The Wong-Viner Envelope Theorem for subdifferentiable functions

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Abstract

The Wong-Viner Envelope Theorem on the equality of long-run and short-run marginal costs (LRMC and SRMC) is reformulated for convex but generally nondifferentiable cost functions. The marginal cost can be formalized as the multi-valued subdifferential a.k.a. the subgradient set but, in itself, this is insufficient to extend the result effectively, i.e., to identify suitable SRMCs as LRMCs. This goal is achieved by equating the profit-imputed values of the fixed inputs to their prices. Thus reformulated, the theorem is proved from a lemma on the sections of the joint subdifferential of a bivariate convex function. The new technique is linked to the Partial Inversion Rule of convex calculus.

Keywords: Wong-Viner Envelope Theorem, nondifferentiable joint costs, profit-imputed valuation of fixed inputs, general equilibrium, public utility pricing.

JEL Nos.: C61, D21, D41.

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1 Introduction

The original Wong-Viner Envelope Theorem on the equality of long-run and short-run marginal costs (LRMC and SRMC) assumes differentiability of the short-run cost (SRC). This assumption turns on the possibility of input substitution, which ensures that, at the optimum, an extra unit of output can be produced as cheaply by increasing the quantity of a variable input as it could be by increasing an input that is actually fixed in the short run. But in reality the fixed inputs are likely to be near-perfect complements of the variable inputs: indeed, a fixed input is usually a productive capacity that can possibly be replaced by another capacity but not by any variable input.[1] An elementary but instructive example of Boiteux's [2, 1.2.2] is a fixed-coefficients technology that produces a quantity $y = \min\{k, v/T\}$ of a homogeneous output good from a quantity k of a fixed input with a price r and a quantity v of a variable input with a price w (where T is a given coefficient). The SRC is $C_{\mathrm{SR}}(y,k,w) = wyT$ for $y \leq k$ (with $C_{\mathrm{SR}} = +\infty$ for $y > k$), and the LRC is $C_{\mathrm{LR}}(y,r,w) = (r + wT)\,y$. The SRMC is wT if $y < k$, but at $y = k$ it is formally $+\infty$ for an output increase (since no extra output can be produced by increasing the variable input without increasing the capacity). The SRC is convex but nondifferentiable and, at $y = k$, its subdifferential $\partial_y C_{\mathrm{SR}}$ is the half-line $[wT, +\infty)$. It contains $r + wT$, which is the LRMC. The example extends to a simple peak-load pricing problem, in which the output y is not a constant but a periodic function of time (with a period T); first noted in [2, 3.3], this is sketched at the end of Section 3 and detailed in [6, Section 2]. Both the LRC and the SRC are then nondifferentiable, but the inclusion between the LRMC and SRMC sets remains true, as it does for any convex technology: $\partial_y C_{\mathrm{LR}}(y,r) \subseteq \partial_y C_{\mathrm{SR}}(y,k)$ when $r \in -\partial_k C_{\mathrm{SR}}(y,k)$, i.e., when the fixed-input bundle k minimizes the total cost of an output bundle y, given the input prices r and w (the latter is suppressed from the notation). For differentiable costs, this reduces to the Wong-Viner equality of gradient vectors, $\nabla_y C_{\mathrm{LR}} = \nabla_y C_{\mathrm{SR}}$.

But for nondifferentiable costs, the inclusion is generally strict ($\partial_y C_{\mathrm{LR}} \subsetneq \partial_y C_{\mathrm{SR}}$), and it shows merely that each LRMC is an SRMC—which is the *reverse* of what is required for the short-run approach to LRMC pricing. What is needed is a result that identifies a suitable SRMC as an LRMC. This is achieved by bringing in the short-run profit (SRP) function Π_{SR}, and by requiring that the given prices for the capital inputs are equal to their profit-imputed values, i.e., that $r = \nabla_k \Pi_{\mathrm{SR}}(p,k)$ or, should the gradient not exist, that $r \in \widehat{\partial}_k \Pi_{\mathrm{SR}}$ (which is the superdifferential of Π_{SR} as a concave function of k). In Boiteux's peak-load pricing problem, this condition simplifies to his "relation between prices and costs" [2, 3.3, p. 76], which is that $r = \int_0^T (p(t) - w)\,\mathrm{d}t$ (with continuous time). In this example and in general, the new condition is—as it must

