

EMPIRICAL LIKELIHOOD FOR HIGH FREQUENCY DATA

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ABSTRACT. With increasing availability of high frequency financial data as a background, various volatility measures and related statistical theory are developed in the recent literature. This paper introduces the method of empirical likelihood to conduct statistical inference on the volatility measures under high frequency data environments. We propose a modified empirical likelihood statistic that is asymptotically pivotal under the infill asymptotics, where the number of high frequency observations in a fixed time interval increases to infinity. Our empirical likelihood approach is extended to be robust to the presence of jumps and microstructure noise. We also provide an empirical likelihood test to detect presence of jumps. Furthermore, we establish Bartlett correction, a higher-order refinement, for a general nonparametric likelihood statistic. Simulation and a real data example illustrate the usefulness of our approach.

1. INTRODUCTION

Realized volatility and its related statistics have become standard tools to explore the behavior of high frequency financial data and to evaluate financial theoretical models including stochastic volatility models. This increase in popularity has been propelled by recent developments of probability and statistical theory and by the increasing availability of high frequency financial data (see, Aït-Sahalia and Jacod, 2014, for a review).

By employing the asymptotic framework so-called the infill asymptotics, where the number of high frequency observations in a fixed time interval (say, a day) increases to infinity, Jacod and Protter (1998) and Barndorff-Nielsen and Shephard (2002) established laws of large numbers and central limit theorems for realized volatility, which were extended to more general setups and statistics by Barndorff-Nielsen *et al.* (2006). Gonçalves and Meddahi (2009) studied higher-order properties of the realized volatility statistic and its bootstrap counterpart. Also various volatility estimation methods are developed to be robust to the presence of jumps (e.g., Barndorff-Nielsen, Shephard and Winkel, 2006, and Andersen, Dobrev and Schaumburg, 2012) and microstructure noise (e.g., Zhang, Mykland and Aït-Sahalia, 2005, Barndorff-Nielsen *et al.*, 2008, and Jacod *et al.*, 2009). Finally, several testing methods for the presence of jumps are developed (e.g., Barndorff-Nielsen and Shephard, 2006, and Aït-Sahalia and Jacod, 2009).

In this paper, we introduce the method of empirical likelihood (see, Owen, 2001, for a review) to conduct statistical inference on the volatility measures under high frequency

data environments. In particular, based on estimating equations for the volatility measures, such as the integrated volatility, modified empirical likelihood statistics are proposed and shown to be asymptotically pivotal under the infill asymptotics. Our empirical likelihood approach is extended to be robust to the presence of jumps and microstructure noise. The proposed statistics share desirable properties of the conventional empirical likelihood, such as range preserving, transformation respecting, and data decided shape for confidence region. We also provide an empirical likelihood test to detect presence of jumps. Our empirical likelihood approach provides useful alternatives to the existing Wald-type inference methods and jump tests. This is illustrated by simulation studies and a real data example.

Another distinguishing feature of (conventional) empirical likelihood is that it admits Bartlett correction, a higher-order refinement (DiCiccio, Hall and Romano, 1991). However, under the infill asymptotics, empirical likelihood is not Bartlett correctable even for the constant volatility case. In order to explore further this issue, we consider a general class of nonparametric likelihood based on Cressie and Read's (1984) power divergence family, which contains empirical likelihood, exponential tilting, and Pearson's χ^2 as special cases. In this general class of likelihood functions, we find some members that admit Bartlett correction under the constant and general non-constant volatility cases. In particular, we show that the second-order refinement to the order $O(n^{-2})$ can be achieved. This Bartlett correctability can be considered as a unique advantage of our nonparametric likelihood approach.

The rest of the paper is organized as follows. In Section 2, we consider a benchmark setup which excludes jumps and microstructure noise, construct the empirical likelihood statistic, and study its first-order asymptotic properties. In Section 3, we propose a jump robust version of the empirical likelihood statistic. Also an empirical likelihood test to detect the presence of jumps is presented. In Section 4, we propose a noise robust version of the empirical likelihood statistic. In Section 5, we conduct second-order analysis for the proposed statistic and establish the Bartlett correctability results. Sections 6 and 7 present some simulation results and real data example, respectively. All proofs of the theorems are contained in the web appendix.

2. BENCHMARK CASE

In this section, we present our methodology in a benchmark setup, which excludes jumps and microstructure noise. Jump and noise robust methods are considered in the following sections. Here we consider a scalar continuous time process (typically a log-price)

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{2.1}$$

for $t \geq 0$, where μ is a drift process, σ is a volatility process, and W is a standard Brownian motion. Suppose we observe high frequency returns $r_i = X_{i/n} - X_{(i-1)/n}$ measured over the period $[(i-1)/n, i/n]$ for $i = 1, \dots, n$. Although our methodology can be applied to other functionals of σ (see Remark 4 below), we focus on the integrated volatility $\theta = \int_0^1 \sigma_u^2 du$ over a fixed interval $[0, 1]$ (say, a day or month).

As a nonparametric measure of volatility, the integrated volatility θ has been drawing considerable attention from researchers who face to high frequency financial data. One popular estimator of θ is so-called the realized volatility $\hat{\theta} = \sum_{i=1}^n r_i^2$. It is known that under certain conditions on the process (2.1), $\hat{\theta}$ is consistent for θ and asymptotically normal under the limit $n \rightarrow \infty$ for increasingly finely sampled returns over the fixed interval $[0, 1]$ (called the infill asymptotics) (e.g., Jacod and Protter, 1998, and Barndorff-Nielsen and Shephard, 2002). In this section, we employ the following setup based on Barndorff-Nielsen *et al.* (2006).

Assumption X. *The process X defined on a filtered probability space follows (2.1), where μ is an adapted predictable locally bounded drift process, and σ is an adapted cadlag volatility process satisfying*

$$\sigma_t = \sigma_0 + \int_0^t a_u^* du + \int_0^t \sigma_{u-}^* dW_u + \int_0^t v_{u-}^* dV_u,$$

where a^* , σ^* , and v^* are adapted cadlag processes, a^* is predictable and locally bounded, and V is a Brownian motion independent of W .

This assumption is general enough to allow for intraday seasonality, long memory, and correlation between σ and W (called the leverage effect). Under this assumption, Barndorff-Nielsen *et al.* (2006) showed that $\hat{\theta}$ is consistent and asymptotically normal

$$\frac{\sqrt{n}(\hat{\theta} - \theta)}{\sqrt{\hat{V}}} \xrightarrow{d} N(0, 1), \quad (2.2)$$

as $n \rightarrow \infty$, where $\hat{V} = \frac{2n}{3} \sum_{i=1}^n r_i^4$. Based on this result, it is customary to construct a Wald-type confidence interval for θ . Also, Gonçalves and Meddahi (2009) proposed bootstrap inference methods on θ . In this paper, we introduce the empirical likelihood approach and propose an alternative inference method for θ .

Based on the estimating equation $\sum_{i=1}^n (nr_i^2 - \theta) = 0$ for the realized volatility $\hat{\theta}$, the empirical likelihood function for θ can be written as

$$EL(\theta) = \max \left\{ \prod_{i=1}^n nw_i \left| \sum_{i=1}^n w_i(nr_i^2 - \theta) = 0, w_i \geq 0, \sum_{i=1}^n w_i = 1 \right. \right\}. \quad (2.3)$$

By the Lagrange multiplier argument, the dual form of (2.3) is written as

$$EL(\theta) = \prod_{i=1}^n \frac{1}{1 + \lambda(nr_i^2 - \theta)},$$

where λ solves $\frac{1}{n} \sum_{i=1}^n \frac{nr_i^2 - \theta}{1 + \lambda(nr_i^2 - \theta)} = 0$. In practice, we employ this dual representation to compute $EL(\theta)$.

Let $R_q = n^{q/2-1} \sum_{i=1}^n |r_i|^q$. The first-order asymptotic distribution of $EL(\theta)$ is obtained as follows. See our web appendix for the proof.

Theorem 1. *Suppose Assumption X holds true. As $n \rightarrow \infty$,*

$$T_{EL}(\theta) = \frac{3}{2} \left(1 - \frac{R_2^2}{R_4} \right) \{-2 \log EL(\theta)\} \xrightarrow{d} \chi_1^2.$$

Remark 1. Based on this theorem, the $100(1 - \alpha)\%$ asymptotic confidence interval for the integrated volatility θ is given by $CI_{EL}^\alpha = \{\theta : T_{EL}(\theta) \leq \chi_{1,\alpha}^2\}$, where $\chi_{1,\alpha}^2$ is the $(1 - \alpha)$ -th quantile of the χ_1^2 distribution.

Remark 2. It should be noted that under the infill asymptotics, the conventional empirical likelihood statistic (i.e., $-2 \log EL(\theta)$) does not converge to the χ^2 distribution. In other words, the empirical likelihood statistic is not internally studentized. This is because the asymptotic variance of the term $\frac{1}{\sqrt{n}} \sum_{i=1}^n (nr_i^2 - \theta)$ does not match to the limit of $\frac{1}{n} \sum_{i=1}^n (nr_i^2 - \theta)^2$ under the infill asymptotics. The correction term $\frac{3}{2} \left(1 - \frac{R_2^2}{R_4} \right)$ is introduced to recover the studentization.

Remark 3. We now discuss advantages of our empirical likelihood confidence interval CI_{EL}^α compared to the conventional Wald-type confidence interval (i.e., $\hat{\theta} \pm z_{\alpha/2} \sqrt{\hat{V}/n}$ for the $(1 - \alpha/2)$ -th quantile $z_{\alpha/2}$ of $N(0, 1)$). First, CI_{EL}^α may be asymmetric around the point estimate $\hat{\theta}$, and its shape is flexibly determined by the data. Second, CI_{EL}^α never contains negative values (called range preserving property). On the other hand, the lower endpoint of the Wald confidence interval may be negative. Third, CI_{EL}^α is transformation respecting (i.e., the confidence interval of $f(\theta)$ is given by $\{f(\theta) : T_{EL}(\theta) \leq \chi_{1,\alpha}^2\}$). However, the Wald confidence interval is not invariant for transformations of θ and may yield different conclusions.

Remark 4. Here we discuss the empirical likelihood method for the integrated volatility $\theta = \int_0^1 \sigma_u^2 du$. Our method can be modified for other objects related to σ . For example, suppose we are interested in the p -th power variation $\theta_p = \int_0^1 \sigma_u^p du$ for $p > 0$. By Barndorff-Nielsen *et al.* (2006), θ_p is consistently estimated by $\hat{\theta}_p = \mu_p^{-1} n^{-1+p/2} \sum_{i=1}^n |r_i|^p$, where $\mu_p = E|z|^p$ with $z \sim N(0, 1)$. Based on the estimating equation for $\hat{\theta}_p$, the empirical

likelihood function for θ_p can be constructed as

$$EL(\theta_p) = \max \left\{ \prod_{i=1}^n n w_i \left| \sum_{i=1}^n w_i (\mu_p^{-1} n^{p/2} |r_i|^p - \theta_p) = 0, w_i \geq 0, \sum_{i=1}^n w_i = 1 \right. \right\}.$$

The asymptotic property of this statistic is established in the same manner (with a different correction term for asymptotic pivotalness). Also, in the next section, we consider empirical likelihood for multipower variation to conduct jump robust inference.

3. JUMP ROBUST INFERENCE AND TEST FOR JUMPS

In this section, we propose a jump robust version of the empirical likelihood statistic and test for presence of jumps. The empirical likelihood function (2.3) proposed in the last section is constructed from the estimating equation for the realized volatility $\hat{\theta} = \sum_{i=1}^n r_i^2$. Our approach can be generalized to other estimating equations for the integrated volatility θ . In particular, it is useful to consider the multipower variation (e.g., Barndorff-Nielsen, Shephard, 2004, and Barndorff-Nielsen, Shephard and Winkel, 2006)

$$\hat{\theta}_{\mathbf{p}} = \sum_{i=m}^n |r_{i-m+1}|^{p_1} \cdots |r_i|^{p_m},$$

for a vector $\mathbf{p} = (p_1, \dots, p_m)$ of positive numbers with $p_1 + \dots + p_m = 2$. Indeed the realized volatility is a special case of the multipower variation (with $m = 1$ and $p_1 = 2$). A remarkable property of the multipower variation is: if p 's are reasonably small, then the estimator $\hat{\theta}_{\mathbf{p}}$ enjoys certain robustness against jumps in the observed process.

