Time Preferences and Bargaining*

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Abstract

This paper presents an analysis of general time preferences in the canonical Rubinstein (1982) model of bilateral alternating-offers bargaining. I derive a simple sufficient structure for optimal punishments and thereby fully characterize (i) the set of equilibrium outcomes for any given preference profile, and (ii) the set of preference profiles for which equilibrium is unique. When both players have a present bias—empirically, a property of most time preferences regarding consumption, and implied, e.g., by any hyperbolic or quasi-hyperbolic discounting—equilibrium is unique, stationary and efficient. When, instead, one player finds a near-future delay more costly than delay from the present—empirically common for time preferences over money—non-stationary equilibria arise that explain inefficiently delayed agreement with gradually increasing offers.

Keywords: alternating offers, time preference, impatience, discounting, dynamic inconsistency, delay, optimal punishment, simple penal codes, non-stationary equilibrium

JEL classification: C78, D03, D74

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1 Introduction

As a mechanism for sharing economic surplus, bargaining is pervasive in decentralized exchange. Understanding this mechanism's functioning, in particular its efficiency properties, is of fundamental importance for any applied work incorporating search frictions (see, e.g., Browning and Chiappori (1998) on household behavior, or Hall and Milgrom (2008) on labor markets) or studying optimal institutional design, when a central authority might impose an allocation instead of leaving it up to decentralized bargaining (e.g., a manager allocating tasks to a team of employees, or a government regulating industry standards).

In the absence of irrevocable commitments, time is the prime variable of bargaining agreements: the parties may agree not only now or never, but also sooner or later. The question of how the parties' attitudes to delay govern their bargaining lies at the heart and the beginning of modern bargaining theory (Ståhl, 1972; Rubinstein, 1982). This paper is the first to provide a general answer to this question, covering also various attitudes to delay other than exponential discounting (ED) while allowing the parties' bargaining strategies to be arbitrarily history-dependent.

Specifically, I fully characterize bilateral alternating-offers bargaining (without a deadline) when each party i evaluates delayed agreements with a continuous utility function $U_i(x_i, t)$, assuming only that she always prefers a greater share x_i of the surplus, holding delay t constant, and a shorter delay t, holding her share $x_i > 0$ constant. With the sole exception of ED, all such preferences are dynamically inconsistent. Hence this paper demonstrates how various forms of dynamic inconsistency are analytically tractable, it sheds light on the robustness of the celebrated conclusions obtained under ED, and it brings the vast body of empirical research on time preferences to bear directly on the study of bargaining.[1]

Dynamically inconsistent preferences require a new analytical approach to this game. As I show, the standard technique of characterizing equilibrium via recursions on the players' equilibrium payoff/utility extrema (see Shaked and Sutton, 1984) generally fails in face of the possibility of multiple and delayed equilibrium agreements because a player may not rank these consistently across different points in time.[2]

I circumvent this problem by directly analyzing the off-path punishments (continuation

[1]The notion of time preferences investigated here also covers bargaining costs beyond general impatience; e.g., $U_i(x_i, t) = x_i - c_i(t)$ means that obtaining no surplus at all is worse under delayed than immediate agreement. This introductory discussion also focuses on preference profiles implying a unique *stationary* equilibrium, which follows from standard concavity assumptions on U_i concerning the surplus share.

[2]Prior work on bargaining with dynamically inconsistent preferences (Burgos, Grant, and Kajii, 2002a; Akin, 2007; Ok and Masatlioglu, 2007; Noor, 2011) has either directly assumed stationary strategies or nonetheless relied on this technique, in any case ruling out preference reversals and characterizing only stationary equilibrium. See, however, the very recent work by Lu (2015) where quasi-hyperbolic discounters bargain over an infinite stream of payoffs.

equilibria) that support equilibrium play, i.e. *optimal penal codes* (cf. Abreu, 1988). I show that it is sufficient to consider *simple* penal codes described by four outcomes.[3] Each of these defines a most severe credible punishment play for *any* deviation by a particular player in a given role, entirely independent of its history; e.g., any deviant proposal by player 1 triggers the exact same continuation equilibrium (upon rejection). Loosely speaking, the strategic complexity becomes twice as important when we allow for dynamic inconsistency: in general, four punishment outcomes are necessary for optimal punishment, but under ED two of them are redundant. This fundamental insight renders the history-dependence of strategies analytically tractable in a unified manner and thus enables me to obtain the paper's core results: a full characterization of both (i) equilibrium outcomes for any given preference profile, and (ii) those preference profiles that imply a unique equilibrium.

Viewed through the lens of the existing evidence on time preferences, this characterization yields the novel prediction that the bargaining mechanism's functioning depends on whether parties share consumption (e.g., a literal cake, effort provision, social esteem) or money. The reason is that people's time preferences differ systematically across these domains, in ways which imply different bargaining equilibria.

Regarding consumption, a rather general *present bias*, where delay is most costly when it takes consumption away from the immediate present, is well-established (e.g., McClure, Ericson, Laibson, Loewenstein, and Cohen, 2007; Brown, Chua, and Camerer, 2009; Augenblick, Niederle, and Sprenger, 2014). When both parties' preferences exhibit this type of bias, the bargaining equilibrium is unique, stationary and efficient. This result proves that the sharp conclusions under ED are robust to various forms of present bias—in particular any hyperbolic (Chung and Herrnstein, 1967; Ainslie, 1975) or quasi-hyperbolic (Phelps and Pollak, 1968; Laibson, 1997) discounting—and finally opens the door to the use of non-cooperative bargaining in applied economic modeling studying such preferences.

Regarding money, a careful examination of the existing experimental evidence (see appendix B.1) reveals that, across studies, at least around a third of individuals display time preferences exhibiting however a *(near-) future bias*.[4] This means that they are most impatient about delay beyond some time in the near future rather than the immediate present;

[3] Mailath, Nocke, and White (2015) present related examples of repeated *sequential* games where no simple penal code is optimal due to incentive trade-offs between within-round and continuation punishment. Here, instead, a single round determines all payoffs.

[4] The terminology for this finding is not uniform: e.g., it has also been called "reverse time-inconsistency" (Sayman and Öncüler, 2009), "increasing impatience" (Attema, Bleichrodt, Rohde, and Wakker, 2010), "hypobolic discounting" (Eil, 2012), or "patient shifts" (Read, Frederick, and Airoldi, 2012). Moreover, due to the focus on (quasi-) hyperbolic discounting in the literature it is often not even made explicit: e.g., Andreoni and Sprenger (2012) estimate a median "beta" greater than one, i.e. the majority exhibits a near-future bias; however, they concentrate on the "absence of present bias".

e.g., near-future bias would be implied under a discounting function which is initially concave (e.g., Ebert and Prelec, 2007). When (at least) one of the parties has a sufficiently strong bias of this type, bargaining has multiple non-stationary equilibria, which necessarily involve inefficient delay. Moreover, as the frequency of offers increases, an arbitrarily small bias becomes sufficient for this result; hence, the bargaining conclusions under ED are not robust to perturbations of this type.

Intuitively, a near-future biased individual does not mind bargaining for a few periods; further, future delay is, however, costly. In contrast to present bias, she therefore does not exert control over the delay she finds most painful. Indeed, after having bargained for a few periods, she will again be willing to bargain for a few more, in return for only a slightly larger share. Given there would subsequently be further delay, her opponent is able to extract a premium for immediate agreement, because that avoids handing over control to her "excessively" patient future selves, and this premium in turn supports the delay. Thus delay is self-enforcing.[5] Although the familiar proposer advantage exerts a countervailing force, it only reduces the premium that the opponent might credibly ask for and becomes negligible when offers are made frequently.

The non-stationary delay equilibria capture elementary strategic considerations. A party does not propose a Pareto-improving division early based on the belief that she would thus lead her opponent to expect an even superior—but not itself Pareto-improving—outcome and, consequently, reject it. To the extent that the set of Pareto-improving divisions shrinks as the parties approach the time of agreement, due to their impatience, those equilibria permit rather rich dynamics such as "gradualism" (see, e.g., Compte and Jehiel, 2004): proposers' offers and respondents' minimally conceded shares increase gradually towards those of the eventually agreed division. Moreover, when the two parties' preferences are not too "asymmetric", the set of non-stationary equilibrium outcomes includes the focal point of equal division (as an immediate agreement).

The multiplicity of equilibrium outcomes for given preferences in this case means that the theory predicts bounds—on agreed divisions as a function of delay—not points. This issue is well-known in the bargaining literature. It arises, *a fortiori*, within the hitherto most successful explanation of inefficient delay through incomplete information (assuming ED), which creates a trade-off between information and time (see, e.g., the survey by Kennan and Wilson, 1993); also, experimental studies of bargaining (over money) have observed both delayed and immediate agreements in objectively identical bargaining situations (see,

[5]This nature of delay is novel, and it means there exist "truly" non-stationary delay equilibria which are non-stationary in *every* subgame; prior constructions in related work rely instead on play of stationary equilibrium off the path (see Avery and Zemsky, 1994).

e.g., the survey by Roth, 1995). Notwithstanding the realism of incomplete information, the model proposed here shows it is not essential to explaining such evidence. It yields similar qualitative implications in terms of observable behavior but has important methodological advantages.[6] First, it provides an explanation that is not only more fundamental (there is no theory of delay without time preferences) but also has an independent empirical foundation. Second, the discipline on beliefs due to perfect information makes it more readily applicable: equilibrium is fully characterized, and experimental researchers are likely to have better control over participants' preferences than their beliefs about others.[7]

The paper is structured as follows: I conclude this introduction with a simple example illustrating how near-future bias invites non-stationary delay equilibria; it is later extended to exhibit also "truly" non-stationary equilibria and long delays (example 3). Section 2 presents the formal model. I fully characterize its equilibrium in section 3 and further investigate equilibrium uniqueness and multiplicity/delay in section 4. Section 5 has concluding remarks. An appendix contains all formal proofs as well as other supplementary material.

Example 1. Consider two parties, Od (player 1) and Eve (player 2), who bargain over how to "split a dollar" and have preferences $U_i(x_i, t) = d_i(t) \cdot x_i$, $i \in \{1, 2\}$, such that $d_i(1) = \delta$ and $d_1(2) = \gamma\delta^2$ for $0 < \delta, \gamma < 1$.[8] Since $\gamma < 1$, Od discounts the second period of delay more heavily than the first, i.e. he has a near-future bias.

Suppose if the two parties fail to agree now, they will reach agreement only after two more rounds. Od is then currently willing to concede a fraction $1 - \gamma\delta$ of her share to reduce delay from two periods to one, but next round—at the time when they would have to agree and the decision is between one period of delay or none—this fraction will be only $1 - \delta$. This wedge means there are divisions that Od would currently want himself to accept in the next round but will subsequently find unacceptable. Eve can extract a "premium" from "current Od" that "next-round's Od" is not wiling to pay; thus delay can be self-enforcing.

Figure 1 explicitly describes equilibrium strategies for a once delayed agreement on division y when Od makes the first offer. The above observation is directly reflected in the off-equilibrium threat for round 2 (bottom), which supports Od's making an unacceptable offer: as the proposer, Eve can obtain share $y_2' \equiv 1 - \gamma\delta^2 y_1$ in an immediate agreement

[6] This comparison is valid only for bargaining without a deadline. With a finite horizon and perfect information, any form of impatience results in a unique backwards induction solution (fully determined by the parties' attitudes to a single period of delay only), with immediate agreement.

[7] These arguments apply as well to recent related approaches based on players' holding incorrect beliefs about their opponent: under "optimism" (see the survey by Yildiz, 2011) they incorrectly believe that they have better knowledge of their proposer advantage, and under "strategic uncertainty" (Friedenberg, 2014) they incorrectly interpret the opponent's deviations as irrationality. The two approaches appear well-suited, however, to explain the effects introduced by deadlines.

[8] Observe that this would be implied if player 1 were a (β, δ')-discounter with $\beta = \frac{1}{\gamma}$ and $\delta' = \gamma\delta$.

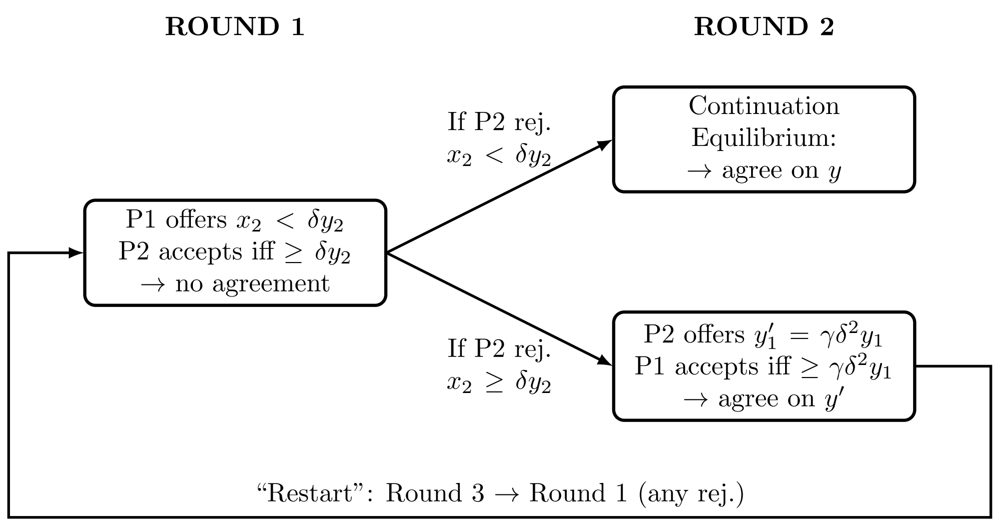

Figure 1: Delay equilibrium in example 1, assuming an equilibrium outcome y with $\frac{1-\delta}{1-\gamma\delta^2} < \delta y_1$ for the subgame beginning with player 2's proposal.

based on subsequent (round-3) delay and may therefore credibly ask for share $\delta y_2'$ as the initial (round-1) respondent. Whenever $1 - \delta y_2' < \delta y_1$, this is indeed more than Od is willing to concede as the proposer who could alternatively obtain y_1 with one period of delay by making an unacceptable offer.

Values of γ for this to hold true exist for any division y with $y_1 \geq \frac{1-\delta}{\delta}$; as $\delta \to 1$, this means *any* division, and in particular that under the unique (symmetric) stationary equilibrium of this game, which has immediate agreement with the relevant (respondent) share equal to $\frac{\delta}{1+\delta}$. In fact, regardless of how small Od's bias is (γ close to one), the strategies then form an equilibrium for sufficiently frequent offers ($\delta < 1$ large enough).

2 Bargaining and Time Preferences

I follow Rubinstein (1982) exactly with regards to the bargaining protocol of (possibly indefinitely) alternating offers and therefore describe this part of the model only informally. My focus is on the generalization of preferences and the equilibrium concept investigated here.

2.1 Bargaining Protocol, Histories and Strategies

There are two players $\{1, 2\} \equiv I$ who bargain over a perfectly divisible surplus of (normalized) size one. Throughout the paper, whenever $i \in I$ denotes one player, $j \equiv 3 - i$ denotes the other. In each round $n \in \mathbb{N}$, player $P(n)$ proposes a surplus division $x \in \left\{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 + x_2 = 1\right\} \equiv X$ to her opponent $R(n)$ (equivalently, $P(n)$ offers $R(n)$

share $x_{R(n)}$) who then responds by either accepting or rejecting the proposal. If it is accepted, the game ends with agreement on x; otherwise, one period of time elapses until the subsequent round $n+1$ takes place, where the roles of proposer and respondent are reversed, i.e. $P(n+1) = R(n)$. This process of alternating offers begins with player 1's proposal—so $P(n) = 1$ if n is odd, and $P(n) = 2$ if n is even—and continues until there is agreement, possibly without ever terminating.

A history of play to the beginning of round $n \in \mathbb{N}$ is a sequence of $n-1$ rejected proposals $h^{n-1} \in X^{n-1}$, where $X^0 \equiv \{\emptyset\}$; in what follows, "history" always refers to such a beginning-of-round history. A strategy σ_i of a player i assigns to every possible such history h^{n-1} an available action: if $i = P(n)$, then $\sigma_i(h^{n-1})$ specifies a proposal $x \in X$, and if $i = R(n)$, then it specifies for every possible proposal whether she accepts or rejects it; $\sigma_{R(n)}(h^{n-1})$ is therefore a response rule which partitions X into a subset of accepted proposals $Y \in \mathcal{P}(X)$ and its complement $X \backslash Y$ of rejected proposals, where I will notationally identify $\sigma_{R(n)}(h^{n-1})$ with Y. A strategy profile σ is a pair of strategies $\left(\sigma_{P(1)}, \sigma_{R(1)}\right)$; for any particular strategy profile σ and history h^{n-1}, $\sigma(h^{n-1}) \equiv \left(\sigma_{P(n)}(h^{n-1}), \sigma_{R(n)}(h^{n-1})\right)$.

A particularly simple type of strategy is one which specifies the same proposal and response rule irrespective of the history of play: if σ_i is such a *stationary strategy*, it is then characterized simply by a pair $\left(x^{(i)}, Y^{(i)}\right) \in X \times \mathcal{P}(X)$, where $x^{(i)}$ is the division that i proposes after any round-n history such that $i = P(n)$, and $Y^{(i)}$ is i's response rule after any round-n history such that $i = R(n)$.

The proposal $x \in X$ of a player i where she offers j the smallest possible share of zero is denoted by $e^{(i)}$.[9] When a player i uses response rule $\{x \in X \,| x_i \geq q\}$, $q \in [0,1]$, I say that i *accepts with threshold* q and denote it by $X_{i,q}$.

If play according to a strategy profile σ results in agreement on division x in round $n \in \mathbb{N} \cup \{\infty\}$, where $n = \infty$ means perpetual disagreement, then, for any $m \leq n$, I let $h^{m-1}(\sigma)$ denote the round-m history it induces, so its induced path is given by $(h^{n-1}(\sigma), x) \in X^n$. Moreover, I define its *play* to be the sequence $\langle\sigma\rangle \equiv \left(\sigma\left(h^{m-1}(\sigma)\right)\right)_{m=1}^{n} \in (X \times \mathcal{P}(X))^n$, where note that $\sigma_{P(m)}(h^{m-1}(\sigma)) \in \sigma_{R(m)}(h^{m-1}(\sigma)) \Leftrightarrow m = n$.[10]

2.2 Outcomes and (Time) Preferences

Players are assumed to care only about the division of the surplus and the delay of an eventual agreement ("consequentialism").[11] I use the term *outcome* for equivalence classes

[9]For certain preferences, the constraint that proposals can only involve non-negative shares requires justification by an outside option of walking away with a zero share (cf. Binmore, 1987, p. 89).

[10]In comparison to a strategy profile's induced path, its play explicitly contains entire response *rules*.

[11]For a model where instead preferences depend also on *how* an agreement is reached, see Li (2007).

of paths given either by some $(x,t) \in X \times T \equiv A$, $T \equiv \mathbb{N}_0$, meaning agreement on division x is reached with a *delay* of t periods, or by D, perpetual disagreement. After any history, in terms of relative time (delay), the set of possible outcomes $A \cup \{D\}$ is therefore identical.

It is notationally convenient to use as the domain of a player i's preferences the set of i's *personal outcomes* $A_i \cup \{(0,\infty)\}$, where $A_i \equiv [0,1] \times T$ is obtained from A after projecting any division x to i's own surplus share $q = x_i \in [0,1]$ (due to the fact that $x_1 + x_2 = 1$, this is without loss of generality), and $(0,\infty)$ is the personal outcome corresponding to perpetual disagreement. The focus of this paper is on the players' time preferences over this domain, under the following assumption.

Assumption 1. *In any round n, player i's preferences over personal outcomes $A_i \cup \{(0,\infty)\}$ are represented by the same utility function $U_i : A_i \cup \{(0,\infty)\} \to \mathbb{R}$ satisfying the following properties, where $(q,q') \in [0,1]^2$ and $(t,t') \in T^2$:*

1. *Disagreement is worst: $U_i(a) \geq U_i(0,\infty)$ for all $a \in A_i$;*
2. *Continuity: $\{a \in A_i | U_i(a) \geq k\}$ and $\{a \in A_i | U_i(a) \leq k\}$ are closed for all $k \in \mathbb{R}$;*[12]
3. *Desirability: $q > q'$ implies $U_i(q,t) > U_i(q',t)$ for all $t \in T$;*
4. *Impatience:*
   (a) *$t > t'$ implies $U_i(q,t) \leq U_i(q,t')$ for all $q \in [0,1]$,*
   (b) *$q > 0$ implies $U_i(q,0) > U_i(q,1)$, and*
   (c) *$\lim_{t\to\infty} U_i(1,t) > U_i(0,0)$ implies there exists $\hat{t} \in T$ such that $\lim_{t\to\infty} U_i(q,t) = U_i\left(q,\hat{t}\right)$ for all $q \in [0,1]$.*

This assumption covers all models of time preferences with impatience put forward in the literature (see Manzini and Mariotti, 2009, and footnote 13). It generalizes the most widely studied class of separable time preferences axiomatized by Fishburn and Rubinstein (1982, thm. 1), where $U_i(q,t) = d(t) \cdot u(q)$ and $d(\cdot)$ is a decreasing "discounting" function to also cover non-separable time preferences such as those proposed by Benhabib, Bisin, and Schotter (2010) or Noor (2011).[13] An *instantaneous-utility function* $u_i : [0,1] \to \mathbb{R}$ can nonetheless be defined by $u_i(q) \equiv U_i(q,0)$ for any $q \in [0,1]$, and it is continuous and increasing by the preferences' continuity and desirability properties, respectively.

[12]Closedness refers to the product topology on $A_i = [0,1] \times T$, where $[0,1]$ and T are endowed with the relative standard and discrete topologies, respectively.

[13]Ok and Masatlioglu (2007) propose a theory of *relative* discounting which relaxes transitivity for comparisons across different delays. Their representation involves an increasing continuous function $u : [0,1] \to \mathbb{R}_+$ and a continuous function $\eta : T^2 \to [0,1]$ with $\eta(\cdot,t)$ decreasing and $\eta(t,t') = \frac{1}{\eta(t',t)}$, where a reward q with

Property (1) extends players' consequentialism to (perpetual) disagreement, continuity (2) is a standard technical assumption, and desirability (3) defines the conflict of interest in the bargaining problem.[14] Property (4) corresponds to a general notion of impatience regarding agreement: for any given division of the surplus, players do not prefer later over sooner agreement (4.a), if a division yields them a positive share they prefer immediate agreement over delayed agreement (4.b), and either they become "overwhelmingly" impatient for delay approaching infinity (the standard case guaranteeing "continuity at infinity"), or they are impatient only regarding a finite horizon (4.c). In what follows, by "impatience", I refer only to the two properties (4.ab). The role of property (4.c) is technical; it is to guarantee existence of a "worst" equilibrium, and I point out explicitly where it is used.

Assumption 1 allows for impatience also regarding a zero share, i.e. $u_i(0) > U_i(0,t)$ for $t > 0$; e.g., consider $U_i(q,t) = q - c(t)$ for $c(\cdot)$ increasing from $c(0) = 0$. Such preferences reflect a cost of bargaining beyond—or in addition to—general impatience about when to receive a reward; in this case party i would quit bargaining altogether if she expected it to last for some time only to eventually result in a very low payoff.

