EFFICIENCY RENTS OF STORAGE PLANTS IN PEAK-LOAD PRICING, II: HYDROELECTRICITY

by

Anthony Horsley and Andrew J Wrobel
London School of Economics and Political Science

Contents:

Discussion Paper
No. TE/99/372
October 1999

The Suntory Centre
Suntory and Toyota International Centres
for Economics and Related Disciplines
London School of Economics and Political Science
Houghton Street
London WC2A 2AE
Tel.: 0171-405 7686

Financial support of the ESRC is gratefully acknowledged (grant 000232822).

Abstract

Duality methods of linear and convex programming are applied to impute definite marginal values to the fixed inputs of a hydroelectric plant from the operating profit. Our earlier analysis of pumped storage (of energy and other cyclically priced goods) is thus extended to valuation of an external inflow to the reservoir. Given a continuous time-of-use price for electricity, the profit-imputed hydro values are uniquely determined – unlike the corresponding values imputed from fuel savings for a mixed hydro-thermal system. In particular the water inflow is assigned a unique, time-dependent shadow price. The short-run profit is then differentiable in all the fixed inputs, so that unique and separate marginal values can be imputed to the reservoir and the turbine capacities (despite their perfect complementarity). The two rents can be expressed in terms of the shadow price for water (which determines the optimal storage policy). In particular, the unit reservoir rent equals the total positive variation of the shadow price over the cycle. Evaluation of profit-imputed rents is shown to be useful not only to a profit-maximising industry but also to a public utility aiming to price its outputs at long-run marginal cost and to optimise its capital stock on the basis of purely short-run calculations. In addition we verify the production set properties that are needed to incorporate such a storage problem into a continuous-time model of general competitive equilibrium with the space of bounded functions of time as the commodity space.

Keywords: storage hydro; rental valuation; peak-load pricing; linear programming.

JEL Nos.: C61, D24, D46, L94, L95.

1. Introduction

This is the second part of our study of storage rents in peak-load pricing. The first part [33] deals with pumped storage. This part treats a production technique whose running input is storable but comes as a fixed, periodic flow—such as the natural water inflow for hydroelectric generation (a.k.a. storage hydro). Although the two cases differ in some basic aspects of economic interest, the analysis uses the same methods. In both cases we use the operating profit function to impute a time-dependent value $\psi(t)$ to the stock in question, and hence also to the relevant fixed inputs. In the hydro case the shadow price function ψ is unique, and so are the associated capacity values (Theorem 4.9).[1] Earlier work is reviewed in Sections 9 and 10.

For its general approach—viz., a treatment of rents in continuous time—this study takes inspiration from Koopmans' paper [39] on optimal water storage policies for a hydro-thermal electricity generating system. Our work is set in recent advances in equilibrium theory, and it takes advantage of modern optimisation techniques. This enables us to use simple and direct methods to derive unique, profit-imputed rental values. By contrast, Koopmans' rents are imputed from savings on the operating costs and are typically nonunique in the most important cases. This does not impede Koopmans' main purpose, which is to verify the cost-optimality of a directly constructed storage policy. But both the nonuniqueness and the fact that his rents are given in terms of a complex operating solution are obstacles to their use in practical investment analysis.

As we have implied, rental indeterminacy can be removed by the switch from a framework of operating-cost minimisation to that of operating-profit maximisation. The problem can be formulated as a linear programme, and the marginal values can be derived by the duality approach. We therefore focus on the dual to the operation problem, viz., on shadow pricing of water.

The operation of hydroelectric plants is perhaps the most-studied of storage problems with cyclically priced goods. Koopmans also addresses this question, but what he sets out is the economics of the problem, as distinct from its engineering and operational research aspects. His work is alone in its use of, and focus on, the concept of efficiency rents: valuation of the fixed inputs is the main conceptual tool he employs; and, indeed, value imputation and its uses can be seen as a theme of his study. He is able to show that the values he imputes (to the reservoir, the turbine, electricity and water) support the policy he constructs as short-run cost minimum for a combined hydro-thermal system. All this is achieved by heuristic uses of what were, at the time, new ideas in mathematical programming. We take the subject matter up by means of the convex calculus, programming and equilibrium theory developed since. Our main purposes are: (i) to give a mathematically complete account of rental values, with

[1] In pumped storage (with imperfect conversion) ψ is nonunique, but even in that case the associated capacity values are unique: see [33].

particular attention paid to the question of the differentiability of the SR profit and SR cost as functions of the fixed inputs; (ii) thus to make clear the feasibility—when the output good is differentiated over continuous time—of imputing separate values to what are perfectly complementary capital inputs (in the sense that no input substitution is possible once the other quantities, here the output and inflow trajectories over the cycle, are fixed); (iii) to put such problems within the compass of equilibrium theory set up in infinite-dimensional commodity and price spaces, viz., spaces of functions of continuous time.

As Koopmans [39, p. 194] emphasises, treating time as a continuous variable is of great help in the handling of integrals, etc. The need for continuous-time modelling is further borne out by our analysis, which shows that in this context time-continuity of output prices is essential for the differentiability of SR profit.

The change of framework from SR cost minimisation to SR profit maximisation means that rental values are imputed by increments to the SR profit Π_{SR} rather than by the corresponding decrements of the SR cost, C_{SR}. Although in equilibrium—i.e., when output prices are SR marginal costs (SRMC's)—SR profit and cost are usually regarded as equivalent for the purpose of rental valuation, this is not always so because profit can be differentiable (in the fixed inputs) when cost is not, as we point out in [32]. In such a case the cost-imputed unit rents are inherently nonunique, in part as a result of their dependence on a nonunique trajectory, p, of the SRMC.

This is so in Koopmans' problem of cost-minimising hydro-thermal despatch: his valuations of the fixed hydro capacities (the reservoir and the turbine) depend on the shadow prices $p(t)$ for electricity and $q(t)$ for water (or, more precisely, for the water's potential energy): see [39, (3.11) and (3.13)]. In Koopmans' analysis both would be unique if the SR cost curve of the thermal generating system were smooth over the entire range of loads, but—as we explain in Remark 7.2—this can fail even for a system in which each plant type is of infinitesimal size, and of course it could never be the case for a system consisting of a finite number of plant types with fixed unit fuel costs.

With a nondifferentiable (but convex) cost, the SR cost reduction from an extra investment $\Delta k = (\Delta k_1, \Delta k_2)$ can still be worked out, to first order, by making suitable choices from the permissible range of marginal values, viz., by taking the minimum of the increment's imputed value $r \cdot \Delta k$ over $r \in -\partial_k C_{\mathrm{SR}}$, where ∂ is the subdifferential: see (9.31). However, such calculations give weaker conclusions and can be considerably more complicated, since in such a case the incremental value of extra investment is only superadditive and *not* additive in the capacity increments, i.e., the cost-imputed value of Δk can be higher than the sum of values of Δk_1 and Δk_2: see (9.32).

For investment decisions, then, rental values give better guidance and are easier to use when they are unique; and our analysis avoids the indeterminacy that stems from the nonuniqueness of shadow output prices by resetting the problem as one of competitive SR profit maximisation, in which the price function p is treated as given.

Another benefit from this formulation is that it allows a production technique with practically no operating cost—such as energy storage—to be analysed independently of the rest of the industry in question. By contrast, in SR cost minimisation such a technique can be studied only in conjunction with others that do have variable costs, such as the thermal fuel cost in Koopmans' treatment of hydro. The profit approach is therefore better suited to the more decentralised and less regulated structure of today's utilities, at least for economies in which market prices accurately reflect resource costs. Recent operational studies of hydro are in fact set up as SR profit maximum problems with given time-of-use (TOU) tariffs—although it has to be said that some of the simplified tariffs employed in [5] and [17] look questionable.

The use of Π_{SR} rather than C_{SR} is also advantageous for a publicly-owned monopoly aiming to price its output at long-run marginal cost (LRMC): this can be achieved through SRMC pricing when the fixed-input prices are equal to their profit-imputed marginal values: the conditions $p \in \partial_y C_{\mathrm{SR}}(y,k)$ and $r = \nabla_k \Pi_{\mathrm{SR}}(p,k)$, or $r \in \partial_k \Pi_{\mathrm{SR}}$ when the gradient fails to exist, imply together that $p \in \partial_y C_{\mathrm{LR}}(y,r)$. With fixed-coefficients techniques C_{SR} is usually nondifferentiable in k, and then $\partial_k C_{\mathrm{SR}}$ cannot replace $\partial_k \Pi_{\mathrm{SR}}$ for this purpose. See [32] for the general results, and Section 8 for the application to hydro.

Changing the framework to one of profit maximisation does not remove all the difficulties, since this does not by itself guarantee the existence of $\nabla_k \Pi_{\mathrm{SR}}$, although it does make it "more likely". In storage problems, although the output price function p is now given, one must also deal with a possible indeterminacy in the shadow price of stock ψ, which cannot be reinterpreted as a given market price. In the hydro context, ψ is the price for both the water in store and the river flow; it corresponds to Koopmans' q. At this point the argument becomes problem-specific; and it requires a detailed examination of the structure of Lagrange multipliers for the capacity constraints over time. This reveals that nonuniqueness of rental values can arise only when the output price p is discontinuous over time. Put another way, if $p(t)$ is continuous in t, then the marginal value of water $\psi(t)$ and the capacity rents are fully determinate. This means that the gradient of SR hydro profit $\Pi_{\mathrm{SR}}^{\mathrm{H}}$ with respect to the river flow e exists, and so do the derivatives of $\Pi_{\mathrm{SR}}^{\mathrm{H}}$ with respect to the storage and turbine capacities, k_{St} and k_{Tu}. Furthermore, $\nabla \Pi_{\mathrm{SR}}^{\mathrm{H}}$ equals ψ, which is independently defined as the dual solution; and both $\partial \Pi / \partial k_{\mathrm{St}}$ and $\partial \Pi / \partial k_{\mathrm{Tu}}$ can be given in terms of p and ψ (Theorem 4.9).

To obtain the rents in terms of the problem's data (p, k_{H}, e) additionally requires solving the dual to the operation problem for ψ or equivalently for the terms of ψ, which are, apart from a constant λ, the cumulatives of two measures κ^{St} and ν^{St} on the time interval that value the reservoir capacity and the nonnegativity constraint on stocks. The dual linear programme for (κ, ν, λ) is stated in Theorem 4.1; it can be handled numerically by standard algorithms. An equivalent convex programme for ψ, given in Proposition 4.4, is also tractable. After spelling out the Kuhn-Tucker Conditions (Proposition 4.3), we use these to establish the differentiability of $\Pi_{\mathrm{SR}}^{\mathrm{H}}$

in k (Theorem 4.9) and to derive the solution, in terms of the optimal ψ, to the primal problem of operation (Proposition 4.6). These are the main results of the paper. The optimal output y has the noteworthy property of being invariant under monotone transformations of the price function p (Remark 4.12). Finally, the dual pair of solutions (ψ, y) is spelt out for the practically most relevant case of a piecewise monotone and continuous p (Proposition 4.10).

Differentiability of Π_{SR} in $k = (k_1, k_2)$ means that the two capital inputs have a well-defined, unique rate of substitution in product value terms, viz., $\partial \Pi / \partial k_1 \div \partial \Pi / \partial k_2$. This is a striking property for inputs which are perfect Allen-Hicks complements, i.e., when the input demands conditional on the other quantities are price-independent, which is the case in hydro: the demands for turbine and reservoir are functions of the output bundle y and the inflow e alone.[2] It may therefore seem surprising that there is scope for capital input substitution at all; and this is possible only when the output is differentiated over time, as in our continuous-time model.

By contrast, in the simplest model of hydro—with discrete time and just two subperiods in a cycle ($t = 0, 1$)—the net energy output $y - e$ is effectively a scalar, since its proportions cannot be varied: $(y - e)(0) \div (y - e)(1) = -1$ always (Example 3.1). As a result, a term of the SR profit $\Pi_{\mathrm{SR}}^{\mathrm{H}}$ has the familiar fixed-coefficients form (3.8), which is of course nondifferentiable (as a function of k and e). The nondifferentiability can disappear only in the limit as subperiods are added and the mesh of discretisation decreases to zero; and the essential assumption of price continuity cannot even be stated neatly in discrete time. Thus the finite-dimensional model, in which p is perforce a step function, creates a wholly misleading impression that $\Pi_{\mathrm{SR}}^{\mathrm{H}}$ is "inherently" nonsmooth.

The continuity assumption on the electricity price p over time is not only a natural one to make, but is also verified for the competitive equilibrium. In [26] we prove that the equilibrium price function is continuous for a class of problems including peak-load pricing of thermally generated electricity; and this result can be extended to the case of hydro-thermal technology. This provides extra motivation to verify, in Lemmas 6.1 and 6.2, those properties of the hydro production set (2.4) which are needed for including the hydro technology in an Arrow-Debreu equilibrium model with $L^{\infty}[0, T]$, or its subspace $\mathcal{C}$ of continuous functions, as the commodity space. We set up such a model in [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [34], [35] and [36], in part by an application of Bewley's [7] framework. It is hoped that, in topics such as energy storage and peak-load pricing, this will lead to an integration of hitherto largely separate economic, engineering and OR studies. For example, the studies of Bauer et al. [5], Gfrerer [17] and Phu [44], though of considerable interest, are all OR work which does not address the economic issues of

[2]The turbine capacity requirement for y is $\check{k}_{\mathrm{Tu}}(y) = \mathrm{EssSup}(y)$. The storage capacity requirement, given y and e, is the infimum of the $\check{k}_{\mathrm{St}}(f)$ defined by (6.1) over f subject to $f \geq y - e$ and $\int_0^T f(t)\, \mathrm{d}t = 0$.

rental valuation and investment. Their adjoint solution seems to be of little interest to them except as a way of deriving the operating solution, and they do not point to its interpretation as the marginal value of water.[3]

The water pricing question is touched upon in some other studies, such as Munasinghe and Warford's [42] and Jacoby's [38], but these authors work with a different kind of model, in which the water price is made to be constant over the cycle. This makes hydro generation analytically similar to a thermal technique. Such models, discussed in Section 10, give very easy operating solutions, but only at the cost of ignoring a major feature of the problem, viz., the cyclic variation of the value of water.

Some aspects left out of the discussion here (such as hydraulic coupling of hydro plants on a common watershed) are included in El-Hawary and Christensen's account of cost-minimising hydro-thermal despatch [13, Chapters 5 and 6]. But theirs is another model in which the value of water, the "water conversion coefficient", is constant in the fixed-head case. When it varies, it is only as a result of head variation. A better treatment of the variable-head case is provided in the Austrian work [5], [17] and [44].

The formal analysis starts with a description of the hydro technology and the plant operation problem in Section 2. The use of duality to derive the rental values is first presented as a heuristic argument, in Section 3. This is formalised in Section 4 by the use of infinite linear programming (LP). The relevant vector spaces, etc., are introduced as needed; for a detailed review see [33].[4] Section 6 gives the supplementary results for setting up the general equilibrium model. In Section 9 we discuss Koopmans' analysis and its relationship to ours. Section 10 presents some simplified models with constant shadow prices for water.

2. The hydro technology and the profit-maximising operation problem

Hydro generation produces electricity, a cyclically priced nonstorable good, from a storable input of water.[5] We assume that a water stock, up to the reservoir's capacity k_{St}, can be held at no running cost or loss of stock. The height at which water is stored, called the *head*, determines its potential energy, which is first converted to

[3]Also, that work does not shows that the adjoint solution ψ is unique, although Phu [43, Satz 2] establishes that ψ is a continuous function of time (on the assumption that the electricity price p is continuously differentiable). As we show, uniqueness and continuity go together in this context.

[4]Some attention is paid to $\mu^{\mathrm{St}} = \kappa^{\mathrm{St}} - \nu^{\mathrm{St}}$ as a measure on a circle rather than on an interval, but this is a minor distinction which disappears when the electricity price p is *cyclically* continuous: if $p(T) = p(0)$, then $\psi(T) = \psi(0)$, where T is the length of the pricing cycle.

[5]The model also applies to other flows of natural energy, e.g., from geothermal sources. It can also be adapted to the case of tidal energy, but this requires changing the assumption that, when $s(t) > 0$, the output rate $y(t)$ is constrained only by k_{Tu} and is therefore independent of the inflow rate $e(t)$. The model is also applicable to supply of other goods, such as water and natural gas (when priced by TOU). In the case of water supply, $e(t)$ means the rainfall (collected in reservoirs), and its conversion to the consumable good consists in water purification and pumping to users.

kinetic energy in penstocks and then to electrical energy with a turbine-generator (or turbine for brevity). We assume the effective head to be fixed.[6] This means that at any time t the energy stock $s(t)$ is in a constant proportion to the water volume in store, and so the water's potential energy can simply be referred to as "water". Similarly the instantaneous rate of river flow into the reservoir, $e(t)$, can be measured in terms of power (instead of volume per unit time).

The turbine-generator's technical efficiency is also taken to be constant.[7] Therefore the water stock can be measured as the output it actually yields upon conversion (i.e., in kWh of electrical energy). The turbine capacity, k_{Tu}, is defined as the maximum output rate (in kW of electrical power); i.e., in unit time a unit turbine can convert 1 unit of stock into 1 unit of output.

The river flow e, varying cyclically over time, is assumed to be known with certainty. A cycle for prices, output and water flows is represented by an interval $[0, T]$ of the real line $\mathbb{R}$. In some applications the cycle can be a week, as in [17], but generally it is a year because of seasonal fluctuations.

The inflow rate e is a periodic function, which can usually be taken to be continuous. It suffices, however, to assume that e is bounded, i.e., that e belongs to $L^{\infty}[0, T]$, which is the vector space of all essentially bounded functions. This space is normed by the supremum norm

$$\|e\|_{\infty} := \mathrm{EssSup}\,|e| = \operatorname*{ess\ sup}_{t \in [0,T]} |e(t)| .$$

The hydro plant's output rate is also a periodic function of time, $y_{\mathrm{H}} \geq 0$, abbreviated to y. A *storage policy* consists in general of output and spillage $\varphi \geq 0$, but, except in Section 5, spillage is excluded here by the assumption that $k_{\mathrm{Tu}} \geq e$. This makes it feasible for the plant to "coast" at any time, i.e., to generate at a rate equal to the current inflow rate, $y(t) = e(t)$. It also means that the whole incentive to use the reservoir comes from the dependence of $p(t)$ on t: if p were a constant, the plant might as well coast all the time.

The net outflow from the reservoir is

$$f = -e + y_{\mathrm{H}} + \varphi \tag{2.1}$$

[6]The effective head is the difference between water surface levels in the reservoir and the tailrace, times penstock efficiency (which is ca. 98%): see, e.g., [13, 2.2.2].

[7]In reality the equipment is not perfectly divisible, and a hydroturbine's efficiency varies with the load, reaching 90% to 95% at full load. At one-quarter load it goes down to 80–85% for movable-blade types, or to 60–70% for fixed-blade types. The generator's efficiency is 90–95%.

and the stock, $s(t)$ at time t, is an absolutely continuous function on $[0,T]$ that satisfies the evolution equation $\dot{s} := \mathrm{d}s/\mathrm{d}t = -f$.[8] This can be restated as

$$s(t) = s(0) - \int_0^t f(\tau)\,\mathrm{d}\tau. \tag{2.2}$$

Since $k_{\mathrm{Tu}} \geq y \geq 0$, the output y belongs to $L^\infty[0,T]$, as do both e and φ (by assumption). So f also belongs to L^∞ by (2.1); and since $\dot{s} := -f$ with $\int_0^T f(t)\,\mathrm{d}t = 0$ (i.e., the flows to/from the reservoir must balance over the cycle), it follows that s belongs to

$$\mathrm{Lip}^{\mathrm{c}}[0,T] := \{s \in \mathrm{Lip}[0,T] : s(0) = s(T)\}, \tag{2.3}$$

the space of cyclic (a.k.a. periodic) Lipschitz functions on $[0,T]$. This is a subspace of $\mathcal{C}[0,T]$, the space of all continuous functions, normed by the maximum norm

$$\|s\|_\infty = \mathrm{Max}\,|s| = \max_{t\in[0,T]} |s(t)|.$$

This space is paired with $\mathcal{M}[0,T]$, the space of all (signed, finite) Borel measures on $[0,T]$, by means of the bilinear form $\langle \mu, s\rangle := \int_{[0,T]} s(t)\,\mu(\mathrm{d}t)$ for $s \in \mathcal{C}$ at $\mu \in \mathcal{M}$; and the norm-dual $\mathcal{C}^*$ is thus identified as $\mathcal{M}$. These and other spaces pertinent to cyclic problems are further discussed in [33, Appendix].

Because of maintenance schedules, etc., the *available* capacities (i.e., the capacities in service) may generally vary over the cycle even though the *installed* capacities are constant over the cycle. But in this analysis the available capacity is, for the most part, taken to equal the installed capacity (or, equivalently, to be a given, constant fraction thereof). The consequent constancy of the available capacity does play a part in some of the main results, including the determinacy of rental values (Theorem 4.9).

However, to exploit fully the framework of sensitivity analysis, the constant existing capacities k are perturbed with increments Δk which are (periodic) functions of time. This is further explained in Subsection 4.2.

On the assumption that the available capacities are constants $k_{\mathrm{H}} = (k_{\mathrm{St}}, k_{\mathrm{Tu}})$, the long-run (LR) production set of the storage hydro technique is the convex cone

$$\begin{aligned}\mathbb{Y}_{\mathrm{H}} := \big\{ &(y, -k_{\mathrm{H}}, -e) \in L^\infty_+ \times \mathbb{R}^2_- \times L^\infty_- : 0 \leq y \leq k_{\mathrm{Tu}}, \\ &\exists s \in \mathrm{Lip}^{\mathrm{c}}\ 0 \leq s \leq k_{\mathrm{St}} \text{ and } \exists \varphi \in [0,e]\ \ \dot{s} = e - y - \varphi \big\}.\end{aligned} \tag{2.4}$$

This formulation imposes the periodicity or balance constraint $s(T) = s(0)$ through (2.3), but the stock level at the beginning or end of cycle is taken to be a costless decision variable. In other words, when it is first commissioned, the reservoir comes filled up to any required level at no extra cost, but its periodic operation thereafter

[8] Since s is absolutely continuous, its derivative $\mathrm{d}s/\mathrm{d}t$ is well defined for almost every (a.e.) t. For these concepts see, e.g., [15], [49] or [56].

is taken to be a technological constraint.[9] As for the constraint $\varphi \leq e$, this is never binding (see Section 5), and it is included only to simplify a proof that $\mathbb{Y}_{\mathrm{H}}$ is weakly* closed.[10]

In these terms, the SR problem of profit-maximising operation of a hydro plant, with capacities $k_{\mathrm{H}} = (k_{\mathrm{St}}, k_{\mathrm{Tu}})$, is:

(2.5) Given $(p, k_{\mathrm{H}}, e) \in L^{\infty *}[0,T] \times \mathbb{R}^2 \times L^{\infty}[0,T]$

(2.6) maximise $\langle p, y \rangle$ over $y \in L^{\infty}$

(2.7) subject to $(y, -k_{\mathrm{H}}, -e) \in \mathbb{Y}_{\mathrm{H}}$ defined by (2.4).

The optimal value of (2.5)–(2.7) is the (maximum) operating profit of the hydro plant, denoted by $\Pi_{\mathrm{SR}}^{\mathrm{H}}(p, k_{\mathrm{H}}, e)$. The (optimal) solution set is $\hat{Y}_{\mathrm{H}}(p, k_{\mathrm{H}}, e)$, abbreviated to $\hat{Y}$. The corresponding lowercase notation $\hat{y}$ is used used *only when the solution is known to be unique* (possibly by assumption in a heuristic argument or a preview of results).

The space $L^{\infty *}$ appearing in (2.5) is the norm-dual of L^{∞}. This contains L^1, the space of all functions integrable with respect to (w.r.t.) the Lebesgue measure, meas. However, much of the analysis applies not only to a time-of-use (TOU) tariff represented by a price function $p \in L^1[0,T]$ but also, more generally, to one represented by a $p \in L^{\infty *}[0,T]$. Such a p can be identified with a finitely additive set function vanishing on meas-null sets, since the integral of any $y \in L^{\infty}$ w.r.t. such a set function defines a bounded linear functional on L^{∞}: see, e.g., [12, III.1–III.2 and IV.8.16] or [57, 2.3]. As an additive set function, a $p \in L^{\infty *}$ has the Hewitt-Yosida decomposition into $p_{\mathrm{CA}} + p_{\mathrm{FA}}$, the sum of its countably additive (c.a.) and purely finitely additive (p.f.a.) parts: see, e.g., [7, Appendix I: (26)–(27)], [12, III.7.8] or [57, 1.23 and 1.24].[11] The c.a. part of p is identified with its density (w.r.t. meas), which exists by the Radon-Nikodym Theorem; so it is a price function $p_{\mathrm{CA}} \in L^1[0,T]$. The p.f.a. part of p can be characterised as a singular element of $L^{\infty *}[0,T]$, i.e., p_{FA} is concentrated on a subset of $[0,T]$ with an arbitrarily small Lebesgue measure. (Formally, a $p \in L^{\infty *}$ is *concentrated on*, or *supported by*, a measurable set A if $\langle p, y \rangle = \langle p, y 1_A \rangle$ for every $y \in L^{\infty}$. A sequence of sets (A_m) is *evanescent* if $A_{m+1} \subseteq A_m$ for every m and $\mathrm{meas}\left(\bigcap_{m=1}^{\infty} A_m\right) = 0$; and p is called *singular* if there exists an evanescent (A_m) such that p is concentrated on A_m for each m. A $p \in L^{\infty *}$ is singular if and only if it is p.f.a. : see [57, 3.1].) This gives p_{FA} the interpretation of an extremely concentrated

[9] When there *is* a price q for the use of initial stock (still subject to the periodicity constraint), the analysis is the same except that $\psi(0) - \psi(T)$ equals q instead of zero, if $p(0) = p(T)$. A more interesting variation is to specify initial and final stocks and a planning interval consisting of a number of cycles. In that case the optimal policy would not, in general, be exactly cyclic, but it would approach a cyclic policy for almost all of a sufficiently long planning interval (a "turnpike" result). In such a setting the initial stock could be valued endogeneously.

[10] The closedness result, Lemma 6.1, holds also without this constraint.

[11] A p.f.a. set function is one that is lattice-disjoint from every c.a. one.

charge. In the hydro context it can arise as a capacity charge for the turbine: see Remark 4.15.

The value of $y \in L^\infty$ at $p \in L^{\infty *}$ is denoted by

$$(2.8) \qquad \langle p, y \rangle_{L^{\infty *}, L^\infty} = \int_0^T p_{\mathrm{CA}}(t)\, y(t)\, \mathrm{d}t + \langle p_{\mathrm{FA}}, y \rangle ,$$

abbreviated to $\langle p, y \rangle$.[12] Although the last term in (2.8) is also an integral, it is one that lacks some basic properties; and we reserve the symbol $\int$ for integration w.r.t. a measure, which is countably additive by definition.[13] The only measures in $L^{\infty *}$ are those having densities, i.e., $L^{\infty *} \cap \mathcal{M} = L^1$.

A $p \in L^{\infty *}$ is, by definition, (strictly) *positive as a linear functional* on L^∞ if $\langle p, \cdot \rangle$ is positive on $L^\infty_+ \setminus \{0\}$. This is the case if and only if $p_{\mathrm{FA}} \geq 0$ and $p_{\mathrm{CA}} > 0$ almost everywhere (a.e.) on $[0, T]$. The latter condition is also written as $p_{\mathrm{CA}} \gg 0$ or as $p_{\mathrm{CA}} \in L^1_{++}$. (For the space $\mathcal{C}$, note that $p \in \mathcal{C}_{++}$ if and only if $\mathrm{Min}(p) > 0$.)

3. Rental valuation of a hydro plant: heuristic solution

Before their formal derivation, the rental values of the fixed inputs, defined as the derivatives of $\Pi^{\mathrm{H}}_{\mathrm{SR}}$ w.r.t. k_{St} and k_{Tu}, are calculated heuristically. To start with, assume that not only the market price of electricity, $p(t)$, but also the shadow price of water, $\psi(t)$, is known.[14] Then the operating decisions can be decentralised within the hydro plant, with the reservoir "buying" water at the price $\psi(t)$ from the river and "selling" it to the turbine, which in turn sells the generated electricity at the market price $p(t)$ outside the plant. The SR profit maximisation separates into problems with obvious solutions, one for the reservoir and one for the turbine. Their maximum profits, $\Pi^{\mathrm{St}}(\psi, k_{\mathrm{St}})$ and $\Pi^{\mathrm{Tu}}(p - \psi, k_{\mathrm{Tu}})$, are both linear in k. A unit turbine can earn the profit flow $(p - \psi)^+$, the nonnegative part of $p - \psi$, by generating when $p(t) > \psi(t)$. The profit is earned only at the times of full capacity utilisation, since the optimum output is $y_{\mathrm{H}}(t) = k_{\mathrm{Tu}}$ when $p(t) > \psi(t)$: see Figures 1a and 1b. In total over the cycle, the rental value of a unit turbine is therefore $\Pi^{\mathrm{Tu}}/k_{\mathrm{Tu}} = \int_0^T (p(t) - \psi(t))^+ \,\mathrm{d}t$. As for the reservoir, a unit can earn a profit of $\psi(\overline{\tau}) - \psi(\underline{\tau})$ by buying stock at time $\underline{\tau}$ and selling it at a later time $\overline{\tau}$ when $\psi(\overline{\tau}) > \psi(\underline{\tau})$. The rental value of a unit reservoir is therefore the sum of all shadow price rises in a cycle. In precise terms: if $\psi(T) \geq \psi(0)$, then $\Pi^{\mathrm{St}}/k_{\mathrm{St}} = \mathrm{Var}^+(\psi)$, which denotes the total

[12] In (2.6) and (2.8) the revenue flow is not discounted because all the prices are in present-value terms. The same applies to the shadow stock prices ψ, introduced formally in (4.15); so the rises of ψ give stock appreciation net of the interest on its value.

[13] The oft-employed term "finitely additive measure" is an oxymoron.

[14] When ψ is introduced formally, as the Lagrange multiplier paired with the parameter e, it is by definition the price for the river flow. To see that prices for the inflowing water and for the water in store should be equal, note that, by assumption, there is no alternative use for the inflow. This is why its price cannot exceed that of the stock. The reverse inequality is obvious.

positive (a.k.a. upper) variation of ψ, i.e., the supremum of $\sum_n (\psi(\overline{\tau}_n) - \psi(\underline{\tau}_n))^+$ over all finite sets of pairwise disjoint subintervals $(\underline{\tau}_m, \overline{\tau}_m)$ of $[0, T]$.

If $\psi(T) < \psi(0)$, this indicates that the reservoir should start the cycle full, and refill towards the end of cycle. This brings an extra profit of $\psi(0) - \psi(T)$ per unit, and the unit rent in question is the *cyclic positive variation*

$$\mathrm{Var}_{\mathrm{c}}^{+}(\psi) := \mathrm{Var}^{+}(\psi) + (\psi(0) - \psi(T))^{+}. \tag{3.1}$$

Later it is shown that actually $\psi(0) = \psi(T)$ if $p(0) = p(T)$ and $p \in \mathcal{C}$, i.e., if $p \in \mathcal{C}^{\mathrm{c}}[0, T]$.

However, the maximum operating profit of the whole hydro plant, $\Pi_{\mathrm{SR}}^{\mathrm{H}}$, is a function *not* of ψ but of the problem's parameters (p, k_{H}, e) alone. This means that ψ is an auxiliary function which must eventually be given in terms of (p, k_{H}, e). The unit rents of the two capacities can then be obtained by substituting the correct ψ into the formulae[15]

$$\frac{\partial \Pi_{\mathrm{SR}}^{\mathrm{H}}}{\partial k_{\mathrm{St}}} = \mathrm{Var}_{\mathrm{c}}^{+}(\psi), \quad \frac{\partial \Pi_{\mathrm{SR}}^{\mathrm{H}}}{\partial k_{\mathrm{Tu}}} = \int_0^T (p(t) - \psi(t))^{+} \, \mathrm{d}t. \tag{3.2}$$

The correct shadow price for water, $\hat{\psi}$, is the marginal value $\nabla_e \Pi_{\mathrm{SR}}^{\mathrm{H}}$ of the inflow. This is a case of differentiating the optimal value function w.r.t. a primal parameter, and a standard result of duality for convex programmes identifies the derivative as the solution to the dual programme: see, e.g., [47, Theorem 16: (b) and (a), with Theorem 15: (e) and (f)] or [37, 7.3: Theorem 1']. Furthermore, the dual programme consists of minimising the value of the primal parameters by the choice of the dual variables. Here this means that $\hat{\psi}$ is that water price function which minimises the total rent of the hydro plant's fixed resources, e and k_{H} (when the unit rents of the capacities are expressed as above in terms of ψ and p). Therefore, given a TOU electricity tariff p, one can find $\hat{\psi}$ by unconstrained minimisation of the fixed-input value

$$k_{\mathrm{St}} \, \mathrm{Var}_{\mathrm{c}}^{+}(\psi) + k_{\mathrm{Tu}} \int_0^T (p(t) - \psi(t))^{+} \, \mathrm{d}t + \int_0^T \psi(t) \, e(t) \, \mathrm{d}t \tag{3.3}$$

over ψ, an arbitrary bounded-variation function on $(0, T)$.

The optimal ψ must be unique if the marginal values of fixed inputs are to exist as the usual partial derivatives (two-sided but single-valued) w.r.t. k_{St} and k_{Tu}, and as the usual gradient vector w.r.t. e. When this is so, the directional derivative of $\Pi_{\mathrm{SR}}^{\mathrm{H}}$ w.r.t. the two capacities and the inflow is a linear function of their increments Δk_{St},

[15] In the heuristic argument the correct shadow price, $\hat{\psi}(k)$, is substituted for ψ after the differentiation, and any marginal effect of k on Π through $\hat{\psi}(k)$ is, justifiably, ignored.

Δk_{Tu} and Δe, i.e.,

$$(3.4)\qquad \mathrm{D}\Pi_{\mathrm{SR}}^{\mathrm{H}}\left(\Delta k_{\mathrm{St}}, \Delta k_{\mathrm{Tu}}\right) = \frac{\partial \Pi_{\mathrm{SR}}^{\mathrm{H}}}{\partial k_{\mathrm{St}}} \Delta k_{\mathrm{St}} + \frac{\partial \Pi_{\mathrm{SR}}^{\mathrm{H}}}{\partial k_{\mathrm{Tu}}} \Delta k_{\mathrm{Tu}} + \left\langle \nabla_e \Pi_{\mathrm{SR}}^{\mathrm{H}}, \Delta e \right\rangle$$
$$= \Delta k_{\mathrm{St}} \operatorname{Var}_{\mathrm{c}}^{+}\left(\hat{\psi}\right) + \Delta k_{\mathrm{Tu}} \int_0^T \left(p\left(t\right) - \hat{\psi}\left(t\right)\right)^{+} \mathrm{d}t + \int_0^T \hat{\psi}\left(t\right) \Delta e\left(t\right) \mathrm{d}t,$$

where all the derivatives and $\hat{\psi}$ are evaluated at the given (k_{H}, e). This means that the *profit*-imputed incremental value of investment is additive in Δk_{St}, Δk_{Tu} and Δe, unlike Koopmans' *cost*-imputed value in (9.32).

The difficulty is, therefore, in calculating ψ and, also, in identifying the case when ψ is unique. The calculation question is addressed rigorously in Theorem 4.1 and Proposition 4.4, and the uniqueness question is addressed in Theorem 4.9 and the supporting lemmas. For a piecewise monotone p, these questions are studied further in Proposition 4.10. But both aspects of the analysis are presented heuristically first.

Minimisation of (3.3) is indeed an equivalent form of the dual to the primal problem of optimal operation (2.5)–(2.7): see Proposition 4.4. An analysis of this form of the dual elucidates the structure of the optimal water price ψ and helps identify the case when the optimal ψ is unique. It is clear that the optimum, $\hat{\psi}$, is obtained from p by "shaving off" the local peaks of p and "filling in" the troughs. The extent of the levelling can be determined from the trade-offs in minimisation of the three terms in (3.3), at least in the case that the market price p is piecewise monotone and $k_{\mathrm{Tu}} > e\left(t\right) > 0$ at all times. (An extension dispensing with the upper bound on e is sketched in Section 5.) The solution, presented graphically in Figure 1, is determined by constancy intervals of $\hat{\psi}$, on each of which the sign of $p\left(t\right) - \hat{\psi}$ stays constant. An interval $\left(\underline{t}, \bar{t}\right)$ around a trough of p, on which $p < \hat{\psi}$ throughout, is characterised by

$$(3.5)\qquad k_{\mathrm{St}} - \int_{\underline{t}}^{\bar{t}} e\left(t\right) \mathrm{d}t = 0$$

whereas an interval around a local peak of p, on which $p > \hat{\psi}$ throughout, is characterised by

$$(3.6)\qquad k_{\mathrm{St}} - \int_{\underline{t}}^{\bar{t}} \left(k_{\mathrm{Tu}} - e\left(t\right)\right) \mathrm{d}t = 0,$$

on the assumption that $k_{\mathrm{St}} / \operatorname{EssInf}\left(e\right)$ and $k_{\mathrm{St}} / \left(k_{\mathrm{Tu}} - \operatorname{EssSup}\left(e\right)\right)$, which have the time dimension, are sufficiently short to ensure that the intervals do not abut on each other. Conditions (3.5)–(3.6) are the first-order conditions (FOC's) for the dual optimum, obtained by equating to zero the increments in the minimand (3.3) that result from shifting the constant values of ψ up or down by an infinitesimal unit, on an interval around a peak or trough of p. Note that the optimal "bang-coast-bang" output ($y = k_{\mathrm{Tu}}$ when $\hat{\psi} < p$, $y = e$ when $\hat{\psi} = p$, and $y = e$ when $\hat{\psi} > p$, as

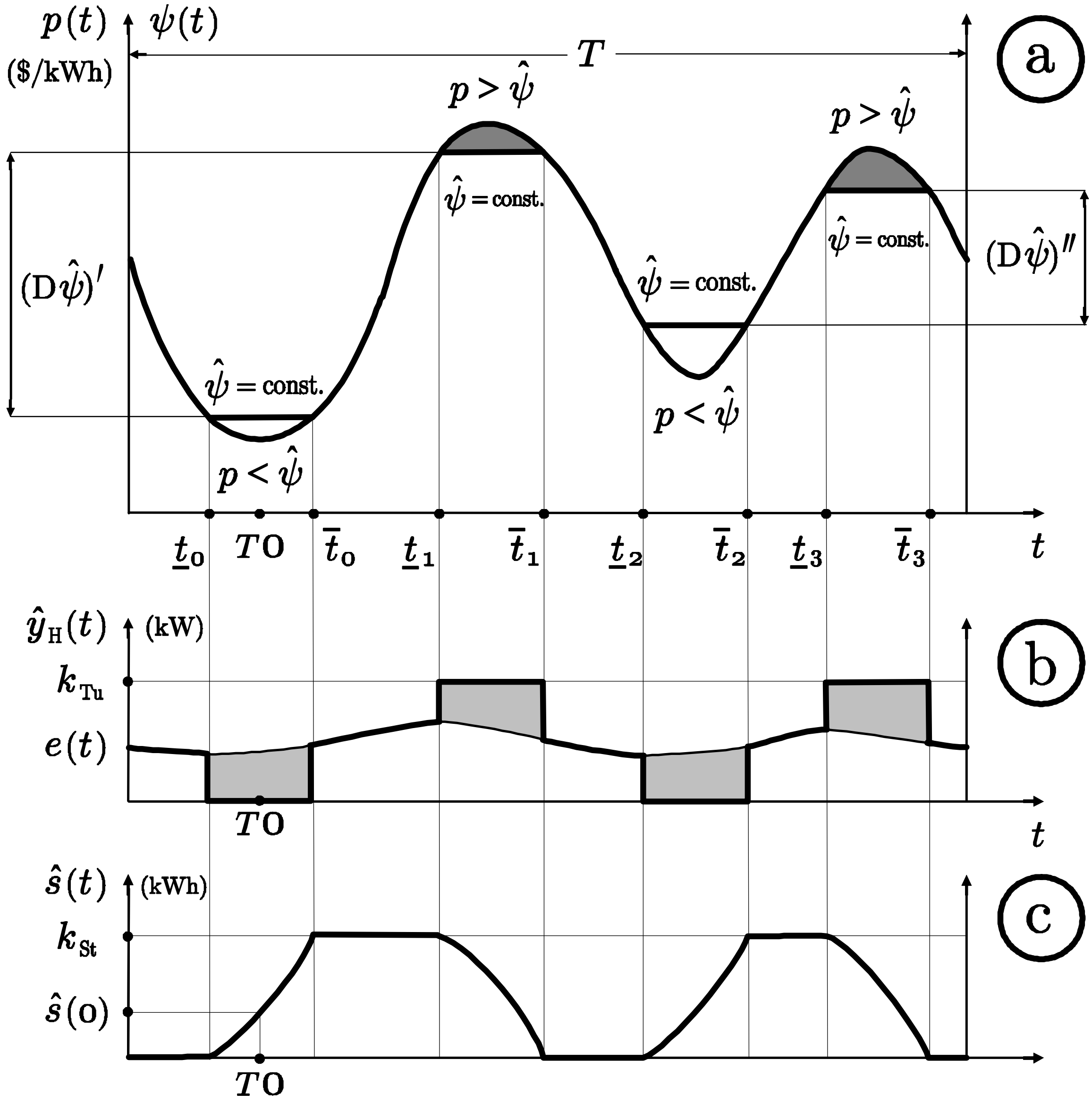

FIGURE 1. Trajectories of: (a) shadow price for water $\hat{\psi}$, (b) profit-maximising hydro output $\hat{y}_{\mathrm{H}}$, (c) water stock, in Theorem 4.9 and Proposition 4.6. Unit rent for storage capacity is $\mathrm{Var}_{\mathrm{c}}^{+}(\hat{\psi}) = (\mathrm{D}\hat{\psi})' + (\mathrm{D}\hat{\psi})''$, the sum of rises of $\hat{\psi}$. Unit rent for turbine capacity is $\int_0^T \left(p - \hat{\psi}\right)^+ (t)\,\mathrm{d}t$, the sum of dark grey areas in (a). In (b), each of the light grey areas equals the reservoir's capacity k_{St}. When $\hat{y}_{\mathrm{H}}(t) \neq e(t)$ in (b), the thin line is the inflow trajectory e, and the thick line is $\hat{y}_{\mathrm{H}}$.

in Figure 1) is made feasible by the fact that, on the intervals of water collection (when $\hat{\psi} > p$) or of discharge at the maximum rate (when $\hat{\psi} < p$), the reservoir goes alternately from empty to full and *vice versa* (Figure 1c).