[1] When the same output good can be produced from a number of plant types, "substitution between the fixed and the variable inputs" is feasible, but only in expenditure terms: a more expensive type of capacity with a lower unit operating cost can be replaced by a cheaper type with a higher operating cost, but the total capacity required for a particular output remains the same.

be—stronger than cost-optimality of the fixed inputs (when the output price system is an SRMC), i.e., if $p \in \partial_y C_{\mathrm{SR}}(y,k)$ then $\widehat{\partial}_k \Pi_{\mathrm{SR}}(p,k) \subseteq -\partial_k C_{\mathrm{SR}}(y,k)$, and the inclusion is generally strict (indeed, $\nabla_k \Pi_{\mathrm{SR}}$ can exist even when $\nabla_k C_{\mathrm{SR}}$ does not, in which case $\nabla_k \Pi_{\mathrm{SR}} \in -\partial_k C_{\mathrm{SR}}$). But this condition ($r \in \widehat{\partial}_k \Pi_{\mathrm{SR}}$) is no stronger than it need be: it is just strong enough to guarantee that if $p \in \partial_y C_{\mathrm{SR}}(y,k)$ then $p \in \partial_y C_{\mathrm{LR}}(y,r)$. This is the Extended Wong-Viner Theorem (Theorem 6). It derives from what we call the Subdifferential Sections Lemma (SSL, i.e., Lemma 2), which gives the joint subdifferential of a bivariate convex function ($\partial_{y,k} C$) in terms of one of *its* partial subdifferentials ($\partial_y C$) and a partial superdifferential, $\widehat{\partial}_k \Pi(p,k)$, of the relevant partial conjugate (which is a saddle function). The SSL is applied twice, to either Π_{SR} or C_{LR} as a saddle function obtained by partial conjugacy from C_{SR} (a jointly convex function of y and k).

So far as we know, the SSL itself is a novelty but, as we show, it can be regarded as a direct precursor of a fundamental principle of convex calculus, viz., the Partial Inversion Rule (PIR, i.e., Lemma 5), which relates the partial sub/super-differentials of a saddle function ($\partial_p \Pi$ and $\widehat{\partial}_k \Pi$) to the joint subdifferential of its bivariate convex "parent" function ($\partial_{y,k} C$). Its applications include the equivalence of the parametric version of Fermat's Rule and the Kuhn-Tucker Saddle-point Condition (see, e.g., [9, 11.39 (d) and 11.50]) and the equivalence of Hamiltonian and Lagrangian systems in variational calculus (see, e.g., [1, 4.8.2] or [8, (10.38) and (10.40)]). Put in general terms, our own use of the SSL relates the marginal optimal values of a programme to those of a subprogramme: in the specific context of extending the Wong-Viner Theorem, SRC minimization is a subprogramme both of SRP maximization and of LRC minimization. The nearly equivalent PIR can serve the same purpose, and this is a new use for what is, in Rockafellar's words, "a striking relationship...at the heart of programming theory" [7, p. 604].