Implicit in assumption 1 is that the parties' time preferences, i.e. their preferences over *delayed* agreements, do not depend on (calendar) time; otherwise, the bargaining problem itself would change over time (e.g., Coles and Muthoo, 2003). When offers are made reasonably frequently, however—the case emphasized in this literature (e.g., Binmore et al., 1986)—such exogenous effects of time on the parties' preferences (such as "ageing") appear less important than their stable component.

Halevy (2015, prop. 4) shows that any of the preferences studied here are dynamically consistent if and only if they satisfy the stationarity axiom. The latter requires that the preference over two delayed rewards (q,t) and (q',t') depend only on their relative delay: $U_i(q,t) \geq U_i(q',t')$ if and only if $U_i(q,t+\tau) \geq U_i(q',t'+\tau)$ for any $\tau \in T$. Additionally imposing stationarity here would yield ED, where $U_i(q,t) = \delta^t \cdot u(q)$ for some $\delta \in (0,1)$ and

delay t is weakly preferred to a reward q' with delay t' if and only if

$$u(q) \geq \eta(t,t')\, u(q').$$

Assumption 1 imposes transitivity, yet the only strategically relevant *inter*-temporal comparisons are of the form $(q,0)$ versus (q',t). Upon setting $\eta(0,t) \equiv d(t)$, relative discounting can be treated here as mere (absolute) discounting, and results apply to any preferences that fall under their theory; e.g., sub-additive discounting (Read, 2001) and similarity-based choice (Rubinstein, 2003).

[14] Absent separability, desirability cannot be formulated entirely independent of the time dimension; specifically, (3) rules out that a player be entirely indifferent regarding her share once delay is "too long". A slight generalization can accommodate such preferences as well, however, without a single change in the results or proofs presented: replace property (3) with "for any $t \in T$, either U_i is constant on $[0,1] \times \{t' \in T | t' \geq t\}$ or $q > q'$ implies $U_i(q,t) > U_i(q',t)$." Property (4.b) then ensures that U_i is non-degenerate, so $U_i(q,0) > U_i(q',0)$. The expositional focus being on impatience, I chose the simpler formulation (3).

some continuous increasing function u (Fishburn and Rubinstein, 1982, thm. 2). With the exception of ED, all time preferences studied here are therefore dynamically inconsistent.

In a nutshell, the central premise of this bargaining model is that bargaining parties are impatient and consequentialist about reaching agreement, where the form of their impatience is stable over time. Impatience about enjoying the fruits of agreement can plausibly be expected in almost any bargaining situation; given a fixed surplus, the assumptions of their preferences' temporal stability and consequentialism allow to focus on this aspect.

2.3 Equilibrium Concept

I assume that the players' preferences are common knowledge. In the terminology proposed by O'Donoghue and Rabin (1999), players are then fully "sophisticated" about their own as well as their opponent's dynamic inconsistency. The equilibrium concept has to incorporate how the intertemporal conflict within a player's preferences is resolved. In single-person decision problems, the standard solution concept for such sophisticated decision makers is that of Strotz-Pollak equilibrium (Strotz, 1956; Pollak, 1968), also known as multiple-selves equilibrium (Piccione and Rubinstein, 1997); it is the subgame-perfect Nash equilibrium (SPNE) of an auxiliary game in which the decision-maker at any point in time is a distinct non-cooperative player. Technically, therefore, one then looks for strategy profiles which are robust to "one-stage deviations", and this formalizes the presumption that a decision-maker cannot "internally" commit to future behavior.

The equilibrium notion employed here is the natural extension of this concept to this game (cf. Chade, Prokopovych, and Smith, 2008). To facilitate its definition, let $z_i^{h^{n-1}}(x, Y|\sigma)$ denote the personal outcome of player i which obtains if, in round n, following history h^{n-1}, $P(n)$ proposes x, $R(n)$ responds using response rule Y, and in case there is no agreement, i.e. $x \notin Y$, both players subsequently adhere to strategy profile σ; for instance, if $\sigma_{P(n+1)}(h^{n-1}, x) = x' \in \sigma_{R(n+1)}(h^{n-1}, x)$, then $z_i^{h^{n-1}}(x, Y|\sigma)$ equals $(x_i, 0)$ whenever $x \in Y$, and $(x'_i, 1)$ otherwise; accordingly, $z_i^{h^{n-1},x}(\sigma(h^{n-1}, x)|\,\sigma) = (x'_i, 0)$.

Definition 1. A strategy profile σ is a **multiple-selves equilibrium ("equilibrium")** if, for any round n, history h^{n-1}, division x and response rule Y,

$$\begin{aligned} U_{P(n)}\left(z_{P(n)}^{h^{n-1}}\left(\sigma_{P(n)}\left(h^{n-1}\right), \sigma_{R(n)}\left(h^{n-1}\right)\middle|\,\sigma\right)\right) &\geq U_{P(n)}\left(z_{P(n)}^{h^{n-1}}\left(x, \sigma_{R(n)}\left(h^{n-1}\right)\middle|\,\sigma\right)\right); \\ U_{R(n)}\left(z_{R(n)}^{h^{n-1}}\left(x, \sigma_{R(n)}\left(h^{n-1}\right)\middle|\,\sigma\right)\right) &\geq U_{R(n)}\left(z_{R(n)}^{h^{n-1}}\left(x, Y|\,\sigma\right)\right). \end{aligned}$$

Observe that this indeed defines the SPNE of the auxiliary game where the set of players is taken to be $I \times \mathbb{N}$. The well-known one-stage deviation principle (e.g., Fudenberg and

Tirole, 1991, thm. 4.2) says that such equilibrium coincides with SPNE of the actual game played by I whenever both players' preferences satisfy ED; hence the model studied in this paper contains that of Rubinstein (1982) as a special case.

Throughout, I consider only pure strategies. This restriction is common in the literature, following Rubinstein (1982), even in models with inherent risk (e.g., Merlo and Wilson, 1995; Cripps, 1998). Permitting randomization devices, while unlikely to enlarge the set of equilibrium outcomes (cf. Binmore, 1987), would here come at the cost of augmenting the domain of preferences by risk, however, adding a layer of cardinality.[15]

3 Equilibrium Characterization

A central property for the analysis of this game is its stationarity: conditional on no agreement, the game repeats itself every two rounds. Hence, ignoring history, all subgames beginning with the very same player i's proposal are identical and, in particular, have the same equilibria; denote this game by G_i.[16] As it stands, the previous section defines G_1; after the sole modification of setting instead $P(1) = 2$, it defines game G_2.

3.1 Agreement and Stationary Equilibrium

In order to first establish equilibrium existence, it is natural to look for an equilibrium which inherits this property, i.e. a stationary equilibrium. Consider the following argument, using, for each player i, a function $\pi_i : U_i(A_i \cup \{(0,\infty)\}) \to [0,1]$, given by

$$\pi_i(U) \equiv \min\{q \in [0,1] \,|u_i(q) \geq U\},$$

to represent her *reservation share* for "rejection value" U; note that π_i is continuous and non-decreasing since u_i is continuous and increasing. Suppose now an equilibrium where players eventually agree (as they clearly would in the terminal round of a truncated game; cf. Binmore, 1987), say on division $\hat{x}$ in a round where player 1 proposes. Immediate agreement on $\hat{x}$ is then an equilibrium outcome of G_1. Positing this as the continuation outcome following (any) rejection in G_2, all offered shares $q \geq \pi_1(U_1(\hat{x}_1, 1))$ are acceptable to responding player 1, and immediate agreement on $\hat{y}$ with $\hat{y}_2 = 1 - \pi_1(U_1(\hat{x}_1, 1))$ is then an equilibrium outcome of G_2. Another such step of backwards induction then yields immediate agreement on $\hat{x}'$ such that $\hat{x}'_1 = 1 - \pi_2(U_2(\hat{y}_2, 1))$ as an equilibrium outcome of G_1, and if

[15] Nonetheless, the model has a straightforward interpretation in terms of bargaining under the shadow of a constant risk of breakdown with non-expected-utility preferences; see appendix B.5.2.

[16] Preferences are non-stationary only with respect to *relative* time (delay), not *absolute* time.

$\hat{x} = \hat{x}'$, then one has obtained a stationary equilibrium.

Once it is shown that players must eventually agree, any stationary equilibrium can be constructed in this manner; a very basic argument indeed establishes this agreement property for *any* equilibrium, stationary or non-stationary. Due to the opponent's impatience, a proposer is certainly able to appropriate some surplus; hence perpetual disagreement cannot be an equilibrium outcome.

Lemma 1. *All equilibrium outcomes are agreement outcomes.*

Given this result, the preceding argument characterizes stationary equilibrium. Consider then the function $f_1 : [0,1] \to [0,1]$ such that

$$f_1(q) \equiv 1 - \pi_2\left(U_2\left(1 - \pi_1\left(U_1(q,1)\right),1\right)\right)$$

and note that $\hat{x} = \hat{x}'$ in the argument above is equivalent to $\hat{x}_1 = f_1(\hat{x}_1)$. It suffices to consider only f_1 (without the analogous f_2) because, following that very argument, stationary equilibria have immediate agreement in every round (on as well as off the equilibrium path), whereby starting from agreement in one where player 1 proposes is without loss of generality.

Proposition 1. *The profile of stationary strategies given by* $\left(x^{(i)}, Y^{(i)}\right)_{i \in I}$ *is an equilibrium if and only if*

$$\left\{ \begin{array}{rcc} x_1^{(1)} & = & f_1\left(x_1^{(1)}\right) \\ x_2^{(2)} & = & 1 - \pi_1\left(U_1\left(x_1^{(1)},1\right)\right) \end{array} \right\} \text{ and, for each } i \in I,\ Y^{(i)} = X_{i,x_i^{(j)}}.$$

Any stationary equilibrium has immediate agreement and, given impatience, is efficient. Existence and uniqueness of stationary equilibrium are equivalent to the analogous properties of fixed points of f_1; more generally, this ensures existence of equilibrium and implies the necessary condition for equilibrium uniqueness of a unique fixed point of f_1.

Corollary 1. *A stationary equilibrium exists. It is unique if and only if* f_1 *has a unique fixed point.*

The function f_1 has a strategic interpretation, but it does not relate directly to *individual* preferences, making it hardly useful for applications. To complete the analysis of stationary equilibrium, I therefore provide a final result which presents a sufficient condition on individual preferences for uniqueness of stationary equilibrium.

Definition 2. The preferences of player $i \in I$ exhibit **initially increasing loss to delay** if $q - \pi_i\left(U_i(q,1)\right)$ is an increasing function of q on the entire interval $[0,1]$.

This property is a straightforward generalization of an axiom proposed in the context of ED by Fishburn and Rubinstein (1982, p. 690), which is used to prove uniqueness in Osborne and Rubinstein (1990, ch. 3); under discounting, it is implied by strict log-concavity—in particular, by concavity—of "instantaneous" utility (Hoel, 1986).[17] The proposition below therefore establishes uniqueness of stationary equilibrium under standard assumptions.

Proposition 2. *If both players' preferences exhibit initially increasing loss to delay, then stationary equilibrium is unique.*

3.1.1 Assuming Stationary Strategies

Proposition 1 generalizes existing results on bilateral alternating-offers bargaining with dynamically inconsistent preferences (Burgos et al., 2002a; Akin, 2007; Ok and Masatlioglu, 2007; Noor, 2011), all of which are effectively based on the *assumption* of stationary strategies (see footnote 19). Since stationary equilibria exhibit very desirable properties—simplicity, existence and (under mild assumptions) also uniqueness—what motivates interest in more general strategy spaces here?

In environmentally rich models of bargaining this assumption is justified by the pragmatic reason to establish at least some tractable equilibrium (e.g., Chatterjee, Dutta, Ray, and Sengupta, 1993). With a highly stylized model such as the one studied here, however, a main objective is to understand fundamental strategic considerations of the players (the infinitely repeated prisoners' dilemma being another example). This conflicts with the severe restrictions this assumption would impose on players' beliefs: however systematically one player has deviated from a given stationary strategy, it restricts the other player to still believing she will subsequently comply with it (see Rubinstein, 1991, p. 912). This point is only reinforced here by the additional presence of *intra*-personal conflict (dynamic inconsistency), where a player's beliefs about her own future behavior are as central as those regarding the opponent (a player may have reason to "doubt herself").

In fact, its simplicity means that a stationary strategy is unfit to exploit the opponent's dynamic inconsistency. Consider a player i who always offers player j a share q and responds with threshold $1-q'$. Given j is impatient and finds perpetual disagreement worst, she would actually suffer from a dynamic preference reversal—alternatively, commitment has value—only if she preferred to wait for q when q' is currently available, implying $q > q'$ by impatience, but subsequently preferred to wait for q' when q is available, implying $q' > q$ by impatience, a contradiction. Assuming stationary strategies therefore amounts to assuming away the

[17] A more general sufficient condition, which applies also under non-separability, is that u_i is strictly log-concave together with $u_i(0) = U_i(0,1)$ and $u_i' > U_i'(\cdot,1)$ (cf. Noor, 2011, sec. 4).

players' incentive to exploit the opponent's dynamically inconsistent time preferences; it is hardly surprising that one thus reproduces the well-known results for ED.

Indeed, the only property of assumption 1 regarding impatience used to derive the results of this section is that conditional on obtaining some surplus, players prefer agreeing immediately over agreeing with a *single* period of delay (4.b). Under stationary equilibrium this suffices to obtain immediate agreement in any round, and attitudes to delay of more than just a single period are immaterial; e.g., proposition 1 would hold yet under "negative time preference" (Loewenstein, 1987, 1991) for longer delays, where immediate agreement is inefficient.

In conclusion, similar to how it would eliminate intertemporal considerations in repeated games, the *assumption* of stationary equilibrium would here unduly constrain the players' strategic interaction, and especially so given that players' preferences are non-stationary.

3.1.2 Failure of the Standard Technique

Nonetheless, given stationary time preferences (ED), stationary equilibrium is indeed without loss of generality (Rubinstein, 1982). As demonstrated with great clarity by Shaked and Sutton (1984), this stems from the fact that—despite possible history-dependence in equilibria—players' minimal and maximal equilibrium payoffs/utilities ("values") each follow the stationary fixed-point structure outlined in the two-step backwards-induction argument of section 3.1; thus they are obtained in a stationary equilibrium, and a unique such equilibrium is unique overall.[18]

Reflecting the above discussion, this argument fails here, however. The reason is that a dynamically inconsistent player's ranking of equilibrium outcomes, in particular her worst and best, may change as one moves her perspective from that of the initial proposer to that of the previous round's respondent who evaluates continuation outcomes in terms of their "rejection value" (with one period of delay added; see also footnote 21).

To illustrate, consider a (β, δ)-discounter with linear instantaneous utility, say player 1, whose minimal equilibrium value as the initial proposer equals v_1^*. Suppose, for the sake of the argument, that we need only entertain the possibility of equilibrium delay of up to one period. Letting x_1 and x_1' denote the (unknown) minimal equilibrium shares among the immediate-agreement and delay equilibria, respectively, we know that $v_1^* = \min\{x_1, \beta\delta x_1'\}$. Moreover, denoting player 1's minimal rejection value by w_1^*, we also know that $w_1^* = \beta\delta \cdot \min\{x_1, \delta x_1'\}$. While it is immediate that $w_1^* = \delta v_1^*$ under ED ($\beta = 1$), more generally, w_1^* cannot be determined from mere knowledge of v_1^* but depends on the underlying shares x_1 and x_1': it equals (i) $\beta\delta v_1^*$ if $x_1 \leq \beta\delta x_1'$, (ii) $\beta\delta x_1$ if $\beta\delta x_1' < x_1 < \delta x_1'$ (in this case $\beta\delta v_1^* < w_1^* < \delta v_1^*$),

[18] For extensions/variations see, e.g., Perry and Reny (1993), Sákovics (1993) or Merlo and Wilson (1995).

and (iii) δv_1^* if $\delta x_1' \leq x_1$; unless delay is ruled out beforehand, simple backwards induction on extreme values is therefore infeasible.[19] In the presence of dynamic inconsistency, any successful approach to a full equilibrium characterization must instead deal explicitly with the possibility of equilibrium delay and, in fact, non-stationary equilibrium.

3.2 Optimal Simple Penal Codes and Simple Play

The approach proposed in this paper directly analyzes the off-path "punishments" (continuation equilibria) which support all equilibrium play. Its basic idea is that the game's stationarity property will nonetheless entail a tractable structure of such punishments, since only two types of round need to be distinguished in terms of deviations: any round in which the same player $i \in \{1, 2\}$ gets to make an offer has the same sets of both equilibrium plays and continuation equilibria. If a particular "optimal" assignment of the latter as punishments deters deviations from *any* equilibrium play, it achieves this in any round where i makes an offer, independent of history. How much tractability is thus gained then depends on how "simple" this optimal assignment is.

Due to the conceptual similarity with the respective definitions by Abreu (1988) for infinitely repeated games, I will call an assignment of punishments which supports all equilibrium play an *optimal penal code* (OPC), and I call it an *optimal simple penal code* (OSPC) if its punishment is history-independent.[20] The main complication relative to Abreu's work on standard repeated games, in which a simultaneous-move game is infinitely repeated, is that an OPC cannot simply assign the worst continuation equilibrium, however. Since moves within a round are sequential, the proposer's punishment for deviant offers is constrained by the respondent's incentives after such a deviation; e.g., the worst continuation equilibrium for the proposer may then weaken the respondent's bargaining position and thus make deviant offers too attractive. Indeed, Mailath et al. (2015) present examples of infinitely repeated *sequential-move* games in which the second mover's incentive constraint causes any OPC to depend on the deviation, i.e. no OSPC exists.

The trade-off between providing incentives within-round and under continuation is, however, less complicated in bargaining than in a repeated game since the respondent's acceptance of an offer ends the game. Punishment therefore takes place only after deviations which result in a rejected offer. As I show below, an OPC can be defined in terms of rejection values for the responding player as follows: any deviant rejection—irrespective of

[19] Nonetheless, the argument has been applied in this context; see Ok and Masatlioglu (2007, p. 230). Although these authors do not formally define a solution concept, it appears to be the one adopted here; in this case example 1 disproves their uniqueness claim ("by continuity").

[20] I am deeply grateful to Can Çeliktemur for pointing out this similarity to me at an early stage.

whether the offer was a deviation or not—is punished such that the respondent obtains her minimal rejection value, and the non-deviant rejection of any deviant offer is punished such that the respondent obtains her maximal rejection value.[21] This ensures maximal compliance by the respondent with her response rule and reduces the set of deviant offers which go unpunished to those which the respondent would accept—thereby ending the game—under *any* continuation equilibrium.

The following notation is useful, given a strategy profile σ in game G_i inducing agreement outcome (x, t), and given a round-n history h^{n-1} of G_i: first, let $\sigma|_{h^{n-1}}$ denote the restriction of σ to continuation histories of h^{n-1}, i.e. histories of the form (h^{n-1}, h^{m-1}) where $h^{m-1} \in X^{m-1}$ for $m \in \mathbb{N}$, and second, let $\sigma|^{h^{n-1}}$ denote the strategy profile in game $G_{P(n)}$ which is obtained from $\sigma|_{h^{n-1}}$ upon replacing h^{n-1} by the initial history h^0.[22] Finally, recalling that all equilibrium outcomes are agreement outcomes (lemma 1), denote a player i's personal equilibrium outcomes in game G_i by A_i^*, which is non-empty by corollary 1.

Fix then any quadruple of strategy profiles $\left(\sigma^{P,i}, \sigma^{R,i}\right)_{i\in I}$ such that, for each $i \in I$, $\sigma^{P,i}$ is a strategy profile in game G_j and $\sigma^{R,i}$ is a strategy profile in game G_i, and define the following mapping $\sigma^*\left(\cdot \middle| \left(\sigma^{P,i}, \sigma^{R,i}\right)_{i\in I}\right)$: for any game G_k, $k \in I$, and any strategy profile σ in that game, it is the unique strategy profile σ^* of G_k such that $\langle\sigma^*\rangle = \langle\sigma\rangle$ and, for any round $m \leq n$ of this game, where n satisfies $\langle\sigma\rangle \in (X \times \mathcal{P}(X))^n$,

$$\sigma^*|^{\left(h^{m-1}(\sigma),x\right)} \quad = \quad \begin{cases} \sigma^{P,P(m)} & x \notin \sigma^*_{R(m)}\left(h^{m-1}(\sigma)\right) \setminus \left\{\sigma^*_{P(m)}\left(h^{m-1}(\sigma)\right)\right\} \\ \sigma^{R,R(m)} & x \in \sigma^*_{R(m)}\left(h^{m-1}(\sigma)\right) \end{cases}. \tag{1}$$

Proposition 3. *Let the quadruple of outcomes* $\left(\left(x^{P,i}, t^{P,i}\right), \left(x^{R,i}, t^{R,i}\right)\right)_{i\in I}$ *be such that, for each* $i \in I$*,*

$$\left(x_j^{P,i}, t^{P,i}\right) \in \arg\max_{(q,t)\in A_j^*} U_j\left(q, t+1\right) \quad \text{and} \quad \left(x_i^{R,i}, t^{R,i}\right) \in \arg\min_{(q,t)\in A_i^*} U_i\left(q, t+1\right). \tag{2}$$

Also, let $q_i^* \equiv \max\left\{\pi_i\left(U_i\left(q, t+1\right)\right) \middle| (q, t) \in A_i^*\right\}$ *for each* $i \in I$*.*

(i) Fix a quadruple of equilibria $\left(\sigma^{P,i}, \sigma^{R,i}\right)_{i\in I}$ *such that, for each* $i \in I$*,* $\sigma^{P,i}$ *is an equilibrium of game* G_j *supporting outcome* $\left(x^{P,i}, t^{P,i}\right)$ *and* $\sigma^{R,i}$ *is an equilibrium of game* G_i *supporting outcome* $\left(x^{R,i}, t^{R,i}\right)$*. Then, for any* $k \in I$ *and strategy profile* $\hat{\sigma}$ *in game* G_k*,* $\langle\hat{\sigma}\rangle$

[21] A continuation equilibrium σ with continuation outcome (x, t) has continuation value $U_i(x_i, t)$ to player i, but rejection value $U_i(x_i, t+1)$. The two rankings of continuation equilibria/outcomes need not coincide when i's preferences are dynamically inconsistent, in which case only rejection values correspond to i's preferences at the time of deciding whether to accept an offer.

[22] Observe that, given h^{n-1}, $\sigma|^{h^{n-1}}$ completely characterizes $\sigma|_{h^{n-1}}$.

is an equilibrium play of G_k *if and only if* $\sigma^*\left(\hat{\sigma}\left|\left(\sigma^{P,i},\sigma^{R,i}\right)_{i\in I}\right.\right)$ *is an equilibrium of* G_k.