Matters complicate when, for relatively large k_{St}, the neighbouring intervals of collection and of discharge abut; but a similar optimality rule applies to such clusters (Proposition 4.10).

The same marginal calculation for the dual problem also shows that an optimum ψ can be nonunique if p is discontinuous over time. Suppose, for example, that p jumps at the beginning, and drops at the end, of an interval $A = \left(\underline{t}, \overline{t}\right)$ meeting (3.6) and the condition

$$p\left(\underline{t}-\right) \vee p\left(\overline{t}+\right) < p\left(\underline{t}+\right) \wedge p\left(\overline{t}-\right) = \inf_{t \in A} p\left(t\right), \tag{3.7}$$

where $\vee$ and $\wedge$ mean the smaller and the larger of the two, and $p\left(t-\right)$ and $p\left(t+\right)$ denote the left and right limits at t. Just before $\underline{t}$ and just after $\overline{t}$, an optimal ψ equals p, i.e., $\psi\left(\underline{t}-\right) = p\left(\underline{t}-\right)$ and $\psi\left(\overline{t}+\right) = p\left(\overline{t}+\right)$. Inside A, $p > \psi = \text{const.}$; but an optimal constant value of ψ on A can be anywhere between the two unequal terms of (3.7): the jump and the drop of p create a "zone of indifference" for $\psi_{|A}$. Figure 2 shows this when $p\left(\overline{t}+\right) \le p\left(\underline{t}-\right) < p\left(\underline{t}+\right) \le p\left(\overline{t}-\right)$ so $p\left(\underline{t}-\right) \le \psi_{|A} \le p\left(\underline{t}+\right)$. Different values from this range divide the same total rent differently between the three fixed inputs of the hydro plant: the jump $\mathrm{D}\psi\left\{\underline{t}\right\} := \psi\left(\underline{t}+\right) - \psi\left(\underline{t}-\right)$, which can be any fraction of $p\left(\underline{t}+\right) - p\left(\underline{t}-\right)$ in Figure 2, is an indeterminate part of the reservoir's rent (per unit); $\int_A \left(p\left(t\right) - \psi\right) \mathrm{d}t$ is the corresponding indeterminate part of the turbine's rent; and the indeterminate $\psi_{|A}$ itself is the river's unit rent, on A. (The case of p dropping at the beginning, and jumping at the end, of an interval $A = \left(\underline{t}, \overline{t}\right)$ that meets Condition (3.5) is similar, except that the turbine's rent on A is of course zero, since $p < \psi_{|A}$.)

Conversely, given a continuous p, the optimum ψ is unique; and then the gradient $\nabla_{k,e}\Pi^{\mathrm{H}}_{\mathrm{SR}}$ exists. For this result, the optimal quantities (the primal solution) are brought into the argument along with the optimal shadow prices (the dual solution). The key principle is that equipment can earn a rent only at a time of full capacity utilisation. In the present context this means that p can exceed ψ only when the turbine is working at full power (i.e., when $y\left(t\right) = k_{\mathrm{Tu}}$); and similarly ψ can exceed p only when the turbine is off (i.e., when $y\left(t\right) = 0$). Therefore $\psi\left(t\right)$ equals $p\left(t\right)$ when the reservoir is either full or empty (since $s\left(t\right) = 0$ or $s\left(t\right) = k_{\mathrm{St}}$ implies that $y\left(t\right) = -\dot{s}\left(t\right) + e\left(t\right) = e\left(t\right)$, which lies *strictly* between 0 and k_{Tu} by assumption). By the same principle, ψ can be rising or falling only when the reservoir is full or empty (respectively); so ψ stays constant on each interval during which the reservoir constraints are inactive (i.e., when $0 < s\left(t\right) < k_{\mathrm{St}}$). Together, these conditions determine the function ψ *almost* completely—except for the possibility of jumps or drops of ψ that may occur at endpoints of a (closed) interval on which the reservoir

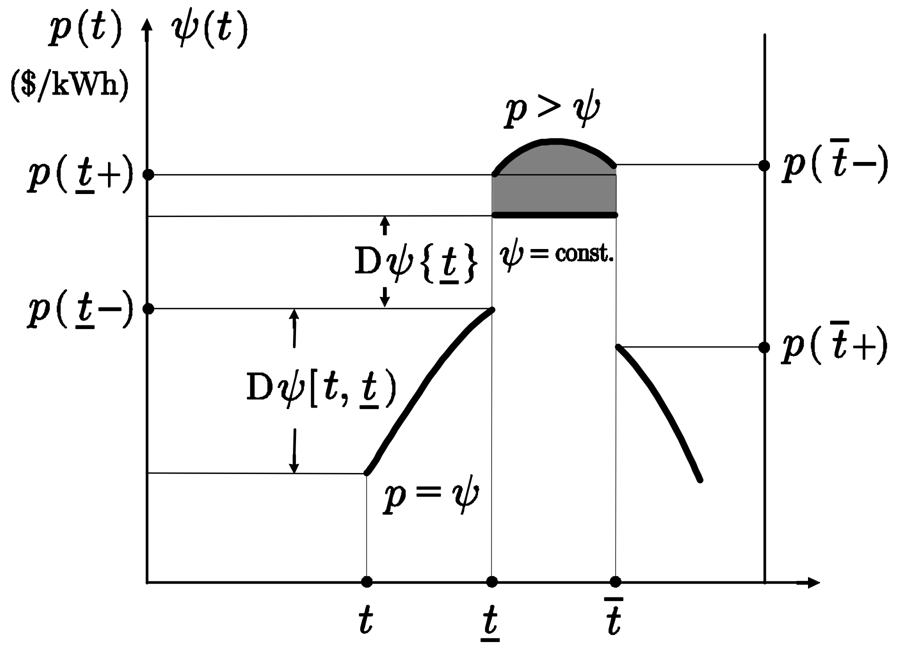

FIGURE 2. Indeterminacy of an optimal shadow price of water ψ when the TOU price of good p is discontinuous. The constant value of ψ on $(\underline{t}, \overline{t})$ can be set at any level between $p(\underline{t}-)$ and $p(\underline{t}+)$ in the case shown. The jump of ψ at $\underline{t}$ is an indeterminate part of the reservoir's unit rent. The dark grey area represents $\int_{\underline{t}}^{\overline{t}} (p(t) - \psi(t))^+ \, dt$, the interval's contribution to the turbine's unit rent.

is either full or empty throughout.[16] Suppose, for example, that the reservoir is full at an instant $\underline{t}$ which is followed by an interval $A = (\underline{t}, \overline{t})$ satisfying (3.6) during which $y_H(t) = k_{Tu}$, i.e., the turbine is working at the maximum rate. Under (3.7), the constant $\psi_{|A}$ is nonunique (within the specified range), or equivalently $D\psi\{\underline{t}\}$ is nonunique.[17] However, this argument also shows that ψ cannot be nonunique without p being discontinuous. Put another way, if p is continuous, then ψ is unique, with $\psi_{|A} = p(\underline{t}) = p(\overline{t})$. Thus fully determinate, separate rental values are imputed to all the fixed inputs, despite the perfect Allen-Hicks complementarity between the reservoir and turbine inputs. This reasoning is formalised in Theorem 4.9.

[16]For simplicity, assume here that the set F of the times when the reservoir is full consists of a finite number of intervals (which may be singletons, as in Example 3.1). Though F can be more complex, this is only a technicality, dealt with in the Proof of Lemma 4.8.

[17]Similar reasoning shows the possibility of an indeterminate jump of ψ at an instant when the reservoir becomes full. And similar drops of ψ are possible at an instant when the reservoir either becomes empty or ceases to be so.

Determinacy of rents obtains *only* in continuous-time analysis. Perforce, discrete-time modelling makes prices discontinuous, and can render it impossible to divide the plant's total rent between the particular fixed inputs on marginalist principles—as the following example shows. It is essentially the two-subperiod model, in which the optimum policy is simply to store, from the low-price subperiod to the high-price subperiod, as much water as the constraints allow.

Example 3.1 (Indeterminacy of hydro rents with a discontinuous price)**.** *The short-run profit function of the hydro technique (2.4) is not everywhere differentiable in* (k_{H}, e)*. To see this, take any numbers* $\overline{\mathsf{p}} > \underline{\mathsf{p}} \geq 0$ *and* e' *and* e'' *with* $k_{\mathrm{Tu}} > \mathsf{e} > 0$ *for* $\mathsf{e} = \mathsf{e}', \mathsf{e}''$*; and define a tariff and an inflow, both piecewise constant, by*

$$p(t) := \begin{cases} \underline{\mathsf{p}} & \text{if } t < T/2 \\ \overline{\mathsf{p}} & \text{if } t \geq T/2 \end{cases}$$

$$e(t) := \begin{cases} \mathsf{e}' & \text{if } t < T/2 \\ \mathsf{e}'' & \text{if } t \geq T/2 \end{cases} .$$

Then a profit-maximising output of a hydro plant with capacities $k_{\mathrm{H}} = (k_{\mathrm{St}}, k_{\mathrm{Tu}})$ *is*[18]

$$y(t) = \begin{cases} \mathsf{y}' := \mathsf{e}' - \epsilon & \text{if } t < T/2 \\ \mathsf{y}'' := \mathsf{e}'' + \epsilon & \text{if } t \geq T/2 \end{cases} ,$$

where

$$\epsilon := \frac{2}{T} k_{\mathrm{St}} \wedge (k_{\mathrm{Tu}} - \mathsf{e}'') \wedge \mathsf{e}' := \min \left\{ \frac{2}{T} k_{\mathrm{St}}, k_{\mathrm{Tu}} - \mathsf{e}'', \mathsf{e}' \right\} . \tag{3.8}$$

So

$$\Pi_{\mathrm{SR}}^{\mathrm{H}}(p, k_{\mathrm{H}}, e) = \frac{T}{2} \left(\underline{\mathsf{p}}\mathsf{e}' + \overline{\mathsf{p}}\mathsf{e}'' + \epsilon \left(\overline{\mathsf{p}} - \underline{\mathsf{p}} \right) \right) . \tag{3.9}$$

Therefore $\Pi_{\mathrm{SR}}^{\mathrm{H}}$ *is nondifferentiable in* (k_{H}, e) *whenever the minimum in (3.8) is attained at more than one of the three terms. (In other words, with this* p *a fixed input's fraction of the total SR profit is determinate* only *when it is either zero or one.)*

Comments:

1. In formal terms, the superdifferential (the set of supergradients) of $\operatorname{Min}(k) := \min_{\phi \in \{1,2,3\}} k_\phi$, as a concave function of k, is

$$\partial_k \operatorname{Min}(k) = \left\{ \left(r^\phi \right)_{\phi=1}^{3} \geq 0 : \sum_{\phi=1}^{3} r^\phi = 1 \text{ and } \forall \phi \, \left(r^\phi = 0 \text{ if } \exists \phi' \; k_{\phi'} < k_\phi \right) \right\} , \tag{3.10}$$

whence $\partial_{k_{\mathrm{H}}, e} \Pi_{\mathrm{SR}}^{\mathrm{H}}$ in Example 3.1 can be worked out by the chain rule.

[18] The optimal y is unique only if $k_{\mathrm{Tu}} - \mathsf{e}'' = \mathsf{e}' \leq 2k_{\mathrm{St}}/T$, although it is always unique in the class of two-valued step functions. Also, it is independent of the two price levels, as long as $\overline{\mathsf{p}} > \underline{\mathsf{p}}$.

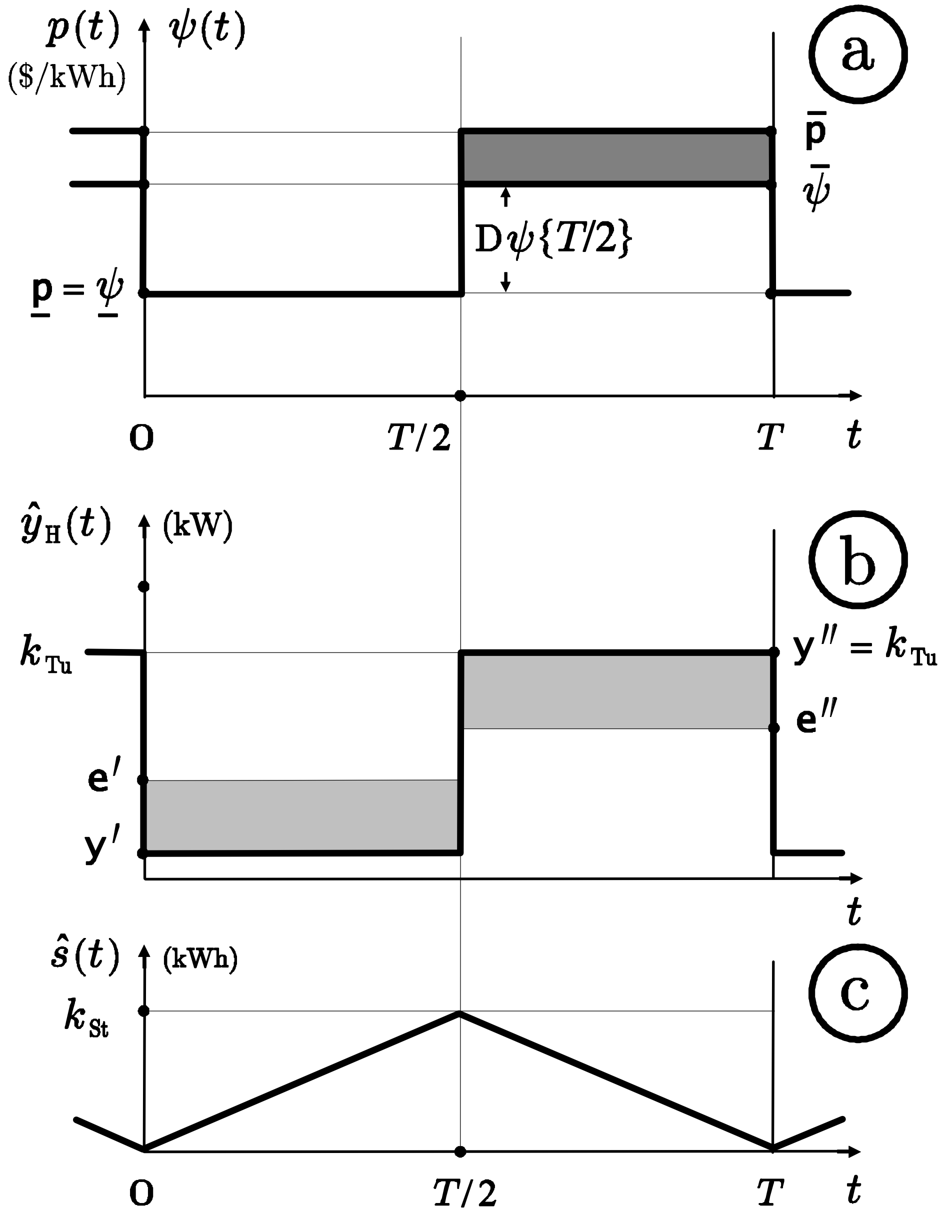

FIGURE 3. Indeterminacy of an optimal shadow price of stock ψ in the case of two subperiods. Its jump $\overline{\psi} - \underline{\psi}$, which equals the reservoir's unit rent, can take any value not exceeding the jump of p. The dark grey area in (a) represents the turbine's unit rent. In (b), each of the light grey areas equals the reservoir's capacity k_{St}.

2. Each element of $\partial_{k_{\mathrm{H}},e}\Pi^{\mathrm{H}}_{\mathrm{SR}}$ can also be obtained from an optimal shadow price of water ψ, found by minimising the fixed-input value (3.3). With the above p and e, an optimum shadow price is any

$$\psi(t) = \begin{cases} \underline{\psi} & \text{if } t < T/2 \\ \overline{\psi} & \text{if } t \geq T/2 \end{cases} \tag{3.11}$$

subject only to the conditions

$$\underline{\mathsf{p}} \leq \underline{\psi} \leq \overline{\psi} \leq \overline{\mathsf{p}} \tag{3.12}$$

$$\mathsf{e}' > \epsilon \Rightarrow \underline{\mathsf{p}} = \underline{\psi} \tag{3.13}$$

$$k_{\mathrm{Tu}} - \mathsf{e}'' > \epsilon \Rightarrow \overline{\psi} = \overline{\mathsf{p}} \tag{3.14}$$

$$\frac{2}{T} k_{\mathrm{St}} > \epsilon \Rightarrow \underline{\psi} = \overline{\psi}. \tag{3.15}$$

(Such a ψ is nonunique unless the minimum in (3.8) is attained at exactly one of the three terms, so that two out of the three implications (3.13)–(3.15) apply. Figure 3 shows this in the case of $\mathsf{e}' > k_{\mathrm{Tu}} - \mathsf{e}'' = 2k_{\mathrm{St}}/T = \epsilon$. Figure 3a is a special case of Figure 2.) The derivative property of $\Pi^{\mathrm{H}}_{\mathrm{SR}}$ means here that $\partial_{k_{\mathrm{H}},e}\Pi^{\mathrm{H}}_{\mathrm{SR}}$ is equal to the set of all those $\left(r^{\mathrm{H}}, \psi\right) = \left(r^{\mathrm{St}}, r^{\mathrm{Tu}}; \psi\right)$ obtained from a ψ meeting (3.11)–(3.15) by substitution into (3.2)–which gives

$$r^{\mathrm{St}} = \mathrm{Var}^{+}_{\mathrm{c}}(\psi) = \overline{\psi} - \underline{\psi} \tag{3.16}$$

$$r^{\mathrm{Tu}} = \int_0^T (p(t) - \psi(t))^{+} \, \mathrm{d}t = \frac{T}{2}\left(\overline{\mathsf{p}} - \overline{\psi}\right). \tag{3.17}$$

The derivative property is later, in (4.52), stated for general price and inflow functions p and e. For the present, two-valued p and e, the derivative property can be readily seen to hold by comparing (3.11)–(3.14) with (3.8)–(3.10) and (3.16)–(3.17).

4. Shadow water price and rents of a hydro plant by linear programming

In this section the SR profit maximum problem and its dual are set up as linear programmes that are doubly infinite because of the continuous-time dating: the primal, for example, contains a continuum of flow variables and also a continuum of capacity constraints on the stock s and the flow f (from the reservoir). The primal and the dual are shown to be soluble, and their solutions are characterised by means of the Kuhn-Tucker Conditions. The dual linear programme is reformulated as a convex programme for shadow pricing of water. The shadow price ψ is shown to be unique if the electricity price p is continuous over time. Formulae are given, in terms of ψ, for the optimal output y and for the unit rents $\nabla_k \Pi^{\mathrm{H}}_{\mathrm{SR}}$ of the reservoir and the turbine.

4.1. **Profit-maximising operation problem.** With the constants k_{St} and k_{Tu} regarded as special cases of cyclically varying capacity functions, and with $k_{\mathrm{Tu}} \geq e$ assumed from here on until Section 5, the LP form of the operation problem for a hydro plant is:

$$\text{Given } (p; k_{\mathrm{St}}, k_{\mathrm{Tu}}; e) \in L_+^{\infty *} \times \mathbb{R}_+^2 \times L_+^{\infty} \subset L_+^{\infty *} \times \mathcal{C}_+ \times L_+^{\infty} \times L_+^{\infty} \tag{4.1}$$

$$\text{with } p_{\mathrm{CA}} \gg 0, \text{ maximise } \langle p, y \rangle \text{ over } y \in L^{\infty} \text{ and } s_0 \in \mathbb{R} \tag{4.2}$$

$$\text{subject to: } \quad 0 \leq y(t) \leq k_{\mathrm{Tu}}(t) \quad \text{for a.e. } t \tag{4.3}$$

$$\int_0^T f(t)\, \mathrm{d}t = 0 \tag{4.4}$$

$$0 \leq s_0 - \int_0^t f(\tau)\, \mathrm{d}\tau \leq k_{\mathrm{St}} \quad \text{for every } t, \tag{4.5}$$

where $f := y - e$, as per (2.1) with $\varphi = 0$.

The two formulations of the operation problem are equivalent in the sense that y solves (2.5)–(2.7) if and only if y together with some s_0 solve (4.1)–(4.5)—in which case y together with the specific value

$$\underline{s}_0(y) := \max_{t \in [0,T]} \left(\int_0^t f(\tau)\, \mathrm{d}\tau \right) = \max_{t \in [0,T]} \left(\int_0^t (y - e)(\tau)\, \mathrm{d}\tau \right) \tag{4.6}$$

is a solution: $\underline{s}_0$ is the lowest initial stock required for $s(t)$ never to fall below 0. (Unless there is spare storage capacity, this is actually the only feasible value for s_0, given y.) One can therefore restrict attention to points (y, s_0) with $s_0 = \underline{s}_0(y)$, and so the stock trajectory associated with a hydro output y is

$$s(t) = \underline{s}_0(y) - \int_0^t f(\tau)\, \mathrm{d}\tau = \underline{s}_0(y) - \int_0^t (y - e)(\tau)\, \mathrm{d}\tau. \tag{4.7}$$

To ensure that Slater's Condition holds for this programme, and hence that its dual is soluble, for the most part it is assumed from here on that

$$k_{\mathrm{Tu}} > \mathrm{EssSup}(e) \geq \mathrm{EssInf}(e) > 0 \text{ and } k_{\mathrm{St}} > 0. \tag{4.8}$$

The "pure coasting" policy—i.e., $y_{\mathrm{H}} = e$ with $\varphi = 0$—is therefore feasible; this assumption is dropped in Section 5.

4.2. **The dual to the operation problem.** As is set out in, e.g., [47], the dual to a convex programme depends on the choice of perturbations for the primal parameters. A choice of admissible perturbations determines the structure of dual variables (a.k.a. Lagrange multipliers) to be paired with the parameter increments. Therefore the dual programme depends not only on the particular values of primal parameters, but also on the vector space of parameter increments or perturbations. This "ambient" space for the given parameter point is chosen to suit one's purpose; and this aspect of duality is relevant to the marginal interpretation of the dual variables (which is spelt out in the Proof of Theorem 4.9).

In the case of (4.1)–(4.5), the programme contains a separate set of capacity constraints for each time t—and therefore, by considering a separate increment $\Delta k_{\mathrm{H}}(t)$ for each t, instantaneous values can be imputed at each time; i.e., a whole trajectory, κ^{H}, of the values of capital services over the cycle can be determined. Thus the value of capacity services can be separated over time, rather than only determined in total for the cycle. By giving an interpretation to the multipliers κ and ν which are terms of the price p as per (4.14)–(4.15) below, this approach—the introduction of cyclically varying increments Δk_{H}—is useful even if the existing capacities k_{H} are actually taken to be constant.

As part of this "variation of constants", we consider a cyclically varying increment $\Delta n_{\mathrm{St}}(t)$ to the zero floor for the water stock in (4.5), and a cyclically varying increment $\Delta n_{\mathrm{Tu}}(t)$ to the zero floor for the turbine output rate in (4.3). Also, a scalar $\Delta\zeta$ is an increment to the zero on the r.h.s. of (4.4); this can be thought of as a quantity of water taken to be available for topping up the reservoir between the cycles.

The SR profit maximisation problem (4.1)–(4.5) is thus embedded in the family of perturbed programmes obtained by adding arbitrary cyclically varying increments Δk_{St}, Δn_{St}, Δk_{Tu}, Δn_{Tu}, Δe and a scalar $\Delta\zeta \in \mathbb{R}$ to the particular parameter point consisting of: the constants k_{St}, $n_{\mathrm{St}} = 0$, k_{Tu}, $n_{\mathrm{Tu}} = 0$, the function e and $\zeta = 0$. This perturbation is termed *refined*, to distinguish it from the coarser perturbation by constant increments to k_{St} and k_{Tu}. The parameter e is "nonstandard" in that it is not the right-hand side of a constraint: see the Appendix for a discussion.

As has already been indicated in (4.1), the function spaces for the resource increments are specified as: $\mathcal{C}[0,T]$ for Δk_{St} and Δn_{St}, and $L^{\infty}[0,T]$ for Δk_{Tu} and Δn_{Tu}. These are paired with $\mathcal{M}[0,T]$ and $L^{\infty *}[0,T]$ as the shadow price (multiplier) spaces. (With an infinite-dimensional parameter space such as L^{∞}, the dual programme depends to some extent also on a specific choice of the dual space, from among those that can be usefully paired with the particular parameter space; and L^{∞} can be paired with $L^{\infty *}$ or L^{1}.) When $p \in L^{1}[0,T]$, the pairing of L^{∞} with its norm-dual $L^{\infty *}$ is needed only for a proof of dual solubility: κ^{Tu} is actually in L^{1} (as is ν^{Tu}).

The marginal value of the services of a unit storage reservoir on an arbitrary interval $A \subset [0,T]$ is therefore given by a measure $\kappa^{\mathrm{St}}(A)$; such a valuation is made possible by limiting the time-varying increment Δk_{St} to A. Another measure $\nu^{\mathrm{St}}(A)$ gives the incremental profit from lowering the stock floor by a unit, on A.

The value of the services, on A, of a unit turbine is the integral of a rental flow $\kappa^{\mathrm{Tu}} \in L^{\infty *}$ (or of $\kappa^{\mathrm{Tu}} \in L^{1}$ if $p \in L^{1}$). The incremental profit from lowering the turbine output floor by a unit is the integral of a $\nu^{\mathrm{Tu}} \in L^{1}$.

The marginal value of water at the beginning (or end) of cycle is a scalar λ (paired with $\Delta\zeta$).

As is spelt out next, the dual to the operation programme (4.1)–(4.5) consists in minimising the value of the fixed hydro resources by an admissible choice of their shadow prices (κ^{St}, ν^{St}; κ^{Tu}, ν^{Tu}; ψ, λ), which are paired with the parameter increments (Δk_{St}, $-\Delta n_{\mathrm{St}}$; Δk_{Tu}, $-\Delta n_{\mathrm{Tu}}$; Δe, $\Delta\zeta$). The main dual constraints (4.14)–(4.15) give a decomposition of the electricity price p into a signed sum of: the turbine capacity charges, the turbine floor values and the shadow price of water ψ (which is the sum of the initial price λ, the cumulative of reservoir capacity charges κ^{St} and the cumulative of ν^{St}).

Theorem 4.1 (Fixed-input value minimisation as the dual). *The dual of the linear programme (4.1)–(4.5), relative to the refined perturbation and the pairing of the parameter spaces $\mathcal{C}$ and L^{∞} with $\mathcal{M}$ and $L^{\infty *}$ respectively, is:*

$$\text{(4.9)} \qquad \text{Given } \left(p; k_{\mathrm{H}}, \underline{s}, \underline{y}, e\right) \text{ as in (4.1)}$$

$$\text{(4.10)} \qquad \text{minimise } k_{\mathrm{St}} \int_{[0,T]} \kappa^{\mathrm{St}}(\mathrm{d}t) + \left\langle \kappa^{\mathrm{Tu}}, k_{\mathrm{Tu}} \right\rangle + \langle \psi, e \rangle$$

$$\text{(4.11)} \qquad \text{over } \lambda \in \mathbb{R}, \psi \in L^{\infty *} \text{ and } \left(\kappa^{\mathrm{St}}, \nu^{\mathrm{St}}; \kappa^{\mathrm{Tu}}, \nu^{\mathrm{Tu}}\right) \in \mathcal{M}^2 \times \left(L^{\infty *}\right)^2$$

$$\text{(4.12)} \qquad \text{subject to: } \left(\kappa^{\mathrm{St}}, \nu^{\mathrm{St}}; \kappa^{\mathrm{Tu}}, \nu^{\mathrm{Tu}}\right) \geq 0$$

$$\text{(4.13)} \qquad \kappa^{\mathrm{St}}[0,T] = \nu^{\mathrm{St}}[0,T]$$

$$\text{(4.14)} \qquad p = \psi + \kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}}$$

$$\text{(4.15)} \qquad \psi = \lambda + \left(\kappa^{\mathrm{St}} - \nu^{\mathrm{St}}\right)[0,\cdot].$$

Comments:

1. Under (4.8), any solution to (4.9)–(4.15) has the disjointness properties that

$$\text{(4.16)} \qquad \kappa^{\phi} \wedge \nu^{\phi} = 0 \quad \text{for } \phi = \mathrm{Tu}, \mathrm{St} \quad \text{and} \quad \kappa^{\mathrm{St}}\{0,T\} \wedge \nu^{\mathrm{St}}\{0,T\} = 0,$$

i.e., it is nonoptimal for the dual variables to overlap and partly cancel each other out. To see this, note that if it were false, then the minimand's value could be decreased by replacing (κ^{St}, ν^{St}; κ^{Tu}, ν^{Tu}) with (μ_+^{St}, μ_-^{St}; μ_+^{Tu}, μ_-^{Tu}) given by

$$\text{(4.17)} \qquad \mu^{\phi} := \kappa^{\phi} - \nu^{\phi} \quad \text{for } \phi = \mathrm{Tu}, \mathrm{St}.$$

2. It follows that the programme (4.10)–(4.15) can be reformulated in terms of the signed variables (4.17), by replacing $\left(\kappa^{\phi}, \nu^{\phi}\right)$ with $\left(\mu_+^{\phi}, \mu_-^{\phi}\right)$ throughout. At an optimum, $\mu^{\mathrm{St}}\{0\}$ and $\mu^{\mathrm{St}}\{T\}$ do not have opposite signs.

3. By the Hewitt-Yosida decomposition, (4.14) can be restated as

$$p_{\mathrm{CA}}(t) = \psi(t) + \kappa_{\mathrm{CA}}^{\mathrm{Tu}} - \nu_{\mathrm{CA}}^{\mathrm{Tu}}(t) \quad \text{for a.e. } t$$

$$\text{(4.18)} \qquad p_{\mathrm{FA}} = \kappa_{\mathrm{FA}}^{\mathrm{Tu}} - \nu_{\mathrm{FA}}^{\mathrm{Tu}}.$$

4. With $p_{\mathrm{FA}} \geq 0$, (4.18)–(4.16) give

$$\text{(4.19)} \qquad \nu_{\mathrm{FA}}^{\mathrm{Tu}} = p_{\mathrm{FA}}^{-} = 0 \quad \text{and} \quad \kappa_{\mathrm{FA}}^{\mathrm{Tu}} = p_{\mathrm{FA}}^{+} = p_{\mathrm{FA}}.$$

So if $p \in L^1[0,T]$, i.e., $p_{\mathrm{FA}} = 0$, then $\kappa^{\mathrm{Tu}} \in L^1$; and in this case the second term of (4.10) can be rewritten as $k_{\mathrm{Tu}} \int_0^T \kappa^{\mathrm{Tu}}(t)\,\mathrm{d}t$.

5. Our formulations of the primal and dual LP's, together with the Kuhn-Tucker Conditions and solubility results to come, extend *mutatis mutandis* to the case of cyclically varying capacities in service $k_{\mathrm{H}}(t)$. In (4.10), k_{St} and k_{Tu} must then be put into the integrands. If the zero floors for $s(t)$ in (4.5) and for $y_{\mathrm{Tu}}(t)$ in (4.3) are replaced by, respectively, a *minimum stock* $n_{\mathrm{St}}(t)$ and a minimum output rate $n_{\mathrm{Tu}}(t)$, then the terms $-\int_{[0,T]} n_{\mathrm{St}}(t)\,\nu^{\mathrm{St}}(\mathrm{d}t)$ and $-\langle \nu^{\mathrm{Pu}}, n_{\mathrm{Tu}} \rangle$ must be added to (4.10): the opposite quantities $-n_{\mathrm{St}}$ and $-n_{\mathrm{Tu}}$ can be regarded as additional fixed inputs. Similarly, $\lambda\zeta$ must be added to (4.10) if ζ replaces the zero on the r.h.s. of (4.4).

The proof that (4.10)–(4.15) is the dual problem is a routine application of the duality framework for optimisation in infinite-dimensional spaces, as expounded in, e.g., [47, Examples 4, 4', 4"] and [3, 3.3–3.7]. This is also true of the proofs that the dual solution exists and gives the marginal values of the primal parameters k and e, but not of the additional arguments showing the uniqueness of these values, i.e., the differentiability of $\Pi^{\mathrm{H}}_{\mathrm{SR}}$ in k and e.

To put the primal constraints in the required operator form, define the integrals I_0 and I_T: $L^\infty[0,T] \to \mathcal{C}[0,T]$ by

$$(4.20) \qquad (I_0 f)(t) := \int_0^t f(\tau)\,\mathrm{d}\tau, \quad (I_T f)(t) := \int_t^T f(\tau)\,\mathrm{d}\tau.$$

The reservoir constraints (4.5) on (y, s_0) can then be rewritten as

$$(4.21) \qquad 0 \le s_0 1_{[0,T]} - I_0 f \le k_{\mathrm{St}}.$$

A formula for the adjoint operation I_0^*: $\mathcal{M}[0,T] \to L^{\infty *}[0,T]$ is needed. (As for the embedding $\mathbb{R} \ni s_0 \mapsto s_0 1_{[0,T]} \in \mathcal{C}$, its adjoint is: $\mathcal{M} \ni \kappa \mapsto \langle \kappa, 1 \rangle = \kappa[0,T]$.)

Lemma 4.2. *The adjoints I_0^*, I_T^* map $\mathcal{M}[0,T]$ into* $\mathrm{BV}[0,T] \subset L^1[0,T]$*; and they are given by*

$$(4.22) \qquad (I_0^* \mu)(t) = \mu[t,T] \quad \text{and} \quad (I_T^* \mu)(t) = \mu[0,t] \quad \text{for a.e. } t,$$

for every $\mu \in \mathcal{M}[0,T]$. If $\mu[0,T] = 0$, then $-I_0^*\mu = \mu[0,\cdot] = I_T^*\mu$.

Proof. This follows from Fubini's Theorem: see [33]. ∎

Proof of Theorem 4.1 (Fixed-input value minimisation as the dual). Since (4.1)–(4.5) is an LP, it would suffice to apply results such as those of [3, 3.3 and 3.6–3.7]. However, to facilitate extensions requiring nonlinear models, this proof is couched in CP terms. The dual to a concave maximisation programme consists in minimising, over the dual variables (the Lagrange multipliers for the primal), the supremum of the Lagrange function over the primal decision variables: see, e.g., [47, (4.6) and (5.13)]. The "cone model" of [47, Example 4'] is applicable, since (4.21) and (4.3)–(4.4) represent the inequality constraints of the primal programme (4.1)–(4.5) by means of the nonnegative cones ($\mathcal{C}_+$ and L^∞_+) and convex constraint maps (which are actually linear).

The dual variables here are the κ^{St}, ν^{St}; κ^{Tu}, ν^{Tu}, ψ and λ of (4.11); and these are paired with the parameter increments Δk_{St}, $-\Delta n_{\mathrm{St}}$, Δk_{Tu}, $-\Delta n_{\mathrm{Tu}}$, Δe and $\Delta\zeta$)—as is discussed in Subsection 4.2.[19] The primal variables are $(y, s_0) \in L^\infty \times \mathbb{R}$, and the Lagrange function is

(4.23)

$$\mathcal{L}^{\mathrm{H}}(y, s_0; \kappa, \nu, \psi, \lambda) = \begin{cases} \Pi^{\mathrm{H}}_{\mathrm{Exc}}(y, s_0; \kappa, \nu, \lambda) + V^{\mathrm{H}}(\kappa, \psi) & \text{if } (\kappa, \nu) \geq 0 \text{ and} \\ & \psi = \lambda - I_0^*\left(\kappa^{\mathrm{St}} - \nu^{\mathrm{St}}\right) \\ +\infty & \text{otherwise} \end{cases}$$

where

$$V^{\mathrm{H}} := \left\langle \kappa^{\mathrm{St}}, k_{\mathrm{St}} \right\rangle_{\mathcal{M},\mathcal{C}} + \left\langle \kappa^{\mathrm{Tu}}, k_{\mathrm{Tu}} \right\rangle_{L^{\infty *}, L^\infty} + \langle \psi, e \rangle_{L^1, L^\infty} \tag{4.24}$$

and, with the notation (4.17),

$$\begin{aligned} \Pi^{\mathrm{H}}_{\mathrm{Exc}} &:= \left\langle p - \mu^{\mathrm{Tu}} - \lambda + I_0^* \mu^{\mathrm{St}}, y \right\rangle - \left\langle \mu^{\mathrm{St}}, s_0 \right\rangle \\ &= \left\langle p - \mu^{\mathrm{Tu}} - \lambda + \mu^{\mathrm{St}}(\cdot, T], y \right\rangle - s_0 \mu^{\mathrm{St}}[0, T], \end{aligned} \tag{4.25}$$

since $I_0^* \mu^{\mathrm{St}} = \mu^{\mathrm{St}}(\cdot, T]$ by Lemma 4.2.