2 Subdifferential Sections Lemma and Partial Inversion Rule

A multi-variate function can be maximized in two (or more) stages: first over a subset of the variables (keeping the rest fixed), then over the other variables (the maximand now being the value function from the first stage); the maximum point can be put together by back substitution. This can be applied to conjugate a bivariate convex function C, i.e., to maximize $\langle p \mid y \rangle - \langle r \mid k \rangle - C(y,k)$ over the two vector variables, y and k. The first-stage optimal-value function is then the sum of $-\langle r \mid k \rangle$ and the partial convex conjugate

$$\Pi(p,k) := \sup_y \left(\langle p \mid y \rangle - C(y,k) \right). \tag{1}$$

This is a saddle (convex-concave) function of p and k: it is convex (like C) in the "conjugated" first variable, but (unlike C) it is concave in the non-conjugated second variable. In terms of the generalized, multi-valued derivatives, viz., the subdifferential ∂ of a convex function and the superdifferential $\widehat{\partial}$ of a concave one, two-stage maximization shows in this case that $(p, -r) \in \partial_{y,k} C_{\mathrm{SR}}(y, k)$ if and only if $p \in \partial_y C_{\mathrm{SR}}(y, k)$ and $r \in \widehat{\partial}_k \Pi_{\mathrm{SR}}(p, k)$. Thus the joint subdifferential of the bivariate convex function is "sliced" along the "p-axis" and, as is next stated formally, the section of the set $\partial C(y, k)$ through any $p \in \partial_y C(y, k)$ is found to be $-\widehat{\partial}_k \Pi(p, k)$.

Definition 1 (Sub/super-gradients) *Let $C\colon Y \to \mathbb{R} \cup \{\pm\infty\}$ be a convex extended-real function on a real vector space Y that is paired with another one, P, by a bilinear form $\langle \cdot \mid \cdot \rangle\colon P \times Y \to \mathbb{R}$. A subgradient of C at a $y \in Y$ is any $p \in P$ such that $C(y + \Delta y) \geq C(y) + \langle p \mid \Delta y \rangle$ for every $\Delta y \in Y$. The set of all subgradients (at y) is the subdifferential $\partial C(y)$. In other words,*

$$p \in \partial C(y) \Leftrightarrow y \text{ maximises } \langle p \mid \cdot \rangle - C. \tag{2}$$

When $\Pi\colon K \to \mathbb{R} \cup \{\pm\infty\}$ is a concave function on a space K paired with another space R, a supergradient of Π at a $k \in K$ is any $r \in R$ such that $\Pi(k + \Delta k) \leq \Pi(k) + \langle r \mid \Delta k \rangle$ for every $\Delta k \in K$. The set of all supergradients (at k) is the superdifferential $\widehat{\partial}\Pi(k)$, which equals $-\partial(-\Pi)(k)$. In other words,

$$r \in \widehat{\partial}\Pi(k) \Leftrightarrow k \text{ maximises } \Pi - \langle r \mid \cdot \rangle. \tag{3}$$

Lemma 2 (Subdifferential sections) *Assume that $C\colon Y \times K \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function*[2] *on the Cartesian product of vector spaces Y and K that are paired (by bilinear forms) with P and R. Let $\Pi\colon P \times K \to \mathbb{R} \cup \{\pm\infty\}$ be the partial convex conjugate of C, i.e., (1) holds for each $p \in P$ and $k \in K$. Then the following conditions are equivalent to each other:*

1. *$(p, -r) \in \partial C(y, k)$.*
2. *$p \in \partial_y C(y, k)$ and $r \in \widehat{\partial}_k \Pi(p, k)$.*

Also, either condition implies that both $C(y, k)$ and $\Pi(p, k)$ are finite.

Proof. This consists in giving the first-order conditions for either simultaneous or sequential maximization over y and k (by convexity, the FOCs are both necessary and sufficient). Formally, by (2), Condition 1 holds if and only if (y, k) maximizes $\langle p, -r \mid \cdot, \cdot \rangle - C$. This holds if and only if: (i) y maximizes $\langle p \mid \cdot \rangle - C(\cdot, k)$ to $\Pi(p, k)$, and (ii) k

[2] Properness of a convex C means that it takes a finite value somewhere, but does not take the value $-\infty$ anywhere. A convex function taking the value $-\infty$ is peculiar: it may take finite values only on the algebraic boundary of its effective domain $\{y : C(y) < +\infty\}$, and it has no finite value at all if it is lower semicontinuous along every straight line. See, e.g., [8, Theorem 4] or [9, 2.5].

