Sales and Collusion in a Market with Storage*

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Abstract

Sales are a widespread and well-known phenomenon that has been documented in several product markets. Regularities in such periodic price reductions appear to suggest that the phenomenon cannot be entirely attributed to random variations in supply, demand, or the aggregate price level. Certain sales are traditional and so well publicized that it is difficult to justify them as devices to separate informed from uninformed consumers. This paper presents a model in which sellers want to reduce prices periodically in order to improve their ability to collude over time. In particular, the study shows that if buyers have heterogeneous storage technologies, periodic sales may facilitate collusion by magnifying intertemporal linking in consumers' decisions. The stability and the profitability of different sale strategies is then explored. The optimal sales discount and timing of sales are characterized. A trade-off between cartel size and aggregate profits arises.

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JEL classification: L11, L12, L13, L41.

1 Introduction

The occurrence of periodic price reductions, or sales, on a variety of items is a pervasive and well-known microeconomic phenomenon that has been documented in several product markets. Typically, a high prices are charged in most periods, but occasionally prices are cut to supply more units to a potentially larger group of consumers. The regular occurrence of such phenomenon appears to suggest that sales cannot be entirely explained by random variations in supply, demand, or the aggregate price level. Moreover, certain sale periods are traditional and so well publicized that it is difficult to justify them as devices to separate informed from uninformed consumers. A growing empirical literature also, appears to suggest that the majority of periodic sales take place for products that are fairly storable, and that storage capacity explains in part the responsiveness of consumers to changes in prices (Bell and Hilber 2006, Hendel and Nevo 2006 & 2010, Erdem et al 2003, Seiler 2010). Such evidence highlights the primary role that storage constraints may play in determining consumers' purchasing behavior and thus, retailers' pricing decisions.

The present paper studies how sale strategies may foster collusion in a market in which goods can be stored. Thus the aim of the study is to provide an additional motive for firms to engage in sale strategies and to shed light on the optimal sales discount and timing. While factors such as informational differences and heterogeneity in willingness to search or pay have received notable attention, the theoretical literature on storage constraints remains scarce despite empirical relevance documented by a growing literature. Notable exceptions are Salop and Stiglitz 1982, Hong, McAfee and Nayyar, 2002, and Dudine, Hendel and Lizzeri 2006.

We consider an industry in which in every period, n firms produce a homogeneous storable good, and sell it to a mass of heterogeneous consumers with unit demand. Consumers differ only by their access to storage. In particular, we restrict attention to economies with two types of consumers: those without storage capacity, and those with storage capacity S.[1] In this context, we examine the effects of heterogeneous storage technologies on firms' incentives to hold periodic sales to support a greater degree of cooperation among firms in a repeated competition setting. Sale strategies will be characterized by a regular price, by a price mark-

[1] The paper assumes that consumers are small and unable to establish reputations. See Nolke and White 2007 for the case of strategic buyers with interdependent demands.

down (on the regular price) and by the frequency of sales. We show that periodic sale strategies sustained by grim trigger punishments allow firms to collude on significant profit levels even when standard no-sale strategies cannot sustain collusion at such profit levels. In such an environment sale strategies may strengthen collusion, as the storage technology intertemporally links consumer demand and thus, reduces the short-run gains from a deviation. In particular, in any such equilibrium firms will charge, in any period of sales, a big enough discount to induce all consumers with open storage capacity to stockpile a quantity sufficient to satisfy all their demand until the next sale. Such behavior reduces incentives to deviate in regular price periods, as only consumers without storage purchase units in such periods. Moreover, the incentives to deviate also, decline in periods with sales both because a lower price is charged in such periods, and because consumers with storage reduce their demand when a deviation is observed in the wake of the imminent price war (implied by the grim trigger punishments).

Often we will refer to the incentives not to deviate from an equilibrium strategy as the stability of a strategy. The first part of the analysis: characterizes consumer demand for the proposed environment; characterizes the set $\mathcal{E}$ of sale strategies which are more stable than any strategy without sales; and provides necessary and sufficient conditions for the non-emptiness of such a set. The second part of the analysis restricts attention to the case of single-unit storage and characterizes the most stable and the most profitable sale strategies in the set $\mathcal{E}$. Within such set a trade-off is proven to emerge between collusive profits and stability (which is measured by the maximal number of firms that can collude on given profits). Such trade-off is explicitly characterized. The second part of the analysis concludes by studying how such trade-off is affected by changes in the environment. In particular, increases in patience (i.e. the frequency of interaction) will lead both to larger cartels at any profit level and to the persistence of the profit-stability trade-off for a larger range of profits. An increase in the profitability of a market instead, will increase both cartel size at intermediate profit levels and the range of profits for which the profit-stability trade-off persists. A larger fraction of consumers without storage will have ambiguous effects on cartel size at high profit levels and will reduce the maximal cartel size at low profit levels. In general, the effect of such a change will remain ambiguous, as a large fraction of consumer with storage could reduce equilibrium profits due to the cost of anticipating production. However, the range of profits for which the profit-stability trade-off exists will be proven to decline in the fraction of consumers without storage. Relationship between the optimal sale markdown and the environment will also be explored. The final part of the analysis consists of two extensions. The first extension considers the multi-unit storage scenario, characterizes the optimal timing of sales, and highlights that access to multi-unit storage technologies can reduce the profit-stability trade-off. The second extension shows that sales do not need to be synchronized when firms compete in multiple markets.

Note the in the proposed model collusion is strengthened at the expense of aggregate profits, since deviation profits decline more than equilibrium profits when sale strategies are employed. Aggregate profits must decline in the proposed setup, since all consumers are homogeneous in their willingness to pay. Note however, that if consumers with higher storage capacity had a lower willingness to pay, a sale strategy may achieve higher profits than the no-sale strategy by price discriminating among different types of consumers, and thus foster collusion even further. We have elected to keep valuations homogeneous across consumers in order to display more explicitly the effects of the intertemporal linking in consumer demand.

Literature Review: One of the first theoretical explanations for sales relates consumer search behavior to price discrimination. Two prominent examples in this literature are Varian 1980 and Salop and Stiglitz 1982. Varian 1980 argues that in the presence heterogeneously informed consumers, retail price variations can arise as a natural outcome of mixed strategy equilibrium in which firms price discriminate consumers with different information. Salop and Stiglitz 1982 instead, considers a model with search costs in which consumers are imperfectly informed about the prices charged by stores and differ in their ability to stockpile. In such framework the authors show that stores have incentives to hold unannounced sales to induce consumers to purchase future consumption. Both models however, are essentially static models and cannot account for correlation in prices. Even though the random sales feature remains a compelling explanation for some erratic price behavior, it appears less suited to account for many of the documented retail markdowns that are predictable, publicly know, and take place in most stores simultaneously (Pesendofer 2002 and Warner and Barsky 1995).

The appealing fashion/clearance paradigm for sales (Lazear 1986, Pashigian 1988, and Pashigian and Bowen 1991) can also, hardly be applied to a wide variety of retail items for which the fashion hypothesis appears a priori less appropriate (either because the items are homogeneous, or because styles change little over time).

A final relevant literature has motivated sales as a form of intertemporal price discrimination (Conlisk, Gerstner and Sobel 1984, Hendel and Nevo 2010, Hong, McAfee and Nayyar 2002, Narasimhan and Jeuland 1985, Sobel 1984). Conlisk, Gerstner and Sobel 1984, and in particular Sobel 1984 study the incentives to hold cyclical simultaneous sales as a means of price discrimination in a durable-good environment. In most periods, prices are kept high to extract surplus from high value consumers, but periodically prices are decreased in order to sell to a larger group of consumers with lower reservation values. A key assumption to generate such price cycles is the constant inflow of new heterogeneous consumers in the market. Hendel and Nevo 2010, Hong, McAfee and Nayyar 2002, Narasimhan and Jeuland 1985, study the incentives to hold periodic sales in a market with storable goods and heterogeneous consumers. In this setup, the incentives to price discriminate consumers over time with sales fully explained by the positive correlation between storage costs and consumers' willingness to pay. We complement these papers by offering a new explanation for the existence of sale strategies in a dynamic storable goods model in which the incentives to hold periodic sales arises even in the absence of such a correlation. Moreover, as in Sobel 1984, we characterize the optimal timing for sales.

The paper is also, closely related to several studies which pointed out importance of the intertemporal linking in decisions to explain collusive behavior (Ausubel and Deneckere 1987, Dana and Fong 2010, Gul 1987, Schiraldi and Nava 2010).

Roadmap: Section 2 introduces the model, defines the relevant class of sale strategies, and presents several preliminary results comparing different classes of sale strategies in terms of their stability. Section 3 restricts attention to a model with unit-storage. For that setup two relevant sale strategies are characterized. Namely, the most stable sale strategy and the most profitable sale strategy in $\mathcal{E}$. The section concludes with the full characterization of the profit-stability trade-off that different sale strategies entail. Comparative static results show how the trade-off is affected by changes in patience, in profitability and in the fraction of consumers with access to storage. Section 4 extends the baseline model in two directions. The first considers the general model with multi-unit storage and again characterizes the most stable and most profitable sale strategies in such a more general framework. It will be shown that sale strategies with infrequent sales may be increase both stability and profits, and may thus, alleviate the trade-off. The second extension shows why synchronization in sales is not necessary when firms operate in several markets. Section 5 concludes. All proofs are relegated to appendix A. Appendix B contains several useful derivations omitted from the main text for clarity.

2 A Model with Storable Goods

This section first introduces a simple economy with storage, defines the class of sale strategies that will be analyzed throughout the paper, and develops several preliminary results on stability.

The Simple Economy with Storage

Consider an infinite-horizon discrete-time model with infinitely lived produces and consumers. Suppose that two goods are traded in the model which we shall refer to as consumption q and money m. In each period, all consumers are endowed with a large amount of money M and with no consumption. The preferences of a consumer are separable over the two goods and satisfy:

$$u(q,m) = \begin{cases} v + m & \text{if} \quad q \geq 1 \\ m & \text{if} \quad q < 1 \end{cases}$$

Hence, the marginal value of consumption is v for the first unit consumed and 0 for any additional unit consumed. The budget constraint faced by each consumer in every period requires that:

$$m = M - p$$

where p denotes the amount of money spent on consumption good. All consumers discount the future at a common factor δ, and their time-preferences over utility sequences $\{u_t\}_{t=0}^{\infty}$ satisfy:

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u_t$$

There is a unit measure of consumers. Consumers differ only in their ability to store the consumption good. In particular, assume that a fraction α_0 of the consumers is unable to store goods, while a fraction α_S can store up to S additional units of consumption. Such units do not depreciate, when stored and can be consumed in any future period.

A finite set of firms, N with cardinality n, supplies consumption good to this market. All firms have a common constant marginal cost of producing consumption good, c. If $\mathbf{p} = (p_1, ..., p_n)$ are the prices set by each firm on the units of consumption sold, and if $d(\mathbf{p})$ denotes the aggregate demand at such prices, the individual demand faced by firm i satisfies:

$$d_i(\mathbf{p}) = \begin{cases} \frac{1}{\left|\arg\min_{j \in N} p_j\right|} d(\mathbf{p}) & \text{if} \quad p_i \leq \min_{j \in N} p_j \\ 0 & \text{if} \quad p_i > \min_{j \in N} p_j \end{cases}$$

The stage profit of firm $i \in N$ given a price vector $\mathbf{p}$ satisfies:

$$v_i(\mathbf{p}) = (p_i - c) d_i(\mathbf{p})$$

All firms discount the future at a common factor δ, and their time-preferences over profit sequences $\{v_t\}_{t=0}^{\infty}$ satisfy:

$$(1 - \delta) \sum_{t=0}^{\infty} \delta^t v_t$$

Sale Strategies and Equilibrium

Firms and consumers observe all the prices quoted in the market in all previous periods. Thus, the set of possible histories in the induced game can be defined as:

$$H = \{\emptyset\} \cup \{\cup_{t=1}^{\infty} \left[\times_{s=1}^{t} \mathbb{R}_{+}^{n}\right]\}$$

A firm's strategy maps histories into a price quoted at a given date. Consumers use the information about past quoted prices and the equilibrium strategy of the firms to form beliefs about future prices in the economy. Since consumers are small we shall assume that their individual decisions are unobservable to any other individual. Consumers will thus, decide how many units to purchase from firms quoting the lowest price, just so to maximize their individual payoff. In particular, consider any sequence of future prices $\mathbf{p}^t = \{\mathbf{p}_z\}_{z=t}^{\infty}$. Let

$\overline{p}_z = \min_{i \in N} \{p_{iz}\}$ denote the market price in period z, and let:

$$T(\mathbf{p}^t) = \min z \;\; \text{subject to} \;\; \overline{p}_t > \delta^z \overline{p}_{t+z}$$

denote how long a consumer has to wait before the current market price exceeds the future discounted price. The next proposition pins down consumer demand for the proposed environment.

Remark 1 *If $\overline{p}_z \leq v$ in any period $z \geq 0$, the demand for consumption good at time t:*
(1) by consumers without storage technology satisfies $d_0(\mathbf{p}^t) = 1$;
(2) by consumers with storage technology and with s units already in storage satisfies:

$$d_S(s, \mathbf{p}^t) = \max \left\{ \min \left\{ T(\mathbf{p}^t), S + 1 \right\} - s, 0 \right\}$$

This is the case since consumers with access to storage purchase multiple units only if they perceive the storage cost δ to be smaller than the cost of future price increases.[2] Since buyers cannot build reputations, all consumers of the same type purchase the same number of units in each period. Hence, the aggregate demand in a period in which all consumers with storage have the same number of units s satisfies:

$$d(s, \mathbf{p}^t) = \alpha_0 + \alpha_S d_S(s, \mathbf{p}^t)$$

The equilibrium strategies that will be analyzed throughout the paper discipline deviations as trigger-strategies would. However, equilibrium prices will vary along the equilibrium path. In particular, consider any strategy in which all firms set prices along the equilibrium path so that for some $\varkappa \in \{2, 3, ...\}$:

$$\overline{p}_t = \begin{cases} (1+\mu)c & \text{if} \;\; \mathrm{mod}(t, \varkappa) \neq 0 \\ (1+\mu\sigma)c & \text{if} \;\; \mathrm{mod}(t, \varkappa) = 0 \end{cases}$$

where $\mathrm{mod}(t, \varkappa) \neq 0$ denotes the $\varkappa$ modulo of the time period t. Such strategy may be interpreted as a cyclical sales policy in which all firms jointly reduce prices every $\varkappa$ periods, where μ denotes the markup in periods without sales, and where $\sigma \in [0, 1]$ denotes the fraction of

[2] The cost of storage coincides with the discount factor, as the rate of time preferences represents the opportunity cost of spending money sooner to stock units.

such a markup charged during sale periods. Deviations from the equilibrium path are punished via reversion to competitive pricing in each future time period. In particular, for any history $h^t \in H$ of length t, all the sale strategies that will be considered will satisfy:

$$\pi(h^t) = \begin{cases} \overline{p}_t & \text{if} \quad p_{iz} = \overline{p}_z \quad \text{for any } i \text{ and any } z \leq t \\ c & \text{if} \qquad\qquad \text{otherwise} \end{cases}$$

The equilibrium punishment strategy is Nash in any subgame in which a deviation has already occurred, since no firm can benefit from a deviation when all the other firms are pricing competitively. Thus, the incentives to comply with a sale strategy will be pinned down by looking only at deviations from the equilibrium path. Let $\Pi_t(\pi)$ denote the aggregate payoff on the equilibrium path, let $\Delta_t(\pi)$ denote the payoff of the most profitable deviation from the equilibrium path, and let the ratio of equilibrium to deviation profits, Π_t/Δ_t, be denoted by $R_t(\pi)$. The following result characterizes the upper-bound on cartel size for which the proposed sale strategy constitutes a Subgame Perfect equilibrium.