To be precise, consider the process

$$Y_t = X_t + J_t, \tag{3.1}$$

for $t \geq 0$, where X is generated by the continuous time process in (2.1) satisfying Assumption X, and J is a jump process, which is assumed to be a Lévy process with no continuous component and index

$$\alpha = \inf \left\{ a \geq 0 : \int_{[-1,1]} |x|^a \Pi(dx) < \infty \right\} \in [0, 2],$$

for the Lévy measure Π . The Lévy process is a convenient and general class of processes to accommodate both finite and infinite activity jumps. Barndorff-Nielsen, Shephard and Winkel (2006, Theorem 1) showed that the limiting distribution of the multipower variation $\hat{\theta}_{\mathbf{p}}$ remains the same regardless of presence of the jump process J as far as

$$\alpha < 1, \quad \frac{\alpha}{2 - \alpha} \leq \min\{p_1, \dots, p_m\} \leq \max\{p_1, \dots, p_m\} < 1. \tag{3.2}$$

A popular choice of \mathbf{p} for the jump robust estimator is the tripower variation (i.e., $m = 3$ and $p_1 = p_2 = p_3 = 2/3$).

Suppose we observe high frequency returns $\tilde{r}_i = Y_{i/n} - Y_{(i-1)/n}$ measured over the period $[(i-1)/n, i/n]$ for $i = 1, \dots, n$. Let $c_{\mathbf{p}} = \prod_{l=1}^m \mu_{p_l}$, where $\mu_p = E|z|^p$ with $z \sim N(0, 1)$. Based on the estimating equation for $\hat{\theta}_{\mathbf{p}}$, we define the jump robust empirical likelihood function for θ as

$$\widetilde{EL}(\theta) = \max \left\{ \prod_{i=1}^n n w_i \left| \sum_{i=1}^n w_i (n |\tilde{r}_{i-m+1}|^{p_1} \cdots |\tilde{r}_i|^{p_m} - c_{\mathbf{p}} \theta) = 0, w_i \geq 0, \sum_{i=1}^n w_i = 1 \right. \right\}. \quad (3.3)$$

Let $\tilde{R}_2 = \hat{\theta}_{\mathbf{p}}$ and $\tilde{R}_4 = n \sum_{i=m}^n |\tilde{r}_{i-m+1}|^{2p_1} \cdots |\tilde{r}_i|^{2p_m}$. Also define the constant

$$d_{\mathbf{p}} = \prod_{l=1}^m \mu_{2p_l} - (2m-1) \prod_{l=1}^m \mu_{p_l}^2 + 2 \sum_{k=1}^{m-1} \prod_{l=1}^k \mu_{p_k} \prod_{l=m-k+1}^m \mu_{p_l} \prod_{l=1}^{m-k} \mu_{p_l+p_l+k}.$$

The first-order asymptotic property of the jump robust empirical likelihood statistic $\widetilde{EL}(\theta)$ is obtained as follows.

Theorem 2. *Suppose Y is generated by (3.1). Assume $p_1 + \cdots + p_m = 2$ and (3.2). Then*

$$\tilde{T}_{EL}(\theta) = \frac{c_{2\mathbf{p}}}{d_{\mathbf{p}}} \left(1 - \frac{\tilde{R}_2^2}{\tilde{R}_4} \right) \{-2 \log \widetilde{EL}(\theta)\} \xrightarrow{d} \chi_1^2,$$

as $n \rightarrow \infty$. This result does not change even if $J = 0$ (the case of no jump).

Remark 5. This theorem says that the empirical likelihood statistic $\tilde{T}_{EL}(\theta)$ has the limiting distribution that is invariant to the presence of jumps. Similar to the benchmark case, we introduce the correction term $\frac{c_{2\mathbf{p}}}{d_{\mathbf{p}}} \left(1 - \frac{\tilde{R}_2^2}{\tilde{R}_4} \right)$ to achieve asymptotic pivotalness. The jump robust confidence interval for θ is obtained in the same manner. We note that the empirical likelihood function (2.3) for the benchmark case (i.e., $m = 1$ and $p_1 = 2$) does not satisfy the condition in (3.2).

We now consider hypothesis testing for presence of jumps in the observed process (i.e., $J = 0$). The basic idea is to compare the two estimating equations for the realized volatility $\hat{\theta}$ and multipower variation $\hat{\theta}_{\mathbf{p}}$. More precisely, we propose the following empirical likelihood statistic

$$\widetilde{EL}^J = \max \left\{ \prod_{i=1}^n n w_i \left| \sum_{i=1}^n w_i (|\tilde{r}_{i-m+1}|^{p_1} \cdots |\tilde{r}_i|^{p_m} - c_{\mathbf{p}} \tilde{r}_i^2) = 0, w_i \geq 0, \sum_{i=1}^n w_i = 1 \right. \right\}. \quad (3.4)$$

The rationale of the above moment restriction is explained as follows. When there is no jump in the process (i.e., $J = 0$), both the multipower variation $\hat{\theta}_{\mathbf{p}}$ (with $p_1 + \cdots + p_m = 2$) and realized volatility $c_{\mathbf{p}} \sum_{i=1}^n \tilde{r}_i^2$ multiplied by $c_{\mathbf{p}}$ are consistent for $c_{\mathbf{p}} \theta$. Therefore, the moment function $\sum_{i=m}^n (|\tilde{r}_{i-m+1}|^{p_1} \cdots |\tilde{r}_i|^{p_m} - c_{\mathbf{p}} \tilde{r}_i^2)$ converges to zero and the statistic \widetilde{EL}^J tends to be small. On the other hand, in the presence of jumps, the moment function typically diverges and so does the statistic.

Let $c_{\tilde{p}_l} = (\mu_{p_l+2}/\mu_{p_l}) \prod_{k=1}^m \mu_{p_k}$. The first-order asymptotic property of the empirical likelihood statistic \widetilde{EL}^J for the presence of jumps is obtained as follows.

Theorem 3. *Suppose $Y = X$ (i.e., no jump in the process), where X satisfies Assumption X. Also assume (3.2). Then*

$$\tilde{T}_{EL}^J = \frac{c_{2\mathbf{p}} + 3c_{\mathbf{p}}^2 - 2c_{\mathbf{p}}c_{\tilde{\mathbf{p}}}}{d_{\mathbf{p}} - 2\sum_{l=1}^m c_{\mathbf{p}}(c_{\tilde{p}_l} - c_{\mathbf{p}}) + 2c_{\mathbf{p}}^2} \{-2\log \widetilde{EL}^J\} \xrightarrow{d} \chi_1^2,$$

as $n \rightarrow \infty$. On the other hand, if Y is generated by (3.1), then the statistic \tilde{T}_{EL}^J diverges.

4. NOISE ROBUST INFERENCE

Our empirical likelihood approach presented above can be also modified to be robust to the presence of microstructure noise. In particular, we adopt the pre-averaging approach of Jacod *et al.* (2009), and construct empirical likelihood based on block averages of the original data. In this section, let us consider the following setup.

Assumption X'. *Observations $\{Z_{i/n}\}_{i=1}^n$ are generated from*

$$Z_{i/n} = X_{i/n} + U_{i/n},$$

where $\{X_{i/n}\}_{i=1}^n$ is drawn from the latent process X satisfying Assumption X, and $\{U_{i/n}\}_{i=1}^n$ is an i.i.d. sequence with zero mean and finite eighth moments and is independent of X .

We are interested in the integrated volatility $\theta = \int_0^1 \sigma_u^2 du$ of the latent process X . It is known that due to the presence of the noise term $U_{i/n}$, the conventional realized volatility based on $\{Z_{i/n}\}_{i=1}^n$ is inconsistent for θ .

In this setup, Jacod *et al.* (2009) developed a noise robust estimator for θ based on the so-called pre-averaging approach. A simplified version of their estimator is described as follows. First, we transform the observed data $\{Z_{i/n}\}_{i=1}^n$ into block averages $\bar{Z}_{i/n} = K^{-1} \sum_{j=0}^{K-1} Z_{(i+j)/n}$ for $i = 0, 1, \dots, n - K + 1$. Second, based on the block averages, compute (half of) the return data $\bar{r}_i = (\bar{Z}_{(i+K)/n} - \bar{Z}_{i/n})/2$ for $i = 1, \dots, n - K + 1$. Finally, we compute the noise robust estimator as

$$\bar{\theta} = \frac{6}{K} \sum_{i=1}^{n_K} \bar{r}_i^2 - \frac{3}{2K^2} \hat{\theta}, \quad (4.1)$$

where $n_K = n - 2K + 2$ and $\hat{\theta} = \sum_{i=1}^n (Z_{i/n} - Z_{(i-1)/n})^2$ is the conventional realized volatility estimator by using the original data. Intuitively, compared to the original $Z_{i/n}$, the variance of the noise in the block average $\bar{Z}_{i/n}$ is reduced by a factor of $1/K$. Thus, the volatility estimator $\bar{\theta}$ based on the block averages are expected to be less sensitive to the presence of the noise term. The second term in (4.1) is a bias correction term. Note that the conventional estimator $\hat{\theta}$ is inconsistent for θ under Assumption X'. Jacod *et al.* (2009) showed that $\bar{\theta}$ is consistent for θ and asymptotically normal with the rate of $n^{-1/4}$.

By utilizing the estimating equation for (4.1), the noise robust empirical likelihood function can be constructed as

$$\overline{EL}(\theta) = \max \left\{ \prod_{i=1}^{n_K} n_K w_i \left| \sum_{i=1}^{n_K} w_i (g_{Ki} - \theta) = 0, w_i \geq 0, \sum_{i=1}^{n_K} w_i = 1 \right. \right\},$$

where

$$g_{Ki} = \frac{6n_K}{K} \bar{r}_i^2 - \frac{3}{2K^2} \hat{\theta}.$$

Choose the block length as $K = \frac{1}{2}cn^{1/2} + o(n^{1/4})$ for some $c > 0$. Then define $\bar{R}_q = n_K^{q/2-1} \sum_{i=1}^{n_K} |\bar{r}_i|^q$ and

$$\begin{aligned} R_4^* &= \frac{4 \cdot 12^4 \Phi_{22}}{3c} \sum_{i=1}^{n_K} \bar{r}_i^4 + \frac{4}{c^3 n} (12^3 \Phi_{12} - 12^4 \Phi_{22}) \sum_{i=1}^{n_K-2K} \bar{r}_i^2 \sum_{j=i+2K}^{i+4K-1} (Z_{j/n} - Z_{(j-1)/n})^2 \\ &\quad + \frac{1}{c^3 n} (12^2 \Phi_{11} - 2 \cdot 12^3 \Phi_{12} + 12^4 \Phi_{22}) \sum_{i=2}^{n-2} (Z_{i/n} - Z_{(i-1)/n})^2 (Z_{(i+2)/n} - Z_{(i+1)/n})^2, \end{aligned}$$

with $\Phi_{11} = \frac{1}{6}$, $\Phi_{12} = \frac{1}{96}$, and $\Phi_{22} = \frac{151}{80640}$. The object R_4^* appeared in Jacod *et al.* (2009, eq. (3.7)) as an estimator of the asymptotic variance of $\bar{\theta}$. Based on this notation, the first-order asymptotic distribution of $\overline{EL}(\theta)$ is obtained as follows.