Formulae (4.23)–(4.25) are interpreted below; for their derivation see the Appendix. But first, to complete the calculation of the dual minimand when $(\kappa, \nu) \geq 0$ and

$$\psi = \lambda - I_0^* \mu_{\mathrm{St}} \tag{4.26}$$

(which are dual constraints, since the minimand is $+\infty$ otherwise), note that

$$\sup_{y, s_0} \mathcal{L} = V + \sup_{y, s_0} \Pi_{\mathrm{Exc}}, \tag{4.27}$$

since V is independent of (y, s_0). By (4.25), Π_{Exc} is linear in these variables, so its supremum is either 0 or $+\infty$; and it is zero if and only if $\partial \Pi_{\mathrm{Exc}} / \partial s_0 = 0$ and $\nabla_y \Pi_{\mathrm{Exc}} = 0$. These conditions are equivalent to the conjunction of (4.13) and

$$p = \lambda + \mu_{\mathrm{St}}[0, \cdot] + \mu_{\mathrm{Tu}}. \tag{4.28}$$

In view of (4.13) and Lemma 4.2, (4.28) with (4.26) are the same as (4.14)–(4.15). So the dual programme is: Given $(p; k, e)$, minimise the $V(\kappa, \psi; k, e)$ of (4.24) over $(\kappa, \nu) \geq 0$, ψ and λ subject to (4.24)–(4.15). ∎

Comments:

1. In (4.24)–(4.25), V is the value of the *available* resources (k, e), priced at (κ, ψ).

[19] Our parameter increments mean what Rockafellar [47] calls "parameters"; i.e., we do not place the origin of the parameter vector space at the original parameter point, which is $(k_{\mathrm{St}}, 0;\ k_{\mathrm{Tu}}, 0;\ e, 0)$.

2. For an entrepreneur buying all the inputs, Π_{Exc} is the excess profit (a.k.a. pure profit) from an output y and the use of an inflow e and an initial stock s_0. To see this, recall from (4.23) that $0 = \left\langle \lambda - \psi - I_0^* \mu^{\mathrm{St}}, e \right\rangle$, add this to (4.25) and use the identities $f(t) = y(t) - e(t)$ and $s(t) = s_0 - I_0 f(t)$ to obtain that

$$\Pi_{\mathrm{Exc}}^{\mathrm{H}} = \langle p, y \rangle - \left\langle \kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}}, y \right\rangle - \left\langle \kappa^{\mathrm{St}} - \nu^{\mathrm{St}}, s \right\rangle - \lambda \langle 1, f \rangle - \langle \psi, e \rangle . \tag{4.29}$$

This sum is the total over the cycle of the revenue from sales to the market minus the cost of all the resources needed at each time t. The resources in question are: the time-varying minimum requirements for the turbine and reservoir capacities (priced at κ), the floors for generation and stock (priced at ν), the required top-up (priced at λ), and the river flow (priced at ψ). The last term in (4.29) can be rewritten as $\int_0^T \psi(t)\, e(t)\, \mathrm{d}t$, since $\psi \in L^1$ by (4.15).

3. By adding and subtracting the value of internal sales (of the outflow y from reservoir to turbine, priced at ψ), (4.29) can be restated as

$$\Pi_{\mathrm{Exc}}^{\mathrm{H}} = \langle p, y \rangle - \left\langle \mu^{\mathrm{Tu}}, y \right\rangle - \langle \psi, y \rangle + \langle \psi, y - e \rangle - \left\langle \mu^{\mathrm{St}}, s_0 - I_0 (y - e) \right\rangle - \langle \lambda, y - e \rangle .$$

This gives Π_{Exc} as the sum of pure profits from the two parts of the plant: the first three terms add up to the excess profit from generation alone, whilst the other three terms add up to the excess profit from storage. The latter sum is equal to the appreciation of s_0 over the cycle because, with $\lambda - \psi = I_0^* \mu^{\mathrm{St}}$ and $f := y - e$ as per (2.1),

$$\langle \psi, f \rangle - \langle \lambda, f \rangle - \left\langle \mu^{\mathrm{St}}, s_0 - I_0 (f) \right\rangle = - \left\langle I_0^* \mu^{\mathrm{St}}, f \right\rangle - \left\langle \mu^{\mathrm{St}}, s_0 - I_0 (f) \right\rangle = -s_0 \left\langle \mu^{\mathrm{St}}, 1 \right\rangle .$$

4.3. **Dual solubility and Kuhn-Tucker Conditions.** The dual programme (4.9)–(4.15) has a solution, in which $\psi \in \mathrm{BV} \subset L^1$ by (4.15) and $\nu^{\mathrm{Tu}} \in L^1$ by (4.19), whilst κ^{Tu} is generally in $L^{\infty *}$ (and κ^{St} and ν^{St} are in $\mathcal{M}$).

Proposition 4.3 (Dual solubility, Kuhn-Tucker Conditions)**.** *Assume (4.8). Then:*

1. The fixed-input value minimisation programme (4.9)–(4.15) has an (optimal) solution
$$\left(\kappa^{\mathrm{St}}, \nu^{\mathrm{St}}; \kappa^{\mathrm{Tu}}, \nu^{\mathrm{Tu}}; \psi, \lambda\right) \in \mathcal{M} \times \mathcal{M} \times L^{\infty *} \times L^1 \times \mathrm{BV} \times \mathbb{R}.$$
The programme's value is finite and equal to the SR profit $\Pi_{\mathrm{SR}}^{\mathrm{H}}(p, k_{\mathrm{H}}, e)$, the optimal value of (4.1)–(4.5). Furthermore, if $p \in L^1[0,T]$, then also $\kappa^{\mathrm{Tu}} \in L^1$ in every solution.
2. Points $(y, \underline{s}_0(y)) \in L^\infty \times \mathbb{R}$ and $\left(\kappa^{\mathrm{St}}, \nu^{\mathrm{St}}; \kappa^{\mathrm{Tu}}, \nu^{\mathrm{Tu}}; \psi, \lambda\right)$ are optimal solutions to, respectively, the primal (4.1)–(4.5) and the dual (4.9)–(4.15) if and only if:
   (a) $(y, \underline{s}_0(y))$ and $\left(\kappa^{\mathrm{St}}, \nu^{\mathrm{St}}; \kappa^{\mathrm{Tu}}, \nu^{\mathrm{Tu}}; \psi, \lambda\right)$ are feasible, i.e., satisfy (4.3)–(4.5) and (4.12)–(4.15).

(b) $\operatorname{supp} \kappa^{\mathrm{St}} \subseteq \{t \in [0,T] : s(t) = k_{\mathrm{St}}\}$, and $\operatorname{supp} \nu^{\mathrm{St}} \subseteq \{t : s(t) = 0\}$, where s is given by (4.6)–(4.7), and supp denotes the support (of the measure).[20].

(c) For every number $\epsilon > 0$, κ^{Tu} is concentrated on $\{t : y(t) \geq k_{\mathrm{Tu}}(t) - \epsilon\}$, whilst $\nu^{\mathrm{Tu}} \in L^1$ is concentrated on $\{t : y(t) = 0\}$. If $p \in L^1$, then also $\kappa^{\mathrm{Tu}} \in L^1$ (and then the above inequality is equivalent to equality with $\epsilon = 0$).[21]

Proof. Like that of Theorem 4.1, this proof is put in CP terms. Since the nonnegative cones in the (primal) parameter spaces ($\mathcal{C}_+$ and L^∞_+) have nonempty interiors (for the supremum norm), the framework of [47, Examples 4, 4', 4"] is applicable. To verify the Generalised Slater's Condition of [47, (8.12)] for the primal constraints (4.3)–(4.5), it suffices to take $y = e$ (so that $f = 0$), setting s_0 at any value strictly between 0 and k_{St}. So the dual has a (proper) solution, and the primal and dual values are equal (and finite): see, e.g., [47, Theorems 18 (a) and 17 (a)]. This proves Part 1.

For Part 2, apply the Kuhn-Tucker saddle-point characterisation of optima—given in, e.g., [47, Theorem 1 (e) and (f)]—to the primal (4.1)–(4.5) and its dual (4.9)–(4.15). This shows that (y, s_0) and $(\kappa, \nu, \psi, \lambda)$ is a dual pair of solutions if and only if they maximise and minimise (respectively) the Lagrange function $\mathcal{L}$ given by (4.23). These max-min conditions are the ones to analyse further.

The minimum in question is characterised by the conditions of: nonnegativity (4.12) and compatibility (4.15) of dual variables, primal feasibility (4.3)–(4.5) and complementary slackness, which here translates into Conditions 2b and 2c. See the Appendix for details. As for the maximum in question, it is characterised by the conditions $\partial \Pi_{\mathrm{Exc}} / \partial s_0 = 0$ and $\nabla_y \Pi_{\mathrm{Exc}} = 0$, i.e., by (4.13)–(4.14). ∎

Comment: The existence of a dual optimum in the norm-dual spaces (κ^{St} and ν^{St} in $\mathcal{M} = \mathcal{C}^*$, and κ^{Tu}, ν^{Tu} and ψ in $L^{\infty *}$) comes automatically from (4.8), which ensures that the Generalised Slater's Condition of [47, (8.12)] holds for the norm topologies of the primal parameter spaces L^∞ and $\mathcal{C}$. The more specific representations of the dual variables come from the structure of the problem at hand: by the constraint (4.15), $\psi \in \mathrm{BV} \subset L^1$; with $p \geq 0$, the optimal ν^{Tu} is in L^1; and if $p \in L^1$ then also the optimal κ^{Tu} is in L^1.

4.4. **Shadow pricing of water as the dual problem.** The variables $\left(\kappa^{\mathrm{St}}, \nu^{\mathrm{St}}\right)$ and $\left(\kappa^{\mathrm{Tu}}, \nu^{\mathrm{Tu}}\right)$ can be eliminated by using (4.15) and (4.14), together with (4.16), to express them in terms of ψ and p. This transforms the dual problem into one of *unconstrained* minimisation over $\psi \in \mathrm{BV}(0,T)$.

The space $\mathrm{BV}(0,T)$ consists of all functions ψ of bounded variation on $(0,T)$ with $\psi(t)$ lying between the left and right limits, $\psi(t-) = \lim_{\tau \nearrow t} \psi(\tau)$ and $\psi(t+) = \lim_{\tau \searrow t} \psi(\tau)$. The one-sided limits exist at every t and are equal *nearly everywhere* (n.e.), i.e., everywhere except for a countable set. (Specification of $\psi(t)$ between $\psi(t-)$ and $\psi(t+)$ is irrelevant; and functions differing only in this way, and therefore

[20] Also known as the carrier, this is the smallest closed set of full measure.

[21] This and Part 2b are stronger forms of the disjointness properties (4.16).

equal n.e., are identified with each other.) Also, a $\psi \in \mathrm{BV}(0,T)$ is extended by continuity to a $\psi \in \mathrm{BV}[0,T]$; i.e.,

$$\psi(0) := \psi(0+) \quad \text{and} \quad \psi(T) := \psi(T-).$$

If finite numbers $\psi(0-)$ and $\psi(T+)$ are additionally specified, then $\psi \in \mathrm{BV}[0-,T+]$.

Together with $\psi(0-) = \psi(T+) = \lambda$, (4.15) defines a one-to-one map of the set of all those $(\lambda, \kappa^{\mathrm{St}}, \nu^{\mathrm{St}})$ satisfying (4.12), (4.13) and (4.16) for $\phi = \mathrm{St}$ onto the set of all those $\psi \in \mathrm{BV}[0-,T+]$ with $\psi(0-) = \psi(T+)$ lying between $\psi(0+)$ and $\psi(T-)$. The inverse map is given by: $\kappa^{\mathrm{St}} = (\mathrm{D}\psi)^+$ and $\nu^{\mathrm{St}} = (\mathrm{D}\psi)^-$, where $\mathrm{D}\psi$ is the measure on $[0,T]$ defined by

$$\mathrm{D}\psi[t', t''] := \psi(t''+) - \psi(t'-) \tag{4.30}$$

for $t' \leq t''$ (and known as the Schwartz distribution derivative of ψ). The Lebesgue-Stieltjes integral of s w.r.t. $\mathrm{D}\psi$ is written as $\int_{[0,T]} s\,\mathrm{d}\psi$. Applied to $(\mathrm{D}\psi)^+$, this gives the $(\mathrm{d}\psi)^+$ below, which must not be misread as $\mathrm{d}(\psi^+)$.

Proposition 4.4 (Shadow pricing of water as the dual)**.** *Assume (4.8). Then the fixed-input value minimisation programme (4.9)–(4.15) is equivalent to the following convex programme (in which the cyclic positive variation of ψ is defined by (3.1):*

(4.31) Given $(p; k_{\mathrm{H}}, e)$ as in (4.1),

$$\text{minimise } k_{\mathrm{St}} \operatorname{Var}_{\mathrm{c}}^{+}(\psi) + k_{\mathrm{Tu}} \left\langle (p-\psi)^+, 1 \right\rangle + \int_0^T \psi(t)\, e(t)\, \mathrm{d}t \tag{4.32}$$

(4.33) over $\psi \in \mathrm{BV}(0,T)$.

The solution set for (4.31)–(4.33) is denoted by $\hat{\Psi}(p, k_{\mathrm{H}}, e) \neq \emptyset$, abbreviated to $\hat{\Psi}$. Again, the corresponding lowercase notation $\hat{\psi}$ is used *only when the dual solution is unique*.

Comment: If $p \in L^1[0,T]$, then the second term of (4.32) can be rewritten as $k_{\mathrm{Tu}} \int_0^T (p-\psi)^+(t)\,\mathrm{d}t$.

Proof of Proposition 4.4. This is a reformulation of Theorem 4.1: substitute the ψ given by (4.15) into (4.14), and note that, given any ψ (and p) the best choices for κ^{Tu} and ν^{Tu} are

$$\left(\kappa^{\mathrm{Tu}}, \nu^{\mathrm{Tu}}\right) = \left((p-\psi)^+, (\psi-p)^+\right) \tag{4.34}$$

because $k_{\mathrm{Tu}} > 0$. This reduces the dual programme (4.9)–(4.15) to minimisation of

$$k_{\mathrm{St}} \int_{[0,T]} (\mathrm{d}\psi)^+ + k_{\mathrm{Tu}} \left\langle (p-\psi)^+, 1 \right\rangle + \int_0^T \psi(t)\, e(t)\, \mathrm{d}t$$

over $\psi \in \mathrm{BV}[0-,T+]$, subject to $\psi(0-) = \psi(T+)$ lying between $\psi(0+)$ and $\psi(T-)$. Hence the first integral equals the sum of $(\psi(0+) - \psi(T-))^+$ and $\int_{(0,T)} (\mathrm{d}\psi)^+$; and this sum is $\operatorname{Var}_{\mathrm{c}}^{+}(\psi)$. ∎

Comment: When $p \in \mathrm{BV}(0,T)$, the shadow pricing problem (4.31)–(4.33) can be reformulated as one of minimising a lower semicontinuous function over a compact subset of $\mathcal{M}^c$ with the weak* topology. (In [33] we prove in detail the corresponding result for pumped storage.) This leads to, e.g., a solubility proof for the dual problem that is based directly on Weierstrass's Theorem, and needs no reference to the primal problem (unlike our earlier Proof of Proposition 4.3, which relies on Slater's Condition for the primal).

4.5. **Primal solubility: existence of optimal storage policy.** The operation problem is soluble for every $p \in L^1$, though not for every $p \in L^{\infty *}$. The assumption that $p \in L^1$ (i.e., $p_{\mathrm{FA}} = 0$) is maintained from here on until Remark 4.15.

Proposition 4.5 (Primal solubility). *Assume that $k_{\mathrm{Tu}} \geq e \geq 0$. If $p \in L^1$, then the SR profit-maximising operation programme (4.1)–(4.5) has an (optimal) solution (y, s_0). It follows that the problem (2.5)–(2.7) has a solution, i.e., $\hat{Y}(p, k_{\mathrm{H}}, e) \neq \emptyset$.*

Proof. With $p \in L^1$, the maximand (4.2) is continuous for the weak* topology $\mathrm{w}(L^\infty, L^1)$. The feasible set is bounded: in y by (4.3), and in s_0 by (4.5) with, e.g., $t = 0$. So, being also weakly* closed, the feasible set is compact by the Banach-Alaoglu Theorem. And it is nonempty, since the point $(y, s_0) = (e, 0)$ is feasible by assumption. ∎

4.6. **Determination of hydro plant's output.** Once the dual is solved, so that an optimal ψ is known, the operation problem largely reduces to maximisation of instantaneous profits (as Part 2c of Proposition 4.3 shows). At each time t with $p(t) \neq \psi(t)$, the optimum output $y_{\mathrm{H}}(t)$ is a "bang-bang control", either k_{Tu} or 0. Any remaining part of y is a "singular control" at a time t when instantaneous optimum is wholly indeterminate because $\psi(t) = p(t)$. This part can be determined on the assumption (4.35) that p has no plateau: this ensures that $p = \psi$ only when the reservoir is either empty or full, and at those times the output rate must equal $e(t)$. This gives a "bang-coast-bang" formula for $\hat{y}$ in terms of any optimal ψ (which not need be unique). See also Figure 1.

Proposition 4.6 (Hydro output with plateau-less price). *In addition to (4.8), assume that $p \in L^1_{++}[0,T]$ and that*

$$\forall \mathsf{p} \in \mathbb{R}_+ \ \operatorname{meas}\{t : p(t) = \mathsf{p}\} = 0. \tag{4.35}$$

If $y \in \hat{Y}(p, k_{\mathrm{H}}, e)$ and $\psi \in \hat{\Psi}(p, k_{\mathrm{H}}, e)$, i.e., y solves (2.5)–(2.7) and ψ solves (4.31)–(4.33), then

$$y(t) = \begin{cases} k_{\mathrm{Tu}} & \text{if } p(t) > \psi(t) \\ e(t) & \text{if } p(t) = \psi(t) \\ 0 & \text{if } p(t) < \psi(t) \end{cases}. \tag{4.36}$$

So (2.5)–(2.7) has a unique solution $\hat{y}(p, k_{\mathrm{H}}, e)$.

The assumption of a plateau-less price p is clearly restrictive, since—leading as it does to $\hat{y}_{\mathrm{H}}$ taking only the values specified in (4.36)—it can never hold in a general equilibrium with a continuous trajectory of hydro output. Such an equilibrium is made possible only by the presence of intervals on which $0 < s(t) < k_{\mathrm{St}}$ and $p = \psi =$ const. and $p = \psi =$ const.: being multivalued, the instantaneous optimum is then compatible with $y_{\mathrm{H}}(t)$ gradually changing in time from $e(t)$ to k_{Tu} or to 0.

For a proof of Proposition 4.6, and for the subsequent arguments, it is useful to introduce a notation for the sets of those times when the reservoir is empty or full or neither, given a hydro output y meeting the balance constraint $\int_0^T f(t)\,\mathrm{d}t = 0$. These sets (which have already appeared in Condition 2b of Proposition 4.3) are:

$$E(f) := \{t \in [0,T] : s(t) = 0\} \tag{4.37}$$

$$F(f, k_{\mathrm{St}}) := \{t \in [0,T] : s(t) = k_{\mathrm{St}}\} \tag{4.38}$$

$$B(f, k_{\mathrm{St}}) := [0,T] \setminus (E \cup F) = \{t : 0 < s(t) < k_{\mathrm{St}}\}, \tag{4.39}$$

where $s(t)$ is given by (4.6)–(4.7) in terms of $f := y - e$, and $k_{\mathrm{St}} \geq \mathrm{Max}(s)$. Since $s(0) = s(T)$, 0 and T are either both in B, or both in E, or both in F.[22] From (4.6), $E \neq \emptyset$. Unless the reservoir constraints are nonbinding, $F \neq \emptyset$ also; and then all the three sets are nonempty. Their connected components are subintervals of $[0,T]$; and, being open, B is the union of a countable (finite or denumerable) sequence of intervals. Those *not* containing 0 or T are denoted by

$$A_m = \left(\underline{t}_m, \bar{t}_m\right) \neq \emptyset$$

for $m = 1, \ldots, M \leq \infty$, where $0 \leq \underline{t}_m < \bar{t}_m \leq T$. If $\{0,T\} \subseteq B$, then B additionally contains two subintervals whose union is

$$A_0 = (\underline{t}_0, T] \cup \left[0, \bar{t}_0\right)$$

for some $0 < \bar{t}_0 < \underline{t}_0 < T$. When $0, T \notin B$, we set for completeness $\underline{t}_0 = T$ and $\bar{t}_0 = 0$, so that $A_0 = \emptyset$ in this case. In either case $B = \bigcup_{m \geq 0} A_m$.

All these sets may be thought of as subsets of the circle that results from "gluing" 0 and T into a single point $T0$. Then $(A_m)_{m \geq 0}$ are the *component arcs* of B (or B-arcs for brevity); A_0 is that arc which contains $T0$ (if $T0 \in B$); and $\underline{t}_m$ and $\bar{t}_m$ are the beginning and the end of arc A_m (w.r.t. the "clockwise" orientation).

The formula for the output y in terms of any $\psi \in \hat{\Psi}$ is proved next. On the set $\{t : p \neq \psi\}$, the optimal y equals unambiguously k_{Tu} or 0. Uniqueness of y on $\{p = \psi\}$ comes from the no-plateau assumption (4.35) on p: this ensures that $\{p = \psi\} \subseteq E \cup F$, up to a null set. And at each $t \in E \cup F$ one has $f(t) = -\dot{s}(t) = 0$ (and hence $y(t) = e(t)$), since, roughly speaking, $s =$ const. "around" t. The latter argument requires, however, a lemma to remove a technical difficulty in differentiating s that arises because (A_m) can be an infinite sequence; and then the set of component intervals of F and/or E can contain uncountably many singletons, in addition to

[22] These cases do not really differ if $p \in \mathcal{C}$ and $p(0) = p(T)$.

a countable set of "proper" intervals of positive length. On the interior of such an interval, $\dot{s} = 0$ obviously; but this must also be shown a.e. on the topological frontier of $F \cup E$. On the interior of such an interval, $\dot{s} = 0$ obviously; but this must also be shown to hold a.e. on the set of all singleton components of $F \cup E$. And the singletons in question *can* form a set of positive measure: indeed, all of $F \cup E$ can be a "fat" Cantor-like set that has a positive measure but contains no proper interval.

Lemma 4.7. *If $s \in \mathrm{Lip}\,[0,T]$ and $s = 0$ on a closed set E, then $\dot{s} = 0$ a.e. on E.*

Proof. See [33]. ∎

Proof of Proposition 4.6. Take any $\psi \in \hat{\Psi}$ (which may be nonunique, unless $p \in \mathcal{C}$). The first and the third lines of (4.36) hold by Part 2c of Proposition 4.3 and (4.14). It remains to show that $0 = y - e =: f$ a.e. on $S := \{t : p = \psi\}$. For each m, one has $\psi = \text{const.}$ on each $A_m(f, k_{\mathrm{St}})$ by Part 2b of Proposition 4.3 and (4.15). Therefore $\mathrm{meas}\,(S \cap A_m) = 0$ by (4.35), and hence $\mathrm{meas}\,(S \cap B\,(f, k_{\mathrm{St}})) = 0$ by countable additivity. This means that S is, up to a null set, contained in the set $F\,(f, k_{\mathrm{St}}) \cup E\,(f)$—on which, by Lemma 4.7, $f = -\dot{s} = 0$, i.e., $y = e$. This completes the proof of (4.36), establishing the uniqueness of y. ∎

4.7. **Rents of an hydroelectric plant.** Optimal values of the coefficients by k_{St}, k_{Tu} and e in the dual minimand (4.32) give the marginal resource values in terms of an optimal water price ψ. The optimal ψ is unique if p, the TOU price of electricity, is continuous over time, i.e., if $p \in \mathcal{C}\,[0,T]$.[23] The plant's operating profit $\Pi^{\mathrm{H}}_{\mathrm{SR}}$ is then differentiable in (k_{H}, e). The result extends to the case of $p \in L^{\infty *}$, if p_{CA} is continuous (Remark 4.15).

Lemma 4.8 (Shadow price uniqueness and continuity). *In addition to (4.8), assume that $p \in \mathcal{C}_{++}\,[0,T]$. Then the dual (4.31)–(4.33) has a unique (optimal) solution $\hat{\psi}\,(p, k_{\mathrm{H}}, e)$. If additionally $p\,(0) = p\,(T)$, then also $\hat{\psi}\,(0) = \hat{\psi}\,(T)$; i.e., if $p \in \mathcal{C}^{\mathrm{c}}\,[0,T]$ then $\hat{\psi} \in \mathcal{C}^{\mathrm{c}}\,[0,T]$.*

Proof. Fix any primal solution $y \in \hat{Y}$, which exists by Proposition 4.3 (though it may be nonunique). To show that there is just one dual solution, we shall express every dual solution $\psi \in \hat{\Psi}$ by the same formula in terms of y.[24]

In the case of $F\,(y, k_{\mathrm{St}}) \neq \emptyset$, which we deal with first, we shall use the Kuhn-Tucker Conditions to show that any $\psi \in \hat{\Psi}$ can be given, in terms of y, as

$$(4.40) \qquad \hat{\psi}\,(p, k_{\mathrm{H}}, e)\,(t) = p\,(t) \quad \text{for every } t \in (E \cup F)\,(f, k_{\mathrm{St}}) \setminus \{0, T\}$$

[23] The optimal ψ is unique as a bounded-variation function on $(0,T)$, extended by continuity to $[0,T]$; as has already been noted, $\psi\,(0-) = \psi\,(T+)$ lies anywhere between $\psi\,(0+)$ and $\psi\,(T-)$.

[24] The basis for this strategy is that every dual solution supports every primal solution (i.e., that the set of saddle points for a dual pair of convex programmes is a Cartesian product).

whereas on the m-th component A_m of $B(f, k_{\mathrm{St}})$, whose endpoints are $\underline{t}_m$ and $\overline{t}_m$, it is the constant

$$\hat{\psi}(t) = \begin{cases} p(\underline{t}_m) & \text{if } \underline{t}_m \neq 0 \\ p(\overline{t}_m) & \text{if } \overline{t}_m \neq T \end{cases} \tag{4.41}$$

for each $m \geq 0$. Since both E and F are nonempty, $A_m \neq (0,T)$, so at least one line of (4.41) applies; and when both do, they are consistent (e.g., for $m = 0$ one has $p(\underline{t}_0) = p(\overline{t}_0)$ if $0, T \in B$). It follows that (4.40)–(4.41) fully determine ψ on $(0,T)$, and hence on $[0,T]$, since at the endpoints ψ is defined by continuity.

To use the Kuhn-Tucker Conditions as stated in Proposition 4.3—in terms of $(\kappa, \nu, \psi, \lambda)$ rather than ψ alone—recall recall from Subsection 4.4 that if a $\psi \in \mathrm{BV}(0,T)$ solves (4.31)–(4.33), then (4.9)–(4.15) is solved by: the same ψ, $\left(\kappa^{\mathrm{Tu}}, \nu^{\mathrm{Tu}}\right) = \left((p-\psi)^+, (\psi - p)^+\right)$, any λ between $\psi(0+)$ and $\psi(T-)$ and $\left(\kappa^{\mathrm{St}}, \nu^{\mathrm{St}}\right) = \left(\mu_+^{\mathrm{St}}, \mu_-^{\mathrm{St}}\right)$, where $\mu^{\mathrm{St}} = \mathrm{D}\psi$ on $(0,T)$ with $\mu\{0\} = \psi(0+) - \lambda$ and $\mu\{T\} = \lambda - \psi(T-)$.

By (4.14)–(4.15),

$$p = \psi + \kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}} = \lambda + \left(\kappa^{\mathrm{St}} - \nu^{\mathrm{St}}\right)[0,\cdot] + \kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}} \quad \text{a.e.} \tag{4.42}$$

It suffices to show that ψ is continuous everywhere on $(0,T)$ and equal to p on $(E \cup F) \setminus \{0,T\}$: (4.41) follows, since ψ is also constant on each B-component A_m, and since $A_m \neq (0,T)$. The main ideas, already set out in Section 3 with Figures 1a and 2, are that nonuniqueness of ψ arise only together with its discontinuity, and that this in turn would imply the discontinuity of p (which is excluded by assumption). And this is because any discontinuities of ψ and $\kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}}$, the two terms of p in (4.42), cannot cancel each other out. A discontinuity of ψ could only be a jump/drop at a time t when the reservoir is full/empty, respectively. If $t \in F$ say, then, being full at t, the reservoir cannot be being discharged just before t.[25] That is, just before t the outflow y cannot exceed the inflow e, which, by assumption, is smaller than k_{Tu}. *A fortiori*, the capacity charge κ^{Tu} must be zero just before t. Similarly, just after a $t \in F$ the reservoir cannot be being charged, i.e., the outflow y cannot be less than the inflow e, which is positive by assumption; so ν^{Tu} must be zero just after t. Therefore $\kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}}$ could change discontinuously at t only in the same direction as ψ (upward if $t \in F$), and not in the opposite direction. So both terms of p must be continuous if their sum is. The "upside down" version of this reasoning applies to $t \in E$.

Since κ^{Tu} and ν^{Tu} are equivalence classes, this argument is formalised by using the essential limit concept—for which see, e.g., [11, IV.36–IV.37] or [54, II.9: p. 90]. It is also convenient to say that an inequality between functions holds *somewhere* on $A \subseteq [0,T]$ to mean that it holds on an $A' \subseteq A$ with $\operatorname{meas} A' > 0$ (i.e., it is *not* the case that the reverse inequality holds a.e. on A).

[25]This, by the way, is where the constancy of k_{St} over t is used.

Together with the $\underline{s}_0$ of (4.6), y solves (4.1)–(4.5). Consider first a $t \in F \backslash \{0, T\}$. For every $\Delta t > 0$, it cannot be that $f > 0$ a.e. on $(t - \Delta t, t)$; i.e., somewhere on $(t - \Delta t, t)$ one has $y \leq e < k_{\mathrm{Tu}}$. So $\kappa^{\mathrm{Tu}} = 0$ somewhere on $(t - \Delta t, t)$, by Part 2c of Proposition 4.3; and, as $\Delta t \to 0$, this shows that the lower left essential limit of κ^{Tu} at t is zero. Similarly, somewhere on $(t, t + \Delta t)$ one has $f \geq 0$, i.e., $y \geq e > 0$. So $\nu^{\mathrm{Tu}} = 0$ somewhere on $(t, t + \Delta t)$. This means that the lower right essential limit of ν^{Tu} at t is zero; i.e.,

$$\operatorname{ess} \liminf_{\tau \searrow t} \nu^{\mathrm{Tu}} (\tau) = 0 = \operatorname{ess} \liminf_{\tau \nearrow t} \kappa^{\mathrm{Tu}} (\tau) \quad \text{for } t \in F \backslash \{0, T\} . \tag{4.43}$$

Given (4.42) as well as continuity of p and nonnegativity of κ^{Pu} and κ^{Tu}, it follows from (4.43) that[26]

$$\begin{aligned} p(t) - \psi(t-) &= \operatorname{ess} \lim_{\tau \nearrow t} \left(\kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}} \right)(\tau) \\ &= \operatorname{ess} \liminf_{\tau \nearrow t} \kappa^{\mathrm{Tu}} (\tau) - \operatorname{ess} \liminf_{\tau \nearrow t} \nu^{\mathrm{Tu}} (\tau) \leq 0 \qquad (4.44) \\ &\leq \operatorname{ess} \liminf_{\tau \searrow t} \kappa^{\mathrm{Tu}} (\tau) - \operatorname{ess} \liminf_{\tau \searrow t} \nu^{\mathrm{Tu}} (\tau) = \operatorname{ess} \lim_{\tau \searrow t} \left(\kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}} \right)(\tau) \qquad (4.45) \\ &= p(t) - \psi(t+) . \end{aligned}$$

Therefore $\psi(t-) \geq \psi(t+)$ from a comparison of the first and the last sums. But also, since $t \in F$,

$$\psi(t-) \leq \psi(t+) \tag{4.46}$$

by Part 2b of Proposition 4.3; so all three inequalities (4.44), (4.45) and (4.46) must actually hold as equalities. This shows that $\psi(t-) = \psi(t+) = p(t)$, i.e., the two-sided limit of ψ at t exists and equals $p(t)$. (Since it exists, it also equals $\psi(t)$ because $\psi(t)$ always lies between $\psi(t-)$ and $\psi(t+)$.) The same can be shown for $t \in E$ (by the "upside down" version of the proof for F); so

$$\psi(t) = \lim_{\tau \to t} \psi(\tau) = p(t) \quad \text{for } t \in (E \cup F) \setminus \{0, T\} \neq \emptyset. \tag{4.47}$$

Nonemptiness of this set follows from the assumption that $F \neq \emptyset$, since $E \neq \emptyset$ always, by (4.6).

By Part 2b of Proposition 4.3, ψ is constant on each A_m. This and (4.47) show that $\psi \in \mathcal{C}(0, T)$. (Equivalently $\psi \in \mathcal{C}[0, T]$, since $\psi(0) := \psi(0+)$ and $\psi(T) := \psi(T-)$.)

It remains to show that the proven properties of ψ imply (4.41). Since $E \cup F \nsubseteq \{0, T\}$, the set B consists of two or more nonempty components A_m. Each of these has at least one endpoint that is neither 0 nor T; i.e., $\underline{t}_m \neq 0$ or $\bar{t}_m \neq T$ ($\underline{t}_m \neq T$ and

[26]This argument uses also the fact that $\liminf (A - B) \leq \liminf A - \liminf B \leq \limsup (A - B)$ whenever the middle term is well defined. It equals $\lim (A - B)$ if the latter exists, as here (although the inequalities suffice). The same holds with $\limsup A - \limsup B$ as the middle term.

$\bar{t}_m \neq 0$ always). Say it is $\underline{t}_m$; then $\underline{t}_m \in (E \cup F) \setminus \{0, T\}$, since $\underline{t}_m \notin A_m$ (A_m is an open arc). So, by (4.47) and the constancy of ψ on A_m,

$$p(\underline{t}_m) = \psi(\underline{t}_m) = \psi(t) \quad \text{for every } t \in A_m. \tag{4.48}$$

If $T \neq \bar{t}_m$, then (4.48) holds with $\bar{t}_m$ in place of $\underline{t}_m$, by the same argument. This also shows that $p(\underline{t}_m) = p(\bar{t}_m)$ if both $\underline{t}_m \neq 0$ and $\bar{t}_m \neq T$. (All this applies to $m = 0$ as well, if $A_0 \neq \emptyset$. Additionally in this case ψ is constant on $A_0 \supset \{0, T\}$; so $\psi(0) = \psi(T)$ even if $p(0) \neq p(T)$.) This fully proves (4.40)–(4.41), when $F \neq \emptyset$.

If $p(0) = p(T)$, then $\psi(0+) = \psi(T-)$ follows by virtually the same argument as that proving (4.47), with 0 and T thought of as a single point of the circle.

Finally, consider the case of $F(f, k_{\mathrm{St}}) = \emptyset$, which is trivial in that the reservoir is never used to capacity, and it earns no rent. Formally, $\kappa^{\mathrm{St}} = \nu^{\mathrm{St}} = 0$ by Part 2b of Proposition 4.3 and (4.13); so ψ is a constant. Its uniqueness is readily shown: ψ minimises (4.32) over $\mathrm{BV}(0, T)$, so, *a fortiori*, it minimises (4.32) over $\mathbb{R}$. Since for $\psi \in \mathbb{R}$ the sum (4.32) simplifies to

$$k_{\mathrm{Tu}} \int_0^T (p - \psi)^+ (t)\, \mathrm{d}t + \int_0^T \psi(t)\, e(t)\, \mathrm{d}t,$$

the minimum in question is characterised by the FOC

$$\mathrm{meas}\{t : p(t) > \psi\} \leq \frac{1}{k_{\mathrm{Tu}}} \int_0^T e(t)\, \mathrm{d}t \leq \mathrm{meas}\{t : p(t) \geq \psi\},$$

which means that ψ is an upper quantile of order $(1/Tk_{\mathrm{Tu}}) \int_0^T e(t)\, \mathrm{d}t$ for the distribution of p with respect to meas/T.[27] And the quantile is unique if $p \in \mathcal{C}$, since the cumulative distribution function of p is then strictly increasing on the interval $(\mathrm{Min}(p), \mathrm{Max}(p))$. ∎

Comment: Although (4.43) suffices for the argument, both inf signs therein can be deleted, i.e., (4.43) can be strengthened to: $\kappa^{\mathrm{Tu}}(t-) = 0 = \nu^{\mathrm{Tu}}(t+)$ with $\nu^{\mathrm{Tu}}(t-) \geq 0$ and $\kappa^{\mathrm{Tu}}(t+) \geq 0$, for $t \in F \setminus \{0, T\}$, whenever $p(t\pm)$ exist.[28] This is because, by (4.16) and the continuity of $\kappa \mapsto \kappa_{\pm} \in \mathbb{R}_+$, the four limits exist and are equal to $\left(\kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}}\right)_{\pm}(t\pm) = (p - \psi)^{\pm}(t\pm)$. All four limits are zero if p is continuous at t.

It follows that $\Pi^{\mathrm{H}}_{\mathrm{SR}}(p; k_{\mathrm{H}}, e)$ is differentiable in (k_{H}, e) if $p \in \mathcal{C}[0, T]$.

Theorem 4.9 (Efficiency rents of a hydro plant)**.** *In addition to (4.8), assume that $p \in \mathcal{C}_{++}[0, T]$. Then the dual problem of water pricing (4.31)–(4.33) has a unique solution $\hat{\psi}(p, k_{\mathrm{H}}, e)$. It follows that the operating profit of a hydro plant—i.e., the value of the primal problem (2.5)–(2.7)—is differentiable with respect to the water*

[27] Note that $0 < \int_0^T e(t)\, \mathrm{d}t < Tk_{\mathrm{Tu}}$ by (4.8).

[28] The abbreviations $\kappa(t\pm)$ for the essential (one-sided) limits should not be mistaken for the ordinary limits of a particular variant of κ, in as much as the ordinary limits may be nonexistent.

inflow function e and the capacities, k_{St} and k_{Tu}, of the reservoir and the turbine. The derivatives defining the unit rents are given by the formulae

$$\hat{r}^{\mathrm{St}}\left(p;k_{\mathrm{H}},e\right):=\frac{\partial\Pi_{\mathrm{SR}}^{\mathrm{H}}}{\partial k_{\mathrm{St}}}\left(p;k_{\mathrm{H}},e\right)=\mathrm{Var}_{\mathrm{c}}^{+}\left(\hat{\psi}\right) \tag{4.49}$$

$$\hat{r}^{\mathrm{Tu}}\left(p;k_{\mathrm{H}},e\right):=\frac{\partial\Pi_{\mathrm{SR}}^{\mathrm{H}}}{\partial k_{\mathrm{Tu}}}\left(p;k_{\mathrm{H}},e\right)=\int_{0}^{T}\left(p-\hat{\psi}\right)^{+}\left(t\right)\mathrm{d}t \tag{4.50}$$

$$\nabla_{e}\Pi_{\mathrm{SR}}^{\mathrm{H}}\left(p;k_{\mathrm{H}},e\right)=\hat{\psi}. \tag{4.51}$$

Comment: Because of its marginal interpretation, a shadow water price $\psi\in\hat{\Psi}\left(p,k_{\mathrm{H}},e\right)$ can be used to decentralise the operating decisions within the storage plant (as already mentioned in Section 3), with the reservoir "buying" water at the price $\psi\left(t\right)$ from the river and "selling" it to the turbine, which in turn sells the generated electricity at the market price $p\left(t\right)$ outside the plant. In this context the complementary slackness conditions mean that, for each of the capital inputs ($\phi=\mathrm{St}$, Tu), its unit rent $\hat{r}^{\phi}k_{\phi}:=\partial\Pi_{\mathrm{SR}}^{\mathrm{H}}/\partial\phi$ equals its unit operating profit from the internal or external sales, $f=y-e$ and $y\in\hat{Y}\left(p,k_{\mathrm{H}},e\right)$. For the turbine this can be seen directly from Part 2c of Proposition 4.3 with (4.34) and (4.50). For the reservoir, by Lebesgue-Stieltjes integration by parts over $[0,T]$,

$$\begin{aligned}\int_{0}^{T}\hat{\psi}\left(t\right)f\left(t\right)\,\mathrm{d}t&=-\int_{0}^{T}\hat{\psi}\left(t\right)\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t=-\left[\hat{\psi}\left(t\right)s\left(t\right)\right]_{t=0-}^{t=T+}+\int_{[0,T]}s\,\mathrm{d}\hat{\psi}\\&=s_{0}\left(\hat{\psi}\left(0-\right)-\hat{\psi}\left(T+\right)\right)+k_{\mathrm{St}}\kappa^{\mathrm{St}}\left[0,T\right]=0+k_{\mathrm{St}}\,\mathrm{Var}_{\mathrm{c}}^{+}\left(\hat{\psi}\right)\\&=k_{\mathrm{St}}\hat{r}^{\mathrm{St}}\end{aligned}$$

by (2.2), Part 2b of Proposition 4.3, (4.15) and (4.49). Of course, reinterpretation of ψ as a "market" price solves nothing by itself: the questions of uniqueness and calculation of ψ (given p, k_{H} and e) still arise.