(ii) The quadruple of equilibria $\left(\sigma^{P,i},\sigma^{R,i}\right)_{i\in I}$ *in (i) can be chosen such that*

$$\sigma^{P,i}=\sigma^*\left(\sigma^{P,i}\left|\left(\sigma^{P,i},\sigma^{R,i}\right)_{i\in I}\right.\right)\quad and\quad \sigma^{R,i}=\sigma^*\left(\sigma^{R,i}\left|\left(\sigma^{P,i},\sigma^{R,i}\right)_{i\in I}\right.\right). \tag{3}$$

(iii) For any $k\in I$*, agreement on division* $\hat{x}$ *with delay* $\hat{t}$ *is an equilibrium outcome of game* G_k *if and only if there exists an equilibrium* σ *of* G_k *play* $\langle\sigma\rangle=\left(\sigma\left(h^{m-1}\left(\sigma\right)\right)\right)_{m=1}^{n}$ *of which is such that* $n=\hat{t}+1$ *and*

$$\sigma\left(h^{m-1}\left(\sigma\right)\right)=\begin{cases}\left(e^{P(m)},X_{R(m),q^*_{R(m)}}\right) & m<n\\ \left(\hat{x},X_{R(m),\hat{x}_{R(m)}}\right) & m=n\end{cases}.$$

The first part of this proposition establishes an OPC, as outlined earlier, with its two components: four *optimal punishments*, where optimality is defined in (2), and an *optimal punishment rule* assigning them to the various possible deviations from a given play, σ^*. This rule (1) distinguishes only between "proposer deviations"—non-deviantly rejected deviant offers—and "respondent deviations"—deviantly rejected offers. Hence there is one punishment per player per role in which a player might deviate.

Given how it identifies the perpetrator, this OPC's punishment is simple in the sense that it need not fit the crime. This is true, however, only as long as no (first) deviation from a given play has occurred. Yet, any given OPC supports, in particular, play of its own constituent punishments. Iteratively applying it also to deviations from first punishment play (second deviations), and then also deviations from second punishment play (third deviations) etc. we can create an OPC in which player i's proposer and respondent deviations are followed by the same respective punishment as well as punishment play (upon rejection), entirely independent of their history, i.e. an OSPC (3); e.g., player 1's proposer deviation from play $\left\langle\sigma^{P,1}\right\rangle$ then simply restarts this punishment play. Part (ii) verifies this argument.

Since the respective continuation equilibria follow from them, an OSPC is fully described by four optimal punishments' plays. However, consequentialist parties care only about outcomes of play, not play itself; e.g., making an offer that is commonly known to be rejected should then be tantamount to not offering anything at all. The third part of the proposition removes such redundancy regarding equivalent types of equilibrium play: without loss of generality, we can restrict attention to *simple play* in which all rejected offers are minimal ones (zero) and all response rules are threshold rules, such that each player applies a single, namely the maximal credible, threshold in all disagreement rounds. Observe that the restriction to simple play implies a stationary structure: all rounds with the same proposer in which players are supposed to delay are played identically.

Given existence of four outcomes as in (2)—to be verified below—proposition 3 drastically reduces the strategic complexity for equilibrium analysis. Any four such punishment outcomes provide sufficient information about equilibrium off-path: there is a unique simple punishment play associated with each of them, and their four punishment plays, applied in the sense of an OSPC, support all equilibrium play; restricting attention to simple play, I will hence notationally identify OSPCs and such quadruples of outcomes $\left(\left(x^{P,i}, t^{P,i}\right), \left(x^{R,i}, t^{R,i}\right)\right)_{i \in I}$. Moreover, checking whether a particular outcome is an equilibrium outcome—e.g., an OSPC outcome—reduces to the straightforward task of checking deviations from simple play only.

Proposition 3 captures the essence of the players' strategic interaction given the game's stationary structure. It holds true under minimal preference assumptions (regarding impatience, it only uses property (4.b), similar to section 3.1's results about stationary equilibrium) and without any restrictions on strategies' history-dependence. Imposing stationary strategies, discussed in section 3.1.1, would amount to requiring both $\sigma^{P,1} = \sigma^{R,2}$ and $\sigma^{P,2} = \sigma^{R,1}$, so continuation play does not depend on current play and there cannot be meaningful punishment. The approach discussed in section 3.1.2 is more general: cast in the framework developed here, it amounts to imposing that optimal punishment does not require delay (note that this would be true if optimal punishments could be found among stationary equilibria). With non-stationary preferences, however, it often does—see example 1. The simplification of the dynamic structure of (non-stationary) delay equilibria is then key to relating the players' "punishment values".

3.3 The Main Result

I will now characterize the set of OSPCs, i.e. outcomes $\left(\left(x^{P,i}, t^{P,i}\right), \left(x^{R,i}, t^{R,i}\right)\right)_{i \in I}$ satisfying (2), and establish their existence in terms of the following values. Define each player i's *optimal punishment values* as her *minimal proposer value* v_i^* and her *minimal rejection value* w_i^*, together with the *supremal delay* t_i^* in game G_i:

$$\begin{aligned} v_i^* &\equiv \min \left\{ U_i\left(q, t\right) \mid \left(q, t\right) \in A_i^* \right\} \\ w_i^* &\equiv \min \left\{ U_i\left(q, t+1\right) \mid \left(q, t\right) \in A_i^* \right\} \\ t_i^* &\equiv \sup \left\{ t \in T \mid \exists q \in \left[0, 1\right], \left(q, t\right) \in A_i^* \right\}. \end{aligned}$$

While there may be multiple optimal punishment outcomes and, consequently, multiple OSPCs, the associated values $\left(v_i^*, w_i^*, t_i^*\right)_{i \in I}$ are unique. The simplified structure of equilibria arrived at above allows to characterize them as the (unique) "extreme" solution to a system of equations, where the ranges of possible delays $(t_i^*)_{i\in I}$ will be both determined by the players' optimal punishment values and key to relating them to each other. From this solution, the set of OSPCs as well as equilibrium outcomes are obtained, for any preference profile under assumption 1. This is the central result of this paper.

Two further definitions, for each $i \in I$, will be useful here: first, the function $\phi_i : u_i([0,1]) \times T \to [0,1]$ such that

$$\phi_i(u,t) \equiv \max\{q \in [0,1] \,|u \geq U_i(q,t)\}$$

gives the (instantaneous-) *u-equivalent share at delay* t, if one exists, and the maximal possible share otherwise; it is non-decreasing in both u and t as well as continuous in u. Second, letting $\mathscr{U}_k \equiv U_k(A_k \cup \{(0,\infty)\})$ for both $k \in I$, the function $\kappa_i : T \times \mathscr{U}_i \times \mathscr{U}_j \times \mathscr{U}_j \to \mathbb{R}_+$ such that

$$\kappa_i(t, v_i, v_j, w_j) \equiv \begin{cases} \phi_i(v_i,t) + \pi_j(w_j) & t = 0 \\ \phi_i(v_i,t) + \max\{\phi_j(v_j, t-1), \phi_j(u_j(0), t)\} & t > 0 \end{cases}$$

measures the *surplus-cost of delay* t in G_i given proposer values v_i and v_j, and rejection value w_j (for the initial respondent j); because of the non-decreasingness of $\phi_i(u,\cdot)$ for any u, $\kappa_i(\cdot, v_i, v_j, w_j)$ is non-decreasing on $T \setminus \{0\}$ for any (v_i, v_j, w_j).

Let then $E \subseteq \prod_{i\in I}(\mathscr{U}_i^2 \times (T \cup \{\infty\}))$ be the set of sextuples $(v_i, w_i, t_i)_{i\in I}$ such that, for each $i \in I$,

$$\begin{aligned} v_i &= u_i(1 - \pi_j(U_j(1 - \pi_i(w_i), 1))) && (4) \\ w_i &= \inf\{U_i(\phi_i(v_i,t), t+1) \,|t \in T,\ t \leq t_i\} && (5) \\ t_i &= \sup\{t \in T | \kappa_i(t, v_i, v_j, w_j) \leq 1\} && (6) \\ 1 &\geq \kappa_i(0, v_i, v_j, w_j). && (7) \end{aligned}$$

Lemma 4 in appendix A.6 shows how each element $(v_i, w_i, t_i)_{i\in I}$ of E corresponds to a quadruple of punishment outcomes that are "constrained" optimal in the following sense: by means of what would be an OSPC if they were actually optimal punishment outcomes, they support a *subset* of equilibrium outcomes, including themselves, on which they are optimal and yield exactly the minimal punishment values $(v_i, w_i)_{i\in I}$ and supremal delays $(t_i)_{i\in I}$ (all relative to this subset, hence constrained). This is reflected by the fixed-point property defining the set E.[23]

If an OSPC exists, its associated values $(v_i^*, w_i^*, t_i^*)_{i\in I}$ are necessarily in E. However, in general, due to the interdependency of punishments—harsher punishments permit longer delays, and longer delays permit harsher punishments—there may be (other) "constrained" OSPCs. In fact, any stationary equilibrium $\left(x^{(i)}, Y^{(i)}\right)_{i\in I}$ is one: regardless of how the parties fail to agree when player i makes an offer, the continuation outcome is $\left(x^{(j)}, 0\right)$, which, by backwards induction, supports the unique outcome $\left(x^{(i)}, 0\right)$ in G_i. The equilibrium values associated with a stationary equilibrium, together with $t_1 = t_2 = 0$, therefore constitute an element of E, proving its non-emptiness.

The six equations 4-6, $i \in I$, already define solutions for six values $(v_i, w_i, t_i)_{i\in I}$. Inequality 7 is typically superfluous (see appendix B.2 for detail); whenever it is not, it rules out those solutions which correspond to mutually incompatible "rewards" instead of punishments supporting some equilibrium play. Thus restricted, E indeed characterizes all constrained OSPCs, and its extreme element corresponds to the most extreme ones, i.e. OSPCs; existence of an OSPC therefore follows from existence of an extreme element of E.

Theorem 1. *The values $(v_i^*, w_i^*, t_i^*)_{i\in I}$ are the unique element of the set E such that $v_i^* \leq v_i$, $w_i^* \leq w_i$ and $t_i^* \geq t_i$ hold true for both $i \in I$ and any $(v_i, w_i, t_i)_{i\in I} \in E$. For each $i \in I$, $\left(x^{P,i}, t^{P,i}\right)$ and $\left(x^{R,i}, t^{R,i}\right)$ are outcomes of player i's optimal proposer and respondent punishment as in (2), respectively, if and only if*

$$\left\{\begin{array}{c} t^{P,i} = 0 \\ x_i^{P,i} = \pi_i\left(w_i^*\right) \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{c} t^{R,i} \in \arg\min\left\{U_i\left(\phi_i\left(v_i^*, t\right), t+1\right) | t \in T,\ t \leq t_i^*\right\} \\ x_i^{R,i} = \phi_i\left(v_i^*, t^{R,i}\right) \end{array}\right\},$$

and the set A_i^* of personal equilibrium outcomes of G_i equals

$$\left\{(q,t) \in A_i \middle| \phi_i\left(v_i^*, t\right) \leq q \leq \begin{cases} 1 - \pi_j\left(w_j^*\right) & t = 0 \\ 1 - \max\left\{\phi_j\left(v_j^*, t-1\right), \phi_j\left(u_j\left(0\right), t\right)\right\} & t > 0 \end{cases}\right\}.$$

The introductory example 1 shows that this characterization neither reduces to uniqueness nor to stationarity of equilibrium nor to stationarity of optimal punishments. In section 4, I present further detail, examples and discussion.

Theorem 1 is partly reminiscient of Merlo and Wilson (1995, thms 7 and 8) who assume ED and analyze bargaining by multiple players under a Markovian process governing the protocol as well as the size of the cake. They also characterize the set of equilibrium values by means of an extremal fixed point, but its nature differs significantly: ED implies that

[23] In analogy to Abreu, Pearce, and Stacchetti (1990) values in E are (exactly) self-generating.

there is a stationary equilibrium outcome which maximizes one player's value at the same time as it minimizes all other players' values. In the two-player case this simple relationship between punishment and reward implies that optimal punishments are efficient, and stationary equilibrium is without loss of generality. Only in the case of more than two players, one player's optimal punishment might necessitate some punishment of another player and some inefficiency, thus complicating the incentive structure (cf. Burgos et al., 2002b).

By contrast, here such a complication arises already with two players, and from a very different source: the non-stationarity of a player's time preferences. Optimal punishment might *require* delay, in which case it is both inefficient and non-stationary. The extreme equilibria are then "truly" non-stationary: their continuation is non-stationary after any history. This distinguishes them from various previously advanced delay equilibrium constructions in variations/extensions of the original Rubinstein (1982) model with a stationary game structure and ED (Haller and Holden, 1990; Muthoo, 1990; van Damme, Selten, and Winter, 1990; Fernandez and Glazer, 1991; Myerson, 1991; Avery and Zemsky, 1994).

The only other instance of a stationary bargaining game with this property I am aware of is that of Busch and Wen (1995).[24] Their model of negotiation extends that of Rubinstein (1982) by a "disagreement game", which is a fixed simultaneous-move game played after any rejected offer and determines a stream of payoffs the players receive before agreement. The purely non-stationary equilibria they construct exploit the resultingly richer preference domain through non-stationary play of the disagreement game similar to folk theorems for repeated games, but constrained by the parties' incentives to reach agreement.

Existence of an OSPC is equivalent to the existence of minimum values v_i^* and w_i^*; this is non-trivial here, as the set of equilibrium outcomes need not be closed.[25] The generality of impatience (4.ab) under assumption 1 means that the length of equilibrium delay might have no upper bound (see example 21 in appendix B.4), despite lemma 1's ruling out perpetual disagreement. While existence of v_i^* follows from standard continuity arguments even then, the (only) role played by impatience property (4.c) is to ensure that w_i^* exists in this case, because the delay of agreement that is required *for optimal punishment* is bounded.

A few features of optimal punishments are noteworthy in view of the often quoted strategic advantage enjoyed by the proposing player. First, a player's optimal proposer punishment is unique and involves no delay: given her impatience, the respondent's rejection value is maximized by the maximal credible share with least delay (see equation 4). Second, an initially proposing player i's least preferred equilibrium outcomes for various delays are nec-

[24] I am indebted to Paola Manzini for drawing my attention to these authors' work.

[25] Although the equilibrium concept introduced in definition 1 is equivalent to a version of subgame-perfect Nash equilibrium, existing results based on the upper hemi-continuity of its equilibrium correspondence (e.g., Börgers, 1991) cannot be applied here, because they assume finitely many players.

essarily indifferent, all yielding her the same minimal value v_i^*; this allows to pin down optimal respondent punishment (see equation 5). Finally, players' incentives in the role of proposer fully determine whether and how long agreement may be delayed (see equation 6).

4 Uniqueness v. Multiplicity, and Delay

Arguably, the main reason for the success of the Rubinstein (1982) model was its unique bargaining prediction under standard preference assumptions, namely exponentially discounted concave utility. His work also included the observation that ED (and therefore impatience) alone fails to guarantee such uniqueness; yet, the degree of the instantaneous utility function's convexity required for equilibrium multiplicity would appear empirically implausible. Analogously, multiplicity of equilibrium outcomes and delay arising from non-stationary time preferences may only arise for time preferences which are, again, empirically implausible; thus we could conclude that the forms of impatience which humans display do not interfere with efficient as well as sharply predictable bargaining.

This section addresses this question based on the following characterization, which is straightforward given the previous section's results.

Theorem 2. *Equilibrium is unique if and only if the set E is a singleton. Its unique element is then $(v_i^*, w_i^*, t_i^*)_{i\in I} = (u_i(q_i^*), U_i(q_i^*, 1), 0)_{i\in I}$ such that q_1^* is the unique fixed point of f_1, and $q_2^* = 1 - \pi_1(U_1(q_1^*, 1))$, with the associated profile of stationary strategies $\left(x^{(i)}, Y^{(i)}\right)_{i\in I}$ such that $x_i^{(i)} = q_i^*$ and $Y^{(i)} = X_{i,1-q_j^*}$ for each player $i \in I$ as the unique equilibrium.*

4.1 Present Bias and Uniqueness

For economic applications, where bargaining arises naturally in various contexts (household decision-making, wage setting, international trade agreements etc.), uniqueness of the bargaining prediction is an important concern. Any uncertainty about this one aspect of a model feeds through all of the conclusions drawn from it. Yet, checking for equilibrium uniqueness on the basis of theorem 2 is cumbersome. I therefore complement it with what I view as the main result concerning uniqueness: a simple and readily testable set of sufficient conditions at the level of individual preferences, which, moreover, establishes uniqueness for a large class of important time preferences.

Since there always exists a stationary equilibrium (corollary 1), sufficient conditions for equilibrium uniqueness must ensure uniqueness of stationary equilibrium. The earlier proposition 2 furnishes such sufficient conditions which can indeed be verified separately for each player's preferences: "initially increasing loss to delay".

A unique stationary equilibrium is the unique equilibrium overall whenever no player's optimal punishment requires delay (see lemma 9 in appendix A.8); the set E is then a singleton, its unique element corresponding to the unique stationary equilibrium. Consider then the following preference property.

Definition 3. The preferences of a player $i \in I$ exhibit a **weak present bias** if, for any two shares $(q, q') \in [0,1]^2$ and any delay $t \in T$,

$$u_i(q) = U_i(q', t) \Rightarrow U_i(q, 1) \leq U_i(q', t+1).$$

If a weakly present biased individual, in a period's time, would be indifferent between receiving a reward q immediately and receiving a reward q' with t periods of delay, she is currently willing to wait for the larger later reward. Because theorem 1 reveals that optimal proposer punishments never involve delay in any case, and, moreover, that a proposing player must be kept indifferent between all of her variously delayed worst equilibrium outcomes, such a weakly present biased player's respondent punishment cannot be made more severe through delaying agreement (see equation 5). If both players' preferences exhibit such a bias, then no optimal punishment requires delay.

Proposition 4. *If both players' preferences exhibit a weak present bias and initially increasing loss to delay, then equilibrium is unique.*

It is straightforward to check weak present bias for any given preferences, and the property is also readily testable empirically, for given period length. Its interpretation is, however, most straightforward for separable time preferences, where $U(q,t) = d(t) \cdot u(q)$. Defining a per-period discounting function $\delta : \mathbb{N} \to [0,1]$ such that $\delta(t) \equiv \frac{d(t)}{d(t-1)}$, the (total) discounting function d can be written as $d(t) \equiv \prod_{s=1}^{t} \delta(s)$, exposing $\delta(s)$ as the discount factor for the s-th period of delay.[26] Weak present bias then reduces to the property that $\delta(1) \leq \delta(t)$, which says that no future period of delay is discounted more heavily than the first one from the immediate present.[27]

Whereas under ED, where $\delta(\cdot)$ is constant and which is thus a limiting case (hence the adjective "weak"), any hyperbolic or quasi-hyperbolic discounting exhibits an actual "bias" toward the present: the (β, δ)-model of quasi-hyperbolic discounting has $\delta(1) = \beta\delta < \delta = \delta(t)$ for any $t > 1$, and hyperbolic discounting has $\delta(\cdot)$ increasing.[28]

[26] I follow the convention that the empty product for $t = 0$ equals one.

[27] Halevy (2008) introduces a strict version of this discounting property, which he calls "diminishing impatience", and goes on to show how non-linear probability weighting regarding the uncertainty inherent in future consumption can explain this finding in intertemporal choice.

[28] The non-separable models of Benhabib et al. (2010) and Noor (2011) were both designed to capture the

Proposition 4 establishes robustness of the wisdom received from the study of ED to various forms of weak present bias: equilibrium is unique as well as efficient, it is easily computed on the basis of only the players' attitudes to a single (the first) period of delay and has familiar comparative statics. If one believes in the essence of present bias but finds the evidence inconclusive as to what exact functional form it assumes, it is comforting that equilibrium is robust to any mis-specification of higher-order delay attitudes. Moreover, the finding that the historically main mode of surplus sharing is efficient under present bias is good news for its evolutionary explanations (e.g., Dasgupta and Maskin, 2005; Netzer, 2009): otherwise, communities without a present bias would have had an evolutionary advantage, making its survival hard to understand.

Most importantly, however, this result expands the scope of applied work, which shows increased interest in the study of present-biased time preferences, in particular (β, δ)-discounting, but has hitherto lacked a strategically founded bargaining solution. Its application requires some caution, however, as the following example indicates—even after putting aside the empirical issue of whether present bias is prevalent on the most commonly considered bargaining domain of monetary rewards (see appendix B.1).

Example 2. Let the two parties' preferences be given by $U_i(q,t) = d_i(t) \cdot q$ with $d_i(0) = 1 > d_i(t) = \beta_i \delta_i^t$ for all $t > 0$, $(\beta_i, \delta_i) \in (0,1)^2$. The unique equilibrium of the game in which player 1 makes the initial offer has immediate agreement on division x such that

$$x_1 = \frac{1 - \beta_2 \delta_2}{1 - \beta_1 \delta_1 \beta_2 \delta_2}.$$

For a given positive period-length, this prediction is indistinguishable from that under ED such that each player i has preferences $U_i(q,t) = \tilde{\delta}_i^t q$ with $\tilde{\delta}_i \equiv \beta_i \delta_i$ (cf. Bernheim and Rangel, 2009, pp. 69-71).

The limiting case of very frequent offers that is commonly focused on in applications becomes somewhat problematic, however, as there is no straightforward continuous-time implementation of (β, δ)-discounting (cf. Harris and Laibson, 2013; Pan, Webb, and Zank, 2015). Either a player's bias is taken to discontinuously differentiate instantaneous from delayed gratification (let $t \in \mathbb{R}_+$ above), in which case $x_1 \to \frac{1-\beta_2}{1-\beta_1\beta_2}$ as $\delta_i \to 1$ (regardless of relative speeds of convergence) and the bargaining outcome is fully determined by the players' very short-run impatience; the initial proposer's advantage then prevails for arbitrarily frequent offers, and—failing to generate an equal split—the model is at odds with the Nash bargaining solution.[29] Or an extended notion of the "present" of length $\tau_i > 0$ is adopted,

very same pattern of preference reversals that hyperbolic and quasi-hyperbolic discounting explain, and it can easily be verified that they, too, exhibit weak present bias.

[29]Notice that, however small a player's bias $\beta_i < 1$, in the limit she obtains none of the surplus in bargaining

such as $d_i(t)$ equal to δ_i^t whenever $t \leq \tau_i$ and $\beta_i \delta_i^t$ otherwise; then, however, as the length of a bargaining period falls below some player's τ_i, the model exhibits multiple equilibria and delay of the type presented in example 1 (there, $1 \leq \tau_1 < 2$).

A related conceptual issue arises concerning the possibly distinct times of agreement and consumption feasibility. If there is an exogenous lag $\hat{\tau}$ between agreement and consumption exceeding the length of time for which there is a "present bias", the unique equilibrium has immediate agreement with player 1's share equal to $x_1 = \frac{1-\delta_2^{\hat{\tau}+1}}{1-\delta_1^{\hat{\tau}+1}\delta_2^{\hat{\tau}+1}}$, where only the "long-run" discounting matters because each player i discounts even immediate agreements with extra factor β_i. Taking a broad perspective on what is being consumed, it could also be a bargainers' relevant others' esteem, proportional to the surplus she fetches; e.g., when a union leader negotiates on behalf of her union. The agreement reached might then differ drastically depending on whether the bargaining is done behind closed doors (there is a lag between agreement and consumption, and only long-run discounting matters) or in the presence of such relevant others (when the timing of agreement and consumption coincide, and the degrees of present bias are the main determinant of the division).[30]

4.2 Multiplicity, Non-Stationarity and Delay

Prior theoretical results regarding multiplicity and delay in the basic Rubinstein (1982) model with complete information are exclusively based on violations of increasing loss to delay that entail multiple stationary equilibria (e.g., Rubinstein, 1982; Hoel, 1986; Noor, 2011). Using these as history-dependent punishments, non-stationary equilibria can be constructed that, in some cases, support delayed agreement. Full equilibrium characterizations have, however, remained elusive even under ED, and the properties of such non-stationary equilibria have not been further investigated.