maximizes $\Pi(p,\cdot) - \langle r \mid \cdot \rangle$. And this pair of conditions is equivalent to Condition 2, again by (2) and (3). Finally, from (i), $\Pi(p,k) = \langle p \mid y \rangle - C(y,k) < +\infty$ (since $C > -\infty$ everywhere). And, from (ii), $\Pi(p,k) = \langle r \mid k \rangle + \sup_{y,k}(\langle p, -r \mid y, k \rangle - C(y,k)) > -\infty$ (since $C < +\infty$ somewhere). So $\Pi(p,k)$ is finite (and hence so is $C(y,k)$). ■

Remark 3 *Under the assumptions of Lemma 2,*

$$\widehat{\partial}_k \Pi(p,k) \subseteq -\partial_k C(y,k) \quad \text{when } p \in \partial_y C(y,k) \tag{4}$$

i.e., when y yields the supremum defining Π in (1).

Proof. Since $\partial C(y,k) \subseteq \partial_y C(y,k) \times \partial_k C(y,k)$, the set $\partial_k C(y,k)$ contains the section of $\partial C(y,k)$ through any $p \in \partial_y C(y,k)$. And this section is $-\widehat{\partial}_k \Pi(p,k)$ by Lemma 2. ■

The inclusion (4) is typically strict: indeed, the set $\widehat{\partial}_k \Pi(p,k)$ may even be a singleton when $\partial_k C(y,k)$ is not, i.e., the ordinary gradient vector $\nabla_k \Pi$ may exist also when $\nabla_k C$ does not. In such a case, (4) becomes: $\nabla_k \Pi(p,k) \in -\partial_k C(y,k)$ if $p \in \partial_y C(y,k)$. See [3, Section 4], [4, Theorem 9] and [6] for examples in the context of peak-load pricing (with Π_{SR} and C_{SR} as Π and C); the examples rely on time-continuity of the price function, which we verify for competitive equilibrium in [5].

The subdifferential correspondences of mutual conjugates are inverse to each other. This rule can be applied to the partial subdifferential ($\partial_y C$) that is the range of the variable (p) indexing the sections of the joint subdifferential (∂C), in Lemma 2. As a result, the saddle-differential correspondence ($\partial_p \Pi \times \widehat{\partial}_k \Pi$) and the joint-subdifferential correspondence ($\partial_{y,k} C$) are shown to be partial inverses of each other: their graphs are identical up to a sign change and the transposition of that pair of variables with respect to which Π and C are mutual conjugates (p and y). Thus the well-known Partial Inversion Rule—given in, e.g., [1, 4.4.14], [7, Lemma 4] and [9, 11.48]—is derived here from the simpler Subdifferential Sections Lemma and the Inversion Rule (which is stated next).

Proposition 4 (Inversion Rule) *Assume that $C \colon Y \to \mathbb{R} \cup \{+\infty\}$ is proper convex, and the space Y is paired with P. Let $\Pi \colon P \to \mathbb{R} \cup \{+\infty\}$ be the convex conjugate of C, i.e., $\Pi(p) := \sup_y (\langle p \mid y \rangle - C(y))$ for each $p \in P$. Then, for every $y \in Y$ and $p \in P$,*

$$p \in \partial C(y) \Leftrightarrow y \in \partial \Pi(p) \text{ and } C \text{ is finite and lower semicontinuous at } y. \tag{5}$$

Either condition implies that $\Pi(p)$ is also finite.

Proof. See, e.g., [1, 4.4.4], [8, Corollary 12A] or [9, 11.3]. ■

Corollary 5 (Partial Inversion Rule) *Under the assumptions of Lemma 2, the following conditions are equivalent to each other:*

1. $(p, -r) \in \partial C(y, k)$.