Theorem 4. *Suppose Assumption X' holds true. As $n \rightarrow \infty$,*

$$\bar{T}_{EL}(\theta) = \frac{36n^{1/2} \bar{R}_4 - \bar{R}_2^2}{n_K K^2 R_4^*} \{-2 \log \overline{EL}(\theta)\} \xrightarrow{d} \chi_1^2.$$

As pointed out by Jacod *et al.* (2009), the pre-averaging estimator $\bar{\theta}$ can be interpreted as a realized kernel estimator in Barndorff-Nielsen *et al.* (2008). Similarly, our empirical likelihood statistic $\bar{T}_{EL}(\theta)$ using block averages may be interpreted as the block empirical likelihood statistic by Kitamura (1997) for weakly dependent time series data. However, here data blocking is employed to reduce the effect of microstructure noise.

In this section, we impose Assumption X' and consider the case of additive and i.i.d. noise for brevity. We conjecture that it is possible to extend our approach to more general setups, such as weakly dependent noise (Aït-Sahalia, Mykland and Zhang, 2011), non-additive noise (Jacod *et al.*, 2009), and endogenous time (Li, Zhang and Zheng, 2013) by modifying the moment function.

5. GENERAL NONPARAMETRIC LIKELIHOOD AND SECOND-ORDER ASYMPTOTICS

In this section, we generalize the construction of nonparametric likelihood for the integrated volatility by using the power divergence family (Cressie and Read, 1984). This family is general enough to accommodate not only the empirical likelihood considered so far, but also other existing likelihood concepts. Based on this general family of nonparametric likelihood functions, we investigate second-order asymptotic properties of the

nonparametric likelihood statistics. In particular, we show that adequate choices of tuning constants lead to Bartlett correctability of the nonparametric likelihood statistic.

5.1. General nonparametric likelihood. We first consider the benchmark setup in Section 2. As a general family of nonparametric likelihood functions, we employ the power divergence family (Cressie and Read, 1984)

$$L_\gamma(w_1, \dots, w_n) = \begin{cases} \frac{2}{\gamma(\gamma+1)} \sum_{i=1}^n \{(nw_i)^{\gamma+1} - 1\} & \text{if } \gamma \neq -1, 0, \\ -2 \sum_{i=1}^n \log(nw_i) & \text{if } \gamma = -1, \\ 2n \sum_{i=1}^n w_i \log(nw_i) & \text{if } \gamma = 0. \end{cases}$$

Based on $L_\gamma(w_1, \dots, w_n)$ and using the estimating equation for the realized volatility $\hat{\theta} = \sum_{i=1}^n r_i^2$, we specify the likelihood function for the integrated volatility θ as

$$\ell_{\gamma, \phi}(\theta) = L_\gamma(w_{\phi,1}, \dots, w_{\phi,n}), \quad (5.1)$$

where the weights $w_{\phi,1}, \dots, w_{\phi,n}$ solve

$$\min_{w_1, \dots, w_n} L_\phi(w_1, \dots, w_n), \quad \text{subject to } \sum_{i=1}^n w_i = 1, \quad \sum_{i=1}^n w_i(nr_i^2 - \theta) = 0. \quad (5.2)$$

Note that the nonparametric likelihood function $\ell_{\gamma, \phi}(\theta)$ contains two tuning constants, γ and ϕ . In the literature, it is commonly assumed $\gamma = \phi$. For example, the empirical likelihood function discussed so far corresponds to $\gamma = \phi = -1$, and Pearson's χ^2 corresponds to $\gamma = \phi = -2$. Also Baggerly (1998) showed that in the class of likelihood functions with $\gamma = \phi$, only empirical likelihood is Bartlett correctable for the mean of i.i.d. data. On the other hand, Schennach (2005, 2007) considered the case of $\gamma \neq \phi$ and studied the exponentially tilted empirical likelihood statistic with $\gamma = -1$ and $\phi = 0$ from Bayesian and frequentist perspectives. In the current setup where we employ the infill asymptotics, it is crucial to consider the general class of $\ell_{\gamma, \phi}(\theta)$ indexed by γ and ϕ to achieve Bartlett correction. Below we will show that even if the volatility process σ is constant, the empirical likelihood statistic (i.e., $\ell_{\gamma, \phi}(\theta)$ with $\gamma = \phi = -1$) is not Bartlett correctable under the infill asymptotics, and the constants γ and ϕ need to be chosen separately to achieve Bartlett correction.

By the Lagrange multiplier argument, the solution of (5.2) is (see, Baggerly, 1998)

$$w_{\phi,i} = \frac{1}{n} (1 + \eta + \lambda(nr_i^2 - \theta))^{\frac{1}{\phi}}, \quad (5.3)$$

for $\phi \neq 0$ and $w_{\phi,i} = \frac{1}{n} \eta \exp(\lambda(nr_i^2 - \theta))$ for $\phi = 0$, where η and λ solve

$$\frac{1}{n} \sum_{i=1}^n (1 + \eta + \lambda(nr_i^2 - \theta))^{\frac{1}{\phi}} = 1, \quad \frac{1}{n} \sum_{i=1}^n (1 + \eta + \lambda(nr_i^2 - \theta))^{\frac{1}{\phi}} (nr_i^2 - \theta) = 0, \quad (5.4)$$

for $\phi \neq 0$ and solve $\frac{1}{n} \sum_{i=1}^n \eta \exp(\lambda(nr_i^2 - \theta)) = 1$ and $\frac{1}{n} \sum_{i=1}^n \eta \exp(\lambda(nr_i^2 - \theta))(nr_i^2 - \theta) = 0$ for $\phi = 0$. In practice, we use (5.3) to compute the likelihood function in (5.1).

The first-order asymptotic distribution of $\ell_{\gamma,\phi}(\theta)$ is obtained as follows.

Theorem 5. *Suppose Assumption X holds true. For each $\gamma, \phi \in \mathbb{R}$, as $n \rightarrow \infty$,*

$$T_{\gamma,\phi}(\theta) = \frac{3}{2} \left(1 - \frac{R_2^2}{R_4} \right) \ell_{\gamma,\phi}(\theta) \xrightarrow{d} \chi_1^2.$$

Note that the first-order asymptotic distribution of the statistic $T_{\gamma,\phi}(\theta)$ is identical to the one in Theorem 1 for empirical likelihood. Moreover, the first-order asymptotic distribution does not depend on the tuning constants γ and ϕ . In the next subsection, we study second-order asymptotic properties of the statistic $T_{\gamma,\phi}(\theta)$ to compare different choices of γ and ϕ . For the first-order asymptotics, similar modifications can be applied to $T_{\gamma,\phi}(\theta)$ to be robust to jumps and microstructure noise.

5.2. Second-order asymptotics. The first-order asymptotic theory for the nonparametric likelihood statistic $T_{\gamma,\phi}(\theta)$ is silent about the choice of tuning constants γ and ϕ . In order to address this issue, we investigate the second-order asymptotic property of $T_{\gamma,\phi}(\theta)$. Following the conventional recipe put forward in DiCiccio, Hall and Romano (1991) and Baggerly (1998), among others, we first derive the signed root of the nonparametric likelihood statistic, and then evaluate the cumulants of the signed root. Based on these cumulants, we seek values of γ and ϕ at which the third and fourth cumulants vanish at sufficiently fast rates to achieve Bartlett correction. Details are provided in the web appendix (proofs of Theorems 6 and 7).

For the second-order analysis, we add the following assumption.

Assumption H. *The process X follows (2.1) with $\mu = 0$ and σ is independent of W and bounded away from zero.*

This assumption is restrictive since it rules out the drift term and leverage effect. Gonçalves and Meddahi (2009, p. 289) imposed a similar but stronger assumption for higher-order analysis of the bootstrap inference. Although the drift term μ is asymptotically negligible at the first-order, it will appear in the higher-order terms and complicates our second-order analysis. Ruling out the leverage effect (i.e., independence between σ and W) also simplifies our second-order analysis since it allows to condition on the path of σ to compute the cumulants of the nonparametric likelihood statistic. Relaxing Assumption H for the second-order analysis is beyond the scope of this paper.

To simplify the exposition of our results, first we consider the simple case where the volatility is constant ($\sigma_t = \sigma$ over $t \in [0, 1]$). In this setting, the higher order properties of the nonparametric likelihood statistic are presented in the next theorem.

Theorem 6. *Suppose Assumptions X and H hold true and $\sigma_t = \sigma$ over $t \in [0, 1]$. Then, for $\gamma = -1$ and $\phi = -1 \pm \frac{\sqrt{5}}{3}$, the nonparametric likelihood statistic $T_{\gamma, \phi}(\theta)$ is Bartlett correctable, i.e., conditionally on the path of σ ,*

$$\Pr \left\{ T_{\gamma, \phi}(\theta) \leq \chi_{1, \alpha}^2 (1 + 3n^{-1}) \right\} = 1 - \alpha + O(n^{-2}).$$

This theorem says that when we choose $\gamma = -1$ and $\phi = -1 \pm \frac{\sqrt{5}}{3}$, the nonparametric likelihood test based on $T_{\gamma, \phi}(\theta)$ using the adjusted critical value $\chi_{1, \alpha}^2 (1 + 3n^{-1})$ provides a refinement to the order $O(n^{-2})$ on the null rejection probability error. It should be noted that the empirical likelihood statistic (i.e., $T_{\gamma, \phi}(\theta)$ with $\gamma = \phi = -1$) is not Bartlett correctable because the fourth cumulant of the signed root does not vanish at the order of $O(n^{-4})$ (see the proof of Theorem 6 in the web appendix). Also note that the Bartlett factor $1 + 3n^{-1}$ does not contain any unknown object.

Finally, we drop the assumption of constant volatility and consider the general case. Although the computations are quite cumbersome, it is possible to estimate some tuning constants $\hat{\gamma}$ and $\hat{\phi}$ such that the nonparametric likelihood statistic $T_{\hat{\gamma}, \hat{\phi}}(\theta)$ is Bartlett correctable. The higher order properties of the nonparametric likelihood statistic in the general case are presented in the next theorem.

Theorem 7. *Suppose Assumptions X and H hold true. Then, for $\hat{\gamma}$, $\hat{\phi}$, and a (defined in (A.12), (A.15), and (A.16), respectively, in the web appendix), the nonparametric likelihood statistic $T_{\hat{\gamma}, \hat{\phi}}(\theta)$ is Bartlett correctable, i.e., conditionally on the path of σ ,*

$$\Pr \left\{ T_{\hat{\gamma}, \hat{\phi}}(\theta) \leq \chi_{1, \alpha}^2 (1 + an^{-1}) \right\} = 1 - \alpha + O(n^{-2}).$$

This theorem says that even for the general case, the nonparametric likelihood statistic $T_{\hat{\gamma}, \hat{\phi}}(\theta)$ with the estimated tuning constants $\hat{\gamma}$ and $\hat{\phi}$ using the adjusted critical value $\chi_{1, \alpha}^2 (1 + an^{-1})$ provides a refinement to the order $O(n^{-2})$ on the null rejection probability error. In the general case, the Bartlett factor a can be estimated by the method of moments or wild bootstrap as in Gonçalves and Meddahi (2009). For the one-sided test, Gonçalves and Meddahi (2009) obtained second-order refinement by the bootstrap to the order $o(n^{-1/2})$. In contrast, we consider the two-sided test and show that our Bartlett correction to the nonparametric likelihood statistic can yield a refinement to the order $O(n^{-2})$.

6. SIMULATION

This section conducts simulation studies in order to evaluate finite sample properties of the empirical likelihood methods presented above.

6.1. Simulation 1: Benchmark case. We adopt simulation designs considered in Gonçalves and Meddahi (2009). In particular, we consider the stochastic volatility model

$$dX_t = \mu_t dt + \sigma_t \left(\rho_1 dW_{1t} + \rho_2 dW_{2t} + \sqrt{1 - \rho_1^2 - \rho_2^2} dW_{3t} \right),$$

where W_{1t} , W_{2t} , and W_{3t} are independent standard Brownian motions.

