{\Lambda}_\phi$, I first have to show that A is defined for functions in this set that do not belong to Λ_ϕ. The proposition below establishes that all functions in $\overline{\Lambda}_\phi$ satisfy conditions 1, 3 and a modified condition 2 in *CI*, and also satisfy condition *CII*.

Proposition 4.2. *If $F = (F_1, F_2, F_3)^{tr} \in \overline{\Lambda}_\phi$, then functions F_i are absolutely continuous, increasing and satisfy $F_i(t_0) = 0$. Also, $F_i'(t) \le \phi'(t)$ a.e. on $[t_0, T]$, $i = 1, 2, 3$.*

Because all functions in $\overline{\Lambda}_\phi$ are absolutely continuous, operator A can be extended from Λ_ϕ to $\overline{\Lambda}_\phi$. The next proposition implies that A is continuous on $\overline{\Lambda}_\phi$.

Proposition 4.3. *For any $F, \tilde{F} \in \overline{\Lambda}_\phi$,*

$$d(A(F), A(\tilde{F})) \le C_0 d(F, \tilde{F}), \tag{4.1}$$

where $C_0 = 3\sqrt{3}(1 + 3\phi(T) - 3\phi(t_0))$; that is, A is Lipschitz on $\overline{\Lambda}_\phi$.

Finally, I establish the continuity of A^{-1} on $A(\Lambda_\phi)$.

Proposition 4.4. *A^{-1} is continuous on $A(\Lambda_\phi)$.*

This proposition follows from the fact that if a continuous operator is defined on a compact set and the inverse operator is defined on the image of that set, then the inverse

operator is continuous. The inverse operator A^{-1} is clearly defined on $A(\Lambda_\phi)$ but I have not shown that it is defined on the set $A(\overline{\Lambda}_\phi)$. However, this does not affect the result (see explanation in Appendix 8).

4.2 Consistent estimation

In this section, I define sieve estimators of the distribution functions F_i and prove their consistency.

Without a loss of generality, assume that the distributions have support $[0, 1]$. Denote the true distribution functions as F_i^* and the corresponding observable functions as G_i^*. That is, $G^* = A(F^*)$, where $F^* = (F_1^*, F_2^*, F_3^*)^{tr}$ and $G^* = (G_1^*, G_2^*, G_3^*)^{tr}$. Let $F^* \in \Lambda_\phi$. The next lemma introduces a function Q on $\overline{\Lambda}_\phi$ that is uniquely minimized by F^*.

Lemma 4.5. F^* *is the unique minimizer of*

$$Q(F) = E(G^* - A(F))^{tr}(G^* - A(F))$$

on $\overline{\Lambda}_\phi$.

The idea of sieve estimation is to use a sample analog of Q and approximate $\overline{\Lambda}_\phi$ with finite-dimensional spaces. For each k, choose base functions $p_{1,k}, \ldots, p_{m(k),k}$ (for example, B-splines with uniform knots or Bernstein polynomials) and introduce the set of linear combinations of these functions:

$$M_k = \{(F_1, F_2, F_3)^{tr} : F_i(t) = \sum_{l=1}^{m(k)} \alpha_l^i p_{l,k}(t), \, t \in [0, 1]\}.$$

In this set of functions, consider only those functions that are in Λ_ϕ:

$$\Sigma_k = \Lambda_\phi \cap M_k.$$

Set Σ_k consists of functions from M_k with certain restrictions on coefficients α_l^i. It is relatively compact and, hence, its closure $\overline{\Sigma}_k$ is compact, and $\overline{\Sigma}_k \subset \overline{\Lambda}_\phi$.

Consider a random sample of n observations $(t_i, w_i)_{i=1}^n$, where t_i is the observed price and w_i is the winner's identity in i's auction. Without a loss of generality, assume that $t_i < t_{i+1}$, $i = 1, \ldots, n-1$. From the sample, find consistent estimators $\hat{G}_{i,n}$ of G_i, for instance, analogs of empirical distribution functions.

Let the sample objective function be

$$\hat{Q}_n(F) = \frac{1}{n} \sum_{i=1}^n (\hat{G}_n(t_i) - A(F)(t_i))^{tr}(\hat{G}_n(t_i) - A(F)(t_i)).$$

Also let $k = k(n)$, and define the following estimator of F^*:

$$\hat{F}_n = \mathrm{argmin}_{F \in \overline{\Sigma}_{k(n)}} \hat{Q}_n(F).$$

The theorem below establishes the estimator consistency when sets $\overline{\Sigma}_k$ well approximate set $\overline{\Lambda}_\phi$.

Theorem 4.6. *If*

$$\forall (F \in \overline{\Lambda}_\phi) \exists (\tilde{F} \in \overline{\Sigma}_k) \quad d(F, \tilde{F}) \xrightarrow{p} 0 \ as \ k \to \infty, \tag{4.2}$$

then estimator $\hat{F}_n$ is consistent:

$$d(\hat{F}_n, F^*) \xrightarrow{p} 0 \ as \ n \to \infty.$$

Condition (4.2) holds if approximating sets are chosen properly $-$ for instance, if base functions $p_{1,k}, \ldots, p_{m(k),k}$ are B-splines with uniform knots, Bernstein polynomials or truncated power series.

5   Identification in generalized competing risks models

The main purpose of this section is to present conditions on observables sufficient to guarantee identification in generalized competing risks models.

In section 2, I gave two examples of these models. First, I explained why we can consider second-price auctions to be a special case of these models. In the other example, I considered widely used classical competing risks models. I now proceed to a more detailed description of generalized competing risks models. For convenience, I use the terminology of reliability theory, which refers to these generalized models as coherent systems.[4] Essentially, a coherent system is a system that collapses because several of its elements fail.

Suppose that a machine with a coherent structure consists of d elements. Denote the elements' lifetimes as $X_1, \ldots, X_d$ and the machine's lifetime as Z; the lifetime Z is a function of $X_1, \ldots, X_d$. Conveniently, Z can be characterized by fatal sets. As defined in section 2, a *fatal set* is a subset of parts such that the failure of all the parts in the subset causes the failure of the machine. Even more conveniently, Z can be characterized by the collection $I_1, \ldots, I_m$ of *minimal fatal sets*, which are fatal sets that do not encompass other fatal sets.

The examples below clarify the structure of a coherent system. To guarantee that the probability of the simultaneous failure of several elements is 0, I suppose that the joint distribution of $X_1, \ldots, X_d$ is absolutely continuous. Also, X_i have the same support $[t_0, T]$.

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[4]The concept of a coherent system was introduced in Barlow and Proschan (1975).

Example 5.1. In a classical competing risks model with d risks, the collection of minimal fatal sets is $I_1 = \{1\}, \ldots, I_d = \{d\}$, and the machine's lifetime is

$$Z = \min\{X_1, \ldots, X_d\}.$$

Clearly, the number of minimal fatal sets coincides with the number of elements. Furthermore, there are no fatal sets other than sets I_i. Take, for instance, set $\{1, 2\}$. Although it is a superset of fatal sets $\{1\}$ and $\{2\}$, it is not fatal itself. Indeed, the death of these two elements could not cause the machine's failure because the death of either of them would have led to failure earlier.

Example 5.2. Consider a button auction with d bidders who have private values. In this case, the fatal sets are the sets of bidders who dropped out before the auction ended. The collection of minimal fatal sets is

$$I_i = \{1, \ldots, i-1, i+1, \ldots, d\}, \quad i = 1, \ldots, d.$$

Here, element lifetimes X_i are bidders' private values, and the lifetime Z is the winning price. Notice that the number of minimal fatal sets is the same as the number of bidders, and there are no fatal sets besides I_i.

Example 5.3. Consider a machine with five parts. Let the collection of minimal fatal sets be $I_1 = \{1, 2, 3\}$, $I_2 = \{1, 2, 4\}$, $I_3 = \{1, 3, 4\}$, $I_4 = \{2, 3, 4\}$, $I_5 = \{1, 3, 5\}$ and $I_6 = \{2, 3, 5\}$. An example of a fatal set that is not a minimal fatal set is $\{1, 2, 3, 5\}$: It causes the failure of the machine when, for instance, the machine's elements break in the order of 5, 1, 2 and 3. Set $\{1, 2, 3, 4\}$, on the other hand, is not fatal because all its three-element subsets are minimal fatal sets.

For coherent systems, the goal is to learn the marginal distributions of element lifetimes X_i from the joint distribution of observed "autopsy" data, which comprise the machine's lifetime Z and a fatal set I that is responsible for the machine's failure. This identification question is raised in Meilijson (1981). Meilijson claims that, under certain restrictions on a coherent system's structure, the distributions of the components' lifetimes are identified if the lifetimes are independent. To formulate the identification result, he introduces an incidence matrix constructed in the following way. Given a collection of minimal fatal sets, the coherent system's incidence matrix is a matrix M such that $M(i, j) = 1$ if $j \in I_i$, and $M(i, j) = 0$ otherwise, $i = 1, \ldots, m$, $j = 1, \ldots, d$.

For example, in the three-bidder auctions considered in Example 5.2, the incidence matrix is

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

In classical competing risks models, on the other hand, the incidence matrix is the $d \times d$ identity matrix.

The main result of Meilijson (1981) says that if X_1, X_2, ..., X_d are non-atomic and independent and possess the same essential infimum and supremum, and if the rank of M is d, then the joint distribution of Z and I uniquely determines the distribution of each X_j, $j = 1, \ldots, d$.

The idea behind Meilijson's proof is (a) to use data only from those cases where set I is a minimal fatal set and (b) to obtain integral equations that relate the distribution functions of components' lifetimes to observable functions, and then apply to them a fixed point theorem for multidimensional functional spaces. Though Meilijson (1981) made important contributions, including the observation that only the data corresponding to minimal fatal sets can be considered, as well as and the rank condition on the incidence matrix, the proofs lack some essential details. First, the author does not discuss necessary conditions on observable data besides mentioning them as a prospect for future research. As we have seen in the auction model, however, such conditions are crucial for obtaining the existence and uniqueness result. Second, he does not explore the existence of underlying distributions that rationalize the observables. A possible reason for this omission is the fact that in the majority of generalized competing risks models, existence cannot be proved and must be assumed, as I explain below. Nevertheless, I show that existence can be established for a special class of competing risks models, and I present conditions on observables that are necessary and sufficient for existence. Another important piece missing from Meilijson's proof is conditions on observables sufficient to guarantee the uniqueness of underlying distributions consistent with the data. I provide these conditions for any generalized competing risks model. Finally, although the author mentions that the locally identified distribution functions can be extended to the whole support, he does not present a proof of this result. As in the auction, such a proof would require the identification result for the case in which all distribution functions have positive values at the initial point.

I suggest a new approach to identification in generalized competing risks models that offers a complete transparent proof of the identification result. I assume that the components' lifetimes have absolutely continuous distributions, even though Meilijson (1981) obtains his result under the weaker assumption that the lifetimes' distributions are merely continuous. The idea behind my method is similar to the case of the auction; namely, I derive a system of non-linear differential equations that relates the underlying distribution functions to observable functions, then examine the existence and uniqueness issues for this system. I use the incidence matrix and assume the rank condition as in Meilijson (1981).

Now I turn to stating the main results for generalized competing risks models. An outline of Meilijson's method are in Appendix B.

For any fatal set D, there is a corresponding observable function G_D:

$$G_D(t) = P(Z \le t, D \text{ causes machine's failure}).$$

Because lifetimes X_i are independent, then

$$G_D(t) = \int_{t_0}^{t} (\prod_{j \in C_D} F_j(s))' \prod_{j \in D^c} (1 - F_j(s)) \prod_{j \in D \setminus C_D} F_j(s) ds, \qquad (5.1)$$

where F_j is the distribution function of X_j, C_D is the intersection of all minimal fatal sets contained in D, and $D^c = \{1, \ldots, d\} \setminus D$.

Let G_i be an observable function corresponding to the minimal fatal set I_i, $i = 1, \ldots, m$:

$$G_i(t) = \int_{t_0}^{t} (\prod_{j \in I_i} F_j(s))' \prod_{j \in I_i^c} (1 - F_j(s)) ds, \quad i = 1, \ldots, m. \qquad (5.2)$$

System (5.2) of integral-differential equations is an analog of system (3.3). The differentiation of the equations in (5.2) yields the following system of non-linear differential equations:

$$(\prod_{j \in I_i} F_j)' = \frac{g_i}{\prod_{j \in I_i^c} (1 - F_j)}, \quad i = 1, \ldots, m. \qquad (5.3)$$

I analyze this system together with initial conditions

$$F_i(t_0) = 0, \quad i = 1, \ldots, d. \qquad (5.4)$$

First, I consider the case in which the number of minimal fatal sets coincides with the number of the machine's components $-$ that is $m = d$. In this instance, the matrix M is quadratic. Let k_{ij} stand for the (i, j) element of the inverse matrix M^{-1}.

In the next theorem I formulate the existence result for problem (5.3)-(5.4) and describe conditions on G_i that guarantee it.

Theorem 5.1. [5] *Let $m = d$. Let functions G_i satisfy the following conditions:*
1. $G_i(t_0) = 0$, $i = 1, \ldots, d$
2. G_i are absolutely continuous on $[t_0, T]$, $i = 1, \ldots, d$
3. G_i are strictly increasing on $[t_0, T]$, $i = 1, \ldots, d$
4. $\lim_{t \downarrow t_0} \prod_{j=1}^{d} G_j^{k_{ij}}(t) = 0$, $i = 1, \ldots, d$
Then problem (5.3)-(5.4) has a solution F on $[t_0, T]$. [6]

Notice that, from the model, conditions 1-4 in this proposition are necessary on G_i. Indeed, 1-3 follow directly from the definition of functions G_i. Given that conditions 1-3

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[5] The proof of this theorem is available upon request.
[6] I consider only positive solutions.

hold, condition 4 can be obtained from (5.3). The interpretation of these conditions is similar to that of conditions (I) and (II) in the auction model.

An important difference between this case and the auction, however, is that even if problem (5.3)-(5.4) possesses a solution F and all F_i in this solution have the properties of distribution functions, the existence of a solution to the model is not guaranteed. Indeed, to satisfy the model, F must solve equation (5.1) for any fatal set D. System (5.3), however, accounts only for the minimal fatal sets. Therefore, after finding a solution to (5.3)-(5.4), we have to substitute it into (5.1) to verify that it solves this equation for any D. Because it is difficult (and perhaps impossible) to find conditions on functions G_D under which the model has a solution, it is common in reliability theory to assume existence. The only situation in which the conditions in Theorem 5.1 guarantee existence of a solution to the model is when $m = d$ and the only fatal sets in the model are minimal fatal sets. Notice that this is the case in the auction model analyzed in this paper.

The next theorem provides conditions on G_i that are sufficient for the uniqueness of a solution to (5.3)-(5.4). The proof of this theorem is in Appendix B.

Theorem 5.2. *Let $m = d$. Suppose that all conditions on G_i in Theorem 5.1 are satisfied. Denote*

$$\Gamma_i(t) = g_i \sum_{l \in I_i^c} \sum_{h=1}^{d} |k_{lh}| (\prod_{j \neq h} G_j^{lj}) G_h^{k_{lh}-1}.$$

If for any $i = 1, \ldots, d$, function

$$\Gamma_i \text{ is Lebesgue integrable in a small neighborhood of } t_0, \tag{5.5}$$

then problem (5.3)-(5.4) has a unique solution on $[t_0, T]$.

Because problem (5.3)-(5.4) has a unique solution, the model cannot have more than one solution. Therefore, the following corollary holds.

Corollary 5.3. *Let $m = d$. Suppose that all conditions in Theorem 5.2 are satisfied. Then a solution to the model, if it exists, is unique.*

When the number of minimal fatal sets exceeds d − that is, $m > d$ − the existence of a solution to the model is always assumed. It is easy, however, to indicate conditions on observable functions that guarantee the uniqueness of a solution to the model when one exists. Consider any $d \times d$ full-rank submatrix of M. Without a loss of generality, suppose that this submatrix is formed by the first d rows in M. The subsystem of (5.3) that comprises the differential equations corresponding to the first d rows in M has only one solution if G_i satisfy the conditions in Theorem 5.2. Consequently, the model has at most one solution. We can find other sufficient conditions by choosing different submatrices of M.

The proofs of theorems 5.1 and 5.2 use the same methods as those of theorems 3.4 and 3.5. First, the existence and uniqueness of a solution are established locally, then globally.

6 Values are not independent

The main goal of this section is to investigate the identification issue in second-price auctions in the absence of independence. Many of the results obtained here can be extended to dependent generalized competing risks models.

For affiliated private values, Athey and Haile (2002) show that when only a subset of bids is observed, the joint distribution of bidders' values is not identified without any additional assumptions. In particular, even if all the bids except for the highest bid are known, there are many distributions consistent with the data.

Even though the distribution of values is not identified, the data are informative and allow finding bounds on distributions. These bounds can exploited in the analysis of counterfactuals and other applications.

Because the auction problem is related to generalized competing risks models, I start by reviewing the competing risks literature regarding partial identification. The study of partial identification in classical competing risks models was initiated by Peterson (1976), who obtained tight point-wise bounds on the joint and marginal survival functions. Crowder (1991) and Bedford and Meilijson (1997) obtained new results on bounds for those functions. We also can look to Manski (1990), who examined partial identification for self-selection models, of which competing risks models are a subset. For generalized competing risks models, several results for bounds on survival functions are established by Deshpande and Karia (1997).

I consider two observational schemes in auctions. The first scheme is the case when only the winner's identity and the transaction price are observed. For this scheme, I derive bounds on the joint distribution of values for any subset of bidders. It is of interest to analyze how these bounds change when more data become available − data on other identities or other bids. That is why I consider the second scheme, which is the situation when all the identities and all the bids except for the highest bid are known. This situation corresponds to continuous monitoring models for coherent systems. For this scheme, I present bounds on the joint distribution of values for the set of all bidders and bounds on the marginal distributions. All the bounds are derived for any type of dependence, not only when private values are affiliated.