Remark 2 *A sale strategy π is a SPE of the infinite repetition of the game if and only if:*

$$n \leq \frac{1}{1-\delta} R_t(\pi) \quad \textit{for any } t \geq 0 \tag{1}$$

The upper-bound on the population size disciplines the largest number of firms that can sustain strategy π as a subgame perfect equilibrium. Note that for any strategy without sales (and thus without storage) such condition would simplify to the common requirement $n \leq \frac{1}{1-\delta}$.

Throughout, we will refer to the strategies defined in this section as sale strategies. Any one of these strategies will be completely pinned down by the three parameters: the sales discount σ, the regular markup μ, and the frequency of sales $\varkappa$. The set of possible sale strategies will be denoted by $\mathcal{S} = \left(0, \frac{v-c}{c}\right] \times [0,1] \times \{2, 3, ...\}$. The upper-bound imposed on the markup requires consumers without storage to be willing to trade in periods without sales. Such restriction is imposed, since any strategy with a higher markup could not be optimal for profits and for stability. Similar considerations show why relaxing any one of the other bounds on $\mathcal{S}$ would not affect any of results developed in what follows.

Preliminary Results on Stability & Profits

This sub-section develops several preliminary results that compare different sale strategies in $\mathcal{S}$ in terms of profits and stability. Necessary and sufficient condition for a strategy with sales to be more stable than a strategy without sales are presented. Necessary and sufficient for the existence of such sale strategies are also presented. To this end, let us begin with a two definitions clarifying the intent of our comparison.

Definition 1 *A sale strategy $\pi \in \mathcal{S}$ is said to be more stable than strategy π' if:*

$$\min_{t\geq 0} R_t(\pi) \geq \min_{t\geq 0} R_t(\pi')$$

Definition 2 *A sale strategy $\pi \in \mathcal{S}$ is said to raise more profits than strategy π' if:*

$$\Pi_0(\pi) \geq \Pi_0(\pi')$$

Note that for a fixed discount factor δ, the definition of stability compares any two strategies by looking at the largest number of firms $n(\pi)$ that can collude on strategy π:

$$n(\pi) = \frac{\min_{t\geq 0} R_t(\pi)}{1-\delta}$$

An alternative, but similar, definition of stability may involve the floor of the map $n(\pi)$. But similar conclusions would hold. Alternatively, one could focus on the the lowest possible discount factor $\delta(\pi)$ needed to collude on strategy π in a population of fixed size n. To do so a fixed point argument would have to be employed to solve for:

$$\delta(\pi) = 1 - \frac{\min_{t\geq 0} R_t(\pi|\delta(\pi))}{n}$$

We refrain from doing so and instead, focus on $n(\pi)$, since the analysis simplifies considerably while the qualitative results coincide.

In order to compare the stability of two sale strategies, the equilibrium and deviation profits must be expressed in terms of the parameters of the strategy. Recursively define the equilibrium demand d_t and storage s_t in each period $t \geq 0$ as follows:

$$\begin{aligned} d_0 &= d(0, \mathbf{p}^0) \;\; \& \;\; d_{S0} = d_S(0, \mathbf{p}^0) \;\; \& \quad s_1 = d_{S0} - 1 \\ d_t &= d(s_t, \mathbf{p}^t) \;\; \& \;\; d_{St} = d_S(s_t, \mathbf{p}^t) \;\; \& \quad s_{t+1} = s_t + d_{St} - 1 \end{aligned}$$

The next remark shows that any sale strategy is cyclical, and that it is without loss to consider only the first $\varkappa$ periods to characterize the entire stream of payoffs.

Remark 3 *If* $\mathrm{mod}(t, \varkappa) = 0$, $s_t = 0$. *If* $\mathrm{mod}(t, \varkappa) = \mathrm{mod}(z, \varkappa)$, $d_t = d_z$.

The claim follows from the properties of the equilibrium pricing path p^* and of the map $d_S(s_t, \mathbf{p}^t)$. It requires consumers not to have any units stored in periods of sale and thus, to have the same demand at congruent dates in the cycle. Thus, for:

$$S(t) = \begin{cases} 0 & \text{if} \quad \mathrm{mod}(t, \varkappa) = 0 \\ \varkappa - \mathrm{mod}(t, \varkappa) & \text{if} \quad \mathrm{mod}(t, \varkappa) \neq 0 \end{cases}$$

equilibrium payoffs must satisfy for $t = \{1, ..., \varkappa - 1\}$:

$$\begin{aligned} \Pi_t(\pi) &= \frac{(1-\delta)}{1-\delta^{\varkappa}} \sum_{z=0}^{\varkappa-1} \delta^z v_{t+z}(\mathbf{p}^{t+z}) = \\ &= \frac{(1-\delta)}{1-\delta^{\varkappa}} \left[\left[\sum_{z=0}^{\varkappa-1} \delta^z d_{t+z} \right] - (1-\sigma) \delta^{S(t)} d_0 \right] \mu c \end{aligned}$$

Since a deviation to price $y \neq p_t^*$ at stage t leads to a conjectured price path $\mathbf{y}^t = \big((y, \mathbf{p}^*_{-it}), \mathbf{c}, \mathbf{c}, \mathbf{c}, ... \big)$, for $y < p_t^*$ the deviation payoffs at each stage must satisfy:

$$\Delta_t(y, \pi) = (y - c) d(s_t, \mathbf{y}^t)$$

For convenience, let us define four classes of sale strategies for which all preliminary results will be developed:

Definition 3 *Let* $\mathcal{N} \subset \mathcal{S}$ *denote those strategies such that* $\sigma = 1$.

Let $\mathcal{V} \subset \mathcal{S}$ *denote those strategies for which* $d_t > \alpha_0$ *and* $\mathrm{mod}(t, \varkappa) \neq 0$ *for some* t.

Let $\mathcal{C} \subset \mathcal{S}$ *denote those strategies for which* $\varkappa \leq S + 1$ *and* $(1 + \mu\sigma) \leq \delta^{\varkappa - 1}(1 + \mu)$.

Let $\mathcal{E} \subset \mathcal{C}$ *denote those strategies for which* $\frac{\alpha_0}{\varkappa \alpha_S + \alpha_0} \leq \sigma$.

The set $\mathcal{N}$ consists of all those sale strategies for which no discount is ever offered along the equilibrium path. The set $\mathcal{V}$ comprises all those strategies for which there is a period along the equilibrium path with no sales and unit demand. The set $\mathcal{C}$ instead, will be proven to consists of all those strategies in which consumers with access to storage purchase only in periods with sales. The rest of the section proves that $\mathcal{E}$ comprises all the sale strategies which are more stable than a strategy without sales, and provides necessary and sufficient conditions for the non-emptiness of $\mathcal{E}$.

The next proposition presents several introductory results on the stability and the profitability of different sale strategies. In particular, it shows that $\mathcal{V}$ and $\mathcal{C}$ partition $\mathcal{S}$ and that any sale strategy in which consumers with storage purchase units during a no-sales period is dominated both in terms of profits and in terms of stability by a policy in which no sales ever take place. Such a result considerably simplifies the analysis of the profit-stability trade-off. The latter observation will be exploited to characterize the set of sale strategies that can be more stable than the revenue maximizing no-sales policy, $\mu = \frac{v-c}{c}$ and $\sigma = 1$.

Proposition 4 *The following claims must hold:*

(1) $\mathcal{S} \backslash \mathcal{C} = \mathcal{V}$

(2) $\mathcal{N} \subseteq \mathcal{V}$

(3) any strategy in $\mathcal{N}$ *is more stable than any strategy in* $\mathcal{V}$*;*

(4) any strategy that sets $\mu = \frac{v-c}{c}$ *and* $\sigma = 1$ *is profit maximizing within* $\mathcal{S}$*;*

(5) for any strategy in $\mathcal{C}$*,* $s_t = S(t)$ *and:*

$$d_t = \begin{cases} \alpha_0 + \varkappa \alpha_S & \text{if} \quad \mathrm{mod}(t, \varkappa) = 0 \\ \alpha_0 & \text{if} \quad \mathrm{mod}(t, \varkappa) \neq 0 \end{cases}$$

The proposition proves that $\mathcal{V}$ and $\mathcal{C}$ partition the set of sale strategies. Moreover, it implies that all strategies with no sales (i.e. strategies in $\mathcal{N}$) are equally stable and more stable than any other sale strategy in which consumers with storage purchase units in periods of sales (i.e. strategies in $\mathcal{V}$). This should also, clarify why sales where assumed to take place in the initial period, since any sale strategy violating such a requirement would belong to $\mathcal{V}$. The last part of the proposition considerably simplifies the expression of the equilibrium payoffs for any strategy in $\mathcal{C}$. In particular, for any $\pi \in \mathcal{C}$ we get:

$$\Pi_t(\pi) = \left[\alpha_0 + \frac{1-\delta}{1-\delta^{\varkappa}}\delta^{S(t)}\left[\sigma(x\alpha_S + \alpha_0) - \alpha_0\right]\right]\mu c$$

The lemma also, pins down the revenue maximizing deviations for such candidate strategies:

$$\Delta_t(\pi) = \max_y \Delta_t(y,\pi) = \begin{cases} \sigma\mu c & \text{if} \quad \text{mod}(t,\varkappa) = 0 \\ \alpha_0\mu c & \text{if} \quad \text{mod}(t,\varkappa) \neq 0 \end{cases}$$

and the corresponding profit ratios:

$$R_t(\pi) = \begin{cases} \frac{\alpha_0}{\sigma} + \frac{1-\delta}{1-\delta^{x}}\left[(\varkappa\alpha_S + \alpha_0) - \frac{\alpha_0}{\sigma}\right] & \text{if} \quad \text{mod}(t,\varkappa) = 0 \\ 1 + \frac{1-\delta}{1-\delta^{\varkappa}}\delta^{S(t)}\left[(\varkappa\alpha_S + \alpha_0)\frac{\sigma}{\alpha_0} - 1\right] & \text{if} \quad \text{mod}(t,\varkappa) \neq 0 \end{cases}$$

The next table displays the effects of marginal changes in the sale strategy on equilibrium and deviation profits and their ratio:

	$d\mu$	$d\sigma$	$d\varkappa$
$d\Pi_t(\pi)$	> 0	> 0	?
$d\Delta_t(\pi)$	≥ 0	≥ 0	0
$dR_t(\pi)$	0	?	?

The sign of $dR_t(\pi)/d\sigma$ is unspecified, since it is negative if $\text{mod}(t,\varkappa) = 0$ and positive otherwise. The signs of $d\Pi_1(\pi)/d\varkappa$ and $dR_1(\pi)/d\varkappa$ coincide. In particular, such derivatives are positive either if $\text{mod}(t,\varkappa) = 0$, or if $\text{mod}(t,\varkappa) \neq 0$ and the fraction of consumers with storage is sufficiently small, and negative otherwise. Appendix B contains the derivation of these derivatives and their signs.

As proven in the next proposition, whenever a strategy belongs $\mathcal{C}$ it is without loss to ignore all, but the first two periods, in order to characterize its stability. Such conclusion coupled with the observation that $\min_t R_t(\pi)$ is independent of μ and single peaked in σ for any given $\varkappa$, implies all strategies in $\mathcal{E}$ must be more stable than a strategy with no sales.

Proposition 5 *Any strategy in $\mathcal{E}$ is more stable than any strategy in $\mathcal{S}\backslash\mathcal{E}$.*

Therefore, whenever the set $\mathcal{E}$ is non-empty sale strategies exist that are more stable than strategies without sales. It is easy to observe that a strategy π in $\mathcal{E}$ will be strictly more stable than a strategy without sales if and only if:

$$\sigma \in \left(\frac{\alpha_0}{\alpha_S \varkappa + \alpha_0}, \frac{\mu + 1}{\mu} \delta^{\varkappa - 1} - \frac{1}{\mu} \right]$$

Such expression and the previous result can be exploited to derive necessary and sufficient conditions for the set $\mathcal{E}$ to be non-empty.

Corollary 6 *$\mathcal{E}$ contains a strategy with a cycle of length $\varkappa \in \{2, ..., S+1\}$ if and only if:*

$$\delta^{\varkappa - 1} \geq \frac{v - c}{v} \frac{\alpha_0}{\varkappa - \alpha_0(\varkappa - 1)} + \frac{c}{v}$$

Thus, if $\delta \geq \frac{v-c}{v} \frac{\alpha_0}{2-\alpha_0} + \frac{c}{v}$, then $\mathcal{E}$ is non-empty.

Such conditions jointly discipline the all free parameters of the model, namely: the fraction of consumers with storage α_0, the profitability of the market $v - c$, and the discount factor δ. The comparative statics results, developed below, discuss in detail how the size of $\mathcal{E}$ the depends on such free parameters. The new bound imposed on the discount rate arises from the consumers' demand. Thus, such condition would only discipline the time preferences of the consumers, if those could differ from time preferences of the firms. However, even when the two coincide, the restriction imposed on δ remains independent of the number of firms in the market.

Before proceeding to the next section, for notational convenience, define $\sigma(\varkappa)$ as the unique positive root of the following quadratic equation:

$$R_1(\sigma(\varkappa), \varkappa) = R_0(\sigma(\varkappa), \varkappa)$$

if such a solution exists in $[0, 1]$, and set $\sigma(\varkappa) = 1$ otherwise. The details of the derivation of $\sigma(\varkappa)$ and the proof of uniqueness are deferred to appendix B. Further, define $\kappa(\varkappa)$ as the smallest discount for which consumers with access to storage would purchase $\varkappa$ units in periods of sales when the regular markup is set at the monopoly level $\mu = (v - c)/c$:

$$\kappa(\varkappa) = \frac{v}{v - c} \delta^{\varkappa - 1} - \frac{c}{v - c}$$

3 Single-Unit Storage and the Profit-Stability Trade-Off

For sake of tractability this section considers environments in which consumers can store at most a single unit, $S = 1$. We do so, since most qualitative results are unaffected by this assumption. Part of the extensions section is devoted to generalizing to arbitrary storage capacities. The section compares sale strategies in $\mathcal{S}$ in terms of profits and stability, and shows that a trade-off can emerge between the two. The next two propositions characterize two strategies particular strategies in $\mathcal{E}$. The former will be the most stable sale strategy, while the latter will be the most profitable of all the sale strategies in $\mathcal{E}$. Whenever the two strategies do not coincide, a trade-off between cartel profits and stability will emerge. The section proceeds with to explicit characterization of the profit-stability trade-off and to several comparative statics.