First, we consider a general case (i.e. with drift and leverage effects) to illustrate the first-order asymptotic theory in Theorem 1 for the nonparametric likelihood statistic $T_{\gamma, \phi}(\theta)$. We consider two different models for the volatility process σ_t . The first model for σ_t is the GARCH(1,1) diffusion

$$d\sigma_t^2 = 0.035(0.636 - \sigma_t^2)dt + 0.144\sigma_t^2 dW_{1t}.$$

The second model is the two-factor diffusion model

$$\sigma_t = f(-1.2 + 0.04\sigma_{1t}^2 + 1.5\sigma_{2t}^2),$$

where $d\sigma_{1t}^2 = -0.00137\sigma_{1t}^2 dt + dW_{1t}$, $d\sigma_{2t}^2 = -1.386\sigma_{2t}^2 dt + (1 + 0.25\sigma_{2t}^2)dW_{2t}$, and

$$f(x) = \begin{cases} \exp(x) & x \leq x_0 \\ \frac{\exp(x_0)}{\sqrt{x_0}} \sqrt{x_0 - x_0^2 + x^2} & x > x_0 \end{cases}$$

with $x_0 = \log(1.5)$. We allow for drift and leverage effects by setting $\mu_t = 0.0314$, $\rho_1 = -0.576$, and $\rho_2 = 0$ for GARCH(1,1) models, and $\mu_t = 0.030$ and $\rho_1 = \rho_2 = -0.30$ for the two-factor diffusion model.

We compare three methods to construct two-sided 95% confidence intervals: (i) the Wald-type interval (Wald), (ii) empirical likelihood (EL) and (iii) nonparametric likelihood (NL) with $\gamma = -1$ and $\phi = -1 + \frac{\sqrt{5}}{3}$.

Table 1 gives the actual coverage rates of all the intervals across 10,000 replications for five different sample sizes: $n = 1152, 288, 48, 24,$ and 12 , corresponding to 1.25-minute, 5-minute, half-hour, 1-hour, and 2-hour returns. The Wald-type intervals tend to undercover for both models. The degree of undercoverage is especially large when sampling is not too frequent. The two-factor model implies overall larger coverage distortions than the GARCH(1,1) model. The nonparametric likelihood intervals (including EL intervals) outperform the Wald-type intervals in all cases.

As we discussed in Remark 3, the nonparametric likelihood intervals are range preserving but the Wald-type confidence interval may contain negative values. To illustrate this point, we report the frequencies of negative left endpoints of the Wald-type confidence intervals in Table 2. This shows that the Wald-type intervals tend to contain negative values particularly for small sample sizes.

Second, we consider two special cases to illustrate the second-order refinements proposed in the last section: (a) a benchmark model where volatility is constant, and (b)

models where volatility is not constant (with no drift term and no leverage effect). Bartlett corrected nonparametric likelihood (BNL) with the Bartlett correction factor $1 + 3/n$ are compared with the above methods. Table 3 shows that the Bartlett corrected nonparametric likelihood intervals outperform all the other intervals even when there is stochastic volatility despite the fact that this correction does not theoretically provide an asymptotic refinement under the non-constant volatility case.

6.2. Simulation 2: Test for jump. In this subsection we evaluate finite sample properties of the nonparametric likelihood tests for the presence of jumps. We adopt simulation designs considered in Dovonon, Gonçalves, Hounyo and Meddahi (2014). In particular, we consider the two-factor diffusion model with diurnality effects.

$$\begin{aligned} d \log S_t &= \mu_t dt + \sigma_{u,t} \sigma_t (\rho_1 dW_{1t} + \rho_2 dW_{2t} + \sqrt{1 - \rho_1^2 - \rho_2^2} dW_{3t}) + dJ_t, \\ \sigma_{u,t} &= 0.88929198 + 0.75 \exp(-10t) + 0.25 \exp(-10(1-t)), \\ \sigma_t &= f(-1.2 + 0.04\sigma_{1t}^2 + 1.5\sigma_{2t}^2), \end{aligned}$$

where $d\sigma_{1t}^2 = -0.00137\sigma_{1t}^2 dt + dW_{1t}$, $d\sigma_{2t}^2 = -1.386\sigma_{2t}^2 dt + (1 + 0.25\sigma_{2t}^2)dW_{2t}$, and

$$f(x) = \begin{cases} \exp(x) & x \leq x_0 \\ \frac{\exp(x_0)}{\sqrt{x_0}} \sqrt{x_0 - x_0^2 + x^2} & x > x_0 \end{cases}$$

with $x_0 = \log(1.5)$. The process $\sigma_{u,t}$ models the diurnal U-shaped pattern in intraday volatility. When $\sigma_{u,t} = 1$ for $t \in [0, 1]$, the return process reduces to the simple case of no diurnally effects. J_t is a finite activity jump process modeled as a compound Poisson process with constant jump intensity λ and random jump size distributed as $N(0, \sigma_{jump}^2)$. Under the null hypothesis of no jumps in the return process, we set $\sigma_{jump}^2 = 0$. Under the alternative hypothesis, we set $\lambda = 0.058$ and $\sigma_{jump}^2 = 1.7241$.

We compare three methods to test for jumps: (i) the Wald-type test (Wald),¹ (ii) (signed root) empirical likelihood (EL) and (iii) (signed root) nonparametric likelihood (NL) with $\gamma = -1$ and $\phi = -1 + \frac{\sqrt{5}}{3}$.² We consider five different sample sizes: $n = 1152$, 576, 288, 96, and 48 corresponding to 1.25-minute, 2.5-minute, 5-minute, 15-minute, and half-hour returns. All results are based on 1,000 Monte Carlo replications.

Table 4 reports the rejection frequencies of tests at the 5% nominal significance level for both cases with and without diurnally effects. The Wald-type test tends to over-reject for the both cases, the degree of which is especially large when sampling is not too

¹We define the statistic $T_n = \frac{\sqrt{n}(RV_n - BV_n)}{\sqrt{\hat{V}_n}}$ where $RV_n = \sum_{i=1}^n r_i^2$, $BV_n = \frac{1}{\mu_1^2} \sum_{i=2}^n |r_i| |r_{i-1}|$ and $\hat{V}_n = \{(\mu_1^{-4} + 2\mu_1^{-2} - 3) - 2\} \frac{n}{\mu_{4/3}^3} \sum_{i=3}^n |r_i|^{4/3} |r_{i-1}|^{4/3} |r_{i-2}|^{4/3}$. Then, the test rejects the null of no jumps at significance level α when $T_n > z_{1-\alpha}$ where $z_{1-\alpha}$ is the 100(1 - α)% percentile of the $N(0, 1)$ distribution.

²We define the signed root of nonparametric likelihood ratio statistic as $NL_* = \text{sgn}(RV_n - TV_n) \bar{\ell}_{\gamma, \phi}^{1/2}$ where $TV_n = \frac{1}{\mu_{2/3}^3} \sum_{i=3}^n |r_i|^{2/3} |r_{i-1}|^{2/3} |r_{i-2}|^{2/3}$. Then, the test rejects the null of no jumps at significance level α when $NL_* > z_{1-\alpha}$ where $z_{1-\alpha}$ is the 100(1 - α)% percentile of the $N(0, 1)$ distribution.

frequent. In all cases, the nonparametric likelihood tests (including EL tests) shows better performance in the null rejection frequencies. The rejection frequencies varies between 7.20% ($n = 1152$) and 17.27% ($n = 48$) for Wald, while it varies between 5.10% ($n=1152$) and 8.54% ($n = 48$) for EL and between 4.68% ($n = 1152$) and 8.83% ($n = 48$) for NL.

We also analyze the power properties of the proposed tests under the alternative hypothesis. We compare the calibrated powers of three tests (i.e., the rejection frequencies of these tests where the critical values are given by the Monte Carlo 95% percentiles of these test statistics under the data generation process satisfying the null hypothesis). Table 5 shows that the nonparametric likelihood tests is slightly less powerful than Wald. Since the nonparametric likelihood tests have better null rejection properties than Wald, these power properties characterize a tradeoff between the null rejection and power properties of Wald-type and nonparametric likelihood tests.

6.3. Simulation 3: Noise robust test. In this subsection we evaluate finite sample properties of the noise robust nonparametric likelihood tests. We adopt simulation designs considered in Jacod *et al.* (2009). In particular, we consider two different models for the X processes. The first model is the constant volatility model

$$X_t = X_0 + \sigma W_t,$$

with $\sigma = 0.2/\sqrt{252}$. The second model is the stochastic volatility model of Heston (1993):

$$\begin{aligned} dX_t &= (\mu - \nu_t/2)dt + \sigma_t dB_t, \\ d\nu_t &= \kappa(\alpha - \nu_t)dt + \gamma\nu_t^{1/2}dW_t, \end{aligned}$$

where $\nu_t = \sigma_t^2$, $\mu = 0.05/252$, $\kappa = 5/252$, $\alpha = 0.04/252$, $\gamma = 0.05/252$ and $Corr(B, W) = -0.5$. As for the microstructure noise, we assume that U_t is i.i.d. and follows $N(0, 0.0005^2)$.

We compare three methods to construct two-sided 95% confidence intervals: (i) the Wald-type interval (Wald),³ (ii) empirical likelihood (EL) and (iii) nonparametric likelihood (NL) with $\gamma = -1$ and $\phi = -1 + \frac{\sqrt{5}}{3}$. For the choice of the block length K , we used $K = \lfloor \frac{n^{1/2}}{6} \rfloor$ so that $c \approx \frac{1}{3}$, following Jacod *et al.* (2009).

Table 6 gives the actual coverage rates of all the intervals across 10,000 replications for eight different sample sizes: $n = 23400, 11700, 7800, 4680, 1560, \text{ and } 780$. The Wald-type intervals tend to undercover for both models. The degree of undercoverage is especially large when sampling is not too frequent. The nonparametric likelihood intervals (including EL intervals) outperform the Wald-type interval in all cases.

³The $100(1 - \alpha)\%$ asymptotic confidence interval for the integrated volatility θ is given by $CI_W^\alpha = \{\theta : T_n(\theta) \leq \chi_{1,\alpha}^2\}$, where $T_n(\theta) = \frac{n^{1/2}(\hat{\theta} - \theta)^2}{R_4^*}$ and $\chi_{1,\alpha}^2$ is the $(1 - \alpha)$ -th quantile of the χ_1^2 distribution.

7. REAL DATA EXAMPLE

We consider tick prices obtained from TickData consisting of intra-day quotes of Alcoa, American Express, Baxter, Citigroup, Dow, Gilead, Goldman Sachs, Intel Corporation, Met, Microsoft, Nike, Pfizer, Verizon and Yahoo from January 2, 2001 to November 15, 2010, which corresponds to 2472 trading days.

Table 7 reports the mean and standard deviation of daily returns based on 5-min intra-day returns. We can observe common features across assets belonging to the same market segment. Negative returns are likely linked to the extraordinary events of the recent financial crisis in 2008-9. Furthermore, Table 8 reports the percentage of days identified with jumps for the period under investigation. As in the Monte Carlo analysis presented in the previous section, we consider three methods to test for jumps: (i) the Wald-type test (Wald), (ii) (signed root) empirical likelihood (EL), and (iii) (signed root) nonparametric likelihood (NL) with $\gamma = -1$ and $\phi = -1 + \frac{\sqrt{5}}{3}$. In line with the Monte Carlo findings, we note that the Wald test tends to over detect the jumps. Indeed, the percentage of days identified with jumps is always larger than 21%. EL and NL implies very similar empirical findings. Using nonparametric likelihood procedures, the percentage of days identified with jumps is always smaller than 13%.