Before a formal proof of Theorem 4.9, it is worth retracing in the present context the familiar argument which establishes the derivative property of the value function when differentiability is taken for granted. With the dual minimand (4.32) denoted by $N\left(k_{\mathrm{H}},e,\psi\right)$, the r.h.s.'s of (4.49)–(4.51) are obviously the partial derivatives of N in $\left(k_{\mathrm{H}},e\right)$ evaluated at the dual optimum $\hat{\psi}\left(k_{\mathrm{H}},e\right)$. And the total derivatives, in $\left(k_{\mathrm{H}},e\right)$, of the dual value $N\left(k_{\mathrm{H}},e,\hat{\psi}\left(k_{\mathrm{H}},e\right)\right)$ are equal to the corresponding partial derivatives, since the partial derivative of N in ψ vanishes by the FOC for the optimality of $\hat{\psi}$. To complete the calculation, note that the dual value equals the primal value $\Pi_{\mathrm{SR}}^{\mathrm{H}}$.[29] This is, indeed, the substance of the first step in the Proof of Theorem 4.9, except that a standard convex duality result is used instead of the above derivation "from first principles". This is necessary because a rigorous application of the chain rule would

[29] Conversely, the equality of SR profit to fixed-input value can be rederived from (4.49)–(4.51) by an application of Euler's Theorem to Π as a jointly homogeneous function of (k,e).

run into difficulties, since it would require the differentiability of $\hat{\psi}$ in $(k_{\rm H}, e)$, and also of N in ψ. This would make their composition $\Pi(k_{\rm H}, e) = N\left(k_{\rm H}, e, \hat{\psi}(k_{\rm H}, e)\right)$ differentiable, but even this should not be presupposed; and the optimal ψ's can actually be nonunique (so $\hat{\psi}$ does not exist) unless $p \in \mathcal{C}$. Differentiability of Π must be proved—by using price continuity, since it is known to fail in general if $p \notin \mathcal{C}$ (Example 3.1). This gap is filled by Lemma 4.8.

Proof of Theorem 4.9 (Efficiency rents of a hydro plant). The first, routine step is to identify the dual variables as marginal values of the primal parameters, with the marginal values formalised as supergradients (of the primal value, a concave function of the parameters): see, e.g., [47, Theorem 16: (b) and (a), with Theorem 15: (e) and (f)] or [37, 7.3: Theorem 1']. This is applied in such a way as to give the marginal interpretation to the optimal κ and ν themselves, rather than only to their totals over the cycle, although the formulae to be proved are for the total values. Therefore the SR profit is considered as a function, $\widetilde{\Pi}^{\rm H}_{\rm SR}$, of all the quantity parameters

$$(\Delta k_{\rm St}, \Delta n_{\rm St}; \Delta k_{\rm Tu}, \Delta n_{\rm Tu}; \Delta e, \Delta\zeta) \in \mathcal{C} \times \mathcal{C} \times L^\infty \times L^\infty \times L^\infty \times \mathbb{R}$$

discussed in Subsection 4.2. It is an extension of the optimal value of the programme (4.1)–(4.5), i.e.,

$$\Pi^{\rm H}_{\rm SR}(p; k_{\rm St}, k_{\rm Tu}, e) = \widetilde{\Pi}^{\rm H}_{\rm SR}(p; k_{\rm St}, 0; k_{\rm Tu}, 0; e, 0) \quad \text{for } (k_{\rm St}, k_{\rm Tu}) \in \mathbb{R}^2,$$

where scalars are identified with constant functions on $[0, T]$. In this setting, the result giving the marginal values of the primal parameters is that[30]

$$\text{(4.52)} \quad \partial_{k_{\rm St}, n_{\rm St}, k_{\rm Tu}, n_{\rm Tu}, e, \zeta} \widetilde{\Pi}^{\rm H}_{\rm SR} = \left\{ \left(\kappa^{\rm St}, -\nu^{\rm St}, \kappa^{\rm Tu}, -\nu^{\rm Tu}, \psi, \lambda\right) : (\kappa, \nu, \psi, \lambda) \text{ meet Conditions 2a, 2b and 2c of Proposition 4.3} \right\}.$$

For differentiation of $\Pi^{\rm H}_{\rm SR}$, with respect to the *constant* capacities and the cyclically varying inflow, it follows from (4.52) that

$$\partial_{k_{\rm St}, k_{\rm Tu}, e} \Pi^{\rm H}_{\rm SR} = \left\{ \left(\int_{[0,T]} \kappa^{\rm St}(\mathrm{d}t), \int_0^T \kappa^{\rm Tu}(t)\,\mathrm{d}t, \psi \right) : \exists \nu\ \exists \lambda\ (\kappa, -\nu, \psi, \lambda) \in \partial_{k_{\rm H}, n_{\rm H}, e, \zeta} \widetilde{\Pi}^{\rm H}_{\rm SR} \right\}$$

$$\text{(4.53)} \qquad = \left\{ \left(\mathrm{Var}^+_{\rm c}(\psi), \int_0^T (p - \psi)^+(t)\,\mathrm{d}t, \psi \right) : \psi \in \hat{\Psi}(p; k_{\rm St}, k_{\rm Tu}, e) \right\},$$

by using (4.34) and the substitution $\kappa^{\rm St} = (\mathrm{D}\psi)^+$. Since the set $\hat{\Psi}$ in (4.53) is actually a singleton by Lemma 4.8, so is $\partial_{k_{\rm H}, e} \Pi^{\rm H}_{\rm SR}(p; k_{\rm H}, e)$; and the proof is complete. ∎

[30] The corresponding result for the marginal values of dual parameters is that $\partial_p \Pi^{\rm H}_{\rm SR} = \hat{Y}$, which is Hotelling's Lemma: see the Appendix for details.

Comment: Some weaker results on the relationship of an optimal ψ to p are much simpler to establish than (4.40)–(4.41), but such results are so weak as to be of little use by themselves. For example:

1. When the number of B-arcs is finite, the equality $\psi = p$ a.e. on $F \cup E$ can be shown by the argument that $\hat{s} =$ const. and so $\hat{y} = e$ a.e. on each F-arc or E-arc R, so $\psi = p$ a.e. on R (and even everywhere on int R if p is continuous, in which case it follows that $\psi = p$ on $F \cup E$, except possibly at the endpoints of F- and E-arcs, whose number is finite). But capacity valuation requires also the values of ψ on the B-arcs—and this necessitates the additional arguments in Proof of Lemma 4.8.
2. By using Lemma 4.7, the equality $\psi = p$ a.e. on $F \cup E$ can be shown for every $p \in L^1_+$. But this may even be vacuous ($F \cup E$ may be a null set), and the stronger result (4.41) does depend on the continuity of p.

4.8. **Case of piecewise monotone electricity price function.** The preceding analysis reduces the problems of rental valuation and plant operation to the water pricing problem (4.31)–(4.33). This in turn reduces to the determination of the constancy arcs of $\hat{\psi}$: on each A_m the shadow price for the water's potential energy is constant and equal to the endpoint value of the electricity price p. These results can be given a more concrete form when p is piecewise monotone, i.e., when

(4.54) there is a finite partition of the time circle into arcs on each of which p is either strictly decreasing or strictly increasing.

The *arc sequence* $(A_m)_{m=0}^{M}$ is then finite, of length not exceeding the number of peaks and troughs of p. (These are defined as *local* maximum or minimum points of p on the circle. The point $T0$, counting always as one, will be taken to be a trough.) A full characterisation of the arc sequence follows. It reduces the continuous-time, infinite-dimensional programmes in question to finite-dimensional ones exactly (and not only approximately like discretisation of time).

Proposition 4.10. *Assume (4.54) with $p \in \mathcal{C}^{c}_{++}[0,T]$ having a local minimum at $T0$. Then, for a hydro plant with capacities $k_{\mathrm{H}} = (k_{\mathrm{St}}, k_{\mathrm{Tu}}) \in \mathbb{R}^2_{++}$ and with an inflow $e \in L^\infty[0,T]$ satisfying (4.8), the shadow price function for water and the profit-maximising output are both unique; i.e., the dual problem (4.31)–(4.33) and the primal problem (2.5)–(2.7) have unique (optimal) solutions $\hat{\psi}(p, k_{\mathrm{H}}, e)$ and $\hat{y}(p, k_{\mathrm{H}}, e) \neq e$. These and the set $B = B\left(\hat{f}, k_{\mathrm{St}}\right)$, defined by (4.39) with $\hat{f} = \hat{y} - e$, have the properties:*

1. *For $m = 0, \ldots, M$,*

$$p\left(\underline{t}_m\right) = p\left(\bar{t}_m\right) \tag{4.55}$$

i.e., the electricity price is the same at both endpoints of A_m, which therefore contains at least one peak or trough of p.

2. *B consists of a finite number, $M+1$, of open arcs $(A_m)_{m=0}^{M}$.*
3. *Every local extremum point (peak or trough) of p belongs to A_m for some m.*
4.

$$\hat{\psi}(p, k_{\mathrm{H}}, e)(t) = \begin{cases} p(\underline{t}_m) = p(\overline{t}_m) & \text{for } t \in A_m \text{ and } m = 0, \dots, M \\ p(t) & \text{for } t \notin B \end{cases} \tag{4.56}$$

$$\hat{y}(p, k_{\mathrm{H}}, e)(t) = \begin{cases} k_{\mathrm{Tu}} & \text{if } p(t) > \hat{\psi}(p, k_{\mathrm{H}}, e)(t) \\ e(t) & \text{if } p(t) = \hat{\psi}(p, k_{\mathrm{H}}, e)(t) \\ 0 & \text{if } p(t) < \hat{\psi}(p, k_{\mathrm{H}}, e)(t) \end{cases} \tag{4.57}$$

5. *For each m, denote by $I^m(t)$ the running total net discharge corresponding to the optimal output; i.e., $I^m(t)$ is the Lebesgue integral over $(\underline{t}_m, t)_{\mathrm{c}}$ of the r.h.s. of (4.57) minus the inflow e, with $\hat{\psi}$ substituted from (4.56). Then the function I^m maps A_m into either $(0, k_{\mathrm{St}})$ or $(-k_{\mathrm{St}}, 0)$, attaining at $\overline{t}_m$ one of its bounds: k_{St} or 0 if $I^m > 0$ on A_m, or $-k_{\mathrm{St}}$ or 0 if $I^m < 0$ on A_m.*[31]
6. *For each $m = 0, \dots, M$ (with the arcs A_m numbered chronologically, starting from $T0 \in A_0$, and with $M+1$ understood as 0), p is above/below $\hat{\psi}(p, k_{\mathrm{H}}, e)$ initially on A_{m+1} if and only if p is finally on A_m below/above $\hat{\psi}(p, k_{\mathrm{H}}, e)$, respectively.*[32]

Conversely, given such (p, k_{H}, e), if $(A_m)_{m=0}^{M}$ is a finite sequence of open arcs having Properties 1, 3, 5 and 6, then the solutions to (2.5)–(2.7) and (4.31)–(4.33) are given by (4.57) and (4.56).

Proof. The primal solution $\hat{y}$ is unique by Proposition 4.6: (4.35) is met, since the set in question is actually finite by (4.54). And $\hat{y} \neq e$ by Remark 4.11 below. The dual solution $\hat{\psi}$ is unique by Lemma 4.8, which also gives (4.55) of Part 1 and (4.56). (The exclusions $\underline{t} \neq 0$, $\overline{t} \neq T$ and $t \neq 0, T$ in (4.41)–(4.42) are unnecessary when $p \in \mathcal{C}^{\mathrm{c}}$, as here.) Substitution of $\hat{\psi}$ for ψ in (4.36) gives (4.57), completing the proof of Part 4.

By Part 1, the number of B-arcs does not exceed the number of peaks and troughs of p. The rest of Part 2 follows readily.

For Part 3, suppose contrarily that p has a local extremum at $t' \notin B$ (so $t' \in F \cup E$). Consider, e.g., the case of a trough $t' \in E$. Take an open arc A (of nonzero length) beginning at t', disjoint from F and sufficiently short for p to be strictly increasing on A. For $t \in A$ one has $\hat{\psi}(t) \le \hat{\psi}(t') = p(t')$, by Part 2b of Proposition 4.3 with (4.15) and by (4.41). Since $p(t') < p(t)$ for $t \in A$, this implies that $\hat{\psi} < p$ on A, so $\mathrm{d}\hat{s}/\mathrm{d}t = -f = e - y = e - k_{\mathrm{Tu}} < 0$ on A, by Part 2c of Proposition 4.3 and (4.8).

[31] It follows that $p - \hat{\psi}$ is initially on A_m, from $\underline{t}_m$ to its next root, of that sign which I^m has throughout A_m; whereas finally on A_m, from its penultimate root to $\overline{t}_m$, $p - \hat{\psi}$ is of the same/opposite sign (and so the number of extrema of p in A_m is odd/even) if respectively $\left|I^m\left(\overline{t}(A_m)\right)\right|$ is k_{St} or 0.

[32] To say that "a function p is above/below ψ initially/finally on an arc A" means, in formal terms, that $\operatorname{sgn}(p - \psi) = \pm 1$ on a sufficiently short open arc starting at $\underline{t}(A)$ or ending at $\overline{t}(A)$, respectively.

But this is not feasible, since $\hat{s}(t') = 0$ (i.e., being empty at t', the reservoir cannot be discharged any further). The other three cases lead to similar contradictions (e.g., that on an A ending at a price peak $t' \in E$ one has $\hat{y} = 0$, and so $\mathrm{d}\hat{s}/\mathrm{d}t = e > 0$), which shows that $t' \in B$.

The four cases listed in Part 5 correspond to the four combinations of $\underline{t}_m \in S$ and $\bar{t}_m \in S'$, where both S and S' are either F or E.

Given (4.57) and the first line of (4.56), Part 6 follows from the fact that if $\bar{t}_m \in S$, then also $\underline{t}_{m+1} \in S$, for $S = F, E$.

The converse can be shown by verification of Conditions 2a–2c of Proposition 4.3. Since this is straightforward, the details are omitted. ∎

4.9. **Miscellaneous remarks.** As has already been noted, the "pure coasting" policy $y = e$—which is trivial in that it makes no use of the reservoir—is feasible by (4.8). But it is not optimal.

Remark 4.11 (Non-coasting hydro output). *For any $p \in L^1_+[0,T]$, $k_{\mathrm{H}} \in \mathbb{R}^2_{++}$ and $e \in L^\infty_{++}[0,T]$ meeting (4.8), if p is nonconstant then $e \notin \hat{Y}(p, k_{\mathrm{H}}, e)$.*

Proof. Note that $e \in \hat{Y}$ is equivalent to $\Pi^{\mathrm{H}}_{\mathrm{SR}} = \int_0^T p(t)\, e(t)\, \mathrm{d}t$, which means (since the dual and the primal values are equal) that $\int_0^T pe\, \mathrm{d}t$ is the value of the programme (4.31)–(4.33). The minimand (4.32) can be rewritten and estimated from below as

$$(4.58)\quad k_{\mathrm{St}} \operatorname{Var}^+_{\mathrm{c}}(\psi) + \int_0^T (p-\psi)^+ (k_{\mathrm{Tu}} - e)\, \mathrm{d}t + \int_0^T \left((p-\psi)^+ + \psi\right) e\, \mathrm{d}t \geq 0 + 0 + \int_0^T p(t)\, e(t)\, \mathrm{d}t.$$

For its minimum to equal $\int_0^T pe\, \mathrm{d}t$ it is therefore necessary (and sufficient) that some ψ meets the conditions: $\psi = \text{const.}$, $p \leq \psi$ and $p \geq \psi$—i.e., $p = \psi = \text{const.}$ ∎

The hydro plant's optimal output is invariant under monotone transformations of the price function p (given k_{H} and e).

Remark 4.12 (Output invariance under monotone price transformations). *Assume that k_{H} and e meet (4.8). If $p \in L^1[0,T]$, and ι is a strictly increasing (real-valued) function on $p[0,T]$ such that $\iota \circ p \in L^1$, then*[33]

$$(4.59)\quad \hat{Y}(\iota \circ p; k_{\mathrm{H}}, e) = \hat{Y}(p; k_{\mathrm{H}}, e) \quad \text{and} \quad \hat{\Psi}(\iota \circ p; k_{\mathrm{H}}, e) = \iota \circ \hat{\Psi}(p; k_{\mathrm{H}}, e),$$

where $\iota \circ \Psi := \{\iota \circ \psi : \psi \in \Psi\}$.

[33]Since a $p \in L^1$ is defined only up to a null set, $p[0,T]$ means here the essential range of p, i.e., the smallest closed set whose inverse image under p has full Lebesgue measure. For $p \in \mathcal{C}$, this is the usual range of p.

Proof. This follows from Conditions 2a–2c of Proposition 4.3 (after recasting them by giving (κ, ν, λ) in terms of ψ as in the Proof of Lemma 4.8): a pair (y, ψ) meets these Kuhn-Tucker Conditions if and only if $(y, \iota \circ \psi)$ meets the same conditions but with $\iota \circ p$ in place of p. ∎

Remark 4.13. *With $k_{\mathrm{St}} > 0$, every shadow price $\psi \in \hat{\Psi}(p, k_{\mathrm{H}}, e)$—i.e., every solution to (4.31)–(4.33)—is always piecewise monotone (also when p is not).*

Proof. This is obvious if the sequence (A_m) of B-arcs is finite (since, by Part 2b of Proposition 4.3, ψ is nonincreasing, constant or nondecreasing on, respectively, each E-arc, B-arc or F-arc). But even when (A_m) is an infinite sequence, the following argument applies.

Since F and E are disjoint closed sets, the (circular) distance between them is positive; specifically

$$\mathrm{dist}_{\mathrm{c}}(E, F) \geq \frac{k_{\mathrm{St}}}{\mathrm{EssSup}(e)} \wedge \frac{k_{\mathrm{St}}}{k_{\mathrm{Tu}}} = \frac{k_{\mathrm{St}}}{k_{\mathrm{Tu}}} > 0,$$

since these are lower bounds for the times needed to fully charge/discharge the reservoir (and since $e \leq k_{\mathrm{Tu}}$). It follows that any B-arc shorter than $\mathrm{dist}_{\mathrm{c}}(E, F)$ has both endpoints in either F or E. For $S = E$, F, denote by S' the union of S and all those B-arcs with *both* endpoints in S. Then F' and E' are also closed sets disjoint from each other, and the complement of $F' \cup E'$ is the union of a finite number of B-arcs (viz., of not more than $T/\mathrm{dist}_{\mathrm{c}}(E, F) \leq T k_{\mathrm{Tu}}/k_{\mathrm{St}}$). So $F' \cup E'$ is a finite union of pairwise disjoint closed arcs. Every such arc, R, is contained in either F' or E' (otherwise R would be partitioned into two nonempty closed sets $R \cap E'$ and $R \cap F'$, which is impossible for a connected set R). So R is disjoint either from $E' \supseteq E$ or from $F' \supseteq F$, and therefore ψ is respectively nonincreasing or nondecreasing on R. ∎

Remark 4.14. *Assume that $p \in \mathcal{C}$, so the unique shadow price $\hat{\psi}$ solving (4.31)–(4.33) is in $\mathcal{C}$. If $p(0) \neq p(T)$, i.e., $p \notin \mathcal{C}^{\mathrm{c}}$, then it can be that $\hat{\psi} \notin \mathcal{C}^{\mathrm{c}}$, i.e., that $\hat{\psi}(0) := \hat{\psi}(0+) \neq \hat{\psi}(T-) =: \hat{\psi}(T)$. In such a case the optimal λ's fill the whole range of values between $\hat{\psi}(0)$ and $\hat{\psi}(T)$. This means that, when the balance constraint is perturbed to $s(0) - s(T) = \zeta$, the right and left partial derivatives of the SR profit w.r.t. ζ (at $\zeta = 0$) are*

$$\frac{\partial \widetilde{\Pi}^{\mathrm{H}}_{\mathrm{SR}}}{\partial_{+}\zeta}\Big|_{\zeta=0} = \hat{\psi}(0) \wedge \hat{\psi}(T) \leq \hat{\psi}(0) \vee \hat{\psi}(T) = \frac{\partial \widetilde{\Pi}^{\mathrm{H}}_{\mathrm{SR}}}{\partial_{-}\zeta}\Big|_{\zeta=0}.$$

The assumption needed for $\nabla_k \Pi^{\mathrm{H}}_{\mathrm{SR}}$ to exist is next weakened to: $p_{\mathrm{CA}} \in \mathcal{C}$.

Remark 4.15 (Case of concentrated charges)**.** *When p has a nonzero p.f.a. term in the decomposition (2.8), this can be interpreted as the "extremely concentrated" part of capacity charges for the turbine—since $p_{\mathrm{FA}} = \kappa^{\mathrm{Tu}}_{\mathrm{FA}}$ at every dual optimum, as (4.19) shows. Such charges can arise in a general equilibrium with uninterruptible demand for the flow in question. And, when the consumption and production rates are* *continuous over time, such charges do have a tractable mathematical representation by singular measures, such as point measures: see* [31, Example 3.1]*.*

The presence of a p.f.a. term $p_{\mathrm{FA}} \neq 0$ *can, however, result in the nonexistence of an optimum* y *for the primal (2.6)–(2.7): see Case (b) in Part 5 below. Except for this, the analysis extends* mutatis mutandis *to the case of* $p \in L^{\infty *}_{++}$*, by replacing* p *with* p_{CA} *and adding rental terms involving* p_{FA}*. This is spelt out below.*

For $p \in L^{\infty *}_{+}$ with $p_{\mathrm{CA}} \gg 0$:

1. The dual problem becomes (4.32)–(4.33) with p_{CA} instead of p and with $k_{\mathrm{Tu}} \|p_{\mathrm{FA}}\|^{*}_{\infty}$ added to the minimand (4.32).[34] Since the extra term is a constant (i.e., is independent of ψ), its addition does not change the dual solution set (which is nonempty).
2. Theorem 4.9 holds with p replaced by p_{CA} and with $\|p_{\mathrm{FA}}\|^{*}_{\infty} = \langle p_{\mathrm{FA}}, 1\rangle$ added to the r.h.s. of (4.50). To see this, it suffices to note that, by Part 1 and the equality of the dual and primal values,
$$\Pi^{\mathrm{H}}_{\mathrm{SR}}(p) - \Pi^{\mathrm{H}}_{\mathrm{SR}}(p_{\mathrm{CA}}) = k_{\mathrm{Tu}} \|p_{\mathrm{FA}}\|^{*}_{\infty}.$$
3. Conditions 2a–2c of Proposition 4.3 imply the same but with p_{CA} in place of p. (The converse is obviously false.) Given Part 1, this means that $\hat{Y}(p) \subseteq \hat{Y}(p_{\mathrm{CA}})$, i.e., if p supports y as a profit maximum, then so does p_{CA} (or equivalently if y solves (2.6)–(2.7), then it also solves (2.6)–(2.7) with p_{CA} in place of p). This can also be established by verifying that the production set meets our Exclusion Condition of [29], which we do for $\mathbb{Y}_{\mathrm{H}}$ in Lemma 6.2 below.
4. Therefore the results on any primal optimum y, such as Proposition 4.6, hold also with p_{CA} in place of p (though they may be vacuous because, at p, there may be no optimal y).
5. To see how the timing of a $p_{\mathrm{FA}} > 0$ matters for the existence of a primal optimum, consider the cases in which such a term is concentrated on each neighbourhood of: either (a) a peak $\bar{t}$, or (b) a trough $\underline{t}$, of a $p_{\mathrm{CA}} \in \mathcal{C}^{\mathrm{c}}$ satisfying (4.54). With $k_{\mathrm{St}} > 0$, Parts 3 and 4 of Proposition 4.10 show that $\hat{y}(p_{\mathrm{CA}}) = k_{\mathrm{Tu}}$ around $\bar{t}$ and $\hat{y}(p_{\mathrm{CA}}) = 0$ around $\underline{t}$. At $p = p_{\mathrm{CA}} + p_{\mathrm{FA}}$ one has: $\hat{y}(p) = \hat{y}(p_{\mathrm{CA}})$ in Case (a), whereas in Case (b) $\hat{Y}(p) = \emptyset$. This can be shown formally by comparing the increments in $\Pi^{\mathrm{H}}_{\mathrm{SR}}$ and in the value of the output $\hat{y}(p_{\mathrm{CA}})$ that result from adding the term p_{FA}: in Case (a) both are equal (to $k_{\mathrm{Tu}} \|p_{\mathrm{FA}}\|$), so $\hat{y}(p_{\mathrm{CA}})$ remains optimal. But in Case (b), the one is $k_{\mathrm{Tu}} \|p_{\mathrm{FA}}\| > 0$ by Part 2, whereas the other is 0; so there is no optimum at p (since $\hat{y}(p_{\mathrm{CA}})$ is the only possibility, by Part 3). In heuristic terms, this is because in Case (b) the extra price term requires a

[34]This also points to a case of the primal value being strictly less—it is never greater—than the dual value. This is when $p_{\mathrm{FA}} > 0$, $p_{\mathrm{CA}} \in \mathrm{BV}$ and $k_{\mathrm{Tu}} > \mathrm{Sup}(e) \geq \mathrm{Inf}(e) > 0$ but $k_{\mathrm{St}} = 0$: the SR profit is then $\langle p, e\rangle$, but the fixed-input value (as found from the dual) is $\int_0^T p_{\mathrm{CA}} e \, \mathrm{d}t + k_{\mathrm{Tu}} \|p_{\mathrm{FA}}\|^{*}_{\infty} > \langle p, e\rangle$, since the dual solution is $\hat{\psi} = p_{\mathrm{CA}}$, and since $k_{\mathrm{Tu}} > \mathrm{Sup}(e)$. When $k_{\mathrm{St}} > 0$, the primal and dual values are of course equal.

brief switch from filling ($y = 0$) to discharging ($y = k_{\mathrm{Tu}}$) around $\underline{t}$—the briefer the better, so no best policy exists.

5. Storage policies with spillage

This section is a brief outline of an extension that dispenses with the condition that $e \le k_{\mathrm{Tu}}$, if spillage is feasible as assumed in (2.4).

To incorporate spillage into the preceding the analysis, one must first of all modify the primal problem (4.1)–(4.5). This means adding the spillage term, $\varphi \in L^{\infty}_{+}$, to the net outflow f, as in (2.1). The extra variable is constrained as in (2.4), i.e., $0 \le \varphi \le e$. There is, however, no real need for an extra Lagrange multiplier corresponding to the constraint $\varphi \ge 0$ because such a multiplier would turn out to be identical to ψ (at the dual optimum). The multiplier must of course be nonnegative, but the constraint $\psi \ge 0$ need not be adjoined to the dual, since it is met anyway by any solution to (4.9)–(4.15) if $p \ge 0$. The multiplier for the constraint $\varphi \le e$ turns out to be zero: the primal value is the same with or without this constraint.[35] This means that free disposal of water is effectively unlimited, as in [39, 1.4a].[36] Finally, an extra slackness condition, that $\psi = 0$ a.e. on $\{t : \varphi(t) > 0\}$,is adjoined to Part 2c of Proposition 4.3.

It can then be proved formally that an optimal storage policy involves no spillage if $p \in L^{1}_{++}$ and $k_{\mathrm{Tu}} \ge e$. This can be shown either by establishing that $\psi \gg 0$, or directly as is sketched next. Suppose contrarily that $\varphi > 0$ on a neighbourhood of some t. If $y(t) < k_{\mathrm{Tu}}(t)$ then the output can be increased around t, so (y, φ) is not optimal. If $y(t) = k_{\mathrm{Tu}}(t)$ then $\dot{s}(t) = (-y + e - \varphi)(t) \le 0 - \varphi(t) < 0$, i.e., the stock is falling around t, and so there is room to store a unit being spilt, to release it at the nearest opportunity (which will come, since $\varphi \ne 0$ implies that $y(\tau) < e(\tau) \le k_{\mathrm{Tu}}$ for some τ). Again, this shows that (y, φ) is not optimal. And although this argument handles y, e and φ as though they were continuous functions (rather than elements of L^{∞}), it can be made fully rigorous by choosing t to be a density point[37] of the set $\{y < k_{\mathrm{Tu}}\}$ or $\{y = k_{\mathrm{Tu}}\}$, respectively.

When spillage φ is assumed feasible, one can drop the condition that $e \le k_{\mathrm{Tu}}$ (whilst retaining $\mathrm{EssInf}(e) > 0$). The primal and dual problems remain feasible, and the Kuhn-Tucker characterisation of optimality (Proposition 4.3) continues to hold, with the above modifications.[38] Some spillage may, however, be unavoidable if the inflow exceeds the turbine's capacity. If this occurs on a relatively short interval, it

[35] If $p \in L^{1}_{+}$, there is an optimum policy with $\varphi(t) \le (e(t) - k_{\mathrm{Tu}})^{+} < e(t)$.

[36] In reality the spillage rate at any time t is constrained—quite apart from the considerations of flood control, etc.—by the spillway capacity below the current water level (unless the reservoir is full at t, in which case there is an automatic overflow "from the top", equal to any excess of $e(t) - y(t)$ over the spillway capacity).

[37] For this concept, which is also used in proving Lemma 4.7, see, e.g., [15, (5.8)] or [49, Exercise 8.11].

[38] Verification of Slater's Condition now requires a different choice of a feasible policy, viz., any (y, φ) with $y + \varphi = e$ and $k_{\mathrm{Tu}} - \epsilon \ge y \ge \epsilon$, $\varphi \ge \epsilon$ (for some number $\epsilon > 0$).

changes the solution in the following way. Consider an inflow increment $(k_{\mathrm{Tu}} - e) + \Delta e$ on an interval $[\underline{t}, \overline{t}]$ on which the reservoir is full in the original operating solution, which corresponds to an inflow $e < k_{\mathrm{Tu}}$. To make room for the total excess inflow, an extra amount $\Delta E = \int_{\underline{t}}^{\overline{t}} \Delta e(t)\, \mathrm{d}t$ of water should be discharged immediately before $\underline{t}$, with the turbine operating at full capacity to sell the extra output at best prices, as close to $p(\underline{t})$ as possible. This solution is supported by the stock price ψ that "freezes" when the discharge starts and stays constant until $\overline{t}$, when it jumps back to the original pricing solution. As ΔE increases, so the discharge period preceding $[\underline{t}, \overline{t}]$ starts earlier. Here we assume that it does not merge with an earlier water collection period (during which $p < \psi$) before ΔE reaches k_{St}.[39] In the borderline case of $\Delta E = k_{\mathrm{St}}$, the reservoir becomes empty at $\underline{t}$ and full again at $\overline{t}$. The no-spillage solution is still feasible, but only just; and the water price on $[\underline{t}, \overline{t}]$ is an arbitrary constant between 0 and $\psi(\underline{t})$.[40] If ΔE is further increased (keeping $\underline{t}$ and $\overline{t}$ fixed), then a total of $\Delta E - k_{\mathrm{St}}$ must be spilt on $[\underline{t}, \overline{t}]$. This can be done in any way but ψ is unique, since $\psi = 0$ on $[\underline{t}, \overline{t}]$.

6. Production set properties for density representation of equilibrium prices

This section verifies the conditions for inclusion of the hydro technology in an equilibrium model with $L^\infty[0,T]$ and $L^1[0,T]$ as the commodity and price spaces. It is shown that the production set $\mathbb{Y}_{\mathrm{H}}$ is weakly* closed. It is also shown to meet our Exclusion Condition of [29], which serves the purpose of L^1-price representation and yet is significantly weaker than the Exclusion Assumption of [7].

Lemma 6.1. *$\mathbb{Y}_{\mathrm{H}}$ is $\mathrm{w}(L^\infty, L^1)$-closed.*

Proof. By the Krein-Smulian Theorem (for which see, e.g., [20, 18E]), it suffices to show that the set $\mathbb{Y}_{\mathrm{H}}$ is closed for the bounded weak* topology of L^∞; and for this it suffices to establish that the set

$$\mathbb{Y}_{\mathrm{H}} \cap \left\{ (y, -k_{\mathrm{H}}, -e) : k_{\mathrm{H}} \leq \overline{k}_{\mathrm{H}}, e \leq \overline{e} \right\}$$

is weakly* compact for each $\overline{k}_{\mathrm{H}} = \left(\overline{k}_{\mathrm{St}}, \overline{k}_{\mathrm{Tu}}\right) \in \mathbb{R}^2_+$ and $\overline{e} \in L^\infty_+$ (since the bound on k_{Tu} bounds y also). The latter set is the image, $\pi(S)$, of the set S of all those $(y, -k_{\mathrm{H}}, -e; s_0, \varphi)$ meeting the conditions: $\varphi \in [0, e]$ and (4.3)–(4.5) with $f = y - e + \varphi$, $k_{\mathrm{H}} \leq \overline{k}_{\mathrm{H}}$ and $e \leq \overline{e}$, under the map π that sends such a point to $(y, -k_{\mathrm{H}}, -e)$. Since S is weakly* compact (by the Banach-Alaoglu Theorem), and since π is weak*-to-weak* continuous, $\pi(S)$ is weakly* compact. ∎

Lemma 6.2. *$\mathbb{Y}_{\mathrm{H}}$ meets the Exclusion Condition of* [29].

[39] If the two do merge, then the two constant values of ψ become one value, which decreases as ΔE continues to increase, with the collection period being reduced.

[40] This indeterminacy is noted in [39, p. 226: last paragraph].

Proof. This follows from the Mackey continuity[41] of the function $\check{k}_{\mathrm{St}}\colon L^{\infty} \to \mathbb{R}$ defined by

$$\check{k}_{\mathrm{St}}(f) := \mathrm{Max}(I_0 f) + \mathrm{Max}(I_T f). \tag{6.1}$$

This gives the *storage capacity requirement* (when f is the net outflow from the reservoir): see [33] for details. To verify the Exclusion Condition, take any $(p, r^{\mathrm{H}}, \psi) \in L^{\infty *} \times \mathbb{R}^2 \times L^{\infty *}$ and an evanescent sequence of measurable sets $A_m \subset [0, T]$ supporting both p_{FA} and ψ_{FA} (so $\mathrm{meas}\, A_m \to 0$ as $m \to \infty$). Take any $(y, -k_{\mathrm{H}}, -e) \in \mathbb{Y}_{\mathrm{H}}$; i.e., $y \in [0, k_{\mathrm{Tu}}]$ and there exists a $\varphi \in [0, e]$ such that

$$\int_0^T f(t)\, \mathrm{d}t = 0 \quad \text{and} \quad \check{k}_{\mathrm{St}}(f) \le k_{\mathrm{St}},$$

where $f := y - e + \varphi$. As can readily be shown, there is a sequence $B_m \supseteq A_m$ with $\mathrm{meas}\, B_m \to 0$ and $\int_{B_m} f(t)\, \mathrm{d}t = 0$. Define $y^m := y 1_{[0,T] \setminus B_m}$ and $e^m := e 1_{[0,T] \setminus B_m}$ and $\varphi^m := \varphi 1_{[0,T] \setminus B_m}$ (so $f^m := f 1_{[0,T] \setminus B_m}$), where 1_A denotes the 0-1 indicator of a set A. Define also

$$k_{\mathrm{St}}^m := k_{\mathrm{St}} - \check{k}_{\mathrm{St}}(f) + \check{k}_{\mathrm{St}}(f^m).$$

Then $\int_0^T f^m = 0$; and $\check{k}_{\mathrm{St}}(f^m) \le k_{\mathrm{St}}^m$ (from the definitions and the inequality $\check{k}_{\mathrm{St}}(f) \le k_{\mathrm{St}}$). Also, $0 \le y^m \le y \le k_{\mathrm{Tu}}$ and $0 \le \varphi^m \le e^m$. Furthermore $f^m \to f$ in $\mathrm{m}(L^{\infty}, L^1)$, so $k_{\mathrm{St}}^m \to k_{\mathrm{St}}$ as $m \to \infty$ by (6.1) and Lemma 4.2. Put together, this shows that the sequence

$$(y^m, -k_{\mathrm{H}}^m, -e) = (y^m, -k_{\mathrm{St}}^m, -k_{\mathrm{Tu}}, -e) \in \mathbb{Y}_{\mathrm{H}}$$

has the required properties, viz.,

$$\left\langle (y^m, -k_{\mathrm{H}}^m, -e), (p, r^{\mathrm{H}}, \psi)_{\mathrm{FA}} \right\rangle = \langle y^m, p_{\mathrm{FA}} \rangle - \langle e^m, \psi_{\mathrm{FA}} \rangle = 0$$

and

$$\begin{aligned} &\left\langle (y^m, -k_{\mathrm{H}}^m, -e) - (y, -k_{\mathrm{H}}, -e), (p, r^{\mathrm{H}}, \psi)_{\mathrm{CA}} \right\rangle \\ &\qquad = \langle y^m - y, p_{\mathrm{CA}} \rangle - \left\langle k_{\mathrm{St}}^m - k_{\mathrm{St}}, r^{\mathrm{St}} \right\rangle - \langle e^m - e, \psi_{\mathrm{CA}} \rangle \to 0 \end{aligned}$$

as $m \to \infty$. ∎

It follows that pure density prices obtain in a general equilibrium model with the hydro technique if the rest of the technology also meets our Exclusion Condition and consumer preferences are Mackey continuous: see [29]. In the context of pricing a continuous-time flow, our Exclusion Condition is also met by the following producer types:

1. a pure supplier, producing a flow of the good in question from a finite number of homogeneous input goods;

[41] It is the upper semicontinuity that is relevant here.

2. a pure user of the flow with a Mackey continuous production function (i.e., a producer of a homogeneous good, the output quantity of which is a Mackey continuous function of the input flow). See [29] for a proof.

For a model of the Electricity Supply Industry with hydro-thermal generation, the equilibrium TOU tariff is therefore a pure price density (i.e., a time-varying rate in \$/kWh) if the demand for electricity is interruptible, since this is the meaning of the Mackey continuity assumptions in the context of continuous-time consumption: see [25].[42]

7. Thermal generation technology

Hydroelectric generation is also studied together with the thermal generating technology, for various purposes. In Section 8 we use profit-based valuation of hydro inputs to characterise LRMC and optimality of a hydro-thermal system in terms of SR functions. Koopmans [39] also studies the hydro-thermal combination of technologies, but his does this to set up the operation problem as one of SR cost minimisation: unlike the hydro technique, thermal techniques have significant variable costs. We comment on his analysis in Section 9. Before discussing either topic, we review the fixed-coefficients model of the thermal technology.[43]

A thermal generating system k_{Th} is specified by the installed capacity of each type of station $\theta \in \Theta$. If finite, the set Θ of thermal techniques can be enumerated as $\{1, 2, \ldots, \#\Theta\}$, and a thermal system is then $k_{\mathrm{Th}} = (k_1, k_2, \ldots, k_{\#\Theta}) \in \mathbb{R}_+^{\#\Theta}$, with a total capacity of $\sum k_{\mathrm{Th}} := k_1 + k_2 + \ldots + k_{\#\Theta}$, where k_θ is the capacity (in kW) of type θ. However, Koopmans [39, pp. 198 ff] assumes a strictly convex SR cost curve, and this necessitates a "continuum" of plant types, since the marginal fuel cost w^θ is constant for each type θ, in the fixed-coefficients model. Both the "continuous" and the finite cases are captured by representing a thermal system by a nonnegative Borel measure, $k_{\mathrm{Th}} \geq 0$, on a compact space Θ of plant types, i.e., by a $k_{\mathrm{Th}} \in \mathcal{M}_+(\Theta)$. The total capacity is then $k_{\mathrm{Th}}(\Theta) = \|k_{\mathrm{Th}}\|_{\mathrm{Var}}$, the usual variation norm of a measure. A system with a finite number of plant types is represented by a point measure.