Notation

Throughout this section it is supposed that d bidders are participating in a second-price auction, and their private values X_1, ..., X_d have continuous marginal distributions on the same support $[t_0, T]$. It is also assumed that $P(X_i = X_j) = 0$, $i \neq j$, so that the

probability of a tie is 0. Instead, it can be assumed that the joint distribution of bidders' values is absolutely continuous, which implies zero probabilities of ties.

Suppose that $D = \{i_1, \ldots, i_r\}$ is a subset that consists of bidders $i_1, \ldots, i_r$. Define Q_D as the distribution function of valuations of bidders in this subset:

$$Q_D(t_{i_1}, \ldots, t_{i_r}) = P(X_{i_1} \leq t_{i_1}, \ldots, X_{i_r} \leq t_{i_r}).$$

For $D = \{1, \ldots, d\}$, I denote Q_D as Q. For $D = \{1, \ldots, m-1, m+1, \ldots, d\}$, I denote Q_D as Q_{-m}. For $D = \{j\}$, Q_D is denoted as F_j.

6.1 Bounds when only the winner's identity and the winning price are observed

To obtain the lower bound on Q_D, I use the fact that if bidder $j \notin D = \{i_1, \ldots, i_r\}$ wins and the price does not exceed t, then all the values $X_{i_1}, \ldots, X_{i_r}$ do not exceed t either. In other words, functions G_j, $j \notin D$, provide information about the lower bound on Q_D. On the other hand, if bidder i_k wins, then it is not known how large the value X_{i_k} is and, consequently, G_{i_k} is not helpful in finding the lower bound on Q_D.

To obtain the upper bound on Q_D, I exploit the fact that if we know an upper bound on value X_{i_k}, then we know an upper bound on the price when bidder i_k wins. If we know upper bounds on values $X_{i_1}, \ldots, X_{i_r}$, and bidder $j \notin D = \{i_1, \ldots, i_r\}$ wins, then in general no conclusion can be made about the price. In other words, only functions $G_{i_1}, \ldots, G_{i_r}$ determine the upper bound on Q_D.

The theorem below formalizes this discussion and presents bounds on distribution functions Q_D.

Theorem 6.1. *Suppose that bidders play their weakly dominant strategy by submitting their true values. Also suppose that only the winner's identity and the transaction price are observed.*
(a) Then Q_D is bounded from below as follows:

$$Q_D(t_{i_1}, \ldots, t_{i_r}) \geq \sum_{l=1}^{r} 1(t_{i_l} = T) G_{i_l}\left(\min_{k=1,\ldots,r} t_{i_k}\right) + \sum_{j \in CD} G_j\left(\min_{k=1,\ldots,r} t_{i_k}\right).$$

(b) Function Q is bounded from above as follows:

$$Q(t_1, \ldots, t_d) \leq 1(\min_i t_i > t_0) \sum_{j=1}^{d} G_j(\min\{t_j, \max_{i \neq j} t_i\}).$$

For any $m = 1, \ldots, d$, distribution function Q_{-m} is tightly bounded from above as follows:

$$Q_{-m}(t_1, \ldots, t_{m-1}, t_{m+1}, \ldots, t_d) \leq 1(\min_{i \neq m} t_i > t_0) \left(\sum_{j \neq m} G_j(t_j) + G_m(\max_{i \neq m} t_i) \right).$$

If D contains at most $d - 2$ elements, then Q_D is tightly bounded from above as follows:

$$Q_D(t_{i_1}, \ldots, t_{i_r}) \leq 1(\min_{i_k \in D} t_{i_k} > t_0) \left(\sum_{i_k \in D} G_{i_k}(t_{i_k}) + \sum_{j \in CD} G_j(T) \right).$$

Theorem 6.1 relies on the fact that bidders submit their true values. However, this condition can be relaxed. Consider the following two assumptions.

Assumption I (AI). Bidders do not bid more than they are willing to pay.

Assumption II (AII). Bidders do not allow an opponent to win at a price they are willing to beat.

These assumptions were introduced in Haile and Tamer (2003). The authors were among the first ones to relax equilibrium conditions and allow other types of bidders' behavior. One of their contributions is the construction of bounds on distributions for this limited structure in certain auction models.

The proposition below shows that the bounds in theorem 6.1 are correct when the equilibrium condition is replaced with assumptions 1 and 2. In addition, I have to assume that the probability of a tie is 0.

Proposition 6.2. *Suppose that only the winner's identity and the transaction price are observed and $P(b_i = b_j) = 0$ for $i \neq j$.*
(a) If AI holds, then Q_D are bounded from above as in theorem 6.1.
(b) If AII holds, then Q_D are bounded from below as in theorem 6.1.

6.2 Bounds when all the identities and all the bids except for the highest bid are observed

Let Π_d denote the set of all the permutations of set $\{1, \ldots, d\}$ and $\rho \in \Pi_d$. Let $\rho(i)$ stand for the ith element of permutation ρ. In the auction context, bidder $\rho(i)$ is the ith highest bidder.

The following $d!$ functions are observed:

$$W_\rho(s_2, \ldots, s_d) = P(\cap_{i \neq 1}(b_{\rho(i)} \leq s_i), b_{\rho(1)} > b_{\rho(2)} > \ldots > b_{\rho(d)}).$$

Notice that

$$G_j(t) = \sum_{\rho \in \Pi_d : \rho(1) = j} W_\rho(t, \ldots, t).$$

Also introduce the following $d(d-1)$ functions:

$$G_{jh}(t) = P(\max_{i \neq j, i \neq h} b_i < b_h, b_h < b_j, b_h \leq t), \quad h \neq j, \;\; j = 1, \ldots, d, \;\; h = 1, \ldots, d.$$

Note that

$$G_{jh}(t) = \sum_{\rho \in \Pi_d : \rho(1) = j, \rho(2) = h} W_\rho(t, \ldots, t).$$

Value $G_{jh}(t)$ is the probability that bidder j wins, bidder h submits the second-highest bid and this bid does not exceed t.

For simplicity, I show bounds only on the joint distribution function Q and marginal distribution functions F_j, $j = 1, \ldots, d$.

Theorem 6.3. *Suppose that bidders play their weakly dominant strategy by submitting their true values. Also suppose all the identities and all the bids except for the highest bid are observed.*

(a) Then function Q is bounded from above and below as follows:

$$\prod_{i=1}^{d} 1(t_i = T) \leq Q(t_1, \ldots, t_d) \leq \sum_{\rho \in \Pi_d} W_\rho(\min\{t_{\rho(1)}, t_{\rho(2)}\}, \ldots, \min_{m=1,\ldots,l} t_{\rho(m)}, \ldots, \min_{i=1,\ldots,d} t_i).$$

(b) The marginal distribution functions F_j, $j = 1, \ldots, d$, are bounded from above and below as follows:

$$\sum_{i \neq j} G_i(t) + 1(t = T) G_j(T) \leq F_j(t) \leq \sum_{i \neq j} G_{ji}(t) + \sum_{i \neq j} G_{ij}(t) + 1(t > t_0) \sum_{i \neq j} \sum_{h \neq i, h \neq j} G_{ih}(T).$$

The example below illustrates the results of theorems 6.1 and 6.3. It is depicted in figures 7 and 8.

Example 6.1. Consider the auction with three buyers. Let $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$ and A be independent random variables distributed on $[0, 1]$ with distribution functions $\tilde{F}_1(t) = t$, $\tilde{F}_2(t) = t^2$, $\tilde{F}_3(t) = \sqrt{t}$ and $\tilde{F}_A(t) = t$. Let private values X_1, X_2 and X_3 of the buyers be

$$X_1 = 0.25\tilde{X}_1 + 0.75A$$
$$X_2 = 0.6\tilde{X}_2 + 0.4A$$
$$X_3 = 0.5\tilde{X}_3 + 0.5A.$$

Figure 7 shows the bounds on the marginal distribution functions F_1, F_2 and F_3 in the first scenario in Theorem 6.1. Figure 8 shows the bounds on F_1, F_2 and F_3 in the second scenario.
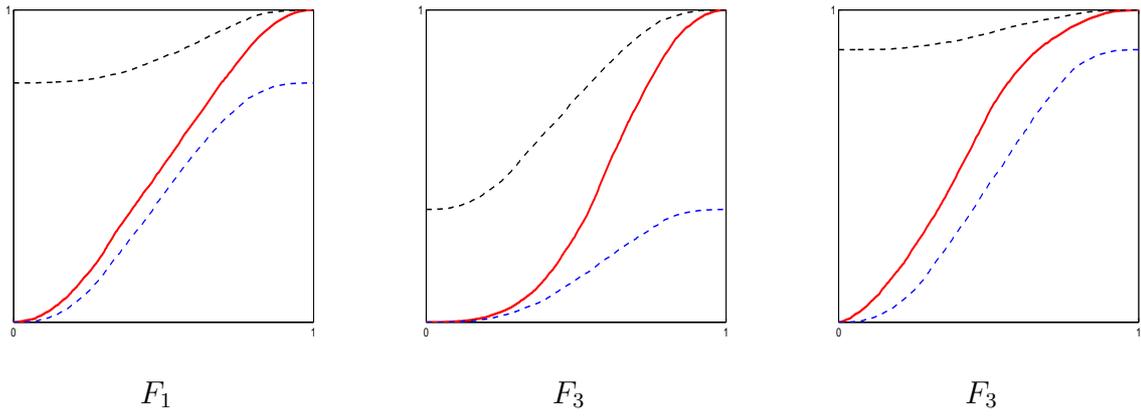
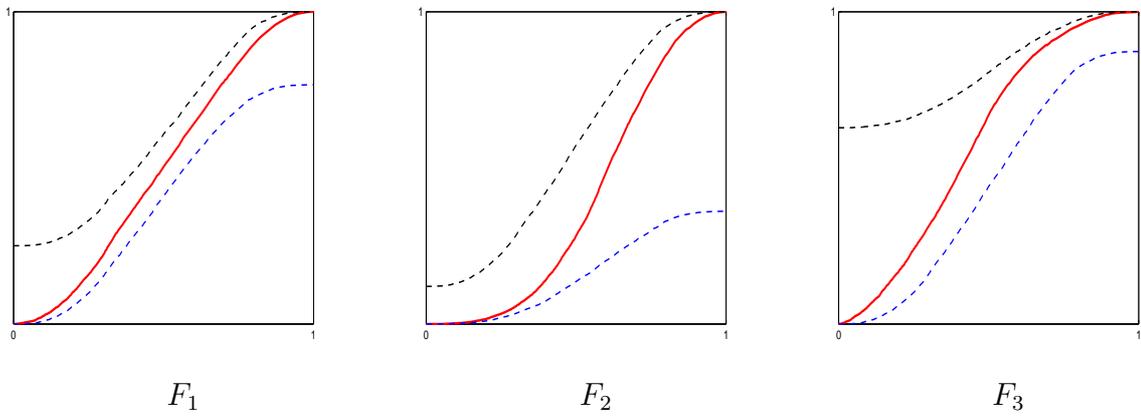Figure 7. Bounds on the marginal distribution functions in the first scenario.

Figure 8. Bounds on the marginal distribution functions in the second scenario.

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7 Appendix A: Proofs of the results in section 3

In the Appendix, I use the following notations. $L^1[\tau, \xi]$ stands for the class of Lebesgue integrable functions on $[\tau, \xi]$. The Euclidian norm of vector $x = (x_1, \ldots, x_d)$ is denoted as $\|x\|$. $\|x\|_1$ stands for the following norm of x: $\|x\|_1 = \sum_{i=1}^{d} |x_i|$. The right derivative of function v at point t is

$$D_R v(t) = \lim_{h \downarrow 0} \frac{v(t+h) - v(t)}{h}.$$

7.1 Proofs of Proposition 3.2 and Corollary 3.3

Proof of Proposition 3.2. It suffices to show that $\lim_{t \downarrow t_0} \frac{F_1}{\sqrt{\frac{G_2 G_3}{G_1}}}(t) = 1$. Let $t_1 > t_0$ be very close to t_0 and let $0 < L < 1$ be such that $F_i(t) \leq L$ for any $t \in (t_0, t_1)$, $i = 1, 2, 3$. Consider the first equation in system (3.3) and use it to obtain that

$$G_1(t_1) \geq \int_{t_0}^{t_1} (F_2 F_3)' (1 - L) ds = (1 - L) F_2(t_1) F_3(t_1),$$

$$G_1(t_1) \leq F_2(t_1) F_3(t_1).$$

Similarly, using the other two equations in (3.3), obtain that

$$(1 - L) F_1(t_1) F_3(t_1) \leq G_2(t_1) \leq F_1(t_1) F_3(t_1),$$

$$(1 - L) F_1(t_1) F_2(t_1) \leq G_3(t_1) \leq F_1(t_1) F_2(t_1).$$

Because $F_1 = \sqrt{\frac{F_1 F_2 F_1 F_3}{F_2 F_3}}$, then

$$F_1(t_1) \leq \frac{1}{1 - L} \sqrt{\frac{G_2 G_3}{G_1}}, \quad F_1(t_1) \geq \sqrt{1 - L} \sqrt{\frac{G_2 G_3}{G_1}}.$$

Because $F_1(t_0) = 0$ and t_1 can be chosen arbitrarily close to t_0, then L can be arbitrarily close to 0. This implies that $\lim_{t \downarrow t_0} \frac{F_1}{\sqrt{\frac{G_2 G_3}{G_1}}}(t) = 1$.

Proof of Corollary 3.3. Conditions (3.2) follow from Proposition 3.2 and the fact that $\lim_{t \downarrow t_0} F_i(t) = 0$, $i = 1, 2, 3$.

7.2 Strategy for proving identification

Theorems 3.4 and 3.5 follow from the proofs in sections 7.3 and 7.4.

As mentioned in section 3.5, my strategy for proving identification consists of two logical steps: first establishing local identification, then global identification.

It can be shown that (DE)-(IC) always has a negative local solution as well as a positive local

solution.[7] Conditions for uniqueness in the theory of differential equations do not let us control the sign of solutions. Therefore, even though I am interested only in a positive solution and can neglect a negative one, sufficient conditions that guarantee uniqueness of a positive local solution cannot be derived from system (DE). To tackle this problem, I use auxiliary tools.

Auxiliary tools

I transform (DE) into a new system by introducing auxiliary functions H_1, H_2, H_3:

$$H_1 = F_2 F_3, \quad H_2 = F_1 F_3, \quad H_3 = F_1 F_2.$$

Clearly, these functions are the distribution functions of $\max\{X_2, X_3\}$, $\max\{X_1, X_3\}$ and $\max\{X_1, X_2\}$, respectively. Functions F_i are expressed through H_i as $F_1^2 = \frac{H_2 H_3}{H_1}$, $F_2^2 = \frac{H_1 H_3}{H_2}$, $F_3^2 = \frac{H_1 H_2}{H_3}$. Taking into account that F_i must be positive, I obtain

$$F_1 = \sqrt{\frac{H_2 H_3}{H_1}}, \quad F_2 = \sqrt{\frac{H_1 H_3}{H_2}}, \quad F_3 = \sqrt{\frac{H_1 H_2}{H_3}}. \tag{7.1}$$

Thus, for any point $t > t_0$, system (DE) can be written as

$$H_1' = \frac{g_1}{1 - \sqrt{\frac{H_2 H_3}{H_1}}}$$

$$H_2' = \frac{g_2}{1 - \sqrt{\frac{H_1 H_3}{H_2}}}$$

$$H_3' = \frac{g_3}{1 - \sqrt{\frac{H_1 H_2}{H_3}}}.$$

Note that initial conditions $H_i(t_0) = 0$ cannot be imposed because the right-hand sides of the equations in this system are undefined when H_i takes value 0. Instead, I can set conditions on the upper limit of H_i at t_0:

$$\lim_{t \downarrow t_0} H_i(t) = 0, \quad i = 1, 2, 3. \tag{IC_H}$$

The right-hand side of the last system is a vector-valued function that depends on t, H_1, H_2 and H_3. Denote it as $J(t, H)$:

$$J(t, H) = \left(\frac{g_1(t)}{1 - \sqrt{\frac{H_2 H_3}{H_1}}}, \frac{g_2(t)}{1 - \sqrt{\frac{H_1 H_3}{H_2}}}, \frac{g_3(t)}{1 - \sqrt{\frac{H_1 H_2}{H_3}}} \right)^{tr}, \tag{7.2}$$

and rewrite the last system as

$$H'(t) = J(t, H(t)). \tag{DE_H}$$

I will refer to (DE_H) as an auxiliary system and to problem (DE_H)-(IC_H) as an auxiliary problem.

Definition 7.1. *Function $H = (H_1, H_2, H_3)^{tr}$ is a solution to (DE_H)-(IC_H) on an interval $(t_0, t_0 + a]$ if H_i are absolutely continuous on $(t_0, t_0 + a]$, satisfy (DE_H) a.e. on $(t_0, t_0 + a]$ and also satisfy (IC_H).*

---

[7]See remark 7.4 for further explanation.

Proof roadmap

Because formulas (7.1) account for the sign of F_i, we automatically consider positive solutions to (DE)-(IC). Thereafter by a solution to (DE)-(IC) I will always mean a positive solution.

The local identification result is proved in steps. In the first step, I show that conditions (I) and (II) are sufficient to guarantee that problem (DE_H)-(IC_H), which is the auxiliary problem, has a local solution. In the second step, I use formulas (7.1) to find F_i from H_i and show that these F_i constitute a local solution to the main problem. Lastly, for the auxiliary problem, I establish that its local solution that was found in the first step is unique. This implies that for the main problem, its local solution that was found in the second step is the unique solution.