To simplify notation in this section let $\alpha = \alpha_0$ and $\rho = c/(v-c)$. Trivially observe that $S = 1$ implies that $\varkappa = 2$ for any policy in $\mathcal{E}$. Note that given the stated assumptions and definitions, $\sigma(2)$ and $\kappa(2)$ respectively satisfy:

$$\begin{aligned} \sigma(2) &= \min\{1, \bar{\sigma}\} \\ \kappa(2) &= \delta(1+\rho) - \rho \end{aligned}$$

where $\bar{\sigma}$ is the unique positive root of the following quadratic equation:

$$\bar{\sigma}^2 (2-\alpha)\delta - \bar{\sigma}\alpha(1-\alpha) - \alpha^2\delta = 0$$

If $\mathcal{E}$ is non-empty, the most stable sale strategy is characterized by the following result.

Proposition 7 *If $\mathcal{E} \neq \emptyset$, no strategy in $\mathcal{S}$ is strictly more stable than strategy $\pi^* \in \mathcal{E}$:*

μ^*	σ^*	$\varkappa^*$
$1/\rho$	$\min\{\sigma(2), \kappa(2)\}$	2

Moreover, π^* is the most profitable of all the strategies in $\mathcal{S}$ with equal stability.

Such a strategy requires firms to set collusive markups in periods without sales and uniquely pins down the optimal discount for the remaining periods. The monopoly markup can be charged in periods without sales since μ has no effect on the stability. The optimal sales discount σ^* is chosen to minimize $\min \{R_0(\pi), R_1(\pi)\}$ within the feasible set $\sigma \in [\alpha/(2-\alpha), \delta(1+\rho)-\rho]$ and crucially depends on the fraction of consumers with storage and on the monopoly markup in the economy. The comparative statics section discusses such dependence in detail. Notice that the optimal sale strategy may depend on all the parameters of the model except for n, since both $\sigma(2)$ and $\kappa(2)$ are independent of n. Thus, the largest number of firms $n(\pi^*)$ willing to collude on any strategy in $\mathcal{S}$ can be found by looking at:

$$n(\pi^*) = \frac{R_1(\pi^*)}{1-\delta}$$

Even though such strategy is optimal in terms of stability, more profitable policies exist in $\mathcal{E}$. The next proposition characterizes, the most profitable sale strategy in $\mathcal{E}$.

Proposition 8 *If $\mathcal{E} \neq \emptyset$, no strategy in $\mathcal{E}$ is strictly more profitable than strategy $\pi^+ \in \mathcal{E}$:*

μ^+	σ^+	$\varkappa^+$
$1/\rho$	$\kappa(2)$	2

As in the previous proposition the strategy requires firms to set collusive markups in periods without sales. However, the profit maximizing sale discount is the uniquely pinned down by the consumer's storage constraint. Obviously, such a discount may be smaller than that of the most stable policy π^*, and no longer depends on the fraction of consumers with storage in the economy. Hence, the upper-bound on the number of firms needed to a collude on π^+ may be smaller than for π^*:

$$n(\pi^+) = \frac{\min \{R_1(\pi^+), R_0(\pi^+)\}}{1-\delta} \leq n(\pi^*)$$

The previous two propositions were meant to highlight the trade-off that may arise between profits and stability in such environments. The first result in fact, showed that sale strategies could be used improve stability at the expense of profits. Strategy π^* was proven to be more stable than any other strategy without sales, but less profitable than full collusion. Similarly π^* was clearly more profitable and less stable than the competitive outcome (i.e. the Nash equilibrium of the stage game). The second result instead, showed that even within $\mathcal{E}$ profit-stability trade-offs would arise whenever $\pi^* \neq \pi^+$.

Since S was fixed to 1, a more stringent characterization of the profit-stability trade-off within $\mathcal{S}$ can be derived. In particular, note that the previous propositions require that:

(1) increasing μ can only benefit profits and cannot harm the stability;

(2) increasing σ can only benefit profits;

(3) increasing σ can harm stability if and only if $\sigma \in [\sigma(2), \kappa(2)]$.

Then, for fixed values of α, ρ, and δ, consider a strategy $(1/\rho, \sigma, 2)$. Let $\bar{R}(\sigma) = \min_t R_t(1/\rho, \sigma, 2)$ denote its stability, let $\bar{\Pi}(\sigma) = \Pi_0(1/\rho, \sigma, 2)$ denote its profits and let:

$$n(\sigma) = \frac{1}{1-\delta}\bar{R}(\sigma)$$

denote the maximal number of firms that can sustain such a strategy in equilibrium. Note that $\bar{\Pi}(\sigma)$ is strictly increasing, and thus invertible, in σ. Finally, for any profit level $\Pi \in \mathbb{R}_+$ let $N(\Pi) = n(\bar{\Pi}^{-1}(\Pi))$ denote the maximal number of firms that can collude on such a profit level while employing a sale strategy. If so, note that a trade-off emerges between profits and stability for $\sigma \in [\sigma(2), \kappa(2)]$:

$$\frac{dN(\Pi)}{d\Pi} = \frac{dn/d\sigma}{d\bar{\Pi}/d\sigma}(\bar{\Pi}^{-1}(\Pi)) = \frac{1}{1-\delta}\frac{dR_0/d\sigma}{d\Pi_0/d\sigma}(1/\rho, \bar{\Pi}^{-1}(\Pi), 2) \leq 0$$

since the denominator is positive, while the numerator is negative (details in Appendix B). In particular, the last expression implies that maximal number of firms that can sustain a collusive sale strategy $(1/\rho, \sigma, 2)$ must decline as profits increase. Further notice that any sale strategy that raises more profits by setting $\sigma > \kappa(2)$ cannot be optimal in terms of stability, since a policy in $\mathcal{N}$ exists that is both more stable and more profitable, as:

$$\frac{1}{1-\delta} > n(\sigma)$$

Similarly, no policy setting $\sigma < \sigma(2)$, could ever be optimal, since it would simultaneously reduce stability and profits. In fact, by continuity a sale strategy $(\mu, \sigma(2), 2)$ could be proven to exist that is equally profitable as $(1/\rho, \sigma, 2)$, but more stable. For convenience let $\Pi^m = v - c$ denote the monopoly profit and let Π^* and Π^+ denote the profits respectively of the most stable and the most profitable strategies in $\mathcal{E}$:

$$\Pi^* = \left[\frac{\delta\alpha}{1+\delta} + \frac{2-\alpha}{1+\delta}\sigma^*\right](v-c)$$
$$\Pi^+ = \left[\frac{\delta\alpha}{1+\delta} + \frac{2-\alpha}{1+\delta}\sigma^+\right](v-c)$$

Note immediately that $\Pi^* \leq \Pi^+$. The next proposition highlights the specific nature of the profit-stability trade-off for economies in which $S = 1$.

Proposition 9 *If $\mathcal{E} \neq \emptyset$, for any profit level $\Pi \in (0, \Pi^m]$ the maximal number firms that can collude on Π while employing a sale strategy satisfies:*

$$N(\Pi) = \begin{cases} \frac{1}{1-\delta} & \textit{if} \quad \Pi \in (\Pi^+, \Pi^m] \\ \frac{1}{1-\delta}\frac{(2-\alpha)\Pi}{(1+\delta)\Pi - \delta\alpha(v-c)} & \textit{if} \quad \Pi \in (\Pi^*, \Pi^+] \\ n(\pi^*) & \textit{if} \quad \Pi \in (0, \Pi^*] \end{cases}$$

In the interval $(\Pi^*, \Pi^+]$, $N(\Pi)$ is decreasing and convex, and by construction satisfies $N(\Pi) > 1/(1-\delta)$. Thus, proposition exactly quantifies the trade-off between profits and cartel size (stability) that different sale strategies imply. Such a trade-off emerges since larger sales discounts may favor stability, but certainly hurt profits.

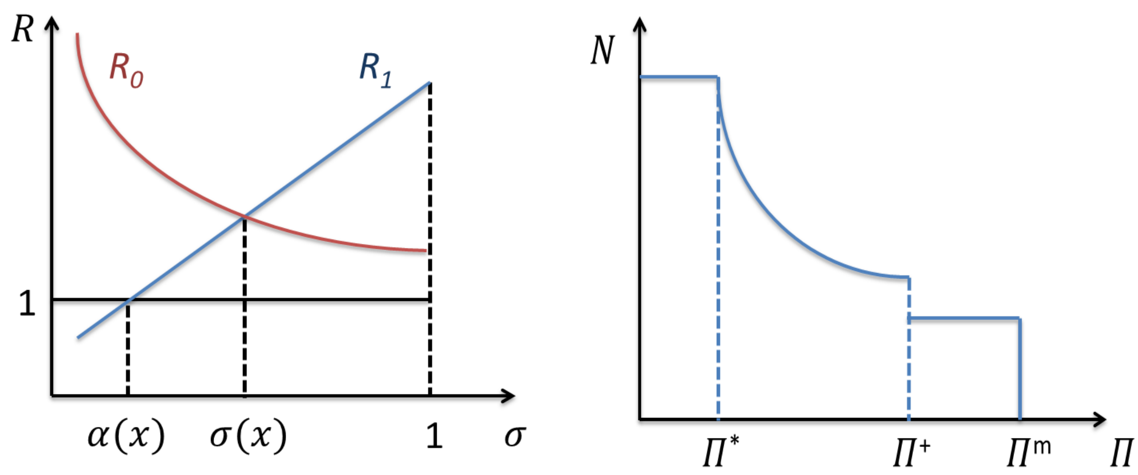

Figure 1: The left plot depicts $R_1(\sigma, \varkappa)$ and $R_0(\sigma, \varkappa)$ as functions of σ, where $\alpha(\varkappa) = \alpha_0/(\alpha_S\varkappa + \alpha_0)$. The right plot depicts the profit-stability trade-off $N(\Pi)$.

The left plot of figure (1) depicts $R_1(\sigma, \varkappa)$ and $R_0(\sigma, \varkappa)$ and shows that for any value of $\varkappa$

a unique discount $\sigma(\varkappa)$ exists which maximizes the size of the cartel. The right plot instead, depicts profit-stability trade-off $N(\Pi)$ derived in the previous proposition.

To highlight the trade-off more explicitly consider an economy in which $\delta = 0.95$, $\alpha = 0.15$, $v = 10$, and $c = 1$. Notice that the maximal cartel size grows from 20 to 29 when firms collude on the most stable sale strategy π^* instead of the monopoly strategy π^m. Profits however, decline significantly from 9 to 2 as firms need to price very aggressively during sales in order to sustain collusion. The most profitable sale strategy instead, marginally improves the cartel size, but raises almost as much profit as the monopoly policy. The following table reports all the relevant variables for the example discussed:

	n	Π	σ	μ	$\varkappa$
π^m	20.0	9.00	1.00	9	$\forall$
π^+	20.5	8.72	0.94	9	2
π^*	28.6	1.96	0.15	9	2

The next section develops comparative statics to highlight how changes in the environment may affect such a trade-off.

Comparative Statics

All the comparative statics are developed for the four relevant free parameters in the model, δ, α, v and c. The first preliminary result characterizes how the size of the set $\mathcal{E}$ changes as such parameters vary. As expected, more sale strategies are stable both when many consumers can store units, and when agents are patient. Increases in the profitability of the market further improve the ability to collude on a given sale strategy.

Proposition 10 *The size of the set $\mathcal{E}$ decreases with c and α, and increases with v and δ.*

The result is proven by studying how the bounds characterizing the stable strategy set $\mathcal{E}$ vary with the free parameters. A larger fraction of consumers with storage increases the size of $\mathcal{E}$, since more sale discounts are stable at any frequency $\varkappa$. Similarly patience δ, and profitability $v - c$, increase the size of $\mathcal{E}$, since the consumer demand constraint $\kappa(\varkappa) \geq \sigma$ is relaxed when such variables grow.

A more compelling result characterizes how the profit-stability trade-off is affected by changes in the free parameters. The next proposition explicitly characterizes this dependence, and shows how equilibrium strategies are affected by changes in the environment.

Proposition 11 *If $\mathcal{E} \neq \emptyset$, for any profit level $\Pi \in (0, \Pi^m]$ the maximal number firms that can collude on Π while employing a sale strategy satisfies:*

$dN(\Pi)$	$d\delta$	$d\alpha$	dv	dc
$\Pi \in (\Pi^+, \Pi^m]$	$+$	0	0	0
$\Pi \in (\Pi^*, \Pi^+]$	$+$	$?$	$+$	$-$
$\Pi \in (0, \Pi^*] \ \cap \ \Pi^* \neq \Pi^+$	$+$	$-$	0	0
$\Pi \in (0, \Pi^*] \ \cap \ \Pi^* = \Pi^+$	$+$	$-$	$+$	$-$

where $dN(\Pi)/d\alpha > 0$ if and only if $\delta > \Pi/\left(2\Pi^m - \Pi\right)$. Moreover, the cut-off profit levels Π^* and Π^+ and the maps σ and κ evaluated at $\varkappa = 2$ and $\sigma(2) \leq \kappa(2)$ further satisfy:

	$d\delta$	$d\alpha$	dv	dc
$d\Pi^+$	$+$	$+$	$+$	$-$
$d\Pi^*$	$-$	$+$	$+$	$-$
$d\sigma$	$-$	$+$	0	0
$d\kappa$	$+$	0	$+$	$-$

The proposition shows that increases in patience (i.e. the frequency of interaction) may lead both to larger equilibrium cartels at any profit level and to the persistence of the profit-stability trade-off on a larger range of profits. Increase in the profitability of a market (i.e. $v-c$) instead, were shown to increase the maximal equilibrium cartel size, but only for intermediate profit levels, as the stability both of the most stable strategy π^* and of the monopoly strategy π^m were proven to be independent of values and costs. Increases in the profitability of a market were also proven to increase the range of profits for which the profit-stability trade-off persists. Finally, increasing the fraction of consumers without storage (i.e. α) was proven to reduce the stability of the most stable sale strategy π^* as intertemporal linking between decisions would decline. The effect of such a change on the maximal cartel size at intermediate profit values was instead, proven ambiguous, as a large fraction of consumer with storage could lead to a decline in equilibrium profits due to the cost of anticipating production. Clearly such a change would have no effect on the stability of strategies without sales. However, the range of profits for which the profit-stability trade-off exists would decline in α and would eventually vanish at some value $\bar{\alpha} < 1$. Note that the discount offered during a period with sales (i.e. $1 - \sigma$) in the most profitable policy π^+ declines with patience and profitability and is unaffected by the fraction of consumers with storage. The discount offered in the most stable strategy π^* (when such strategy does not coincide with π^+) instead, grows with patience and the fraction of consumers with storage and is unaffected by profitability. Figure 2 below provides a visual characterization of the comparative statics results presented in the previous proposition.