For each plant type θ, its *unit variable cost* w^θ (a.k.a. unit fuel cost, operating or running cost, in \$/kWh) is determined by the relevant fuel price and the plant's fuel consumption coefficient. Here w^θ is assumed to be a continuous and strictly positive function of θ, i.e., $w \in \mathcal{C}_{++}(\Theta)$. So $0 < \mathrm{Min}(w) := \min_{\theta \in \Theta} w^\theta$, and $\mathrm{Max}(w) < +\infty$.[44]

A system k_{Th} defines a certain distribution of capacity over the unit variable costs. Formally this is the direct image, $k_{\mathrm{Th}} \circ w^{-1}$, of the measure k_{Th} on Θ under the map $w\colon \Theta \to \mathbb{R}$, a.k.a. the distribution of the function $w\colon \Theta \to \mathbb{R}$ w.r.t. the measure k_{Th} on Θ; and it is defined by $(k_{\mathrm{Th}} \circ w^{-1})(S) := k_{\mathrm{Th}}(w^{-1}(S))$ for every Borel set

[42]The extra assumptions are unnecessary for the density representation of water prices: that $\psi_{\mathrm{FA}} = 0$ follows from $\psi \in \partial_e \Pi_{\mathrm{SR}}^{\mathrm{H}} \subset \mathrm{BV} \subset L^1$.

[43]For a general discussion of fixed-coefficients techniques with multiple outputs, see [32].

[44]The range of w is the interval $[\mathrm{Min}(w), \mathrm{Max}(w)]$ if Θ is a connected topological space, e.g., an interval of $\mathbb{R}$.

$S \subseteq \mathbb{R}$. Its cumulative distribution function (c.d.f.) is a nondecreasing function from $[\mathrm{Min}(w), \mathrm{Max}(w)]$ into $[0, \|k_{\mathrm{Th}}\|]$. The inverse of this c.d.f. is the *the system's marginal variable cost*, i.e., the unit variable cost of the system's marginal station on line. This is a nondecreasing function, denoted here by $w_{\uparrow}(\mathsf{y}, k_{\mathrm{Th}})$, of the system load $\mathsf{y} \in [0, \|k_{\mathrm{Th}}\|]$. Its graph is also known as the *capacity-incremental operating cost curve*: see, e.g., [6, Figure 5(a)].[45]

Comments:

1. We denote the inverse of the c.d.f. (of $k_{\mathrm{Th}} \circ w^{-1}$) by $w_{\uparrow}(\cdot, k_{\mathrm{Th}})$ because it is the *nondecreasing rearrangement* of the function w: $\Theta \to \mathbb{R}$ w.r.t. the measure k_{Th} on Θ, i.e., it is the nondecreasing function on $[0, \|k_{\mathrm{Th}}\|]$ whose distribution w.r.t. meas is the same as the distribution of w w.r.t. k_{Th}. For a discussion, see, e.g., [41, 1.D] and [50, Lemma 1].
2. The one-sided (left and right) limits of $w_{\uparrow}$ exist at every $\mathsf{y} \in (0, \|k_{\mathrm{Th}}\|)$, with $w_{\uparrow}(\mathsf{y}-) = w_{\uparrow}(\mathsf{y}+)$ for nearly every y. Where the two limits differ, $w_{\uparrow}(\mathsf{y})$ is best viewed as a correspondence, i.e., $w_{\uparrow}(\mathsf{y}) = [w_{\uparrow}(\mathsf{y}-), w_{\uparrow}(\mathsf{y}+)]$, as in Figure 5c (where $\check{k}$ is the optimal system, discussed below). It is also convenient to define $w_{\uparrow}(0-) := 0$ and $w_{\uparrow}(\|k_{\mathrm{Th}}\|+) := \mathrm{Max}(w)$.

The thermal system's *instantaneous SR cost* (a.k.a. the system's fuel cost per unit time, in \$/h) is the integrated inverse of the system's c.d.f. of the unit variable cost, i.e.,

$$c_{\mathrm{SR}}(\mathsf{y}, k_{\mathrm{Th}}, w) := \int_0^{\mathsf{y}} w_{\uparrow}(\mathsf{x}, k_{\mathrm{Th}})\, \mathrm{d}\mathsf{x}. \tag{7.1}$$

This is a convex and increasing function of the output rate $\mathsf{y} \in [0, \|k_{\mathrm{Th}}\|]$, with $c_{\mathrm{SR}}(\mathsf{y}) = 0$ for $\mathsf{y} \le 0$ and $c_{\mathrm{SR}}(\mathsf{y}) = +\infty$ for $\mathsf{y} > \|k_{\mathrm{Th}}\|$, as in Figure 5d.[46] Thus c_{SR} represents the capacity constraint as well as the variable cost actually incurred. The SR cost curve has a kink at full capacity $\mathsf{y} = \|k_{\mathrm{Th}}\|$, and it can also have an offpeak kink at some $\mathsf{y} < \|k_{\mathrm{Th}}\|$, in which case the curve has two different slopes to the immediate right and left of y. All the intermediate slopes form the subdifferential $\partial c(\mathsf{y})$ at y; and the graph of the correspondence $\mathsf{y} \mapsto \partial c(\mathsf{y})$ is the *thermal short-run marginal cost* (SRMC) curve, a.k.a. the perfectly competitive SR supply curve (Figure 5).

Formally, c_{SR} has the *left* derivative $\mathrm{d}c/\mathrm{d}_{-}\mathsf{y}$ and the *right* derivative $\mathrm{d}c/\mathrm{d}_{+}\mathsf{y}$ at each $\mathsf{y} \le \|k_{\mathrm{Th}}\|$; and the ordinary two-sided derivative $\mathrm{d}c/\mathrm{d}\mathsf{y}$ exists if the two one-sided derivatives are equal. If not, then $\partial c(\mathsf{y}) = [\mathrm{d}c/\mathrm{d}_{-}\mathsf{y}, \mathrm{d}c/\mathrm{d}_{+}\mathsf{y}]$. This is also equal to $w_{\uparrow}(\mathsf{y}, k_{\mathrm{Th}})$ for $\mathsf{y} \in [0, \|k_{\mathrm{Th}}\|)$, with $\partial c(\|k_{\mathrm{Th}}\|) = [w_{\uparrow}(\|k_{\mathrm{Th}}\|-), +\infty)$.

[45] This is the operating cost for merit-order loading, which, given the constancy of w^{θ} for a fixed θ, is optimal and identical to incremental loading. For a discussion of the latter when the two differ, see, e.g., [55, 3.4, 5.2: Figure 5.1].

[46] If Θ is finite, then c_{SR} is piecewise linear in y.

The total SR cost of the thermal system's output $y_{\mathrm{Th}} \in L^{\infty}_{+}[0,T]$ over the cycle is

$$C^{\mathrm{Th}}_{\mathrm{SR}}(y_{\mathrm{Th}}, k_{\mathrm{Th}}, w) = \int_0^T c_{\mathrm{SR}}(y_{\mathrm{Th}}(t), k_{\mathrm{Th}}, w)\,\mathrm{d}t. \tag{7.2}$$

For a finite set Θ of plant types, the determination of an optimal thermal system is discussed in detail detail in, e.g., [6, 61–65: Figure 7], [42, pp. 37–40: Figure 3-4] and [55, 6.2: Figure 6.1]. The following supplementary remarks focus on Koopmans' case of a continuum Θ.

Given a unit fixed cost r^{θ} (a.k.a. capacity cost, in \$/kW) for each $\theta \in \Theta$, the station type that is optimal for meeting a load of duration τ per cycle is that θ which minimises $r^{\theta} + \tau w^{\theta}$.[47] The unit variable cost of the optimal station type is denoted by $\check{w}(\tau)$, and its unit fixed cost by $\check{r}(\tau)$. The (minimum) thermal LR cost of a unit load of duration $\tau > 0$ is

$$c_{\mathrm{LR}}(\tau) := \min_{\theta}\left(r^{\theta} + \tau w^{\theta}\right) = \check{r}(\tau) + \tau \check{w}(\tau). \tag{7.3}$$

This is a concave, increasing function of τ, with $c_{\mathrm{LR}}(0) := 0$ and

$$\frac{\mathrm{d}c_{\mathrm{LR}}}{\mathrm{d}\tau}(\tau) = \check{w}(\tau) \quad \text{for nearly every } \tau \in (0,T]; \tag{7.4}$$

see Figure 4c. The graph of c_{LR} is known as the *total cost-duration curve*. The exceptional τ's in (7.4) are those for which $\check{w}(\tau)$ is actually a proper interval (rather than a single number), viz.,

$$\check{w}(\tau) := w^{\check{\Theta}(\tau)}, \text{ where } \check{\Theta}(\tau) := \operatorname*{ArgMin}_{\theta}\left(r^{\theta} + \tau w^{\theta}\right) \tag{7.5}$$

is the nonempty set of all optimal station types. However, the set of such τ's *is* at most countable because $\check{w}$ is a nonincreasing u.h.c. correspondence from $(0,T]$ into $\mathbb{R}$: see Figure 4b.

The points $\left(r^{\theta}, w^{\theta}\right)_{\theta\in\Theta}$ form the *ex ante capital-fuel substitution curve* (a.k.a. the efficient technological frontier or the cost characteristics curve), shown in Figure 4a. For simplicity it is assumed that:

1. Each station type is fully identified by its unit variable cost, which ranges over an interval $[\mathsf{w}', \mathsf{w}''] = [\mathrm{Min}(w), \mathrm{Max}(w)]$. That is, $w^{\theta} := \theta$ for each $\theta \in \Theta := [\mathsf{w}', \mathsf{w}'']$.
2. No station type is redundant for LR cost minimisation. That is, r^{w} is a *strictly* convex (and decreasing) function of $\mathsf{w} \in [\mathsf{w}', \mathsf{w}'']$, with $r^{\mathsf{w}} = +\infty$ for $\mathsf{w} < \mathsf{w}'$ and $r^{\mathsf{w}} = r^{\mathsf{w}''}$ for $\mathsf{w} > \mathsf{w}''$. Then $\check{w}(\tau)$, which is the solution for w to the inclusion $\tau \in -\partial r(\mathsf{w})$, is a continuous and nonincreasing function of $\tau \in (0,T]$. (Without the strict convexity, $\check{w}$ would be a u.h.c. correspondence. Also, if r^{w} is differentiable in w, then $\check{w}(\tau)$ decreases in τ strictly.)

[47] Like w^{θ}, r^{θ} is taken to be a continuous function of θ.

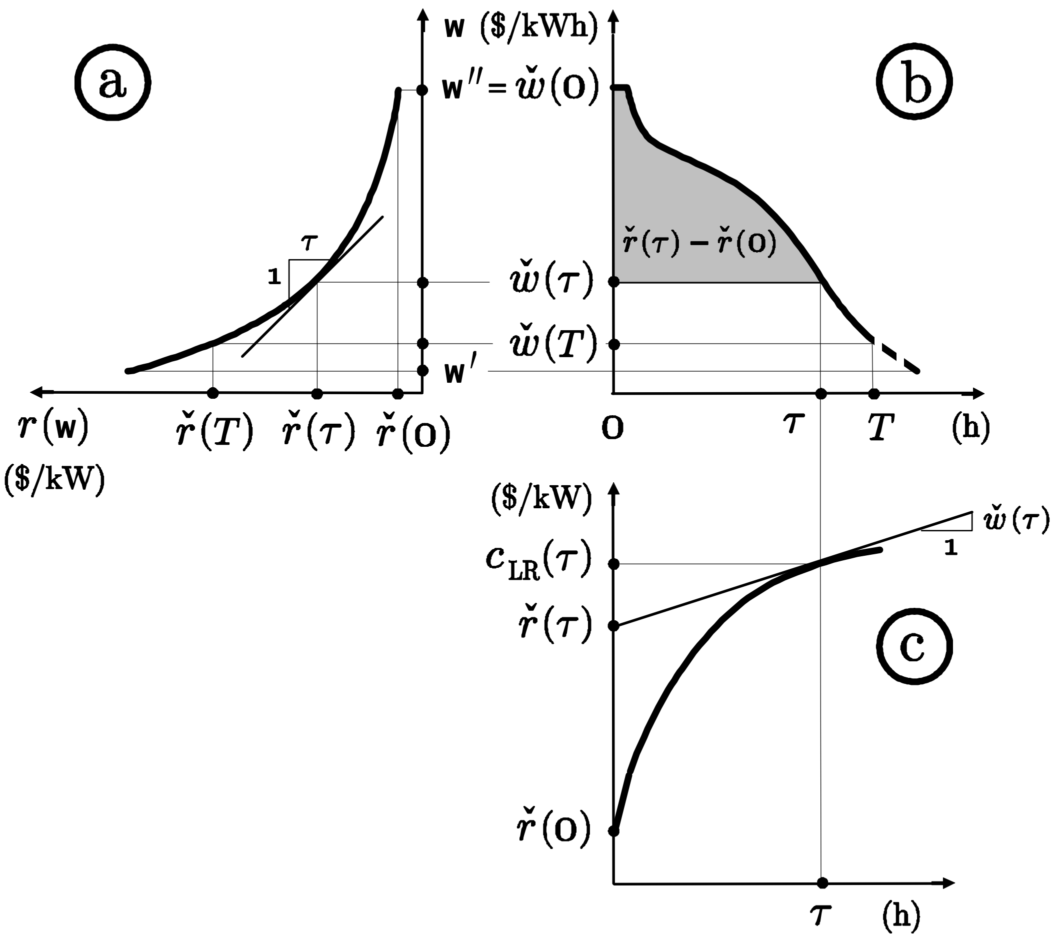

FIGURE 4. Part (c) shows the thermal LR unit cost c_{LR} (as a function of load duration τ). This is constructed from the capital-fuel substitution curve—shown in (a)—through $\check{w}(\tau)$, the unit fuel cost of the station optimal for a load of duration τ, shown in (b).

The LR cost-minimising system $\check{k}_{\mathrm{Th}}$ is then unique for each output y_{Th}, and it can be given in terms of the thermal *load-duration curve* (LDC), which is the graph of the nonincreasing rearrangement $y_{\mathrm{Th}}^{\downarrow}$ of y_{Th} (w.r.t. meas). Namely, $\check{k}_{\mathrm{Th}}$ contains $y_{\mathrm{Th}}^{\downarrow}(\tau + \mathrm{d}\tau) - y_{\mathrm{Th}}^{\downarrow}(\tau)$ units (kW's) of those station types between $\check{w}(\tau)$ and $\check{w}(\tau + \mathrm{d}\tau)$. Formally $\check{k}_{\mathrm{Th}}$ is the image, under the map $\check{w}$: $(0, T] \to [\mathsf{w}', \mathsf{w}'']$, of the measure on $(0, T]$ whose c.d.f. is $-y_{\mathrm{Th}}^{\downarrow} + \mathrm{EssSup}(y_{\mathrm{Th}})$.[48] See Figure 5.

[48] With the notation (4.30), $\check{k}_{\mathrm{Th}}\left(y_{\mathrm{Th}}; r^{\mathrm{Th}}, w\right)(S) = \left(-\mathrm{D}y_{\mathrm{Th}}^{\downarrow}\right)\left(\check{w}^{-1}(S)\right)$ for $S \subseteq [\mathsf{w}', \mathsf{w}'']$.

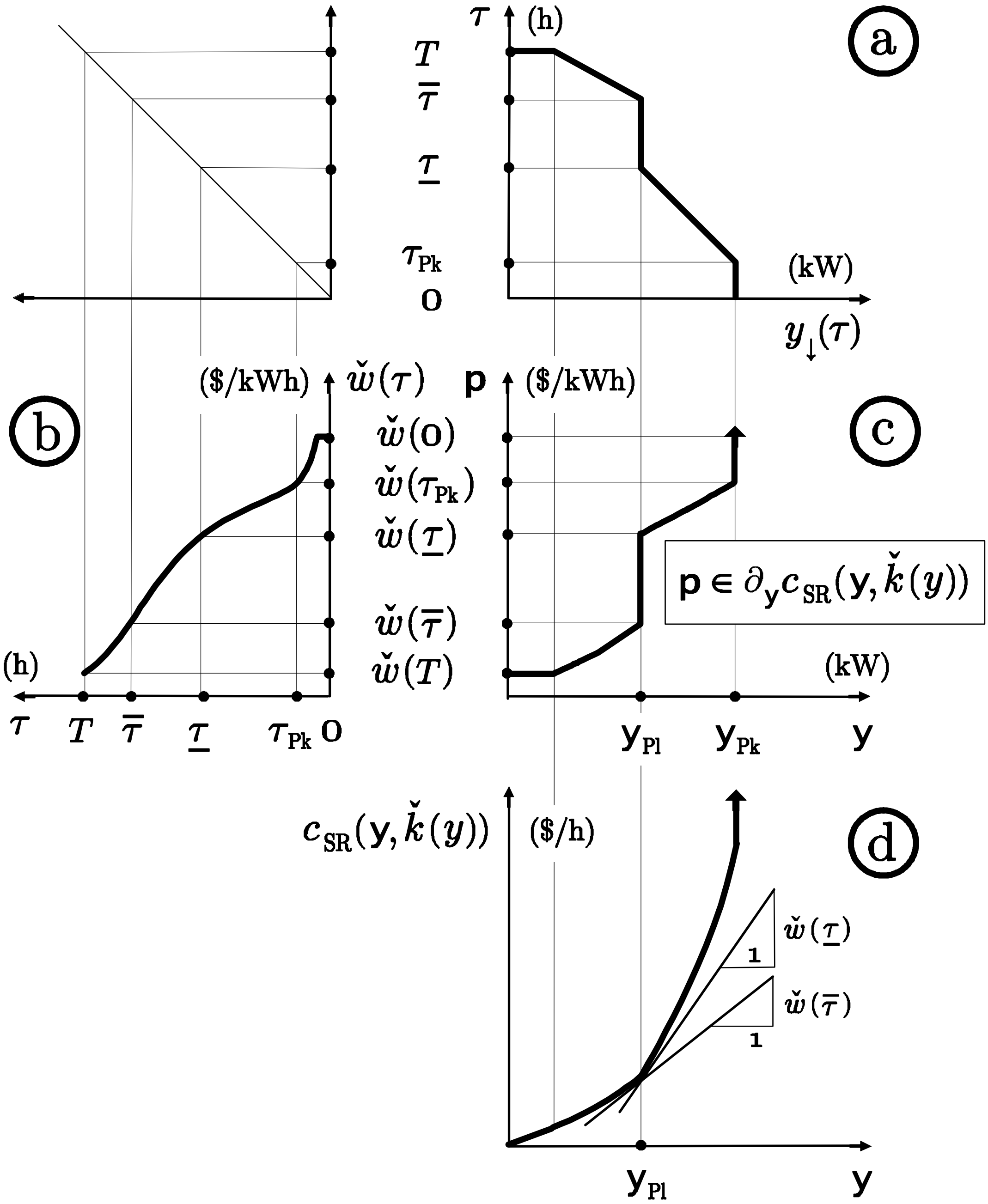

FIGURE 5. Parts (c) and (d) show the SRMC and the SRC curves for the optimal thermal system, $\check{k}$. These are constructed from the load-duration curve—shown in (a)—and from $\check{w}(\tau)$, the unit fuel cost of the station optimal for a load of duration τ, shown in (b).

Koopmans' strict convexity assumption on the SR cost curve can now be derived for the optimal thermal system. However, the curve is generally nonsmooth.

Remark 7.1 (Strictly convex SRC for optimal thermal system). *Assume that r^{w} is differentiable in* w. *If y_{Th} is continuous on $[0,T]$, then $c_{\mathrm{SR}}\left(\mathsf{y},\check{k}_{\mathrm{Th}}\left(y_{\mathrm{Th}}\right)\right)$ is strictly convex in* y *on the interval* $[\mathrm{EssInf}\left(y_{\mathrm{Th}}\right),\mathrm{EssSup}\left(y_{\mathrm{Th}}\right)]$, *with $c_{\mathrm{SR}}\left(\mathsf{y}\right)=\check{w}\left(T\right)\mathsf{y}$ for* $\mathsf{y}\in[0,\mathrm{EssInf}\left(y_{\mathrm{Th}}\right)]$.[49]

Proof. Note first that $y_{\mathrm{Th}}^{\downarrow}$ is also continuous on $(0,T)$, so the measure $\mathrm{D}y_{\mathrm{Th}}^{\downarrow}$ it defines by (4.30) is nonatomic on $(0,T)$. And since $\check{w}$ is strictly decreasing (so $\check{w}^{-1}$ is a function rather than a correspondence), it follows that $\check{k}_{\mathrm{Th}}$—the optimal system or, equivalently, the system's distribution of unit variable cost—has no atom except possibly at $\check{w}\left(T\right)$. Since $y_{\downarrow}\left(T+\right):=0$, the point mass of $\check{k}_{\mathrm{Th}}$ at $\check{w}\left(T\right)$—which equals the point mass of $-\mathrm{D}y_{\mathrm{Th}}^{\downarrow}$ at T—is $y_{\downarrow}\left(T-\right)=\mathrm{EssInf}\left(y\right)$. This is the required base-load capacity. Therefore the inverse of its c.d.f. has no interval of constancy, other than $[0,\mathrm{EssInf}\left(y_{\mathrm{Th}}\right)]$, where it equals $\check{w}\left(T\right)$. Therefore its integral, c_{SR}, is strictly convex (except for the one linear segment). ∎

As is shown next, the SRC curve of the optimal system can have an offpeak kink, at some load $\mathsf{y}<\|k_{\mathrm{Th}}\|$, also in the continuum model of plant types. This is because LR cost minimisation can result in a system containing only some, and not all, of the station types available in the technology; and this is so when the output has an offpeak plateau.

Remark 7.2 (Nonsmooth SRC for optimal thermal system). *If $y_{\mathrm{Th}}^{\downarrow}$ (thermal LDC) stays constant and equal to some* y, *on an interval $[\underline{\tau},\overline{\tau}]$, as in Figure 5a, then the optimal plant mix $\check{k}_{\mathrm{Th}}\left(y_{\mathrm{Th}}\right)$ contains no stations with unit variable costs between $\check{w}\left(\overline{\tau}\right)$ and $\check{w}\left(\underline{\tau}\right)$. So, as in Figure 5c, $\partial_{\mathsf{y}}c_{\mathrm{SR}}\left(\mathsf{y},\check{k}_{\mathrm{Th}}\left(y_{\mathrm{Th}}\right)\right)=\left[\check{w}\left(\overline{\tau}\right),\check{w}\left(\underline{\tau}\right)\right]$. This is a proper interval, except for two extreme cases.*[50]

Proof. By the formula for the optimal system $\check{k}_{\mathrm{Th}}$, its marginal variable cost $w_{\uparrow\check{k}}$ jumps at y from $\check{w}\left(\overline{\tau}\right)$ to $\check{w}\left(\underline{\tau}\right)$; in the notation (4.30),

$$\check{k}_{\mathrm{Th}}\left(\left(\check{w}\left(\overline{\tau}\right),\check{w}\left(\underline{\tau}\right)\right)\right)=-\mathrm{D}y_{\mathrm{Th}}^{\downarrow}\left(\underline{\tau},\overline{\tau}\right)=0.$$

This means that $\partial c_{\mathrm{SR}}\left(\mathsf{y}\right)=\left[\check{w}\left(\overline{\tau}\right),\check{w}\left(\underline{\tau}\right)\right]$, by (7.1). It remains to show that this interval is not a single point. Since r^{w} is differentiable in w, $\check{w}\left(\tau\right)$ is *strictly* decreasing in τ (except possibly around 0 or to T, where $\check{w}$ can be constant, as in Figures 4b and

[49] The continuity assumption cannot be dropped: Figure 5 shows how a discontinuity of $y_{\mathrm{Th}}^{\downarrow}$ (a jump from $\underline{\mathsf{y}}$ to $\overline{\mathsf{y}}$) results in a linear segment of the optimal system's SRC curve.

[50] The exceptions are: (i) if $\overline{\tau}\leq-\left(\mathrm{d}r/\mathrm{dw}\right)\left(\mathsf{w}''-\right)$ then $\left(\mathrm{d}c_{\mathrm{SR}}/\mathrm{dy}\right)\left(\mathsf{y}\right)=\check{w}\left(0\right)=\mathrm{Max}\left(w\right)$, and (ii) if $\underline{\tau}\geq-\left(\mathrm{d}r/\mathrm{dw}\right)\left(\mathsf{w}'+\right)$ then $\left(\mathrm{d}c_{\mathrm{SR}}/\mathrm{dy}\right)\left(\mathsf{y}\right)=\check{w}\left(T\right)$; in the latter case $\check{w}\left(T\right)=\mathrm{Min}\left(w\right)$. These cases cannot arise if the capital-fuel substitution rate has a sufficient range, viz., if the extreme values of $-\mathrm{d}r/\mathrm{dw}$ are 0 and T or greater. The conditions—the second of which, but not the first, is met in Figure 4a—mean that $\check{w}$ cannot stay constant around 0 or T.

5b for τ close to 0). So $[\check{w}(\overline{\tau}), \check{w}(\underline{\tau})]$ is a proper interval (except in the two extreme cases). ∎

Given the unit capacity costs $r^{\mathrm{Th}} = \left(r^\theta\right)_{\theta\in\Theta}$ as well as w, the (minimum) LR thermal cost of an output y_{Th} can be given in terms of (7.3) as

$$C_{\mathrm{LR}}^{\mathrm{Th}}\left(y_{\mathrm{Th}}; r^{\mathrm{Th}}, w\right) = -\int_{[0,T]} c_{\mathrm{LR}}(\tau; r, w)\, y_{\mathrm{Th}}^{\downarrow}(\mathrm{d}\tau) \tag{7.6}$$

$$= \mathrm{Min}(r)\,\mathrm{EssSup}(y_{\mathrm{Th}}) + \int_0^T \check{w}(\tau)\, y_{\mathrm{Th}}^{\downarrow}(\tau)\,\mathrm{d}\tau. \tag{7.7}$$

Formula (7.7) follows from (7.6) by Lebesgue-Stieltjes integration by parts: see [21, (5)].

8. LRMC pricing by the SR approach with hydro-thermal technology

Profit-based valuation of capital inputs is of obvious interest to a privately-owned industry, but it turns out to be relevant also for a publicly-owned (or regulated) utility aiming to price its outputs at LRMC and optimise its capital stock. This can be achieved on the basis of purely short-run calculations by the use of the Wong-Viner Envelope Theorem. The original version of this theorem—for the case of differentiable costs—is that $p = \nabla_y C_{\mathrm{LR}}(y, r)$ if $p = \nabla_y C_{\mathrm{SR}}(y, k)$ and the fixed-input k is optimal. The optimality condition is equivalent to $k = \nabla_r C_{\mathrm{LR}}(y, r)$ by Shepard's Lemma, and hence it is also equivalent to $r = \nabla_k C_{\mathrm{SR}}(y, k)$ by conjugate duality. (As usual, C_{LR}, C_{SR} and Π_{SR} denote the long-run cost, the short-run cost and the short-run profit, as functions of: the output bundle y and its price system p, the fixed input quantities k and their prices r. The variable-input prices, w, are suppressed, since they are kept unchanged throughout.)

However, in applications joint-cost functions are rarely differentiable: as Littlechild [40, p. 324] puts it, in the peak-load pricing problem "cost functions...lose differentiability at certain crucial points". For convex functions one can use the *subdifferential* a.k.a. subgradient set (denoted by ∂) as a generalised, multi-valued gradient: see [37] or [47] for subdifferential calculus. This provides a mathematical language, but it does not by itself solve the problem. It is equally essential to replace the SR *cost*-imputed valuation of fixed inputs (which is an equivalent form of the fixed-input optimality condition) by the *profit*-imputed valuation, i.e., to replace the condition $r \in \partial_k C_{\mathrm{SR}}(y, k)$ by $r \in \partial_k \Pi_{\mathrm{SR}}(p, k)$. This works whether Π_{SR} is differentiable or not, i.e., if $p \in \partial_y C_{\mathrm{SR}}(y, k)$ and $r \in \partial_k \Pi_{\mathrm{SR}}(p, k)$, then $p \in \partial_y C_{\mathrm{LR}}(y, r)$ and, also, $k \in \partial_r C_{\mathrm{LR}}(y, r)$: see [32]. Another advantage of using Π_{SR} is that it can be differentiable in k even when C_{SR} is not—as is indeed the case with the hydro-thermal generation technology, to which our extension of the Wong-Viner Theorem is next applied.

The system we consider consists of thermal plants (of types $\theta = 1, 2, \ldots, \Theta$) and one hydro plant. In the long run the river flow $e(t)$ is, like the other inputs, assumed to be a choice variable, with a known price $\psi(t)$. (The alternative of assuming that e is fixed, even in the long run, is equally manageable.) The objective, then, is to give a set of conditions that involve only the SR functions and ensure that:

1. p is an LRMC electricity tariff, for a system output y_{ThH}, based on the input prices, viz., the unit fuel costs $\left(w^{\theta}\right)_{\theta=1}^{\Theta}$ for the thermal stations, unit generating capacity costs $r^{\mathrm{Th}} = \left(r^{\theta}\right)_{\theta=1}^{\Theta}$ and r^{Tu} (for the hydro turbine), a unit reservoir cost r^{St} and a TOU water price ψ.
2. For the output y_{ThH}, the generating system $(k_{\mathrm{Th}}; k_{\mathrm{St}}, k_{\mathrm{Tu}}, e)$ is optimal, with an optimised river flow.
3. y_{ThH} is scheduled optimally (i.e., so as to minimise the thermal fuel cost) as the sum of the thermal output y_{Th} (from the system $k_{\mathrm{Th}} = (k_{\theta})_{\theta=1}^{\Theta}$) and the hydro output y_{H} (from the hydro plant with capacities $(k_{\mathrm{St}}, k_{\mathrm{Tu}})$ and the river flow e).

Conditions 1 and 2 can of course be stated directly in LR terms—as $p \in \partial_y C_{\mathrm{LR}}^{\mathrm{ThH}}$ and $k_{\mathrm{ThH}} \in \partial_r C_{\mathrm{LR}}^{\mathrm{ThH}}$, where $C_{\mathrm{LR}}^{\mathrm{ThH}}$ is the LR cost function derived from the algebraic sum $\mathbb{Y}_{\mathrm{H}} + \mathbb{Y}_{\mathrm{Th}}$ of the production sets for the hydro and thermal technologies—but one reason for using the SR approach is that direct LR calculations are not feasible. This is because, by contrast to the purely thermal case, *no* explicit formulae for either the LRMC or the optimal system are available for the hydro-thermal combination.[51] The SR problems, although far from being simple, are much more tractable. Koopmans [39] finds the cost-minimising hydro-thermal despatch, i.e., the $(y_{\mathrm{Th}}, y_{\mathrm{H}})$ meeting Condition 3 above. But if the other conditions (LRMC pricing and system optimality) are also to be met on the basis of SR calculations, then the extended Wong-Viner Theorem is what is required, and this uses profit-imputed values. In an analysis based on this theorem the despatch problem can be dealt with indirectly, by using the profit-maximising solution for the hydro operation. That is, along with the other two, Condition 3 is deduced from simpler SR conditions (which include profit-maximising hydro operation and profit-imputed valuation of hydro inputs). This approach leads to the following set of necessary and sufficient conditions—stated entirely in terms of the SR functions—for LRMC pricing, system optimality and optimal despatch.

Theorem 8.1 (Envelope Theorem for hydro-thermal technology)**.** *The above set of Conditions 1 to 3 on: the system output y_{ThH}, the hydro output y_{H}, the thermal output y_{Th}, a time-continuous electricity tariff p satisfying (4.35), thermal capacities $k_{\theta} > 0$ (for each θ), the storage capacity $k_{\mathrm{St}} > 0$, the hydro turbine capacity $k_{\mathrm{Tu}} > 0$, an inflow e satisfying (4.8), and the corresponding rental prices $r^{\theta} \geq 0$, $r^{\mathrm{St}} \geq 0$, $r^{\mathrm{Tu}} \geq 0$ and $\psi \geq 0$, with thermal fuel prices w, is equivalent to the following set of conditions:*

$$y_{\mathrm{ThH}} = y_{\mathrm{Th}} + y_{\mathrm{H}} \tag{8.1}$$

$$p(t) \in \partial_y c_{\mathrm{SR}}^{\mathrm{Th}}\left(y_{\mathrm{Th}}(t), k_{\mathrm{Th}}, w\right), \tag{8.2}$$

[51] The sets $\mathbb{Y}_{\mathrm{H}}$ and $\mathbb{Y}_{\mathrm{Th}}$ are spelt out in (2.4) and [32], but this does not lead to a workable formula for $C_{\mathrm{LR}}^{\mathrm{ThH}}$.

where $\partial_{\mathrm{y}} c$ is the scalar subdifferential given explicitly after (7.1),

$$y_{\mathrm{H}}(t)=\begin{cases} k_{\mathrm{Tu}} & \text{if } p(t)>\psi(t) \\ e(t) & \text{if } p(t)=\psi(t) \\ 0 & \text{if } p(t)<\psi(t) \end{cases} \tag{8.3}$$

$$r^{\mathrm{St}}=\mathrm{Var}_{\mathrm{c}}^{+}(\psi) \tag{8.4}$$

$$r^{\mathrm{Tu}}=\int_0^T (p(t)-\psi(t))^{+}\,\mathrm{d}t \tag{8.5}$$

$$\psi=\hat{\psi}(p,k_{\mathrm{St}},k_{\mathrm{Tu}},e), \tag{8.6}$$

where $\hat{\psi}$ is the unique solution to (4.31)–(4.33), and

$$r^{\theta}=\int_0^T \left(p(t)-w^{\theta}\right)^{+}\mathrm{d}t \tag{8.7}$$

for each $\theta=1,2,\ldots,\Theta$.

Proof. This can be proved in the same way as the corresponding result for thermal generation with pumped storage: see [32]. ∎

Comments:

1. The assumption (4.35) that p has no plateau can be expected to fail in general equilibrium, as we have pointed out after Proposition 4.6. Without this assumption, the problem of profit-maximising hydro operation may have multiple solutions, and Condition (8.3) has to be replaced by: y_{H} solves (4.1)–(4.4). This certainly implies that

$$y_{\mathrm{H}}(t)=\begin{cases} k_{\mathrm{Tu}} & \text{if } p(t)>\psi(t) \\ 0 & \text{if } p(t)<\psi(t) \end{cases} \tag{8.8}$$

   and that

$$0\le y_{\mathrm{H}}(t)\le k_{\mathrm{Tu}} \quad \text{if } p(t)=\psi(t), \tag{8.9}$$

   i.e., at those times t with $p(t)\neq\psi(t)$ the hydro plant is operated just like a thermal plant with a time-varying "fuel" price $\psi(t)$.
2. This idea can also be expressed by placing the hydro plant in the system's instantaneous merit order, i.e., by constructing for each instant t a new SRC curve $c_{\mathrm{SR}}^{\mathrm{ThH}}$ for the *whole* system, just as it is done for the thermal subsystem. (The $c_{\mathrm{SR}}^{\mathrm{ThH}}$ varies with t and contains a linear segment of slope $\psi(t)$ and length k_{Tu} along the load axis.) Then, under (8.1) and (8.2), Conditions (8.8)–(8.9) are equivalent to $p(t)\in\partial_{\mathrm{y}} c_{\mathrm{SR}}^{\mathrm{ThH}}(y_{\mathrm{ThH}}(t))$.

3. What Conditions (8.8)–(8.9) and (8.1)–(8.2) do *not* guarantee is feasibility of y_{H}: it may fail to satisfy the reservoir and water balance constraints (4.5) and (4.4). This is why Conditions (8.8)–(8.9), or their equivalent, cannot replace (8.3) when the tariff p has plateaux. With such a p, the hydro operation problem may not be solved completely by water pricing alone.
4. In applying this SR approach to LR equilibrium, the generating capacity costs r^{θ} and r^{Tu} may be regarded as given. The case of the marginal reservoir cost, r^{St}, is rather different: this is an increasing function of k_{St}, i.e., the supply cost of storage capacity is a convex function G_{St} of k_{St}. (Similarly the cost of procuring a river flow e is a convex function of e. A fixed river flow $\overline{e}$ that cannot be improved is a special case, in which the cost is formally 0 for $e \leq \overline{e}$ and $+\infty$ otherwise.)

9. Shadow prices and rents in Koopmans' model of hydro-thermal generation

Koopmans [39] studies hydroelectric generation in the framework of SR cost minimisation for a combined thermal-hydroelectric system. In this section we give a programming formulation of Koopmans' despatch problem and sketch his solution method. Our purpose is not to present the details of Koopmans' construction of the optimal water storage policy, but rather to spell out the relationship between his analysis and ours, and, also, to show how his approach can be put in the formal framework of duality for convex programming. As we show in Remark 9.5, Koopmans' cost-imputed rents become the same as our profit-imputed rents once a particular variant of his (time-dependent) shadow price for electricity p_{Ko} has been chosen. But there are different variants of p_{Ko}, so Koopmans' shadow prices and hydro rents are to some extent indeterminate. Although this does not matter for his cost-optimality proof, it does limit the usefulness of Koopmans' rents as investment guides because it means that the incremental value of investments is *not* an additive function of increments to the capacities: see Remark 9.6 and (9.32). This drawback is inherent in value imputation by the cost minimisation approach when the SR cost function is convex (and therefore subdifferentiable) but not differentiable.

9.1. **Koopmans' despatch problem.** For a thermal-hydroelectric system, SR cost minimisation (a.k.a. optimal despatch) consists in splitting a given output to be generated by the combined system, y_{ThH}, into the sum of a thermal output, y_{Th} and a hydro output, y_{H}. Koopmans studies this as a problem over a planning period for which the initial and the final stocks of water are both given. We give a purely cyclic version of his model, in which the only constraint on stocks at the beginning or the end of cycle, apart from the reservoir's capacity k_{St}, is the periodicity constraint

$s(T) = s(0)$, as in (2.4). The despatch problem is then:

(9.1) Given $y_{\mathrm{ThH}} \in L^{\infty}_{+}$, $(k_{\mathrm{Th}}; k_{\mathrm{St}}, k_{\mathrm{Tu}}) \in \mathcal{M}_{+}(\Theta) \times \mathbb{R}^2_{+}$, $w \in \mathcal{C}_{+}(\Theta)$ and $e \in L^{\infty}_{+}$,

(9.2) minimise $C^{\mathrm{Th}}_{\mathrm{SR}}(y_{\mathrm{Th}}, k_{\mathrm{Th}}, w)$ over $(y_{\mathrm{Th}}, y_{\mathrm{H}}) \in L^{\infty} \times L^{\infty}$ and $s_0 \in \mathbb{R}$

(9.3) subject to: $y_{\mathrm{Th}} + y_{\mathrm{H}} = y_{\mathrm{ThH}}$

(9.4) $0 \le y_{\mathrm{Th}} \le \|k_{\mathrm{Th}}\|$

(9.5) $0 \le y_{\mathrm{H}} \le k_{\mathrm{Tu}}$

(9.6) $\int_0^T f(t)\,\mathrm{d}t = 0$

(9.7) $0 \le s_0 - \int_0^t f(\tau)\,\mathrm{d}\tau \le k_{\mathrm{St}}$ for every t,

where $f := y_{\mathrm{H}} - e$.[52] Spillage is ruled out because it is assumed here (though not in Koopmans' paper) that $e \le k_{\mathrm{Tu}} \wedge y_{\mathrm{ThH}}$, i.e., that the water inflow rate never exceeds either the turbine capacity or the demanded system's output. The optimal value of (9.1)–(9.7) is the (minimum) SR cost of the combined system, denoted by $C^{\mathrm{ThH}}_{\mathrm{SR}}(y_{\mathrm{ThH}}, k_{\mathrm{H}}, e)$. It depends also on k_{Th} and w, but these are fixed from now on.