APPENDIX A. TABLES

| n | Wald | EL | NL | Wald | EL | NL |
|------|----------------------|-------|-------|----------------------|-------|-------|
| | GARCH(1,1) diffusion | | | Two-factor diffusion | | |
| 12 | 80.83 | 84.80 | 84.48 | 73.24 | 78.45 | 78.02 |
| 24 | 86.97 | 90.34 | 90.03 | 80.61 | 85.65 | 85.23 |
| 48 | 90.41 | 92.76 | 92.46 | 85.76 | 89.38 | 89.04 |
| 288 | 94.55 | 94.98 | 94.92 | 93.52 | 94.50 | 94.35 |
| 1152 | 94.72 | 94.83 | 94.79 | 94.91 | 95.31 | 95.22 |

TABLE 1. Coverage probabilities of nominal 95% confidence intervals for integrated volatility with leverage and drift

| | 95% | 99% | 99.9% | 95% | 99% | 99.9% |
|------|----------------------|-------|-------|----------------------|-------|-------|
| n | GARCH(1,1) diffusion | | | Two-factor diffusion | | |
| 12 | 80.64 | 99.85 | 100 | 91.86 | 99.96 | 100 |
| 24 | 6.94 | 43.60 | 95.44 | 40.61 | 79.32 | 99.02 |
| 48 | 0.01 | 0.47 | 7.21 | 8.75 | 25.59 | 58.14 |
| 288 | 0 | 0 | 0 | 0 | 0.04 | 0.23 |
| 1152 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE 2. Frequencies (measured by percentages) of negative left endpoints of 95%, 99%, and 99.9% Wald confidence intervals for integrated volatility with leverage and drift

| n | Wald | EL | NL | BNL |
|----------------------|-------|-------|-------|-------|
| Constant volatility | | | | |
| 12 | 81.20 | 85.18 | 84.77 | 87.46 |
| 24 | 87.63 | 90.66 | 90.35 | 92.00 |
| 48 | 91.04 | 93.54 | 93.14 | 94.08 |
| 288 | 94.24 | 94.85 | 94.78 | 94.89 |
| 1152 | 95.27 | 95.39 | 95.34 | 95.40 |
| GARCH(1,1) diffusion | | | | |
| 12 | 81.39 | 85.29 | 85.02 | 87.73 |
| 24 | 87.51 | 90.89 | 90.61 | 92.04 |
| 48 | 90.98 | 93.51 | 93.19 | 93.89 |
| 288 | 94.44 | 94.97 | 94.87 | 94.97 |
| 1152 | 95.07 | 95.18 | 95.14 | 95.18 |
| Two-factor diffusion | | | | |
| 12 | 73.74 | 77.97 | 77.63 | 80.87 |
| 24 | 80.90 | 85.72 | 85.33 | 87.06 |
| 48 | 86.05 | 86.69 | 89.45 | 90.32 |
| 288 | 92.83 | 94.08 | 93.95 | 94.08 |
| 1152 | 94.22 | 95.04 | 94.99 | 95.02 |

TABLE 3. Coverage probabilities of nominal 95% confidence intervals for integrated volatility with no drift and no leverage

| n | Wald | EL | NL | Wald | EL | NL |
|------|-------------------------|------|------|----------------------|------|------|
| | without diurnal effects | | | with diurnal effects | | |
| 48 | 17.27 | 8.54 | 8.83 | 16.24 | 7.82 | 8.04 |
| 96 | 13.34 | 6.79 | 6.98 | 12.56 | 6.57 | 6.80 |
| 288 | 9.83 | 5.94 | 6.08 | 9.76 | 5.70 | 5.80 |
| 576 | 8.70 | 5.79 | 5.85 | 8.58 | 5.59 | 5.42 |
| 1152 | 7.53 | 5.38 | 5.27 | 7.20 | 5.10 | 4.68 |

TABLE 4. Rejection frequencies of tests at 5% level

| n | Wald | EL | NL | Wald | EL | NL |
|------|-------------------------|-------|-------|----------------------|-------|-------|
| | without diurnal effects | | | with diurnal effects | | |
| 48 | 78.40 | 72.59 | 72.59 | 75.86 | 68.05 | 68.23 |
| 96 | 84.93 | 80.94 | 80.94 | 82.75 | 75.86 | 76.22 |
| 288 | 90.19 | 86.75 | 86.93 | 87.47 | 83.12 | 83.30 |
| 576 | 92.01 | 88.92 | 89.65 | 91.28 | 85.66 | 85.66 |
| 1152 | 93.10 | 90.38 | 90.74 | 91.65 | 88.38 | 88.56 |

TABLE 5. Calibrated power

| n | Wald | EL | NL | Wald | EL | NL |
|-------|---------------------|-------|-------|--------------|-------|-------|
| | constant volatility | | | Heston Model | | |
| 780 | 85.79 | 89.33 | 89.18 | 85.41 | 89.52 | 89.30 |
| 1560 | 90.41 | 92.40 | 92.28 | 91.36 | 93.55 | 93.33 |
| 4680 | 93.19 | 94.40 | 94.26 | 94.06 | 95.26 | 95.15 |
| 7800 | 93.84 | 94.53 | 94.46 | 94.88 | 95.50 | 95.41 |
| 11700 | 94.34 | 95.19 | 95.11 | 95.18 | 95.74 | 95.66 |
| 23400 | 94.78 | 95.26 | 95.22 | 95.54 | 95.79 | 95.77 |

TABLE 6. Coverage probabilities of nominal 95% confidence intervals for integrated volatility

| | Mean $\times 10^{-4}$ | SD $\times 10^{-4}$ |
|-------------------|-----------------------|---------------------|
| Alcoa | -6.17 | 215.13 |
| American Express | 3.48 | 201.84 |
| Baxter | 3.16 | 135.96 |
| Citigroup | -1.82 | 266.59 |
| Dow | 4.37 | 184.85 |
| Gilead | -7.68 | 207.68 |
| Goldman Sachs | 2.67 | 200.02 |
| Intel Corporation | -2.12 | 195.60 |
| Met | -5.90 | 216.37 |
| Microsoft | -5.47 | 153.43 |
| Nike | 7.36 | 149.27 |
| Pfizer | -6.70 | 139.92 |
| Verizon | 7.44 | 148.23 |
| Yahoo | -3.69 | 249.70 |

TABLE 7. Mean and standard deviation of daily returns based on 5-min intra-day returns for the period 2001-2010.

| | Wald | EL | NL |
|-------------------|-------|-------|-------|
| Alcoa | 23.68 | 8.70 | 9.95 |
| American Express | 22.15 | 8.07 | 9.18 |
| Baxter | 25.29 | 9.90 | 11.21 |
| Citigroup | 24.32 | 10.87 | 12.04 |
| Dow | 25.13 | 9.74 | 11.01 |
| Gilead | 23.64 | 7.16 | 8.28 |
| Goldman Sachs | 21.59 | 5.68 | 6.67 |
| Intel Corporation | 22.99 | 8.45 | 9.87 |
| Met | 25.68 | 11.49 | 12.60 |
| Microsoft | 23.51 | 9.82 | 11.03 |
| Nike | 24.80 | 8.90 | 9.86 |
| Pfizer | 23.11 | 10.63 | 12.04 |
| Verizon | 24.77 | 8.66 | 9.94 |
| Yahoo | 25.25 | 9.98 | 11.13 |

TABLE 8. Percentage of days identified with jumps for the period 2001-2010.

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WEB APPENDIX TO “EMPIRICAL LIKELIHOOD FOR HIGH FREQUENCY DATA”

ABSTRACT. In this appendix, we present proofs of the theorems in the paper.

APPENDIX A. MATHEMATICAL APPENDIX

Throughout the appendix, let $R_q = n^{q/2-1} \sum_{i=1}^n |r_i|^q$, $\bar{\sigma}_q = \int_0^1 \sigma_u^q du$, and $\mu_q = E|z|^q$ with $z \sim N(0, 1)$ for $q > 0$.

A.1. Proof of Theorems 1 and 5. Theorem 1 is a special case of Theorem 5. Since the proof is similar, we focus on the case of $\gamma, \phi \neq -1, 0$. From Barndorff-Nielsen *et al.* (2006, Theorem 1), Assumption X guarantees

$$R_q \xrightarrow{p} \mu_q \bar{\sigma}_q, \quad (\text{A.1})$$

for any $q > 0$. This implies $R_2 \xrightarrow{p} \bar{\sigma}_2$, $R_4 \xrightarrow{p} \mu_4 \bar{\sigma}_4$, and

$$\frac{3}{2} \left(1 - \frac{R_2^2}{R_4} \right) \xrightarrow{p} \frac{3}{2} \left(\frac{\mu_4 \bar{\sigma}_4 - \bar{\sigma}_2^2}{\mu_4 \bar{\sigma}_4} \right). \quad (\text{A.2})$$

Let $g_i = nr_i^2 - \theta$, $\bar{g} = \frac{1}{n} \sum_{i=1}^n g_i$, and $\bar{V} = \frac{1}{n} \sum_{i=1}^n g_i^2$. By (A.1) and Barndorff-Nielsen *et al.* (2006), we obtain

$$\left(\frac{2}{3} \mu_4 \bar{\sigma}_4 \right)^{-1/2} \sqrt{n} \bar{g} = \left(\frac{2}{3} \mu_4 \bar{\sigma}_4 \right)^{-1/2} \sqrt{n} (R_2 - \bar{\sigma}_2) \xrightarrow{d} N(0, 1), \quad (\text{A.3})$$

$$\bar{V} = R_4 - 2\bar{\sigma}_2 R_2 + \bar{\sigma}_2^2 \xrightarrow{p} \mu_4 \bar{\sigma}_4 - \bar{\sigma}_2^2. \quad (\text{A.4})$$

By these results combined with $E[g_i^2] < \infty$ for all $i = 1, \dots, n$, we can apply the same argument to Owen (1988) to show $\max_{1 \leq i \leq n} |\eta + \lambda g_i| \xrightarrow{p} 0$. Thus, by expanding

$$\frac{1}{n} \sum_{i=1}^n (1 + \eta + \lambda(nr_i^2 - \theta))^{\frac{1}{\phi}} = 1, \quad \frac{1}{n} \sum_{i=1}^n (1 + \eta + \lambda(nr_i^2 - \theta))^{\frac{1}{\phi}} (nr_i^2 - \theta) = 0, \quad (\text{A.5})$$

around $(\eta, \lambda) = (0, 0)$, we obtain

$$\begin{aligned} \lambda &= -\phi \bar{V}^{-1} \bar{g} + O_p(n^{-1}), \\ \eta &= \frac{1}{2} \phi (\phi + 1) \bar{V}^{-1} \bar{g}^2 + O_p(n^{-2}). \end{aligned}$$

Based on these results, an expansion of $\ell_{\gamma, \phi}(\theta)$ around $(\eta, \lambda) = (0, 0)$ yields

$$\ell_{\gamma, \phi}(\theta) = \frac{2}{\gamma(\gamma + 1)} \sum_{i=1}^n \left\{ (1 + \eta + \lambda g_i)^{\frac{\gamma+1}{\phi}} - 1 \right\} = \bar{V}^{-1} (\sqrt{n} \bar{g})^2 + O_p(n^{-1}).$$

Therefore, the conclusion follows by (A.2)-(A.4).

A.2. Proofs of Theorems 2 and 3. Proofs of these theorems are similar to that of Theorem 5 above.

First, we show Theorem 2. Let $\tilde{g}_i = n|\tilde{r}_{i-m+1}|^{p_1} \cdots |\tilde{r}_i|^{p_m} - c_{\mathbf{p}}\theta$. By Barndorff-Nielsen, Shephard and Winkel (2006, Theorem 1), we can replace (A.3) and (A.4) with

$$(d_{\mathbf{p}}\bar{\sigma}_4)^{-1/2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{g}_i \xrightarrow{d} N(0, 1),$$

$$\frac{1}{n} \sum_{i=1}^n \tilde{g}_i^2 \xrightarrow{p} c_{2\mathbf{p}}\bar{\sigma}_4 - c_{\mathbf{p}}^2\bar{\sigma}_2^2,$$

respectively. The remaining part is similar.