The global identification result is obtained from the local identification result by showing how the unique local solution to (DE)-(IC) can be extended to the unique solution on the whole support. The idea is to extend this local solution to small intervals progressively farther to the right until the upper support point T is reached.

7.3 Local identification

Proving local identification is the most difficult part of the identification proof. I show that to establish the existence of a local solution, I only need conditions (I) and (II). To obtain local uniqueness, I use condition (III) as well as (I) and (II).

7.3.1 Existence of a local solution

I start by finding an interval on which a local solution to the auxiliary problem (DE_H)-(IC_H) and a local solution to the main problem (DE)-(IC) exist. Then I prove local existence for (DE_H)-(IC_H) and use this result to establish local existence for (DE)-(IC).

Before moving on, I must introduce some notations and carry out preliminary technical work. First of all, I have to indicate the domain of function $J(t, H)$. Take into account formulas (7.1), which express F through H, and note that for the auxiliary problem, we want to prove not only that there is a local solution but also that this solution is such that functions $\frac{H_2 H_3}{H_1}$, $\frac{H_1 H_3}{H_2}$, $\frac{H_1 H_2}{H_3}$ take values less than 1 and the following conditions hold:

$$\lim_{t \downarrow t_0} \frac{H_2 H_3}{H_1}(t) = 0, \quad \lim_{t \downarrow t_0} \frac{H_1 H_3}{H_2}(t) = 0, \quad \lim_{t \downarrow t_0} \frac{H_1 H_2}{H_3}(t) = 0.$$

This accords with the fact that for function $J(t, H)$ to be well defined, the denominators in $J(t, H)$ must be separated from 0. To do that, choose any $\delta \in (0, 1)$ and allow H to take values only in the following sets:

$$\bar{H}_0(\delta) = (0, \infty)^3 \cap \{(h_1, h_2, h_3)^{tr} : h_2 h_3 \leq \delta h_1, h_1 h_3 \leq \delta h_2, h_2 h_3 \leq \delta h_1\}.$$

Let $\bar{D}_0(\delta) = [t_0, T] \times \bar{H}_0(\delta)$ be the domain of $J(t, H)$ (a.e. with respect to t). As we can see, δ guarantees that the denominators in $J(t, H)$ are separated from 0 by the value $1 - \sqrt{\delta}$.

To determine an interval of existence for a local solution, I use conditions (II). Choose $\gamma > 0$

such that $\gamma/(1 - \sqrt{\delta})^2 \leq \delta$. Let $t_0 + a$, $a > 0$, be a point from $[t_0, T]$ such that

$$\forall (t \in [t_0, t_0 + a]) \quad \frac{G_2 G_3}{G_1}(t) \leq \gamma, \quad \frac{G_1 G_3}{G_2}(t) \leq \gamma, \quad \frac{G_1 G_2}{G_3}(t) \leq \gamma. \qquad (7.3)$$

Conditions (II) guarantee that such $t_0 + a$ exists. Interval $[t_0, t_0 + a]$ is an interval on which a solution to problem (DE_H)-(IC_H) exists.

Auxiliary system with ϵ

The right-hand side $J(t, H)$ of the auxiliary system (DE_H) has singularities in H when $H_1 = 0$ or $H_2 = 0$ or $H_3 = 0$. These singularities can be handled by using a very small $\epsilon > 0$ and considering an auxiliary system with $\epsilon > 0$:

$$H_1' = \frac{g_1}{1 - \sqrt{\frac{H_2 H_3}{H_1 + \epsilon}}}$$

$$H_2' = \frac{g_2}{1 - \sqrt{\frac{H_1 H_3}{H_2 + \epsilon}}}$$

$$H_3' = \frac{g_3}{1 - \sqrt{\frac{H_1 H_2}{H_3 + \epsilon}}},$$

together with initial conditions

$$H_i(t_0) = 0, \quad i = 1, 2, 3. \qquad (IC_{H,\epsilon})$$

Denote

$$J^\epsilon(t, H) = \left(\frac{g_1(t)}{1 - \sqrt{\frac{H_2 H_3}{H_1 + \epsilon}}}, \frac{g_2(t)}{1 - \sqrt{\frac{H_1 H_3}{H_2 + \epsilon}}}, \frac{g_3(t)}{1 - \sqrt{\frac{H_1 H_2}{H_3 + \epsilon}}} \right)^{tr}$$

and rewrite the system with ϵ as

$$H'(t) = J^\epsilon(t, H(t)). \qquad (DE_{H,\epsilon})$$

The definition of a solution to $(DE_{H,\epsilon})$-$(IC_{H,\epsilon})$ is analogous to Definition 7.1 and defines a solution on $[t_0, t_0 + a]$ instead of $(t_0, t_0 + a]$.

Introduce

$$\bar{H}(\delta) = [0, \infty)^3 \cap \{(h_1, h_2, h_3)^{tr} : h_2 h_3 \leq \delta h_1, h_1 h_3 \leq \delta h_2, h_2 h_3 \leq \delta h_1\}$$

and let $\bar{D}(\delta) = [t_0, T] \times \bar{H}(\delta)$ be the domain of $J^\epsilon(t, H)$ (a.e. with respect to t). The difference between $\bar{H}(\delta)$ and $\bar{H}_0(\delta)$ is that $\bar{H}(\delta)$ allows H_i to take value 0.

Lemma 7.1. *Let observable functions G_i satisfy conditions (I) and (II). Let $J^\epsilon(t, H)$ be defined on $\bar{D}(\delta)$. Then $(DE_{H,\epsilon})$-$(IC_{H,\epsilon})$ has a solution on $[t_0, t_0 + a]$.*

Proof. To prove this result, I use a Tonelli approximation approach, which builds special approximations of a solution on very small intervals. These approximations have an important property − when the lengths of the intervals go to zero, the sequence of approximations has a subsequence converging to a solution to $(DE_{H,\epsilon})$-$(IC_{H,\epsilon})$.

Tonelli approximations are constructed step by step according to a specified rule. Consider, for example, intervals $[t_0, t_0 + \frac{1}{k}]$, $[t_0 + \frac{1}{k}, t_0 + \frac{2}{k}]$, ..., $[t_0 + \frac{r}{k}, t_0 + a]$, where $a \leq \frac{r+1}{k}$, and k is very large. For these intervals an approximation is built in the following way. First, an approximation is found on $[t_0, t_0 + \frac{1}{k}]$, then it is extended to interval $(t_0 + \frac{1}{k}, t_0 + \frac{2}{k}]$. Next, the approximation is extended to $(t_0 + \frac{2}{k}, t_0 + \frac{3}{k}]$ and so on. This process is continued until the approximation is constructed on the whole interval $[t_0, t_0 + a]$.

A special feature of the Tonelli approach is that the extension of the approximation to $(t_0 + \frac{i}{k}, t_0 + \frac{i+1}{k}]$ is completely determined by the values of the approximating function on $[t_0 + \frac{i-1}{k}, t_0 + \frac{i}{k}]$ and therefore does not require any knowledge about the approximation on $[t_0, t_0 + \frac{i-1}{k})$.

Now I turn to describing the rule of constructing approximations. The integration of both sides in $(DE_{H,\epsilon})$ yields $H(t) = \int_{t_0}^t J^\epsilon(s, H)ds$. For a given k, denote a corresponding Tonelli approximation as $H^k = (H_1^k, H_2^k, H_3^k)$. Function H^k is defined according to the following rule:

$$H^k(t) = \int_{t_0}^t J^\epsilon \left(s, H^k \left(s - \frac{1}{k} \right) \right) ds, \quad t \in [t_0, t_0 + a]. \tag{7.4}$$

Choose a k that is large enough. To carry out the first step of constructing an approximation on $[t_0, t_0 + \frac{1}{k}]$, let

$$H_i^k(t) = 0, \quad t \in [t_0 - 1, t_0], \quad i = 1, 2, 3.$$

Let me show that formula (7.4) is meaningful. In the first step, it defines $H^k(t)$ for $t \in [t_0, t_0 + \min\{\frac{1}{k}, a\}]$. Because $J^\epsilon(s, H^k(s - \frac{1}{k})) = (g_1(s), g_2(s), g_3(s))^{tr}$ for any $s \in [t_0, t_0 + \min\{\frac{1}{k}, a\}]$ and $g_i \in L^1[t_0, t_0 + a]$, then the integral on the right-hand side exists. For the next step to be well defined, I have to check that for $t \in [t_0, t_0 + \min\{\frac{1}{k}, a\}]$, the values of the constructed function $H^k(t) = (H_1^k, H_2^k, H_3^k)^{tr}$ belong to $\bar{H}(\delta)$. Indeed, $H_i^k(t) = G_i(t)$. Properties $\frac{H_2^k(t)H_3^k(t)}{H_1^k(t)} \leq \delta$, $\frac{H_1^k(t)H_3^k(t)}{H_2^k(t)} \leq \delta$ and $\frac{H_1^k(t)H_2^k(t)}{H_3^k(t)} \leq \delta$ follow from (7.3) and the fact that $\gamma < \delta$. Therefore, $H^k(t) \in \bar{H}(\delta)$.

In the second step, formula (7.4) defines H^k on $[t_0 + \frac{1}{k}, t_0 + \min\{\frac{2}{k}, a\}]$. For $t \in [t_0 + \frac{1}{k}, t_0 + \min\{\frac{2}{k}, a\}]$, the Lebesgue integral on the right-hand side exists because function $J^\epsilon \left(s, H^k \left(s - \frac{1}{k} \right) \right) ds$ is evidently measurable and bounded by a Lebesgue integrable function:

$$\left| J_i^\epsilon \left(s, H^k \left(s - \frac{1}{k} \right) \right) ds \right| \leq \frac{g_i(s)}{1 - \sqrt{\delta}} \in L^1[t_0, t_0 + a], \quad s \in [t_0, t_0 + \min\{\frac{2}{k}, a\}].$$

Clearly, $H_i^k(t) > 0$. Because $H_2^k(t) \leq \frac{G_2(t)}{1 - \sqrt{\delta}}$, $H_3^k(t) \leq \frac{G_3(t)}{1 - \sqrt{\delta}}$ and $H_1^k(t) \geq G_1(t)$, then

$$\frac{H_2^k(t)H_3^k(t)}{H_1^k(t)} \leq \frac{G_2(t)G_3(t)}{(1 - \sqrt{\delta})^2 G_1(t)} \leq \frac{\gamma}{(1 - \sqrt{\delta})^2} \leq \delta.$$

Likewise,

$$\frac{H_1^k(t)H_3^k(t)}{H_2^k(t)} \leq \frac{G_1(t)G_3(t)}{(1 - \sqrt{\delta})^2 G_2(t)} \leq \frac{\gamma}{(1 - \sqrt{\delta})^2} \leq \delta,$$

$$\frac{H_1^k(t)H_2^k(t)}{H_3^k(t)} \leq \frac{G_1(t)G_2(t)}{(1 - \sqrt{\delta})^2 G_3(t)} \leq \frac{\gamma}{(1 - \sqrt{\delta})^2} \leq \delta.$$

Therefore, $H^k(t) \in \bar{H}(\delta)$ for $t \in [t_0 + \frac{1}{k}, t_0 + \min\{\frac{2}{k}, a\}]$.

All subsequent steps are similar to the second step. By continuing to construct approximations in this manner, I can eventually define function H^k on the whole interval $[t_0, t_0 + a]$.

I take progressively smaller intervals and obtain a sequence of approximations $\{H^k\}$. Because for any k

$$\|H^k(t)\|_1 \leq \frac{G_1(t) + G_2(t) + G_3(t)}{1 - \sqrt{\delta}} \leq \frac{G_1(t_0 + a) + G_2(t_0 + a) + G_3(t_0 + a)}{1 - \sqrt{\delta}}, \qquad (7.5)$$

functions H^k in this sequence are uniformly bounded. Moreover, sequence $\{H^k\}$ is equicontinuous, a property that is implied by inequality (7.6) and the absolute continuity of G_i on $[t_0, t_0 + a]$:

$$\|H^k(t) - H^k(\tau)\|_1 \leq \frac{\|G(t) - G(\tau)\|_1}{1 - \sqrt{\delta}}, \quad t, \tau \in [t_0, t_0 + a]. \qquad (7.6)$$

According to the Arzela-Ascoli theorem, sequence $\{H^k\}$ is relatively compact in $C([t_0, t_0 + a], \bar{H})$, so it contains a subsequence $\{H^{k_m}\}$ such that for some function H^ϵ

$$\sup_{t \in [t_0, t_0 + a]} \|H^\epsilon(t) - H^{k_m}(t)\|_1 \to 0$$

as $m \to \infty$. Because

$$J^\epsilon\left(t, H^{k_m}\left(t - \frac{1}{k_m}\right)\right) \to J^\epsilon(t, H^\epsilon(t)) \quad a.e. \quad [t_0, t_0 + a]$$

as $m \to \infty$, and a.e. on $[t_0, t_0 + a]$

$$\left\| J^\epsilon\left(t, H^{k_m}\left(t - \frac{1}{k_m}\right)\right) \right\|_1 \leq \frac{g_1(t) + g_2(t) + g_3(t)}{1 - \sqrt{\delta}} \in L^1[t_0, t_0 + a],$$

then according to the Lebesgue dominated convergence theorem, H solves

$$H^\epsilon(t) = \int_{t_0}^t J^\epsilon(s, H^\epsilon(s)) ds, \quad t \in [t_0, t_0 + a].$$

The last equation implies that H^ϵ is absolutely continuous and solves $(DE_{H,\epsilon})$-$(IC_{H,\epsilon})$ a.e. on $[t_0, t_0 + a]$. $\qquad \square$

Local existence for the auxiliary problem

The next proposition formulates the local existence result for the auxiliary problem.

Proposition 7.2. *Let observable functions G_i satisfy conditions (I) and (II). Let $J(t, H)$ be defined on $\bar{D}_0(\delta)$. Then (DE_H)-(IC_H) has a solution on $(t_0, t_0 + a]$.*

Proof. Choose a sequence ϵ_m such that $\epsilon_m \to 0$ as $m \to \infty$. For every ϵ_m, denote a solution constructed under Proposition 7.1 for this ϵ_m as H^{ϵ_m}. As I proved, for every ϵ_m, function H^{ϵ_m} is absolutely continuous on $[t_0, t_0 + a]$ and $H_i^{\epsilon_m}(t) > 0$, $t \in (t_0, t_0 + a]$.

Notice that the bounds in (7.5) and (7.6) do not depend on the value of ϵ, therefore,

$$\|H^{\epsilon m}(t)\|_1 \leq \frac{\|G(t_0 + a)\|_1}{1 - \sqrt{\delta}}, \quad t \in [t_0, t_0 + a],$$

and

$$\|H^{\epsilon m}(t) - H^{\epsilon m}(\tau)\|_1 \leq \frac{\|G(t) - G(\tau)\|_1}{1 - \sqrt{\delta}}, \quad t, \tau \in [t_0, t_0 + a].$$

The last two inequalities and the Arzela-Ascoli theorem imply that sequence $\{H^{\epsilon m}\}$ is relatively compact in $C([t_0, t_0 + a], \bar{H})$. Hence, it has a subsequence $H^{\epsilon m_l}$ such that for some function H

$$\sup_{t \in [t_0, t_0 + a]} \|H(t) - H^{\epsilon m_l}(t)\|_1 \to 0$$

as $l \to \infty$. Because

$$J^\epsilon\left(t, H^{\epsilon m_l}(t)\right) \to J(t, H(t)) \quad a.e. \quad [t_0, t_0 + a]$$

as $l \to \infty$, and a.e. on $[t_0, t_0 + a]$

$$\left\|J^{\epsilon m_l}\left(t, H^{\epsilon m_l}(t)\right)\right\|_1 \leq \frac{g_1(t) + g_2(t) + g_3(t)}{1 - \sqrt{\delta}} \in L^1[t_0, t_0 + a],$$

the Lebesgue dominated convergence theorem yields

$$H(t) = \int_{t_0}^t J(s, H(s)) ds, \quad t \in [t_0, t_0 + a].$$

From the last equation, it can be concluded that H_i are absolutely continuous on $[t_0, t_0 + a]$ and constitute a solution to (DE_H)-(IC_H) on $(t_0, t_0 + a]$. $\qquad\square$

It is remarkable that this existence result does not require any assumptions on observable G_i besides necessary conditions, which are satisfied in the model.

The proof of this proposition implies that if we take a solution H to (DE_H)-(IC_H) on $(t_0, t_0+a]$ and define the function for t_0 as $H(t_0) = (0,0,0)^{tr}$, then this extended function is absolutely continuous on $[t_0, t_0 + a]$ and clearly satisfies (DE_H)-(IC_H) a.e. on $[t_0, t_0 + a]$. In other words, a solution H can be extended from $(t_0, t_0 + a]$ to $[t_0, t_0 + a]$.

The following explanation shows why I cannot use standard existence theorems to prove Proposition 7.2. A general form of a system of differential equations is

$$x'(t) = v(t, x(t)),$$

where x and v are vector-valued functions. Let the initial condition be

$$x(t_0) = x_0.$$

In our problem, x is function H, and $v(t, x)$ is $J(t, H)$.[8] Existence theorems are usually proved

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[8] Even though initial conditions (IC_H) characterize the limit at t_0 rather than the value at t_0, this does not matter because, as I mentioned above, solution H can be extended from $(t_0, t_0 + a]$ to $[t_0, t_0 + a]$.

for the situation in which the domain of v is $[t_0 - h, t_0 + h] \times B(x_0)$ or $[t_0, t_0 + h] \times B(x_0)$, where $B(x_0)$ is an open ball with the center in x_0.[9] This property implies, for example, that x_0 is an interior point in the domain of v with respect to x. Existence theorems are also proved for some more general cases, but all require, at the very least, x_0 to be an interior point in the domain of v with respect to x, and this domain must satisfy certain properties. Because of the specificity of sets $\bar{H}_0(\delta)$ and $\bar{H}(\delta)$ and the fact that the point of the initial conditions $(0, 0, 0)^{tr}$ is on the border of these sets, I cannot apply any of those results. The method of Tonelli approximation allows me to take into account the specificity of $\bar{H}_0(\delta)$ and $\bar{H}(\delta)$ by verifying at each step that the values of the constructed Tonelli function belong to the domain $\bar{H}(\delta)$.