If the programme (9.1)–(9.7) is feasible, then an (optimal) solution exists, since the constraint set is weakly* compact, and the minimand, being convex and continuous for the Mackey topology, is weakly* lower semicontinuous. The optimum is unique if the instantaneous SR cost c_{SR} (7.1) is *strictly* convex in y, which is the case if and only if the thermal system's distribution of unit variable cost (i.e., the image measure $k_{\mathrm{Th}} \circ w^{-1}$ on $w(\Theta) \subset \mathbb{R}$) is nonatomic.

9.2. **Koopmans' optimal policy and shadow prices.** Under the assumption that y_{ThH} and e are piecewise monotone, Koopmans [39, pp. 201–219] solves the optimal despatch problem by a direct construction of what he calls the *target rate of thermal generation*, denoted here by $y^{\dagger}_{\mathrm{Th}}(t)$, with $y^{\dagger}_{\mathrm{H}} = y_{\mathrm{ThH}} - y^{\dagger}_{\mathrm{Th}}$ as the target rate of hydrogeneration. The *actual* rate of hydrogeneration is the target rate truncated to meet the turbine constraint (9.5), i.e.,

$$y^{\mathrm{Ko}}_{\mathrm{H}} = \left(y^{\dagger}_{\mathrm{H}} \wedge k_{\mathrm{Tu}}\right)^{+} = \left(y^{\dagger}_{\mathrm{H}}\right)^{+} \wedge k_{\mathrm{Tu}}. \tag{9.8}$$

The *actual* thermal output is therefore

$$y^{\mathrm{Ko}}_{\mathrm{Th}} = y_{\mathrm{ThH}} - y^{\mathrm{Ko}}_{\mathrm{H}}, \tag{9.9}$$

see [39, (2.25) with (2.53) ff.]. A key property of $y^{\dagger}_{\mathrm{Th}}(t)$ is that it rises or falls only when the water stock is $s^{\mathrm{Ko}}(t) = k_{\mathrm{St}}$ or $s^{\mathrm{Ko}}(t) = 0$, respectively.

Comments:

[52] The constraints (9.5)–(9.7) are the same as (4.3)–(4.5).

1. Koopmans' construction (of $y_{\mathrm{Th}}^{\dagger}$, etc.) can readily be adapted to the purely cyclic case. The extra variable $s^{\mathrm{Ko}}(0)$, and hence the whole stock trajectory s^{Ko}, is determined from $y_{\mathrm{H}}^{\mathrm{Ko}}$ by (4.7)–(4.6).
2. A more significant difference between (9.1)–(9.7) and Koopmans' original problem is his assumption of unlimited thermal capacity; and Koopmans' solution may of course become infeasible once the finiteness of the total capacity $\|k_{\mathrm{Th}}\|$ is taken into account, as in (9.4).
3. However, the solution is independent of the particular shape of the convex SRMC curve, as Koopmans points out in [39, p. 225, footnote]. This can actually be proved without reference to his construction, by using the Kuhn-Tucker Conditions as in the Proof of Remark 4.12.
4. It follows that Koopmans' solution remains feasible (and hence optimal) if the problem remains feasible with a finite $\|k_{\mathrm{Th}}\|$. In intuitive terms, this is because $y_{\mathrm{Th}}^{\mathrm{Ko}}$ uses the thermal capacity sparingly, and it will satisfy this constraint if possible. A rigorous proof can be based on the independence property: consider a sequence of increasing, convex and finite extensions $c^m \colon \mathbb{R}_+ \to \mathbb{R}_+$ of the given c_{SR} with $c^m(\mathsf{y}) \to +\infty$ as $m \to \infty$ for $\mathsf{y} > \|k_{\mathrm{Th}}\|$. If $y_{\mathrm{Th}}^{\mathrm{Ko}}$ were infeasible for (9.1)–(9.7) but a feasible y_{Th} did exist, then, for large enough m, such a y_{Th} would be better than $y_{\mathrm{Th}}^{\mathrm{Ko}}$ for Koopmans' problem with c^m in place of c_{SR} (since $\int_0^T c^m\left(y_{\mathrm{Th}}^{\mathrm{Ko}}\right)$ would go to $+\infty$ as $m \to \infty$, whilst $\int_0^T c^m(y_{\mathrm{Th}})$ would stay constant, being equal to $\int_0^T c_{\mathrm{SR}}(y_{\mathrm{Th}})$ because $c^m = c_{\mathrm{SR}}$ on $[0, \|k_{\mathrm{Th}}\|]$ for each m).

To establish cost-optimality of y^{Ko}, Koopmans [39, (3.4) and (3.6)] first defines shadow prices of electricity $p_{\mathrm{Ko}}(t)$ and of water $\psi_{\mathrm{Ko}}(t)$ as the instantaneous SRMC's at the actual and the target rates of thermal generation, i.e., as slopes of the graph of $c_{\mathrm{SR}}(\cdot, k_{\mathrm{Th}}, w)$ at $y_{\mathrm{Th}}^{\mathrm{Ko}}(t)$ and $y_{\mathrm{Th}}^{\dagger}(t)$, respectively. Formally,

$$p_{\mathrm{Ko}}(t) \in \partial_{\mathsf{y}} c_{\mathrm{SR}}\left(y_{\mathrm{Th}}^{\mathrm{Ko}}(t), k_{\mathrm{Th}}, w\right) \tag{9.10}$$

$$\psi_{\mathrm{Ko}}(t) \in \partial_{\mathsf{y}} c_{\mathrm{SR}}\left(y_{\mathrm{Th}}^{\dagger}(t), k_{\mathrm{Th}}, w\right), \tag{9.11}$$

where each subdifferential $\partial_{\mathsf{y}} c_{\mathrm{SR}}$ can be an interval of $\mathbb{R}$ rather than a single number. That is, these prices are nonunique when the thermal SRC curve has kinks. That is, these prices are nonunique when the thermal SRC curve has kinks. There is always a kink at the full thermal capacity $\|k_{\mathrm{Th}}\|$, but this gives no trouble in the valuation of *increases* to hydro capacities. A troublesome kink—resulting in the afore-mentioned nonadditivity of the incremental value—is an offpeak kink, at some $\mathsf{y} < \|k_{\mathrm{Th}}\|$. And the SRC curve does usually have offpeak kinks, even when each thermal station is "infinitesimal": see Remark 7.2. However, this does not spoil Koopmans' optimality proof, since a careful selection of $(p_{\mathrm{Ko}}, \psi_{\mathrm{Ko}})$ will ensure the properties needed for the saddle-point inequalities (9.20) below.

Remark 9.1 (Nonuniqueness and selection of Koopmans' shadow prices). *When the thermal SRC curve is kinked at some load* y, *neither* p_{Ko} *nor* ψ_{Ko} *is always fully determined by (9.10)–(9.11), and they must be selected so as to ensure that:*

1. ψ_{Ko} *rises only on the set* $\left\{t : s^{\mathrm{Ko}}(t) = k_{\mathrm{St}}\right\}$, *and falls only on* $\left\{t : s^{\mathrm{Ko}}(t) = 0\right\}$.
2. $p_{\mathrm{Ko}}(t) - \psi_{\mathrm{Ko}}(t)$ *is nonnegative, nonpositive or zero if* $y_{\mathrm{H}}^{\mathrm{Ko}}(t)$ *is* k_{Tu}, 0 *or strictly between* 0 *and* k_{Tu} *(respectively).*

To meet Condition 1, it suffices to make ψ_{Ko} stay constant when $y_{\mathrm{Th}}^{\dagger}$ does (since $y_{\mathrm{Th}}^{\dagger}$ has the monotonicity property required of ψ_{Ko}, and since the two are linked by (9.11)). This can be done by choosing, for any such y, a constant $\mathsf{w} \in \partial c_{\mathrm{SR}}(\mathsf{y})$ and by setting $\psi_{\mathrm{Ko}}(t) = \mathsf{w}$ whenever $y_{\mathrm{Th}}^{\dagger}(t) = \mathsf{y}$. (For example, either the highest or the lowest permissible value will do, as is pointed out in [39, p. 222: lines 1–4].) Condition 2 can be met by a subsequent selection of p_{Ko} when this is necessary, i.e., when $y_{\mathrm{Th}}^{\dagger}(t) = y_{\mathrm{Th}}^{\mathrm{Ko}}(t) = \mathsf{y}$. (Such a choice is shown in Figure 6f.)

Given the electricity and water prices (9.10)–(9.11), the reservoir's imputed value on an infinitesimal interval $(t, t + \mathrm{d}t)$ is

$$\kappa_{\mathrm{Ko}}^{\mathrm{St}}(\mathrm{d}t) = \mathrm{d}\psi_{\mathrm{Ko}}(t)^{+} = \left(\psi_{\mathrm{Ko}}(t + \mathrm{d}t) - \psi_{\mathrm{Ko}}(t)\right)^{+} \tag{9.12}$$

and the turbine's value is

$$\kappa_{\mathrm{Ko}}^{\mathrm{Tu}}(t)\,\mathrm{d}t = \left(p_{\mathrm{Ko}}(t) - \psi_{\mathrm{Ko}}(t)\right)^{+} \mathrm{d}t. \tag{9.13}$$

Similarly $\nu_{\mathrm{Ko}}^{\mathrm{St}}(\mathrm{d}t) = \mathrm{d}\psi_{\mathrm{Ko}}(t)^{-}$, $\nu_{\mathrm{Ko}}^{\mathrm{Tu}}(t) = \left(p_{\mathrm{Ko}}(t) - \psi_{\mathrm{Ko}}(t)\right)^{-}$ and $\lambda_{\mathrm{Ko}} = \psi_{\mathrm{Ko}}(T0)$, as in Section 4.

The totals of $\kappa_{\mathrm{Ko}}^{\mathrm{St}}$ and $\kappa_{\mathrm{Ko}}^{\mathrm{Th}}$ over the cycle are, as in (3.2),

$$r_{\mathrm{Ko}}^{\mathrm{St}} = \mathrm{Var}_{\mathrm{c}}^{+}(\psi_{\mathrm{Ko}}) \quad \text{and} \quad r_{\mathrm{Ko}}^{\mathrm{Tu}} = \int_0^T \left(p_{\mathrm{Ko}}(t) - \psi_{\mathrm{Ko}}(t)\right)^{+} \mathrm{d}t; \tag{9.14}$$

these are the R and Q of [39, (3.9) to (3.13)]. Having thus priced the fixed quantities (which in his despatch problem include the demanded system's output y_{ThH}), Koopmans sets up a "fictitious" profit maximisation problem in which both output components (thermal and hydro) as well as all the hydro inputs—but not the thermal capacities—are treated as decision variables. In more precise terms, the "fictitious entrepreneur" faces the "market" prices p_{Ko} and ψ_{Ko} for electricity and water, as well as the implied rental prices ($\kappa_{\mathrm{Ko}}^{\mathrm{St}}$ and $\kappa_{\mathrm{Ko}}^{\mathrm{Tu}}$) for the hydro capacity services. (He is free to vary his demand for capacity services with time.) He determines *freely* both the thermal output y_{Th} and the hydro output y_{H}, and sells the outputs. He must buy or rent all the hydro inputs (viz., the river flow, the reservoir services and the turbine services) in quantities required for the y_{H} he has chosen; and he must also pay the operating costs, but not the capital costs, of generating the chosen thermal output y_{Th} (from the given thermal system k_{Th}). Koopmans shows that the profit in question, $\Pi_{\mathrm{Ko}}(y_{\mathrm{Th}}, y_{\mathrm{H}})$, is maximised at the solution $\left(y_{\mathrm{Th}}^{\mathrm{Ko}}, y_{\mathrm{H}}^{\mathrm{Ko}}\right)$ he has constructed.

To establish cost-optimality, he proves further that

$$C_{\mathrm{SR}}^{\mathrm{Th}}\left(y_{\mathrm{Th}}\right)-C_{\mathrm{SR}}^{\mathrm{Th}}\left(y_{\mathrm{Th}}^{\mathrm{Ko}}\right)\geq\Pi\left(y_{\mathrm{Th}}^{\mathrm{Ko}},y_{\mathrm{H}}^{\mathrm{Ko}}\right)-\Pi_{\mathrm{Ko}}\left(y_{\mathrm{Th}},y_{\mathrm{H}}\right)\geq 0$$

for any feasible policy $(y_{\mathrm{Th}}, y_{\mathrm{H}})$ for the despatch problem (9.1)–(9.7). This shows that $\left(y_{\mathrm{Th}}^{\mathrm{Ko}}, y_{\mathrm{H}}^{\mathrm{Ko}}\right)$ minimises the combined system's SR cost, i.e., it solves the despatch problem. It also follows, as we explain below, that p_{Ko}, ψ_{Ko}, $r_{\mathrm{Ko}}^{\mathrm{St}}$ and $r_{\mathrm{Ko}}^{\mathrm{Tu}}$ are the marginal values of electricity, water and the hydro capacities, i.e., that they are the derivatives of the mimimum cost $C_{\mathrm{SR}}^{\mathrm{ThH}}$ w.r.t. y_{ThH}, e, k_{St} and k_{Tu} (Remark 9.4).

9.3. **The Lagrangian and Koopmans' optimality proof.** Koopmans' argument can be expounded in the duality framework of, e.g., [47]. For this, the Lagrange function for (9.1)–(9.7) identified as Koopmans' "enterpreneurial profit" Π_{Ko} (a function of the hydro and thermal outputs y_{H} and y_{Th}, which are the primal variables) plus a term V_{Ko} which is independent of y_{H} and y_{Th}. As in Subsection 4.2, the refined perturbation is employed, i.e., the primal programme is perturbed by cyclically varying increments Δk_{St}, Δn_{St}, Δk_{Tu}, Δn_{Tu}, Δe and a scalar $\Delta\zeta\in\mathbb{R}$ to the particular parameter point consisting of: the constants k_{St}, $n_{\mathrm{St}}=0$, k_{Tu}, $n_{\mathrm{Tu}}=0$, the function e and $\zeta=0$. These increments are again paired with the dual variables κ^{St}, ν^{St}, κ^{Tu}, ν^{Tu}, ψ and λ. Since there is also the additional parameter y_{ThH}, there is an extra dual variable p, paired with an increment Δy_{ThH} and interpreted as the shadow price for electricity. (This is specific to Koopmans' cost-minimisation problem. In our profit-maximisation framework of Section 2, p is a datum and *not* a dual variable.) The notation is abbreviated to

$$y:=\left(y_{\mathrm{Th}},y_{\mathrm{H}},s_0\right)\in L^{\infty}\times L^{\infty}\times\mathbb{R}\tag{9.15}$$

$$\rho:=\left(p;\kappa^{\mathrm{St}},\nu^{\mathrm{St}};\kappa^{\mathrm{Tu}},\nu^{\mathrm{Tu}};\psi,\lambda\right)\in L^{\infty*}\times\mathcal{M}^2\times\left(L^{\infty*}\right)^2\times L^{\infty*}\times\mathbb{R}.\tag{9.16}$$

Remark 9.2 (Lagrange function for Koopmans' despatch problem)**.** *After reorienting (9.1)–(9.7) to maximisation of $-C_{\mathrm{SR}}^{\mathrm{Th}}$, the Lagrange function, for the refined perturbation of this programme, is*

(9.17)

$$\mathcal{L}_{\mathrm{Ko}}\left(y,\rho\right)=\begin{cases}\Pi_{\mathrm{Ko}}\left(y,\rho\right)+V_{\mathrm{Ko}}\left(\rho\right) & \begin{array}{l}\text{if } 0\leq y_{\mathrm{Th}}\leq\|k_{\mathrm{Th}}\| \text{ and}\\ (\kappa,\nu)\geq 0 \text{ and } \psi=\lambda-\left(\kappa^{\mathrm{St}}-\nu^{\mathrm{St}}\right)[\cdot,T]\end{array}\\ +\infty & \begin{array}{l}\text{if } 0\leq y_{\mathrm{Th}}\leq\|k_{\mathrm{Th}}\| \text{ but at least one}\\ \text{of the conditions on } (\kappa,\nu,\psi,\lambda) \text{ fails}\end{array}\\ -\infty & \text{if the above condition on } y_{\mathrm{Th}} \text{ fails}\end{cases}$$

where

$$\begin{aligned}V_{\mathrm{Ko}}\left(p,\kappa,\psi\right)&:=V^{\mathrm{H}}\left(\kappa,\psi\right)-\left\langle p,y_{\mathrm{ThH}}\right\rangle\\&:=k_{\mathrm{St}}\kappa^{\mathrm{St}}\left[0,T\right]+k_{\mathrm{Tu}}\kappa^{\mathrm{Tu}}\left[0,T\right]+\left\langle\psi,e\right\rangle_{L^1,L^{\infty}}-\left\langle p,y_{\mathrm{ThH}}\right\rangle_{L^{\infty*},L^{\infty}}\end{aligned}\tag{9.18}$$

and

$$\text{(9.19)}\quad \Pi_{\mathrm{Ko}}(y,\rho) := -\int_0^T c_{\mathrm{SR}}(y_{\mathrm{Th}}(t), k_{\mathrm{Th}})\,\mathrm{d}t + \langle p, y_{\mathrm{Th}} + y_{\mathrm{H}}\rangle - \left\langle \kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}} + \psi, y_{\mathrm{H}}\right\rangle - s_0\left(\kappa^{\mathrm{St}} - \nu^{\mathrm{St}}\right)[0,T].$$

Proof. This is essentially a case of the argument set out in the Appendix. It applies because the maximand, $-C_{\mathrm{SR}}^{\mathrm{Th}}$, is a concave function of y_{Th}; and although $C_{\mathrm{SR}}^{\mathrm{Th}}$ is finite only for $y_{\mathrm{Th}} \leq \|k_{\mathrm{Th}}\|$, it has a (convex) finite extension to the whole of L^∞: see (9.40). The only additional aspect is the primal constraint (9.4), imposed by the total of the thermal capacities k_{Th}. These are not among the parameters being perturbed, so they remain unpriced, and the constraint (9.4) is not removed from the Lagrange function. Hence the case of $\mathcal{L} = -\infty$ arises in (9.17) as in [47, (4.4)]. ∎

Comments :

1. Given that $\psi = \lambda - \left(\kappa^{\mathrm{St}} - \nu^{\mathrm{St}}\right)[\cdot, T]$, $s(t) = s_0 - \int_0^t f(\tau)\,\mathrm{d}\tau$ and $f(t) = y_{\mathrm{H}}(t) - e$, an equivalent form of (9.19) is

$$\Pi_{\mathrm{Ko}}(y,\rho) = \langle p, y_{\mathrm{Th}} + y_{\mathrm{H}}\rangle - \int_0^T c_{\mathrm{SR}}(y_{\mathrm{Th}}(t), k_{\mathrm{Th}})\,\mathrm{d}t - \left\langle \kappa^{\mathrm{Tu}} - \nu^{\mathrm{Tu}}, y_{\mathrm{H}}\right\rangle - \int_0^T s(t)\left(\kappa^{\mathrm{St}} - \nu^{\mathrm{St}}\right)(\mathrm{d}t) - \int_0^T \psi(t)\,e(t)\,\mathrm{d}t - \lambda\int_0^T f(t)\,\mathrm{d}t.$$

2. The above formula corresponds to [39, (3.15)]. It gives $\Pi_{\mathrm{Ko}}(y,\rho)$ the interpretation of the fictitious entrepreneur's profit: the sum on the r.h.s. is the revenue from electricity sales minus the thermal system's fuel cost and minus the cost of all the hydro resources needed at each time t. As in (4.29), the hydro resources are the requirements for the turbine and reservoir capacities (priced at κ), the floors for generation and stock (priced at ν), the river flow (priced at ψ) and the required top-up (priced at λ).
3. The shadow prices $(\kappa, \nu, \psi, \lambda)$ in (9.17) meet the same compatibility condition as in (4.23); this is a case of (A.10) in the Appendix. It follows that Π_{Ko} can be given in terms of y alone (i.e., without involving e), as in (9.19) here and in [39, (3.17)]; this is a case of (A.13). This is why the set of decision variables of Koopmans' "fictitious market" problem can indeed be reduced to $y = (y_{\mathrm{Th}}, y_{\mathrm{H}}, s_0)$.[53]
4. Unlike our $\Pi_{\mathrm{SR}}^{\mathrm{H}}$, which is the operating profit of the hydro plant, the maximum of Π_{Ko} over y is the operating profit of the *thermal* system: see (9.30).

To prove that the y^{Ko} given by (9.8)–(9.9) is cost-optimal, Koopmans [39, pp. 222–224] establishes the following inequalities, in which y is any feasible point of (9.1)–(9.7) and ρ_{Ko} is obtained from $(p_{\mathrm{Ko}}, \psi_{\mathrm{Ko}})$ by (9.12)–(9.13):

$$\text{(9.20)}\qquad C(y_{\mathrm{Th}}) - C\left(y_{\mathrm{Th}}^{\mathrm{Ko}}\right) \geq \Pi\left(y^{\mathrm{Ko}}, \rho_{\mathrm{Ko}}\right) - \Pi(y, \rho_{\mathrm{Ko}}) \geq 0$$

[53] It is impossible to eliminate y_{Th} or y_{H} because their sum is not fixed in *this* problem.

(where C, Π, V and $\mathcal{L}$ are abbreviations for in which $C_{\mathrm{SR}}^{\mathrm{Th}}$, Π_{Ko}, V_{Ko} and $\mathcal{L}_{\mathrm{Ko}}$). We next make clear that this argument amounts to showing that $\left(y^{\mathrm{Ko}}, \rho_{\mathrm{Ko}}\right)$ is a saddle point for the Lagrange function (9.17), i.e., that

$$\mathcal{L}\left(y^{\mathrm{Ko}}, \rho\right) \geq \mathcal{L}\left(y^{\mathrm{Ko}}, \rho_{\mathrm{Ko}}\right) \geq \mathcal{L}\left(y, \rho_{\mathrm{Ko}}\right) \tag{9.21}$$

for every y and ρ.[54]

Remark 9.3. *The first inequality of (9.21) implies the first inequality of (9.20). The second inequality of (9.21) is equivalent to the second inequality of (9.20).*

Proof. Note first that (9.20) can be restated as

$$-C\left(y_{\mathrm{Th}}^{\mathrm{Ko}}\right) - \left(-C\left(y_{\mathrm{Th}}\right)\right) \geq \mathcal{L}\left(y^{\mathrm{Ko}}, \rho_{\mathrm{Ko}}\right) - \mathcal{L}\left(y, \rho_{\mathrm{Ko}}\right) \geq 0, \tag{9.22}$$

since $\mathcal{L}\left(y, \rho\right) = \Pi\left(y, \rho\right) + V\left(\rho\right)$ with V independent of y, for every y and ρ meeting the conditions in (9.17).

The second inequality of (9.22) is the same as the second inequality of (9.21).

For the other part, recall that $\inf_{\rho} \mathcal{L}\left(y, \rho\right) = -C\left(y_{\mathrm{Th}}\right)$ for every feasible point y. (This is a general property which comes purely from the definition of $\mathcal{L}$ as the Lagrange function for maximisation of $-C$.) By this identity, the first part of (9.21) is equivalent to

$$-C\left(y_{\mathrm{Th}}^{\mathrm{Ko}}\right) = \mathcal{L}\left(y^{\mathrm{Ko}}, \rho_{\mathrm{Ko}}\right). \tag{9.23}$$

Given that $-C\left(y_{\mathrm{Th}}\right) \leq \mathcal{L}\left(y, \rho_{\mathrm{Ko}}\right)$, this implies the first inequality of (9.22).[55] ∎

9.4. **Koopmans' shadow prices as marginal values.** The saddle-point inequalities imply the usual derivative property, i.e., that the dual variables ρ_{Ko} are the marginal values of the "refined" primal parameters (Δy_{ThH}, Δk_{St}, Δn_{St}, Δk_{Tu}, Δn_{Tu}, Δe, $\Delta \zeta$). As in the Proof of Theorem 4.9, this implies that $r_{\mathrm{Ko}}^{\mathrm{St}}$, $r_{\mathrm{Ko}}^{\mathrm{Tu}}$, ψ_{Ko} and p_{Ko} are the subgradients of the combined system's SR cost $C_{\mathrm{SR}}^{\mathrm{ThH}}$ with respect to the *constant* hydro capacities and the cyclically varying inflow and system's output.

Remark 9.4 (Koopmans' shadow prices as cost-imputed values). *The subdifferential* $\partial C_{\mathrm{SR}}^{\mathrm{ThH}}\left(y_{\mathrm{ThH}}, k_{\mathrm{H}}, e\right)$ *consists of all those* $\left(p_{\mathrm{Ko}}, -r_{\mathrm{Ko}}^{\mathrm{H}}, -\psi_{\mathrm{Ko}}\right)$ *meeting Conditions (9.10), (9.11), (9.14) and the conditions of Remark 9.1.*[56]

Proof. This follows from (9.21): see, e.g., [47, Theorem 16: (b) and (a), with Theorem 15: (e) and (f)]. ∎

An equivalent derivative property helps relate Koopmans' analysis to ours.

[54] (9.21) implies that $\mathcal{L}\left(y^{\mathrm{Ko}}, \rho_{\mathrm{Ko}}\right)$ is finite.

[55] Although (9.20) is what Koopmans states, he establishes (9.23), and thus proves (9.21) in full.

[56] The fixed arguments, k_{Th} and w, of $C_{\mathrm{SR}}^{\mathrm{ThSH}}$ are suppressed from the notation.

Remark 9.5 (Cost- and profit-imputed values)**.** *The inclusion*

$$(p_{\mathrm{Ko}}, -r^{\mathrm{H}}_{\mathrm{Ko}}, -\psi_{\mathrm{Ko}}) \in \partial_{y,k_{\mathrm{H}},e} C^{\mathrm{ThH}}_{\mathrm{SR}} (y_{\mathrm{ThH}}, k_{\mathrm{H}}, e) . \tag{9.24}$$

is equivalent to the following pair of conditions:

$$p_{\mathrm{Ko}} \in \partial_y C^{\mathrm{ThH}}_{\mathrm{SR}} (y_{\mathrm{ThH}}, k_{\mathrm{H}}, e) \tag{9.25}$$

$$(r^{\mathrm{H}}_{\mathrm{Ko}}, \psi_{\mathrm{Ko}}) \in \partial_{k_{\mathrm{H}},e} \Pi^{\mathrm{H}}_{\mathrm{SR}} (p_{\mathrm{Ko}}, k_{\mathrm{H}}, e) . \tag{9.26}$$

Proof. This is a case of the differential equivalences between a bivariate convex function and its partial or total conjugates, for which see, e.g., [45, Lemma 4], [46, 37.5] or [4, 4.4.14].

In detail, the partial convex conjugate of $C^{\mathrm{ThH}}_{\mathrm{SR}} (y_{\mathrm{ThH}}, k_{\mathrm{H}}, e)$ with respect to the variable y_{ThH} is the combined system's SR profit

$$\Pi^{\mathrm{ThH}}_{\mathrm{SR}} (p; k_{\mathrm{H}}, e) = \Pi^{\mathrm{H}}_{\mathrm{SR}} (p; k_{\mathrm{H}}, e) + \Pi^{\mathrm{Th}}_{\mathrm{SR}} (p) , \tag{9.27}$$

which is a saddle function, convex in p and concave in (k_{H}, e). Therefore (9.24) is equivalent to the conjunction

$$y_{\mathrm{ThH}} \in \partial_p \Pi^{\mathrm{ThH}}_{\mathrm{SR}} (p_{\mathrm{Ko}}; k_{\mathrm{H}}, e) \text{ and } (r^{\mathrm{H}}_{\mathrm{Ko}}, \psi_{\mathrm{Ko}}) \in \partial_{k_{\mathrm{H}},e} \Pi^{\mathrm{ThH}}_{\mathrm{SR}} (p_{\mathrm{Ko}}; k_{\mathrm{H}}, e) . \tag{9.28}$$

The second inclusion of (9.28) is the same as (9.26), since $\Pi^{\mathrm{Th}}_{\mathrm{SR}}$ in (9.27) is independent of (k_{H}, e). And the first inclusion of (9.28) is equivalent to (9.25), again because $\Pi^{\mathrm{ThH}}_{\mathrm{SR}}$ and $C^{\mathrm{ThH}}_{\mathrm{SR}}$ are conjugate to each other as functions of p and y_{ThH}. ∎

This shows that the indeterminacy of Koopmans' rents ψ_{Ko} and $r^{\mathrm{H}}_{\mathrm{Ko}}$ is largely a consequence of their dependence on indeterminate shadow prices of output p_{Ko}. Once a p_{Ko} meeting (9.25) has been chosen and fixed, the corresponding ψ_{Ko} and $r^{\mathrm{H}}_{\mathrm{Ko}}$ can be viewed as *profit*-imputed rents, by (9.26); and such rents are fully determinate if $p_{\mathrm{Ko}} \in \mathcal{C}[0, T]$ by our preceding analysis (Lemma 4.8 and Theorem 4.9). In other words, a continuous choice of p_{Ko} leaves just one choice for ψ_{Ko}, viz., the profit-imputed value $\hat{\psi} (p_{\mathrm{Ko}}, k_{\mathrm{H}}, e)$. To grasp this in terms of Koopmans' analysis, recall that a particular choice of p_{Ko} restricts the admissible choices for ψ_{Ko} through the conditions on p_{Ko} and ψ_{Ko} jointly (Remark 9.1).

Comment: Inclusion (9.24) is also equivalent to the following case of Hotelling's Lemma:

$$(y_{\mathrm{ThH}}, -k_{\mathrm{H}}, -e) \in \partial_{p,r^{\mathrm{H}},\psi} \Pi_{\mathrm{SL}} (p_{\mathrm{Ko}}, r^{\mathrm{H}}_{\mathrm{Ko}}, \psi_{\mathrm{Ko}}) , \tag{9.29}$$

where Π_{SL}, a jointly convex function of $\left(p, r^{\mathrm{H}}, \psi\right)$, is defined by[57]

$$\text{(9.30)} \qquad \Pi_{\mathrm{SL}} := \Pi_{\mathrm{SR}}^{\mathrm{Th}}\left(p; k_{\mathrm{Th}}, w\right) + \Pi_{\mathrm{LR}}^{\mathrm{H}}\left(p, r^{\mathrm{H}}, \psi\right) = \begin{cases} \Pi_{\mathrm{SR}}^{\mathrm{Th}}\left(p\right) & \text{if } \left(p, r^{\mathrm{H}}, \psi\right) \text{ meets (9.14) and } \psi \geq 0 \\ +\infty & \text{if not} \end{cases} .$$

(The above condition on $\left(p, r^{\mathrm{H}}, \psi\right)$ means that it belongs to Y_{H}°, the polar cone of Y_{H}.) Like Remark 9.5, the equivalence of (9.29) to (9.24) follows from conjugacy: the convex conjugate of Π_{SL} w.r.t. all three variables $\left(p, r^{\mathrm{H}}, \psi\right)$ is $C_{\mathrm{SR}}^{\mathrm{ThH}}\left(y_{\mathrm{ThH}}, k_{\mathrm{H}}, e\right)$ as a function of $\left(y_{\mathrm{ThH}}, -k_{\mathrm{H}}, -e\right)$. This can be established in stages, by using the partial conjugacy between $C_{\mathrm{SR}}^{\mathrm{ThH}}$ and $\Pi_{\mathrm{SR}}^{\mathrm{ThH}}$, and by showing that the partial concave conjugate of $\Pi_{\mathrm{SR}}^{\mathrm{H}}\left(p; k_{\mathrm{H}}, e\right)$ w.r.t. $\left(k_{\mathrm{H}}, e\right)$ is $-\Pi_{\mathrm{LR}}^{\mathrm{H}}$ (and so the corresponding conjugate of $\Pi_{\mathrm{SR}}^{\mathrm{ThH}}$ is $-\Pi_{\mathrm{SL}}$).

For the purpose of marginal valuation a subdifferential such as $\partial_k C_{\mathrm{SR}}$ is mainly of interest as a representation of the directional derivative $\mathrm{D}C\left(k, \Delta k\right)$, which approximates the cost decrement $C\left(k + \Delta k\right) - C\left(k\right)$ resulting from an extra investment Δk. As a function of the increment vector Δk, the directional derivative is the support function of $\partial C\left(k\right)$. This means that the thermal SR cost change resulting from changes to the hydro inputs and/or the combined system's output is

$$\text{(9.31)} \quad -\mathrm{D}C_{\mathrm{SR}}^{\mathrm{ThH}}\left(\Delta y_{\mathrm{ThH}}, \Delta k_{\mathrm{H}}, \Delta e\right) = \inf_{p, r^{\mathrm{H}}, \psi} \left\{ \int_0^T \left(-p \Delta y_{\mathrm{ThH}} + \psi \Delta e\right) \mathrm{d}t + r^{\mathrm{H}} \cdot \Delta k_{\mathrm{H}} : \left(p, -r^{\mathrm{H}}, -\psi\right) \in \partial_{y_{\mathrm{ThH}}, k_{\mathrm{H}}, e} C_{\mathrm{SR}}^{\mathrm{ThH}} \right\},$$

where both $\mathrm{D}C$ and ∂C are evaluated at the given $\left(y_{\mathrm{ThH}}, k_{\mathrm{H}}, e\right)$. The formula, essentially the same as in [39, (3.23) ff.], is of most interest in the case of an expansion of the hydro system (including watershed investment to improve the river flow), i.e., in the case of $\left(\Delta k_{\mathrm{H}}, \Delta e\right) > 0$ with $\Delta y_{\mathrm{ThH}} = 0$ (so that $\mathrm{D}C \leq 0$). However, the subdifferential $\partial C_{\mathrm{SR}}^{\mathrm{ThH}}$ does not reduce to a single, ordinary gradient vector;[58] and this is why the incremental value $-\mathrm{D}C$ is generally superadditive but *not* additive in the increments. This means that the incremental saving or cost resulting from extra inputs or outputs has to be calculated jointly for all the quantities being varied: the calculation cannot be split up either by the three groups of variables (y_{ThH}, k_{H} and e) or within a group.

Nor is the cost decrement additive for parameter increments of definite signs, such as increases in the hydro inputs. As an example of particular interest, the saving (on thermal fuel cost) from extra investment into *both* of the hydro capacities can

[57] In the profit function Π_{SL} the only fixed quantities are k_{Th}, as in Koopmans fictitious profit maximisation. Indeed, given the relationships (9.14), etc., between the shadow prices, $\Pi_{\mathrm{SL}}\left(p, r^{\mathrm{SH}}, \psi\right)$ is the maximum of $\Pi_{\mathrm{Ko}}\left(y; p, \left(\kappa, \nu\right)^{\mathrm{SH}}, \psi, \lambda\right)$ over y.

[58] Furthermore, the set ∂C_{SR} does not factorise into the Cartesian product of partial subdifferentials, as we note in [32].

be greater than the sum of such savings from each capacity increment on its own. In symbols, with ∂/∂_+ denoting the right partial derivative, if $\Delta k_{\mathrm{St}} > 0$ and $\Delta k_{\mathrm{Tu}} > 0$ then

$$(9.32) \qquad -\mathrm{D}C_{\mathrm{SR}}^{\mathrm{ThH}}\left(\Delta k_{\mathrm{St}}, \Delta k_{\mathrm{Tu}}\right) \geq -\left(\frac{\partial C_{\mathrm{SR}}^{\mathrm{ThH}}}{\partial_+ k_{\mathrm{St}}}\Delta k_{\mathrm{St}} + \frac{\partial C_{\mathrm{SR}}^{\mathrm{ThH}}}{\partial_+ k_{\mathrm{Tu}}}\Delta k_{\mathrm{Tu}}\right).$$

As we show next, the above inequality is generally strict—by contrast to the equality (3.4) for profit increments.

Remark 9.6 (Nonadditivity of cost-imputed incremental values). *If $\partial c_{\mathrm{SR}}(\mathsf{y})$ is a proper (nonsingleton) interval, then Koopmans' shadow prices $p_{\mathrm{Ko}}(t)$ and $\psi_{\mathrm{Ko}}(t)$ are nonunique at those times when $y_{\mathrm{Th}}^{\mathrm{Ko}}(t) = \mathsf{y} = y_{\mathrm{Th}}^{\dagger}(t)$, and such plateaux do arise in the competitive equilibrium.*

Proof. It might at first seem an unlikely coincidence for $y_{\mathrm{Th}}^{\mathrm{Ko}}$ and $y_{\mathrm{Th}}^{\dagger}$ to remain simultaneously equal and constant for some time, and precisely at the level of a kink of the SRC curve. But, as is shown below, with a continuum of plants in the thermal technology such a plateau is actually typical of the competitive equilibrium if the marginal utility of electricity to consumers (or its productivity to industrial users) is time-continuous.