Next, we show Theorem 3. Let $\bar{g}_i = n(|\tilde{r}_{i-m+1}|^{p_1} \cdots |\tilde{r}_i|^{p_m} - c_{\mathbf{p}}\tilde{r}_i^2)$. Under the null hypothesis of no jump (i.e. $Y = X$), we can apply the limit theorems in Barndorff-Nielsen *et al.* (2006). Thus we can replace (A.3) and (A.4) with

$$\left(\{d_{\mathbf{p}} - 2 \sum_{l=1}^m c_{\mathbf{p}}[c_{\bar{\mathbf{p}}_l} - c_{\mathbf{p}}] + 2c_{\mathbf{p}}^2\} \bar{\sigma}_4 \right)^{-1/2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \bar{g}_i \xrightarrow{d} N(0, 1),$$

$$\frac{1}{n} \sum_{i=1}^n \bar{g}_i^2 \xrightarrow{p} (c_{2\mathbf{p}} + 3c_{\mathbf{p}}^2 - 2c_{\mathbf{p}}c_{\bar{\mathbf{p}}})\bar{\sigma}_4,$$

respectively. The remaining part is similar. Under the alternative of the presence of jumps, Barndorff-Nielsen, Shephard and Winkel (2006, Theorem 1) implies that $\frac{1}{n} \sum_{i=1}^n \bar{g}_i$ converges to a non-zero constant. Therefore, the test statistic \tilde{T}_{EL}^J diverges as $n \rightarrow \infty$.

A.3. Proof of Theorem 4. The proof is similar to that of Theorem 5 above. By Jacod *et al.* (2009, Theorem 3.1), we obtain

$$\frac{n^{1/2}}{n_K^2 R_4^*} \left\{ \sum_{i=1}^{n_K} (g_{Ki} - \theta) \right\}^2 = \frac{n^{1/2}(\bar{\theta} - \theta)^2}{R_4^*} \xrightarrow{d} \chi_1^2. \quad (\text{A.6})$$

Also, by inspection of the derivations in Jacod *et al.* (2009), we can obtain

$$\frac{3}{2K^2} \hat{\theta} \xrightarrow{p} \frac{12}{c^2} \int_0^1 U_s^2 ds, \quad \frac{6}{K} \bar{R}_2 \xrightarrow{p} \theta + \frac{12}{c^2} \int_0^1 U_s^2 ds.$$

These results imply

$$\frac{1}{n_K} \sum_{i=1}^{n_K} (g_{Ki} - \theta)^2 - \frac{36}{K^2} (\bar{R}_4 - \bar{R}_2^2) = -2 \frac{6}{K} \bar{R}_2 \left(\frac{3}{2K^2} \hat{\theta} + \theta \right) + \left(\frac{3}{2K^2} \hat{\theta} + \theta \right)^2 + \frac{36}{K^2} \bar{R}_2^2 \xrightarrow{p} 0. \quad (\text{A.7})$$

By (A.6) and (A.7), a similar argument to the proof of Theorem 5 yields

$$\bar{T}_{EL}(\theta) = \frac{36n^{1/2}}{n_K K^2} \frac{\bar{R}_4 - \bar{R}_2^2}{R_4^*} \frac{\left\{ \sum_{i=1}^{n_K} (g_{Ki} - \theta) \right\}^2}{\sum_{i=1}^{n_K} (g_{Ki} - \theta)^2} + o_p(1) \xrightarrow{d} \chi_1^2.$$

A.4. Proof of Theorem 6. Due to independence between σ and W , the symbols such as $E[\cdot]$ and $O_p(\cdot)$ mean the conditional expectation and stochastic order given the path of σ , respectively. Before analyzing Bartlett correctability of the nonparametric likelihood statistic, we introduce further notation. We transform the moment function as $m_i = V^{-1/2}(nr_i^2 - \theta)$ with $V = E[n^{-1} \sum_{i=1}^n (nr_i^2 - \theta)^2]$ and define

$$\bar{A}_k = \frac{1}{n} \sum_{i=1}^n m_i^k, \quad \alpha_k = E[\bar{A}_k], \quad A_k = \bar{A}_k - \alpha_k,$$

for $k = 1, 2, \dots$. Note that Assumption H implies

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad A_k = O_p(n^{-1/2}),$$

for each $k = 1, 2, \dots$, where the first equality follows from $E[r_i^2] = \int_{(i-1)/n}^{i/n} \sigma_u^2 du$, the second equality follows by construction, and the third equality follows from Barndorff-Nielsen *et al.* (2006, Theorem 2).

Based on the above notation, the nonparametric likelihood statistic is rewritten as $\ell_{\gamma, \phi}(\theta) = L_{\gamma}(w_{\phi,1} \dots, w_{\phi,n})$, where

$$w_{\phi,i} = \frac{1}{n}(1 + \eta + \tilde{\lambda}m_i)^{\frac{1}{\phi}},$$

and η and $\tilde{\lambda}$ solves

$$\frac{1}{n} \sum_{i=1}^n (1 + \eta + \tilde{\lambda}m_i)^{\frac{1}{\phi}} = 1, \quad \frac{1}{n} \sum_{i=1}^n (1 + \eta + \tilde{\lambda}m_i)^{\frac{1}{\phi}} m_i = 0.$$

Expansions of these equations around $\eta + \tilde{\lambda}w_i = 0$ and repeated substitutions yield expansions of η and $\tilde{\lambda}$ as follows

$$\begin{aligned} \eta &= \frac{1}{2}\phi(1 + \phi)A_1^2 + \frac{1}{6}\phi(1 + \phi)(1 - \phi)\alpha_3A_1^3 - \frac{1}{2}\phi(1 + \phi)A_1^2A_2 \\ &\quad + \frac{1}{2}\phi(1 + \phi)A_1^2A_2^2 - \frac{1}{2}\phi(1 - \phi)(1 + \phi)\alpha_3A_1^3A_2 + \frac{1}{6}\phi(1 - \phi)(1 + \phi)A_1^3A_3 \\ &\quad + \frac{1}{8}\phi \left\{ (1 + \phi)^3 + (1 - \phi)^2(1 + \phi)\alpha_3^2 - \frac{1}{3}(1 - \phi)(1 + \phi)(1 - 2\phi)\alpha_4 \right\} A_1^4 + O_p(n^{-5/2}). \end{aligned}$$

and

$$\begin{aligned} \tilde{\lambda} &= -\phi A_1 - \frac{1}{2}\phi(1 - \phi)\alpha_3A_1^2 + \phi A_1A_2 \\ &\quad - \phi A_1A_2^2 + \frac{3}{2}\phi(1 - \phi)\alpha_3A_1^2A_2 - \frac{1}{2}\phi(1 - \phi)A_1^2A_3 \\ &\quad - \frac{1}{2}\phi \left\{ \phi(1 + \phi) + (1 - \phi)^2\alpha_3^2 - \frac{1}{3}(1 - \phi)(1 - 2\phi)\alpha_4 \right\} A_1^3 + O_p(n^{-2}). \end{aligned}$$

By inserting these formulae to an expansion of $n^{-1}\ell_{\gamma,\phi}(\theta)$ around $\eta + \tilde{\lambda}w_i = 0$, we obtain

$$\begin{aligned}
& n^{-1}\ell_{\gamma,\phi}(\theta) \\
&= A_1^2 + \frac{1}{3}(1-\gamma)\alpha_3 A_1^3 - A_1^2 A_2 + A_1^2 A_2^2 - (1-\gamma)\alpha_3 A_1^3 A_2 + \frac{1}{3}(1-\gamma)A_1^3 A_3 \\
&+ \left\{ \left(\frac{1}{4} + \frac{\gamma}{2} + \frac{\gamma\phi}{2} - \frac{\phi^2}{4} \right) + \left(\frac{1}{4} - \frac{\gamma}{2} + \frac{\gamma\phi}{2} - \frac{\phi^2}{4} \right) \alpha_3^2 + \left(-\frac{1}{12} + \frac{\gamma}{4} + \frac{\gamma^2}{12} - \frac{\gamma\phi}{2} + \frac{\phi^2}{4} \right) \alpha_4 \right\} A_1^4 \\
&+ O_p(n^{-5/2}). \tag{A.8}
\end{aligned}$$

Let $\bar{\sigma}_{q,n} = n^{q/2-1} \sum_{i=1}^n \left(\int_{(i-1)/n}^{i/n} \sigma_u^2 du \right)^{q/2}$. For the term $\frac{3}{2} \left(1 - \frac{R_2^2}{R_4} \right)$, expansions around $R_4 = 3\bar{\sigma}_{4,n}$ and $R_2 = \theta$ yield

$$\begin{aligned}
\frac{3}{2} \left(1 - \frac{R_2^2}{R_4} \right) &= \frac{1}{2} \frac{3\bar{\sigma}_{4,n} - \theta^2}{\bar{\sigma}_{4,n}} + \frac{1}{2} \frac{\theta^2}{\bar{\sigma}_{4,n}} \left(\frac{R_4}{3\bar{\sigma}_{4,n}} - 1 \right) - \frac{1}{2} \frac{\theta^2}{\bar{\sigma}_{4,n}} \left(\frac{R_4}{3\bar{\sigma}_{4,n}} - 1 \right)^2 \\
&- \frac{V^{1/2}\theta}{\bar{\sigma}_{4,n}} A_1 - \frac{1}{2} \frac{V}{\bar{\sigma}_{4,n}} A_1^2 + \frac{2}{3} \frac{V\theta^2}{\bar{\sigma}_{4,n}^2} A_1^2 + \frac{1}{3} \frac{V^{3/2}\theta}{\bar{\sigma}_{4,n}^2} A_1 A_2 + O_p(n^{-3/2}). \tag{A.9}
\end{aligned}$$

where $\frac{R_4}{3\bar{\sigma}_{4,n}} - 1 = \frac{2}{3} \frac{V^{1/2}\theta}{\bar{\sigma}_{4,n}} A_1 + \frac{1}{3} \frac{V}{\bar{\sigma}_{4,n}} A_2$.

Consider the constant volatility case $\sigma_t = \sigma$ over $t \in [0, 1]$. In this case, it holds

$$\sigma^2 = \theta, \quad \bar{\sigma}_{4,n} = \theta^2, \quad V = 2\theta^2, \quad \alpha_3 = 2\sqrt{2}, \quad \alpha_4 = 15. \tag{A.10}$$

Then by (A.8) and (A.9), the expansion of the nonparametric likelihood statistic $n^{-1}T_{\gamma,\phi}(\theta)$ is written as

$$\begin{aligned}
n^{-1}T_{\gamma,\phi}(\theta) &= A_1^2 - \frac{2\sqrt{2}}{3}\gamma A_1^3 - \frac{2}{3}A_1^2 A_2 + \frac{4}{9}A_1^2 A_2^2 - \frac{2\sqrt{2}}{9}(19 - 8\gamma)A_1^3 A_2 \\
&+ \frac{1}{3}(1-\gamma)A_1^3 A_3 + \left(\frac{41}{36}\gamma - 3\gamma\phi + \frac{5}{4}\gamma^2 + \frac{3}{2}\phi^2 \right) A_1^4 + O_p(n^{-5/2}).
\end{aligned}$$

As in Baggerly (1998), to achieve Bartlett correction, we investigate the conditions of γ and ϕ where the third and fourth cumulants of the signed root of the above expansion vanish at sufficiently fast rates.

First, we consider the third cumulant. After some algebra, the signed root form is obtained as $n^{-1}T_{\gamma,\phi}(\theta) = (S_1 + S_2 + S_3)^2 + O_p(n^{-5/2})$, where

$$S_1 = A_1, \quad S_2 = -\frac{1}{3}A_1 A_2 - \frac{\sqrt{2}}{3}\gamma A_1^2,$$

and $S_3 = O_p(n^{-3/2})$ is not displayed since it is not used to compute the third cumulant. Based on this form, the third cumulant of $S_1 + S_2 + S_3$ is obtained as

$$\kappa_3(\gamma, \phi) = E[S_1^3] + 3E[S_1^2 S_2] - 3E[S_1^2]E[S_2] + O(n^{-3}),$$

where by Lemma 2,

$$\begin{aligned} E[S_1^3] &= 2\sqrt{2}n^{-2} + O(n^{-3}), & E[S_1^2 S_2] &= -\sqrt{2}(\gamma + 2)n^{-2} + O(n^{-3}), \\ E[S_1^2]E[S_2] &= -\frac{\sqrt{2}}{3}(\gamma + 2)n^{-2} + O(n^{-3}). \end{aligned}$$

Therefore, if $\gamma = -1$, then the dominant term of the third cumulant vanishes and it holds $\kappa_3(-1, \phi) = O(n^{-3})$.