Local existence for the main problem

Now that I have established the local existence result for the auxiliary problem (DE_H)-(IC_H), I can turn to proving that the main problem (DE)-(IC) has a local solution. This result is easy to obtain if we recall how H and F are related in formulas (7.1).

Theorem 7.3. *Let observable functions G_i satisfy conditions (I) and (II). Then (DE)-(IC) has a solution on $[t_0, t_0 + a]$.*

Proof. Let H be a solution to (DE_H)-(IC_H) on $(t_0, t_0 + a]$. For $t > t_0$, define F_i according to formulas (7.1), and let $F_i(t_0) = 0$, $i = 1, 2, 3$. It follows from (DE_H) that the ratios $\frac{H_i}{G_i}$ have finite positive limits when $t \downarrow t_0$. Therefore, functions F_i are continuous at t_0 because for some constant $C \neq 0$,

$$\lim_{t \downarrow t_0} F_1(t) = \lim_{t \downarrow t_0} \sqrt{\frac{H_2 H_3}{H_1}}(t) = C \lim_{t \downarrow t_0} \sqrt{\frac{G_2 G_3}{G_1}}(t) = 0$$

and, similarly, $\lim_{t \downarrow t_0} F_2(t) = \lim_{t \downarrow t_0} F_3(t) = 0$. Because functions F_i are absolutely continuous on $[t_0 + \Delta, t_0 + a]$ for any $\Delta \in (0, a)$, and continuous at point t_0, they are absolutely continuous on $[t_0, t_0 + a]$. It is evident that F_i solve equations (DE) a.e. on $[t_0, t_0 + a]$. $\qquad \square$

Observe that because $J(t, H)$ is defined on $\bar{D}_0(\delta)$ and therefore a solution H to (DE_H)-(IC_H) takes values only in $\bar{H}_0(\delta)$, the values of the corresponding functions F_i belong to $[0, \sqrt{\delta}]$ only. The goal, however, is to identify F_i for all values in $[0, 1]$. This will be possible because δ can be arbitrarily close to 1.

Remark 7.4. *The last thing about the local existence that is worth mentioning concerns the comment made in section 7.2 about the existence of a negative function F that satisfies (DE) a.e. in a neighborhood of t_0 and also satisfies (IC). Note that functions F_i are expressed through H_i as $F_1^2 = \frac{H_2 H_3}{H_1}$, $F_2^2 = \frac{H_1 H_3}{H_2}$, $F_3^2 = \frac{H_1 H_2}{H_3}$, as follows from the definition of functions H_i. Taking into account that F_i are positive, I obtained (7.1) and substituted these formulas into (DE) to obtain the auxiliary system (DE_H). However, if I were looking for negative solutions, I would substitute formulas*

$$F_1 = -\sqrt{\frac{H_2 H_3}{H_1}}, \quad F_2 = -\sqrt{\frac{H_1 H_3}{H_2}}, \quad F_3 = -\sqrt{\frac{H_1 H_2}{H_3}}$$

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[9] For systems with discontinuous right-hand sides, this result is illustrated in Filippov (1988).

into (DE) and obtain a different form of the auxiliary system:

$$H_1' = \frac{g_1}{1 + \sqrt{\frac{H_2 H_3}{H_1}}}$$

$$H_2' = \frac{g_2}{1 + \sqrt{\frac{H_1 H_3}{H_2}}} \qquad (7.7)$$

$$H_3' = \frac{g_3}{1 + \sqrt{\frac{H_1 H_2}{H_3}}}.$$

Using the techniques of this section, it can be shown that (7.7) with initial conditions (IC_H) has a local solution H. This implies there is a negative function F that solves (DE) a.e. in a neighborhood of t_0.

7.3.2   Uniqueness of a local solution

The next step in the proof of local identification is to show that (DE)-(IC) has only one local solution. Local existence was proved without imposing any assumptions on G_i besides necessary conditions (I) and (II). To establish local uniqueness, I will assume that condition (III) is also satisfied. In fact, condition (III) is the most important condition for proving uniqueness.

I start by stating the local uniqueness result. It relies mostly on conditions (3.1), which find the rate of convergence of F_i at t_0 in terms of observable functions G_i.

Theorem 7.5. *Let observable functions G_i satisfy conditions (I), (II) and (III). Then (DE)-(IC) has only one solution in a neighborhood of t_0.*

The idea of the proof of this theorem is to take two local solutions to problem (DE)-(IC) and show that they coincide on their common interval of existence.

Suppose that F and $\tilde{F}$ are two local solutions to (DE)-(IC) with a common interval of existence $[t_0, t_0 + c]$, $c > 0$. Let H_i and $\tilde{H}_i$ be corresponding to them auxiliary functions:

$$H_1 = F_2 F_3, \quad H_2 = F_1 F_3, \quad H_3 = F_1 F_2,$$

$$\tilde{H}_1 = \tilde{F}_2 \tilde{F}_3, \quad \tilde{H}_2 = \tilde{F}_1 \tilde{F}_3, \quad \tilde{H}_3 = \tilde{F}_1 \tilde{F}_2.$$

Clearly, if functions H and $\tilde{H}$ are identical, then F and $\tilde{F}$ coincide.

The lemma below is key to proving that functions H and $\tilde{H}$ are identical.

Lemma 7.6. *Functions H and $\tilde{H}$ satisfy the following inequality a.e. on $[t_0, t_0 + c]$:*

$$\|H'(t) - \tilde{H}'(t)\|_1 \leq \Gamma_0(t)\|H(t) - \tilde{H}(t)\|_1, \qquad (7.8)$$

where

$$\Gamma_0(t) = C \left(\frac{g_1}{G_1}(t) + \frac{g_2}{G_2}(t) + \frac{g_3}{G_3}(t) \right) \left(\sqrt{\frac{G_2 G_3}{G_1}(t)} + \sqrt{\frac{G_1 G_3}{G_2}(t)} + \sqrt{\frac{G_1 G_2}{G_3}(t)} \right)$$

and $C > 0$ is some constant.

Proof. From (DE_H) obtain

$$H_i' - \tilde{H}_i' = \frac{g_i(F_i - \tilde{F}_i)}{(1 - F_i)(1 - \tilde{F}_i)}, \quad i = 1, 2, 3. \tag{7.9}$$

From equalities

$$H_1 - \tilde{H}_1 = F_2(F_3 - \tilde{F}_3) + \tilde{F}_3(F_2 - \tilde{F}_2)$$
$$H_2 - \tilde{H}_2 = F_1(F_3 - \tilde{F}_3) + \tilde{F}_3(F_1 - \tilde{F}_1)$$
$$H_3 - \tilde{H}_3 = F_1(F_2 - \tilde{F}_2) + \tilde{F}_2(F_1 - \tilde{F}_1),$$

find that on $(t_0, t_0 + c]$

$$F_1 - \tilde{F}_1 = -\frac{F_1}{\tilde{F}_3(F_2 + \tilde{F}_2)}(H_1 - \tilde{H}_1) + \frac{F_2}{\tilde{F}_3(F_2 + \tilde{F}_2)}(H_2 - \tilde{H}_2) + \frac{1}{F_2 + \tilde{F}_2}(H_3 - \tilde{H}_3)$$

$$F_2 - \tilde{F}_2 = \frac{\tilde{F}_2}{\tilde{F}_3(F_2 + \tilde{F}_2)}(H_1 - \tilde{H}_1) - \frac{F_2\tilde{F}_2}{F_1\tilde{F}_3(F_2 + \tilde{F}_2)}(H_2 - \tilde{H}_2) + \frac{F_2}{F_1(F_2 + \tilde{F}_2)}(H_3 - \tilde{H}_3) \tag{7.10}$$

$$F_3 - \tilde{F}_3 = \frac{1}{F_2 + \tilde{F}_2}(H_1 - \tilde{H}_1) + \frac{\tilde{F}_2}{(F_2 + \tilde{F}_2)F_1}(H_2 - \tilde{H}_2) - \frac{\tilde{F}_3}{(F_2 + \tilde{F}_2)F_1}(H_3 - \tilde{H}_3).$$

According to (3.1), there exist constants $C_1 > 0$, $C_2 > 0$ such that on $(t_0, t_0 + c]$

$$C_1 \le \frac{F_1}{\sqrt{\frac{G_2 G_3}{G_1}}} \le C_2, \quad C_1 \le \frac{F_2}{\sqrt{\frac{G_1 G_3}{G_2}}} \le C_2, \quad C_1 \le \frac{F_3}{\sqrt{\frac{G_1 G_2}{G_3}}} \le C_2$$

$$C_1 \le \frac{\tilde{F}_1}{\sqrt{\frac{G_2 G_3}{G_1}}} \le C_2, \quad C_1 \le \frac{\tilde{F}_2}{\sqrt{\frac{G_1 G_3}{G_2}}} \le C_2, \quad C_1 \le \frac{\tilde{F}_3}{\sqrt{\frac{G_1 G_2}{G_3}}} \le C_2$$

$(t_0 + c$ can be taken close enough to t_0). Then on $(t_0, t_0 + c]$,

$$|F_1 - \tilde{F}_1| \le K\frac{1}{G_1}\sqrt{\frac{G_2 G_3}{G_1}}|H_1 - \tilde{H}_1| + K\sqrt{\frac{G_3}{G_1 G_2}}|H_2 - \tilde{H}_2| + K\sqrt{\frac{G_2}{G_1 G_3}}|H_3 - \tilde{H}_3|$$

$$|F_2 - \tilde{F}_2| \le K\sqrt{\frac{G_3}{G_1 G_2}}|H_1 - \tilde{H}_1| + K\frac{1}{G_2}\sqrt{\frac{G_1 G_3}{G_2}}|H_2 - \tilde{H}_2| + K\sqrt{\frac{G_1}{G_2 G_3}}|H_3 - \tilde{H}_3| \tag{7.11}$$

$$|F_3 - \tilde{F}_3| \le K\sqrt{\frac{G_2}{G_1 G_3}}|H_1 - \tilde{H}_1| + K\sqrt{\frac{G_1}{G_2 G_3}}|H_2 - \tilde{H}_2| + K\frac{1}{G_3}\sqrt{\frac{G_1 G_2}{G_3}}|H_3 - \tilde{H}_3|,$$

where $K > 0$ is a constant expressed in terms of C_1 and C_2. Let $L > 0$ be a constant that bounds F_i and $\tilde{F}_i$ from above on $[t_0, t_0 + c]$. Denote $C = \frac{K}{(1-L)^2}$. Inequalities (7.11) and equations (7.9) imply that a.e. on $[t_0, t_0 + c]$

$$\|H' - \tilde{H}'\|_1 \le C\left(\frac{g_1}{G_1} + \frac{g_2}{G_2} + \frac{g_3}{G_3}\right)\left(\sqrt{\frac{G_2 G_3}{G_1}} + \sqrt{\frac{G_1 G_3}{G_2}} + \sqrt{\frac{G_1 G_2}{G_3}}\right)\|H - \tilde{H}\|_1.$$

$\square$

Establishing inequality (7.8) is the most challenging part of proving local uniqueness.

Notice that because H and $\tilde{H}$ solve the auxiliary problem (DE_H)-(IC_H), then a.e. on $(t_0, t_0+c]$

$$H'(t) = J(t, H(t)),$$
$$\tilde{H}'(t) = J(t, \tilde{H}(t)).$$

Therefore, inequality (7.8) can be rewritten as

$$\|J(t, H(t)) - J(t, \tilde{H}(t))\|_1 \le \Gamma_0(t)\|H(t) - \tilde{H}(t)\|_1.$$

This last inequality is a generalized local Lipschitz condition for function $J(t, H)$ with respect to variable H. It is worth emphasizing that this inequality holds only for the values of functions H and $\tilde{H}$ at the same point t but not for any two arbitrary values of variable H.

The following two lemmas prove that inequality (7.8) together with condition (III) yield that H and $\tilde{H}$ are identical functions and, therefore, prove Theorem 7.5.

Lemma 7.7. *Let* $z : [\tau, \xi] \to \Re^n$ *be an absolutely continuous function. Then* $\|z\|_1$ *has the right derivative* $D_R\|z\|_1$ *a.e. on* $[\tau, \xi]$, *and*

$$D_R\|z(t)\|_1 \le \|z'(t)\|_1 \quad a.e.\ on\ [\tau, \xi].$$

Proof. Hartman (1964) proves a similar lemma for smooth functions for the maxnorm and the euclidian norm. First, for any fixed i consider function $|z_i|$. Since z_i is absolutely continuous, $|z_i|$ is absolutely continuous too. $D_R|z_i(t)|$ then exists a.e. on $[\tau, \xi]$.

Let $t \in [\tau, \xi]$ be a point in which z_i has derivative. Use the definition of the right derivative:

$$D_R|z_i(t)| = \lim_{h \to +0} \frac{|z_i(t+h)| - |z_i(t)|}{h}$$

to conclude that $D_R|z_i(t)| = z_i'(t)$ if $z_i(t) > 0$ and $D_R|z_i(t)| = -z_i'(t)$ if $z_i(t) < 0$. Indeed, if $z_i(t) > 0$, then $z_i(t+h) > 0$ for small enough h, and $D_R|z_i(t)| = z_i'(t)$. In a similar way we consider the case $z_i(t) < 0$. If $z_i(t) = 0$, then

$$D_R|z_i(t)| = \lim_{h \to 0+} \frac{|z_i(t+h)|}{h} = \left|\lim_{h \to 0+} \frac{z_i(t+h)}{h}\right| = |z_i'(t)|.$$

In all three cases $D_R|z_i(t)| \le |z_i'(t)|$.

Function $\|z\|_1$ is the sum of absolutely continuous function and, hence, absolutely continuous. Then a.e. on $[\tau, \xi]$

$$D_R\|z(t)\|_1 = D_R(\sum_{i=1}^n |z_i(t)|) = \sum_{i=1}^n D_R|z_i(t)| \le \sum_{i=1}^n |z_i'(t)| = \|z'(t)\|_1.$$

$\square$

Lemma 7.8. *Let function* $v : [\tau, \xi] \to \Re$ *be absolutely continuous. Suppose that* $v(\tau) = 0$, *and a.e. on* $[\tau, \xi]$

$$D_R v(t) \le \Gamma(t)v(t), \quad where \quad \Gamma \in L^1[\tau, \xi].$$

Then

$$v(t) \leq 0, \quad t \in [\tau, \xi].$$

Proof. Results similar to the one in this lemma have been obtained by researchers on a more general level. However, it is easier to prove this lemma directly than to show how it follows from more general results.

Function $\phi(t) = v(t)e^{-\int_\tau^t \Gamma(s)ds}$ is absolutely continuous as the product of two absolutely continuous function and

$$D_R\phi(t) = D_R(v(t))e^{-\int_\tau^t \Gamma(s)ds} - \Gamma(t)v(t)e^{-\int_\tau^t \Gamma(s)ds} \leq 0 \quad a.e. \ [\tau, \xi].$$

Szarski (1965) uses Zygmund's lemma to show that if ϕ is absolutely continuous and $D_R\phi(t) \leq 0$ a.e. on $[\tau, \xi]$, then ϕ is non-increasing on $[\tau, \xi]$. Since $\phi(\tau) = 0$, then $\phi(t) \leq 0$ on $[\tau, \xi]$ and, hence, $v(t) \leq 0$ on $[\tau, \xi]$. $\square$

Let me explain in more detail how these two lemmas imply that functions H and $\tilde{H}$ coincide on $[t_0, t_0 + c]$. Consider $[\tau, \xi] = [t_0, t_0 + c]$. In the first lemma, take $z(t) = H(t) - \tilde{H}(t)$ and use inequality (7.8) to obtain

$$D_R\|H(t) - \tilde{H}(t)\|_1 \leq \Gamma_0(t)\|H(t) - \tilde{H}(t)\|_1.$$

In the second lemma, let $v(t) = \|H(t) - \tilde{H}(t)\|_1$ and $\Gamma(t) = \Gamma_0(t)$. Because condition (III) holds, then according to this lemma, $\|H(t) - \tilde{H}(t)\|_1 \leq 0$, $t \in [t_0, t_0 + c]$. This means that $\|H(t) - \tilde{H}(t)\|_1 = 0$, $t \in [t_0, t_0 + c]$, or, in other words, functions H and $\tilde{H}$ coincide on $[t_0, t_0 + c]$. In its turn, this implies that functions F and $\tilde{F}$ coincide on $[t_0, t_0 + c]$ too.

To summarize, I have shown that, given conditions (I), (II) and (III) on observable functions G_i, problem (DE)-(IC) has the unique solution F in a neighborhood of t_0. As mentioned in section 3, this solution is assumed to be monotone.

7.4   Global identification

Now I establish that the local solution to (DE)-(IC) can be extended to a solution on the entire interval $[t_0, T]$, and that such extension is unique.

Consider Figure 3 and the local solution F on $[t_0, t_0 + c]$ depicted on the left in this figure. Notice that all functions F_i take positive values at $t_0 + c$ and these values are known. Denote them as $v_i = F_i(t_0 + c), v_i > 0$. To extend the local solution to the right, I need to solve system (DE) in a right-hand side neighborhood of $t_0 + c$ given that functions F_i in a solution to this system take values v_i at $t_0 + c$. Clearly, results of theorems 7.3 and 7.5 cannot be used for this problem because the methods in these theorems were developed for the situation when all initial values of F_i are 0. Therefore, to carry out the extension process I first need to prove the local existence and uniqueness result for the case when when all the initial values of F_i are positive.