Under this continuity assumption on the demand side, the equilibrium electricity price $p^{\star}$ is also continuous over time. This is because a jump in the price would make the consumption rate drop, while the output rate could only go up: see [26]. It follows that the consumption rate $x^{\star}$, equal to the combined output $y_{\mathrm{ThH}}^{\star} = y_{\mathrm{Th}}^{\mathrm{Ko}} + y_{\mathrm{H}}^{\mathrm{Ko}}$, is also continuous over time. To show how this leads to nonunique p_{Ko} and ψ_{Ko} (at least if $y_{\mathrm{Th}}^{\mathrm{Ko}}$ and $y_{\mathrm{H}}^{\mathrm{Ko}}$ are continuous), we examine the price and quantity changes over a time interval $\left[\underline{t}, \overline{t}\right]$ in which the hydro plant's reservoir goes from being full at $\underline{t}$ to being empty at $\overline{t}$ (Figure 6). On such an interval $y_{\mathrm{Th}}^{\dagger}$ is a constant, denoted by $\mathsf{y}^{\dagger}$ (Figure 6a). The marginal utility is also taken to be piecewise monotone (for any constant consumption rate), rising over an interval that includes $\underline{t}$ and then falling over an interval that includes $\overline{t}$. As we show below, production-supporting prices p_{Ko} and ψ_{Ko} are then nonunique (even though the equilibrium prices $p^{\star}$ and $\psi^{\star}$ may be unique). In this argument a star ($\star$) indicates those, and only those, of the equilibrium prices and quantities which have to be distinguished from non-equilibrium values. (The y^{Ko} are also equilibrium quantities. As for (k_{H}, e), these can be thought of either as long-run equilibrium quantities or as given in the short run. The same goes for w, the thermal fuel prices. It is assumed that $e < k_{\mathrm{Tu}}$.) Note also that, since $p^{\star}$ is continuous, $\psi^{\star}$ is the unique shadow price of water associated with $p^{\star}$, i.e., $\psi^{\star} = \hat{\psi}\left(p^{\star}, k_{\mathrm{H}}, e\right)$ in the notation of Section 4.[59]

Just before $\underline{t}$ the reservoir remains full and the hydro plant is "coasting" (i.e., $y_{\mathrm{H}}^{\mathrm{Ko}}(t) = e(t)$), whilst the prices $p^{\star}(t) = \psi^{\star}(t)$ and the (actual) thermal output rate $y_{\mathrm{Th}}^{\mathrm{Ko}}(t)$ are all rising (on the assumption that any rise of e, which equals $y_{\mathrm{H}}^{\mathrm{Ko}}$ at the

[59] The $\psi^{\star}$ is the same as the ψ_{Ko} corresponding to the choice of $p^{\star}$ for p_{Ko}.

time, is by itself insufficient to meet the strengthening demand). At $\underline{t}$ the hydro plant starts generating above the coasting rate; the water stock falls and so $\psi^\star$ becomes constant (Figures 6b to 6d). This means that also $p^\star$ must stay constant, at the same value $\psi^\star(\underline{t})$, from $\underline{t}$ until such t' when $y_{\mathrm{H}}^{\mathrm{Ko}}$ first reaches k_{Tu}. (This is because if $p^\star(t) > \psi^\star(\underline{t})$ before t', then, at the infimum of such t's, $y_{\mathrm{H}}^{\mathrm{Ko}}$ would jump from e to k_{Tu}, whilst $y_{\mathrm{Th}}^{\mathrm{Ko}}$ could not fall; and the resulting jump in $y_{\mathrm{ThH}}^\star$ could not be matched by a continuous rise in $x^\star$.) Since $p^\star(t) \in \partial_y c_{\mathrm{SR}}\left(y_{\mathrm{Th}}^{\mathrm{Ko}}(t), k_{\mathrm{Th}}^\star\right)$ and $\psi^\star(\underline{t}) \in \partial_y c_{\mathrm{SR}}\left(\mathsf{y}^\dagger, k_{\mathrm{Th}}^\star\right)$ by (9.10)–(9.11), it follows from $p^\star(t) = \psi^\star(\underline{t})$ that $y_{\mathrm{Th}}^{\mathrm{Ko}} = \mathsf{y}^\dagger$ on $[\underline{t}, t']$ at least. (This is because $c_{\mathrm{SR}}(\cdot, k_{\mathrm{Th}})$ is *strictly* convex if y_{Th} is continuous and k_{Th} is optimal for y_{Th}, on the assumption of a smooth ex ante capital-fuel substitution curve: see Remark 7.1.) The constancy of $y_{\mathrm{Th}}^{\mathrm{Ko}}$ on $[\underline{t}, t']$ implies in turn that $\partial_y c_{\mathrm{SR}}\left(\mathsf{y}^\dagger, k_{\mathrm{Th}}^\star\right) = \left[\underline{\mathsf{w}}^\dagger, \overline{\mathsf{w}}^\dagger\right]$ for some $\overline{\mathsf{w}}^\dagger \neq \underline{\mathsf{w}}^\dagger = \psi^\star(\underline{t})$; i.e., the equilibrium SRC curve has a kink at $\mathsf{y}^\dagger$ by Remark 7.2: see Figure 6e. Given that $p^\star(t) \in \partial c_{\mathrm{SR}}\left(y_{\mathrm{Th}}^{\mathrm{Ko}}(t)\right)$, it follows that $y_{\mathrm{Th}}^{\mathrm{Ko}}$ must continue to equal $\mathsf{y}^\dagger$ also after t' until, at some t'', $p^\star$ reaches $\overline{\mathsf{w}}^\dagger$. (From t' until t'' the demand $x^\star$, equal to $y_{\mathrm{Th}}^{\mathrm{Ko}} + y_{\mathrm{H}}^{\mathrm{Ko}} = \text{const.}$, is kept constant by the increase of price $p^\star$.) From t'' until some t''', $p^\star$ stays above $\overline{\mathsf{w}}^\dagger$, and $y_{\mathrm{Th}}^{\mathrm{Ko}}$ is above $\mathsf{y}^\dagger$ (with $y_{\mathrm{H}}^{\mathrm{Ko}} = k_{\mathrm{Tu}}$). Thereafter the trajectories follow a similar pattern in the reverse order.

The point is that even if the equilibrium price functions $p^\star$ and $\psi^\star$ are unique, the production-supporting prices p_{Ko} and ψ_{Ko} are not. On the interval $\left[\underline{t}, \overline{t}\right]$, ψ_{Ko} can be set at any constant value between $\underline{\mathsf{w}}^\dagger$and $\overline{\mathsf{w}}^\dagger$; and although this also determines $p_{\mathrm{Ko}} = \psi_{\mathrm{Ko}}$ on $[\underline{t}, t']$, the values of p_{Ko} on $[t', t'']$ are subject only to the constraints $\psi_{\mathrm{Ko}} \leq p_{\mathrm{Ko}}(t) \leq \overline{\mathsf{w}}^\dagger$. On $[t'', t''']$, p_{Ko} equals $p^\star$. A "general" admissible choice of $(p_{\mathrm{Ko}}, \psi_{\mathrm{Ko}})$ is shown in Figure 6f. With reference to (9.31) and (9.14), the choice that minimises $\left(\psi_{\mathrm{Ko}} - \underline{\mathsf{w}}^\dagger\right)^+$, which is the interval's contribution to $\mathrm{Var}^+(\psi_{\mathrm{Ko}}) = r_{\mathrm{Ko}}^{\mathrm{St}}$, is $\psi_{\mathrm{Ko}} = \underline{\mathsf{w}}^\dagger$ with any p_{Ko} (Figure 6g). But it is a different choice that minimises $\int_{\underline{t}}^{t'''} (p_{\mathrm{Ko}} - \psi_{\mathrm{Ko}})^+ \,\mathrm{d}t$, which is the interval's contribution to $r_{\mathrm{Ko}}^{\mathrm{Tu}}$: it is $p_{\mathrm{Ko}} = \psi_{\mathrm{Ko}} = \overline{\mathsf{w}}^\dagger$, on $[\underline{t}, t'']$: see Figure 6h. (A lower value for ψ_{Ko} would not do because it would mean a higher integral of $(p_{\mathrm{Ko}} - \psi_{\mathrm{Ko}})^+$ over $[t'', t''']$, where p_{Ko} is given, equal to $p^\star$.) It is therefore impossible to minimise both capacity values, $r_{\mathrm{Ko}}^{\mathrm{St}}$ and $r_{\mathrm{Ko}}^{\mathrm{Tu}}$, by the same choice of the shadow prices p_{Ko} and ψ_{Ko}; i.e., the set $\partial_k C_{\mathrm{SR}}^{\mathrm{ThH}}$ does *not* have a least point. By (9.31), this means that the incremental value $-\mathrm{D}C$ is not additive even if both increments $(\Delta k_{\mathrm{St}}, \Delta k_{\mathrm{Tu}})$ are positive. ∎

9.5. **The dual of Koopmans' despatch problem.** An alternative to Koopmans' approach is to formulate the dual, solve it *first*, and then use the optimal shadow prices p and ψ to obtain the primal (operating) solution by (4.36), as in our profit-maximisation framework. Such an approach seems workable also in Koopmans' cost-minimisation framework; and, unlike the primal, the dual reduces to unconstrained minimisation. However, by comparison with our dual minimand (4.32), in which p is a vector of *data* rather than variables, the minimand (9.38) contains the extra variables p and an extra nonlinear term $\Pi_{\mathrm{SR}}^{\mathrm{Th}}(p)$. It is not as simple as the minimand

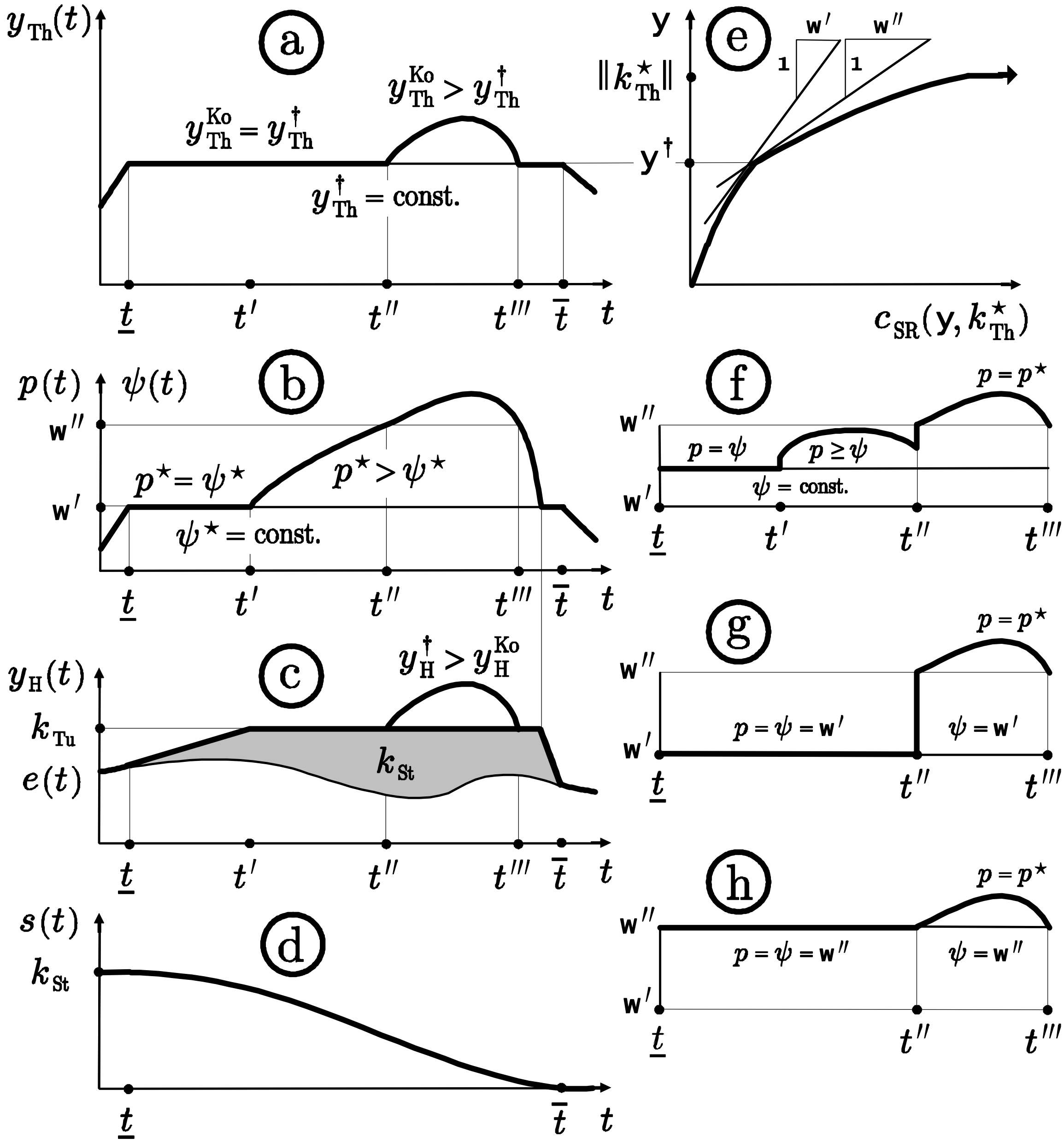

FIGURE 6. Nonuniqueness of Koopmans' shadow prices for water ψ and for electricity p (Remark 9.6).

(4.31), and it does not seem possible to get a simple picture of the dual solution for Koopmans' problem.

The dual is in this case the problem of shadow pricing for both electricity and water so as to minimise the value of *all* the available resources—viz., the maximum operating profit (the total rent) $\Pi_{\mathrm{SR}}^{\mathrm{Th}}$ of the thermal system k_{Th}, plus the value of the hydroelectric resources (k_{H}, e), minus the value of the demanded hydro-thermal output y_{ThH}. Formally the dual minimand, to be minimised over the dual variables ρ of (9.16), is the supremum of the Lagrange function (9.17) over the primal variables y of (9.15). The supremum of $\Pi_{\mathrm{Ko}}(y, \rho)$ is $\Pi_{\mathrm{SR}}^{\mathrm{Th}}(p)$, the (maximum) SR profit of the thermal system; so the dual of the programme (9.1)–(9.7) is:

(9.33) Given $(y_{\mathrm{ThH}}, k_{\mathrm{Th}}, k_{\mathrm{St}}, k_{\mathrm{Tu}}, e)$ and w as in (9.1)

(9.34) minimise $V^{\mathrm{H}}\left(\kappa^{\mathrm{St}}, \kappa^{\mathrm{Tu}}, \psi; k_{\mathrm{St}}, k_{\mathrm{Tu}}, e\right) + \Pi_{\mathrm{SR}}^{\mathrm{Th}}(p, k_{\mathrm{Th}}, w) - \langle p, y_{\mathrm{ThH}} \rangle$

(9.35) over $p \in L^{\infty *}$ and $\left(\kappa^{\mathrm{St}}, \nu^{\mathrm{St}}; \kappa^{\mathrm{Tu}}, \nu^{\mathrm{Tu}}; \psi, \lambda\right)$ as in (4.11)

(9.36) subject to (4.12), (4.13), (4.14) and (4.15),

where V^{H} is the value of the hydro resources, given by (4.24).

Comments:

1. At any dual optimum one has $p \geq 0$.
2. In terms of the Hewitt-Yosida decomposition of a $p \geq 0$, the thermal system's SR profit is

(9.37)
$$\begin{aligned} \Pi_{\mathrm{SR}}^{\mathrm{Th}}(p) &= \Pi_{\mathrm{SR}}^{\mathrm{Th}}(p_{\mathrm{CA}}; k_{\mathrm{Th}}, w) + \|p_{\mathrm{FA}}\|_{\infty}^{*} \|k_{\mathrm{Th}}\|_{\mathrm{Var}} \\ &= \int_0^T \pi_{\mathrm{SR}}(p_{\mathrm{CA}}(t); k_{\mathrm{Th}}, w)\, \mathrm{d}t + \|p_{\mathrm{FA}}\|_{\infty}^{*} \|k_{\mathrm{Th}}\|_{\mathrm{Var}}, \end{aligned}$$

   where $\pi_{\mathrm{SR}}(\mathsf{p})$ is the thermal system's instantaneous SR profit per unit time when the price rate is p (in $/kWh).
3. Formally π_{SR} is defined as the convex conjugate of c_{SR} as a function of the output rate y. Therefore $\partial_{\mathsf{y}} c$ and $\partial_{\mathsf{p}} \pi$ have the same graph in the (y, p)-plane—viz., the capacity-incremental operating cost curve, which is the graph of the system's c.d.f. of the unit variable cost, discussed in Section 7. Furthermore $\pi_{\mathrm{SR}}(\mathsf{p}) + c_{\mathrm{SR}}(\mathsf{y}) \geq \mathsf{py}$, with equality if and only if $\mathsf{p} \in \partial_{\mathsf{y}} c$ (or equivalently $\mathsf{y} \in \partial_{\mathsf{p}} \pi$). This is Young's inequality: see, e.g., [53, 1.17 (b)]. It follows that $\pi_{\mathrm{SR}}(\mathsf{p})$ can be calculated by integrating over $[0, \mathsf{p}]$ the inverse of $\partial_{\mathsf{y}} c$, i.e., by integrating the afore-mentioned c.d.f. This calculation consists in decomposing $[0, \mathsf{p}]$ into infinitesimal tranches $[\mathsf{w}, \mathsf{w} + \mathsf{dw}]$, and summing all the terms dw multiplied by the total capacity of those stations with unit fuel costs below w: for such a station this tranche of p is part of the operating profit.
4. By eliminating the other variables as in Proposition 4.4, the dual (9.33)–(9.36) is reduced to unconstrained minimisation of

(9.38)
$$\Pi_{\mathrm{SR}}^{\mathrm{Th}}(p, k_{\mathrm{Th}}) + k_{\mathrm{St}} \operatorname{Var}_{\mathrm{c}}^{+}(\psi) + k_{\mathrm{Tu}} \left\langle (p - \psi)^{+}, 1 \right\rangle + \int_0^T \psi(t)\, e(t)\, \mathrm{d}t - \langle p, y_{\mathrm{ThH}} \rangle$$

   over $p \in L^{\infty *}$ and $\psi \in \mathrm{BV}(0, T) \subset L^1$.

5. If (p, ψ) is a dual optimum, then so is its density part (p_{CA}, ψ). This follows from (9.37) and from the generating capacity constraint

$$y_{ThH} \leq \|k_{Th}\| + k_{Tu}. \tag{9.39}$$

9.6. **Slater's Condition for Koopmans' despatch problem.** The Generalised Slater's Condition of [47, (8.12)] guarantees the fullest duality results for convex programmes, viz., solubility of the dual, equality of the primal and dual values, the Kuhn-Tucker characterisation of optima as saddle points, and the derivative property of the value function (i.e., that its subdifferential w.r.t. the primal parameters is equal to the dual solution set). In our profit maximisation framework, Slater's Condition (4.8) is not seriously restrictive. But in Koopmans' cost minimisation problem (9.1)–(9.7) Slater's Condition means, *inter alia*, strict inequalities in *both* (9.4) and (9.5), which implies an excess of generating capacity (i.e., a strict inequality in (9.39)). Although such an excess is costly and unjustifiable in the long-run with perfect competition, Koopmans [39, pp. 193 and 197–198] does assume unlimited thermal generating capacity. This assumption is actually not as questionable as it at first seems, since it can be explicated as a purely formal extension of the SRC curve to meet Slater's Condition without positing any real overcapacity. To spell this out, consider the finite, convex extension of c_{SR} to the half-line $(-\infty, \|k_{Th}\| + 1]$ or larger, defined by

$$c_{SR}^{Ex}(\mathsf{y}; k_{Th}, w) := c_{SR}(\mathsf{y} \wedge \|k_{Th}\|; k_{Th}, w) + (\mathsf{y} - \|k_{Th}\|)^{+} \frac{\mathrm{d}c_{SR}}{\mathrm{d}_{-}\mathsf{y}}(\|k_{Th}\|). \tag{9.40}$$

This means continuing the SRC curve in a straight line at the curve's maximum slope, which it has to the immediate left of the total thermal capacity $\|k_{Th}\|$.[60] The point is that if the original hydro-thermal despatch problem (9.1)–(9.7) is feasible, then the operating cost cannot be lowered by adding an extra thermal station with the highest unit operating cost (of all those already in the system); so the extension (9.40) does not change the programme's value. In formal terms, an "extended" primal programme is obtained by replacing $C_{SR}^{Th}(\cdot, k_{Th}, w)$ in (9.2) with its extension C_{SR}^{Ex} defined on $L^{\infty}[0,T]$ by (7.2) with c_{SR}^{Ex} instead of c_{SR}. Imposed on the "extended" primal, Slater's Condition ensures the derivative property of the value function, i.e., that $\partial_{y_{ThH}, k_H, e} C_{SR}^{ExH}$ equals the set of all those (p, κ^{H}, ψ) solving the "extended" dual.[61] The incremental values of the hydro inputs are then given by (9.31) with C_{SR}^{ExH} instead of C_{SR}^{ThH}. But in the case of input increases $(\Delta k_H, \Delta e) \geq 0$, the "extended" values are actually the same as the original incremental values (since C_{SR}^{ExH} equals the original value C_{SR}^{ThH} if the original problem is feasible, and since input increases preserve feasibility).

[60] The slope equals $\max_{\theta} \{w^{\theta} : \theta \in \operatorname{supp} k_{Th}\} = w_{\uparrow}(\|k_{Th}\| -)$.

[61] The dual is (9.33)–(9.36) with k_{Th} replaced by the sum of k_{Th} and a unit measure concentrated on $\operatorname{ArgMax}(w_{|\operatorname{supp} k_{Th}})$.

Since $c_{\mathrm{SR}}^{\mathrm{Ex}}$ is differentiable at $\mathsf{y} = \|k_{\mathrm{Th}}\|$, the preceding argument also shows that the kink of c_{SR} at $\|k_{\mathrm{Th}}\|$ does *not* contribute to the nonadditivity (9.32) of incremental values for *increases* of the hydro capacities.[62] (This is confirmed by recalling that Remark 9.6, which shows the nonadditivity when c_{SR} has an *offpeak* kink at some $\mathsf{y}^{\dagger}$, does rely on the presence of an interval $[t'', t''']$ on which $y_{\mathrm{Th}}^{\mathrm{Ko}}(t) > \mathsf{y}^{\dagger}$.)

9.7. **Koopmans' solution and majorisation.** As Koopmans notes [39, p. 225, footnote], his optimal hydro output $y_{\mathrm{H}}^{\mathrm{Ko}}$ does not depend on the shape of the convex curve c_{SR}, which is determined by the thermal fuel prices w. It follows that $y_{\mathrm{H}}^{\mathrm{Ko}}$ is, with any w, better (or at least not worse) than the "pure coasting" policy $y_{\mathrm{H}} = e$, on the assumption that the latter is feasible in the problem (9.1)–(9.7), i.e., that

$$k_{\mathrm{Tu}} \geq e \geq 0 \quad \text{and} \quad \|k_{\mathrm{Th}}\| \geq y_{\mathrm{ThH}} - e \geq 0. \tag{9.41}$$

Those policies which, like $y_{\mathrm{H}}^{\mathrm{Ko}}$, always improve on pure coasting can be characterised in terms of the Hardy-Littlewood-Polya majorisation order $\prec_{\mathrm{HLP}}$, abbreviated to $\prec$. This is actually a partial preorder on $L^1[0,T]$. It can be defined, along with the lower weak majorisation $\prec_{\mathrm{w}}$ and upper weak majorisation $\prec^{\mathrm{w}}$, in terms of the nondecreasing rearrangement $x_{\uparrow}$ of x (introduced in Section 7). Namely,

$$\begin{aligned}
x \prec_{\mathrm{w}} y &\Leftrightarrow \forall \tau \in [0,T] \int_{\tau}^{T} x_{\uparrow}(t)\,\mathrm{d}t \leq \int_{\tau}^{T} y_{\uparrow}(t)\,\mathrm{d}t \\
x \prec y &\Leftrightarrow \left(x \prec_{\mathrm{w}} y \text{ and } \int_0^T x(t)\,\mathrm{d}t = \int_0^T y(t)\,\mathrm{d}t \right) \\
x \prec^{\mathrm{w}} y &\Leftrightarrow \forall \tau \in [0,T] \int_0^{\tau} y_{\uparrow}(t)\,\mathrm{d}t \leq \int_0^{\tau} x_{\uparrow}(t)\,\mathrm{d}t
\end{aligned}$$

If $\int_0^T x(t)\,\mathrm{d}t = \int_0^T y(t)\,\mathrm{d}t$, then $x \prec^{\mathrm{w}} y$ is equivalent to $x \prec y$ and to $x \prec_{\mathrm{w}} y$.

Comments:

1. These concepts were first used for finite-dimensional vectors, and then extended to functions: see [19, 2.18] and [41, 1.D].
2. The definitions of $\prec^{\mathrm{w}}$, $\prec$ and $\prec_{\mathrm{w}}$ apply also to integrable functions on any finite measure space Ω, instead of $[0,T]$. For example, this can be a probability space; and in the context of risk aversion, upper weak majorisation is known as the second-degree stochastic dominance (discussed in, e.g., [14, 2.14 and 2.16]). More precisely, one order is the reverse of the other, i.e., x second-degree dominates y if and only if $x \prec^{\mathrm{w}} y$. (So $x \prec y$ means the second-degree stochastic dominance, of y by x, for the special case of equal means.)

[62] The peak kink of the SRC curve is relevant for (9.31) in the case of input decreases that make the primal infeasible. For example, if $\mathrm{Sup}(y_{\mathrm{ThSH}}) = \|k_{\mathrm{Th}}\| + k_{\mathrm{Tu}}$, then $\partial C_{\mathrm{SR}}^{\mathrm{ThSH}} / \partial_{-} k_{\mathrm{Tu}} = -\infty$, and this is expressed in an arbitrarily high shadow price $p_{\mathrm{Ko}}(t)$ when $y_{\mathrm{Th}}^{\mathrm{Ko}}(t) = \|k_{\mathrm{Th}}\|$ in (9.10).

3. For x and y in $L^1(\Omega)$, x second-degree dominates y (i.e., $x \prec^{\mathrm{w}} y$) if and only if $\int_\Omega u(y(\omega))\,\mathrm{d}\omega \le \int_\Omega u(x(\omega))\,\mathrm{d}\omega$ for every nondecreasing, concave function $u\colon \mathbb{R} \to \mathbb{R}$. Also, $x \prec y$ if and only if $\int_\Omega c(x(\omega))\,\mathrm{d}\omega \le \int_\Omega c(y(\omega))\,\mathrm{d}\omega$ for every convex function $c\colon \mathbb{R} \to \mathbb{R}$. A similar characterisation of $\prec_{\mathrm{w}}$ is used below, in the Proof of Remark 9.7.

Roughly speaking, $x \prec y$ means that the distribution of x (w.r.t. the Lebesgue measure here) is "more concentrated about the average" than the distribution of y. By using $\prec$ one can give a precise meaning to the notion that policies better than coasting use storage to reduce the variability of thermal output over the cycle.

Remark 9.7. *In addition to (9.41), assume that the measure k_{Th} on Θ (representing the thermal system) is nonatomic. Then a policy y_{H} (with no spillage) is not worse than pure coasting at every fuel price system $w \in \mathcal{C}_+(\Theta)$ if and only if:*

1. *y_{H} is feasible (which depends on y_{ThH}, k_{Th}, k_{St}, k_{Tu} and e, but not on w).*
2. *The corresponding thermal output $y_{\mathrm{Th}} = y_{\mathrm{ThH}} - y_{\mathrm{H}}$ is majorised by the "coasting" thermal output $y_{\mathrm{Cs}} := y_{\mathrm{ThH}} - e$, i.e.,*

$$y_{\mathrm{Th}} \prec_{\mathrm{HLP}} y_{\mathrm{Cs}}.$$

Proof. That y_{H} improves on "coasting" at any w means that y_{H} is feasible and

$$\int_0^T c(y_{\mathrm{Th}}(t))\,\mathrm{d}t \le \int_0^T c(y_{\mathrm{Cs}}(t))\,\mathrm{d}t \qquad (9.42)$$

for *every* nondecreasing, convex function $c\colon [0, \|k_{\mathrm{Th}}\|] \to \mathbb{R}$ such that $(\mathrm{d}c/\mathrm{d}_+\mathrm{y})(0) > 0$ and $(\mathrm{d}c/\mathrm{d}_-\mathrm{y})(\|k_{\mathrm{Th}}\|) < +\infty$. (This is because any convex shape of c_{SR} can be obtained from some w, if k_{Th} is nonatomic.) And Condition (9.42) is equivalent to $y_{\mathrm{Th}} \prec_{\mathrm{w}} y_{\mathrm{Cs}}$ by Chong's variant [9, Theorem 2.3] of a theorem of Hardy et al. [19, 3.17: 108], which is also given in [41, 4.B.2].[63] Finally, $y_{\mathrm{Th}} \prec_{\mathrm{w}} y_{\mathrm{Cs}}$ is equivalent to $y_{\mathrm{Th}} \prec y_{\mathrm{Cs}}$, since $\int_0^T y_{\mathrm{Th}}(t)\,\mathrm{d}t = \int_0^T y_{\mathrm{Cs}}(t)\,\mathrm{d}t$. ∎

10. Models of hydro-thermal generation with a constant shadow price for water

This section describes the models of Jacoby [38] and Munasinghe and Warford [42], who over-simplify the formulation of the problem by making the water price constant throughout the production cycle. Jacoby, like Koopmans, uses a purely short-run formulation. Munasinghe and Warford use a mixture of SR and LR concepts, and they do not address the operation of a *given* hydro-thermal system.

[63] The notation $\prec_{\mathrm{w}}$ is that of [41, 1.A.2]; it corresponds to $\prec\!\prec$ in [9, p. 1324].

10.1. **Jacoby's despatch problem.** Jacoby's method of determining the hydro output applies to a model of thermal cost minimisation based on the assumption that the *only* constraint on the hydro output y_{H}, apart from the turbine constraint (9.5), is imposed by the total volume W of water available for hydro generation in one cycle. The despatch problem can then be formulated as:

(10.1) Given $(y_{\mathrm{ThH}}, k_{\mathrm{Th}}, k_{\mathrm{Tu}}, w)$ as in (9.1) and $W \in \mathbb{R}_+$

(10.2) minimise $C_{\mathrm{SR}}^{\mathrm{Th}}(y_{\mathrm{Th}}, k_{\mathrm{Th}}, w)$ over $(y_{\mathrm{Th}}, y_{\mathrm{H}})$

(10.3) subject to: $y_{\mathrm{ThH}} = y_{\mathrm{Th}} + y_{\mathrm{H}}$

(10.4) $0 \le y_{\mathrm{Th}} \le \|k_{\mathrm{Th}}\|$

(10.5) $0 \le y_{\mathrm{H}} \le k_{\mathrm{Tu}}$

(10.6) $$\int_0^T y_{\mathrm{H}}(t)\, \mathrm{d}t \le W.$$

This model is also expounded by Anderson [2, pp. 276–280], who points out that it can be a realistic description of hydroelectric operation in the case of a large reservoir built primarily for other purposes (such as irrigation or flood control), i.e., when hydro generation is only a "fringe benefit". The model can also apply to a purely hydroelectric scheme, though only in the idealised case of an instantaneous "downpour" inflow. Such an inflow can be represented by a one-point measure, of some mass E concentrated at the beginning of a cycle. Any excess $(\mathsf{E} - k_{\mathrm{St}})^+$ of the inflow over the reservoir capacity has to be spilt, and the amount of water available for hydro generation is

(10.7) $$W = \mathsf{E} \wedge k_{\mathrm{St}}.$$

Anderson [2, pp. 280–282] also outlines a dynamic programme in which the cycle $[0, T]$ is divided into subperiods with different total water inflows. The water volumes to be used in each subperiod are then decision variables. This extension of Jacoby's model can be viewed as a discrete-time version of Koopmans' model (with Anderson's upper bound $\hat{S}$ on the stock interpreted as the reservoir's capacity k_{St}). However, except for computation purposes, discretisation is analytically disadvantageous in this context, as is shown by Koopmans' work and ours.

10.2. **Jacoby's optimal policy and shadow price.** Jacoby's solution to (10.1)–(10.6) uses the load-duration curve for the whole hydro-thermal system to place the hydro plant in the system's merit order. The range of loads $[\underline{\mathsf{y}}, \overline{\mathsf{y}}]$ to be met from hydro is found on the LDC, which formally is the nonincreasing rearrangement $y_{\mathrm{ThH}}^{\downarrow}$

of the demanded hydro-thermal output, from the conditions:

$$W = \int_{\underline{\mathsf{y}}}^{\overline{\mathsf{y}}} \left(y_{\mathrm{ThH}}^{\downarrow} \right)^{-1} (\mathsf{y}) \, \mathrm{d}\mathsf{y} \tag{10.8}$$

$$\overline{\mathsf{y}} - \underline{\mathsf{y}} = k_{\mathrm{Tu}}, \tag{10.9}$$

where both $\underline{\mathsf{y}}$ and $\overline{\mathsf{y}}$ are constrained to lie between 0 and $k_{\mathrm{Tu}} + \|k_{\mathrm{Th}}\|$.[64] See Figure 7a or [2, Figure 3b]. The optimal hydro output, unique if $y_{\mathrm{ThH}} \in \mathcal{C}$, is

$$\check{y}_{\mathrm{H}} = \left(y_{\mathrm{ThH}} - \underline{\mathsf{y}} \right)^{+} \wedge k_{\mathrm{Tu}} = \left(y_{\mathrm{ThH}} - \underline{\mathsf{y}} \right)^{+} \wedge \left(\overline{\mathsf{y}} - \underline{\mathsf{y}} \right). \tag{10.10}$$

The corresponding thermal output is[65]

$$\check{y}_{\mathrm{Th}} = \left(y_{\mathrm{ThH}} - \overline{\mathsf{y}} \right)^{+} + y_{\mathrm{ThH}} \wedge \underline{\mathsf{y}}. \tag{10.11}$$

See Figure 7b.

The system (10.8)–(10.9) is soluble unless: either W is smaller than the r.h.s. of (10.8) for $\underline{\mathsf{y}} = \|k_{\mathrm{Th}}\|$ and $\overline{\mathsf{y}} = k_{\mathrm{Tu}} + \|k_{\mathrm{Th}}\|$, or W is larger than the r.h.s. of (10.8) for $\underline{\mathsf{y}} = 0$ and $\overline{\mathsf{y}} = k_{\mathrm{Tu}}$. In the latter case, with a "large" amount of water W, Formulae (10.10)–(10.11) still apply, with $\underline{\mathsf{y}} = 0$: water is then in excess supply, its imputed value is zero, and the hydro plant is the first in the merit order, i.e., it is the base-load plant. In the other case W is too small for the despatch programme (10.1)–(10.6) to be feasible.

In Jacoby's problem the marginal value of water is a constant ψ_{Ja}, which is formally defined as the negative of a subgradient w.r.t. W of the optimal value C_{Ja} of (10.1)–(10.6). It is the imputed unit running cost of the hydro plant, ψ_{Ja}. This always lies between the two unit variable costs, denoted by $\underline{\mathsf{w}} \leq \overline{\mathsf{w}}$, of the two thermal stations which are adjacent to the hydro plant in the merit order. However, ψ_{Ja} cannot be determined any further: the relevant left and right derivatives are $-\partial C_{\mathrm{Ja}}/\partial_{-} W = \overline{\mathsf{w}}$ and $-\partial C_{\mathrm{Ja}}/\partial_{+} W = \underline{\mathsf{w}}$. The two adjacent plant types are usually different when the thermal system is optimal for its output $\check{y}_{\mathrm{Th}} = y_{\mathrm{ThH}} - \check{y}_{\mathrm{H}}$. This contains a plateau at the level $\underline{\mathsf{y}}$, of duration equal to $\mathrm{meas} \left\{ t : \overline{\mathsf{y}} \geq y_{\mathrm{ThH}} (t) \geq \underline{\mathsf{y}} \right\}$: see Figure 7b. The unit variable costs of the two thermal stations are $\underline{\mathsf{w}} := \check{w} (\overline{\tau})$ and $\overline{\mathsf{w}} := \check{w} (\underline{\tau})$, defined by (7.3) with

$$\underline{\tau} := \mathrm{meas} \left\{ t : y_{\mathrm{ThH}} (t) > \overline{\mathsf{y}} \right\} \leq \mathrm{meas} \left\{ t : y_{\mathrm{ThH}} (t) < \underline{\mathsf{y}} \right\} =: \overline{\tau}, \tag{10.12}$$

and actually $\underline{\tau} < \overline{\tau}$, at least if y_{ThH} is continuous. Hence $\underline{\mathsf{w}} < \overline{\mathsf{w}}$ typically (for example, when the thermal technology is a continuum of station types with a smooth capital-fuel substitution curve, as in Figure 4a and Remark 7.2). Then ψ_{Ja} is indeterminate

[64]The integrand of (10.8) can also be given as $\mathrm{meas} \{ t : y (t) > \mathsf{y} \}$.

[65]This means that $\check{y}_{\mathrm{Th}}$ stays constant, equal to $\underline{\mathsf{y}}$, whenever the hydroturbine constraints (10.5) are inactive; i.e., the instantaneous thermal SRMC's are equalised as much as possible. This resembles a consumer's choice with rationing; and indeed problem (10.1)–(10.6) is formally equivalent to maximisation of the "utility" $- \int_0^T c_{\mathrm{SR}} \left(y_{\mathrm{ThH}} (t) - y_{\mathrm{H}} (t) \right) \mathrm{d}t$ over y_{H} subject to the "rationing" and "budget" constraints (10.5)–(10.6).

within the range

$$\Psi_{\mathrm{Ja}} := -\partial_W C_{\mathrm{Ja}} = [\underline{\mathsf{w}}, \overline{\mathsf{w}}] \,. \tag{10.13}$$

See Figure 7c (which shows the same curve as Figure 4b) and Figure 7d.

10.3. **Munasinghe-Warford's problem.** Munasinghe and Warford [42, pp. 62–65] use the same model of hydro generation that is constrained by k_{Tu} and W alone. However, what they consider is the position after the hydro investment but before the thermal investment. Therefore, instead of the capacities k_{Th} of Jacoby's problem, their data include the vector of unit thermal capacity costs r^{Th}; and their problem can be formulated as:

$$\text{Given } \left(y_{\mathrm{ThH}}, k_{\mathrm{Tu}}; r^{\mathrm{Th}}, w\right) \text{ and } W \in \mathbb{R}_+ \tag{10.14}$$

$$\text{minimise } C_{\mathrm{LR}}^{\mathrm{Th}}\left(y_{\mathrm{Th}}, r^{\mathrm{Th}}, w\right) \text{ over } \left(y_{\mathrm{Th}}, y_{\mathrm{H}}\right) \tag{10.15}$$

$$\text{subject to: } \; y_{\mathrm{ThH}} = y_{\mathrm{Th}} + y_{\mathrm{H}} \tag{10.16}$$

$$0 \le y_{\mathrm{H}} \le k_{\mathrm{Tu}} \tag{10.17}$$

$$\int_0^T y_{\mathrm{H}}(t)\,\mathrm{d}t \le W, \tag{10.18}$$

with $C_{\mathrm{LR}}^{\mathrm{Th}}$ given by (7.6) or (7.7). The optimal hydro output $\check{y}_{\mathrm{H}}$ is found exactly as in Jacoby's problem, from (10.8)–(10.10); it depends only on y_{ThH}, k_{Tu} and W. The marginal value of water, defined as the derivative in W of the optimal value C_{Mu} of the problem (10.14)–(10.18), is again a constant, ψ_{Mu}. Furthermore

$$\psi_{\mathrm{Mu}} = \frac{c_{\mathrm{LR}}(\overline{\tau}) - c_{\mathrm{LR}}(\underline{\tau})}{\overline{\tau} - \underline{\tau}} = \frac{1}{\overline{\tau} - \underline{\tau}} \int_{\underline{\tau}}^{\overline{\tau}} \breve{w}(\tau)\,\mathrm{d}\tau, \tag{10.19}$$

with $\overline{\tau}$ and $\underline{\tau}$ given by (10.12). This means that ψ_{Mu} can be found graphically by referring the points on the combined system's LDC whose load-coordinates are $\underline{\mathsf{y}}$ and $\overline{\mathsf{y}}$ to the graph of the thermal unit LR cost c_{LR} as a function of load duration, which is given by (7.3). More precisely, (10.19) means that the imputed value of water is the slope of the secant line through the resulting pair of points on the c_{LR} curve.[66] The secant's intercept on the cost axis is the imputed value of a unit hydro turbine (in \$/kW). See Figure 7e or [42, Figure 4.5].

Comments:

1. What makes ψ_{Mu} determinate, by contrast to ψ_{Ja}, is that in Munasinghe and Warford's problem the thermal system k_{Th} is, implicitly, being re-optimised to each value of W (instead of being fixed as in Jacoby's problem). The indeterminacy (10.13) of ψ_{Ja} is consistent with the determinacy of ψ_{Mu} because it is generally impossible to impute a unique ψ_{Ja} through the unique ψ_{Mu} because the latter depends on r^{Th}, and the r^{Th}'s imputed to a given k_{Th} are nonunique.

[66] And it equals the average of unit fuel costs for those thermal types that will *not* be invested in, given the hydro plant.