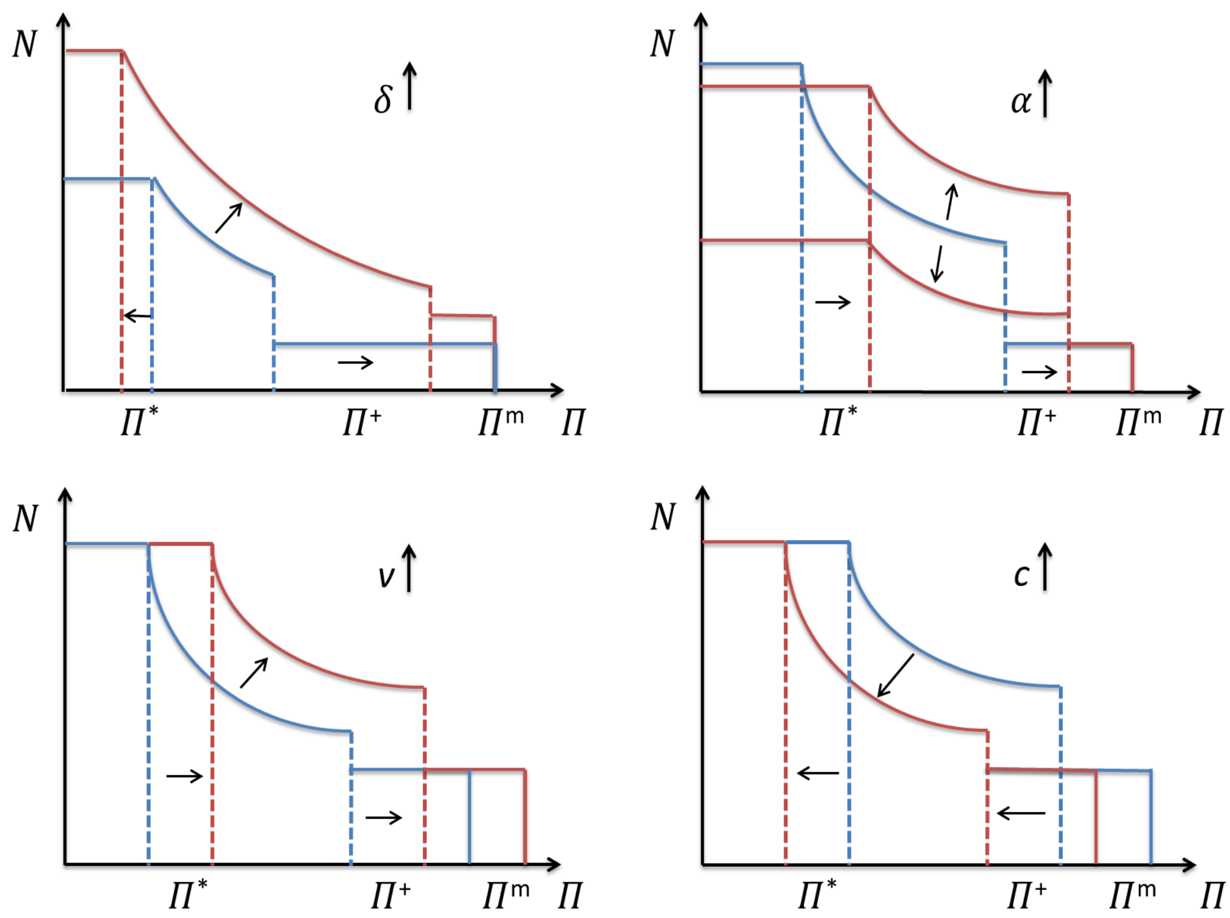

Figure 2: Comparative statics on the trade-off $N(\Pi)$ with respect to an increase: in δ (top left), in α (top right), in v (bottom left), and in c (bottom right).

4 Multi-Unit Storage and Asynchronized Sales

This section considers two extensions of the baseline model. In the first consumers have access to multi-unit storage technologies, $S > 1$, while in the second firms compete in multiple markets.

General Storage $S > 1$

The first extension characterizes the optimal timing of sales, and highlights that access to multi-unit storage technologies can reduce the profit-stability trade-off. This will be the case: since the most stable sale strategy may both display infrequent sales and be both more profitable, than the most stable policy with frequency 2; and because the most profitable sale strategy will not be affected by the change in storage constraints.

As in the previous section, the next two propositions characterize the most stable and the most profitable sale strategies in $\mathcal{E}$. For convenience, let us identify two particular sale frequencies which will be employed in the characterization of the most stable sale strategies:

$$\begin{aligned} \bar{\varkappa} &= \arg \max_{\varkappa \in \{2,\ldots,S+1\}} R_0(\sigma(\varkappa), \varkappa) \quad \text{s.t.} \quad \kappa(\varkappa) \geq \sigma(\varkappa) \\ \breve{\varkappa} &= \arg \min_{\varkappa \in \{2,\ldots,S+1\}} \varkappa \quad \text{s.t.} \quad \kappa(\varkappa) \leq \sigma(\varkappa) \end{aligned}$$

Proposition 12 *Assume that $\mathcal{E} \neq \emptyset$. If $\bar{\varkappa}$ is defined and if $R_0(\sigma(\bar{\varkappa}), \bar{\varkappa}) > R_1(\kappa(\breve{\varkappa}), \breve{\varkappa})$ whenever $\breve{\varkappa}$ is also defined, then no strategy in $\mathcal{S}$ is strictly more stable than strategy $\pi^* \in \mathcal{E}$:*

μ^*	σ^*	$\varkappa^*$
$1/\rho$	$\sigma(\bar{\varkappa})$	$\bar{\varkappa}$

otherwise, no strategy in $\mathcal{S}$ is strictly more stable than strategy $\pi^* \in \mathcal{E}$:

μ^*	σ^*	$\varkappa^*$
$1/\rho$	$\kappa(\breve{\varkappa})$	$\breve{\varkappa}$

Moreover, π^* is the most profitable of all the strategies in $\mathcal{S}$ with equal stability.

Such a strategy requires firms to set collusive markups in periods without sales and uniquely pins down the optimal discount for the remaining periods. Compared to the policy found in the previous section, the most stable strategy may display less frequent sales, $\varkappa^* > 2$. In particular, a necessary condition for this phenomenon to occur is that the critical ratio, $R_0(\sigma(\varkappa), \varkappa)$, be increasing with $\varkappa$ at $\varkappa = 2$. The optimal sales discount and the optimal frequency of sales crucially depend on the fraction of consumers with storage, on the discount rate, and on the monopoly markup in the economy. In many common scenarios, the sale strategy π^* can be simplified to $(1/\rho, \sigma(\breve{\varkappa} - 1), \breve{\varkappa} - 1)$ if $\breve{\varkappa} > 2$, and to $(1/\rho, \kappa(2), 2)$ otherwise.

The most profitable sale strategy in $\mathcal{E}$ remains unaffected when consumers can store multiple units and thus, the proposition characterizing such strategy coincides with the one presented in the previous section.

Proposition 13 *If $\mathcal{E} \neq \emptyset$, no strategy in $\mathcal{E}$ is strictly more profitable than strategy $\pi^+ \in \mathcal{E}$:*

μ^+	σ^+	$\varkappa^+$
$1/\rho$	$\kappa(2)$	2

The profit maximizing sale strategy cannot change even if consumers can store multiple units, since a lower sales frequency would lead to a bigger discount and thus to lower profits. Again, the optimal sales discount is the uniquely pinned down by the consumer's storage constraint.

As in the previous section, such propositions would display the profit-stability trade-off for this more general environment. The first result in fact, showed that sale strategies with infrequent sales could be used to further improve stability at the expense of profits. In fact, strategy π^* was proven to be more stable than any other strategy without sales, but less profitable than full collusion. Similarly π^* was clearly more profitable and less stable than the competitive outcome (i.e. the Nash equilibrium of the stage game). The second result instead, showed that the profit-stability trade-off would persist when π^* would entail infrequent sales.

The comparative static results on the size of the set $\mathcal{E}$ developed in the previous section also hold without further modifications in this more complex environment. As expected, more sale strategies will be stable: when consumers are more patient; when more consumers have access to storage; or when the market becomes more profitable.

Proposition 14 *The size of the set $\mathcal{E}$ decreases with c and α, and increases with v and δ.*

Obviously, all the comparative statics developed in the previous section on the most profitable policy π^+ hold without any further modification. As for the most stable policy π^*, results would have to be adjusted to allow optimal frequency $\varkappa^*$ to respond to changes in the environment. However, results with a flavor similar to the ones developed in the previous section would hold at any given frequency $\varkappa^*$. We refrain from developing such comparative statics in full generality as the additional intuition gained is limited.

To conclude this part of the analysis consider again the economy in which $\delta = 0.95$, $\alpha = 0.15$, $v = 10$, and $c = 1$. Suppose that consumers with access to storage can store up to thirty units, $S = 30$. Notice that the maximal cartel size is achieved with infrequent sales which take place every 21 periods. Maximal cartel size grows from to 37 when firms collude on the most stable sale strategy $\pi^*(S)$. Since sales occur less frequently smaller discounts are necessary to sustain the maximal cartel size. Hence, in this environment the profits of the most stable strategy $\pi^*(S)$ can be larger than those associated to the most stable strategy $\pi^*(1)$ of an economy in which at most a single unit can be stored $S = 1$, as is the case in the example reported below. Thus, the profit-stability trade-off can decrease when consumers gain access to more efficient storage technologies. The following table reports all the relevant variables for the example discussed:

	n	Π	σ	μ	$\varkappa$
π^m	20.0	9.00	1.00	9	$\forall$
π^+	20.5	8.72	0.94	9	2
$\pi^*(1)$	28.6	1.96	0.15	9	2
$\pi^*(S)$	37.4	4.62	0.27	9	21

Multiple Markets and Asynchronized Sales

From the previous discussion, it may appear that coordination in sales is necessary to achieve any stability gain. In contrast, we provide a simple example to argue that sale strategies do not need to synchronized. In particular, we will argue that when firms operate in multiple markets, sales do not need to be simultaneous and symmetric either within or across markets. Consider a variation on the previously described economy in which there are two identical markets A and B, each with a mass 1/2 of consumers, and an even number $n \geq 4$ of firms operating in both markets.[3] Corollary 9 provides sufficient conditions for the existence of a stable sale strategy

[3] Notice that since markets are symmetric, since firms' objective functions are not strictly concave, and since returns to scale are constant, there are no stability gains due to the multi-market setup (Bernheim and Whinston

$\pi_k \in \mathcal{E}$ in each market $k \in \{A, B\}$. Each of these strategies still requires firms to charge the fixed markup, μ_k, in almost every period, and to periodically hold sales by reducing the markup to $\mu_k \sigma_k$ every $\varkappa_k$ periods. The most stable sale strategy in such an environment still prescribes set $\pi_k = \pi^*$ in each market $k \in \{A, B\}$. Such a sale strategy sustains collusion if in any period $t \in \{0, 1\}$:

$$n \leq \frac{1}{1-\delta} \frac{\Pi_t(\pi_A^*) + \Pi_t(\pi_B^*)}{\Delta_t(\pi_A^*) + \Delta_t(\pi_B^*)} = \frac{1}{1-\delta} R_t(\pi^*) \tag{2}$$

since $\Pi_t(\pi_k) = \Pi_t(\pi^*)$ and $\Delta_t(\pi_k) = \Delta_t(\pi^*)$ for any $k \in \{A, B\}$.

Now consider a strategy $\bar{\pi}^*$ in which the markup in each market is fixed to $\bar{\mu}_k = \mu^*$, but different firms hold sales in different markets every $\varkappa^*$ periods. In particular, consider a strategy in which sales occurring along the equilibrium path satisfy in every period t:

(1) if $\mathrm{mod}(t, 2\varkappa^*) = 0$, firms $\{1, 2, ..., n/2\}$ set a discount $\bar{\sigma}_A = \sigma^*$ in market A and $\bar{\sigma}_B > \sigma^*$ in market B, while all the remaining firms set $\bar{\sigma}_B = \sigma^*$ in market B and $\bar{\sigma}_A > \sigma^*$ in market A;

(2) if $\mathrm{mod}(t, 2\varkappa^*) = \varkappa^*$, firms $\{n/2 + 1, ..., n\}$ set a discount $\bar{\sigma}_A = \sigma^*$ in market A and $\bar{\sigma}_B > \sigma^*$ in market B, while all the remaining firms set $\bar{\sigma}_B = \sigma^*$ in market B and $\bar{\sigma}_A > \sigma^*$ in market A;

(3) if $\mathrm{mod}(t, 2\varkappa^*) \neq 0, \varkappa^*$, $\bar{\sigma}_A = \bar{\sigma}_B = 1$ for every firm in every market.

Note that any firm charging $\bar{\sigma}_k > \sigma^*$ in market k does not collect profits in that market during a sales period. Also, observe that the total profit across markets is constant for each firm and equal to the total profit achieved in case of simultaneous sales. Without loss consider period 0 and a firm $i \in \{1, 2, ..., n/2\}$ and note that:

$$\begin{aligned}
\Pi_{0i}(\bar{\pi}_A) &= \frac{1}{2}\left[\frac{\delta - \delta^x}{1-\delta^x}\alpha_0 + \frac{1-\delta}{1-\delta^{2x}}\left[2\left(\varkappa\alpha_S + \alpha_0\right)\sigma\right]\right]\mu c \\
\Pi_{0i}(\bar{\pi}_B) &= \frac{1}{2}\left[\frac{\delta - \delta^x}{1-\delta^x}\alpha_0 + \frac{\delta^x - \delta^{x+1}}{1-\delta^{2x}}\left[2\left(\varkappa\alpha_S + \alpha_0\right)\sigma\right]\right]\mu c \\
&\Rightarrow \Pi_{0i}(\bar{\pi}_A) + \Pi_{0i}(\bar{\pi}_B) = \Pi_0(\pi_A^*) + \Pi_0(\pi_B^*)
\end{aligned}$$

Moreover, note that in any period t the deviation profits of each player coincide in each market k with those of the most stable sale strategy since $\Delta_t(\bar{\pi}_k) = \Delta_t(\pi_k^*)$. The few last observations in turn imply that strategy $\bar{\pi}^*$ is as stable as most stable strategy π^*. Without loss of generality consider the incentives to deviate in period 0 of a for a firm $i \in \{1, 2, ..., n/2\}$ holding a sales

1990, and Spagnolo 1999).

in market A:

$$n \leq \frac{1}{1-\delta} \frac{\Pi_0(\bar{\pi}_A) + \Pi_0(\bar{\pi}_B)}{\Delta_0(\bar{\pi}_A) + \Delta_0(\bar{\pi}_B)} = \frac{1}{1-\delta} R_0(\pi^*)$$

which is equivalent to condition (2). Similarly, incentives to comply with the equilibrium strategy remain unaffected in periods without sales. Thus, maximal cartel size under which a sale strategy sustains collusion remains unaffected even with asynchronized sales. Hence, an asynchronized sale strategy would strictly dominate a simultaneous sale strategy for any arbitrarily small menu cost incurred by firms while changing prices.