7.4.1   Positive initial values

Let $t_1 \in (t_0, T)$ and functions F_i satisfy initial conditions

$$F_i(t_1) = v_i, \quad i = 1, 2, 3, \tag{7.12}$$

where v_i are known, $0 < v_i < 1$. Notice that the values of $G_i(t_1)$ are known.

I first consider the auxiliary system (DE_H). The initial conditions on functions H_i are obviously

$$H_1(t_1) = v_2 v_3, \quad H_2(t_1) = v_1 v_3, \quad H_3(t_1) = v_1 v_2. \tag{7.13}$$

Proposition 7.9. *Let observable functions G_i satisfy conditions (I). Then (DE_H)-(7.13) has a solution in a right-hand neighborhood of t_1.*

Proof. The proof uses the Tonelli approximations approach. It is similar to the proof of Lemma 7.1 and differs from it by technical details.

Let me first specify the domain of the right-hand side $J(t, H)$ of the auxiliary system (DE_H) and find a solution's interval of existence. Let $\Delta > 0$ be any number such that $\Delta < \min\{1 - v_1, 1 - v_2, 1 - v_3\}$. Define set

$$\bar{H}(\Delta) = [0, \infty)^3 \cap \{(h_1, h_2, h_3)^{tr} : h_2 h_3 \leq (v_1 + \Delta)^2 h_1, h_1 h_3 \leq (v_2 + \Delta)^2 h_2, h_2 h_3 \leq (v_3 + \Delta)^2 h_1\}.$$

Let the domain of $J(t, H)$ be $\bar{D}(\Delta) = [t_1, T] \times \bar{H}$. For a given Δ, I can always choose a $\gamma > 0$ small enough so that

$$(1 + \gamma)^2 v_1^2 \leq (v_1 + \Delta)^2, \quad (1 + \gamma)^2 v_2^2 \leq (v_2 + \Delta)^2, \quad (1 + \gamma)^2 v_3^2 \leq (v_3 + \Delta)^2.$$

Because $\lim_{t \downarrow t_1} G_i(t) = G_i(t_1)$, there exists a point $t_1 + a_1$, $a_1 > 0$, from $[t_1, T]$ such that

$$G_1(t_1 + a_1) - G_1(t_1) \leq \gamma v_2 v_3 (1 - v_1 - \Delta)$$
$$G_2(t_1 + a_1) - G_2(t_1) \leq \gamma v_1 v_3 (1 - v_2 - \Delta)$$
$$G_3(t_1 + a_1) - G_3(t_1) \leq \gamma v_1 v_2 (1 - v_3 - \Delta).$$

Interval $[t_1, t_1 + a_1]$ is an interval on which a local solution exists.

Now I construct Tonelli approximations. For any natural number k let

$$H_1^k(t) = v_2 v_3, \quad H_2^k(t) = v_1 v_3, \quad H_3^k(t) = v_1 v_2$$

for $t \in [t_1 - 1, t_1]$. Denote $v_0 = (v_2 v_3, v_1 v_3, v_1 v_2)^{tr}$ and let v_0^i be the i's coordinate of v_0, $i = 1, 2, 3$. Define function

$$H^k(t) = v_0 + \int_{t_1}^{t} J\left(s, H^k\left(s - \frac{1}{k}\right)\right) ds, \quad t \in [t_1, t_1 + a_1]. \tag{7.14}$$

This formula is meaningful. In the first step it defines H on $[t_1, t_1 + \min\{\frac{1}{k}, a_1\}]$. For t from this interval the Lebesgue integral on the right-hand side exists because the integrand is bounded from

above by functions from $L^1[t_1, t_1 + a_1]$:

$$\left| J_i\left(s, H^k\left(s - \frac{1}{k}\right)\right) \right| \leq \frac{g_i(s)}{1 - v_i}, \quad s \in [t_1, t_1 + \min\{\tfrac{1}{k}, a_1\}].$$

Evidently, for $t \in [t_1, t_1 + \min\{\tfrac{1}{k}, a_1\}]$

$$H_1^k(t) = v_2 v_3 + \frac{G_1(t) - G_1(t_1)}{1 - v_1}$$
$$H_2^k(t) = v_1 v_3 + \frac{G_2(t) - G_2(t_1)}{1 - v_2}$$
$$H_3^k(t) = v_1 v_2 + \frac{G_3(t) - G_3(t_1)}{1 - v_3}.$$

Let me show that $H^k(t) \in \bar{H}$ for $t \in [t_1, t_1 + \min\{\tfrac{1}{k}, a_1\}]$. Consider, for instance, $\frac{H_2^k H_3^k}{H_1^k}$. Because

$$H_2^k(t) \leq v_1 v_3 + \frac{G_2(t_1 + a_1) - G_2(t_1)}{1 - v_2} \leq v_1 v_3 + \frac{\gamma v_1 v_3 (1 - v_2 - \Delta)}{1 - v_2} \leq (1 + \gamma) v_1 v_3,$$

$$H_3^k(t) \leq (1 + \gamma) v_1 v_2,$$

$$H_1^k(t) \geq v_2 v_3,$$

then

$$\frac{H_2^k(t) H_3^k(t)}{H_1^k(t)} \leq (1 + \gamma)^2 v_1^2 \leq (v_1 + \Delta)^2.$$

Likewise,

$$\frac{H_1^k(t) H_3^k(t)}{H_2^k(t)} \leq (v_2 + \Delta)^2, \quad \frac{H_1^k(t) H_2^k(t)}{H_3^k(t)} \leq (v_3 + \Delta)^2.$$

In the second step formula (7.14) defines H on $[t_1 + \tfrac{1}{k}, t_1 + \min\{\tfrac{2}{k}, a_1\}]$. For t from this interval the Lebesgue integral on the right-hand side exists because

$$\left| J_i\left(s, H^k\left(s - \frac{1}{k}\right)\right) \right| \leq \frac{g_i(s)}{1 - v_i - \Delta} \in L^1[t_1, t_1 + a_1], \quad s \in [t_1, t_1 + \min\{\tfrac{2}{k}, a_1\}].$$

Note that $H^k(t) \in \bar{H}$ for $t \in [t_1 + \tfrac{1}{k}, t_1 + \min\{\tfrac{2}{k}, a_1\}]$. Indeed,

$$H_2^k(t) \leq v_1 v_3 + \frac{G_2(t_1 + a_1) - G_2(t_1)}{1 - v_2 - \Delta} \leq v_1 v_3 + \gamma v_1 v_3 = (1 + \gamma) v_1 v_3,$$

$$H_3^k(t) \leq (1 + \gamma) v_1 v_2,$$

$$H_1^k(t) \geq v_2 v_3.$$

Therefore,

$$\frac{H_2^k(t) H_3^k(t)}{H_1^k(t)} \leq (1 + \gamma)^2 v_1^2 \leq (v_1 + \Delta)^2.$$

In a similar way I can show that for $t \in [t_1 + \frac{1}{k}, t_1 + \min\{\frac{2}{k}, a_1\}]$

$$\frac{H_1^k(t) H_3^k(t)}{H_2^k(t)} \leq (v_2 + \Delta)^2, \quad \frac{H_1^k(t) H_2^k(t)}{H_3^k(t)} \leq (v_3 + \Delta)^2.$$

This process continues and defines function H^k on the whole interval $[t_1, t_1 + a_1]$.

Now let me obtain the properties of sequence $\{H^k\}$. Inequality

$$\|H^k(t)\|_1 \leq (1 + \gamma)(v_2 v_3 + v_1 v_3 + v_1 v_2)$$

for all $t \in [t_1, t_1 + a_1]$ implies that sequence $\{H^k\}$ is uniformly bounded.

Because for any $t, \tau \in [t_1, t_1 + a_1]$

$$
\begin{aligned}
\|H^k(t) - H^k(\tau)\|_1 &\leq \frac{|G_1(t) - G_1(\tau)|}{1 - v_1 - \Delta} + \frac{|G_2(t) - G_2(\tau)|}{1 - v_2 - \Delta} + \frac{|G_3(t) - G_3(\tau)|}{1 - v_3 - \Delta} \\
&\leq \frac{\|G(t) - G(\tau)\|_1}{1 - \max\{v_1 + \Delta, v_2 + \Delta, v_3 + \Delta\}},
\end{aligned}
$$

and G_i are absolutely continuous on $[t_1, t_1 + a_1]$, then sequence $\{H^k\}$ is equicontinuous. According to the Arzela-Ascoli theorem, $\{H^k\}$ is relatively compact in $C([t_1, t_1 + a_1], \bar{H})$. Hence, it contains a subsequence H^{k_m} such that for some function H,

$$\sup_{[t_1, t_1 + a_1]} \|H(t) - H^{k_m}(t)\|_1 \to 0 \text{ as } m \to \infty.$$

Because

$$J\left(t, H^{k_m}\left(t - \frac{1}{k_m}\right)\right) \to J(t, H(t)) \quad \text{a.e. on } [t_1, t_1 + a_1]$$

and a.e. on $[t_1, t_1 + a_1]$

$$\left|J\left(t, H^{k_m}\left(t - \frac{1}{k_m}\right)\right)\right| \leq \frac{g_1(t) + g_2(t) + g_3(t)}{1 - \max\{v_1 + \Delta, v_2 + \Delta, v_3 + \Delta\}} \in L^1[t_1, t_1 + a],$$

then by the Lebesgue dominated convergence theorem, $H(t)$ solves

$$H(t) = v_0 + \int_{t_1}^{t} J(s, H(s)) ds, \quad t \in [t_1, t_1 + a_1],$$

which implies that H_i are absolutely continuous and solve (DE_H)-(7.13) on $[t_1, t_1 + a_1]$. $\qquad \square$

The existence result of Proposition 7.2 also required G_i to satisfy conditions (II). Note that because the values of the underlying distribution functions F_i at t_1 are separated from 0, then the result of Proposition 7.9 does not require any conditions on the behavior of G_i around t_1.

The next theorem establishes the local existence and uniqueness result for problem (DE)-(7.12). It is noteworthy that conditions $F_i(t_1) > 0$ guarantee uniqueness result without any additional conditions on functions G_i.

Theorem 7.10. *Let observable functions G_i satisfy conditions (I). Then (DE)-(7.12) has only one solution in a right-hand neighborhood of t_1.*

Proof. According to Proposition 7.9, problem (DE_H)-(7.13) has a solution H on $[t_1, t_1 + a_1]$, $a_1 > 0$. Use this solution to find functions

$$F_1 = \sqrt{\frac{H_2 H_3}{H_1}}, \quad F_2 = \sqrt{\frac{H_1 H_3}{H_2}}, \quad F_3 = \sqrt{\frac{H_1 H_2}{H_3}}.$$

Clearly, $F = (F_1, F_2, F_3)^{tr}$ is absolutely continuous and solves (DE)-(7.12) on $[t_1, t_1 + a_1]$.

The uniqueness proof is based on obtaining a generalized local Lipschitz condition (7.8). Let F and $\tilde{F}$ be two local solutions of (DE)- (7.12). Without a loss of generality, assume that $[t_1, t_1 + a_1]$ is their common interval of existence. Let H and $\tilde{H}$ be their corresponding auxiliary functions:

$$H_1 = F_2 F_3, \quad H_2 = F_1 F_3, \quad H_3 = F_1 F_2,$$

$$\tilde{H}_1 = \tilde{F}_2 \tilde{F}_3, \quad \tilde{H}_2 = \tilde{F}_1 \tilde{F}_3, \quad \tilde{H}_3 = \tilde{F}_1 \tilde{F}_2.$$

Functions H and $\tilde{H}$ solve the auxiliary system (DE_H) a.e. on $[t_1, t_1 + a_1]$.

The proof of the uniqueness part of this theorem is much easier than the proof for problem (DE)-(IC). Indeed, for (DE)-(IC), the difficulty of proving uniqueness for stemmed from the fact that all F_i had values 0 at t_0. Now all $F_i(t_1)$ are positive. Use (7.10) and the fact that F_i are separated from 0 in a neighborhood of t_1 (without a loss of generality, a_1 is small enough) to obtain

$$|F_i - \tilde{F}_i| \leq K \|H - \tilde{H}\|_1$$

on $[t_1, t_1 + a_1]$ for some constant K. Exploit (7.9) and establish that for some constant C,

$$\|H'(t) - \tilde{H}'(t)\|_1 \leq C(g_1(t) + g_2(t) + g_3(t)) \|H(t) - \tilde{H}(t)\|_1.$$

a.e. on $[t_1, t_1 + a_1]$. Because $g_i \in L^1[t_1, t_1 + a_1]$, then lemmas 7.7 and 7.8 imply that H and $\tilde{H}$ coincide on $[t_1, t_1 + a_1]$. Hence, F and $\tilde{F}$ coincide on this interval too. $\qquad \square$

7.4.2 Extension of the local solution to the whole support

Now I turn to the final element of the identification proof. I demonstrate how the unique local solution to (DE)-(IC) can be uniquely extended to a solution on the whole support. Throughout this section, I assume that functions F_i obtained from H_i are strictly monotone − that is, the ratios $\frac{H_2 H_3}{H_1}$, $\frac{H_1 H_3}{H_2}$, $\frac{H_1 H_2}{H_3}$ are strictly increasing.

To begin, recall that in the proof of the existence result in section 7.3.1, function $J(t, H)$ was defined on $\bar{D}_0(\delta)$ and the values of function H were restricted to set $\bar{H}(\delta)$ for a chosen $0 < \delta < 1$:

$$\bar{H}_0(\delta) = (0, \infty)^3 \cap \{(h_1, h_2, h_3)^{tr} : h_2 h_3 \leq \delta h_1, h_1 h_3 \leq \delta h_2, h_2 h_3 \leq \delta h_1\}.$$

Because the local solution to the auxiliary problem takes values only in this set, the functions F_i in the corresponding local solution to the main problem (DE_H)-(IC_H) take values in $[0, \sqrt{\delta}]$ only. However, we also want to identify F_i when these functions take values above $\sqrt{\delta}$. Notice that $\delta < 1$ could be chosen arbitrarily close to 1, and this is what will allow extending the local

solution to the whole support.

Fix δ, $0 < \delta < 1$, and let the domain of $J(t, H)$ be $\bar{D}_0(\delta) = [t_0, T] \times \bar{H}_0(\delta)$ (a.e. with respect to t). Theorem 7.5 proved that given conditions (I), (II) and (III), system (DE_H) with initial conditions (IC_H) has the unique solution $H = (H_1, H_2, H_3)$ on some interval $[t_0, t_0 + c]$. Denote $t_1 = t_0 + c$, and calculate

$$x_{i1} = H_i(t_1), \quad i = 1, 2, 3.$$

Because H_i are strictly increasing functions, then $x_{i1} > 0$. Note that $H(t_1) \in \bar{H}_0(\delta)$. If $H(t_1)$ is an interior point in $\bar{H}_0(\delta)$ − that is, if

$$\frac{x_{11} x_{21}}{x_{31}} < \delta, \quad \frac{x_{11} x_{31}}{x_{21}} < \delta, \quad \frac{x_{21} x_{31}}{x_{11}} < \delta -$$

then $(t_1, H(t_1))$ is an interior point of $\bar{D}_0(\delta)$, therefore $J(t, H)$ is defined in a neighborhood of this point. This means that the auxiliary system (DE_H), considered for $t \geq t_1$, with initial conditions

$$H_i(t_1) = x_{i1}, \quad i = 1, 2, 3,$$

is a well-defined problem. In light of the results of Proposition 7.9 and Theorem 7.10, this problem has a unique solution H on some interval $[t_1, t_1 + \mu]$, $\mu > 0$. Thus, I can uniquely extend the local solution found on $[t_0, t_1]$ to a solution on the interval $[t_0, t_1 + \mu]$. Note that the value of $H(t_1 + \mu)$ belongs to $\bar{H}_0(\delta)$. If this value is in the interior of set $\bar{H}_0(\delta)$, I can extend the solution even farther to the right and continue this process until I reach a point in which the value of function H becomes located on the border of set $\bar{H}_0(\delta)$. This point determines the solution's right maximal interval of existence for the given value of δ.

Definition 7.2. *An interval $[t_0, \xi]$ is the maximal interval of existence of solution H to (DE_H)-(IC_H) if there does not exist an extension of H over an interval $[t_0, \xi + \eta]$ such that $\eta > 0$ and H remains a solution to (DE_H)-(IC_H).*

In the case that I am currently considering, the solution's maximal interval of existence is determined by the value of δ that was chosen to define set $\bar{H}_0(\delta)$. The proposition below yields an explicit formula for this interval.

Proposition 7.11. *Let function $J(t, H)$ be defined on $\bar{D}_0(\delta)$. Assume that all conditions on G_i that guarantee existence and uniqueness of a local solution to (DE_H)-(IC_H) are satisfied. The maximal interval of existence of solution H to (DE_H)-(IC_H) is $[t_0, T_\delta]$, where T_δ is such that*

$$\max \left\{ \frac{H_2(T_\delta) H_3(T_\delta)}{H_1(T_\delta)}, \ \frac{H_1(T_\delta) H_3(T_\delta)}{H_2(T_\delta)}, \ \frac{H_1(T_\delta) H_2(T_\delta)}{H_3(T_\delta)} \right\} = \delta.$$

This proposition follows from the discussion above and therefore it is left without a proof.