While the characterization of theorem 1 covers all of these cases—appendix B.3 characterizes equilibrium for the best-known example of multiplicity under ED—a central contribution of this paper is to demonstrate the emergence of non-stationary delay equilibria for empirically plausible time preferences (under standard "curvature" assumptions), independent of the number of stationary equilibria. Before discussing the qualitative features of these preferences, I first argue for the relevance of such equilibria by showing how they can capture two stylized tendencies in real bargaining: gradual agreement and equal surplus division.

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against an exponential discounter.

[30] I thank Erik Eyster and David Cooper for independently pointing out the following: any (common) lag between time of agreement and time of consumption does not affect the unique bargaining outcome under ED (this can be seen from the functions π_i), but under (β, δ)-discounting would shift bargaining power toward the player who is more patient in the long-run.

All non-stationary delay equilibria share the same fundamental strategic reasoning. Although Pareto-improvements are available, none of them gets proposed, because a proposing player believes that, by doing so, she would induce the opponent to expect an even superior (and non-Pareto-improving) agreement and, accordingly, reject such a proposal. This belief leads to offers that are, in turn, unfavourable vis-à-vis the delayed outcome for the respondent. As the time of agreement draws closer, the set of Pareto-improvements shrinks at the rate of the players' impatience, and they may reason and behave in this way while still making ever greater concessions, thus agreeing gradually.

Formally, for any equilibrium play $(x^n, Y^n)_{n=1}^{\hat{t}+1}$ which has agreement on division $x^{\hat{t}+1}$ reached with delay $\hat{t}$, define a player i's *concession* in round n, denoted b_i^n, as her offer x_j^n if she is the proposing player, $i = P(n)$, and the supremal share of the opponent she would accept, i.e. $\sup \{x_j \in [0,1] \,|\, x \in Y^n\}$, if she is the responding player, $i = R(n)$. Call an equilibrium with outcome $\left(\hat{x}, \hat{t}\right)$ a *gradual-agreement equilibrium* if its play has both players' concessions b_i^n increasing in n. Gradual agreement meaningfully applies only to equilibria with delay, all of which are non-stationary (proposition 1); then, however, the above requirement is rather strong, treating a player's offers and response rules symmetrically in terms of concessions (it clearly implies increasing offers by each player).

Proposition 5. *If both players are strictly impatient in the sense that, for both $i \in I$, $t < t'$ implies $U_i(q,t) > U_i(q,t')$ whenever $q > 0$, then every equilibrium outcome is the outcome of a gradual-agreement equilibrium.*[31]

Under gradual agreement, a player's concession has the interpretation of the credible promise that she will subsequently always be willing to give up at least this share, as long as the other player keeps to her promise. The fact that this promise has no material counterpart makes it distinct from the commitment mechanisms in related work explaining such "gradualism" (Admati and Perry, 1991; Compte and Jehiel, 2004).[32] These authors also provide anecdotal evidence for this tendency; it is, however, evident also in the laboratory (Zwick, Rapoport, and Howard, 1992; Kahn and Murnighan, 1993).

Another prominent empirical finding is that bargaining parties, especially in payoff-symmetric bargaining problems, tend to share the surplus equally, typically without delay (e.g., Weg, Rapoport, and Felsenthal, 1990; Roth, 1995). This coincides with the Nash bargaining solution, and it is also the limiting outcome of a unique stationary equilibrium as offers become arbitrarily frequent under ED (Binmore et al., 1986, prop. 4). In terms of the

[31] If the requirement for gradual agreement were weakened to *non-decreasing* concessions, this proposition would hold true for any preference profile.

[32] In these papers, the value of a player's outside option increases in the opponent's past concessions. Li (2007) obtains a similar effect with history-dependent preferences.

more general time preferences considered here, the following holds true.

Proposition 6. *If the two bargaining parties' preferences are identical, then an immediate equal split is an equilibrium outcome whenever delay is an equilibrium outcome.*

If (non-stationary) delay equilibria exist for identical preferences, they imply threats that support an immediate equal split. The proposition holds true without recourse to a limiting argument, even for non-negligible costs of disagreement; as offers become more frequent, delay equilibria are, however, more likely to exist (see example 3 below).

The unified explanation for these different, incompatible, tendencies through non-stationary delay equilibria comes with equilibrium multiplicity, as a stationary equilibrium always exists. Yet, upon taking the perspective of a bargainer facing such indeterminacy, one may reasonably expect behavior to become prone to influence by commonly shared norms of behaviour (promise-keeping) or focal points (symmetry), coordinating beliefs. Such auxiliary assumptions can therefore serve to sharpen predictions.

This section's results thus far hold true for any preference profile which implies existence of a delay equilibrium. Contrary to violations of increasing loss to delay, there is significant evidence for violations of weak present bias (on the domain of monetary rewards; see appendix B.1). The remainder of this section shows that the broad features of time preferences corresponding to these observed violations match exactly those which are theoretically conducive to the emergence of non-stationary delay equilibria.

Qualitatively, the preference property which invites delay equilibria is that one party is more averse to *near-future delay* than initial delay. Under discounting, using the decomposition $d_i(t) \equiv \prod_{s=1}^{t} \delta_i(s)$, this means that $\delta_i(s) < \delta_i(1)$ for $s > 1$ not too large; i.e., a near-future period of delay is discounted more heavily than the first one. Graphically, one should think of discounting functions that are initially concave (for functional forms see Ebert and Prelec 2007; Bleichrodt, Rohde, and Wakker 2009; Takeuchi 2011). The resulting type of dynamic inconsistency makes a player vulnerable: she becomes *relatively*, say in terms of her average discount rate, more patient about a given delayed (near-future) agreement as time passes and therefore takes an increasingly tougher stance in bargaining; in fact, more than she would want to initially, when she is therefore willing to make significant concessions to avoid the painful delay. This strengthens her opponent's position enabling delay to be *self-enforcing*: the "bad" deal a player is willing to accept in order to avoid this delay supports the tough play that leads to it.

To illustrate, consider the simpler characterization of theorem 1 for "discounted shares" where $U_i(q,t) = d_i(t) \cdot q$: then $w_i^* = \Delta_i(t_i^* + 1)\, v_i^*$ for $\Delta_i(t) \equiv \inf\{\delta_i(s)\,|s \in T,\ s \leq t\}$ the minimal per-period discount factor over a delay-horizon of t, and the remaining values

$(v_i^*, t_i^*)_{i\in I}$ are the unique extreme element of the solutions $(v_i, t_i)_{i\in I}$ to

$$\begin{aligned} v_i &= \frac{1-\Delta_j(1)}{1-\Delta_i(t_i+1)\Delta_j(1)} \\ t_i &= \sup\{t\in T|\kappa_i(t, v_i, v_j, \Delta_j(t_j+1)v_j)\le 1\}, \end{aligned} \tag{8}$$

for both $i \in I$.[33] Equation 8, necessarily satisfied by a player i's minimal proposer value v_i^*, is reminiscent of the familiar solution under ED. In this case $\delta_i(\cdot)$ and hence $\Delta_i(\cdot)$ is a constant δ_i, and v_i^* equals $(1-\delta_j)/(1-\delta_i\delta_j)$. The sole difference is the term $\Delta_i(t_i^*+1)$ that reflects that a player's punishment is possibly made more severe through delay; it exploits i's most painful equilibrium delay, which is not necessarily the longest. Since Δ_i is the smallest per-period discount factor over a given horizon of potential delays, non-stationary delay equilibria might emerge due to only relatively small fluctuations in how various periods are discounted.[34] The following example concludingly illustrates various points of this section.

Example 3. Let the two players' preferences be given by $U_i(q,t) = d(t)\cdot q$ such that

$$d(t) = \begin{cases} \delta^t & t\le\tau \\ \gamma\delta^t & t>\tau \end{cases}, \ (\delta,\gamma)\in(0,1)^2 \text{ and } \tau>0.$$

The $\tau+1$-th period of delay is discounted most heavily: whereas the per-period discount factors are $\delta(t)=\delta$ for all $t\in T\setminus\{0,\tau+1\}$, for that period it is $\delta(\tau+1)=\gamma\delta$; since $\tau>0$, weak present bias is violated, and there is instead a bias toward not experiencing more than τ periods of delay. Hence $\Delta(t)$ equals δ for all $t\le\tau$ and $\gamma\delta$ for all $t>\tau$; given Δ determines whether non-stationary delay equilibria emerge, this minimal deviation of ED is made only for convenience, to keep the number of parameters down to a mere three, $\{\delta,\gamma,\tau\}$.

Due to identical preferences, the player subscript is omitted in what follows. Suppose there is an equilibrium in which agreement is delayed by τ periods: then $v^* = \frac{1-\delta}{1-\gamma\delta^2}$ (from equation 8) and $w^* = \gamma\delta v^*$; a delay $\tau>0$ is "self-enforcing" if and only if $1 \ge \kappa(\tau, v^*, v^*, w^*) = \frac{v^*}{\delta^\tau} + \frac{v^*}{\delta^{\tau-1}}$, which reduces to

$$\delta^\tau \ge (1+\delta)\cdot\frac{1-\delta}{1-\gamma\delta^2} \tag{9}$$

[33] Inequality 7 becomes superfluous (see appendix B.2). For per-period discount factors to be well-defined, $d_i(t)>0$ is required, which is implied by the version of desirability given in assumption 1; for a generalization, set $\delta_i(s)=1$ whenever $d_i(s-1)=0$. $\triangle_i(t_i^*+1)$ is indeed a minimum due to impatience property (4.c).

[34] More precisely, "relative" refers to how the size of a drop from $\Delta_i(t)$ to $\Delta_i(t+1)$ compares to players' overall patience for delays of $t-1$ and t periods, i.e. $d_i(t)$ and $d_j(t-1)$. Greater patience in these terms means smaller drops suffice for non-stationary delay equilibria.

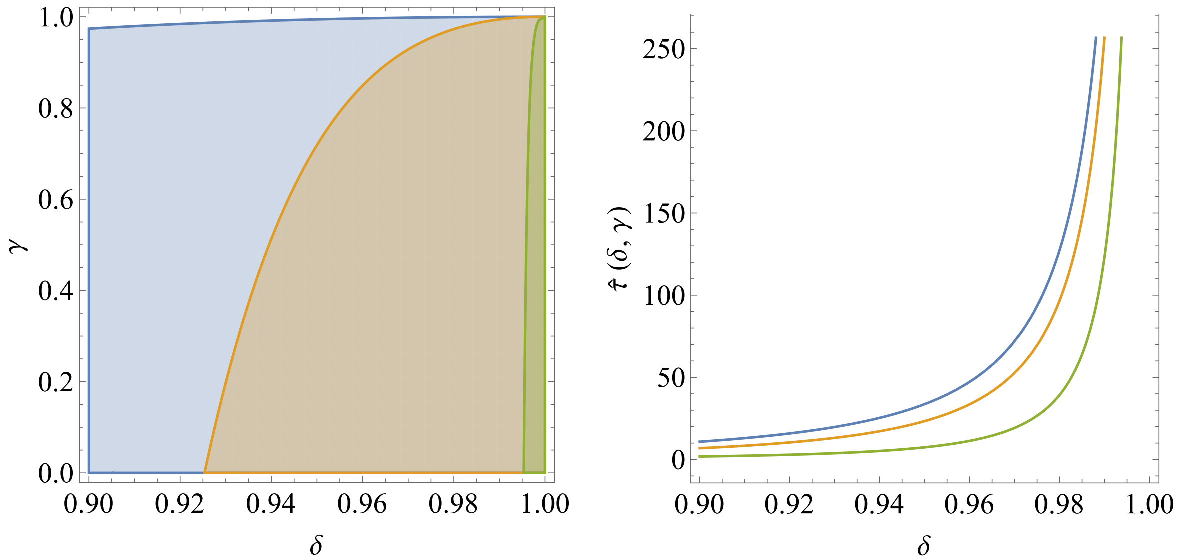

Figure 2: Graphs regarding equilibrium delay in example 3. The panel on the left shows the parametric regions (δ, γ) such that delay equilibria exist for three given values of τ, which are 1 (blue, brown and green), 25 (brown and green) and 1000 (green). The panel on the right plots $\hat{\tau}(\delta, \gamma)$ as a function of δ for three given values of γ, which are 0.5 (blue), 0.75 (brown) and 0.99 (green).

after substituting for v^*. The left-hand side is the present value of the surplus, and the right-hand side is the present value of the incentive cost of a delay of τ periods: each proposer requires $v^* = \frac{1-\delta}{1-\gamma\delta^2}$, and the factor $(1+\delta)$ is due to the fact that the initial proposer does so immediately whereas the other player does so only next round. Observe that, for any given $\tau > 0$ and $\gamma < 1$, there exist large enough values of δ such that inequality 9 is satisfied (the left-hand side limits to one whereas the right-hand side limits to zero as $\delta \to 1$); generally, as δ increases, the set of parameters γ and τ for which delay equilibria exist expands, as the left-hand-side panel of figure 2 illustrates. Whenever such delay equilibria exist, the minimal proposer and rejection values are obtained only by means of a "truly" non-stationary delay equilibrium, using optimal punishments.

Notice also that inequality 9 implies $w^* < v^* < \frac{v^*}{\delta^{\tau-1}} \leq \frac{1}{2}$, and an equal split with any delay up to $\tau - 1$ periods is then an equilibrium outcome (in particular under immediate agreement). It may also be reached gradually, say with delay $\hat{t}$, $0 < \hat{t} < \tau$: define a sequence $(b^n)_{n=1}^{\hat{t}+1}$ of "concessions" such that $b^1 \equiv 0$ and $b^n \equiv \frac{1}{2}\left(b^{n-1} + \delta^{\hat{t}+1-n} \cdot \frac{1}{2}\right)$, noting that the sequence is increasing, and that b^n falls short of a player's present value of agreeing on an equal split with the delay $\hat{t} + 1 - n$ that remains as of the n-th round, which is $\delta^{\hat{t}+1-n} \cdot \frac{1}{2}$. It is straightforward to verify that the following describes equilibrium play with gradual agreement: in any (disagreement) round n, $1 \leq n < \hat{t} + 1$, the proposing player $P(n)$ offers the share b^n, and the responding player $R(n)$ accepts with threshold $1-b^{n+1}$ ($b^n < 1-b^{n+1}$ follows from $b^n < b^{n+1} < \frac{1}{2}$); in (agreement) round $n = \hat{t}+1$, the proposing player $P\left(\hat{t}+1\right)$ offers the share $\frac{1}{2}$, and the responding player $R\left(\hat{t}+1\right)$ accepts with threshold $\frac{1}{2}$.

Solving for τ, inequality 9 becomes

$$\tau \leq \frac{\ln\left(1-\delta^2\right) - \ln\left(1-\gamma\delta^2\right)}{\ln\left(\delta\right)} \equiv \hat{\tau}\left(\delta,\gamma\right),$$

and if it is satisfied, the maximal delay t^* equals $\lfloor \hat{\tau}(\delta,\gamma) \rfloor$ (the greatest integer not exceeding $\hat{\tau}(\delta,\gamma)$). For any $\gamma < 1$, it approaches infinity as $\delta \to 1$, showing once more how small deviations from ED result in the emergence of delay equilibria as offers become very frequent; e.g., $\lfloor \hat{\tau}(\delta,\gamma) \rfloor = 404$ in case $\delta = \gamma = 0.999$. The right-hand-side panel of figure 2 shows how $\hat{\tau}(\delta,\gamma)$ approaches infinity as $\delta \to 1$.

The resulting delays can be very costly. The present value of the surplus in an equilibrium where agreement is maximally delayed equals $\gamma\delta^{t^*}$ whenever $\tau \leq \hat{\tau}(\delta,\gamma)$. As $\delta \to 1$, for any given $\gamma < 1$, not only is $\tau \leq \hat{\tau}(\delta,\gamma)$ going to be satisfied, but the entire surplus vanishes. For instance, while in the case of $\delta = \gamma = 0.99$ the maximal surplus loss amounts to roughly one third of the surplus, for values of γ that fall short of δ, the loss can be dramatic: up to 99.8% of the surplus can be lost through delay when $\delta = 0.99999$ and $\gamma = 0.99$.

When players discount the future only up to a finite number of delays, equilibrium delay can even be unbounded. Example 6 in appendix B.4 demonstrates this point, by only slightly modifying the example given here.

5 Concluding Remarks

This paper has revisited a central question in bargaining theory: how do bargaining outcomes, in particular the incidence of inefficient delay, depend on the parties' attitudes to delay? Prior results were predicated either on the assumption that these attitudes conform to ED or on the assumption that the parties' behavior is unresponsive to observed past behavior (i.e. stationary). Here, I have provided the first general answer to this question, fully characterizing equilibrium in the canonical Rubinstein (1982) model of alternating-offers bargaining under minimal assumptions on the parties' attitudes to delay, and for arbitrarily history-dependent behavior.

Together, the generality of preferences and strategies considered in this paper required a novel analytical approach. I derived a simple, yet sufficiently general structure for off-path punishments that makes non-stationarity of equilibria tractable in this game also under dynamic inconsistency. To the best of my knowledge, this is the first stationary sequential-move game where such optimal simple penal codes were shown to be both existing and useful. Future research should clarify how useful this approach is in other, related games, especially regarding analyses of dynamically inconsistent preferences.

When at least one of the parties finds a delay beyond some time in the near future more costly than initial delay from the present, this basic and otherwise disciplined model provides a rich descriptive theory of bargaining. Although there is no material reason for why consequentialist bargainers should ever care about how they have failed to agree in the past—they can therefore always ignore history and stick to a stationary equilibrium—such dynamic inconsistency provides a motive for treating observed past behavior as indicative of future behavior. This allows to capture the main stylized tendencies in real bargaining.

Although preferences of this type could be peculiar to bargaining situations, they are not: the recent empirical research on time preferences suggests that such near-future bias is a more general phenomenon concerning how many people evaluate delayed *monetary* rewards (appendix B.1). It is a recent "discovery" mainly because this empirical research is only beginning to move on from disproving the universal nature of present bias, especially of the quasi-hyperbolic type, towards investigating what forms time preferences over money actually take. Concerning a hedonic notion of utility, a present bias (which I take to mean a bias toward *instantaneous* gratification) is intuitive, and the quasi-hyperbolic model of discounting has proven enormously useful in explaining many important behaviors (e.g., Laibson, 1997; O'Donoghue and Rabin, 1999; Bénabou and Tirole, 2002). Money would seem to be only a means to pleasure, however, in which case only people's borrowing and lending rates could be elicited using monetary rewards. What drives the observation that the vast majority of participants nonetheless reveal biases—either present or near-future bias—is currently not understood and awaits further investigation. Whatever we will come to conclude from such research, however, it seems fair to assume that any useful model of time preferences will be covered by this paper's minimal assumptions, whereby its basic bargaining implications will have been characterized here.

In fact, for many settings—certainly those in which intertemporal trade-offs occur mainly across or with sufficiently long delays (beyond a week, say; cf. Sayman and Öncüler, 2009, p. 470)—a genuine present bias is indistinguishable from a near-future bias. Bargaining is somewhat special in that the parties may interact with high frequency, and this distinction becomes behaviorally relevant. Since present bias implies immediate agreement, this observation suggests an intervention to restore bargaining efficiency under near-future bias: limiting the frequency of offers. Conditional on failing to agree, parties would thus be committed to not agreeing for sufficiently long, so that the first period of delay is most costly; thus present bias is induced, which leads to immediate agreement.

I have conducted the analysis under the assumption that each party has perfect knowledge of her own as well as her opponent's preferences. A near-future biased bargainer suffers from her dynamic inconsistency because she knows she will take a tougher stance in the future than she currently would like to; present bias is here a force toward immediate agreement because it is also correctly anticipated. The question arises how much sophistication is required for these results, especially in view of potential pitfalls of the intervention just suggested for the case of near-future bias. Assuming quasi-hyperbolic discounting, Akin (2007) has already shown that when the parties are persistently naïve about their own present bias but sophisticated about that of the opponent, there can be severe delays.

Although this paper has emphasized time preferences in a strict sense, there certainly exist sources other than pure time which induce various dynamically inconsistent time preferences—of both the present- and future-biased type—in bargaining. In appendix B.5 I sketch bargaining models for two of them where the results of this paper directly apply: imperfect altruism regarding future generations' bargaining outcomes (cf. Phelps and Pollak, 1968), and nonlinear probability weighting when bargaining takes place under the shadow of breakdown risk (cf. Barberis, 2012). In conclusion, both the analytical approach and the mechanisms discovered in this paper for reduced-form time preferences should therefore serve as a useful guide for further analyses of psychologically enriched preferences in strategic bargaining, and—plausibly—even beyond.

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Appendix

A Proofs

A.1 Lemma 1

Proof. By contradiction. Suppose D is an equilibrium outcome of G_i for some $i \in I$, and consider its first round. In any equilibrium, respondent j accepts any offered share $x_j > \pi_j(U_j(1,1))$ because $U_j(1,1)$ is the maximal possible rejection value. Proposing division x such that $\pi_j(U_j(1,1)) < x_j < 1$, which exists due to j's impatience, i obtains utility $u_i(x_i) > u_i(0) \geq U_i(0,\infty)$. □

A.2 Proposition 1

Define function $f_i : [0,1] \to [0,1]$ for each $i \in I$ as

$$f_i(q) \equiv 1 - \pi_j(U_j(1 - \pi_i(U_i(q,1)), 1)); \tag{10}$$

the next lemma prepares for the proposition by means of a formal investigation of the fixed-point properties of f_i.

Lemma 2. *For each $i \in I$, the set of fixed points of function f_i is non-empty and closed, and any such fixed point is a positive share; moreover, if q is a fixed point of f_i, then $1 - \pi_i(U_i(q,1))$ is a fixed point of f_j, and f_1 has a unique fixed point if and only if f_2 has a unique fixed point.*

Proof. Take any $i \in I$. The function f_i is continuous and satisfies $0 < f_i(0) \leq f_i(1) \leq 1$ by the continuity and impatience of preferences; hence, it has a fixed point by the intermediate-value theorem. Since the set of fixed points is the intersection of two closed graphs—those of the continuous function f_i and the identity function which is continuous as well—it is closed. Moreover, f_i is non-decreasing by desirability, so any fixed point is a positive share.

Now suppose q is a fixed point of f_i for some $i \in I$ and let $q' = 1 - \pi_i(U_i(q,1))$; then

$$\begin{aligned} f_j(q') &= 1 - \pi_i(U_i(1 - \pi_j(U_j(1 - \pi_i(U_i(q,1)), 1)), 1)) \\ &= 1 - \pi_i(U_i(f_i(q), 1)) \\ &= 1 - \pi_i(U_i(q,1)) \\ &= q'. \end{aligned}$$

For the last claim, suppose q is in fact the unique fixed point of f_i; it remains to show that then q' is the unique fixed point of f_j. Suppose not, and that f_j had another fixed point $q'' \neq q'$. Then also $q''' = 1 - \pi_j (U_j (q'', 1))$ would be a fixed point of f_i; because q is the unique fixed point of f_i, $q''' = q$ has to hold. Moreover, since q' is a fixed point of f_j, it must be that also $1 - \pi_j (U_j (q', 1)) = q$ from repeating the above argument. But this leads to a contradiction as follows:

$$1 - q = \pi_j (U_j (q', 1)) = \pi_j (U_j (q'', 1)) \Rightarrow f_j (q') = f_j (q'') \Rightarrow q' = q''.$$

□

We are then ready to prove the proposition.