The previous argument required the number of firms operating in each market to exceed four. This was necessary, since the deviation payoff $\Delta_0(\bar{\pi}_k)$ would increase, if a single firm held sales in market k, as $\bar{\sigma}_k > \sigma^*$ for any firm not holding sales. If so, the largest sustainable cartel with asynchronized sales would smaller than with synchronized sales, as stability is inversely related to the lowest price charged by a competing firm. Note that the straightforward extension of the multi-market model to asymmetric markets would generate sale strategies which are not synchronized across markets as well as within markets.

Let $\bar{\pi}^1$ denote the variant of strategy $\bar{\pi}^*$ in which a single firm has sales in market A in periods $\text{mod}(t, 2\varkappa^*) = 0$ and in market B in periods $\text{mod}(t, 2\varkappa^*) = \varkappa^*$. Again consider an economy in which $S = 1$, $\delta = 0.95$, $\alpha = 0.15$, $v = 10$, and $c = 1$. Fix the threat discount of all the firms not selling in a market k during a sales period to $\bar{\sigma}_k = 0.2$. As expected, the stability of strategy $\bar{\pi}^1$ is smaller compared to $\bar{\pi}^*$ as cartel size declines whenever deviation profits grow:

	n	Π	σ	μ	$\varkappa$
π^m	20.0	9.00	1.00	9	$\forall$
π^+	20.5	8.72	0.94	9	2
π^*	28.6	1.96	0.15	9	2
$\bar{\pi}^*$	28.6	1.96	0.15	9	2
$\bar{\pi}^1$	24.7	1.96	0.15	9	2

5 Conclusion

The analysis presented a novel rationale for sales in an industry in which a homogeneous storable good is produced by n firms, and sold to consumers with access to heterogeneous storage technologies. In this context, the paper examined the effects of heterogeneity in storage on firms' incentives to hold periodic sales to support a greater degree of collusion. In such an environment sale strategies were proven to strengthen the incentives to collude, as storage would intertemporally link consumer demand and thus, reduce the short-run gains from a deviation. In particular, in any stable equilibrium firms would charge in any period of sales a big enough discount to induce all consumers with open storage capacity to stockpile a quantity sufficient to satisfy all their demand until the next sale. Such behavior was shown to reduce incentives to deviate both in regular price periods (as only consumers without storage would purchase units in such periods) and in periods with sales (both because a lower price would be charged in such periods, and because consumers with storage would reduce their demand if a deviation were observed in the wake of an imminent price war).

The first part of the analysis: characterized consumer demand; characterized the set $\mathcal{E}$ of sale strategies which are more stable than any strategy without sales; and presented necessary and sufficient conditions for the non-emptiness of such a set. The second part of the analysis focused on single-unit storage and characterized the most stable and the most profitable sale strategies in the set $\mathcal{E}$. Within such set a trade-off was proven to emerge between collusive profits and cartel size. Such trade-off and its dependence on the environment were explicitly characterized. Relationship between the optimal sale markdown and the environment was also explored. The final part of the analysis extended the baseline model in two directions. The first extension considered the multi-unit storage scenario, characterized the optimal timing of sales, and showed that access to multi-unit storage technologies could reduce the profit-stability trade-off. The second extension instead, proved that synchronization in sales would not be necessary in multi-market setup.

In the proposed model collusion was strengthened at the expense of aggregate profits, since deviation profits would decline more than equilibrium profits if sale strategies were employed. Aggregate profits however, had to decline when firms held sales, as consumers were homogeneous in their willingness to pay. Note however, that if consumers with higher storage capacity had a lower willingness to pay, a sale strategy could achieve higher profits than the no-sale strategy by price discriminating among different types of consumers, and could thus foster collusion even further. Valuations were kept homogeneous across consumers only to display more explicitly the effects of the intertemporal linking in consumer demand.

Finally, note that the rationality imposed on consumer demand required that all buyers would understand the consequences of a deviation on future prices. This assumption could easily be relaxed by introducing behavioral buyers who can only imperfectly forecast future prices. All results would still hold qualitatively, even though the stability of any given strategy may decline.

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6 Appendix

Part A: Proofs

Remark 1 *If $\overline{p}_z \leq v$ in any period $z \geq 0$, the demand for consumption good at time t:*
(1) by consumers without storage technology satisfies $d_0(\mathbf{p}^t) = 1$;
(2) by consumers with storage technology and with s units already in storage satisfies:

$$d_S(s, \mathbf{p}^t) = \max\left\{\min\left\{T(\mathbf{p}^t), S+1\right\} - s, 0\right\}$$

Proof. Part (1) of the claim is trivial. To prove (2) notice that by construction $d_S(s, \mathbf{p}^t) \in [0, S-1+s]$. The upper-bound must hold, since no consumer can store more than S units. The lower-bound must hold because no player can benefit by disposing already purchased units given that $\overline{p}_z \geq 0$ for any z. Also notice that only profiles of demand such that guarantee a consumption stream of a unit in every period can be optimal, since prices satisfy $\overline{p}_z \leq v$. Thus, payoff stream can be compared by looking only at the total expenditure on consumption good.

Then, consider the case in which $T(\mathbf{p}^t) \leq S+1$. By contradiction consider a profile of demand for the successive $T(\mathbf{p}^t)$ periods, $\{d_{t+z}\}_{z=0}^{T(\mathbf{p}^t)}$ and suppose that $d_t \neq \max\{T(\mathbf{p}^t) - s, 0\}$. If so, there exists a profile of demands $\left\{d'_{t+z}\right\}_{z=0}^{T(\mathbf{p}^t)}$ that costs less and that leaves the consumer with exactly as many units stored in period $t + T(\mathbf{p}^t)$. In fact, consider:

$$\begin{aligned} d'_t &= \max\left\{T(\mathbf{p}^t) - s, 0\right\} \\ d'_{t+z} &= 0 \ \text{ if } \ z \in \left(0, T(\mathbf{p}^t)\right) \\ d'_{t+T(\mathbf{p}^t)} &= \left[\sum_{z=0}^{T(\mathbf{p}^t)} d_z\right] - d'_t \end{aligned}$$

by construction the profile leaves the consumer with exactly as many units stored in period $t + T(\mathbf{p}^t)$. Moreover d' costs less, since:

$$\begin{aligned} \sum_{z=0}^{T(\mathbf{p}^t)} \delta^z \overline{p}_z d_z &= \sum_{z=0}^{T(\mathbf{p}^t)-1} \delta^z \overline{p}_z d_z + \delta^{T(\mathbf{p}^t)} p_{t+T(\mathbf{p}^t)} d_{t+T(\mathbf{p}^t)} \geq \\ &\geq \overline{p}_t \sum_{z=0}^{T(\mathbf{p}^t)-1} d_z + \delta^{T(\mathbf{p}^t)} p_{t+T(\mathbf{p}^t)} d_{t+T(\mathbf{p}^t)} = \\ &= \overline{p}_t d'_t + \overline{p}_t \left[\sum_{z=0}^{T(\mathbf{p}^t)-1} d_z - d'_t\right] + \delta^{T(\mathbf{p}^t)} p_{t+T(\mathbf{p}^t)} d_{t+T(\mathbf{p}^t)} = \\ &= \overline{p}_t d'_t + \overline{p}_t \left[d'_{t+T(\mathbf{p}^t)} - d_{t+T(\mathbf{p}^t)}\right] + \delta^{T(\mathbf{p}^t)} p_{t+T(\mathbf{p}^t)} d_{t+T(\mathbf{p}^t)} = \\ &= p_t d'_t + \delta^{T(\mathbf{p}^t)} p_{t+T(\mathbf{p}^t)} d'_{t+T(\mathbf{p}^t)} + \left[\delta^{T(\mathbf{p}^t)} p_{t+T(\mathbf{p}^t)} - \overline{p}_t\right]\left[d_{t+T(\mathbf{p}^t)} - d'_{t+T(\mathbf{p}^t)}\right] \geq \\ &\geq p_t d'_t + \delta^{T(\mathbf{p}^t)} p_{t+T(\mathbf{p}^t)} d'_{t+T(\mathbf{p}^t)} \end{aligned}$$

given that: (i) $\sum_{z=0}^{T(\mathbf{p}^t)-1} d_z \geq d_t' \ \& \ d_{t+T(\mathbf{p}^t)} \leq d_{t+T(\mathbf{p}^t)}'$, since that consumers consume one unit in every period; and (ii) $\delta^{T(\mathbf{p}^t)} p_{t+T(\mathbf{p}^t)} < \overline{p}_t \leq \delta^z \overline{p}_{t+z}$ for any $z \in (0, T(\mathbf{p}^t))$. Thus a contradiction is established. A very similar and omitted argument works also for the case in which $T(\mathbf{p}^t) > S+1$ and establishes the claim. ■

Remark 2 *Strategy π is a SPE of the infinite repetition of the game if and only if:*

$$\delta \geq 1 - \frac{1}{n}\frac{\Pi_t}{\Delta_t}$$

Proof. The proof of the result is trivial. No player benefits from a deviation along the equilibrium path if:

$$\frac{\Pi_t}{n} \geq (1-\delta)\Delta_t$$

where Δ_t denotes the most profitable deviation. Such condition is exploited to pin down the requirement on the critical discount rate. Moreover no deviation can be profitable off the equilibrium path, since all players make at most zero profits when all competitors quote prices at marginal cost. ■

Remark 3 *If* $\operatorname{mod}(t, \varkappa) = 0$*,* $s_t = 0$*. If* $\operatorname{mod}(t, \varkappa) = \operatorname{mod}(z, \varkappa)$*,* $d_t = d_z$*.*

Proof. The first claim is proven by induction. Note that $\operatorname{mod}(0, \varkappa) = 0$ and $s_0 = 0$. We show that if the claim is true for any $t \leq T$ such that $\operatorname{mod}(t, \varkappa) = 0$ it is true for any $t \leq T + \varkappa$ such that $\operatorname{mod}(t, \varkappa) = 0$. In fact consider the largest date t such that $t \leq T$ and $\operatorname{mod}(t, \varkappa) = 0$. Such date exists by the initial condition and the induction hypothesis. At such date the demand of an individual with storage satisfies:

$$d_S(0, \mathbf{p}^t) = \min\left\{T(\mathbf{p}^t), S+1\right\}$$

Moreover $T(\mathbf{p}^t) \leq \varkappa$, since $(1 + \mu\sigma) > \delta^{\varkappa}(1 + \mu\sigma)$. Hence, $d_S(0, \mathbf{p}^t) \leq \varkappa$ and $s_t < \varkappa$ given that one unit will be consumed. Moreover, in any period $z \in \{t+1, ..., t+\varkappa\}$, since $(1+\mu) > \delta(1+\mu) > \delta(1+\mu\sigma)$, we have that $T(\mathbf{p}^t) = 1$ and consequently:

$$d_S(s_t, \mathbf{p}^t) = \begin{cases} 0 & \text{if} \quad s_t > 0 \\ 1 & \text{if} \quad s_t = 0 \end{cases} \Rightarrow s_{t+1} = \begin{cases} s_t - 1 & \text{if} \quad s_t > 0 \\ 0 & \text{if} \quad s_t = 0 \end{cases}$$

Which establishes that $s_{t+\varkappa} = 0$, since $s_t < \varkappa$. The second claim follows immediately, since from the previous part of the proof it is straightforward to observe that:

$$d_S(s_t, \mathbf{p}^t) = \begin{cases} d_S(0, \mathbf{p}^0) & \text{if} \quad \text{mod}(t, \varkappa) = 0 \\ 0 & \text{if} \quad s_t > 0 \;\&\; \text{mod}(t, \varkappa) \neq 0 \\ 1 & \text{if} \quad s_t = 0 \;\&\; \text{mod}(t, \varkappa) \neq 0 \end{cases} \tag{3}$$

■

Proposition 4 *The following claims must hold:*

(1) $\mathcal{S}\backslash\mathcal{C} = \mathcal{V}$

(2) $\mathcal{N} \subseteq \mathcal{V}$

(3) any strategy in $\mathcal{N}$ *is more stable than any strategy in* $\mathcal{V}$*;*

(4) any strategy that sets $\mu = \frac{v-c}{c}$ *and* $\sigma = 1$ *is profit maximizing within* $\mathcal{S}$*;*

(5) for any strategy in $\mathcal{C}$*,* $s_t = S(t)$ *and:*

$$d_t = \begin{cases} \alpha_0 + \varkappa\alpha_S & \textit{if} \;\; \text{mod}(t, \varkappa) = 0 \\ \alpha_0 & \textit{if} \;\; \text{mod}(t, \varkappa) \neq 0 \end{cases}$$

Proof. To prove part (1), we begin by arguing that $\mathcal{S}\backslash\mathcal{C} \subseteq \mathcal{V}$. Consider a strategy $\pi \in \mathcal{S}\backslash\mathcal{C}$ for which the constraint $(1 + \mu\sigma) \leq \delta^{\varkappa-1}(1 + \mu)$ is violated. By the demand structure established in condition (3) a period exists in which $\text{mod}(t, \varkappa) \neq 0$ and $d_{St} > 1$, since $d_{0S} < \varkappa$. Therefore, every strategy violating $(1 + \mu\sigma) \leq \delta^{\varkappa-1}(1 + \mu)$ must belong to $\mathcal{V}$. Similarly, if $\varkappa > S + 1$ a period would exists in which $d_{St} > 1$ and $\text{mod}(t, \varkappa) \neq 0$, and the strategy would again belong to $\mathcal{V}$. Thus, $\mathcal{S}\backslash\mathcal{C} \subseteq \mathcal{V}$. Further note that condition (5) requires $\mathcal{C} \cap \mathcal{V} = \emptyset$, and thus establishes (1). Note that (2) is immediate, because $\sigma = 1$ implies $d_S(s_t, \mathbf{p}^t) = 1$ for any t as $\overline{p}_{t+1} = \overline{p}_t$ requires $\overline{p}_t > \delta\overline{p}_{t+1}$.