2. $y \in \partial_p \Pi(p, k)$ *and* $r \in \widehat{\partial}_k \Pi(p, k)$, *and* $C(\cdot, k)$ *is finite and lower semicontinuous at* y *(for any locally convex topology on* Y *that makes* P *the continuous dual space).*

Also, either condition implies that both $C(y, k)$ *and* $\Pi(p, k)$ *are finite.*

Proof. This follows from Lemma 2 and Proposition 4 with $C(\cdot, k)$ in place of C. ■

Comment: There is a structural difference between the Subdifferential Sections Lemma and the Partial Inversion Rule. The SSL turns the condition $(p, -r) \in \partial_{y,k} C$ into a pair of conditions, $p \in \partial_y C$ and $r \in \widehat{\partial}_k \Pi$, that involve two functions but use partial subdifferentials with respect to the *same* variables as in the joint subdifferential. The PIR turns the condition $(p, -r) \in \partial_{y,k} C$ into the pair of conditions $y \in \partial_p \Pi$ and $r \in \widehat{\partial}_k \Pi$. These use a single function Π, but only one of its arguments (k) is the same as in the original function C: the other argument (y) is replaced by its dual (p) in inverting $\partial_y C$ into $\partial_p \Pi$. This step requires the semicontinuity of C with respect to y—and this is why the PIR is not purely algebraic like the SSL.

3 Extended Wong-Viner Envelope Theorem

With the variable-input prices w kept fixed and suppressed from the notation, the long-run cost C_{LR} is a function of the output bundle $y \in Y$ and the fixed-input prices $r \in R$, and the short-run profit Π_{SR} is a function of the output prices $p \in P$ and the fixed-input bundle $k \in K$ (where Y and K are the commodity spaces for outputs and the fixed inputs, and are paired with price spaces P and R). By definition, both C_{LR} and Π_{SR} are partial conjugates of the short-run cost function C_{SR}: more precisely, Π_{SR} is, as a function of p, the convex conjugate of C_{SR} as a function of y (with k and w fixed), and C_{LR} is, as a function of r, the concave conjugate of $-C_{\mathrm{SR}}$ as a function of k (with y and w fixed), i.e.,

$$C_{\mathrm{LR}}(y, r, w) = \inf_k \{\langle r \mid k \rangle + C_{\mathrm{SR}}(y, k, w)\} \tag{6}$$

$$\Pi_{\mathrm{SR}}(p, k, w) = \sup_y \{\langle p \mid y \rangle - C_{\mathrm{SR}}(y, k, w)\}. \tag{7}$$

The SSL can be applied to each of these relationships, with the following result.

Theorem 6 (Extended Wong-Viner Theorem) *Assume that* $C_{\mathrm{SR}}\colon Y \times K \to \mathbb{R} \cup \{+\infty\}$ *is a proper convex function, which defines* C_{LR} *and* Π_{SR} *by (6) and (7). Then the following conditions are equivalent to one another:*

1. $p \in \partial_y C_{\mathrm{SR}}(y, k)$ *and* $r \in \widehat{\partial}_k \Pi_{\mathrm{SR}}(p, k)$.

2. $(p, -r) \in \partial C_{\mathrm{SR}}(y, k)$.

3. $p \in \partial_y C_{\mathrm{LR}}(y, r)$ *and* $r \in -\partial_k C_{\mathrm{SR}}(y, k)$.

Comment (comparison with the differentiable case): The usual Wong-Viner Envelope Theorem for differentiable costs is:

$$r = -\nabla_k C_{\mathrm{SR}}(y, k), \text{ i.e., } k \text{ yields the } \inf \text{ in (6)} \Rightarrow \nabla_y C_{\mathrm{SR}}(y, k) = \nabla_y C_{\mathrm{LR}}(y, r). \quad (8)$$

The equivalence of Conditions 1 and 3 in Theorem 6 extends this result because, by Remark 3,

$$\widehat{\partial}_k \Pi_{\mathrm{SR}}(p, k) \subseteq -\partial_k C_{\mathrm{SR}}(y, k) \quad \text{when } p \in \partial_y C_{\mathrm{SR}}(y, k) \quad (9)$$

i.e., when y yields the supremum in (7). In the differentiable case, the inclusion (9) reduces to the equality $\nabla_k \Pi_{\mathrm{SR}} = -\nabla_k C_{\mathrm{SR}}$ (when $p = \nabla_y C_{\mathrm{SR}}$), and thus the equivalence of Conditions 1 and 3 reduces to (8).