Proof. Consider a profile of stationary strategies $\left(x^{(i)}, Y^{(i)}\right)_{i \in I}$ which constitutes equilibrium. If $x^{(1)} \notin Y^{(2)}$ then the outcome under this profile in G_1 must be $\left(x^{(2)}, 1\right)$, since D is not an equilibrium outcome by lemma 1. Because this outcome obtains irrespective of play in the initial round of G_1, its responding player 2 must accept any proposal x with $x_2 > \pi_2 \left(U_2 \left(x_2^{(2)}, 1\right)\right)$. Impatience property (4.b) of player 2's preferences implies that either (i) $\pi_2 \left(U_2 \left(x_2^{(2)}, 1\right)\right) < x_2^{(2)}$ or (ii) $\pi_2 \left(U_2 \left(x_2^{(2)}, 1\right)\right) = x_2^{(2)} = 0$. In case of (i) there exist values $\epsilon > 0$ such that $\epsilon < x_2^{(2)} - \pi_2 \left(U_2 \left(x_2^{(2)}, 1\right)\right)$, and any of them satisfy

$$u_1 \left(1 - \pi_2 \left(U_2 \left(x_2^{(2)}, 1\right)\right) - \epsilon\right) > U_1 \left(1 - \pi_2 \left(U_2 \left(x_2^{(2)}, 1\right)\right) - \epsilon, 1\right) \geq U_1 \left(1 - x_2^{(2)}, 1\right)$$

by impatience property (4.b) and desirability of player 1's preferences, applied in this sequence. In case of (ii), impatience property (4.b) together with continuity of player 1's preferences imply existence of $\epsilon > 0$ such that $u_1 (1 - \epsilon) > U_1 (1, 1)$. In any case therefore, player 1 can propose immediately accepted divisions that yield a payoff greater than proposing $x^{(1)}$, contradicting equilibrium. After a symmetric argument, it is then proven that $x^{(i)} \in Y^{(j)}$ for both $i \in I$.

Given this immediate-agreement property of stationary equilibrium, by desirability, (i) a responding player j must accept any offered share $x_j > \pi_j \left(U_j \left(x_j^{(j)}, 1\right)\right)$ as well as reject any $x_j < \pi_j \left(U_j \left(x_j^{(j)}, 1\right)\right)$, and (ii) there cannot exist a proposal x by player i with $x_i > x_i^{(i)}$ such that $x \in Y^{(j)}$, whereby

$$x_i^{(i)} = 1 - \pi_j \left(U_j \left(x_j^{(j)}, 1\right)\right) \text{ and } Y^{(i)} = X_{i, x_i^{(j)}},$$

and substituting the expression for $x_2^{(2)}$ into that for $x_1^{(1)}$ yields $x_1^{(1)} = f_1 \left(x_1^{(1)}\right)$, establishing necessity. Sufficiency is easily verified, and its proof omitted here. □

A.3 Corollary 1

Proof. Proposition 1 implies that there are as many distinct stationary equilibria as there are distinct fixed points of f_1, and lemma 2 thus establishes existence. $\square$

A.4 Proposition 2

Proof. Consider the following difference:

$$\begin{aligned} q - f_1(q) &= q - 1 + \pi_2\left(U_2\left(1 - \pi_1\left(U_1(q,1)\right),1\right)\right) \\ &= \left[q - \pi_1\left(U_1(q,1)\right)\right] - \left[\left(1 - \pi_1\left(U_1(q,1)\right)\right) - \pi_2\left(U_2\left(1 - \pi_1\left(U_1(q,1)\right),1\right)\right)\right]. \end{aligned}$$

If player 1's preferences exhibit initially increasing loss to delay, the first term in square brackets is increasing in q. The second such term is increasing in $1 - \pi_1(U_1(q,1))$ if player 2's preferences exhibit initially increasing loss to delay, and since $1 - \pi_1(U_1(q,1))$ is non-increasing in q, overall the two terms' difference is increasing. Then $q - f_1(q)$ has at most one root, and application of lemma 2 yields the result that f_1 has a unique fixed point. Finally, apply corollary 1. $\square$

A.5 Proposition 3

Proof. **Part (i).** Recalling lemma 1, let $\langle\hat{\sigma}\rangle$ be an equilibrium play of G_k with outcome $\left(\hat{x},\hat{t}\right)$. This implies existence of an equilibrium of G_k with the same play and outcome; without loss of generality, assume $\hat{\sigma}$ is itself such an equilibrium. Also, let $\sigma^* = \sigma^*\left(\langle\hat{\sigma}\rangle \middle| \left(\sigma^{P,i},\sigma^{R,i}\right)_{i\in I}\right)$ and recall that, by construction, $\langle\sigma^*\rangle = \langle\hat{\sigma}\rangle$. Continuation play under σ^* following any deviation from its path is an equilibrium of the resulting subgame. In order to verify that σ^* is an equilibrium it therefore suffices to verify that there are no profitable one-stage deviations at the histories $h^{n-1}(\sigma^*) = h^{n-1}(\hat{\sigma})$ along its path of play.

Take then such a history $h = h^{n-1}(\sigma^*)$, where player P makes an offer to player R, and $\sigma^*(h) = \hat{\sigma}(h) = \left(\tilde{x},\tilde{Y}\right)$. Consider any proposal $x' \in \tilde{Y}$; $\hat{\sigma}$'s being an equilibrium and the construction of σ^* imply that

$$u_R\left(x'_R\right) \geq U_R\left(z_R^h\left(x',\emptyset \middle| \hat{\sigma}\right)\right) \geq \min\left\{U_R\left(x_R,t+1\right) \middle| (x_R,t) \in A_R^*\right\} = U_R\left(z_R^h\left(x',\emptyset \middle| \sigma^*\right)\right),$$

whereby acceptance is optimal for R under σ^*.

Next, consider any proposal $x' \notin \tilde{Y} \setminus \{\tilde{x}\}$; $\hat{\sigma}$'s being an equilibrium and the construction of σ^* imply that

$$u_R(x'_R) \le U_R\left(z_R^h(x', \emptyset|\, \hat{\sigma})\right) \le \max\{U_R(x_R, t+1)\,|(x_R, t) \in A_R^*\} = U_R\left(z_R^h(x', \emptyset|\, \sigma^*)\right),$$

whereby rejection is optimal for R under σ^*.

The only remaining case at the responding stage is that of proposal $\tilde{x}$ such that $\tilde{x} \notin \tilde{Y}$; this implies that $n < \hat{t} + 1$, and then $\hat{\sigma}$'s being an equilibrium play and the construction of σ^* imply that

$$u_R(\tilde{x}_R) \le U_R\left(z_R^h(\tilde{x}, \emptyset|\, \hat{\sigma})\right) = U_R\left(\hat{x}_R, \hat{t} + 1 - n\right) = U_R\left(z_R^h(\tilde{x}, \emptyset|\, \sigma^*)\right),$$

whereby rejection is optimal for R under σ^*.

Finally, consider the proposing player P's incentive to propose $x' \neq \tilde{x}$: if $x' \in \tilde{Y}$, then $u_P(x'_P) \le U_P\left(z_P^h\left(\tilde{x}, \tilde{Y}\middle|\, \hat{\sigma}\right)\right)$ by $\hat{\sigma}$'s being an equilibrium, and because of $z_P^h\left(\tilde{x}, \tilde{Y}\middle|\, \sigma^*\right) = z_P^h\left(\tilde{x}, \tilde{Y}\middle|\, \hat{\sigma}\right) = \left(\hat{x}_P, \hat{t} + 1 - n\right)$ such deviations are not profitable to P under σ^*.

Given $q_R^* = \pi_R\left(U_R\left(x_R^{P,P}, t^{P,P} + 1\right)\right)$, it follows from $\hat{\sigma}$'s being an equilibrium that $\{x \in X\,|x_R > q_R^*\} \subseteq \tilde{Y}$ and $u_P(1 - q_R^*) \le U_P\left(\hat{x}_P, \hat{t} + 1 - n\right)$: R must accept any offer which exceeds her maximal credible reservation share, and if $u_P(1 - q_R^*) > U_P\left(\hat{x}_P, \hat{t} + 1 - n\right)$ were true, then, because $u_P(\cdot)$ is continuously increasing and $q_R^* < 1$ due to R's impatience, there would exist $\epsilon > 0$ such that P's offering the accepted share $q_R^* + \epsilon$ would be a profitable deviation under $\hat{\sigma}$. Because, under σ^*, any deviant proposal $x' \notin \tilde{Y}$ yields utility $U_P\left(x_P^{P,P}, t^{P,P} + 1\right)$, which is less than $u_P(1 - q_R^*)$ by impatience, also no such deviation is profitable for P. While this proves necessity in part (i), sufficiency is immediate from $\langle\sigma^*\rangle = \langle\hat{\sigma}\rangle$.

Part (ii). Fix any four plays $\left(\left\langle\sigma^{P,i}\right\rangle, \left\langle\sigma^{R,i}\right\rangle\right)_{i\in I}$, where, for each $i \in I$, $\left\langle\sigma^{P,i}\right\rangle$ is a play of game G_j and $\left\langle\sigma^{R,i}\right\rangle$ is a play of game G_i, and define the mapping $\sigma^{**}\left(\cdot\,\middle|\left(\left\langle\sigma^{P,i}\right\rangle, \left\langle\sigma^{R,i}\right\rangle\right)_{i\in I}\right)$ such that, it assigns a strategy profile σ^{**} in game G_k to any play $\langle\sigma^0\rangle$ of this game, for any $k \in I$, as follows. Identifying any play $\langle\sigma\rangle \in \{\langle\sigma^0\rangle\} \cup \left\{\left\langle\sigma^{P,i}\right\rangle, \left\langle\sigma^{R,i}\right\rangle\right\}_{i\in I}$ with a sequence of states $(\langle\sigma\rangle_m)_{m=1}^n$, where n is such that $\langle\sigma\rangle \in (X \times \mathcal{P}(X))^n$, and letting $\chi(\langle\sigma\rangle_m) \equiv \sigma_{P(m)}(h^{m-1}(\sigma))$ and $\Upsilon(\langle\sigma\rangle_m) \equiv \sigma_{R(m)}(h^{m-1}(\sigma))$ for any $m \le n$ (note that it is possible to extract this information from play only), σ^{**} is defined such that:

(1) round 1 is in state $\langle\sigma^0\rangle_1$,

(2) if round n is in state $\langle\sigma\rangle_m$, then $P(n)$ makes proposal $\chi(\langle\sigma\rangle_m)$ and $R(n)$ uses response rule $\Upsilon(\langle\sigma\rangle_m)$, and

(3) if round n is in state $\langle\sigma\rangle_m$ and $R(n)$ rejects proposal x, then round $n+1$ is in state

$$\tau\left(\langle\sigma\rangle_m, x\right) = \begin{cases} \langle\sigma\rangle_{m+1} & x = \chi\left(\langle\sigma\rangle_m\right) \notin \Upsilon\left(\langle\sigma\rangle_m\right) \\ \left\langle\sigma^{P,P(n)}\right\rangle_1 & x \neq \chi\left(\langle\sigma\rangle_m\right) \notin \Upsilon\left(\langle\sigma\rangle_m\right) \\ \left\langle\sigma^{R,R(n)}\right\rangle_1 & x \neq \chi\left(\langle\sigma\rangle_m\right) \in \Upsilon\left(\langle\sigma\rangle_m\right) \end{cases}.$$

It is straightforward to verify that σ^{**} is indeed a strategy profile in game G_k. Now take any quadruple of equilibria $\left(\sigma^{P,i}, \sigma^{R,i}\right)_{i\in I}$ as in part (i) of the proposition and construct, for each $i \in I$, the strategy profiles

$$\hat{\sigma}^{P,i} = \sigma^{**}\left(\left\langle\sigma^{P,i}\right\rangle \middle| \left(\left\langle\sigma^{P,i}\right\rangle, \left\langle\sigma^{R,i}\right\rangle\right)_{i\in I}\right) \quad \text{and} \quad \hat{\sigma}^{R,i} = \sigma^{**}\left(\left\langle\sigma^{R,i}\right\rangle \middle| \left(\left\langle\sigma^{P,i}\right\rangle, \left\langle\sigma^{R,i}\right\rangle\right)_{i\in I}\right).$$

By construction, $\left(\hat{\sigma}^{P,i}, \hat{\sigma}^{R,i}\right)_{i\in I}$ are equilibria as in part (i) of the proposition and, moreover, satisfy (3).

Part (iii). To ease notation, I consider only equilibrium outcomes of G_1; a mere exchange of player indices yields the proof for G_2. Let $\left(\sigma^{P,i}, \sigma^{R,i}\right)_{i\in I}$ be a quadruple of equilibria as in part (i). Define also each player i's minimal reservation share $q_i^{**} \equiv \pi_i\left(U_i\left(x_i^{R,i}, t^{R,i}+1\right)\right)$, where, clearly, $q_i^{**} \leq q_i^*$.

The first step is show that there exists a strategy profile σ' of game G_1 such that its play $\langle\sigma'\rangle$ equals that of the proposition's statement. While trivial for $\hat{t} = 0$, if $\hat{t} > 0$, it requires that $q_2^* > 0$, and if $\hat{t} > 1$, it requires additionally that $q_1^* > 0$. Now observe that if $q_i^* = 0$ then in any equilibrium, in particular $\hat{\sigma}$, whenever i responds, she accepts any positive offer. Hence there cannot be an equilibrium in which i rejects a zero offer by player j because for offers $\epsilon > 0$ small enough, $u_j(1-\epsilon) > U_j(1,1)$ by j's impatience.

The second step is to prove that for any such strategy profile σ', the strategy profile $\sigma = \sigma^*\left(\sigma' \middle| \left(\sigma^{P,i}, \sigma^{R,i}\right)_{i\in I}\right)$ is an equilibrium, where it suffices to verify that there are no profitable one-stage deviations at the histories $h^{n-1}(\sigma)$, $n \leq \hat{t}+1$, since the continuation strategy profiles $\left(\sigma^{P,i}, \sigma^{R,i}\right)_{i\in I}$ are all equilibria of their respective subgames. Consider then any such history $h = h^{n-1}(\sigma)$, where player P is supposed to propose division $\tilde{x}$ and player is supposed to accept with threshold $\tilde{q}$; if $n < \hat{t}+1$ then $(\tilde{x}, \tilde{q}) = \left(e^{(P)}, q_R^*\right)$, if $n = \hat{t}+1$ then $(\tilde{x}, \tilde{q}) = (\hat{x}, \hat{x}_R)$. Observe that, by construction, $\tilde{x}_R \leq q_R^*$: either $\tilde{x}_R = 0$, or $\tilde{x}_R$ equals $\hat{x}_R$, which, otherwise, could not be an equilibrium offer by P, since R must accept any offer $q > q_R^*$ in any equilibrium, and P could therefore profitably deviate to offering a lower share $q \in (q_R^*, \hat{x}_R)$. Similarly, $\tilde{q} \geq q_R^{**}$: either $\tilde{q} = q_R^*$, or $\tilde{q}$ equals $\hat{x}_R$, which, otherwise, could not be an equilibrium offer which R accepts, since R would gain from rejection for any continuation equilibrium (note that $\tilde{q} < q_R^{**}$ implies $U_R\left(x_R^{R,R}, t^{R,R}+1\right) = u_R\left(q_R^{**}\right)$).

R's rejection of any *deviant* offer $q < \tilde{q}$ is optimal: whenever such offers exist, $q_R^* > 0$ and rejection yields $U_R\left(x_R^{P,P}, t^{P,P}+1\right) = u_R\left(q_R^*\right)$ which is greater than the value of acceptance $u_R(q)$ since $q_R^* \geq \tilde{q} > q$. Since R's impatience implies that $x_R^{P,P} > q_R^*$, $U_P\left(x_P^{P,P}, t^{P,P}+1\right) < u_P\left(1-q_R^*\right) \leq u_P\left(1-\tilde{q}\right)$; hence such deviations could not be profitable for P.

R's acceptance of any offer $q \geq \tilde{q}$ is optimal since it yields at least value $u_R(\tilde{q})$ whereas rejection yields at most $u_R\left(q_R^{**}\right)$. Among these offers, $\tilde{q}$ is clearly the best accepted offer for P. If $n = \hat{t}+1$, this shows there is no profitable deviation for either player.

Finally, consider then the case of $n < \hat{t}+1$: regarding R's incentives, it remains to verify optimality of rejecting the non-deviant zero offer which yields $U_R\left(\hat{x}_R, \hat{t}-n+1\right)$; if this were less than $u_R(0)$, however, there cannot be an equilibrium with outcome $\left(\hat{x}, \hat{t}\right)$ because there is no equilibrium offer which R would optimally reject. Regarding P's incentives, it remains to verify that making the zero offer which yields $U_P\left(\hat{x}_P, \hat{t}-n+1\right)$ is no worse than making the lowest accepted offer of q_R^*; if this were true, however, there cannot be an equilibrium with outcome $\left(\hat{x}, \hat{t}\right)$ because there is no equilibrium offer which P would optimally make that is rejected. □

A.6 Theorem 1

The proof proceeds via a series of lemmas, two of which—lemmas 4 and 8—jointly imply theorem 1. The first lemma makes an important observation concerning elements $(v_i, w_i, t_i)_{i \in I}$ of the set E: w_i is obtained as i's lowest *rejection* value from a *finite* sequence of personal outcomes with the same *continuation* value v_i. It is here that impatience property (4.c) plays its role: it allows to bound this sequence in terms of the delay of outcomes, ensuring existence of a personal outcome yielding a minimal rejection value for any value of v_i and thus of an optimal respondent punishment outcome. Otherwise the associated sequence of rejection values might decrease forever in delay, and there would be no optimal such punishment.

Lemma 3. *If $(v_i, w_i, t_i)_{i \in I} \in E$, then, for both $i \in I$: for any delay $t \in T$ such that $t \leq t_i$,*

$$U_i\left(\phi_i\left(v_i, t\right), t\right) = v_i,$$

and there exists $\bar{t}_i \in T$ such that

$$w_i = \min\left\{U_i\left(\phi_i\left(v_i, t\right), t+1\right) | t \in T,\ t \leq \bar{t}_i\right\}.$$

Proof. To reserve index i for general statements, I will prove the two claimed properties for $i = 1$; the proof for $i = 2$ follows from mere relabeling.

Observe that $v_i \geq \underline{v}_i \equiv u_i\left(1-\pi_j\left(U_j\left(1,1\right)\right)\right)$ for both i; since $\pi_j\left(U_j\left(1,1\right)\right)<1$ due to j's impatience property (4.b), this implies that $v_i > u_i\left(0\right)$, which is equivalent to $\phi_i\left(v_i,0\right)>0$ and thus yields that $\phi_i\left(v_i,t\right)>0$ for all $t\in T$.

The first claim holds true by definition for $t=0$, so consider it for $0<t\leq t_1$. Inequality 7 implies that t_1 is the supremum taken over a non-empty set, and therefore

$$\kappa_1\left(t,v_1,v_2,w_2\right)\leq 1 \Leftrightarrow \phi_1\left(v_1,t\right)\leq 1-\max\left\{\phi_2\left(v_2,t-1\right),\phi_2\left(u_2\left(0\right),t\right)\right\};$$

because, $\phi_2\left(v_2,t-1\right)>0$, it follows that $\phi_1\left(v_1,t\right)<1$; hence $U_1\left(\phi_1\left(v_1,t\right),t\right)=v_1$.

The second claim holds true if t_1 is finite: simply set $\bar{t}_1=t_1$ and recall that any finite set of real numbers has a minimum value. Consider it then for $t_1=\infty$, where the first claim implies that $U_1\left(\phi_1\left(v_1,t\right),t\right)=v_1$ for all $t\in T$, and distinguish the two possible cases according to impatience property (4.c). Suppose, first, that player 1's preferences satisfy $\lim_{t\to\infty}U_1\left(1,t\right)\leq u_1\left(0\right)$. There then exists a finite delay $\hat{t}$ such that $t\geq\hat{t}$ implies $U_1\left(1,t\right)<\underline{v}_1$; since $v_1\geq\underline{v}_1$, $U_1\left(\phi_1\left(v_1,t\right),t\right)<v_1$ follows, a contradiction. The alternative case is that there exists $\hat{t}\in T$ such that $\lim_{t\to\infty}U_1\left(q,t\right)=U_1\left(q,\hat{t}\right)$ for all $q\in[0,1]$; combined with impatience property (4.a), this means that $U_1\left(q,t\right)=U_1\left(q,\hat{t}\right)$ for all shares q and delays $t\geq\hat{t}$, implying $U_1\left(\phi_1\left(v_1,t\right),t+1\right)=U_1\left(\phi_1\left(v_1,\hat{t}\right),\hat{t}+1\right)$, which proves the claim for $\bar{t}_1=\hat{t}$. □

The following lemma forms the core of theorem 1: it shows that every element $\left(v_i,w_i,t_i\right)_{i\in I}$ of E corresponds to four credible punishment outcomes which, in a manner similar to OSPCs, support exactly the punishment values as well as delays which compose $\left(v_i,w_i,t_i\right)_{i\in I}$. To facilitate the statement, for any punishment values $\left(v_1,w_1,v_2,w_2\right)\in\mathscr{U}_1^2\times\mathscr{U}_2^2$ and any i, denote by $A_i\left(v_1,w_1,v_2,w_2\right)$ the set

$$\left\{\left(q,t\right)\in A_i \middle| \phi_i\left(v_i,t\right)\leq q\leq\begin{cases}1-\pi_j\left(w_j\right) & t=0\\ 1-\max\left\{\phi_j\left(v_j,t-1\right),\phi_j\left(u_j\left(0\right),t\right)\right\} & t>0\end{cases}\right\}.$$

Also, based on proposition 3, say the quadruple $\left(\left(x^{P,i},t^{P,i}\right),\left(x^{R,i},t^{R,i}\right)\right)_{i\in I}\in A^4$ of punishment outcomes **permit a simple penal code** if the two implied shares

$$\left(\hat{q}_1,\hat{q}_2\right)\equiv\left(\pi_1\left(U_1\left(x_1^{P,2},t^{P,2}+1\right)\right),\pi_2\left(U_2\left(x_2^{P,1},t^{P,1}+1\right)\right)\right)$$

are such that, for each $i\in I$, the sequence $s^{P,i}\equiv\left(s_m^{P,i}\right)_{m=1}^{n}\in\left(X\times\mathcal{P}\left(X\right)\right)^n$ such that

$n = t^{P,i} + 1$ and, for any $m \leq n$,

$$s_m^{P,i} = \begin{cases} \left(e^{(P(m))}, X_{R(m),\hat{q}_{R(m)}}\right) & m < n \\ \left(x^{P,i}, X_{R(m),x^{P,i}_{R(m)}}\right) & m = n \end{cases},$$

is a play of game G_j (where $P(1) = j$), and the sequence $s^{R,i} \equiv \left(s_m^{R,i}\right)_{m=1}^{n} \in (X \times \mathcal{P}(X))^n$ such that $n = t^{R,i} + 1$ and, for any $m \leq n$,

$$s_m^{R,i} = \begin{cases} \left(e^{(P(m))}, X_{R(m),\hat{q}_{R(m)}}\right) & m < n \\ \left(x^{R,i}, X_{R(m),x^{P,i}_{R(m)}}\right) & m = n \end{cases},$$

is a play of game G_i (where $P(1) = i$). For any such quadruple with associated plays $\left(s^{P,i}, s^{R,i}\right)_{i \in I}$ as above, define the mapping $\sigma^s\left(\cdot \middle| \left(\left(x^{P,i}, t^{P,i}\right), \left(x^{R,i}, t^{R,i}\right)\right)_{i \in I}\right)$ such that it assigns to any outcome $(x^0, t^0) \in A$ a strategy profile σ^s in game G_k, for any $k \in I$, as follows: taking the sequence $s \equiv (s_m)_{m=1}^{n'} \in (X \times \mathcal{P}(X))^{n'}$ such that $n' = t^0 + 1$ and, for any $m \leq n'$,

$$s_m = \begin{cases} \left(e^{(P(m))}, X_{R(m),\hat{q}_{R(m)}}\right) & m < n' \\ \left(x^0, X_{R(m),x^0_{R(m)}}\right) & m = n' \end{cases},$$

and letting s^0 be the play $(s_m^0)_{m=1}^{n}$ such that $s = (s^0, s_m)_{m=n+1}^{n'}$, $\sigma^s = \sigma^{**}\left(s^0 \middle| \left(s^{P,i}, s^{R,i}\right)_{i \in I}\right)$ where the disagreement-round reservation shares (q_1^*, q_2^*) are replaced by $(\hat{q}_1, \hat{q}_2)$. Note that either s is itself a play, in which case $s^0 = s$, or it is not, in which case $n < n'$ and play s^0 has an outcome in which there is agreement with delay $n - 1$ and one player obtains the entire surplus.