Proposition 7.11 implies that for the given δ, $[t_0, T_\delta]$ is the maximal interval of existence of a corresponding solution F to problem (DE)-(IC). Also, the values of functions F_i on $[t_0, T_\delta]$ belong to $[0, \sqrt{\delta}]$, and for point T_δ,

$$\max \{F_1(T_\delta), \ F_2(T_\delta), \ F_3(T_\delta)\} = \sqrt{\delta}.$$

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$$F_1, F_2, F_3 \text{ on } [t_0, T_{\delta_1}] \qquad\qquad\qquad F_1, F_2, F_3 \text{ on } [t_0, T_{\delta_2}]$$
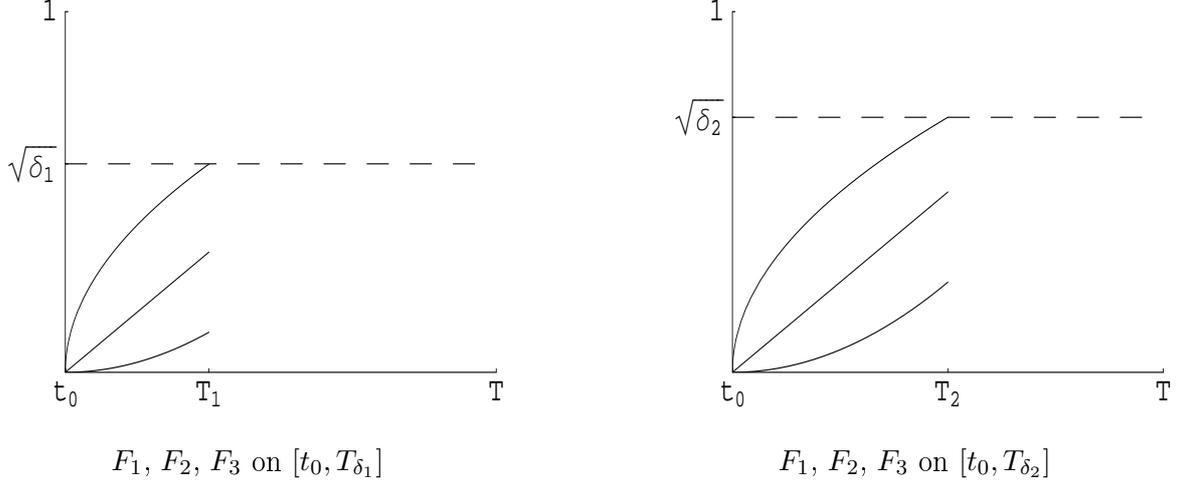
Figure 9. Maximal intervals of existence of a solution to the main problem: $[t_0, T_{\delta_1}]$ corresponds to δ_1 (left), $[t_0, T_{\delta_2}]$ corresponds to δ_2, where $\delta_2 > \delta_1$ (right).

Figure 9 depicts maximal intervals of existence of a solution F for values δ_1 and δ_2, where $\delta_2 > \delta_1$. Maximal interval $[t_0, T_{\delta_1}]$ corresponds to δ_1, and maximal interval $[t_0, T_{\delta_2}]$ corresponds to δ_2. Because functions F_i are strictly increasing, then $T_{\delta_2} > T_{\delta_1}$. Intuitively, if δ approaches 1, then the maximal interval of existence approaches support $[t_0, T]$. The theorem below establishes this fact.

Theorem 7.12. *Consider a strictly increasing sequence δ_n, $n \geq 1$, such that $\delta_n < 1$, and $\delta_n \to 1$ as $n \to \infty$. Assume that all conditions on G_i that guarantee the existence and uniqueness of a local solution to problem (DE$_H$)-(IC$_H$) are satisfied. Let $[t_0, T_{\delta_n}]$ be the maximal interval of existence for the solution to (DE$_H$)-(IC$_H$) when $J(t, H)$ is defined on $\bar{D}_0(\delta_n)$. Then T_{δ_n} is determined from the equation*

$$\max\left\{ \frac{H_2(T_{\delta_n}) H_3(T_{\delta_n})}{H_1(T_{\delta_n})}, \ \frac{H_1(T_{\delta_n}) H_3(T_{\delta_n})}{H_2(T_{\delta_n})}, \ \frac{H_1(T_{\delta_n}) H_2(T_{\delta_n})}{H_3(T_{\delta_n})} \right\} = \delta_n, \qquad (7.15)$$

and T_{δ_n} is a strictly increasing sequence. If

$$F_i(T) = 1, \quad i = 1, 2, 3, \qquad (7.16)$$

then $T_{\delta_n} \to T$ as $n \to \infty$.

Proof. Proposition 7.11 clearly implies equation (7.15). Because functions $\frac{H_2 H_3}{H_1}$, $\frac{H_1 H_3}{H_2}$, $\frac{H_1 H_2}{H_3}$ and sequence δ_n are strictly increasing, equation (7.15) implies that sequence T_{δ_n} is strictly increasing. Because T_{δ_n} increases and is bounded from above by T, it converges to some point $\bar{T} \leq T$. If $\bar{T} < T$, then we get a contradiction with the condition $\delta_n \to 1$ and conditions (7.16). Thus, $\bar{T} = T$. $\qquad\square$

Taking into account that $F_1^2 = \frac{H_2 H_3}{H_1}$, $F_2^2 = \frac{H_1 H_3}{H_2}$, $F_3^2 = \frac{H_1 H_2}{H_3}$, we can see that Theorem 7.12 guarantees that by choosing δ arbitrarily close to 1, we will identify F_i on the whole support $[t_0, T]$. This completes the proof of identification.

7.5   Auctions with any number of bidders

Proofs of propositions 3.6 and 3.7 and Corollary 3.8 are similar to those of propositions 3.1 and 3.2 and Corollary 3.3.

Proof of Theorem 3.9. I can use the same approach as in the case of three bidders. System (3.6) can be rewritten in a convenient form by introducing d auxiliary functions H_1, H_2, ..., H_d that stand for the distribution functions of $\max\{X_2, X_3, \ldots, X_d\}$, $\max\{X_1, X_3, \ldots, X_d\}$, ..., $\max\{X_1, X_2, \ldots, X_{d-1}\}$, respectively:

$$H_1 = F_2 F_3 \ldots F_d, \quad H_2 = F_1 F_3 \ldots F_d, \quad \ldots \quad , H_d = F_1 F_2 \ldots F_{d-1}.$$

For $t > t_0$ functions F_i can be expressed through H_i as

$$F_1 = \left(\frac{H_2 H_3 \ldots H_d}{H_1^{d-2}} \right)^{\frac{1}{d-1}}, \quad \ldots \quad , F_d = \left(\frac{H_1 H_2 \ldots H_{d-1}}{H_d^{d-2}} \right)^{\frac{1}{d-1}}, \tag{7.17}$$

therefore, (3.6) can be rewritten in the following way:

$$H_i' = \frac{g_i}{1 - \left(\frac{H_1 \ldots H_{i-1} H_{i+1} \ldots H_d}{H_i^{d-2}} \right)^{\frac{1}{d-1}}}, \quad i = 1, \ldots, d. \tag{7.18}$$

This system together with initial conditions

$$\lim_{t \downarrow t_0} H_i(t) = 0, \quad i = 1, \ldots, d. \tag{7.19}$$

constitutes an auxiliary problem. To deal with discontinuities in H on the right-hand side in (7.18), I introduce a very small number $\epsilon > 0$ and obtain an auxiliary system with ϵ:

$$H_i' = \frac{g_i}{1 - \left(\frac{H_1 \ldots H_{i-1} H_{i+1} \ldots H_d}{H_i^{d-2} + \epsilon} \right)^{\frac{1}{d-1}}}, \quad i = 1, \ldots, d.$$

As in the case of three bidders, first I can establish local existence for the auxiliary system with ϵ. Then I can show the existence of a local solution to the auxiliary problem (7.18)-(7.19) by letting $\epsilon \to 0$. After that, I can use formulas (7.17), which express F through H, to prove that the main problem (3.6)-(3.7) has a local solution.[10]

Proof of Theorem 3.10. The existence part of this theorem follows from Theorem 3.9. To prove the uniqueness part, let F and $\tilde{F}$ be two solutions to (3.6)-(3.7) with a common interval of existence $[t_0, t_0 + c]$, $c > 0$. Let

$$H_i = F_1 \ldots F_{i-1} F_{i+1} \ldots F_d, \quad \tilde{H}_i = \tilde{F}_1 \ldots \tilde{F}_{i-1} \tilde{F}_{i+1} \ldots \tilde{F}_d, \quad i = 1, \ldots, d.$$

The idea is to derive an inequality similar to (7.8). Use (7.17) and (7.18) to obtain that a.e. on

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[10]A detailed proof of Theorem 3.9 is available upon request.

$[t_0, t_0 + c]$

$$H_i' - \tilde{H}_i{}' = \frac{g_i(F_i - \tilde{F}_i)}{(1 - F_i)(1 - \tilde{F}_i)}. \tag{7.20}$$

The definitions of H and $\tilde{H}$ allow me to express $H - \tilde{H}$ through $F - \tilde{F}$ as follows:

$$H - \tilde{H} = B(F, \tilde{F})(F - \tilde{F}),$$

where a $d \times d$ matrix $B(F, \tilde{F})$ depends on F and $\tilde{F}$ in this way:

$$B(F, \tilde{F}) = \begin{pmatrix} 0 & F_3 F_4 \ldots F_d & \tilde{F}_2 F_4 \ldots F_d & \tilde{F}_2 \tilde{F}_3 F_5 \ldots F_d & \ldots & \tilde{F}_2 \tilde{F}_3 \ldots \tilde{F}_{d-1} \\ F_3 F_4 \ldots F_d & 0 & \tilde{F}_1 F_4 \ldots F_d & \tilde{F}_1 \tilde{F}_3 F_5 \ldots F_d & \ldots & \tilde{F}_1 \tilde{F}_3 \ldots \tilde{F}_{d-1} \\ \ldots & \ldots & \ldots & \ldots & & \\ F_2 F_3 \ldots F_{d-1} & \tilde{F}_1 F_3 \ldots F_{d-1} & \tilde{F}_1 \tilde{F}_2 F_4 \ldots F_d & \tilde{F}_1 \tilde{F}_2 \tilde{F}_3 \ldots F_d & \ldots & 0 \end{pmatrix}.$$

The result of Proposition 3.7 implies that $\lim_{t \downarrow t_0} \frac{F_i}{\tilde{F}_i}(t) = 1$. Therefore, for a t close enough to t_0 (without a loss of generality, I can assume that $t_0 + c$ is close enough to t_0), matrix $B(F, \tilde{F})$ can be written as

$$B(F, \tilde{F}) = (I + M_{o(1)}(F, \tilde{F})) B_0(F),$$

where I is the $d \times d$ identity matrix, $M_{o(1)}(F, \tilde{F})$ is a $d \times d$ matrix such that each of its elements is $o(1)$ as $t \to t_0$, and $B_0(F) = B(F, F)$:

$$B_0(F) = \begin{pmatrix} 0 & F_3 F_4 \ldots F_d & F_2 F_4 \ldots F_d & F_2 F_3 F_5 \ldots F_d & \ldots & F_2 F_3 \ldots F_{d-1} \\ F_3 F_4 \ldots F_d & 0 & F_1 F_4 \ldots F_d & F_1 F_3 F_5 \ldots F_d & \ldots & F_1 F_3 \ldots F_{d-1} \\ \ldots & \ldots & \ldots & \ldots & & \\ F_2 F_3 \ldots F_{d-1} & F_1 F_3 \ldots F_{d-1} & F_1 F_2 F_4 \ldots F_d & F_1 F_2 F_3 \ldots F_d & \ldots & 0 \end{pmatrix}.$$

Matrix $B_0(F)$ is symmetric and invertible at any point $t \neq t_0$. The inverse matrix is

$$B_0^{-1}(F) = \frac{1}{(d-1) F_1 F_2 \ldots F_d} \begin{pmatrix} -(d-2) F_1^2 & F_1 F_2 & F_1 F_3 & F_1 F_4 & \ldots & F_1 F_d \\ F_1 F_2 & -(d-2) F_2^2 & F_2 F_3 & F_2 F_4 & \ldots & F_2 F_d \\ \ldots & \ldots & \ldots & \ldots & & \\ F_1 F_d & F_2 F_d & F_3 F_d & F_4 F_d & \ldots & -(d-2) F_d^2 \end{pmatrix}.$$

Thus, $F - \tilde{F}$ can be expressed through $H - \tilde{H}$ as

$$F - \tilde{F} = B_0^{-1}(F)(I + M_{o(1)}(F, \tilde{F}))^{-1}(H - \tilde{H}). \tag{7.21}$$

The next step is to bound on $[t_0, t_0 + c]$ the absolute values of the elements in $B_0^{-1}(F)$ by observable functions. This is achieved by using the result of Proposition 3.7. Take, for instance, the element

$B_0^{-1}(F)_{11}$ in the first row and the first column:

$$|B_0^{-1}(F)_{11}| = \left| \frac{(d-2)F_1}{(d-1)F_2 \ldots F_d} \right| \leq K_{11} \frac{\left(\frac{G_2 \ldots G_d}{G_1^{d-2}} \right)^{\frac{1}{d-1}}}{\prod_{i=2}^{d} \left(\frac{G_1 \ldots G_{i-1} G_{i+1} \ldots G_d}{G_i^{d-2}} \right)^{\frac{1}{d-1}}}$$

for some constant K_{11}. Consider another cell in $B_0^{-1}(F)$, for example, the element $B_0^{-1}(F)_{12}$ in the first row and the second column:

$$|B_0^{-1}(F)_{12}| = \left| \frac{1}{(d-1)F_3 \ldots F_d} \right| \leq K_{12} \prod_{i=3}^{d} \left(\frac{G_1 \ldots G_{i-1} G_{i+1} \ldots G_d}{G_i^{d-2}} \right)^{-\frac{1}{d-1}}$$

for some constant K_{12}. For the other elements, bounds are found in a similar way. Then equations (7.20) and (7.21) yield that a.e. on $[t_0, t_0 + c]$

$$\|H' - \tilde{H}'\|_1 \leq C \sum_{i=1}^{d} \left(\frac{G_1 G_2 \ldots G_{i-1} G_{i+1} \ldots G_d}{G_i^{d-1}} \right)^{\frac{1}{d-2}} \cdot \sum_{i=1}^{d} \frac{g_i}{G_i} \|H - \tilde{H}\|_1$$

for some constant C. The last inequality and lemmas 7.7 and 7.8 imply that H and $\tilde{H}$ coincide on $[t_0, t_0 + c]$, and hence, F and $\tilde{F}$ coincide on $[t_0, t_0 + c]$.

7.6   Auctions with two types of bidders

Proof of Theorem 3.11.

Necessity is obvious. Sufficiency follows from theorem 3.12 by considering $\psi(x) = -\ln x$, $x \in (0, 1]$.

Proof of Theorem 3.12.

Let

$$F_I(t) = P(\text{value of type I bidder} \leq t), \quad F_{II}(t) = P(\text{value of type II bidder} \leq t)$$

and introduce functions

$$\Sigma_I = C(1, \underbrace{F_I(t), \ldots, F_I(t)}_{k-1}, \underbrace{F_{II}(t), \ldots, F_{II}(t)}_{d-k}) = \psi^{-1} \left((k-1)\psi(F_I(t)) + (d-k)\psi(F_{II}(t)) \right),$$

$$\Sigma_{II} = C(\underbrace{F_I(t), \ldots, F_I(t)}_{k}, \underbrace{F_{II}(t), \ldots, F_{II}(t)}_{d-k-1}, 1) = \psi^{-1} \left(k\psi(F_I(t)) + (d-k-1)\psi(F_{II}(t)) \right).$$

The system of differential equations that determines Σ_I and Σ_{II} is

$$\Sigma_I' = \frac{g_I}{1 - \frac{\psi'(\Sigma_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\Sigma_I) + \frac{d-k}{d-1}\psi(\Sigma_{II})))}} \tag{7.22}$$

$$\Sigma_{II}' = \frac{g_{II}}{1 - \frac{\psi'(\Sigma_{II})}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\Sigma_I) + \frac{d-k}{d-1}\psi(\Sigma_{II})))}}.$$

I analyze it together with initial conditions

$$\Sigma_I(t_0) = \Sigma_{II}(t_0) = 0. \tag{7.23}$$

It is enough to show that problem (7.22)-(7.23) can have only one solution in a neighborhood of t_0. Then the extension of this solution along the whole support will be unique too.