To prove (3), first observe that all strategies in $\mathcal{N}$ are equally stable. Note that, by the proof of (2), for any strategy $\pi \in \mathcal{N}$ equilibrium payoffs simplify to $\Pi_t(\pi) = \mu c$. Thus, a deviating player can capture at most such a profit by undercutting the price marginally. Any deviation to a price $y \in (c, (1+\mu)c)$, must satisfy $d(s_t, \mathbf{y}^t) \leq 1$, since $\overline{p}_t > \delta c$ and therefore $\Delta_t(y, \pi) \leq (y - c)$. Hence, $\Delta_t(\pi) = \mu c$ and $R_t(\pi) = 1$ for any $t \in \{0, 1, ...\}$ and any $\pi \in \mathcal{N}$. Now, consider a strategy $\pi \in \mathcal{V}$ and a period t in which $d_t > \alpha_0$ and $\overline{p}_t = (1 + \mu)c$. Note such conditions imply that

$s_t = 0$ and $d_t \geq 1$. If so, by $\overline{p}_t = (1+\mu)c$, we get that $\Delta_t(y,\pi) = (y-c)d(s_t, \mathbf{y}^t) = (y-c)$, and $\Delta_t(\pi) = \mu c$. Moreover, if such a period exists, it must be that:

$$d_{S0} = \min\left\{T(\mathbf{p}^0), S+1\right\} < \varkappa$$

because of the evolution of savings and demand discussed in the pervious lemma (condition 3). In turn this requires that $\delta^{t-1}(1+\mu) \geq (1+\mu\sigma) > \delta^t(1+\mu)$ for some $t \in \{1, ..., \varkappa - 1\}$. If so, pick the smallest t for which $(1+\mu\sigma) > \delta^t(1+\mu)$ and notice that:

$$\begin{aligned}
\Pi_t(\pi) &= \frac{(1-\delta)}{1-\delta^{\varkappa}}\left[\left[\sum_{z=0}^{\varkappa-1}\delta^z d_{t+z}\right] - (1-\sigma)\delta^{S(t)}d_0\right]\mu c = \\
&= \frac{(1-\delta)}{1-\delta^{\varkappa}}\left[\left[\sum_{z=0}^{S(t)-1}\delta^z + \alpha_0\sum_{z=S(t)+1}^{\varkappa-1}\delta^z\right] + \sigma\delta^{S(t)}(\alpha_0 + \alpha_S t)\right]\mu c = \\
&= \alpha_0\Pi_{0t}(\pi) + \alpha_S\Pi_{St}(\pi) \leq \mu c
\end{aligned}$$

where the last inequality must hold since:

$$\begin{aligned}
\Pi_{0t}(\pi) &= \frac{\left(1-\delta^{S(t)} + \delta^{S(t)+1} - \delta^{\varkappa}\right) + \sigma(\delta^{S(t)} - \delta^{S(t)+1})}{1-\delta^{\varkappa}}\mu c \leq \mu c \\
\Pi_{St}(\pi) &= \frac{\left(1-\delta^{S(t)}\right) + \sigma t(\delta^{S(t)} - \delta^{S(t)+1})}{1-\delta^{\varkappa}}\mu c = \\
&= \frac{\left(1-\delta^{\varkappa-t}\right) + \sigma t(\delta^{\varkappa-t} - \delta^{\varkappa-t+1})}{1-\delta^{\varkappa}}\mu c \leq \mu c \quad \Leftrightarrow \quad \sigma \leq \frac{\left(1-\delta^t\right)}{t(1-\delta)}
\end{aligned}$$

The inequality bounding $\Pi_{0t}(\pi)$ must hold, since it cannot be profitable to cut prices on consumers that do not alter their demand. The inequality bounding $\Pi_{St}(\pi)$ must hold instead, since firms prefer to delay production costs and because $\delta^{t-1}(1+\mu) \geq (1+\mu\sigma)$ requires:

$$\sigma \leq \frac{1+\mu\sigma}{1+\mu} \leq \delta^{t-1} \leq \frac{\sum_{z=0}^{t-1}\delta^z}{t} = \frac{\left(1-\delta^t\right)}{t(1-\delta)}$$

Hence, a strategy $\pi \in \mathcal{V}$ cannot be more stable than a strategy in $\mathcal{N}$, since $R_t(\pi) \leq 1$.

The proof of (4) is trivial. The proposed strategy raises a profit of $v - c$, since $d_t = 1$ for any t. No strategy in which $d_t = 1$ for any t can do better, since v is the highest price that a buyer willing pay for a unit of consumption. But any other strategy such that $d_t \neq 1$ for some t must satisfy $d_{S0} > 1$, by the properties of the demand function derived in condition (3). In turn, if $d_{S0} > 1$, it must be that $(1+\mu\sigma) \leq \delta(1+\mu)$. Thus by (3), we get that profits can be expressed as follows for some $d_{S0} \in (1, \varkappa]$:

$$\begin{aligned}\Pi_0(\pi) &= \frac{(1-\delta)}{1-\delta^{\varkappa}}\left[\left[\sum_{z=0}^{\varkappa-1}\delta^z d_z\right] - (1-\sigma)d_0\right]\mu c = \\ &= \frac{(1-\delta)}{1-\delta^{\varkappa}}\left[\left[\alpha_0\sum_{z=1}^{d_{S0}-1}\delta^z + \sum_{z=d_{0S}}^{\varkappa-1}\delta^z\right] + \sigma(\alpha_0+\alpha_S d_{S0})\right]\mu c\end{aligned}$$

An argument similar to the one developed in the previous part of the proof shows that $\Pi_0(\pi) \leq \mu c$. In particular, write profits as $\Pi_0(\pi) = \alpha_0\Pi_{00}(\pi) + \alpha_S\Pi_{S0}(\pi)$ and notice that for the same reason described in part (3) $\Pi_{00}(\pi) \leq \mu c$. Then let $t = d_{0s}$ notice that:

$$\Pi_{S0}(\pi) = \frac{\left(\delta^t - \delta^{\varkappa}\right) + \sigma t(1-\delta)}{1-\delta^{\varkappa}}\mu c \leq \mu c \quad \Leftrightarrow \quad \sigma \leq \frac{\left(1-\delta^t\right)}{t(1-\delta)}$$

where the inequality bounding $\Pi_{S0}(\pi)$ is established by $\delta^{t-1}(1+\mu) \geq (1+\mu\sigma)$ as in part (3). This establishes (4), since $\mu c \leq v - c$ is necessary for profits to be maximal by the properties of the demand function.

Part (5) follows trivially from condition (3) and the demand functions of both types of consumers discussed in the text. ■

Proposition 5 *Any strategy in $\mathcal{E}$ is more stable than any strategy in $\mathcal{S}\backslash\mathcal{E}$.*

Proof. First we establish that a strategy $\mathcal{E}$ in is more stable than a strategy in $\mathcal{N}$. Consider a strategy $\pi \in \mathcal{E}$. By definition of $\mathcal{E}$, $\sigma \geq \frac{\alpha_0}{\varkappa\alpha_S+\alpha_0}$, and therefore we get that $R_1(\pi) \geq 1$ and that for any $t \in \{1, 2, ..., \varkappa - 2\}$:

$$R_t(\pi) = 1 + \frac{(1-\delta)}{1-\delta^{\varkappa}}\delta^{\varkappa-t}\left[\frac{\varkappa\alpha_S+\alpha_0}{\alpha_0}\sigma - 1\right] \leq R_{t+1}(\pi)$$

Hence, the stability of a strategy in $\pi \in \mathcal{E}$ will be pinned down by the minimum between $R_0(\pi)$ and $R_1(\pi)$. Moreover, $R_0(\pi) > 1$ since for any μ and for any $\varkappa > 1$:

$$\begin{aligned}R_0(\pi) &= \frac{\alpha_0}{\sigma} + \frac{1-\delta}{1-\delta^x}\left[(\varkappa\alpha_S+\alpha_0) - \frac{\alpha_0}{\sigma}\right] \geq \\ &\geq \alpha_0 + \frac{(1-\delta)\varkappa}{1-\delta^x}\alpha_S = \alpha_0 + \frac{\varkappa}{1+\delta+...+\delta^{x-1}}\alpha_S > 1\end{aligned}$$

where the first inequality holds since $dR_0(\pi)/d\sigma < 0$. Which establishes that if a strategy π belongs to $\mathcal{E}$ then it must be more stable than any strategy in $\mathcal{N}$, since $\min_{t\geq 0} R_t(\pi) \geq 1$. Since any strategy in $\mathcal{N}$ is more stable than any strategy in $\mathcal{V}$, what remains to be proven is that any strategy in $\mathcal{E}$ is more stable than strategies in $\mathcal{C}\backslash\mathcal{E}$. But this is immediate since $\pi \in \mathcal{C}\backslash\mathcal{E}$ implies $\sigma < \frac{\alpha_0}{\varkappa\alpha_S+\alpha_0}$, and thus $R_1(\pi) < 1$. ■

Corollary 6 *$\mathcal{E}$ contains a strategy with a cycle of length $\varkappa \in \{2, ..., S+1\}$ if and only if:*

$$\delta^{\varkappa-1} \geq \frac{v-c}{v}\frac{\alpha_0}{\varkappa - \alpha_0(\varkappa-1)} + \frac{c}{v} \tag{4}$$

Thus, if $\delta \geq \frac{v-c}{v}\frac{\alpha_0}{2-\alpha_0} + \frac{c}{v}$, then $\mathcal{E}$ is non-empty.

Proof. First, we establish that 4 implies the existence of a strategy with a cycle of length $\varkappa \in \{2, ..., S+1\}$ in $\mathcal{E}$. Let constraint 4 hold for some $\varkappa \in \{2, ..., S+1\}$. Take any strategy that sets $\mu = \frac{v}{c} - 1$ and

$$\sigma \in \left[\frac{\alpha_0}{\alpha_S\varkappa + \alpha_0}, \delta^{\varkappa-1}\left(1+\frac{1}{\mu}\right) - \frac{1}{\mu}\right] \tag{5}$$

at the given value $\varkappa$. The strategy obviously belongs to $\mathcal{E}$. Moreover, such a strategy exists since the interval in which σ was chosen is non-empty, whenever 4 holds at $\varkappa$.

Next we establish the necessity of 4. Any strategy in $\mathcal{E}$ must satisfy 5 by construction. Consider any one of these strategies, and notice that:

$$\left[\frac{\alpha_0}{\alpha_S\varkappa + \alpha_0}, \delta^{\varkappa-1}\left(1+\frac{1}{\mu}\right) - \frac{1}{\mu}\right] \subseteq \left[\frac{\alpha_0}{\alpha_S\varkappa + \alpha_0}, \delta^{\varkappa-1}\frac{v}{v-c} - \frac{c}{v-c}\right]$$

Since the non-emptiness of the bigger interval is equivalent to 4, we get that 4 being violated prevent the existence of a policy with cycle length $\varkappa$ in $\mathcal{E}$. This establishes the necessity. The last observation is a trivial corollary. ■

Proposition 7 *If $\mathcal{E} \neq \emptyset$, no strategy in $\mathcal{S}$ is strictly more stable than strategy $\pi^* \in \mathcal{E}$:*

μ^*	σ^*	$\varkappa^*$
$1/\rho$	$\min\{\sigma(2), \kappa(2)\}$	2

Moreover, π^* is the most profitable of all the strategies in $\mathcal{S}$ with equal stability.

Proof. This follows from the proof of proposition (12). ■

Proposition 8 *If $\mathcal{E} \neq \emptyset$, no strategy in $\mathcal{E}$ is strictly more profitable than strategy $\pi^+ \in \mathcal{E}$:*

μ^+	σ^+	$\varkappa^+$
$v/c-1$	$\kappa(2)$	2

Proof. This follows from the proof of proposition (13). ■

Proposition 9 *If $\mathcal{E} \neq \emptyset$, for any profit level $\Pi \in (0, \Pi^m]$ the maximal number firms that can collude on Π while employing a sale strategy satisfies:*

$$N(\Pi) = \begin{cases} \frac{1}{1-\delta} & \text{if } \Pi \in (\Pi^+, \Pi^m] \\ \frac{1}{1-\delta}\frac{(2-\alpha)\Pi}{(1+\delta)\Pi - \delta\alpha(v-c)} & \text{if } \Pi \in (\Pi^*, \Pi^+] \\ n(\pi^*) & \text{if } \Pi \in (0, \Pi^*] \end{cases}$$

Proof. First note that if $\Pi > \Pi^+$ no strategy in $\mathcal{E}$ is more profitable than Π^+. Thus, no such profit level can be sustained by a sale strategy belonging to $\mathcal{E}$. If so, the most stable strategy is one without sales. However, all strategies in $\mathcal{N}$ are equally stable by proposition (7) and thus, $N(\Pi) = \frac{1}{1-\delta}$ for any such strategy.

Then suppose that $\Pi \leq \Pi^*$ and consider any strategy $(\mu, \sigma, 2)$ with profits Π. Note that $\Pi \leq \Pi^*$ implies that either $\mu \leq \mu^*$ or $\sigma \leq \sigma^*$. Also note that a different strategy $(\bar{\mu}, \sigma^*, 2)$ exists which raises exactly the same profits, since any profit level $\Pi \leq \Pi^*$ can be obtained by picking $\bar{\mu} \in (0, \mu^*]$. Thus observe that strategy $(\bar{\mu}, \sigma^*, 2)$ is equally stable to strategy $\pi^* = (\mu^*, \sigma^*, 2)$ and thus more stable than $(\mu, \sigma, 2)$.

Finally consider the case in which $\Pi \in (\Pi^*, \Pi^+]$. Note that for this to be the case it must be that $\Pi^* < \Pi^+$, which in turn requires

$$\sigma^* = \sigma(2) < \kappa(2) = \sigma^+$$

Note that setting $\mu = \mu^*$ is always optimal for both profits and stability. Thus, for any profit level $\Pi \in (\Pi^*, \Pi^+]$ a corresponding sales discount exists $\sigma(\Pi) \in (\sigma^*, \sigma^+]$ which sustains profit level Π. Such a discount is found by solving the following equality with respect to σ:

$$\Pi = \left[\alpha\frac{\delta}{1+\delta} + \sigma(2-\alpha)\frac{1}{1+\delta}\right]\mu^* c$$
$$\Rightarrow \quad \sigma(\Pi) = \frac{1}{2-\alpha}\left[\frac{\Pi}{v-c}(1+\delta) - \alpha\delta\right]$$

The value of $N(\Pi)$ in such interval can then be found by computing $n(\sigma(\Pi))$:

$$\begin{aligned} N(\Pi) &= n(\sigma(\Pi)) = \frac{1}{1-\delta}R_0(\mu^*, \sigma(\Pi), 2) = \frac{1}{1-\delta}\frac{\Pi}{\sigma(\Pi)\mu^* c} = \\ &= \frac{1}{1-\delta}\frac{(2-\alpha)\,\Pi}{(1+\delta)\,\Pi - \delta\alpha\,(v-c)} \end{aligned}$$

which establishes the desired result. ■

Proposition 10 *The size of the set $\mathcal{E}$ decreases with c and α, and increases with v and δ.*

Proof. First note that by corollary 9 the sufficient condition for the existence of a sale strategy with period 2 requires:

$$\delta \geq \frac{\alpha}{2-\alpha} + \frac{c}{v}\left[1 - \frac{\alpha}{2-\alpha}\right] = h \tag{6}$$

Further notice that such condition is easier to satisfy when the right hand side of (6) is smaller. This in turn implies the desired results since:

$$\begin{aligned} \frac{dh}{d\alpha} &= \frac{v-c}{v}\frac{2}{(2-\alpha)^2} > 0 \\ \frac{dh}{dv} &= -\frac{c}{v^2}\left[1 - \frac{\alpha}{2-\alpha}\right] < 0 \\ \frac{dh}{dc} &= \frac{1}{v}\left[1 - \frac{\alpha}{2-\alpha}\right] > 0 \end{aligned}$$

The final observation on δ is trivial the left hand side of (6) increases in δ. ■

Proposition 11 *If $\mathcal{E} \neq \emptyset$, for any profit level $\Pi \in (0, \Pi^m]$ the maximal number firms that can* *collude on* Π *while employing a sale strategy satisfies:*