Lemma 4. *Fix any values $(v_i, w_i, t_i)_{i \in I} \in E$. Then the quadruple $\left(\left(y^{(i)}, 0\right), \left(x^{(i)}, t^{(i)}\right)\right)_{i \in I}$ of outcomes such that, for each $i \in I$,*

$$y_i^{(i)} = \pi_i(w_i) \quad \text{and} \quad \left\{ \begin{array}{c} t^{(i)} \in \arg\min \left\{U_i\left(\phi_i(v_i, t), t+1\right) | t \in T,\ t \leq t_i\right\} \\ x_i^{(i)} = \phi_i\left(v_i, t^{(i)}\right) \end{array} \right\},$$

permits a simple penal code; moreover, for any $k \in I$ and outcome $\left(\hat{x}, \hat{t}\right) \in A$, the strategy profile $\sigma^s\left(\left(\hat{x}, \hat{t}\right) \middle| \left(\left(y^{(i)}, 0\right), \left(x^{(i)}, t^{(i)}\right)\right)_{i \in I}\right)$ is an equilibrium of game G_k supporting outcome $\left(\hat{x}, \hat{t}\right)$ if and only if $\left(\hat{x}_k, \hat{t}\right) \in A_k(v_1, w_1, v_2, w_2)$; thereby, for each $i \in I$,

$$\left\{\left(1 - y_j^{(j)}, 0\right), \left(x_i^{(i)}, t^{(i)}\right)\right\} \subseteq A_i(v_1, w_1, v_2, w_2) \subseteq A_i^*,$$

and the following equalities hold true:

$$\begin{aligned}
v_i &= \min\left\{U_i\left(q,t\right) \mid (q,t) \in A_i\left(v_1,w_1,v_2,w_2\right)\right\} \\
&= u_i\left(1-\pi_j\left(U_j\left(1-y_i^{(i)},1\right)\right)\right) \\
w_i &= \min\left\{U_i\left(q,t+1\right) \mid (q,t) \in A_i\left(v_1,w_1,v_2,w_2\right)\right\} \\
&= U_i\left(x_i^{(i)},t^{(i)}+1\right) \\
t_i &= \sup\left\{t \in T \mid \exists q \in [0,1],\ (q,t) \in A_i\left(v_1,w_1,v_2,w_2\right)\right\}.
\end{aligned}$$

Proof. Take any $(v_i, w_i, t_i)_{i\in I} \in E$ and note that a player i's personal outcome $\left(x_i^{(i)}, t^{(i)}\right)$ is well-defined: by lemma 3, such a (finite) $t^{(i)} \in T$ exists. The claim that

$$\left\{\left(1-y_j^{(j)},0\right),\left(x_i^{(i)},t^{(i)}\right)\right\} \subseteq A_i\left(v_1,w_1,v_2,w_2\right)$$

and all of the five equalities, for each i, are then straightforward from the construction of the respective two outcomes $\left(y^{(i)},0\right)$ and $\left(x^{(i)},t^{(i)}\right)$ as well as the set $A_i\left(v_1,w_1,v_2,w_2\right)$ given $(v_i,w_i,t_i)_{i\in I} \in E$.

The following two observations, each for both $i \in I$, will be helpful in proving the central claims here. First,

$$v_i > \max\left\{u_i\left(0\right), w_i\right\}, \tag{11}$$

where $v_i > u_i(0)$ follows from equation 4 due to j's impatience and implies $\phi_i\left(v_i,0\right) > 0$, whereby $v_i > w_i$ follows from equation 5 by i's impatience. Second, for any $t \in T$,

$$\kappa_i\left(t+1,v_i,v_j,w_j\right) \geq \kappa_j\left(t,v_j,v_i,w_i\right); \tag{12}$$

to see this, simply note that $\phi_i\left(v_i,t+1\right) \geq \max\left\{\phi_i\left(v_i,t-1\right),\phi_i\left(u_i\left(0\right),t\right)\right\}$using inequality 11 together with i's impatience, and that $\max\left\{\phi_j\left(v_j,t\right),\phi_j\left(u_j\left(0\right),t+1\right)\right\} \geq \phi_j\left(v_j,t\right)$.

I will now show that the outcomes $\left(\left(y^{(i)},0\right),\left(x^{(i)},t^{(i)}\right)\right)_{i\in I}$ permit a simple penal code. Define, for each $i \in I$, $\hat{q}_i \equiv \pi_i\left(U_i\left(1-y_j^{(j)},1\right)\right)$, and note that equation 4 implies that

$$\hat{q}_i = 1-\phi_j\left(v_j,0\right). \tag{13}$$

There is nothing to check regarding outcomes without delay, so suppose that $\left(x^{(1)},t^{(1)}\right)$ has $t^{(1)} > 0$. This implies $t_1 > 0$ and therefore, via equations 6 and 7, for $i=1$, $\phi_1\left(v_1,1\right) + \max\left\{\phi_2\left(v_2,0\right),\phi_2\left(u_2\left(0\right),1\right)\right\} \leq 1$. By inequality 11 for $i=2$, we obtain that $\phi_1\left(v_1,1\right) < 1$, which implies that also $\phi_1\left(v_1,0\right) < 1$, and hence, via equation 4 for $i=1$, that $\hat{q}_2 > 0$. A similar argument shows that $t^{(2)} > 0$ implies $\hat{q}_1 > 0$. While both of these implications are necessary, they are only sufficient in case no $t^{(i)}$ exceeds one. However, if $t^{(i)} > 1$ and hence $t_i > 1$ for some i then by inequality 12 $t_j > 0$, and therefore both $\hat{q}_j > 0$ and $\hat{q}_i > 0$. This establishes the first claim.

Consider then for any $\left(\hat{x}, \hat{t}\right) \in A$ the strategy profile $\sigma^s\left(\left(\hat{x}, \hat{t}\right) \middle| \left(\left(y^{(i)}, 0\right), \left(x^{(i)}, t^{(i)}\right)\right)_{i \in I}\right)$ in game G_1, which I will simply denote by σ^s in what follows. I will show that it is an equilibrium of G_1 supporting outcome $\left(\hat{x}, \hat{t}\right)$ if and only if $\left(\hat{x}_1, \hat{t}\right) \in A_1\left(v_1, w_1, v_2, w_2\right)$, under the assumption that, for each $i \in I$, $\left\{\left(1 - y_j^{(j)}, 0\right), \left(x_i^{(i)}, t^{(i)}\right)\right\} \subseteq A_i^*$; in other words, under this assumption, I will prove that $\left(\hat{x}_1, \hat{t}\right) \in A_1\left(v_1, w_1, v_2, w_2\right)$ characterizes those outcomes for which both of the following hold: (i) there are no profitable one-stage deviations from play $\langle \sigma^s \rangle$ and (ii) play $\langle \sigma^s \rangle$ has outcome $\left(\hat{x}, \hat{t}\right)$. However, since $\left\{\left(1 - y_2^{(2)}, 0\right), \left(x_1^{(1)}, t^{(1)}\right)\right\} \in A_1\left(v_1, w_1, v_2, w_2\right)$ and the case of G_2 follows from mere relabeling, the proof itself verifies the underlying assumption.

First, consider immediate-agreement outcomes $(\hat{x}, 0)$. Play of σ^s then clearly has outcome $(\hat{x}, 0)$. Player 2's accepting all offers $q \geq \hat{x}_2$ is optimal if and only if $\hat{x}_2 \geq \pi_2\left(w_2\right)$, because deviantly rejecting such an offer would trigger her respondent punishment, which has continuation outcome $\left(x^{(2)}, t^{(2)}\right)$ and associated rejection value w_2; her rejecting all other offers is optimal if and only if $\hat{x}_2 \leq \hat{q}_2$ because non-deviantly rejecting such a deviant offer would trigger player 1's proposer punishment, which has continuation outcome $\left(y^{(1)}, 0\right)$ and associated rejection value $U_2\left(1 - \pi_1\left(w_1\right), 1\right)$; using equation 13, $\hat{x}_2 \leq \hat{q}_2$ is equivalent to $\phi_1\left(v_1, 0\right) \leq \hat{x}_1$. To summarize, in terms of player 1's share in $\hat{x}$, player 2's response rule is optimal if and only if $\phi_1\left(v_1, 0\right) \leq \hat{x}_1 \leq 1 - \pi_2\left(w_2\right)$; this is equivalent to $\left(\hat{x}_1, 0\right) \in A_1\left(v_1, w_1, v_2, w_2\right)$.

Given player 2 optimally accepts with threshold $\hat{x}_2$, this is the lowest immediately accepted offer, and there is no profitable deviation for player 1 if and only if $u_1\left(\hat{x}_1\right) \geq U_1\left(\pi_1\left(w_1\right), 1\right)$, because any deviation to a rejected offer triggers her proposer punishment which has continuation outcome $\left(y^{(1)}, 0\right)$ and associated rejection value $U_1\left(\pi_1\left(w_1\right), 1\right)$; inequality 11 implies $\phi_1\left(v_1, 0\right) > \pi_1\left(w_1\right)$, whereby $v_1 \geq U_1\left(\pi_1\left(w_1\right), 1\right)$ from player 1's impatience, and there is no profitable deviation for proposing player 1 whenever there is none for responding player 2. Hence, there is no profitable deviation from $\langle \sigma^s \rangle$ if and only if $\left(\hat{x}_1, 0\right) \in A_1\left(v_1, w_1, v_2, w_2\right)$.

Next, consider outcomes $(\hat{x}, 1)$ where agreement is delayed by a single period. Play $\langle \sigma^s \rangle$ then has outcome $(\hat{x}, 1)$ if and only if it reaches the second round, which is equivalent to $\hat{q}_2 > 0$ and independent of $\hat{x}$. Observe that $\hat{q}_2 = 0$, which, by equation 13, is equivalent to $\phi_1\left(v_1, 0\right) = 1$, and jointly with inequality 11 for $i = 2$, would indeed mean that $A_1\left(v_1, w_1, v_2, w_2\right)$ contains no delayed-agreement outcomes. In deriving the remaining restrictions on $\hat{x}$, from optimality of $\langle \sigma^s \rangle$, we can restrict ourselves to the case where $\hat{q}_2 > 0$.

If play of σ^s reaches the second round, the above finding for the case of immediate-agreement outcomes—by mere relabeling—shows that there are then no profitable one-stage deviations if and only if $\phi_2(v_2, 0) \leq \hat{x}_2 \leq 1 - \pi_1(w_1)$. In terms of player 1's share this is equivalent to

$$\pi_1(w_1) \leq \hat{x}_1 \leq 1 - \phi_2(v_2, 0).$$

In the first round σ^s specifies that player 2 respond to offers by accepting with threshold $\hat{q}_2$. Accepting offers $q \geq \hat{q}_2$ is optimal if and only if $u_2(\hat{q}_2) \geq U_2\left(x_2^{(2)}, t^{(2)} + 1\right) = w_2$, since the (deviant) rejection of any such offer is followed by continuation outcome $\left(x^{(2)}, t^{(2)}\right)$; however, since $u_2(\pi_2(w_2)) \geq w_2$, this is implied by $\hat{q}_2 \geq \pi_2(w_2)$, which, using equation 13, is equivalent to $1 \geq \phi_1(v_1, 0) + \pi_2(w_2)$ and guaranteed by inequality 7 for $i = 1$. Rejection of all (deviant) offers q such that $0 < q < \hat{q}_2$ is followed by continuation outcome $\left(y^{(1)}, 0\right)$ and optimal by construction since $\hat{q}_2 > 0$ implies that $u_2(\hat{q}_2) = U_2(1 - \pi_1(w_1), 1)$ is the associated rejection value. Rejecting the zero offer specified for the proposer in this round is optimal if and only if $u_2(0) \leq U_2(\hat{x}_2, 1)$; now either $u_2(0) \leq U_2(1, 1)$, in which case $u_2(0) \leq U_2(\hat{x}_2, 1)$ is equivalent to $\hat{x}_1 \leq 1 - \phi_2(u_2(0), 1)$, or $u_2(0) > U_2(1, 1)$ in which case there is no $\hat{x}$ such that rejecting a zero offer is optimal when its value is $U_2(\hat{x}_2, 1)$. However, then $\phi_2(u_2(0), 1) = 1$ and this, together with inequality 11 for $i = 1$, would indeed imply that $A_1(v_1, w_1, v_2, w_2)$ contains no outcomes with delayed agreement.

By equation 13, the initial proposer 1 can obtain at most the value v_1 from making a deviant accepted offer $q \geq \hat{q}_2$; making a deviant rejected offer, i.e. an offer q such that $0 < q < \hat{q}_2$ yields value $U_1(\pi_1(w_1), 1)$ which is no greater than v_1 due to inequality 11; hence making her supposed (rejected) offer of a zero share is optimal if and only if $v_1 \leq U_1(\hat{x}_1, 1)$. This is equivalent to $\hat{x}_1 \geq \phi_1(v_1, 1)$ unless $v_1 > U_1(\phi_1(v_1, 1), 1)$; the latter is, however, equivalent to $v_1 > U_1(1, 1)$, and in this case there is no $\hat{x}$ such that offering a zero share is optimal; because then $\phi_1(v_1, 1) = 1$, in combination with inequality 11 for $i = 2$, this would indeed mean that $A_1(v_1, w_1, v_2, w_2)$ contains no delayed-agreement outcomes. In summary, using that $\pi_1(w_1) < \phi_1(v_1, 1)$ from inequality 11, and noting that $\min\{1 - \phi_2(v_2, 0), 1 - \phi_2(u_2(0), 1)\}$ equals $1 - \max\{\phi_2(v_2, 0), \phi_2(u_2(0), 1)\}$, we obtain also for this case that $\langle\sigma^s\rangle$ has outcome $(\hat{x}, 1)$ and no profitable deviations if and only if $(\hat{x}_1, 1) \in A_1(v_1, w_1, v_2, w_2)$.

Finally, consider outcomes $\left(\hat{x}, \hat{t}\right)$ such that $\hat{t} > 1$. For play $\langle\sigma^s\rangle$ to have outcome $\left(\hat{x}, \hat{t}\right)$ it must be that both $\hat{q}_1 > 0$ and $\hat{q}_2 > 0$. Since we have already seen that $\hat{q}_2 = 0$ would imply that there is no delayed outcome in $A_1(v_1, w_1, v_2, w_2)$, it remains to verify that $\hat{q}_1 = 0$ would imply that there is outcome with delay beyond one period in $A_1(v_1, w_1, v_2, w_2)$: however, this follows immediately from its definition upon using that $\hat{q}_1 = 0$ is equivalent to $\phi_2(v_2, 0) = 1$, by equation 13, in combination with inequality 11 for $i = 1$. In deriving the remaining restrictions on $\hat{x}$ and $\hat{t}$, from optimality of $\langle\sigma^s\rangle$, we can restrict ourselves to the case where $\hat{q}_i > 0$ for both players i.

In the last round of play $\langle\sigma^s\rangle$, which is then round $\hat{t}+1$ of G_1, we can use the previous findings to conclude that there is no profitable deviation if and only if

$$\begin{cases} \pi_1(w_1) \leq \hat{x}_1 \leq 1 - \phi_2(v_2, 0) & \hat{t} \text{ odd} \\ \phi_1(v_1, 0) \leq \hat{x}_1 \leq 1 - \pi_2(w_2) & \hat{t} \text{ even} \end{cases}. \tag{14}$$

Consider then any round $n < \hat{t}+1$ of play $\langle\sigma^s\rangle$ in G_1 and denote, for simplicity its proposer by P and its respondent by R. Optimality of R's response rule is characterized in a manner similar to optimality of initial respondent 2's response rule when we considered agreement-outcomes with one round of delay; it is therefore characterized by $u_R(0) \leq U_R\left(\hat{x}_R, \hat{t}+1-n\right)$. Since $U_R\left(\hat{x}_R, \hat{t}+1-n\right)$ is non-decreasing in n, this yields only two restrictions, namely those for the first two rounds' respondent stages, which are $u_2(0) \leq U_2\left(\hat{x}_2, \hat{t}\right)$ and $u_1(0) \leq U_1\left(\hat{x}_1, \hat{t}-1\right)$, respectively. These two inequalities are equivalent to

$$\phi_1\left(u_1(0), \hat{t}-1\right) \leq \hat{x}_1 \leq 1 - \phi_2\left(u_2(0), \hat{t}\right)$$

whenever both $u_2(0) \leq U_2\left(1, \hat{t}\right)$ and $u_1(0) \leq U_1\left(1, \hat{t}-1\right)$ hold true; otherwise, there is no $\hat{x}$ such that play of σ^s with outcome $\left(\hat{x}, \hat{t}\right)$ is optimal, and $A_1(v_1, w_1, v_2, w_2)$ indeed contains no such outcome.

Again, similar to optimality for initial proposer 1 when we considered one round of delay, proposer P's zero offer is here optimal if and only if $v_P \leq U_P\left(\hat{x}_P, \hat{t}+1-n\right)$. Since $U_P\left(\hat{x}_P, \hat{t}+1-n\right)$ is non-decreasing in n, this yields only two restrictions, namely those for the first two rounds' proposer stages, which are $v_1 \leq U_1\left(\hat{x}_1, \hat{t}\right)$ and $v_2 \leq U_2\left(\hat{x}_2, \hat{t}-1\right)$, respectively. These two inequalities are equivalent to

$$\phi_1\left(v_1, \hat{t}\right) \leq \hat{x}_1 \leq 1 - \phi_2\left(v_2, \hat{t}-1\right)$$

whenever both $v_1 \leq U_1\left(1, \hat{t}\right)$ and $v_2 \leq U_2\left(1, \hat{t}-1\right)$ hold true; otherwise, there is no $\hat{x}$ such that play of σ^s with outcome $\left(\hat{x}, \hat{t}\right)$ is optimal, and $A_1(v_1, w_1, v_2, w_2)$ indeed contains no such outcome. Now observe that $\phi_1\left(v_1, \hat{t}\right)$ is at least as large as any of $\pi_1(w_1)$, $\phi_1(v_1, 0)$ or $\phi_1\left(u_1(0), \hat{t}-1\right)$, due to 1's impatience and inequality 11; moreover, also $\phi_2\left(v_2, \hat{t}-1\right)$ is at least as large as both $\phi_2(v_2, 0)$ and $\pi_2(w_2)$ due to 2's impatience and inequality 11. Hence we can summarize this case by the condition that $\left(\hat{x}, \hat{t}\right)$ is such that

$$\phi_1\left(v_1, \hat{t}\right) \leq \hat{x}_1 \leq 1 - \max\left\{\phi_2\left(v_2, \hat{t}-1\right), \phi_2\left(u_2(0), \hat{t}\right)\right\},$$

and this is indeed equivalent to $\left(\hat{x}_1, \hat{t}\right) \in A_1\left(v_1, w_1, v_2, w_2\right)$. □

The next lemma proves that E is non-empty; more specifically, it shows how every stationary equilibrium corresponds to an element of E.

Lemma 5. *Let (q_1, q_2) be a pair of positive shares such that $q_1 = f_1(q_1)$ and $q_2 = 1 - \pi_1(U_1(q_1, 1))$. The values $(u_i(q_i), U_i(q_i, 1), 0)_{i \in I}$ are an element of the set E, which is therefore non-empty.*

Proof. Existence of such a pair, and also the property that $q_2 = f_2(q_2)$, are guaranteed by lemma 2 (the functions f_i are defined by equation 10).

Let then $(v_i, w_i, t_i)_{i \in I} = (u_i(q_i), U_i(q_i, 1), 0)_{i \in I}$. Note, first, that $\phi_i(v_i, 0) = q_i$, implying that $w_i = U_i(\phi_i(v_i, 0), 1)$, from which equation 4 is equivalent to $q_i = f_i(q_i)$ and holds by construction. Given $t_1 = t_2 = 0$, the values $(v_i, w_i, t_i)_{i \in I}$ clearly also satisfy equation 5.

Observe next that

$$\kappa_i(0, v_i, v_j, w_j) = q_i + \pi_j(U_j(q_j, 1)) = 1; \tag{15}$$

this equality follows directly from the construction for $i = 2$, and substituting $f_1(q_1)$ for q_1 and $1 - \pi_1(U_1(q_1, 1))$ for q_2 yields it also for the case of $i = 1$. Thus inequality 7 is shown to hold true. Finally, note that $q_j > \pi_j(U_j(q_j, 1))$ follows from impatience property (4.b) because $q_j > 0$; using this inequality and the non-decreasingness of the functions $\phi_k(u, \cdot)$ for any $u \in u_k([0,1])$ and any $k \in I$ in equation 15 yields

$$\begin{aligned} 1 &< q_i + q_j \\ &= \phi_i(v_i, 0) + \phi_j(v_j, 0) \\ &\leq \kappa_i(1, v_i, v_j, w_j) \\ &\leq \kappa_i(t, v_i, v_j, w_j) \end{aligned}$$

for any $t \in T$, implying that the constructed values $(v_i, w_i, t_i)_{i \in I}$ also satisfy equation 6. □

Now define each player i's infimal punishment values $(\tilde{v}_i, \tilde{w}_i)$ as

$$\begin{aligned} \tilde{v}_i &\equiv \inf\left\{U_i(q, t) \mid (q, t) \in A_i^*\right\} \\ \tilde{w}_i &\equiv \inf\left\{U_i(q, t+1) \mid (q, t) \in A_i^*\right\}. \end{aligned}$$

Contrary to (v_i^*, w_i^*), these values certainly exist, and so does t_i^*, defined as a supremum. The following basic observation is subsequently useful in relating $\tilde{v}_i$ and $\tilde{w}_i$.