System (7.22) implies that for any point from the support

$$k\Sigma_I + (d-k)\Sigma_{II} - (d-1)\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_I) + \frac{d-k}{d-1}\psi(\Sigma_{II})\right) = kG_I + (d-k)G_{II}.$$

Suppose that problem (7.22)-(7.23) has two solutions (Σ_I, Σ_{II}) and $(\tilde{\Sigma}_I, \tilde{\Sigma}_{II})$ with a common interval of existence $[t_0, t_0 + a]$. I want to show that for any $t \in [t_0, t_0 + a]$, $\Sigma_I(t) \geq \tilde{\Sigma}_I(t)$ iff $\Sigma_{II}(t) \leq \tilde{\Sigma}_{II}(t)$. Fix $t \in (t_0, t_0 + a]$. From the equation

$$k\Sigma_I + (d-k)\Sigma_{II} - (d-1)\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_I) + \frac{d-k}{d-1}\psi(\Sigma_{II})\right) =$$
$$= k\tilde{\Sigma}_I + (d-k)\tilde{\Sigma}_{II} - (d-1)\psi^{-1}\left(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{II})\right)$$

obtain that

$$k\left(1 - \frac{\psi'(\Sigma_I^*)}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_I^*) + \frac{d-k}{d-1}\psi(\Sigma_{II})\right)\right)}\right)(\Sigma_I - \tilde{\Sigma}_I) =$$
$$= (d-k)\left(1 - \frac{\psi'(\Sigma_{II}^*)}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}\psi(\Sigma_{II}^*)\right)\right)}\right)(\tilde{\Sigma}_{II} - \Sigma_{II}), \tag{7.24}$$

where $\Sigma_I^* = \alpha\Sigma_I + (1-\alpha)\tilde{\Sigma}_I$ for some $\alpha = \alpha(\Sigma_I(t), \tilde{\Sigma}_I(t), \Sigma_{II}(t)) \in [0,1]$, and $\Sigma_{II}^* = \beta\Sigma_{II} + (1-\beta)\tilde{\Sigma}_{II}$ for some $\beta = \beta(\tilde{\Sigma}_I(t), \Sigma_{II}(t), \tilde{\Sigma}_{II}(t)) \in [0,1]$. Note that for $t < T$,

$$\frac{\psi'(\Sigma_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\Sigma_I) + \frac{d-k}{d-1}\psi(\Sigma_{II})))} < 1, \quad \frac{\psi'(\tilde{\Sigma}_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}.\psi(\tilde{\Sigma}_{II})))} < 1.$$

Because $\frac{\tilde{\Sigma}_I}{\Sigma_I} \to 1$, $\frac{\tilde{\Sigma}_{II}}{\Sigma_{II}} \to 1$ as $t \to t_0$, then for t close enough to t_0 (and we can choose a to be small enough),

$$\frac{\psi'(\Sigma_I^*)}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\Sigma_I^*) + \frac{d-k}{d-1}\psi(\Sigma_{II})\right)\right)} < 1, \quad \frac{\psi'(\Sigma_{II}^*)}{\psi'\left(\psi^{-1}\left(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}\psi(\Sigma_{II}^*)\right)\right)} < 1.$$

Therefore, from (7.24) I obtain that $\Sigma_I(t) \geq \tilde{\Sigma}_I(t)$ iff $\Sigma_{II}(t) \leq \tilde{\Sigma}_{II}(t)$. Now I want show that this and the fact that the function $\frac{\psi''(x)}{(\psi'(x))^2}$ is increasing imply that

$$(\Sigma_I' - \tilde{\Sigma}_I')(\Sigma_I - \tilde{\Sigma}_I) \leq 0, \quad (\Sigma_{II}' - \tilde{\Sigma}_{II}')(\Sigma_{II} - \tilde{\Sigma}_{II}) \leq 0 \quad a.e. \ [t_0, t_0 + a].$$

Suppose that for a given point $t \in (t_0, t_0 + a]$, at which the derivatives Σ_I' and $\tilde{\Sigma}_I'$ exist, it holds that $\Sigma_I \geq \tilde{\Sigma}_I$. Let us prove that $\Sigma_I' - \tilde{\Sigma}_I' \leq 0$. From (7.22) obtain that

$$\Sigma_I' - \tilde{\Sigma}_I' = \frac{g_I}{W_I} \left(\frac{\psi'(\Sigma_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\Sigma_I) + \frac{d-k}{d-1}\psi(\Sigma_{II})))} - \frac{\psi'(\tilde{\Sigma}_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{II})))} \right),$$

where

$$W_I = \left(1 - \frac{\psi'(\Sigma_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\Sigma_I) + \frac{d-k}{d-1}\psi(\Sigma_{II})))} \right) \left(1 - \frac{\psi'(\tilde{\Sigma}_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{II})))} \right).$$

Because $\Sigma_{II} \leq \tilde{\Sigma}_{II}$, then

$$\Sigma_I' - \tilde{\Sigma}_I' \leq \frac{g_I}{W_I} \left(\frac{\psi'(\Sigma_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\Sigma_I) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{II})))} - \frac{\psi'(\tilde{\Sigma}_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{II})))} \right).$$

Now we want to show that the difference in the parenthesis is non-positive.

Because $\frac{\psi'(\tilde{\Sigma}_I)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{II})))} < 1$, then $\tilde{\Sigma}_I > \psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{II}))$, and there-fore, $\Sigma_I > \psi^{-1}(\frac{k}{d-1}\psi(\tilde{\Sigma}_I) + \frac{d-k}{d-1}\psi(\tilde{\Sigma}_{II}))$.

Thus, if we show that the function

$$\frac{\psi'(y_1)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)))}$$

is decreasing in y_1 when y_1 and y_2 close to 0 and $y_1 > \psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2))$, then we establish that $\Sigma_I' - \tilde{\Sigma}_I' \leq 0$.

The derivative of this function with respect to y_1 is

$$\frac{\psi''(y_1)}{\psi'(\psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)))} - \frac{k(\psi'(y_1))^2\psi''(\psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)))}{(d-1)\left(\psi'(\psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)))\right)^3}.$$

For this derivative to be non-positive, it is sufficient that

$$\frac{\psi''(y_1)}{(\psi'(y_1))^2} \geq \frac{\psi''(\psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2)))}{(\psi'(\psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2))))^2}.$$

The last inequality holds because of the assumption that $\frac{\psi''(x)}{(\psi'(x))^2}$ is increasing and the condition $y_1 > \psi^{-1}(\frac{k}{d-1}\psi(y_1) + \frac{d-k}{d-1}\psi(y_2))$.

To summarize, we have established that

$$(\Sigma_1' - \tilde{\Sigma}_1', \Sigma_1 - \tilde{\Sigma}_1) \leq 0 \quad a.e. \ [t_0, t_0 + a],$$

that is,

$$\frac{d}{dt}(\Sigma_1 - \tilde{\Sigma}_1)^2 \leq 0 \quad a.e. \ [t_0, t_0 + a].$$

This inequality and (7.23) imply that Σ_1 and $\tilde{\Sigma}_1$ coincide in a neighborhood of t_0.

In a similar way, it can be shown that

$$\frac{d}{dt}(\Sigma_2 - \tilde{\Sigma}_2)^2 \leq 0 \quad a.e. \ [t_0, t_0 + a]$$

and, therefore, Σ_2 and $\tilde{\Sigma}_2$ coincide in a neighborhood of t_0.

8   Appendix B: Proofs of the results in section 4

Proof of Proposition 4.1. Let F, $\tilde{F} \in \Lambda$, and $G = A(F)$, $\tilde{G} = A(\tilde{F})$. For convenience, I temporarily use the following metric:

$$d_1(F, \tilde{F}) = \sup_{t \in [0,1]} \sum_{j=1}^{3} |F_j(t) - \tilde{F}_j(t)|$$

$$d_1(G, \tilde{G}) = \sup_{t \in [0,1]} \sum_{j=1}^{3} |G_j(t) - \tilde{G}_j(t)|.$$

From the definition of A,

$$G_1(t) - \tilde{G}_1(t) = \int_{t_0}^{t} (F_2 F_3)^{'}(1 - F_1)ds - \int_{t_0}^{t} (\tilde{F}_2 \tilde{F}_3)^{'}(1 - \tilde{F}_1)ds =$$

$$= F_2 F_3 - \tilde{F}_2 \tilde{F}_3 - \int_{t_0}^{t} (F_2 F_3)^{'}(F_1 - \tilde{F}_1)ds + \int_{t_0}^{t} \tilde{F}_1((\tilde{F}_2 \tilde{F}_3)^{'} - (F_2 F_3)^{'})ds.$$

Integration by parts yields

$$G_1(t) - \tilde{G}_1(t) = (F_2 F_3 - \tilde{F}_2 \tilde{F}_3)(1 - \tilde{F}_1) - \int_{t_0}^{t} (F_2 F_3)^{'}(F_1 - \tilde{F}_1)ds + \int_{t_0}^{t} \tilde{F}_1^{'}(F_2 F_3 - \tilde{F}_2 \tilde{F}_3)ds.$$

Knowing that $\tilde{F}_1$ and $F_2 F_3$ are distribution functions, obtain that for any $t \in [t_0, T]$,

$$|G_1(t) - \tilde{G}_1(t)| \leq 2 \sup_{[t_0, T]} |F_2 F_3 - \tilde{F}_2 \tilde{F}_3| + \sup_{[t_0, T]} |F_1 - \tilde{F}_1| \leq 3d_1(F, \tilde{F}).$$

After arriving at similar inequalities for $G_2 - \tilde{G}_2$ and $G_3 - \tilde{G}_3$,

$$\sum_{j=1}^{3} |G_j(t) - \tilde{G}_j(t)| \leq 9d_1(F, \tilde{F}), \quad t \in [t_0, T],$$

and, hence,

$$d_1(G, \tilde{G}) \leq 9d_1(F, \tilde{F}).$$

Because
$$d_1(F, \tilde{F}) \le \sqrt{3}d(F, \tilde{F}) \text{ and } d_1(G, \tilde{G}) \ge d(G, \tilde{G}), \tag{8.1}$$

then
$$d(G, \tilde{G}) \le 9\sqrt{3}d(F, \tilde{F}),$$

or, equivalently,
$$d(A(F), A(\tilde{F})) \le 9\sqrt{3}d(F, \tilde{F}).$$

Proof of Proposition 4.2. Because $F \in \overline{\Lambda}_\phi$, then there exists a sequence $F_n \in \Lambda_\phi$ such that $d(F_n, F) \to 0$ as $n \to \infty$. Take any two points $t_1, t_2 \in [t_0, T]$ and any F_i, $i = 1, 2, 3$. Convergence in metric d implies point-wise convergence. Therefore,

$$|F_j(t_1) - F_j(t_2)| = \lim_{n \to \infty} |F_{n,j}(t_1) - F_{n,j}(t_2)| \le |\phi(t_1) - \phi(t_2)|.$$

The last inequality and the absolute continuity of ϕ imply that F_i is absolutely continuous. Functions $F_{n,i}$ are strictly increasing and converge to F_i point-wise, so F_i are increasing. $F_{n,i}(t_0)$ converge to $F_i(t_0)$. Hence, $F_i(t_0) = 0$. In a similar way, it can proved that $F_i(T) = 1$.

Because F_i are absolutely continuous, they can differentiated a.e. on $[t_0, T]$. Let t be a point at which both F_i and ϕ have derivatives. For any fixed h,

$$\frac{F_i(t+h) - F_i(t)}{h} = \lim_{n \to \infty} \frac{F_{n,i}(t+h) - F_{n,i}(t)}{h} \le \frac{\phi(t+h) - \phi(t)}{h}.$$

Taking the limit as $h \to 0$, we obtain that $F_i'(t) \le \phi'(t)$.

Proof of Proposition 4.3. This proof is similar to the proof of Proposition 4.1. Let $F, \tilde{F} \in \overline{\Lambda}_\phi$ and $G = A(F)$, $\tilde{G} = A(\tilde{F})$. Integration by parts yields

$$G_1(t) - \tilde{G}_1(t) = (F_2 F_3 - \tilde{F}_2 \tilde{F}_3)(1 - \tilde{F}_1) - \int_{t_0}^t (F_2 F_3)'(F_1 - \tilde{F}_1)ds + \int_{t_0}^t \tilde{F}_1'(F_2 F_3 - \tilde{F}_2 \tilde{F}_3)ds.$$

Therefore, for any $t \in [t_0, T]$,

$$|G_1(t) - \tilde{G}_1(t)| \le (1 + \phi(T) - \phi(t_0)) \sup_{[t_0,T]} |F_2 F_3 - \tilde{F}_2 \tilde{F}_3| + 2(\phi(T) - \phi(t_0)) \sup_{[t_0,T]} |F_1 - \tilde{F}_1| \le$$
$$\le (1 + 3\phi(T) - 3\phi(t_0))d_1(F, \tilde{F}).$$

Similar inequalities for $G_2 - \tilde{G}_2$ and $G_3 - \tilde{G}_3$ imply that

$$d_1(G, \tilde{G}) \le 3(1 + 3\phi(T) - 3\phi(t_0))d_1(F, \tilde{F}).$$

Taking into account (8.1),

$$d(A(F), A(\tilde{F})) \le C_0 d(F, \tilde{F}), \quad \text{where} \quad C_0 = 3\sqrt{3}(1 + 3\phi(T) - 3\phi(t_0)).$$

Proof of Proposition 4.4. Let $G_0 \in A(\Lambda_\phi)$ and $d(G_n, G_0) \to 0$ as $n \to \infty$ for $G_n \in A(\Lambda_\phi)$. Denote $F_0 = A^{-1}G_0$, $F_n = A^{-1}G_n$. Clearly, F_0, $F_n \in \Lambda_\phi$. I want to show that $d(F_n, F_0) \to 0$ as $n \to \infty$. Notice that the sequence F_n is equicontinuous, as all functions in the sequence are bounded and

$$|F_n(t_1) - F_n(t_2)| \le |\phi(t_1) - \phi(t_2)|$$

for any $t_1, t_2 \in [t_0, T]$. According to the Arzela-Ascoli theorem, there is a convergent subsequence F_{n_k}. Let F^* be the limit of F_{n_k}. Because $F^* \in \overline{\Lambda}_\phi$ and A is continuous on $\overline{\Lambda}_\phi$,

$$d(AF_{n_k}, AF^*) \to 0.$$

Thus, $AF^* = G_0$. Given that on $A(\Lambda_\phi)$ inverse A^{-1} is defined, $F^* = F_0$.

Proof of Lemma 4.5. $Q(F^*) = 0$. Because the inverse operator A^{-1} exists on $A(\Lambda_\phi)$, then $A(F) \ne G^*$ and, hence, $Q(F) > 0$ for any $F \in \Lambda_\phi$, $F \ne F^*$. Now consider $F \in \overline{\Lambda}_\phi \backslash \Lambda_\phi$. Taking into account the result of Proposition 4.2, conclude that there is a function F_i in F that is constant on some interval in $[t_0, T]$. There are two possible cases for F: when (a) $F_i(t) > 0$ for $t > t_0$, $i = 1, 2, 3$, and (b) some F_i takes value 0 in a right-hand side neighborhood of t_0. In the first case, $A(F) \ne G^*$ because the uniqueness result was proved without the assumption of the strict monotonicity of F_i. In the second case, without a loss of generality assume that $F_1(t) = 0$, $t \in [t_0, t_0 + \omega)$. Then $G_2(t) = 0$ and $G_3(t) = 0$, $t \in [t_0, t_0 + \omega)$, for the corresponding $G = A(F)$. Because $G_i^*(t) > 0$ for $t > t_0$, $i = 1, 2, 3$, then obviously $A(F) \ne G^*$.

Proof of Theorem 4.6. To prove this theorem, I use lemmas A1 and A2 from Newey and Powell (2003). Consistency will hold if all conditions in Lemma A1 are satisfied. I divide these conditions into three groups, as in Newey and Powell (2003).

(i) According to Lemma 4.5, F^* is the unique minimizer of Q on $\overline{\Lambda}_\phi$.

(ii) Set $\overline{\Lambda}_\phi$ is compact. Let me show that Q and $\hat{Q}_n$ are continuous on $\overline{\Lambda}_\phi$ and

$$\sup_{F \in \overline{\Lambda}_\phi} |\hat{Q}_n(F) - Q(F)| \xrightarrow{p} 0. \tag{8.2}$$

The continuity of Q and $\hat{Q}_n$ will follow from the properties of A on $\overline{\Lambda}_\phi$. First, consider Q. For any $F, \tilde{F} \in \overline{\Lambda}_\phi$

$$|Q(F) - Q(\tilde{F})| = |E\{(G^* - A(F))^{tr}(G^* - A(F)) - (G^* - A(\tilde{F}))^{tr}(G^* - A(\tilde{F}))\}| =$$

$$= |E \sum_{j=1}^{3} (A(\tilde{F})_j - A(F)_j)(2G_j^* - A(F)_j - A(\tilde{F})_j)|.$$

For any $t \in [t_0, T]$, $A(F)_j(t) \le 1$ and $G_j^*(t) \le 1$, $j = 1, 2, 3$, therefore

$$|Q(F) - Q(\tilde{F})| \le 4E \sum_{j=1}^{3} |A(\tilde{F})_j - A(F)_j|.$$

Applying the Cauchy-Schwartz inequality and (4.1),

$$|Q(F) - Q(\tilde{F})| \leq 4\sqrt{3} E \sqrt{(A(\tilde{F}) - A(F))^{tr}(A(\tilde{F}) - A(F))} \leq 4\sqrt{3} d(A(F), A(\tilde{F})) \leq 4\sqrt{3} C_0 d(F, \tilde{F}).$$

Thus, function Q is Lipschitz and therefore continuous.

Now consider function $\hat{Q}_n$. Similar to the methods described above,

$$|\hat{Q}_n(F) - \hat{Q}_n(\tilde{F})| \leq \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{3} |(\hat{G}_{n,j}(t_i) - A(F)_j(t_i))^2 - (\hat{G}_{n,j}(t_i) - A(\tilde{F})_j(t_i))^2| = \qquad (8.3)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{3} |(A(\tilde{F})_j(t_i) - A(F)_j(t_i))(2\hat{G}_{n,j}(t_i) - A(\tilde{F})_j(t_i) - A(F)_j(t_i))| \leq$$

$$\leq \frac{4\sqrt{3}}{n} \sum_{i=1}^{n} \sqrt{(A(\tilde{F})(t_i) - A(F)(t_i))^{tr}(A(\tilde{F})(t_i) - A(F)(t_i))} \leq$$

$$\leq 4\sqrt{3} d(A(F), A(\tilde{F})) \leq 4\sqrt{3} C_0 d(F, \tilde{F}).$$

Property (8.2) will follow from Lemma A2 in Newey and Powell (2003). Indeed, it is clear that

$$\forall (F \in \overline{\Lambda}_\phi) \quad \hat{Q}_n(F) \xrightarrow{p} Q(F).$$

This fact combined with (8.3) implies (8.2).

(iii) This condition follows from assumption (4.2).

Conditions (i)-(iii) imply the consistency property (4.6).