Comments (failure of naive extension):

1. The Wong-Viner Theorem *cannot* be extended to the general, subdifferentiable case simply by transcribing the ∇'s to ∂'s in (8) because, even when $r \in -\partial_k C_{\mathrm{SR}}(y, k)$,

$$p \in \partial_y C_{\mathrm{SR}}(y, k) \not\Rightarrow p \in \partial_y C_{\mathrm{LR}}(y, r). \quad (10)$$

   It is the *reverse* inclusion that always holds,[3] i.e.,

$$\text{if } r \in -\partial_k C_{\mathrm{SR}}(y, k) \text{ then } \partial_y C_{\mathrm{LR}}(y, r) \subseteq \partial_y C_{\mathrm{SR}}(y, k) \quad (11)$$

   but the inclusion is generally strict, i.e., $\partial_y C_{\mathrm{LR}} \neq \partial_y C_{\mathrm{SR}}$—and thus it fails to attain the goal of identifying an SRMC as an LRMC.

2. Our extension (Theorem 6) succeeds because it strengthens the insufficient condition $r \in -\partial_k C_{\mathrm{SR}}$ in (10) to $r \in \widehat{\partial}_k \Pi_{\mathrm{SR}}$ (which is stronger because the inclusion in (9) is usually strict, when C_{SR} is nondifferentiable).

3. This can be illustrated in the context of pricing, over the demand cycle, the services of a homogeneous productive capacity with a unit capital cost r and a unit running cost w. The technology can be interpreted as, e.g., electricity generation from a single type of thermal station with a fuel cost w (in \$/kWh) and a capacity cost r (in \$/kW per period). The cycle is represented here by a continuous time interval $[0, T]$, but the same arguments apply with discrete time. The long-run cost is $C_{\mathrm{LR}}(y, r)$ $= w \int_0^T y(t)\, \mathrm{d}t + r \sup_{t \in [0,T]} y(t)$. The short-run cost is $C_{\mathrm{SR}}(y, k) = w \int_0^T y(t)\, \mathrm{d}t$ if

[3] The inclusion (11) follows from (6) by Remark 3 (applied to the saddle function C_{LR} as a partial conjugate of C_{SR}).

$0 \leq y \leq k$ (with $C_{\mathrm{SR}} = +\infty$ otherwise) for $y \in L^{\infty}[0,T]$, the space of all essentially bounded functions (paired with the space of integrable functions $L^1[0,T]$). The short-run cost is nondifferentiable whenever $\sup_t y(t) = k$ (i.e., whenever there is no spare capacity), and then the condition $r \in -\partial_k C_{\mathrm{SR}}(y,k)$ says nothing about r (except that $r \geq 0$)—so it obviously cannot ensure that an SRMC price system is an LRMC. By contrast, the condition $r = \partial \Pi_{\mathrm{SR}} / \partial k = \int_0^T (p(t) - w)^+ \, \mathrm{d}t$ does specify r, and thus it is much stronger (if $p \in \partial_y C_{\mathrm{SR}}(y,k)$, i.e., if: $y(t) = k$ when $p(t) > w$, $0 \leq y(t) \leq k$ when $p(t) = w$ and $y(t) = 0$ when $p(t) < w$). It is strong enough, by Theorem 6, to ensure that if $p \in \partial_y C_{\mathrm{SR}}(y,k)$ then $p \in \partial_y C_{\mathrm{LR}}(y,r)$. For this example, one can also check this by calculating both subdifferentials directly.

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