Lemma 6. *For each $i \in I$, $(q,t) \in A_i^*$ implies $(1-\pi_i(U_i(q,t+1)),0) \in A_j^*$.*

Proof. Let σ be an equilibrium of game G_i which supports i's personal outcome (q,t), denote the share $1-\pi_i(U_i(q,t+1))$ by $\hat{q}$ and the division such that j's share equals $\hat{q}$ by $\hat{x}$. The strategy profile $\hat{\sigma}$ in game G_j such that $\hat{\sigma}(h^0) = (\hat{x}, X_{i,\hat{q}})$ and $\hat{\sigma}(x,h) = \sigma(h)$ for any division x and history h, is an equilibrium supporting j's personal outcome $(1-\hat{q},0)$: in the subgame G_i which ensues after an initial proposal x is rejected, regardless of x, $\hat{\sigma}$ specifies equilibrium σ, which induces personal outcome (q,t) for player i and thus implies that the initial response rule of accepting with threshold $\hat{q}$ is optimal for i; the initial proposer j best-responds by offering this share, because this is the lowest accepted offer and, moreover, satisfies $u_j(1-\hat{q}) \geq U_j(1-q,t+1)$, due to $\hat{q} \leq q$, which follows from i's impatience, together with the desirability and impatience properties of j's preferences. □

The next lemma prepares for the final lemma, which will bound the values $(\tilde{v}_i, \tilde{w}_i, t_i^*)_{i\in I}$ by an element of the set E, providing the second part to the proof of theorem 1, in addition to lemma 4.

Lemma 7. *The following hold true for each $i \in I$:*

$$\tilde{v}_i = u_i(1-\pi_j(U_j(1-\pi_i(\tilde{w}_i),1))) \tag{16}$$

$$\tilde{w}_i \geq \inf\{U_i(\phi_i(\tilde{v}_i,t),t+1) | t \in T,\ t \leq t_i^*\} \tag{17}$$

$$t_i^* \leq \sup\{t \in T | \kappa_i(t,\tilde{v}_i,\tilde{v}_j,\tilde{w}_j) \leq 1\} \tag{18}$$

$$1 \geq \kappa_i(0,\tilde{v}_i,\tilde{v}_j,\tilde{w}_j). \tag{19}$$

Proof. I will prove all four conditions 16-19 for the case of $i=1$; mere relabeling yields them for $i=2$.

To show that $(\tilde{v}_1, \tilde{w}_1)$ satisfies equation 16, first, combine lemma 6, for $i=2$, with the fact that any equilibrium of game G_1 must have the initial respondent 2 accept all offers greater than $\sup\{\pi_2(U_2(q,t+1)) \,|\, (q,t) \in A_2^*\}$, to obtain

$$\tilde{v}_1 = u_1(1-\sup\{\pi_2(U_2(q,t+1)) \,|\, (q,t) \in A_2^*\}).$$

It remains to prove that $\pi_2(U_2(1-\pi_1(\tilde{w}_1),1)) = \sup\{\pi_2(U_2(q,t+1)) \,|\, (q,t) \in A_2^*\}$. For this, also combine lemma 6, now for $i=1$, with the fact that any equilibrium of G_2 must have the initial respondent 1 reject all offers less than $\pi_1(\tilde{w}_1)$, which yields that

$$1-\pi_1(\tilde{w}_1) = \sup\{q \in [0,1] \,|\, (q,0) \in A_2^*\}.$$

To complete this first part of the proof, observe then that any $(q,t) \in A_2^*$ with $t > 0$ satisfies $U_1(1-q,t) \geq \tilde{w}_1$, which implies $1-q \geq \pi_1(U_1(1-q,t)) \geq \pi_1(\tilde{w}_1)$ by 1's impatience and the non-decreasingness of π_1, and therefore

$$\pi_2(U_2(q,t+1)) \leq \pi_2(U_2(1-\pi_1(\tilde{w}_1),t+1)) \leq \pi_2(U_2(1-\pi_1(\tilde{w}_1),1))$$

by the desirability and impatience properties of 2's preferences, together with the non-decreasingness of π_2.

Regarding the proof that $(\tilde{v}_1, \tilde{w}_1, t_1^*)$ satisfies inequality 17, simply note that $(q,t) \in A_1^*$ implies $U_1(q,t) \geq \tilde{v}_1$ by the definition of $\tilde{v}_1$, and thus $q \geq \phi_1(\tilde{v}_1, t)$; the claim then follows from the desirability property of 1's preferences.[35]

Inequality 18 certainly holds true if $t_1^* = 0$. Consider therefore the case of $t_1^* > 0$, where construct the sequence k_t such that, for all $t \in T \backslash \{0\}$ with $t \leq t_1^*$, $k_t \equiv \kappa_1(t, \tilde{v}_1, \tilde{v}_2, \tilde{w}_2)$. Since the function $\kappa_1(\cdot, v_1, v_2, w_2)$ is non-decreasing on $T \backslash \{0\}$, so is the sequence, and due boundedness of κ_1 from above, the sequence possesses a limit. Suppose that $\lim_{t \to t_1^*} k_t > 1$: this means there exists a (finite) $\hat{t} \in T \backslash \{0\}$ with $\hat{t} \leq t_1^*$ such that $\kappa_1(t, \tilde{v}_1, \tilde{v}_2, \tilde{w}_2) > 1$ holds whenever $t \geq \hat{t}$, implying that for any $q \in [0,1]$, $q < \phi_1(\tilde{v}_1, t)$ or $1-q < \max\{\phi_2(\tilde{v}_2, t-1), \phi_2(u_2(0), t)\}$, and thus, from desirability, that $U_1(q,t) < U_1(\phi_1(\tilde{v}_1,t),t) \leq \tilde{v}_1$ or $U_2(1-q,t-1) < U_2(\phi_2(\tilde{v}_2,t-1),t-1) \leq \tilde{v}_2$ or $U_2(1-q,t) < U_2(\phi_2(u_2(0),t),t) \leq u_2(0)$; each of these says that there is no $q \in [0,1]$ such that $(q,t) \in A_1^*$, a contradiction.[36] Hence, k_t is a non-decreasing sequence with $\lim_{t \to t_1^*} k_t \leq 1$, proving the claim.

Finally, $\kappa_1(0, \tilde{v}_1, \tilde{v}_2, \tilde{w}_2) \leq 1$ holds true, because any pair (q_1, q_2) as in lemma 5 satisfies that $(q_i, 0) \in A_i^*$, and therefore $\tilde{v}_i \leq u_i(q_i)$ as well as $\tilde{w}_i \leq U_i(q_i, 1)$, for both i; non-decreasingness of $\kappa_1(0, v_1, v_2, w_2)$ in both v_1 and w_2, and lemma 5's equation 15 then yield

$$\kappa_1(0, \tilde{v}_1, \tilde{v}_2, \tilde{w}_2) \leq \kappa_1(0, u_1(q_1), u_2(q_2), U_2(q_2, 1)) = 1.$$

□

Lemma 8. *There exist values $(v_i, w_i, t_i)_{i \in I} \in E$ such that $v_i \leq \tilde{v}_i$, $w_i \leq \tilde{w}_i$ and $t_i \geq t_i^*$ for both $i \in I$.*

Proof. Consider the following sequence $(v_i^n, w_i^n, t_i^n)_{i \in I}$: $(w_1^1, w_2^1) \equiv (\tilde{w}_1, \tilde{w}_2)$ and, for any

[35]Under the weakening of desirability suggested in footnote 14, the observation $\tilde{v}_1 > u_1(0)$, which follows from equation 16, means that no equilibrium delay t can be such that player 1 does not care about share: otherwise there would exist $(q,t) \in A_1^*$ with $U_1(q,t) = U_1(0,t)$, but $U_1(0,t) \leq u_1(0)$ by impatience; hence $U_1(q,t) < \tilde{v}_1$, a contradiction.

[36]See footnote 35 for why this is true even under the weakening of desirability suggested in footnote 14.

$n \in \mathbb{N}$ and each i,

$$\begin{aligned} v_i^n &\equiv u_i\left(1-\pi_j\left(U_j\left(1-\pi_i\left(w_i^n\right),1\right)\right)\right) \\ t_i^n &\equiv \sup\left\{t \in T \middle| \kappa_i\left(t, v_i^n, v_j^n, w_j^n\right) \leq 1\right\} \\ w_i^{n+1} &\equiv \inf\left\{U_i\left(\phi_i\left(v_i^n, t\right), t+1\right) \middle| t \in T,\ t \leq t_i^n\right\}. \end{aligned}$$

Note that $v_i^1 = \tilde{v}_i$, $\kappa_i\left(0, v_i^1, v_j^1, w_j^1\right) = \kappa_i\left(0, \tilde{v}_i, \tilde{v}_j, \tilde{w}_j\right) \leq 1$ and $t_i^1 \geq t_i^*$, by lemma 7. It is straightforward that $w_i^{n+1} \leq w_i^n$, $v_i^{n+1} \leq v_i^n$ and $t_i^{n+1} \geq t_i^n$. I will establish the claim by proving that the sequence $\left(v_i^n, w_i^n, t_i^n\right)_{i \in I}$ possesses a limit in $\prod_{i \in I}\left(\mathscr{U}_i^2 \times \left(T \cup \{\infty\}\right)\right)$ which is an element of E.

The first step is to prove that the sequence $\left(w_1^n, w_2^n\right)$ converges: since each component sequence w_i^n is non-increasing and bounded from below by $U_i\left(0, \infty\right) \in \mathbb{R}$, it converges. Denoting this limit by $\left(\hat{w}_1, \hat{w}_2\right)$, the continuity properties of the functions involved imply the following convergence properties of the sequences v_i^n and t_i^n, for each i:

$$\begin{aligned} v_i^n &\rightarrow u_i\left(1-\pi_j\left(U_j\left(1-\pi_i\left(\hat{w}_i\right),1\right)\right)\right) \equiv \hat{v}_i \\ t_i^n &\rightarrow \sup\left\{t \in T \middle| \kappa_i\left(t, \hat{v}_i, \hat{v}_j, \hat{w}_j\right) \leq 1\right\} \equiv \hat{t}_i; \end{aligned}$$

since also $\kappa_i\left(0, \hat{v}_i, \hat{v}_j, \hat{w}_j\right) \leq \kappa_i\left(0, \tilde{v}_i, \tilde{v}_j, \tilde{w}_j\right) \leq 1$, $\left(\hat{v}_i, \hat{w}_i, \hat{t}_i\right)_{i \in I} \in E$. □

Theorem 1 follows from combining lemmas 4 and 8: by lemma 4, $\left(v_i, w_i, t_i\right)_{i \in I} \in E$ implies $v_i \geq \tilde{v}_i$, $w_i \geq \tilde{w}_i$ and $t_i \leq t_i^*$ for both i, and, by lemma 8, there exists an element $\left(v_i, w_i, t_i\right)_{i \in I}$ in E such that $v_i \leq \tilde{v}_i$, $w_i \leq \tilde{w}_i$ and $t_i \geq t_i^*$ hold for both i; hence $\left(\tilde{v}_i, \tilde{w}_i, t_i^*\right)_{i \in I}$ is the unique extreme element of E. Since lemma 4 also shows that $\left(v_i, w_i, t_i\right)_{i \in I} \in E$ implies existence of both $\left(y_i, 0\right) \in A_i^*$ such that $v_i = u_i\left(x_i\right)$ and $\left(x_i, t\right) \in A_i^*$ such that $w_i = U_i\left(x_i, t+1\right)$, for both i, $\left(\tilde{v}_i, \tilde{w}_i, t_i^*\right)_{i \in I} = \left(v_i^*, w_i^*, t_i^*\right)_{i \in I}$.

A.7 Theorem 2

Proof. Lemma 4 implies that E's being a singleton is necessary for equilibrium uniqueness.

Concerning its sufficiency, lemma 5 implies that whenever E is a singleton, its unique element is the values $\left(u_i\left(q_i^*\right), U_i\left(q_i^*, 1\right), 0\right)_{i \in I}$, as in theorem 2's statement. Characterization theorem 1 then implies that $\left(v_i^*, w_i^*, t_i^*\right)_{i \in I}$ equals $\left(u_i\left(q_i^*\right), U_i\left(q_i^*, 1\right), 0\right)_{i \in I}$ and that each A_i^* equals the singleton $\left\{\left(q_i^*, 0\right)\right\}$. Consider then any round in which player P makes an offer to responding player R: since any equilibrium has the outcome that offer q_P^* is accepted, it must be that P indeed offers q_P^*, and R accepts it; moreover, since any equilibrium has the same continuation outcome following any rejection in which R has offer q_R^* accepted, with associated rejection value $U_R(q_R^*, 1)$, any optimal response rule must have R accept any offer $q > \pi_R(U_R(q_R^*, 1))$ as well as reject any offer $q < \pi_R(U_R(q_R^*, 1))$, pinning down a unique equilibrium; now observe that the pair (q_1^*, q_2^*) satisfies $q_i^* = 1 - \pi_j\left(U_j\left(q_j^*, 1\right)\right)$ not only for $i = 2$ but also for $i = 1$, by lemma 2, to conclude that it is indeed the stationary equilibrium of theorem 2. □

A.8 Proposition 4

The proof uses the following lemma which shows that a unique stationary equilibrium is the unique equilibrium overall whenever no player's optimal punishment requires delay.

Lemma 9. *If the values* $(v_i^*, w_i^*, t_i^*)_{i\in I}$ *are such that*

$$0 \in \arg\min\left\{U_i\left(\phi_i\left(v_i^*, t\right), t+1\right) | t \in T,\ t \leq t_i^*\right\}$$

for both $i \in I$, *then equilibrium is unique if and only if stationary equilibrium is unique.*

Proof. Theorem 1 implies that the outcome $\left(x^{R,i}, 0\right)$ such that $x_i^{R,i} = \phi_i(v_i^*, 0)$ is an optimal respondent punishment outcome for player i. Hence $w_i^* = U_i\left(\phi_i\left(v_i^*, 0\right), 1\right)$ and her optimal proposer punishment has outcome $\left(x^{P,i}, 0\right)$ such that $x_i^{P,i} = \pi_i\left(U_i\left(\phi_i\left(v_i^*, 0\right), 1\right)\right)$. Using equation 4,

$$\begin{aligned}\phi_i\left(v_i^*, 0\right) &= 1 - \pi_j\left(U_j\left(1 - \pi_i\left(U_i\left(\phi_i\left(v_i^*, 0\right), 1\right)\right), 1\right)\right) \\ &= f_i\left(\phi_i\left(v_i^*, 0\right)\right),\end{aligned}$$

which, by lemma 2 and proposition 1, reveals that $x_i^{R,i} = f_i\left(x_i^{R,i}\right)$ as well as $x_j^{P,i} \equiv 1 - x_i^{P,i} = 1 - \pi_i\left(U_i\left(x_i^{R,i}, 1\right)\right)$ are the two player's respective proposer shares in one particular stationary equilibrium. If there is a unique stationary equilibrium, then $\left(x^{R,1}, 0\right) = \left(x^{R,2}, 0\right)$ and $\left(x^{P,1}, 0\right) = \left(x^{R,2}, 0\right)$ such that $x_1^{R,1} = x_1^{P,2} = \phi_1\left(v_1^*, 0\right) = q_1^*$ and $x_2^{P,1} = x_2^{R,2} = 1 - \pi_1\left(U_1\left(\phi_1\left(v_1^*, 0\right), 1\right)\right) = q_2^*$ for q_1^* the unique fixed point of f_1 and $q_2^* \equiv 1 - \pi_1\left(U_1\left(q_1^*, 1\right)\right)$. Then $(v_i^*, w_i^*, t_i^*)_{i\in I} = \left(u_i\left(q_i^*\right), U_i\left(q_i^*, 1\right), 0\right)_{i\in I}$, and theorem 1 implies that $A_i^* = \{(q_i^*, 0)\}$ for each i. It is straightforward that the two subgames' respectively unique equilibrium outcomes imply a unique equilibrium, which is the unique stationary one associated with q_1^*. This proves sufficiency. Necessity holds trivially. □

The following proof then establishes proposition 4.

Proof. I prove the following claim: if a player i's preferences exhibit a weak present bias, $i \in I$, then

$$0 \in \arg\min\left\{U_i\left(\phi_i\left(v_i^*, t\right), t+1\right) | t \in T,\ t \leq t_i^*\right\}.$$

Lemma 9 and proposition 2 then jointly imply the claim's truth.

Lemma 3 and theorem 1 imply a finite delay $\bar{t}_i$ such that

$$w_i^* = \min \left\{ U_i \left(\phi_i \left(v_i^*, t \right), t+1 \right) | t \in T,\ t \leq \bar{t}_i \right\},$$

where $U_i \left(\phi_i \left(v_i^*, t \right), t \right) = v_i^*$ holds true for any $t \in T$ such that $t \leq \bar{t}_i$. A weak present bias then implies that $U_i \left(\phi_i \left(v_i^*, 0 \right), 1 \right) \leq U_i \left(\phi_i \left(v_i^*, t \right), t+1 \right)$ for all such t, and hence $w_i^* = U_i \left(\phi_i \left(v_i^*, 0 \right), 1 \right)$, proving the claim. □

A.9 Proposition 5

Proof. The proposition holds trivially for immediate-agreement equilibrium outcomes. Suppose therefore that $\left(\hat{x}, \hat{t} \right)$ with $\hat{t} > 0$ is an equilibrium outcome of game G_1; the case of game G_2 follows from mere relabeling of players. Theorem 1 implies that $\hat{x}$ is an interior division, since

$$0 < \phi_1 \left(v_1^*, \hat{t} \right) \leq \hat{x}_1 \leq \max \left\{ \phi_2 \left(v_2^*, \hat{t} - 1 \right), \phi_2 \left(u_2 \left(0 \right), \hat{t} \right) \right\} < 1. \tag{20}$$

For every round $n \leq \hat{t} + 1$, define each player i's reservation share for the rejection value corresponding to agreement on $\hat{x}$ with remaining delay $\hat{t}+1-n$: $\pi_i^n \equiv \pi_i \left(U_i \left(\hat{x}_i, \hat{t} + 1 - n \right) \right)$. The inequalities in 20 imply $u_i \left(\pi_i^n \right) = U_i \left(\hat{x}_i, \hat{t} + 1 - n \right)$ because of $U_i \left(\hat{x}_i, \hat{t} + 1 - n \right) \geq u_i \left(0 \right)$, and the stronger impatience property assumed in the proposition yields that π_i^n is increasing, since $\hat{x}_i > 0$.

Define a play as follows: in round 1, player 1 offers a share of $b_1^1 = 0$, and player 2 accepts with threshold $1 - b_2^1$ such that $b_2^1 = \phi_1 \left(v_1^*, 0 \right)$; in round n such that $1 < n < \hat{t} + 1$, player $P \left(n \right)$ offers a share of $b_{P(n)}^n = \frac{1}{2} \left(b_{P(n)}^{n-1} + \pi_{R(n)}^n \right)$ and player $R \left(n \right)$ accepts with threshold $1 - b_{R(n)}^n$ such that $b_{R(n)}^n = \frac{1}{2} \left(b_{R(n)}^{n-1} + \pi_{P(n)}^n \right)$, with the sole exception that $b_1^2 = \phi_2 \left(v_2^*, 0 \right)$; in round $n = \hat{t} + 1$, player $P \left(n \right)$ offers a share $b_{P(n)}^n = \hat{x}_{R(n)}$ and player $R \left(n \right)$ accepts with threshold $1 - b_{R(n)}^n$ such that $b_{R(n)}^n = \hat{x}_{P(n)}$.

First, verify that each sequence $\left(b_i^n \right)_{n=1}^{\hat{t}+1}$ is increasing since $b_i^{n-1} < \pi_j^n$: this is true for $n - 1 = 1$, because $b_i^1 \leq \pi_j^1 < \pi_j^2$, and if it is true for $n - 1 \geq 1$ such that $n < \hat{t} + 1$, it is true for n, because $b_i^n = \frac{1}{2} \left(b_i^{n-1} + \pi_j^n \right) < \pi_j^n < \pi_j^{n+1}$. Second, observe that $b_{P(n)}^n < 1 - b_{R(n)}^n$ for all $n < \hat{t} + 1$: since $\pi_1^n + \pi_2^n < 1$ for all such n, this follows from $b_i^n \leq \pi_j^n$; hence this indeed defines a play with outcome $\left(\hat{x}, \hat{t} \right)$.

The final step is to show that this defines equilibrium play. Taken then any strategy profile σ of game G_1 such that $\langle \sigma \rangle$ equals the above play (clearly, one exists) and define the strategy profile $\hat{\sigma} \equiv \sigma^* \left(\sigma \middle| \left(\sigma^{P,i}, \sigma^{R,i} \right)_{i \in I} \right)$, where $\left(\sigma^{P,i}, \sigma^{R,i} \right)_{i \in I}$ is an OPC, as in proposition 3, part (i). Hence $\langle \hat{\sigma} \rangle = \langle \sigma \rangle$ and $\hat{\sigma}$ is an equilibrium if and only if there are no profitable one-stage deviations from its play $\langle\hat{\sigma}\rangle$.

Consider then any round $n \leq \hat{t}+1$ of play $\langle\hat{\sigma}\rangle$. Rejecting an offer $q \geq 1-b^n_{R(n)}$ is no better than accepting it for $R(n)$, since it yields the minimal credible rejection value $w^*_{R(n)}$ due to optimal punishment, but

$$w^*_{R(n)} \leq U_{R(n)}\left(\hat{x}_{R(n)}, \hat{t}+1-n\right) = u_{R(n)}\left(\pi^n_{R(n)}\right) \leq u_{R(n)}\left(1-\pi^n_{P(n)}\right) \leq u_{R(n)}\left(1-b^n_{R(n)}\right),$$

using that $\left(\hat{x}, \hat{t}-n\right)$ is a continuation equilibrium outcome (by assumption), that $\pi^n_1+\pi^n_2 \leq 1$ and that $b^n_{R(n)} \leq \pi^n_{P(n)}$; accepting an offer $q < 1-b^n_{R(n)}$ such that $q \neq b^n_{P(n)}$ is no better than rejecting it, since

$$u_{R(n)}\left(1-b^n_{R(n)}\right) \leq u_{R(n)}\left(1-\phi_{P(n)}\left(v^*_{P(n)}, 0\right)\right) = U_{R(n)}\left(1-\pi_{P(n)}\left(w^*_{P(n)}\right), 1\right),$$

using that any *responding* player i's concession is at least $\phi_j\left(v^*_j, 0\right)$, by construction, and theorem 1, which shows that continuation with optimal punishment of a proposing player i has rejection value $U_j\left(1-\pi_i\left(w^*_i\right), 1\right)$ for respondent j, and that this is equal to $u_j\left(1-\phi_i\left(v^*_i, 0\right)\right)$; finally, accepting offer $q = b^n_{P(n)} < 1-b^n_{R(n)}$, which can only be the case for $n < \hat{t}+1$, is no better than rejecting it, since

$$u_{R(n)}\left(b^n_{P(n)}\right) \leq u_{R(n)}\left(\pi^n_{R(n)}\right) = U_{R(n)}\left(\hat{x}_{R(n)}, \hat{t}+1-n\right).$$

Consider then the proposer's incentives, given the respondent's behavior and punishments for deviations: the minimal offer which the respondent accepts equals $b^n_{R(n)}$, which is no greater than $\pi^n_{P(n)}$, whereby

$$u_{P(n)}\left(b^n_{R(n)}\right) \leq u_{P(n)}\left(\pi^n_{P(n)}\right) = U_{P(n)}\left(\hat{x}_{P(n)}, \hat{t}+1-n\right),$$

so there is no profitable deviation to any (alternative) accepted offer; any other deviant offer has (rejection) value $U_{P(n)}\left(\pi_{P(n)}\left(w^*_{P(n)}\right), 1\right)$ which is no greater than $v^*_{P(n)}$ by theorem 1, and since $U_{P(n)}\left(\hat{x}_{P(n)}, \hat{t}+1-n\right) \geq v^*_{P(n)}$, because $\left(\hat{x}, \hat{t}\right)$ is an equilibrium outcome, there is no profitable deviation to a rejected offer either. □

A.10 Proposition 6

Proof. Omitting player indices due to symmetry, by theorem 1, if there exists an equilibrium with delayed agreement, then $\kappa\left(1, v^*, v^*, w^*\right) \leq 1$, which, using $v^* > u(0)$, is equivalent to

$$\phi\left(v^*, 1\right) + \phi\left(v^*, 0\right) \leq 1.$$

Since $\phi(v^*, 1) \geq \phi(v^*, 0)$, this implies $\frac{1}{2} \geq \phi(v^*, 0) \geq \pi(w^*)$, and hence $\left(\frac{1}{2}, 0\right) \in A^*$. $\square$

B Supplementary Material

B.1 Empirical Evidence on Time Preferences

Early evidence on time preferences comes mainly from psychological research and is summarized by Frederick, Loewenstein, and O'Donoghue (2002). They conclude that "virtually every assumption underlying the [exponential-discounting] model has been tested and found to be descriptively invalid in at least some situations" (p. 352). The most compelling refutation of ED is a direct violation of its stationarity axiom which requires that, holding amounts constant, choice between two delayed rewards depends only on their *relative* delay; yet, a typical experimental subject would, e.g., choose \$45 now over \$52 in 20 days but also \$52 in 130 over \$45 in 110 days (Kirby and Herrnstein, 1995).