$dN(\Pi)$	$d\delta$	$d\alpha$	dv	dc
$\Pi \in (\Pi^{+}, \Pi^{m}]$	$+$	0	0	0
$\Pi \in (\Pi^{*}, \Pi^{+}]$	$+$	$?$	$+$	$-$
$\Pi \in (0, \Pi^{*}] \;\cap\; \Pi^{*} \neq \Pi^{+}$	$+$	$-$	0	0
$\Pi \in (0, \Pi^{*}] \;\cap\; \Pi^{*} = \Pi^{+}$	$+$	$-$	$+$	$-$

where $dN(\Pi)/d\alpha > 0$ *if and only if* $\delta > \Pi/\left(2\Pi^{m} - \Pi\right)$. *Moreover, the cut-off profit levels* Π^{*} *and* Π^{+} *and the maps* σ *and* κ *evaluated at* $\varkappa = 2$ *and* $\sigma(2) \leq \kappa(2)$ *further satisfy:*

	$d\delta$	$d\alpha$	dv	dc
$d\Pi^{+}$	$+$	$+$	$+$	$-$
$d\Pi^{*}$	$-$	$+$	$+$	$-$
$d\sigma$	$-$	$+$	0	0
$d\kappa$	$+$	0	$+$	$-$

Proof. First note when $\Pi \in (\Pi^{+}, \Pi^{m}]$ the sign of all the derivatives of $N(\Pi) = 1/(1-\delta)$ is trivial. Next, consider the case in which $\Pi \in (\Pi^{*}, \Pi^{+}]$. Note that within such interval $\sigma \in (\sigma^{*}, \sigma^{+}]$ and:

$$\begin{aligned}
\frac{dN(\Pi)}{d\alpha} &= \frac{1}{1-\delta} \frac{\left[2\delta\left(v-c\right) - \left(1+\delta\right)\Pi\right]\Pi}{\left(\left(1+\delta\right)\Pi - \delta\alpha\left(v-c\right)\right)^{2}} > 0 \quad \Leftrightarrow \quad 2\delta\left(v-c\right) > \left(1+\delta\right)\Pi \\
\frac{dN(\Pi)}{d\delta} &= \frac{1}{\left(1-\delta\right)^{2}} \left[\frac{\left(2-\alpha\right)\left[2\delta\left(\Pi - \alpha\left(v-c\right)\right) + \alpha\left(v-c\right)\right]\Pi}{\left(\left(1+\delta\right)\Pi - \delta\alpha\left(v-c\right)\right)^{2}}\right] > 0 \\
\frac{dN(\Pi)}{dv} &= -\frac{dN(\Pi)}{dc} = \frac{1}{1-\delta} \frac{\left(2-\alpha\right)\alpha\delta\Pi}{\left(\left(1+\delta\right)\Pi - \delta\alpha\left(v-c\right)\right)^{2}} > 0
\end{aligned}$$

The second inequality holds, since $\Pi > \sigma\left(v-c\right)$ and $\sigma > \alpha$ together imply $\Pi > \alpha\left(v-c\right)$ (where the first condition holds since $\Pi_0 > \Delta_0$ for the strategy to belong to $\mathcal{E}$, and where the second condition holds since $\sigma > \sigma^{*}$ and since the only positive root of $\sigma(2)$ satisfies $\sigma^{*} > \alpha$, as explained in appendix B).

Before we proceed final scenario $\Pi \in (0, \Pi^{*}]$, let us prove all the remaining results. First, observe that $d\sigma(2)/dv = d\sigma(2)/dc = 0$, since both R_0 and R_1 are independent of values and costs (see appendix B). Further note that by the implicit function theorem applied to the map $\sigma(2)$:

$$\begin{aligned}
\sigma_\delta &= \frac{d\sigma(2)}{d\delta} = -\frac{R_{1\delta} - R_{0\delta}}{R_{1\sigma} - R_{0\sigma}} = -\frac{(1-\alpha)\alpha\sigma^*}{\delta\left[2\delta(2-\alpha)\sigma^* - (1-\alpha)\alpha\right]} \\
\sigma_\alpha &= \frac{d\sigma(2)}{d\alpha} = -\frac{R_{1\alpha} - R_{0\alpha}}{R_{1\sigma} - R_{0\sigma}} = \frac{\sigma^*(\sigma^*\delta + 1) + 2\alpha(\delta - \sigma^*)}{\left[2\delta(2-\alpha)\sigma^* - (1-\alpha)\alpha\right]}
\end{aligned}$$

Moreover, note that $d\sigma(2)/d\delta < 0$, since $2\delta(2-\alpha)\sigma^* > (1-\alpha)\alpha$ by definition of σ^*; and that in the only relevant scenario (i.e. $\kappa(2) > \sigma(2)$) $d\sigma(2)/d\alpha > 0$, since $\delta > \kappa(2) > \sigma(2) = \sigma^*$. Also, note that $R_{1\delta} > 0$, $R_{0\delta} < 0$, $R_{1\alpha} < 0$ and $R_{0\alpha} < 0$. The sign of the derivatives of the map $\kappa(2)$ follow trivially from its definition.

Then note that Π^+ and its derivatives with respect to the relevant parameters satisfy:

$$\begin{aligned}
\Pi^+ &= \frac{\delta\alpha}{1+\delta}(v-c) + \frac{2-\alpha}{1+\delta}(\delta v - c) \\
\frac{d\Pi^+}{d\delta} &= \frac{2v + 2(1-\alpha)c}{(1+\delta)^2} > 0 \quad \& \quad \frac{d\Pi^+}{d\alpha} = \frac{c(1-\delta)}{1+\delta} > 0 \\
\frac{d\Pi^+}{dv} &= \frac{2\delta}{1+\delta} > 0 \quad \& \quad \frac{d\Pi^+}{dc} = -\frac{2-\alpha(1-\delta)}{1+\delta} < 0
\end{aligned}$$

To compute the derivatives of Π^*, consider the case in which $\kappa(2) > \sigma(2)$. Or else, Π^* and Π^+ and their respective derivatives would coincide. If so:

$$\begin{aligned}
\Pi^* &= \left[\frac{\delta\alpha}{1+\delta} + \frac{2-\alpha}{1+\delta}\sigma(2)\right](v-c) \\
\frac{d\Pi^*}{d\delta} &= \frac{1}{(1+\delta)^2}\left[\alpha - (2-\alpha)\sigma(2) + (2-\alpha)(1+\delta)\frac{d\sigma(2)}{d\delta}\right](v-c) < 0 \\
\frac{d\Pi^*}{d\alpha} &= \left[\frac{\delta - \sigma(2)}{1+\delta} + \frac{2-\alpha}{1+\delta}\frac{d\sigma(2)}{d\alpha}\right](v-c) > 0 \\
\frac{d\Pi^*}{dv} &= -\frac{d\Pi^*}{dc} = \left[\frac{\delta\alpha}{1+\delta} + \frac{2-\alpha}{1+\delta}\sigma(2)\right] > 0
\end{aligned}$$

where the first inequality holds since $\mathcal{E} \neq \emptyset$ implies $\alpha - (2-\alpha)\sigma(2) \leq 0$, and the second inequality holds since $\kappa(2) > \sigma(2)$ implies $\delta > \sigma(2)$. At last, consider the case in which

$\Pi \in (0, \Pi^*]$. Suppose that $\Pi^* \neq \Pi^+$. If so, $\kappa(2) > \sigma(2)$ and therefore:

$$\frac{dN(\Pi)}{d\delta} = \frac{1}{(1-\delta)^2}\left[R_0 + (1-\delta)\left[R_{0\delta} + R_{0\sigma}\sigma_\delta\right]\right] > 0 \tag{7}$$

$$\frac{dN(\Pi)}{d\alpha} = \frac{1}{1-\delta}\left[R_{0\alpha} + R_{0\sigma}\sigma_\alpha\right] < 0 \tag{8}$$

$$\frac{dN(\Pi)}{dv} = \frac{dN(\Pi)}{dc} = 0 \tag{9}$$

where (7) is positive since $R_{0\sigma}\sigma_\delta > 0$ and because:

$$R_0 + (1-\delta) R_{0\delta} = \frac{\alpha}{\sigma} + \frac{2\delta}{(1+\delta)^2}\left[(2-\alpha) - \frac{\alpha}{\sigma}\right] > 0$$

where (8) is negative since:

$$\begin{aligned} R_{0\alpha} + R_{0\sigma}\sigma_\alpha &= \frac{R_{1\sigma}R_{0\alpha} - R_{0\sigma}R_{1\alpha}}{R_{1\sigma} - R_{0\sigma}} = \\ &= -\frac{1}{R_{1\sigma} - R_{0\sigma}}\frac{\delta}{\sigma\alpha(1+\delta)^2}\left[\alpha\delta + \sigma(2-\alpha)\right] < 0 \end{aligned}$$

and where (9) holds trivially.

Finally, consider the case in which $\Pi \in (0, \Pi^*]$ and $\Pi^* = \Pi^+$. If so, $\kappa(2) \leq \sigma(2)$ and:

$$\begin{aligned} \frac{dN(\Pi)}{d\delta} &= \frac{1}{(1-\delta^2)^2}\left[2\delta + \frac{2-\alpha}{\alpha}\left[(1+\delta^2)\kappa + (1-\delta^2)\kappa_\delta\right]\right] > 0 \\ \frac{dN(\Pi)}{d\alpha} &= -\frac{\delta}{1-\delta^2}\frac{2\kappa}{\alpha^2} < 0 \\ \frac{dN(\Pi)}{dv} &= \frac{\delta}{1-\delta^2}\frac{2-\alpha}{\alpha}\kappa_v > 0 \quad \& \quad \frac{dN(\Pi)}{dc} = \frac{\delta}{1-\delta^2}\frac{2-\alpha}{\alpha}\kappa_c < 0 \end{aligned}$$

which concludes the proof. ■

Proposition 12 *Assume that $\mathcal{E} \neq \emptyset$. If $\bar{\varkappa}$ is defined and if $R_0(\sigma(\bar{\varkappa}), \bar{\varkappa}) > R_1(\kappa(\breve{\varkappa}), \breve{\varkappa})$ whenever $\breve{\varkappa}$ is also defined, then no strategy in $\mathcal{S}$ is strictly more stable than strategy $\pi^* \in \mathcal{E}$:*

μ^*	σ^*	$\varkappa^*$
$1/\rho$	$\sigma(\bar{\varkappa})$	$\bar{\varkappa}$

otherwise, no strategy in $\mathcal{S}$ is strictly more stable than strategy $\pi^* \in \mathcal{E}$:

μ^*	σ^*	$\varkappa^*$
$1/\rho$	$\kappa(\breve{\varkappa})$	$\breve{\varkappa}$

Moreover, π^* is the most profitable of all the strategies in $\mathcal{S}$ with equal stability.

Proof. To prove the claim it suffices to show that π^* is more stable than any other strategy in $\mathcal{E}$. Consider any other strategy $\pi = (\mu, \sigma, \varkappa) \in \mathcal{S}$. First let us establish that if $\pi \in \mathcal{E}$, then the sale strategy $\pi(\varkappa) = (1/\rho, \min\{\kappa(\varkappa), \sigma(\varkappa)\}, \varkappa)$ also belongs to $\mathcal{E}$ and is more stable than π. Note that:

$$\alpha(\varkappa) \equiv \frac{\alpha_0}{\alpha_S \varkappa + \alpha_0} \leq \delta^{\varkappa-1}\left(1 + \frac{1}{\mu}\right) - \frac{1}{\mu} \leq \kappa(\varkappa)$$

where the first inequality holds by $\pi \in \mathcal{E}$, and the second since $\mu \leq 1/\rho$. Moreover,

$$\min\{\kappa(\varkappa), \sigma(\varkappa)\} \in [\alpha(\varkappa), \kappa(\varkappa)]$$

since $\sigma(\varkappa) \geq \alpha(\varkappa)$ given that $R_1(\sigma, \varkappa) < 1 < R_0(\sigma, \varkappa)$ for any $\sigma < \sigma(\varkappa)$. Thus, $\pi(\varkappa) \in \mathcal{E}$. To prove that $\pi(\varkappa)$ is more stable than π, first note that the markup μ does not affect $\min_{t\geq 0} R_t(\pi)$ and increases $\kappa(\varkappa)$, which in turn implies that setting μ to its upper-bound cannot reduce the stability. Then note that $\min_{t\geq 0} R_t(\sigma, \varkappa)$ is single peaked in $\sigma \in [0, 1]$, since $R_0(\sigma, \varkappa)$ decreases in σ, since $R_1(\sigma, \varkappa)$ increases in σ, and since $R_0(0, \varkappa) > R_1(0, \varkappa)$ (see appendix part B for detail). Also notice that the peak $\min_{t\geq 0} R_t(\sigma, \varkappa)$ with respect to σ is achieved exactly at $\sigma = \sigma(\varkappa)$. Thus, if $\sigma(\varkappa) \leq \kappa(\varkappa)$, no strategy with the same cycle length can be more stable than $\pi(\varkappa)$. If however, $\sigma(\varkappa) > \kappa(\varkappa)$, the most stable strategy must satisfy $\sigma = \sigma(\varkappa)$, since $\min_{t\geq 0} R_t(\sigma, \varkappa)$ increases in σ for $\sigma < \sigma(\varkappa)$.

Next observe that by the implicit function theorem we get that:

$$\sigma'(\varkappa) = -\frac{R_{1\varkappa} - R_{0\varkappa}}{R_{1\sigma} - R_{0\sigma}} \geq 0$$

Note that the denominator is trivially positive (see appendix B), and that the numerator is negative since at $\sigma = \sigma(\varkappa)$:

$$R_{0\varkappa} - R_{1\varkappa} = \frac{1-\delta}{1-\delta^{\varkappa}} \left[\alpha_S \left(1 - \delta^{\varkappa-1} \frac{\sigma}{\alpha_0} \right) + \frac{\delta^{\varkappa-1} \log \delta}{1-\delta^{\varkappa}} \left(\left(\frac{\delta \alpha_0}{\sigma} - 1 \right) \left(\frac{\sigma}{\alpha(\varkappa)} - 1 \right) \right) \right] \geq 0$$

where the first term is positive since $\sigma(\varkappa) \leq \alpha_0/\delta^{\varkappa-1}$, and where the second term is positive since $\sigma(\varkappa) \geq \min\{\alpha(\varkappa), \alpha_0\}$ (see appendix B for details). Hence, since $\kappa'(\varkappa) < 0$, there exists a unique value $\bar{\varkappa}$ such that $\sigma(\bar{\varkappa}) = \kappa(\bar{\varkappa})$.