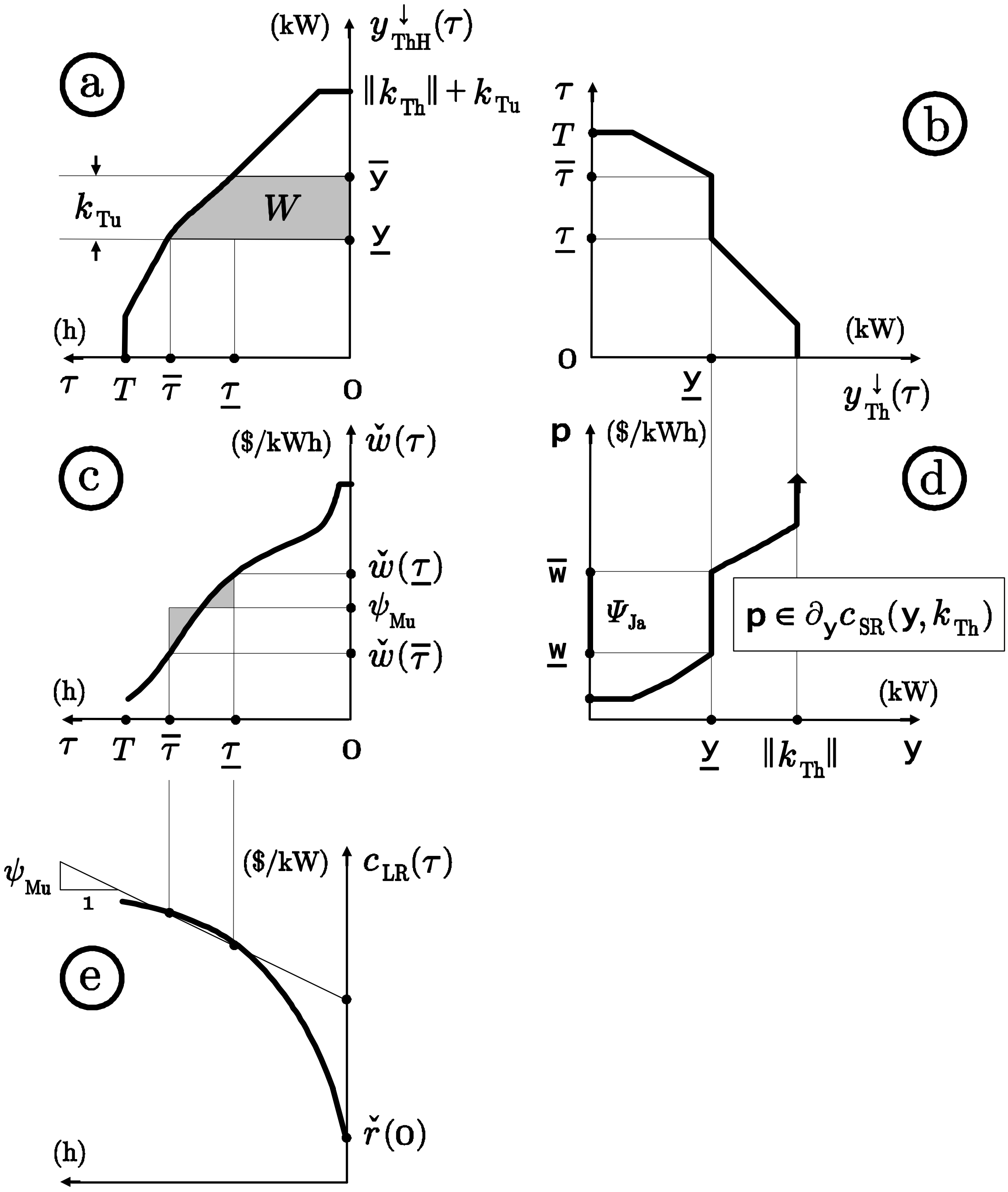

FIGURE 7. Part (a) shows the determination of the hydro output in Jacoby's and Munasinghe-Warford's problems. Part (d) shows the indeterminacy of the shadow water price ψ_{Ja}, when the thermal system is optimal for thermal load-duration curve in (b). Part (e), or (c) with the two gray areas equalised, shows the determination of the shadow water price ψ_{Mu}.

In precise terms, given W and $(y_{\mathrm{ThH}}, k_{\mathrm{Tu}}, w)$, the interval $\Psi_{\mathrm{Ja}}(k_{\mathrm{Th}})$ consists of all the values of $\psi_{\mathrm{Mu}}(r^{\mathrm{Th}})$ for $r^{\mathrm{Th}} \in \partial_{k_{\mathrm{Th}}} C_{\mathrm{SR}}^{\mathrm{Th}}(\check{y}_{\mathrm{Th}}, k_{\mathrm{Th}})$. Since $\check{y}_{\mathrm{Th}}$ has a plateau, such r^{Th}'s are nonunique (and they do not just differ by an additive constant, i.e., by a term independent of θ, which obviously would not influence ψ_{Mu}): see [32].

2. The concept of the reservoir's marginal value r^{St} makes sense in Jacoby's problem if the available water is interpreted as the stored part of a point inflow as in (10.7); and if $\mathsf{E} > k_{\mathrm{St}}$ then $r^{\mathrm{St}} \in -\partial_{k_{\mathrm{St}}} C_{\mathrm{Ja}} = -\partial_W C_{\mathrm{Ja}} =: \Psi_{\mathrm{Ja}}$. However, the formula $r^{\mathrm{St}} = \mathrm{Var}^+(\psi)$ fails for $\psi \in \Psi_{\mathrm{Ja}}$, since ψ is constant over time (unlike the ψ_{Ko} and $\hat{\psi}$ of Sections 9 and 4). This does not, of course, contradict Koopmans' analysis or ours: the formula applies only to a gradual inflow at a finite rate, and not to a point inflow. The latter is a limiting case, though: the point inflow of $\mathsf{E} > k_{\mathrm{St}}$ can be approximated by, e.g., a two-valued step function $e_\epsilon := (\mathsf{E}/\epsilon)\, 1_{[0,\epsilon]}$. For small ϵ Koopmans' model gives $\psi_{\mathrm{Ko}}^\epsilon = 0$ on $[0, \epsilon]$, whilst on $[\epsilon, T]$ the $\psi_{\mathrm{Ko}}^\epsilon$ is a constant ψ^ϵ that converges to ψ_{Ja} as $\epsilon \searrow 0$; so the variation formula (applied with $\epsilon > 0$ and followed by passage to the limit), gives the same answer as Jacoby's model:

$$r_\epsilon^{\mathrm{St}} = \mathrm{Var}_{\mathrm{c}}^+(\psi_{\mathrm{Ko}}^\epsilon) = \psi^\epsilon \to \psi_{\mathrm{Ja}} \quad \text{as } \epsilon \searrow 0.$$

11. Conclusions

By combining and developing useful features of earlier work on hydroelectricity, this analysis gives a sound basis for valuation of existing hydro plants and for investment decisions. Definite marginal values can can be imputed to the hydro inputs, including the water inflow, when the hydro operation problem set up as (short-run) profit maximisation, given a continuous TOU price for electricity. This is much simpler than the SR cost-minimum problem for a combined hydro-thermal system. On the basis of the operating profit, the storage reservoir and the turbine can be assigned separate rental values despite their perfect complementarity. In this framework there is a unique solution to the dual programme, which has the interpretation of (TOU) shadow pricing of water. This also gives the marginal values of the capacities, which can therefore be calculated by standard LP methods.[67] Evaluation of such efficiency rents turns out to be useful not only to decentralised industries but also to monopolistic public utilities.

Extension to the case of stochastic river flows is a subject for future work.

[67]The rents are expressed in terms of the given market price p and the shadow price ψ, whose component terms are found from (4.9)–(4.15). Numerical solution may in practice require discretisation of time to transform this infinite LP into a finite one; in this respect our uniqueness result for the continuous-time LP ensures that the approximate solutions converge as the discretisation is refined.

Appendix A. Duality for a class of convex programmes with nonstandard parameters

A.1. **A primal programme with standard and nonstandard parameters.** To facilitate the applications of duality in Sections 4 and 9, we spell out the Lagrange function and the dual programme, along with the associated marginal-value results and characterisations of optima, when the primal programme has the form:

$$\text{Given } s = (a, b, c) \in \mathcal{S} := \mathcal{A} \times \mathcal{B} \times \mathcal{C} \text{ and } e \in \mathcal{E}, \tag{A.1}$$

$$\text{maximise } M(y) \text{ over } y \in \mathcal{Y} \tag{A.2}$$

$$\text{subject to: } A(y) \leq a \tag{A.3}$$

$$B(y, e) = b \tag{A.4}$$

$$C(y, e) \leq c. \tag{A.5}$$

The maximand M: $\mathcal{Y} \to \mathbb{R}$ is assumed to be a norm-continuous, concave and finite (real-valued) function on a Banach lattice $\mathcal{Y}$, which is the primal variables space. Similarly A: $\mathcal{Y} \to \mathcal{A}$, B: $\mathcal{Y} \times \mathcal{E} \to \mathcal{B}$ and C: $\mathcal{Y} \times \mathcal{E} \to \mathcal{C}$ are continuous convex maps of $\mathcal{Y} \times \mathcal{E}$ into the Banach lattices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively. These are primal parameter spaces, as is the Banach lattice $\mathcal{E}$. The Banach dual $\mathcal{Y}^*$ serves as the dual parameter space (paired with the primal variables space $\mathcal{Y}$). The other four Banach duals ($\mathcal{A}^*$, etc.) serve as the dual variables spaces (paired with the primal parameter spaces).[68] In this and in other respects, the exposition of [47, Examples 4, 4', 4"] is followed.

The constraint maps dependent on e (viz., B and C) are actually taken to be linear in $(y, e) \in \mathcal{Y} \times \mathcal{E}$, i.e.,

$$B(y, e) = Fy + Ge \quad \text{and} \quad C(y, e) = Hy + Je \tag{A.6}$$

for some linear maps $(F, H): \mathcal{Y} \to \mathcal{B} \times \mathcal{C}$ and $(G, J): \mathcal{E} \to \mathcal{B} \times \mathcal{C}$. Another simplifying structural assumption is that M is independent of the nonstandard parameter e. It is also independent of a, b and c, since these are the standard constraint parameters. (A *standard* a.k.a. ordinary parameter, a, is the right-hand side of a constraint $A(y) \leq a$, or $A(y) = a$, on a decision variable y. A standard Lagrange multiplier is one that is paired with a standard parameter. Of course, it is always possible to recast the problem as one with standard parameters only, by replacing each nonstandard parameter e with a new variable z that is constrained by an extra equality $z = e$, in which e is a standard parameter. But this does not simplify the analysis.)

One of the objectives in this Appendix is to derive the linear dependence condition that expresses compatibility of the multipliers β, γ and ψ paired with the parameters b, c and e. A heuristic argument can be based on the usual marginal interpretation of the multipliers: from (A.4)–(A.6) it follows that an increment Δe has the same effect

[68] For the hydro application in Section 4, $\mathcal{E}$ is L^∞, and its Banach predual L^1 is a sufficient space for the relevant dual variables; also, the weak* topology on $\mathcal{E} = L^\infty$ is adequate. This follows from Lemma 4.2. By contrast, although $\mathcal{Y} = L^\infty$ as well, L^1 will not do as the dual parameter space unless $p \in L^1$.

on the feasible set (and therefore on the solution and the programme's value) as the pair of increments $\Delta b = -G\Delta e$ and $\Delta c = -J\Delta e$. Since the effects on the value are measured by the multipliers, this means that for every $\Delta e \in \mathcal{E}$

$$\langle \psi, \Delta e \rangle = \langle \beta, -G\Delta e \rangle + \langle \gamma, -J\Delta e \rangle .$$

In terms of the adjoint operators, $\psi + G^* \beta + J^* \gamma = 0$; and this is one of the dual constraints. To derive the complete dual programme we use the framework of [47].

As for the choice of topologies, these must be consistent with the pairings. Furthermore, the norm topology has to be put on the primal parameter spaces $\mathcal{A}$ and $\mathcal{C}$ if the strongest form (A.21) of Slater's Condition is to be met. Topologies on the primal variables space $\mathcal{Y}$ and on $\mathcal{E}$ (along with those on $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$) must be chosen so as to make the maximand M upper semicontinuous (u.s.c.) and the constraint maps (A, B, C) lower semicontinuous (l.s.c.); i.e., the epigraphs of A, B and C must be closed sets. Given the convexity and the Banach-space assumptions, the l.s. continuity is actually equivalent to continuity (for the norm topology). On the dual variables spaces ($\mathcal{A}^*$, ..., $\mathcal{E}^*$) the weak* topologies will do.[69] Topology on the dual parameter space is, as usual, a matter of choice; and the Mackey topology $\mathrm{m}(\mathcal{Y}^*, \mathcal{Y})$ is best if continuity of the dual value function is sought.[70]

The primal value, i.e., the optimal value of (A.1)–(A.5), is denoted by $\Pi(s, e)$. This is because in the Proof of Theorem 4.9 it is the SR profit $\widetilde{\Pi}^{\mathrm{H}}_{\mathrm{SR}}$, a function of the nonstandard parameter e (the inflow) and of the standard parameters (the other available resources k_{St}, n_{St}, k_{Tu}, n_{Tu}, ζ). With regard to the $\widetilde{\Pi}^{\mathrm{H}}_{\mathrm{Exc}}$ of (4.25) and (4.29), its counterpart in this Appendix is the R of (A.13) and (A.14) below.

Also, in the hydro problem $A(y) = y$; and whenever A is linear like C, Constraint (A.3) can be included in (A.5), but keeping it separate makes the application more transparent.

A.2. **The Lagrange function.** Increments $(\Delta s, \Delta e) = (\Delta a, \Delta b, \Delta c; \Delta e)$ to the given primal parameter point

$$(s, e) = (a, b, c; e)$$

are paired with the dual variables (a.k.a. primal Lagrange multipliers) denoted by

$$(\sigma, \psi) = (\alpha, \beta, \gamma; \psi) .$$

Then, by definition of the Lagrange function in [47, (4.2) or (5.12)], after reorienting the primal problem to maximisation,

$$\text{(A.7)} \qquad \mathcal{L}(y; \sigma, \psi) := \sup_{\Delta s, \Delta e} \left\{ M(y) - \delta(y; s + \Delta s, e + \Delta e) - \langle \sigma, \Delta s \rangle - \langle \psi, \Delta e \rangle \right\} ,$$

[69] The weak topologies do not enter the analysis explicitly, but they make the adoint operators continuous: see, e.g., [20, 16C].

[70] When $\mathcal{Y}$ has a Banach predual $\mathcal{P}$, the restriction of $\mathrm{m}(\mathcal{Y}^*, \mathcal{Y})$ to $\mathcal{P}$ is the norm topology of $\mathcal{P}$.

where δ is the 0-∞ indicator function of the constraint set, i.e.,

$$\delta(y; s, e) = \begin{cases} 0 & \text{if } y \text{ meets (A.3), (A.4) and (A.5)} \\ +\infty & \text{otherwise} \end{cases} .$$

Note for clarity that a perturbation consists here in adding increments $(\Delta s, \Delta e)$ to the original parameter point (s, e), which is generally nonzero: unlike [47], we do not place the origin of the (primal) parameter vector space at the unperturbed parameter point. This helps keep track of the dependence on (s, e) of all the concepts (viz., the primal/dual programmes and values, the Lagrange function, etc.).

Maximisation on the right-hand side of (A.7) over the *standard* parameter increment Δs gives

$$\begin{aligned} \mathcal{L}(y; \sigma, \psi) = M(y) - \langle \alpha, A(y) - a \rangle + \langle \beta, b \rangle + \langle \gamma, c \rangle \\ - \inf_{\Delta e} \{ \langle \psi, \Delta e \rangle + \langle \beta, B(y, e + \Delta e) \rangle + \langle \gamma, C(y, e + \Delta e) \rangle \} \end{aligned} \quad \text{(A.8)}$$

if $(\alpha, \gamma) \geq 0$ (otherwise $\mathcal{L} = +\infty$): see, e.g., [47, (4.4)]. So calculation of $\mathcal{L}$ reduces to that of the infimum in (A.8). Upon splitting the adjoint operators, B^*: $\mathcal{B}^* \to \mathcal{Y}^* \times \mathcal{E}^*$ and C^*: $\mathcal{C}^* \to \mathcal{Y}^* \times \mathcal{E}^*$, into $B^* = (F^*, G^*)$ and $C^* = (H^*, J^*)$, the last two terms of (A.8) can be expressed as

(A.9)
$$\langle B^* \beta, (y, e + \Delta e) \rangle + \langle C^* \gamma, (y, e + \Delta e) \rangle = \langle F^* \beta + H^* \gamma, y \rangle + \langle G^* \beta + J^* \gamma, e + \Delta e \rangle$$

and so the infimum over $\Delta e \in \mathcal{E}$ in (A.8) is finite (i.e., not $-\infty$) if and only if

$$\psi + G^* \beta + J^* \gamma = 0. \quad \text{(A.10)}$$

On this multiplier compatibility condition, the minimum value in (A.8) is attained at any Δe, and it equals

$$\langle F^* \beta + H^* \gamma, y \rangle + \langle G^* \beta + J^* \gamma, e \rangle = - \langle \psi, e \rangle + \langle F^* \beta + H^* \gamma, y \rangle . \quad \text{(A.11)}$$

(Setting $\Delta e = 0$ in (A.8)–(A.9) shows the minimum to be equal to the left-hand sum; and then the right-hand sum is obtained by adding and subtracting $\langle \psi, e \rangle$ and applying (A.10).) Therefore, upon defining

$$V(\alpha, \beta, \gamma; \psi) := \langle \alpha, a \rangle + \langle \beta, b \rangle + \langle \gamma, c \rangle + \langle \psi, e \rangle \quad \text{(A.12)}$$

and

$$\begin{aligned} R(y; \alpha, \beta, \gamma; \psi) &:= M(y) - \langle \alpha, A(y) \rangle - \langle F^* \beta + H^* \gamma, y \rangle && \text{(A.13)} \\ &= M(y) - \langle \alpha, A(y) \rangle - \langle \beta, B(y, e) \rangle - \langle \gamma, C(y, e) \rangle - \langle \psi, e \rangle , && \text{(A.14)} \end{aligned}$$

with the last equality following from (A.11) and (A.6), one obtains that

(A.15)

$$\mathcal{L}(y;\sigma,\psi) = \begin{cases} R(y;\sigma,\psi) + V(\sigma,\psi) & \text{if } (\alpha,\gamma) \geq 0 \text{ and } \psi + G^*\beta + J^*\gamma = 0 \\ +\infty & \text{otherwise} \end{cases}$$

In view of (A.12) and (A.14), the sum $\mathcal{L} = R + V$ is independent of ψ and e under (A.10); this comes from the linearity of constraints and the independence of the maximand M on e. It is nevertheless useful to split $\mathcal{L}$ into $R + V$: (A.13) is used below to formulate the dual, whilst (A.14) gives an interpretation of R when (A.1)–(A.5) is an SR profit maximisation problem. In that case, V is the value of fixed resources (s, e) at shadow prices (σ, ψ); and, with $M(y) := \langle p, y \rangle$, (A.14) gives R as the excess profit (a.k.a. pure profit) of an entrepreneur selling the output y at the given market price system p and buying the input e as well the minimum quantities of the other inputs which are required for y, given e. (In Section 4 the excess profit is denoted by $\widetilde{\Pi}^{\mathrm{H}}_{\mathrm{Exc}}$.)

A.3. **The dual programme.** The dual to a concave maximisation programme consists in minimising, over the dual variables, the supremum of the Lagrange function over the primal variables: see, e.g., [47, (4.6), (5.13)]. Since V is independent of y, in the case of (A.1)–(A.5) the dual minimand is

$$\sup_y \mathcal{L}(y;\sigma,\psi) = V(\sigma,\psi) + \sup_y R(y;\sigma,\psi)$$

on the conditions (A.10) and $(\alpha, \gamma) \geq 0$, which become dual constraints (because otherwise $\mathcal{L}$, and hence also the dual minimand, equals $+\infty$).

When A is a linear map and $M(y) = \langle p, y \rangle$, (A.1)–(A.5) is a linear programme (LP), and Formula (A.13) shows that R is linear in y. So its supremum over y is either 0 or $+\infty$; and it is zero if and only if $\nabla_y R = 0$. This is the other dual constraint, (A.20) below. So, with V defined by (A.12), the dual LP is:

(A.16) Given $s = (a, b, c) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C}$ and $e \in \mathcal{E}$ (and $p \in \mathcal{Y}^*$),

(A.17) minimise $V(\alpha, \beta, \gamma; \psi)$ over $(\alpha, \beta, \gamma; \psi) \in \mathcal{S}^* = \mathcal{A}^* \times \mathcal{B}^* \times \mathcal{C}^* \times \mathcal{E}^*$

(A.18) subject to: $(\alpha, \gamma) \geq 0$

(A.19) $\psi + G^*\beta + J^*\gamma = 0$

(A.20) $A^*\alpha + F^*\beta + H^*\gamma = p.$

When the primal (A.1)–(A.5) is an SR profit maximisation programme, the dual can be interpreted as shadow pricing of the fixed resources so as minimise their total value. In the hydro context the primal (4.1)–(4.5) is a case of (A.1)–(A.5); and Theorem 4.1, which identifies the dual, follows from the above by formal substitution. It would be tedious to spell this out in full, but (4.24)–(4.25) correspond to (A.12)–(A.13), whilst

(4.23) and (4.29) correspond to (A.15) and (A.14); and the dual (4.9)–(4.15) is a case of (A.16)–(A.20), with (4.22) giving the adjoint of I_0.

A.4. **The Kuhn-Tucker Conditions.** These characterise a pair of optimal primal and dual solutions as a saddle-point of the Lagrange function: see, e.g., [47, Theorem 15 (e) and (f)]. For the LP at hand (when $M(y) = \langle p, y\rangle$ and A is linear), this means that y and (σ, ψ) solve the primal (A.1)–(A.5) and the dual (A.16)–(A.20) if and only if they maximise and minimise, respectively, the $\mathcal{L}(y; \sigma, \psi)$ of (A.15). The minimum in question is characterised by the complementary slackness conditions

$$\langle \alpha, A(y) - a\rangle = 0 = \langle \gamma, C(y, e) - c\rangle$$

in addition to the conditions of: primal feasibility (A.3)–(A.5), multiplier nonnegativity and multiplier compatibility in (A.15). The maximum in question is characterised by (A.20).

A characterisation of primal solutions follows when the dual is soluble, which is the case under the Generalised Slater's Condition of [47, (8.12)] for the primal constraints (A.3)–(A.5) at (s, e)—i.e., when there exists a $y \in \mathcal{Y}$ with

$$a - A(y) \in \operatorname{int}(\mathcal{A}_+), \quad b = B(y, e) \quad \text{and} \quad c - C(y, e) \in \operatorname{int}(\mathcal{C}_+), \tag{A.21}$$

where $\operatorname{int}(\mathcal{A}_+)$ is the norm-interior of the nonnegative cone in the parameter space $\mathcal{A}$. Nonemptiness of the interiors of $\mathcal{A}_+$ and $\mathcal{C}_+$ is part of the assumption; and it implies that each space, $\mathcal{A}$ or $\mathcal{C}$, can be equivalently renormed so as to be isomorphic (as a normed lattice) to the space of continuous functions $\mathcal{C}(\mathfrak{T})$ on a compact $\mathfrak{T}$: see, e.g., [8, Theorem XV.28 with Lemma XV.16.3 and Exercise XV.12.4] or [51, V.8.5 with V.8.4] for this (the Kakutani-Krein-Krein Theorem). Our two uses of the symbol $\mathcal{C}$ are therefore consistent. In the hydro application in Section 4, the $\mathcal{C}$ of (A.1) is simply $\mathcal{C}[0, T]$, whilst $\mathcal{A} = L^{\infty}[0, T]$; and in either space the nonnegative cone (L^{∞}_{+} or $\mathcal{C}_+$) has a nonempty interior.[71]

A.5. **Primal marginal values.** One reason for solving the dual is to obtain the derivatives of the primal optimal value Π w.r.t. the parameters, s and e. Since Π is concave in (s, e), its superdifferential $\partial\Pi$ serves as a generalised—i.e., possibly multivalued—derivative. For the infinite-dimensional case it is simplest to adopt the algebraic concept of the superdifferential, as in, e.g., [20]. In general $\partial\Pi$ is then a subset of the algebraic dual (which is larger than the norm-dual), but actually $\partial_{s,e}\Pi(s, e) \subset \mathcal{S}^* \times \mathcal{E}^*$ if Π is norm-continuous at (s, e).[72] This is the case here on Slater's Condition (A.21). Furthermore, the superdifferential in question is then equal to the dual solution set; i.e.,

$$\partial_{s,e}\Pi(s, e) = \{(\sigma, \psi) : (\sigma, \psi) \text{ solves (A.16) to (A.20)}\}. \tag{A.22}$$

[71] $L^{\infty}[0, T]$ is also isomorphic to some $\mathcal{C}(\mathfrak{T})$, but such a $\mathfrak{T}$ is extremally disconnected.

[72] Therefore $\partial\Pi$ is equal to the topological superdifferential of [47], which is in general defined as $\partial_{s,e}\Pi(s, e) \cap \mathcal{S}^* \times \mathcal{E}^*$.

See, e.g., [47, Theorem 16 (b) and (a)]. It follows that

$$\partial_e \Pi(s,e) = \{\psi : \exists \sigma \ (\sigma,\psi) \text{ solves (A.16) to (A.20)}\}. \tag{A.23}$$

Comments:

1. The formulation (A.22) of the derivative property relies on assigning a dual variable to each primal parameter, standard or not, as is done in [47]. This means that there is an explicit price variable for each resource—and this is convenient, although it also results in the linear dependence (A.19) between the multipliers, since there is the nonstandard multiplier ψ, in addition to a standard one for each constraint.[73]
2. Similar results can in principle be achieved by using just the standard multipliers, even when there are nonstandard parameters. In the case of (A.1)–(A.5), this approach would lead to the result that $\nabla_e \Pi = \nabla_e \mathcal{L} = -G^* \beta - J^* \gamma$, if the gradient exists. This formula for $\nabla_e \Pi$ is equivalent to (A.23), by (A.19). The first equality of $\nabla_e \Pi$ to $\nabla_e \mathcal{L}$ (which is evaluated at the primal optimum and the supporting multiplier) is known as the General Envelope Theorem. In smooth calculus it can be proved, together with the existence of $\nabla_e \Pi$, from the Implicit Function Theorem: see, e.g., [1, (10.8)] for the case of finite-dimensional spaces.[74] For an extension to general Banach spaces see, e.g., [37, 7.2: p. 298: first equality in last line].
3. However, smooth analysis relies on assumptions that fail in our applications. A basic obstacle is that no constraint qualification whatsoever can ensure differentiability of the hydro SR profit function $\Pi^{\mathrm{H}}_{\mathrm{SR}}$ in (k,e), or of $\Pi^{\mathrm{PS}}_{\mathrm{SR}}$ in k for the case of pumped storage in [33]. As is shown by the positive result and the counterexample (Theorem 4.9 and Example 3.1), differentiability of Π_{SR} depends on the continuity over time of the market price function p, which of course does not even appear in the (primal) constraints. In such a case, the existence of the value function's gradient $\nabla \Pi$ can be established by showing that the set $\partial \Pi$ is actually a singleton; and this can be achieved by analysing the Kuhn-Tucker Conditions and using the equality of $\partial \Pi$ and the dual solution set.
4. For a framework which uses only the standard multipliers but deals with a possibly nondifferentiable, convex value function, see, e.g., [18, Theorem 17]; this gives the directional derivatives. But Rockafellar's framework [47, Examples 4, 4', 4"] is preferable for convexly parameterised convex problems.

A.6. **Dual marginal values.** It is also of interest to spell out the derivative property of the dual optimal value. The parameterised dual minimand, with $v \in \mathcal{Y}^*$ as the dual parameter paired with the primal variable y, is

$$\sup_y \left(\mathcal{L}(y;\sigma,\psi) + \langle v, y\rangle\right) = V(\sigma,\psi) + \sup_y \left(R(y;\sigma,\psi) + \langle v, y\rangle\right) \tag{A.24}$$

[73] For the finite-dimensional case this means that there are more multipliers than constraints.

[74] This is also outlined in [52, 1.F.b], but without a proof of differentiability.

when $(\alpha, \gamma) \geq 0$ and (A.10) holds (otherwise $\mathcal{L} = +\infty$): see, e.g., [47, (4.15)]. The dual value is denoted here by $\Pi' (v; s, e)$. Since it is convex in v, its generalised derivative is the subdifferential $\partial_v \Pi'$. The equality of the primal and dual optimal values means that $\Pi (s, e) = \Pi' (0; s, e)$; and this holds under Slater's Condition (A.21) on the primal.

In the case of $M (y) = \langle p, y \rangle$ and a linear A, the supremum in (A.24) is either 0 or $+\infty$; and it is zero if and only if $\nabla_y R = v$. So the parameterised dual LP consists of (A.16)–(A.19) and the constraint

$$A^* \alpha + F^* \beta + H^* \gamma = p + v,$$

which is (A.20) with $p + v$ instead of p. Therefore $\Pi' (p, v; s, e)$ depends on the arguments p and v only through $p+v$. Since $\Pi' (p, 0) = \Pi (p)$, with s and e suppressed from the notation, it follows that at $v = 0$ (and any $p \in \mathcal{Y}^*$)

$$\begin{aligned} \partial_p \Pi (p) \cap \mathcal{Y} &= \partial_p \Pi' (p, 0) \cap \mathcal{Y} = \partial_v \Pi' (p, 0) \cap \mathcal{Y} \\ &= \{y : y \text{ solves (A.1)} - \text{(A.5) with } M (y) = \langle p, y \rangle\} . \end{aligned} \tag{A.25}$$

See [47, Theorem 16']. The dual value's derivative property is thus identified as Hotelling's Lemma, when (A.1)–(A.5) is a profit maximisation problem.

Comments:

1. Although p and v are both paired with y and have the same incremental effect on Π', their roles in the duality framework are formally different: whilst v is a dual parameter, p is a primal datum (defining the maximand M), but it is not a parameter.
2. $\partial_p \Pi (p) \subseteq \mathcal{Y}_+^{**}$ if the constraint (A.3) implies that $y \geq 0$. This is because Π is then nondecreasing in p; so any element of $\partial_p \Pi$ is a nonnegative—and hence norm-continuous—linear functional on the Banach lattice $\mathcal{Y}^*$. By the same argument, $\partial_p \Pi_{|\mathcal{P}} (p) \subseteq \mathcal{Y}$ if $\mathcal{Y}$ has a Banach predual $\mathcal{P}$, to which Π can then be restricted as a function of p (as in the hydro problem, where $\mathcal{Y} = L^\infty = L^{1*}$).
3. If (A.3) imposes also an upper bound on y, then Π is norm-continuous in $p \in \mathcal{Y}^*$. (Since Π is finite, $\partial \Pi (p) \neq \emptyset$: see, e.g., [20, 6D]. Furthermore $\partial \Pi (p) \subseteq \mathcal{Y}^{**}$, since Π is monotone. So Π is $\mathrm{w} (\mathcal{Y}^*, \mathcal{Y}^{**})$-l.s.c. and hence also norm-l.s.c. on $\mathcal{Y}^*$. And a finite, l.s.c., convex function on a Banach lattice is actually norm-continuous: see, e.g., [20, Exercise 3.50].

References

[1] Afriat, S. N. (1971): "Theory of maxima and the method of Lagrange", *SIAM Journal on Applied Mathematics*, 20, 343–357.

[2] Anderson, D. (1976): "Models for determining least-cost investments in electricity supply", *Bell Journal of Economics*, 7, 267–299.

[3] Anderson, E. J., and P. Nash (1987): *Linear programming in infinite-dimensional spaces*. New York-Chichester-Brisbane-Toronto-Singapore: Wiley.

[4] Aubin, J. P., and I. Ekeland (1984): *Applied nonlinear analysis*. New York: Wiley.

[5] Bauer, W., H. Gfrerer, and H. Wacker (1984): "Optimization strategies for hydro energy storage plants", *Zeitschrift fur Operations Research*, Series B, 28, 103–131.

[6] Berrie, T. (1967): "The economics of system planning in bulk electricity supply", *Electrical Review*, 181. (Also in: *Public enterprise*, ed. by R. Turvey, pp. 173–211. Harmondsworth-Baltimore, MD-Ringwood, Victoria: Penguin Books, 1968.)

[7] Bewley, T. (1972): "Existence of equilibria in economies with infinitely many commodities", *Journal of Economic Theory*, 4, 514–540.

[8] Birkhoff, G. (1967): *Lattice theory*. Providence, RI: American Mathematical Society.

[9] Chong, K. M. (1974): "Some extensions of a theorem of Hardy, Littlewood and Polya and their applications", *Canadian Journal of Mathematics*, 26, 1321–1346.

[10] Day, P. W. (1973): "Decreasing rearrangements and doubly stochastic operators", *Transactions of the American Mathematical Society*, 178, 383–392.

[11] Dellacherie, C., and P. A. Meyer (1978): *Probabilities and potential*. Amsterdam-New York-Oxford: North-Holland.

[12] Dunford, N., and J. T. Schwartz (1958): *Linear operators, Part I: General theory*. New York: Interscience.

[13] El-Hawary, M. E., and G. S. Christensen (1979): *Optimal economic operation of electric power systems*. New York-San Francisco-London: Academic Press.

[14] Fishburn, P. C., and R. G. Vickson (1980): "Theoretical foundations of stochastic dominance", in *Stochastic dominance: an approach to decision-making under risk*, G. A. Whitmore and M. C. Findlay (eds), 37–113. Lexington, MA-Toronto: Heath & Co.

[15] Foran, J. (1991): *Fundamentals of real analysis*. New York-Basel-Hong Kong: Dekker.

[16] Gemignani, M. C. (1972): *Elementary topology*. Reading, MA: Addison-Wesley.

[17] Gfrerer, H. (1984): "Optimization of hydro energy storage plants by variational methods", *Zeitschrift fur Operations Research*, Series B, 28, 87–101.

[18] Golstein, E. G. (1972): *Theory of convex programming*. (Translations of Math. Monographs vol. 36.) Providence, RI: AMS.

[19] Hardy, G. H., J. E. Littlewood, and G. Polya (1959): *Inequalities*. Cambridge: Cambridge University Press.

[20] Holmes, R. B. (1975): *Geometric functional analysis and its applications*. Berlin–Heidelberg-New York: Springer-Verlag.

[21] Horsley, A. (1982): "Electricity pricing for large supply systems", STICERD Discussion Paper TE/82/48, LSE.

[22] Horsley, A., and A. J. Wrobel (1986): "The Mackey continuity of the monotone rearrangement", *Proceedings of the American Mathematical Society*, 97, 626–628.

[23] Horsley, A., and A. J. Wrobel (1987): "The extreme points of some convex sets in the theory of majorization", *Indagationes Mathematicae* (Nederl. Akad. Wetensch. Proc. Ser. A), 90, 171–176.

[24] Horsley, A., and A. J. Wrobel (1988): "Subdifferentials of convex symmetric functions: An application of the inequality of Hardy, Littlewood and Polya", *Journal of Mathematical Analysis and Applications*, 135, 462–475.
[25] Horsley, A., and A. J. Wrobel (1989): "The existence of an equilibrium density for marginal cost prices, and a solution to the shifting-peak problem", STICERD Discussion Paper TE/89/186, LSE; CentER Discussion Paper 9012, Tilburg University.
[26] Horsley, A., and A. J. Wrobel (1990): "The continuity of the equilibrium price density: The case of symmetric joint costs, and a solution to the shifting-pattern problem", CentER Discussion Paper 9014, Tilburg University. (Presented at the World Congress of the Econometric Society, Barcelona, 1990. Revised in 1997 to include storage technologies.)
[27] Horsley, A., and A. J. Wrobel (1991): "The closedness of the free-disposal hull of a production set", *Economic Theory*, 1, 386–391.
[28] Horsley, A. and A. J. Wrobel (1991): "Marginal analysis and Bewley equilibria: the use of subgradients", STICERD Discussion Paper TE/91/231, LSE.
[29] Horsley, A., and A. J. Wrobel (1991): "Price singularities: representation vs. exclusion", STICERD Discussion Paper TE/91/232, LSE.
[30] Horsley, A., and A. J. Wrobel (1992): "Continuity of demand and the direct approach to equilibrium existence for Banach commodity spaces", STICERD Discussion Paper TE/92/246, LSE.
[31] Horsley, A., and A. J. Wrobel (1993): "Uninterruptible consumption, concentrated charges, and equilibrium in the commodity space of continuous functions", STICERD Discussion Paper TE/96/300, LSE (presented at ESEM93, Uppsala).
[32] Horsley, A., and A. J. Wrobel (1996): "The Wong-Viner Envelope Theorem for nonsmooth joint costs, rental valuation and the short-run approach to long-run equilibrium", STICERD Discussion Paper TE/96/299, LSE.
[33] Horsley, A., and A. J. Wrobel (1996): "Efficiency rents of storage plants in peak-load pricing, I: pumped storage", STICERD Discussion Paper TE/96/301, LSE.
[34] Horsley, A., T. Van Zandt, and A. J. Wrobel (1998): "Berge's Maximum Theorem with two topologies on the action set", *Economics Letters*, 61, 285–291.
[35] Horsley, A., and A. J. Wrobel (1999): "Efficiency rents of hydroelectric storage plants in continuous-time peak-load pricing", in *The Current State of Economic Science* (Volume 1, pp. 453–480), ed. by S. B. Dahiya. Spellbound Publications: Rohtak.
[36] Horsley, A., and A. J. Wrobel (1999 or 2000): "Localisation of continuity to bounded sets for nonmetrisable vector topologies and its applications to economic equilibrium theory", to appear in *Indagationes Mathematicae* (Nederl. Akad. Wetensch. Proc. Ser. A).
[37] Ioffe, A. D., and V. M. Tihomirov (1979): *Theory of extremal problems*. Amsterdam-New York-Oxford: North-Holland.
[38] Jacoby, H. P. (1967): "Analysis of investments in electric power", Economic Development Series, Center for International Affairs, Harvard University.
[39] Koopmans, T. C. (1957): "Water storage policy in a simplified hydroelectric system", in: *Proceedings of the First International Conference on Operational Research*, pp. 193–227. London-Baltimore, 1957. (Cowles Foundation Paper No. 115. Also in: *The collected papers of T. C. Koopmans*, pp. 282–316.)
[40] Littlechild, S. C. (1970): "Marginal cost pricing with joint costs", *Economic Journal*, 80, 323–335.
[41] Marshall, A. W., and I. Olkin (1979): *Inequalities: theory of majorization and its applications*. New York-London-Toronto-Sydney-San Francisco: Academic Press.
[42] Munasinghe, M., and J. J. Warford (1982): *Electricity pricing: theory and case studies*. Baltimore-London: The Johns Hopkins University Press (for the World Bank).

[43] Phu, H. X. (1984): "Zur Stetigkeit der Losung der adjungierten Gleichung bei Aufgaben der optimalen Steuerung mit Zustandbeschrankungen", *Zeitschrift fur Analysis und ihre Anwendungen*, 3, 527–539.

[44] Phu, H. X. (1987): "On the optimal control of a hydroelectric power plant", *Systems and Control Letters*, 8, 281–288.

[45] Rockafellar, R. T. (1968): "A general correspondence between dual minimax problems and convex programs", *Pacific Journal of Mathematics*, 25, 597–611.

[46] Rockafellar, R.T. (1970): *Convex analysis*. Princeton, NJ: Princeton University Press.

[47] Rockafellar, R.T. (1974): *Conjugate duality and optimization*. Philadelphia, PA: SIAM.

[48] Rudin, W. (1973): *Functional analysis*. New York: McGraw-Hill.

[49] Rudin, W. (1974): *Real and complex analysis*. New York: McGraw-Hill.

[50] Ryff, J. V. (1965): "Orbits of L^1-functions under doubly stochastic transformations", *Transactions of the American Mathematical Society*, 117, 92–100.

[51] Schaefer, H. H. (1971): *Topological vector spaces*. Berlin-Heidelberg-New York: Springer-Verlag (Graduate Texts in Mathematics, vol. 3).

[52] Takayama, A. (1985): *Mathematical economics*. Cambridge-London-New York: Cambrigde University Press.

[53] Tiel, J. van (1984): *Convex analysis*. Chichester-New York-Brisbane: Wiley.

[54] Tjur, T. (1972): *On the mathematical foundations of probability* (Lecture Notes 1). Institute of Mathematical Statistics, University of Copenhagen.

[55] Vardi, J., and B. Avi-Itzhak (1981): *Electric energy generation: economics, reliability and rates*. Cambridge, MA-London: The MIT Press.

[56] Weir, A. (1973): *Lebesgue integration and measure*. Cambridge: Cambridge University Press.

[57] Yosida, K., and E. Hewitt (1952): "Finitely additive measures", *Transactions of the American Mathematical Society*, 72, 46–66.

(Anthony Horsley and Andrew J. Wrobel) Department of Economics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom

E-mail address: `LSEecon123@msn.com`