Next, we set $\gamma = -1$ and analyze the fourth cumulant. After some algebra, the signed root form of $n^{-1}T_{\gamma, \phi}(\theta)$ with $\gamma = -1$ is obtained as $n^{-1}T_{-1, \phi}(\theta) = (T_1 + T_2 + T_3)^2 + O_p(n^{-5/2})$, where

$$\begin{aligned} T_1 &= A_1, & T_2 &= -\frac{1}{3}A_1A_2 + \frac{\sqrt{2}}{3}A_1^2, \\ T_3 &= \frac{1}{6}A_1A_2^2 - \frac{11\sqrt{2}}{9}A_1^2A_2 + \frac{1}{3}A_1^2A_3 + \left(\frac{3}{4}\phi^2 + \frac{3}{2}\phi - \frac{1}{18}\right)A_1^3. \end{aligned}$$

Then the fourth cumulant of $T_1 + T_2 + T_3$ is obtained as

$$\begin{aligned} \kappa_4(-1, \phi) &= E[T_1^4] + 4E[T_1^3T_2] + 4E[T_1^3T_3] - 3(E[T_1^2])^2 \\ &\quad + 6E[T_1^2T_2^2] - 4E[T_1^3]E[T_2] - 12E[T_1^2T_2]E[T_2] - 6E[T_1^2]E[T_2^2] \\ &\quad + 12E[T_1^2](E[T_2])^2 - 12E[T_1^2]E[T_1T_2] - 12E[T_1^2]E[T_1T_3] + O(n^{-4}), \end{aligned}$$

where by Lemma 2,

$$\begin{aligned} E[T_1^4] &= 3n^{-2} + 12n^{-3} + O(n^{-4}), & E[T_1^3T_2] &= -\frac{76}{3}n^{-3} + O(n^{-4}), \\ E[T_1^3T_3] &= \left\{ \frac{74}{3} + \frac{15}{4} \left(3\phi^2 + 6\phi - \frac{2}{9} \right) \right\} n^{-3} + O(n^{-4}), & (E[T_1^2])^2 &= n^{-2}, \\ E[T_1^2T_2^2] &= \frac{16}{3}n^{-3} + O(n^{-4}), & E[T_1^3]E[T_2] &= -\frac{4}{3}n^{-3} + O(n^{-4}), \\ E[T_1^2T_2]E[T_2] &= \frac{2}{3}n^{-3} + O(n^{-4}), & E[T_1^2]E[T_2^2] &= \frac{4}{3}n^{-3} + O(n^{-4}), \\ E[T_1^2](E[T_2])^2 &= \frac{2}{9}n^{-3} + O(n^{-4}), & E[T_1^2]E[T_1T_2] &= -\frac{10}{3}n^{-3} + O(n^{-4}), \\ E[T_1^2]E[T_1T_3] &= \left\{ \frac{16}{3} + \frac{3}{4} \left(3\phi^2 + 6\phi - \frac{2}{9} \right) \right\} n^{-3} + O(n^{-4}). \end{aligned}$$

Therefore, if

$$9\phi^2 + 18\phi + 4 = 0,$$

i.e. $\phi = -1 \pm \frac{\sqrt{5}}{3}$, then the dominant term of the fourth cumulant vanishes and it holds $\kappa_4\left(-1, -1 \pm \frac{\sqrt{5}}{3}\right) = O(n^{-4})$.

Finally, by setting $\gamma = -1$ and $\phi = -1 \pm \frac{\sqrt{5}}{3}$, it holds

$$\begin{aligned} E[T_1^2] &= n^{-1}, & E[T_1 T_2] &= -\frac{10}{3}n^{-2} + O(n^{-3}), \\ E[T_2^2] &= \frac{4}{3}n^{-2} + O(n^{-3}), & E[T_1 T_3] &= \frac{25}{6}n^{-2} + O(n^{-3}), \end{aligned}$$

and thus the second cumulant used to compute the Bartlett correction factor is obtained as

$$nE[(T_1 + T_2 + T_3)^2] = 1 + 3n^{-1} + O(n^{-2}).$$

A.5. Proof of Theorem 7. We now drop the assumption of constant volatility. In the general case, the identities in (A.10) do not apply. Thus the objects such as V , α_3 , and α_4 become unknown and need to be estimated. In this case, by (A.8) and (A.9), the expansion of the nonparametric likelihood statistic $n^{-1}T_{\gamma,\phi}(\theta)$ is written as

$$\begin{aligned} n^{-1}T_{\gamma,\phi}(\theta) &= \frac{1}{2}cA_1^2 + \frac{1}{6}c^{1/2} \{c^{1/2}(1-\gamma)\alpha_3 + 2d^{3/2} - 6d^{1/2}\} A_1^3 + \frac{1}{6}c(d-3)A_1^2A_2 \\ &\quad + \frac{1}{18}c(9-3d-cd)A_1^2A_2^2 + \frac{1}{6}c(1-\gamma)A_1^3A_3 \\ &\quad + \frac{1}{18}c^{1/2} \{c^{1/2}(d-9)(1-\gamma)\alpha_3 + 18d^{1/2} + 9cd^{1/2} - 6d^{3/2} - 4cd^{3/2}\} A_1^3A_2 \\ &\quad + \frac{1}{18} \{2c^{1/2}d^{1/2}(d-3)(1-\gamma)\alpha_3 - 9c + 12cd - 4cd^2 + 9cf\} A_1^4 + O_p(n^{-5/2}), \end{aligned}$$

where

$$\begin{aligned} c &= \frac{V}{\bar{\sigma}_{4,n}}, & d &= \frac{\theta^2}{\bar{\sigma}_{4,n}}, \\ f &= \left(\frac{1}{4} + \frac{\gamma}{2} + \frac{\gamma\phi}{2} - \frac{\phi^2}{4} \right) + \left(\frac{1}{4} - \frac{\gamma}{2} + \frac{\gamma\phi}{2} - \frac{\phi^2}{4} \right) \alpha_3^2 + \left(-\frac{1}{12} + \frac{\gamma}{4} + \frac{\gamma^2}{12} - \frac{\gamma\phi}{2} + \frac{\phi^2}{4} \right) \alpha_4. \end{aligned}$$

First, we consider the third cumulant. After some algebra, the signed root form is obtained as $n^{-1}T_{\gamma,\phi}(\theta) = (S_1 + S_2 + S_3)^2 + O_p(n^{-5/2})$, where

$$S_1 = \frac{\sqrt{2}}{2}c^{1/2}A_1, \quad S_2 = \frac{\sqrt{2}}{12} \{c^{1/2}(1-\gamma)\alpha_3 - 6d^{1/2} + 2d^{3/2}\} A_1^2 + \frac{\sqrt{2}}{12}c^{1/2}(d-3)A_1A_2,$$

and $S_3 = O_p(n^{-3/2})$ is not displayed since it is not used to compute the third cumulant.

Based on this form, the third cumulant of $S_1 + S_2 + S_3$ is obtained as

$$\kappa_3(\gamma, \phi) = E[S_1^3] + 3E[S_1^2S_2] - 3E[S_1^2]E[S_2] + O(n^{-3}),$$

where by Lemma 1,

$$\begin{aligned} E[S_1^3] &= \frac{2\sqrt{2}}{15} \{c^{3/2}\alpha_3 + 9d^{1/2} - 2d^{3/2}\} n^{-2} + O(n^{-3}), \\ E[S_1^2 S_2] &= \frac{\sqrt{2}}{10} \{5c^{-1/2}(1-\gamma)\alpha_3 - 2c^{3/2}\alpha_3 - 18d^{1/2} + 4d^{3/2}\} n^{-2} + O(n^{-3}), \\ E[S_1^2]E[S_2] &= \frac{\sqrt{2}}{30} \{5c^{-1/2}(1-\gamma)\alpha_3 - 2c^{3/2}\alpha_3 - 18d^{1/2} + 4d^{3/2}\} n^{-2} + O(n^{-3}). \end{aligned}$$

Therefore, if we set γ as

$$\gamma^* = 1 - \frac{4}{15}c^2 - \frac{12}{5} \frac{c^{1/2}d^{1/2}}{\alpha_3} + \frac{8}{15} \frac{c^{1/2}d^{3/2}}{\alpha_3}, \quad (\text{A.11})$$

then it holds $\kappa_3(\gamma^*, \phi) = O(n^{-3})$. Note that under the constant volatility case, the equation (A.11) reduces to $\gamma^* = -1$. In the general case, however, γ^* depends on unknown objects c , d , and α_3 . By replacing these objects with consistent estimators, we propose the data-dependent value of γ :

$$\hat{\gamma} = 1 - \frac{4}{15}\hat{c}^2 - \frac{12}{5} \frac{\hat{c}^{1/2}\hat{d}^{1/2}}{\hat{\alpha}_3} + \frac{8}{15} \frac{\hat{c}^{1/2}\hat{d}^{3/2}}{\hat{\alpha}_3}, \quad (\text{A.12})$$

where $\hat{c} = \frac{\hat{V}}{\hat{\sigma}_{4,n}}$, $\hat{d} = \frac{\theta^2}{\hat{\sigma}_{4,n}}$, $\hat{\alpha}_3 = \hat{V}^{-3/2} \frac{1}{n} \sum_{i=1}^n (nr_i^2 - \theta)^3$, $\hat{V} = \frac{1}{n} \sum_{i=1}^n (nr_i^2 - \theta)^2$, and $\hat{\sigma}_{4,n} = \frac{1}{3}(\hat{V} + \theta^2)$. Since $\hat{\gamma} - \gamma^* = O_p(n^{-1/2})$, we need to take the estimation error of $\hat{\gamma}$ into account for the second-order analysis below.