9 Appendix C: Identification in generalized competing risks models

First, I outline Meilijson's approach. From (5.3), Meilijson obtains a system of integral equations that do not contain the derivatives of F_j:

$$F(t) = \exp\left\{ \tilde{T} \log \int_{t_0}^{t} \exp\{-\bar{M} \log(1 - F(s)) dG(s)\} \right\},$$

where matrix $\bar{M}$ is such that $\bar{M}(i,j) = 1 - M(i,j)$ and $\tilde{T} = (M^{tr}M)^{-1}M^{tr}$. He suggests applying to these equations a fixed point theorem for multidimensional functional spaces. As I mentioned, however, his proofs miss important parts.

I now turn to describing my method. The rank condition implies that $m \geq d$ − that is, there are at least as many minimal fatal sets as the number of the elements in a coherent system. First, I consider the case of $m = d$ and assume that the rank condition for the incidence matrix M holds − that is, M is invertible. Introduce auxiliary functions

$$H_i = \prod_{j \in I_i} F_j, \quad i = 1, \ldots, d,$$

and denote $H = (H_1, \ldots, H_d)^{tr}$. The rank condition guarantees that functions F_i, $i = 1, \ldots, d$, taking into account that they are positive, are uniquely expressed through functions H_i, $i = 1, \ldots, d$, via multiplication, division and taking a rational root. Indeed,

$$\log H_i = \sum_{j \in I_i} \log F_j, \quad i = 1, \ldots, d.$$

These equations can be rewritten as $\log H = M \log F$, therefore $F = \exp\{M^{-1} \log H\}$, that is,

$$F_i = \prod_{j=1}^{d} H_j^{k_{ij}}, \quad i = 1, \ldots, d. \tag{9.1}$$

Similar to the auction problem, I obtain an auxiliary system of differential equations by rewriting (5.3) in terms of H:

$$H_i' = \frac{g_i}{\prod_{j \in I_i^c} \left(1 - \prod_{l=1}^{d} H_l^{k_{jl}} \right)}, \quad i = 1, \ldots, d. \tag{9.2}$$

Functions H_i satisfy initial conditions

$$\lim_{t \downarrow t_0} H_i(t) = 0, \quad i = 1, \ldots, d. \tag{9.3}$$

As with the auction, the existence and uniqueness theorems 5.1 and 5.2 can be proved in steps. First, the results are obtained locally, then globally.

The existence of a local solution to (5.3)-(5.4) can be proved in the following way. First, to avoid discontinuities in H, I can modify the auxiliary system (9.2) by introducing a very small number ϵ when necessary. Using Tonelli approximations, I can establish the existence of a local solution for the auxiliary system with ϵ. After that, I can take the limit as $\epsilon \to 0$ and show the existence of a local solution for (9.2)-(9.3). Then I can use formulas (9.1) to obtain the existence of a local solution to problem (5.3)-(5.4). To establish local uniqueness, I obtain a generalized local Lipschitz condition on H_i.

Finally, I can show that the unique local solution can be extended to the whole support, and that such extension is unique. Again, the monotonicity of F_i in this solution has to be assumed.

Below I prove the local uniqueness part of Theorem 5.2.

Proof of Theorem 5.2. Let F and $\tilde{F}$ be two local solutions to (5.3)-(5.4) with a common interval of existence $[t_0, t_0 + c]$. Let H and $\tilde{H}$ be the corresponding auxiliary functions. Then H and $\tilde{H}$ solve auxiliary system (9.2) a.e. on $(t_0, t_0 + c]$. Denote the right-hand side of (9.2) as

$$J(t, H) = \left(\frac{g_1(t)}{\prod_{j \in I_1^c} \left(1 - \prod_{l=1}^{d} H_l^{k_{jl}} \right)}, \ldots, \frac{g_d(t)}{\prod_{j \in I_d^c} \left(1 - \prod_{l=1}^{d} H_l^{k_{jl}} \right)} \right).$$

A plan is to derive a generalized local Lipschitz condition on H_i and then use lemmas 7.7 and 7.8 to establish that H and $\tilde{H}$ coincide. This will imply that F and $\tilde{F}$ coincide. Consider $H_i - \tilde{H}_i$ for

any i and let $|I_i^c|$ be the number of elements in I_i^c. Then a.e. on $[t_0, t_0 + c]$

$$|H_i' - \tilde{H}_i'| = \frac{g_i}{\prod_{j \in I_i^c}(1 - F_j)} - \frac{g_i}{\prod_{j \in I_i^c}(1 - \tilde{F}_j)} =$$

$$= \frac{g_i}{\prod_{j \in I_i^c}(1 - F_j)\prod_{j \in I_i^c}(1 - \tilde{F}_j)}\left(\prod_{j \in I_i^c}(1 - F_j) - \prod_{j \in I_i^c}(1 - F_j + (F_j - \tilde{F}_j))\right) \le$$

$$\le \frac{g_i}{\prod_{j \in I_i^c}(1 - F_j)\prod_{j \in I_i^c}(1 - \tilde{F}_j)}2^{|I_i^c|-1}\sum_{j \in I_i^c}|F_j - \tilde{F}_j| \le C_i g_i \sum_{j \in I_i^c}|F_j - \tilde{F}_j|$$

for some constant C_i. Differences $|F_j - \tilde{F}_j|$ can be bounded from above by expressions of $|H_j - \tilde{H}_j|$. According to (9.1), for $t > t_0$,

$$F_j - \tilde{F}_j = \prod_{l=1}^{d} H_l^{k_{jl}} - \prod_{l=1}^{d} \tilde{H}_l^{k_{jl}},$$

therefore

$$F_j - \tilde{F}_j = \sum_{h=1}^{d}\prod_{l<h} H_l^{k_{jl}} \prod_{m>h} \tilde{H}_m^{k_{jm}}(H_h^{k_{jh}} - \tilde{H}_h^{k_{jh}})$$

For $x_1, x_2 > 0$, by the mean value theorem

$$x_l^\alpha - x_2^\alpha = \alpha(\theta x_1 + (1 - \theta)x_2)^{\alpha-1}(x_1 - x_2),$$

where $\theta = \theta(x_1, x_2) \in [0, 1]$. If $\alpha \ge 1$, then

$$|x_l^\alpha - x_2^\alpha| \le \alpha(\max\{x_1, x_2\})^{\alpha-1}|x_1 - x_2|.$$

If $\alpha < 1$, then

$$|x_l^\alpha - x_2^\alpha| \le |\alpha|(\min\{x_1, x_2\})^{\alpha-1}|x_1 - x_2|.$$

Because $H_h(t), \tilde{H}_h(t) > 0$ for $t > t_0$, then for $t > t_0$,

$$|H_h^{k_{jh}}(t) - \tilde{H}_h^{k_{jh}}(t)| \le W_{jh}(t)|H_h(t) - \tilde{H}_h(t)|,$$

where

$$W_{jh}(t) = \left(1(k_{jh} \ge 1)\max\{H_h(t), \tilde{H}_h(t)\} + 1(k_{jh} < 1)\min\{H_h(t), \tilde{H}_h(t)\}\right)^{k_{jh}-1}.$$

Because

$$\lim_{t\downarrow t_0}\frac{H_h(t)}{G_h(t)} = 1 \text{ and } \lim_{t\downarrow t_0}\frac{\tilde{H}_h(t)}{G_h(t)} = 1,$$

then

$$\lim_{t\downarrow t_0}\frac{W_{jh}(t)}{G_h^{k_{jh}-1}(t)} = 1.$$

Thus, for $t > t_0$,

$$|F_j - \tilde{F}_j| \leq L_j \sum_{h=1}^{d} (\prod_{l \neq h} G_l^{k_{jl}}) G_h^{k_{jh}-1} |k_{jh}| |H_h - \tilde{H}_h|$$

for some constants $L_j > 0$. Thus, a.e. on $[t_0, t_0 + c]$

$$|H_i'(t) - \tilde{H}_i'(t)| \leq D_i g_i \sum_{j \in I_i^c} \sum_{h=1}^{d} (\prod_{l \neq h} G_l^{k_{jl}}(t)) G_h^{k_{jh}-1}(t) |k_{jh}| |H_h(t) - \tilde{H}_h(t)|$$

for some constants $D_i > 0$ and, hence,

$$||H'(t) - \tilde{H}'(t)||_1 \leq C(\Gamma_1(t) + \ldots \Gamma_d(t)) ||H(t) - \tilde{H}(t)||_1$$

for some constant $C > 0$. This inequality and lemmas 7.7 and 7.8 imply that $H(t) = \tilde{H}(t)$, $t \in [t_0, t_0 + c]$.

10 Appendix D: Bounds on distributions

Proof of Theorem 6.1.

(a) First, I prove the result for the lower bound. Suppose that $\max_{k=1,\ldots,r} t_{i_k} < T$. Then

$$Q_D(t_{i_1}, \ldots, t_{i_r}) = P(\cap_{k=1}^{r}(X_{i_k} \leq t_{i_k})) = P(\cap_{k=1}^{r}(b_{i_k} \leq t_{i_k})) =$$

$$= \sum_{m=1}^{r} P(\cap_{k=1}^{r}(b_{i_k} \leq t_{i_k}), i_m \text{ wins}) + \sum_{j \in CD} P(\cap_{k=1}^{r}(b_{i_k} \leq t_{i_k}), j \text{ wins}) \geq$$

$$\geq \sum_{j \in CD} P(\text{price} \leq \min_{k=1,\ldots,r} t_{i_k}, j \text{ wins}) = \sum_{j \in CD} G_j(\min_{k=1,\ldots,r} t_{i_k}).$$

Now consider the case when at least one of t_{i_k} takes value T. It is enough to consider the case when $t_{i_1} = T$ and $\max_{k=2,\ldots,r} t_{i_k} < T$. Denote $\tilde{D} = \{i_2, \ldots, i_r\}$. From what I have shown above, it follows that

$$Q_D(T, \ldots, t_{i_r}) = Q_{\tilde{D}}(t_{i_2}, \ldots, t_{i_r}) \geq \sum_{j \in C\tilde{D}} G_j(\min_{k=2,\ldots,r} t_{i_k}) = G_{i_1}(\min_{k=2,\ldots,r} t_{i_k}) + \sum_{j \in CD} G_j(\min_{k=2,\ldots,r} t_{i_k}) =$$

$$= G_{i_1}(\min_{k=1,\ldots,r} t_{i_k}) + \sum_{j \in CD} G_j(\min_{k=1,\ldots,r} t_{i_k}) = \sum_{l=1}^{r} 1(t_{i_l} = T) G_{i_l}(\min_{k=1,\ldots,r} t_{i_k}) + \sum_{j \in CD} G_j(\min_{k=1,\ldots,r} t_{i_k}).$$

(b) For any $D = \{t_{i_1}, \ldots, t_{i_r}\}$, if $\min_{k=1,\ldots,r} t_{i_k} = t_0$ then $Q_D(t_{i_1}, \ldots, t_{i_r}) = 0$ because by assumption all marginal distributions X_i are continuous and, therefore, do not have mass points. Suppose that $\min_{k=1,\ldots,r} t_{i_k} > t_0$.

$$Q(t_1, \ldots, t_d) = P(\cap_{i=1}^{d}(X_i \leq t_i)) = P(\cap_{i=1}^{d}(b_i \leq t_i)) = \sum_{j} P(\cap_{i \neq j}(b_i \leq t_i), b_j \leq t_j, j \text{ wins}) \leq$$

$$\leq \sum_{j} P(\text{price} \leq \min\{\max_{i \neq j} t_i, t_j\}, j \text{ wins}) = \sum_{j} G_j(\min\{\max_{i \neq j} t_i, t_j\}).$$

To obtain an upper bound for Q_{-m}, use the upper bound for Q and the fact that

$$Q_{-m}(t_1, \ldots, t_{m-1}, t_{m+1}, \ldots, t_d) = Q(t_1, \ldots, t_{m-1}, T, t_{m+1}, \ldots, t_d).$$

Clearly,

$$Q_{-m}(t_1, \ldots, t_{m-1}, t_{m+1}, \ldots, t_d) \leq \sum_{j \neq m} G_j(\min\{T, t_j\}) + G_m(\min\{\max_{i \neq m} t_i, T\}) =$$
$$= \sum_{j \neq m} G_j(t_j) + G_m(\max_{i \neq m} t_i).$$

To obtain an upper bound for $D = \{t_{i_1}, \ldots, t_{i_r}\}$ that contains at most $d - 2$ elements, use the upper bound for Q and substitute values of t_j, $j \notin D$, with the value of T.

Proof of Proposition 6.2.

(a) According to AI, for any $D = \{t_{i_1}, \ldots, t_{i_r}\}$, the event $\{\cap_{k=1}^r (X_{i_k} \leq t_{i_k})\}$ implies the event $\{\cap_{k=1}^r (b_{i_k} \leq t_{i_k})\}$. Therefore,

$$Q_D(t_{i_1}, \ldots, t_{i_r}) = P(\cap_{k=1}^r (X_{i_k} \leq t_{i_k})) \leq P(\cap_{k=1}^r (b_{i_k} \leq t_{i_k})).$$

The rest of the proof for the upper bounds is the same as in Theorem 6.1.

(b) Suppose that $\max_{k=1,\ldots,r} t_{i_k} < T$. Then

$$Q_D(t_{i_1}, \ldots, t_{i_r}) = P(\cap_{k=1}^r (X_{i_k} \leq t_{i_k})) = \sum_{k=1}^r P(\cap_{k=1}^r (X_{i_k} \leq t_{i_k}), i_k \text{ wins}) +$$
$$+ \sum_{j \in CD} P(\cap_{k=1}^r (X_{i_k} \leq t_{i_k}), j \text{ wins}) \geq \sum_{j \in CD} P(\cap_{k=1}^r (X_{i_k} \leq t_{i_k}), j \text{ wins}).$$

According to AII, for $j \in CD$, the event $\{\text{price} \leq \min_{k=1,\ldots,r} t_{i_k}, j \text{ wins}\}$ implies the event $\{\cap_{k=1}^r (X_{i_k} \leq t_{i_k}), j \text{ wins}\}$. Indeed, if some X_{i_k} was larger than t_{i_k}, then bidder i_k would not allow bidder j to win at a price less or equal than $\min_{k=1,\ldots,r} t_{i_k}$. Therefore,

$$Q_D(t_{i_1}, \ldots, t_{i_r}) \geq \sum_{j \in CD} P(\text{price} \leq \min_{k=1,\ldots,r} t_{i_k}, j \text{ wins}).$$

The rest of the proof for the lower bounds is the same as in Theorem 6.1.

Proof of Theorem 6.3.

(a) The lower bound is obvious. Let me obtain the upper bound.

$$Q(t_1, \ldots, t_d) = \sum_{\rho \in \Pi_d} P(\cap_{i=1,\ldots d}(b_i \leq t_i), b_{\rho(1)} > b_{\rho(2)} > \ldots > b_{\rho(d)}) \leq$$
$$\leq \sum_{\rho \in \Pi_d} P(\cap_{l=2,\ldots,d}(b_{\rho(l)} \leq \min_{m=1,\ldots,l} t_{\rho(m)}), b_{\rho(1)} > b_{\rho(2)} > \ldots > b_{\rho(d)}) =$$
$$= \sum_{\rho \in \Pi_d} W_\rho(\min\{t_{\rho(1)}, t_{\rho(2)}\}, \ldots, \min_{m=1,\ldots,l} t_{\rho(m)}, \ldots, \min_{i=1,\ldots,d} t_i).$$

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(b) Without a loss of generality, consider function F_1.

$$F_1(t) = P(X_1 \le t) = P(b_1 \le t) = \sum_i \sum_{h \ne i} P(\max_{l \ne h, l \ne i} b_l < b_h, b_h < b_i, b_1 \le t) =$$

$$= \sum_{h \ne 1} P(\max_{l \ne h, l \ne 1} b_l < b_h, b_h < b_1, b_1 \le t) + \sum_{i \ne 1} P(\max_{l \ne 1, l \ne i} b_l < b_1, b_1 < b_i, b_1 \le t) +$$

$$+ \sum_{i \ne 1} \sum_{h \ne i, h \ne 1} P(\max_{l \ne h, l \ne i} b_l < b_h, b_h < b_i, b_1 \le t).$$

Let $t \in (t_0, T]$. For the upper bound,

$$P(b_1 \le t) \le \sum_{h \ne 1} P(\max_{l \ne h, l \ne 1} b_l < b_h, b_h < b_1, b_h \le t) + \sum_{i \ne 1} P(\max_{l \ne 1, l \ne i} b_l < b_1, b_1 < b_i, b_1 \le t) +$$

$$+ \sum_{i \ne 1} \sum_{h \ne i, h \ne 1} P(\max_{l \ne h, l \ne i} b_l < b_h, b_h < b_i) = \sum_{h \ne 1} G_{1h}(t) + \sum_{i \ne 1} G_{i1}(t) + \sum_{i \ne 1} \sum_{h \ne i, h \ne 1} G_{ih}(T).$$

Let $t \in [t_0, T)$. For the lower bound,

$$P(b_1 \le t) \ge 0 + \sum_{i \ne 1} P(\max_{l \ne 1, l \ne i} b_l < b_1, b_1 < b_i, b_1 \le t) +$$

$$+ \sum_{i \ne 1} \sum_{h \ne i, j \ne 1} P(\max_{l \ne h, l \ne i} b_l < b_h, b_h < b_i, b_h \le t) = \sum_{i \ne 1} G_{i1}(t) + \sum_{i \ne 1} \sum_{h \ne i, h \ne 1} G_{ih}(t) =$$

$$= \sum_{i \ne 1} \left(G_{i1}(t) + \sum_{h \ne i, h \ne 1} G_{ih}(t) \right) = \sum_{i \ne 1} \sum_{h \ne i} G_{ih}(t) = \sum_{i \ne 1} G_i(t).$$

Evidently, $F_1(t_0) = 0$ and $F_1(T) = 1$.