Let $f(\varkappa) = (\kappa(\varkappa)/\alpha(\varkappa)) - 1$. Note that for a sale strategy to be more stable than a strategy without sales it must be that $f(\varkappa) \geq 0$. Furthermore:

$$\begin{aligned}
\frac{\partial}{\partial \varkappa} R_1(\kappa(\varkappa), \varkappa) &= \frac{\delta^{\varkappa-1} - \delta^{\varkappa}}{1-\delta^{\varkappa}} \left[f'(\varkappa) + \frac{\log \delta}{1-\delta^{\varkappa}} f(\varkappa) \right] \leq \\
&\leq \frac{\delta^{\varkappa-1} - \delta^{\varkappa}}{1-\delta^{\varkappa}} \left[f'(\varkappa) - \frac{1-\delta}{1-\delta^{\varkappa}} f(\varkappa) \right] \leq \\
&\leq \frac{\delta^{\varkappa-1} - \delta^{\varkappa}}{(1-\delta^{\varkappa})\,\varkappa} \left[\varkappa f'(\varkappa) - f(\varkappa) \right] \leq 0
\end{aligned}$$

where the first inequality holds since $\log \delta \leq \delta - 1$, and the second since $1 - \delta^{\varkappa} \leq \varkappa(1-\delta)$. The third inequality instead, holds since:

$$\begin{aligned}
\varkappa f'(\varkappa) - f(\varkappa) &= \frac{1}{\alpha(\varkappa)} \left[\varkappa \kappa'(\varkappa) - \frac{\varkappa \alpha'(\varkappa) \kappa(\varkappa)}{\alpha(\varkappa)} - \kappa(\varkappa) + \alpha(\varkappa) \right] = \\
&= \frac{1}{\alpha(\varkappa)} \left[\log \delta^{\varkappa} (\kappa(\varkappa) + 1/\mu) + \kappa(\varkappa)(1 - \alpha(\varkappa)) - \kappa(\varkappa) + \alpha(\varkappa) \right] = \\
&= \frac{1}{\alpha(\varkappa)} \left[\log \delta^{\varkappa} \left(\kappa(\varkappa) + \frac{1}{\mu} \right) + \alpha(\varkappa)(1 - \kappa(\varkappa)) \right] = \\
&= \frac{1}{\alpha(\varkappa)} \left(1 + \frac{1}{\mu} \right) \left[\log \delta^{\varkappa} \left(\delta^{\varkappa-1} \right) + \alpha(\varkappa) \left(1 - \delta^{\varkappa-1} \right) \right] \leq \\
&\leq \frac{1}{\alpha(\varkappa)} \left(1 + \frac{1}{\mu} \right) \left[(\delta^{\varkappa} - 1)\delta^{\varkappa-1} + \alpha(\varkappa) \left(1 - \delta^{\varkappa-1} \right) \right] \leq \\
&\leq \frac{1}{\alpha(\varkappa)} \left(1 + \frac{1}{\mu} \right) \left[\left(\alpha(\varkappa) - \delta^{\varkappa-1} \right) \left(1 - \delta^{\varkappa-1} \right) \right] < 0
\end{aligned}$$

where the first inequality holds, since $\log \delta^{\varkappa} \leq \delta^{\varkappa} - 1$, where the second holds trivially, and where the last inequality holds since $\kappa(\varkappa) \geq \alpha(\varkappa)$ is equivalent to:

$$\delta^{\varkappa-1} \geq \alpha(\varkappa) + \frac{\mu+1}{\mu}(1-\alpha(\varkappa)) > \alpha(\varkappa)$$

The last few observations together established that if $\sigma(\varkappa) \geq \kappa(\varkappa)$ for some $\varkappa$, then increasing the cycle length would only reduce the stability of the sale strategy $\pi(\varkappa)$. In turn this establishes that setting $\varkappa > \breve{\varkappa}$ cannot improve stability.

Finally, note that, if $\bar{\varkappa}$ exits, no strategy with period $\varkappa < \breve{\varkappa}$ can be more stable than $(1/\rho, \sigma(\bar{\varkappa}), \bar{\varkappa})$ by definition of $\bar{\varkappa}$. Thus, the most stable sale strategy will be either $(1/\rho, \sigma(\bar{\varkappa}), \bar{\varkappa})$ or $(1/\rho, \kappa(\breve{\varkappa}), \breve{\varkappa})$ depending on the relative stability of the two.

The observation about profits follows trivially, since changing σ and $\varkappa$ would necessarily reduce stability by construction of π^* and because $\mu^* = \frac{v}{c} - 1$ raises the highest profit and cannot lower stability. ■

Proposition 13 *If $\mathcal{E} \neq \emptyset$, no strategy in $\mathcal{E}$ is strictly more profitable than strategy $\pi^+ \in \mathcal{E}$:*

μ^+	σ^+	$\varkappa^+$
$v/c-1$	$\kappa(2)$	2

Proof. By the properties of the time zero profit function discussed in appendix B, profits at time 0 increase in μ, σ, and $\varkappa$. Thus, the most profitable strategy in $\mathcal{E}$ with a cycle of length $\varkappa$ must trivially satisfy $\mu = \frac{v}{c} - 1$ and $\sigma = \kappa(\varkappa)$, since $\delta^{\varkappa-1}\left(1+\frac{1}{\mu}\right) - \frac{1}{\mu}$ increases in μ. Thus, (2) follows immediately since $\varkappa$ is chosen by definition so to maximize profits in $\mathcal{E}$ and since:

$$\begin{aligned}
\frac{\partial \Pi_0(\mu, \kappa(\varkappa), \varkappa)}{\partial \varkappa} &= \alpha_0 \frac{c(1-\delta)^2\delta^{\varkappa-1}(1+\mu)}{(1-\delta^{\varkappa})^2}\log\delta + \\
&\quad \alpha_S \frac{c(1-\delta)}{(1-\delta^{\varkappa})^2}\left[(1-\delta^{\varkappa})(\delta^{\varkappa-1}(1+\mu)-1) + \delta^{\varkappa-1}((1+\mu)-\delta)\log\delta^{x}\right] < \\
&< \alpha_S \frac{c(1-\delta)}{(1-\delta^{\varkappa})^2}\left[(\delta^{\varkappa-1}(1+\mu)-1)(1-\delta^{\varkappa}) + \delta^{\varkappa-1}((1+\mu)-\delta)\log\delta^{x}\right] \leq \\
&\leq \alpha_S \frac{c(1-\delta)}{(1-\delta^{\varkappa})^2}\left[(\delta^{\varkappa-1}(1+\mu)-1) - \delta^{\varkappa-1}((1+\mu)-\delta)\right](1-\delta^{\varkappa}) = \\
&= -\alpha_S \frac{c(1-\delta)^2}{(1-\delta^{\varkappa})} < 0
\end{aligned}$$

where the second inequality holds $\log \delta^x \leq \delta^x - 1$ and the rest is simple algebra. ■

Proposition 14 *The size of the set $\mathcal{E}$ decreases with c and α, and increases with v and δ.*

Proof. First note that by corollary 9 the sufficient condition for the existence of a sale strategy with period $\varkappa$ requires:

$$\delta^{\varkappa-1} \geq \frac{1}{1+\rho}\frac{\alpha}{\varkappa-\alpha(\varkappa-1)} + \frac{\rho}{1+\rho} = h(\alpha,\rho) \tag{10}$$

Further notice that such condition is harder to satisfy when either ρ or α increase since:

$$\begin{aligned}\frac{dh(\alpha,\rho)}{d\alpha} &= \frac{1}{1+\rho}\frac{\varkappa}{(\varkappa-\alpha(\varkappa-1))^2} > 0 \\ \frac{dh(\alpha,\rho)}{d\rho} &= \frac{1}{(1+\rho)^2}\left[\frac{(1-\alpha)\varkappa}{\varkappa-\alpha(\varkappa-1)}\right] > 0\end{aligned}$$

Thus, the size of the set $\mathcal{E}$ decreases with both ρ and α. To establish the comparative statics on c and v, simply note that $d\rho/dv < 0$ and that $d\rho/dc > 0$. The final observation on δ is trivial the left hand side of (10) increases in δ. ■

Part B: Derivatives and Signs

Recall that for any strategy $\pi \in \mathcal{C}$ equilibrium and deviation profits in the two critical periods respectively satisfy:

$$\begin{aligned}\Pi_0(\pi) &= \left[\alpha_0 + \frac{1-\delta}{1-\delta^x}\left[\sigma(\varkappa\alpha_S + \alpha_0) - \alpha_0\right]\right]\mu c \\ \Pi_1(\pi) &= \left[\alpha_0 + \frac{\delta^{\varkappa-1}-\delta^{\varkappa}}{1-\delta^{\varkappa}}\left[\sigma(\varkappa\alpha_S + \alpha_0) - \alpha_0\right]\right]\mu c \\ \Delta_0(\pi) &= \sigma\mu c \\ \Delta_1(\pi) &= \alpha_0\mu c\end{aligned}$$

Derivatives at $t = 0$:

$$\begin{aligned}
\frac{d\Pi_0(\pi)}{d\mu} &= \left[\alpha_0 + \frac{1-\delta}{1-\delta^x}\left[\sigma(\varkappa\alpha_S + \alpha_0) - \alpha_0\right]\right] c > 0 \\
\frac{d\Pi_0(\pi)}{d\sigma} &= \frac{1-\delta}{1-\delta^\varkappa}(\varkappa\alpha_S + \alpha_0)\mu c > 0 \\
\frac{d\Pi_0(\pi)}{d\varkappa} &= \left[\frac{1-\delta}{1-\delta^\varkappa}\sigma\alpha_S + \log\delta\frac{\delta^\varkappa - \delta^{\varkappa+1}}{(1-\delta^\varkappa)^2}\left[\sigma(\varkappa\alpha_S + \alpha_0) - \alpha_0\right]\right]\mu c > 0 \\
\frac{d\Delta_0(\pi)}{d\mu} &= \sigma c \geq 0 \quad \& \quad \frac{d\Delta_0(\pi)}{d\sigma} = \mu c > 0 \quad \& \quad \frac{d\Delta_0(\pi)}{d\varkappa} = 0
\end{aligned}$$

To sign $d\Pi_0(\pi)/d\varkappa$ consider harder case, namely $\alpha_S = 1$. If so:

$$\frac{d\Pi_0(\pi)}{d\varkappa} = \frac{1-\delta}{1-\delta^\varkappa}\left[1 + \varkappa\log\delta\frac{\delta^\varkappa}{1-\delta^\varkappa}\right]\sigma\mu c \geq 0$$

which is positive, since:

$$x\log\delta = \log\delta^x \geq 1 - \frac{1}{\delta^\varkappa}$$

Similarly, derivatives at $t = 1$, satisfy:

$$\begin{aligned}
\frac{d\Pi_1(\pi)}{d\mu} &= \left[\alpha_0 + \frac{\delta^{\varkappa-1} - \delta^\varkappa}{1-\delta^\varkappa}\left[\sigma(\varkappa\alpha_S + \alpha_0) - \alpha_0\right]\right] c > 0 \\
\frac{d\Pi_1(\pi)}{d\sigma} &= \frac{\delta^{\varkappa-1} - \delta^\varkappa}{1-\delta^\varkappa}(\varkappa\alpha_S + \alpha_0)\mu c > 0 \\
\frac{d\Pi_1(\pi)}{d\varkappa} &= \left[\frac{\delta^{\varkappa-1} - \delta^\varkappa}{1-\delta^\varkappa}\sigma\alpha_S + \log\delta\frac{\delta^{\varkappa-1} - \delta^\varkappa}{(1-\delta^\varkappa)^2}\left[\sigma(\varkappa\alpha_S + \alpha_0) - \alpha_0\right]\right]\mu c \\
\frac{d\Delta_1(\pi)}{d\mu} &= \alpha_0 c \quad \& \quad \frac{d\Delta_1(\pi)}{d\sigma} = \frac{d\Delta_1(\pi)}{d\varkappa} = 0
\end{aligned}$$

Again, to sign $d\Pi_1(\pi)/d\varkappa$ consider harder case, namely $\alpha_S = 1$. If so:

$$\frac{d\Pi_1(\pi)}{dx} = \frac{\delta^{\varkappa-1} - \delta^\varkappa}{1-\delta^\varkappa}\left[1 + \varkappa\log\delta\frac{1}{1-\delta^\varkappa}\right]\sigma\mu c \leq 0$$

which is negative, since:

$$\log\delta^\varkappa \leq \delta^\varkappa - 1$$

Moreover $d\Pi_1(\pi)/d\varkappa > 0$, when $\alpha_0 = 1$. Thus, the sign of $d\Pi_1(\pi)/d\varkappa$ depends on the fraction of consumers with storage in the economy. Notice that the resulting critical ratios are independent of μ:

$$\begin{aligned} R_0(\pi) &= \frac{\alpha_0}{\sigma} + \frac{1-\delta}{1-\delta^x}\left[(\varkappa\alpha_S + \alpha_0) - \frac{\alpha_0}{\sigma}\right] \\ R_1(\pi) &= 1 + \frac{\delta^{\varkappa-1} - \delta^{\varkappa}}{1-\delta^{\varkappa}}\left[(\varkappa\alpha_S + \alpha_0)\frac{\sigma}{\alpha_0} - 1\right] \end{aligned}$$

Derivatives at $t = 0$:

$$\begin{aligned} \frac{dR_0(\pi)}{d\sigma} &= -\left[\frac{\delta - \delta^x}{1-\delta^x}\right]\frac{\alpha_0}{\sigma^2} < 0 \\ \frac{dR_1(\pi)}{d\sigma} &= \left[\frac{\delta^{\varkappa-1} - \delta^{\varkappa}}{1-\delta^{\varkappa}}\right]\left[\frac{\varkappa\alpha_S + \alpha_0}{\alpha_0}\right] > 0 \\ \frac{dR_0(\pi)}{d\varkappa} &= \frac{1-\delta}{1-\delta^{\varkappa}}\alpha_S + \log\delta\frac{\delta^{\varkappa} - \delta^{\varkappa+1}}{(1-\delta^{\varkappa})^2}\left[\varkappa\alpha_S + \alpha_0 - \frac{\alpha_0}{\sigma}\right] \geq 0 \\ \frac{dR_1(\pi)}{d\varkappa} &= \frac{\delta^{\varkappa-1} - \delta^{\varkappa}}{1-\delta^{\varkappa}}\frac{\alpha_S}{\alpha_0}\sigma + \log\delta\frac{\delta^{\varkappa-1} - \delta^{\varkappa}}{(1-\delta^{\varkappa})^2}\left[\frac{\varkappa\alpha_S + \alpha_0}{\alpha_0}\sigma - 1\right] \end{aligned}$$

where the sign of $dR_1(\pi)/d\varkappa$ coincides with that of $d\Pi_1(\pi)/d\varkappa$.

Notice that $R_1(\pi) - R_0(\pi) = 0$ requires:

$$\sigma^2\left[\delta^{x-1}\frac{\alpha_S\varkappa + \alpha_0}{\alpha_0}\right] + \sigma\left[\frac{1-\delta^{\varkappa-1}}{1-\delta} - [\alpha_S\varkappa + \alpha_0]\right] - \alpha_0\left[\frac{\delta - \delta^{\varkappa}}{1-\delta}\right] = 0$$

Such condition always has unique positive solution which satisfies $\sigma^\circ \in [\alpha_0, \alpha_0/\delta^{\varkappa-1}]$, since it is negative both at $\sigma = 0$ and at $\sigma = \alpha_0$, and positive at $\sigma = \alpha_0/\delta^{\varkappa-1}$. However, the solution could in principle require $\sigma^\circ > 1$. If so, the solution to the general program $\max_{\sigma\in[0,1]} \min_t R_t(\pi) = R_1(\pi)$ will satisfy $\sigma(\varkappa) = \min\{1, \sigma^\circ\}$.