Next, we rederive the stochastic expansion of $n^{-1}T_{\hat{\gamma},\phi}(\theta)$ with $\hat{\gamma}$ in (A.12). By expanding $\hat{\gamma}$ around $(\hat{c}, \hat{d}, \hat{\alpha}_3) = (c, d, \alpha_3)$, it holds

$$\hat{\gamma} = \gamma^* + gA_2 + hA_3 + O_p(n^{-1}),$$

where

$$\begin{aligned} g &= -\frac{8}{15}c^2 + \frac{8}{45}c^3 - \frac{6}{5} \frac{c^{1/2}d^{1/2}}{\alpha_3} + \frac{4}{5} \frac{c^{3/2}d^{1/2}}{\alpha_3} + \frac{4}{15} \frac{c^{1/2}d^{3/2}}{\alpha_3} - \frac{16}{45} \frac{c^{3/2}d^{3/2}}{\alpha_3}, \\ h &= \frac{4}{15}(9 - 2d) \frac{c^{1/2}d^{1/2}}{\alpha_3^2}. \end{aligned}$$

By using this expansion of $\hat{\gamma}$, we can rewrite the expansion of the nonparametric likelihood statistic as

$$\begin{aligned} &n^{-1}T_{\hat{\gamma},\phi}(\theta) \\ &= \frac{1}{2}cA_1^2 + \frac{1}{6}c^{1/2} \{c^{1/2}(1-\gamma^*)\alpha_3 + 2d^{3/2} - 6d^{1/2}\} A_1^3 + \frac{1}{6}c(d-3)A_1^2A_2 \\ &\quad + \frac{1}{18}c(9-3d-cd)A_1^2A_2^2 + \frac{1}{6}c(1-\gamma^* - h\alpha_3)A_1^3A_3 \\ &\quad + \frac{1}{18}c^{1/2} \{c^{1/2}((d-9)(1-\gamma^*) - 3g)\alpha_3 + 18d^{1/2} - 6d^{3/2} + 9cd^{1/2} - 4cd^{3/2}\} A_1^3A_2 \\ &\quad + \frac{1}{18} \{2c^{1/2}d^{1/2}(d-3)(1-\gamma^*)\alpha_3 - 9c + 12cd - 4cd^2 + 9cf\} A_1^4 + O_p(n^{-5/2}). \end{aligned} \quad (\text{A.13})$$

After some algebra, the signed root form is obtained as $n^{-1}T_{\hat{\gamma},\phi}(\theta) = (T_1 + T_2 + T_3)^2 + O_p(n^{-5/2})$, where

$$\begin{aligned} T_1 &= \frac{\sqrt{2}}{2}c^{1/2}A_1, & T_2 &= jA_1^2 + kA_1A_2, \\ T_3 &= lA_1A_2^2 + qA_1^2A_3 + mA_1^2A_2 + \zeta(\phi)A_1^3, \end{aligned}$$

and

$$\begin{aligned} j &= \frac{\sqrt{2}}{12} \{c^{1/2}(1-\gamma)\alpha_3 + 2d^{3/2} - 6d^{1/2}\}, & k &= \frac{\sqrt{2}}{12}c^{1/2}(d-3), \\ l &= \frac{\sqrt{2}}{36}c^{1/2}(9-3d-cd) - \frac{\sqrt{2}}{144}c^{1/2}(d-3)^2, \\ m &= \frac{\sqrt{2}}{72} \{c^{1/2}(d-15)(1-\gamma)\alpha_3 + 18d^{1/2} - 2d^{5/2} + 18cd^{1/2} - 8cd^{3/2} - 6c^{1/2}g\alpha_3\}, \\ q &= \frac{\sqrt{2}}{12}c^{1/2}(1-\gamma-h\alpha_3), \\ \zeta(\phi) &= \frac{\sqrt{2}}{36} \left\{ 9c^{1/2}f - 9c^{1/2} + 12c^{1/2}d - 4c^{1/2}d^2 - \frac{1}{4}c^{1/2}(1-\gamma)^2\alpha_3^2 - c^{-1/2}d^3 - 9c^{-1/2}d + 6c^{-1/2}d^2 \right\}. \end{aligned}$$

By the definition of γ^* , we can show that the third cumulant of $T_1 + T_2 + T_3$ satisfies $\kappa_3(\hat{\gamma}, \phi) = O(n^{-3})$. After lengthy calculations, by using the expectations in Lemma 1, the fourth cumulant

$$\begin{aligned} \kappa_4(\hat{\gamma}, \phi) &= E[T_1^4] + 4E[T_1^3T_2] + 4E[T_1^3T_3] - 3(E[T_1^2])^2 \\ &\quad + 6E[T_1^2T_2^2] - 4E[T_1^3]E[T_2] - 12E[T_1^2T_2]E[T_2] - 6E[T_1^2]E[T_2^2] \\ &\quad + 12E[T_1^2](E[T_2])^2 - 12E[T_1^2]E[T_1T_2] - 12E[T_1^2]E[T_1T_3] + O(n^{-4}) \end{aligned}$$

is written in the form of

$$\kappa_4(\hat{\gamma}, \phi) = \xi_1\zeta(\phi) + \xi_2 + O(n^{-4}), \quad (\text{A.14})$$

where ξ_1 and ξ_2 are implicitly defined and do not depend on ϕ . Although $\zeta(\phi)$, ξ_1 and ξ_2 contain unknown objects c , d , and α_3 , they can be estimated by \hat{c} , \hat{d} , and $\hat{\alpha}_3$, respectively (denote by $\hat{\zeta}(\phi)$, $\hat{\xi}_1$ and $\hat{\xi}_2$). Then if the solution exists, the ideal value $\hat{\phi}$ is given by a solution of

$$\hat{\xi}_1\hat{\zeta}(\hat{\phi}) + \hat{\xi}_2 = 0. \quad (\text{A.15})$$

It should be noted that in the expansion (A.13), ϕ appears only in the term f . Therefore, the estimation error $\hat{\phi} - \phi$ is of negligible order $O_p(n^{-5/2})$, and it holds $\kappa_4(\hat{\gamma}, \hat{\phi}) = O(n^{-4})$, i.e., the dominant term of the fourth cumulant vanishes if we choose $\hat{\gamma}$ and $\hat{\phi}$ as in (A.12) and (A.15), respectively.

Finally, we compute the second cumulant and Bartlett factor. Using the expectations in Lemma 1, we have

$$\begin{aligned} E[T_1^2] &= \frac{c}{2}E[A_1^2] = n^{-1}, & E[T_1T_2] &= rn^{-2} + O(n^{-3}), \\ E[T_2^2] &= sn^{-2} + O(n^{-3}), & E[T_1T_3] &= tn^{-2} + O(n^{-3}), \end{aligned}$$

where (recall $\bar{\sigma}_{q,n} = n^{q/2-1} \sum_{i=1}^n \left(\int_{(i-1)/n}^{i/n} \sigma_u^2 du \right)^{q/2}$)

$$\begin{aligned} r &= 4\sqrt{2}c^{1/2}V^{-2} \{j\bar{\sigma}_{6,n}V^{1/2} + k(9\bar{\sigma}_{8,n} - 2\theta\bar{\sigma}_{6,n})\}, \\ s &= 12j^2\bar{\sigma}_{4,n}^2V^{-2} + 2jk(72\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} - 24\theta\bar{\sigma}_{4,n}^2)V^{-5/2} \\ &\quad + k^2(192\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} + 288\bar{\sigma}_{6,n}^2 - 288\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + 48\theta^2\bar{\sigma}_{4,n}^2)V^{-3}, \\ t &= 24\sqrt{2}c^{1/2}l(4\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} + 6\bar{\sigma}_{6,n}^2 - 6\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + \theta^2\bar{\sigma}_{4,n}^2)V^{-3} \\ &\quad + 18\sqrt{2}c^{1/2}q(15\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} - 6\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + \theta^2\bar{\sigma}_{4,n}^2)V^{-3} \\ &\quad + 6\sqrt{2}c^{1/2}V^{-5/2} \{m(6\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} - 2\theta\bar{\sigma}_{4,n}^2) + \zeta(\phi)\bar{\sigma}_{4,n}V^{1/2}\}. \end{aligned}$$

Thus, the second cumulant used to compute the Bartlett correction factor is obtained as

$$nE[(T_1 + T_2 + T_3)^2] = 1 + an^{-1} + O(n^{-2}), \quad (\text{A.16})$$

where $a = 2(r + t) + s$.

A.6. Lemmas . Here we present some approximation formulae for the moments of A_k . Lemma 1 is derived under Assumptions X and H, which allows non-constant volatility. Lemma 2 is derived for the constant volatility case. The proofs are available from the authors upon request.

Lemma 1. *Suppose Assumptions X and H hold true. Then*

$$\begin{aligned}
E[A_1] &= 0, & E[A_1^2] &= 2\bar{\sigma}_{4,n}V^{-1}n^{-1}, & E[A_1^3] &= 8\bar{\sigma}_{6,n}V^{-3/2}n^{-2}, \\
E[A_1^4] &= 12\bar{\sigma}_{4,n}^2V^{-2}n^{-2} + 48\bar{\sigma}_{8,n}V^{-2}n^{-3}, & E[A_1^5] &= 160\bar{\sigma}_{4,n}\bar{\sigma}_{6,n}V^{-5/2}n^{-3} + O(n^{-4}), \\
E[A_1^6] &= 120\bar{\sigma}_{4,n}^3V^{-3}n^{-3} + O(n^{-4}), & E[A_1A_2] &= (12\bar{\sigma}_{6,n} - 4\theta\bar{\sigma}_{n,4})V^{-3/2}n^{-1} \\
E[A_1^2A_2] &= (72\bar{\sigma}_{8,n} - 16\theta\bar{\sigma}_{6,n})V^{-2}n^{-2}, \\
E[A_1^3A_2] &= (72\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} - 24\theta\bar{\sigma}_{4,n}^2)V^{-5/2}n^{-2} + O(n^{-3}), \\
E[A_1^4A_2] &= (384\bar{\sigma}_{6,n}^2 + 864\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} - 320\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n})V^{-3}n^{-3} + O(n^{-4}), \\
E[A_1^5A_2] &= (720\bar{\sigma}_{4,n}^2\bar{\sigma}_{6,n} - 240\theta\bar{\sigma}_{4,n}^3)V^{-7/2}n^{-3} + O(n^{-4}), \\
E[A_1A_3] &= (90\bar{\sigma}_{8,n} - 36\theta\bar{\sigma}_{6,n} + 6\theta^2\bar{\sigma}_{4,n})V^{-2}n^{-1}, \\
E[A_1^3A_3] &= (540\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} - 216\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + 36\theta^2\bar{\sigma}_{4,n}^2)V^{-3}n^{-2} + O(n^{-3}), \\
E[A_1^5A_3] &= (5400\bar{\sigma}_{4,n}^2\bar{\sigma}_{8,n} - 2160\theta\bar{\sigma}_{4,n}^2\bar{\sigma}_{6,n} + 360\theta^2\bar{\sigma}_{4,n}^3)V^{-4}n^{-3} + O(n^{-4}), \\
E[A_1^2A_2^2] &= (192\bar{\sigma}_{4,n}\bar{\sigma}_{8,n} + 288\bar{\sigma}_{6,n}^2 - 288\theta\bar{\sigma}_{4,n}\bar{\sigma}_{6,n} + 48\theta^2\bar{\sigma}_{4,n}^2)V^{-3}n^{-2} + O(n^{-3}), \\
E[A_1^4A_2^2] &= (1152\bar{\sigma}_{4,n}^2\bar{\sigma}_{8,n} + 3456\bar{\sigma}_{4,n}\bar{\sigma}_{6,n}^2 - 2880\theta\bar{\sigma}_{4,n}^2\bar{\sigma}_{6,n} + 480\theta^2\bar{\sigma}_{4,n}^3)V^{-4}n^{-3} + O(n^{-4}).
\end{aligned}$$

Lemma 2. *Suppose Assumptions X and H hold true. Furthermore, assume that $\sigma_t = \sigma$ over $t \in [0, 1]$. Then*

$$\begin{aligned}
E[A_1] &= 0, & E[A_1^2] &= n^{-1}, & E[A_1^3] &= \alpha_3n^{-2}, & E[A_1^4] &= 3n^{-2} + (\alpha_4 - 3)n^{-3}, \\
E[A_1^5] &= 10\alpha_3n^{-3} + O(n^{-4}), & E[A_1^6] &= 15n^{-3} + O(n^{-4}), \\
E[A_1A_2] &= \alpha_3n^{-1}, & E[A_1^2A_2] &= (\alpha_4 - 1)n^{-2}, \\
E[A_1^3A_2] &= 3\alpha_3n^{-2} + O(n^{-3}), & E[A_1^4A_2] &= (6\alpha_4 + 4\alpha_3^2 - 6)n^{-3} + O(n^{-4}), \\
E[A_1^5A_2] &= 15\alpha_3n^{-3} + O(n^{-4}), & E[A_1A_3] &= \alpha_4n^{-1}, \\
E[A_1^3A_3] &= 3\alpha_4n^{-2} + O(n^{-3}), & E[A_1^5A_3] &= 15\alpha_4n^{-3} + O(n^{-4}), \\
E[A_1^2A_2^2] &= (\alpha_4 + 2\alpha_3^2 - 1)n^{-2} + O(n^{-3}), & E[A_1^4A_2^2] &= (3\alpha_4 + 12\alpha_3^2 - 3)n^{-3} + O(n^{-4}).
\end{aligned}$$

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