Great effort has gone into uncovering what alternative forms of impatience humans display. For primary rewards (consumption, in a broad sense), a "present bias" towards instantaneous gratification appears rather uncontested; more specifically, this seems to be the consequence of some form of hyperbolic discounting (Chung and Herrnstein, 1967; Ainslie, 1975), and the present bias is well-captured by the (β, δ)-model of quasi-hyperbolic discounting (Phelps and Pollak, 1968; Laibson, 1997). The large survey of mainly psychologists' studies by Frederick et al. (2002) makes this point (for humans as well as other animals), and the more recent evidence from experimental economics confirms it (McClure, Ericson, Laibson, Loewenstein, and Cohen, 2007; Brown, Chua, and Camerer, 2009; Augenblick, Niederle, and Sprenger, 2014). The implied dynamic preference reversal takes the following form: a person may prefer a larger later reward (LL) over a smaller sooner one (SS) when both are in the future, but once sooner becomes now, she will prefer SS; the rare longitudinal designs of Ainslie and Haendel (1983), Read and van Leeuwen (1998) and Augenblick, Niederle, and Sprenger (2014) indeed find such "impatient switches" for primary rewards.

Conclusion 1. Regarding primary rewards, most subjects exhibit present bias.

By far, most of the empirical evidence on time preferences has been collected from the study of inter-temporal trade-offs in monetary amounts, however. Cubitt and Read (2007) theoretically explain why the link between revealed time preferences over such monetary rewards and those over consumption is likely to be tenuous, given people have access to external credit markets; indeed, they should only reveal the market interest rates which individuals face, in an unbiased manner (cf., however, Harrison and Swarthout, 2011).[37]

[37] Relatedly, Chabris, Laibson, and Schuldt (2008) present a whole list of potential confounds in the

Indeed, although the overall degree of impatience seems positively correlated across the consumption and money domains (Reuben, Sapienza, and Zingales, 2010), the evidence regarding present bias is much weaker for monetary rewards. Most remarkably, Augenblick et al. (2014) compare time preferences over effort and money within a single experimental paradigm and find much stronger evidence of present bias regarding effort than money. However, neither is present bias entirely absent, nor is it the only significant bias on the money domain.

Table 1 lists representative experimental studies of *individual* time preferences over monetary rewards not included in Frederick et al. (2002).[38] It reveals a striking amount of individual heterogeneity in terms of basic *qualitative* preference properties, apparent in any of the various experimental designs. In particular, all studies—employing very different subject pools as well as methods—find a significant proportion of subjects who violate stationarity or dynamic inconsistency in the opposite direction of present bias, "future bias". Overall, we obtain the following conclusion.

Conclusion 2. Regarding monetary rewards, subjects split into three similarly sized groups: a third exhibits no bias, a third exhibits present bias, and a third exhibits (near-) future bias.

In longitudinal designs future bias also turns into actual dynamic preference reversals: e.g., 19 out of the 38 participants in Sayman and Öncüler (2009, experiment 1) chose 7 euros the next day over 10 euros in three days, but reversed their choice the next day (when it was 7 euros now v. 10 euros in two days). While the incidence of future bias fluctuates across designs, it appears particularly strong when both the delay to SS and that between SS and LL are relatively short (Sayman and Öncüler, 2009, suggest less than a week, p. 470); such designs have been investigated only more recently but are most relevant for bargaining applications. Salience of the time dimension seems to be another strongly promoting factor (see Eil, 2012).

For the purpose of using this evidence in the present bargaining application, two difficulties in qualitatively classifying participants' choices should be mentioned. First, due to the discreteness of the choice problems posed, small biases go undetected. A rather extreme example is the longitudinal design of Read et al. (2012): all participants received only a

estimation of utility-discount rates when studying monetary rewards.

[38]The table is not exhaustive of the large number of recent studies. Rather, it is meant to be representative. However, I exclude studies (e.g., Read, 2001; Meier and Sprenger, 2015) or parts of studies (e.g., Eil, 2012) where *qualitative* results would be distorted by the failure to control for utility curvature (see Andersen, Harrison, Lau, and Rutström, 2008). Moreover, I exclude studies which provide too little information on qualitative individual heterogeneity; e.g., between-subjects designs (e.g., Rubinstein, 2003; Cohen, Tallon, and Vergnaud, 2011) or fits of mixture models (e.g., Andersen, Harrison, Lau, and Rutström, 2014) as well as models with individual random effects on parameters (e.g., Abdellaoui, Bleichrodt, and l'Haridon, 2013).

Study	N	Method	Delays (t/Δ/unit)	% PB	% FB	Remarks
Classifications from Choice						
Ahlbrecht and Weber (1997, part 2)	132	BC	0-24/6/Ms	25	25	Also study loss domain
Ashraf, Karlan, and Yin (2006)	1777	BC*	0, 6/1/Ms	28	20	Clients of rural Philippine banks
Sayman and Öncüler (2009, study 1)	38	BC, L	0-7/2-7/Ds	13	29	Up to 50% FB for t & Δ small
Sayman and Öncüler (2009, study 2a)	72	BC*	0-14/2-14/Ds	6	13	
Attema, Bleichrodt, Rohde, and Wakker (2010)	55	M-D*	0/n.a./Ms	15	65	
Meier and Sprenger (2010)	541	BC	0, 6/1/Ms	36	9	
Takeuchi (2011)	55	M-D	0/n.a./Ds	17	66	Also estimates "concave" discounting
Dohmen, Falk, Huffman, and Sunde (2012)	344	BC	0, 6/6, 12/Ms	34	32	% of those who were classified
Eil (2012, "WTP" task)	95	BC	0, 6/1, 6/Ms	36	43	Comparison for fixed $\Delta = 1$
Eil (2012, "WTW" task)	95	BC-D	0, 6/0.5-48/Ms	31	69	Comparison for fixed $t = 0$
Read, Frederick, and Airoldi (2012, exp. 1)	128	BC, L	0-5/1/Ws	11	11	
Read, Frederick, and Airoldi (2012, exp. 2)	201	BC, L	0, 3/2/Ws	9	10	US residents (email recruits)
Dupas and Robinson (2013)	185	BC*	0, 1/1/Ms	35	20	Poor working Kenyans
Augenblick, Niederle, and Sprenger (2014)	75	CTB, L	0, 3/3, 6/Ws	37	20	Contrast results with those for effort
Giné, Goldberg, Silverman, and Yang (2014)	661	CTB, L	1-61/30/Ds	34	31	Malawian farmers
Halevy (2015, larger stakes)	176	BC, L	0, 4/1/Ws	34	20	Many violate (time-) invariance
Halevy (2015, smaller stakes)	176	BC, L	0, 4/1/Ws	31	17	Many violate (time-) invariance
(β, δ)-Estimations (PB as $\beta < 1$, FB as $\beta > 1$)						
Benhabib, Bisin, and Schotter (2010)	27	M-A	0/3-181/Ds	Median $\beta > 1$		Suggest fixed cost of delay
Andreoni and Sprenger (2012)	97	CTB	0-35/35-98/Ds	Median $\beta > 1$		
Augenblick, Niederle, and Sprenger (2014)	75	CTB, L	0, 3/3, 6/Ws	≥ 33	≥ 17	Contrast results with those for effort
Olea and Strzalecki (2014)	336	BC*	0/1-60/Ys	>10	>30	Options have three payoff dates

N: number of participants analyzed.

*Method: BC binary choices (D means choose delay), M matching (choose indifferent amount A or delay D), CTB convex time budgets, L longitudinal, * hypothetical.*

Delays: ranges if more than two (t delay to SS; Δ delay LL minus delay SS; D day, W week, M month, Y year).

PB present bias, FB future bias; for L design choice classification % refers to dynamic preference reversals.

Table 1: Experimental studies of individual time preferences over monetary rewards not included in Frederick et al. (2002) (discounting estimations are only included if they control for curvature and allow for future bias).

few identical binary choice problems, and 79% of the participants either always chose SS or always chose LL. Second, future bias over a short horizon might very well be coded as present bias when "immediate" ($t = 0$ in the table) does not refer to the delay in *receiving* the payment, as is the case under designs with "front-end delay". This is indeed very common to equalize the credibility of receiving "immediate" and later payments; e.g., in Meier and Sprenger (2010) "no delay" ($t = 0$) means having a mail order sent off the same day rather than receiving cash immediately. As important as this procedure is for ruling out confounds regarding present bias, it also makes it impossible to distinguish a bias for immediate gratification from one regarding gratification within only a few days (cf. examples 1 and 2). Given that small biases and very-short run attitudes matter for the present application under sufficiently frequent offers, the above conclusion is very conservative regarding the incidence of (near-) future bias.

Finally, a few authors have also investigated the separability into discounting and instantaneous utility functions (on the money domain). Benhabib et al. (2010) find that a fixed-cost of delay in addition to discounting greatly improves their estimation results. Echenique et al. (2014) analyze the data of Andreoni and Sprenger (2012) and reject separability for almost one half of the participants on the basis of its revealed preference implications; using their own method and data, Ericson and Noor (2015) reject separability for almost 70% of their participants.

Conclusion 3. On the money domain, the separability of preferences into discounting and instantaneous utilities tends to be violated when tested.

Why time preferences over monetary rewards reveal various biases when, theoretically, they should not reveal any bias at all is not well understood. Current work in experimental economics (e.g., Carvalho, Prina, and Sydnor, 2014; Ambrus, Ásgeirsdóttir, Noor, and Sándor, 2015; Carvalho, Meier, and Wang, 2015) explores the role of fluctuations in liquidity (present or anticipated), which has theoretically been shown to be potentially important (Noor, 2009; Gerber and Rohde, 2010, 2015). The reason may also have to do with how the human brain processes monetary rewards: e.g., the recent meta-analysis by Sescousse, Caldú, Segura, and Dreher (2013) finds monetary rewards to engage areas of the brain which are active also for different primary rewards, but at the same time also a distinct, evolutionarily more recent, one.[39] Money may therefore act, at least partially, as a learned cue for immediate or near-future consumption (cf. Fudenberg and Levine, 2006, pp. 1457-8); in a similar vein, it may produce anticipatory utility, which can lead to initially concave discounting (Loewenstein, 1987).

[39]In prior work Kable and Glimcher (2007) had demonstrated that individual time preferences over monetary rewards have a neural correlate.

In any case, given the three conclusions of this section, I impose only minimal assumptions on time preferences in the present study. Essentially, these are only that more is better for any given delay, and sooner is better for any given (positive) amount. Thus I cover the entire spectrum of suggested time preferences and can investigate differential implications of very broad qualitative features, in particular present and (near-) future bias.[40]

B.2 The Role of Inequality 7 in Theorem 1

The following example shows how inequality 7 matters in the equilibrium characterization of theorem 1. It is required both for the set E to contain only values corresponding to "exactly self-enforcing" punishments, and for $(v_i^*, w_i^*, t_i^*)_{i\in I}$ to be the extreme element of the set E; i.e., without imposing it in the definition of E, neither of these two properties holds, in general. The counterexample below relies on the coincidence of multiple stationary equilibria and perfect patience regarding delays beyond a few periods. Indeed, whenever the two players' preferences imply a unique stationary equilibrium or satisfy impatience property (4.c) in the sense of $\lim_{t\to\infty} U_i(1,t) \leq u_i(0)$ for both $i \in I$, inequality 7 can safely be omitted.

Example 4. Let the two players' preferences be identically given by $U_i(q,t) = q - c(q,t)$, where

$$c(q,t) = \begin{cases} 0 & t = 0 \\ \gamma(1 + (1-q)) & t > 0 \end{cases}, \ \gamma \in (0,1).$$

Due to symmetry, player indices are omitted in what follows. The preferences are clearly covered by assumption 1 (for completeness, let $U(0,\infty) = -2\gamma$), where impatience property (4.c) is satisfied since there is no cost to any delay beyond the first one.

The set of stationary equilibria is characterized by the fixed points $q = f(q)$ of which there exist three, namely $\left\{\gamma, \frac{1+2\gamma}{2+\gamma}, 1\right\}$. Consider then the values $\left(\hat{v}, \hat{w}, \hat{t}\right) = (1, 1-\gamma, \infty)$ and note that $\phi(\hat{v}, t) = 1$ for all delays t, whereby

$$\kappa(t, \hat{v}, \hat{v}, \hat{w}) = \begin{cases} 1 + (1-\gamma) & t = 0 \\ 1 + 1 & t > 0 \end{cases} > 1$$

shows that equation 6 is satisfied. From $U(\phi(\hat{v}, t), t+1) = U(1, t+1) = 1 - \gamma$ for all $t \in T$ and $\phi(\hat{v}, 0) = f(\phi(\hat{v}, 0))$, respectively, it is straightforward to see that also equations 5 and 4 are satisfied. Although the values $\left(\hat{v}, \hat{w}, \hat{t}\right)$ solve all of the equations 4-6, $\hat{v}$ and

[40] Some findings (e.g., Read, 2001; Rubinstein, 2003; Dohmen et al., 2012) suggest that transitivity may yet be violated in comparisons across different delays (cf. Manzini and Mariotti, 2007). Ok and Masatlioglu (2007) propose a model of "relative" discounting, which maintains separability but accommodates those violations. Footnote 13 explains why also these preferences are covered here.

$\hat{w}$ are not punishment values supporting equilibrium outcomes, however. Instead, they are the maximal proposer values and rejection values, respectively, and therefore correspond to mutually inconsistent (credible) "rewards"; the set of outcomes they support as punishments is empty (see lemma 4).

More importantly, however, the equilibrium values (v^*, w^*, t^*) are not an extreme element of the set of solutions to equations 4-6 for any value of γ greater than $\frac{1}{5}$. The preferences clearly exhibit a weak present bias (only the first period of delay is costly), which implies that $v^* = \gamma$ and $w^* = 0$, but $t^* < \infty = \hat{t}$: using that $\phi(u,t) = \phi(u,1) = \min\left\{\frac{u+2\gamma}{1+\gamma}, 1\right\}$ for all $t > 0$, $\kappa(2,\gamma,\gamma,0) \geq 2 \cdot \frac{3\gamma}{1+\gamma}$ which exceeds one for any $\gamma > \frac{1}{5}$.

B.3 Multiplicity under Exponential Discounting

The following example is one of ED, which exhibits multiple stationary equilibria due to a violation of initially increasing loss to delay and, possibly, delay. It was presented already by Rubinstein (1982, conclusion I), but its set of equilibria has not yet been characterized.

Example 5. Let the two parties' preferences be identically given by $U_i(q,t) = q - ct$, for $c \in (0,1)$. Due to symmetry, player indices are omitted in what follows. The preferences are covered by assumption 1 once $U(0,\infty) = -\infty$ is specified; in particular, impatience property (4.c) is satisfied: $U(1,t)$ tends to minus infinity, whereas $u(0) = 0$. In the assumed absence of uncertainty, they actually satisfy ED, albeit with "strongly" convex instantaneous utility: $U(q,t) = \ln(\delta^t u(q))$ for $\delta \equiv \exp(-c)$ and $u(q) \equiv \exp(q)$. Hence they exhibit a weak present bias but violate initially increasing loss to delay, because $q - \pi(U(q,1)) = q - \max\{q-c, 0\}$ is constant on $[c,1] \neq \emptyset$.[41]

This results in a multiplicity of stationary equilibrium: any $q \in [c,1]$ is a proposer's equilibrium share in some stationary equilibrium (with immediate agreement, of course). Applying the characterization of theorem 1, $v^* = c$ and $w^* = 0$, where both of these minimal proposer and rejection values correspond to a player's least preferred stationary equilibrium. Using these two least preferred stationary equilibria as optimal punishments, non-stationary delay equilibria can be constructed, and equation 6 offers a formula to compute the maximal

[41] One may interpret such preferences as there being a cost to bargaining. To justify the non-negativity of each player's share in any proposal, assume then that players have an "outside option" of leaving bargaining forever which is equivalent to obtaining a zero share immediately.

such delay for any $c \in (0,1)$:

$$\begin{aligned} \kappa(t,c,c,0) &= \min\{c+ct,1\}+\min\{ct,1\} \\ &= \begin{cases} (2t+1)\,c & t \leq \frac{1-c}{c} \\ 1+ct & \frac{1-c}{c} \leq t \leq \frac{1}{c} \\ 2 & \frac{1}{c} \leq t \end{cases} \\ \Rightarrow t^* &= \sup\{t \in T\,|\kappa(t,c,c,0) \leq 1\} \\ &= \max\left\{t \in T \,\middle|\, t \leq \frac{1}{2}\cdot\frac{1-c}{c}\right\} \\ &= \left\lfloor \frac{1}{2}\cdot\frac{1-c}{c}\right\rfloor. \end{aligned}$$

For instance, if $c = \frac{1}{100}$ so that the cost per bargaining round equals one percent of the surplus, then the maximal equilibrium delay is 49 periods, with an associated efficiency loss of 98 percent of the surplus. To determine the values of c for which delayed agreement is an equilibrium outcome, simply solve $\kappa(1,c,c,0) \leq 1$ for c, yielding $c \leq \frac{1}{3}$. The set of equilibrium divisions with a given delay $t \leq t^*$ in game G_1 equals $\{x \in X|c+ct \leq x_1 \leq 1-ct\}$ and is monotonically shrinking in t.

B.4 Unbounded Equilibrium Delay

The following example slightly modifies example 3 to exhibit unbounded equilibrium delay.

Example 6. Let the two players' preferences be given by $U_i(q,t) = d(t)\cdot q$ such that

$$d(t) = \begin{cases} \delta^t & t \leq \tau \\ \gamma\delta^{\tau+1} & t > \tau \end{cases}, \; (\delta,\gamma) \in (0,1)^2 \text{ and } \tau > 0.$$

Due to identical preferences, the player subscript is omitted in what follows.

Observe that $\Delta(t)$ equals δ for all $t \leq \tau$ and $\gamma\delta$ for all $t > \tau$, exactly as in example 3. Hence, whenever there is an equilibrium in which agreement is delayed by τ periods, $v^* = \frac{1-\delta}{1-\gamma\delta^2}$ and $w^* = \gamma\delta v^*$, as was found there.

The absence of discounting beyond a delay of $\tau+1$ periods implies that equilibrium delay is unbounded if and only if $1 \geq \kappa(\tau+2,v^*,v^*,w^*) = 2\frac{v^*}{\gamma\delta^{\tau+1}}$, which reduces to

$$\delta^\tau \geq \frac{2}{\gamma\delta}\cdot\frac{1-\delta}{1-\gamma\delta^2} \tag{21}$$

after substituting for v^*. Notice that this inequality is more stringent than example 3's

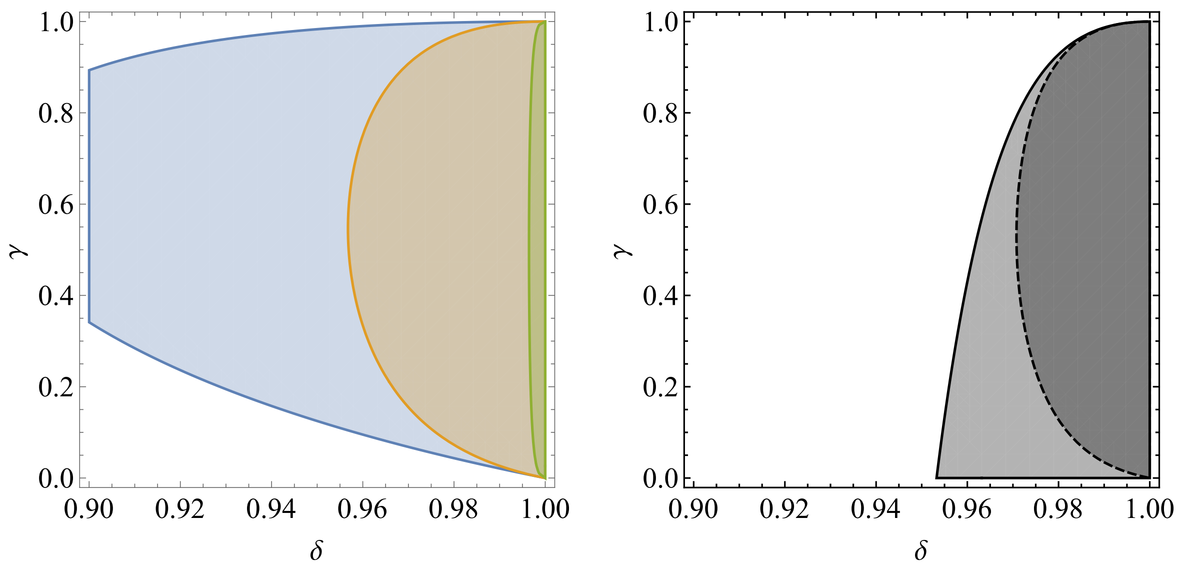

Figure 3: Graphs regarding unbounded equilibrium delay in example 3. The panel on the left shows the parametric regions (δ, γ) such that equilibrium delay is unbounded for three given values of τ, which are 1 (blue, brown and green), 25 (brown and green) and 1000 (green). The panel on the right illustrates how the respective parametric regions for existence of delay equilibria (superset, bounded by solid line) and unbounded equilibrium delay (subset, bounded by dashed line) are related for the case of $\tau = 50$.

inequality 9, which showed when delay equilibria exist; in particular, $\gamma > 0$ is here required. Indeed, γ might be too low: despite existence of an equilibrium with delay τ, which fully determines the optimal punishments, proposing players would then require too large a compensation for longer delays, as those would have additional discount factor γ. Nonetheless, for any given $\tau > 0$ and $\gamma < 1$, there again exist large enough values of δ such that also inequality 21 is satisfied, with the set of parameters γ and τ such that equilibrium delay is unbounded expanding as δ increases. Figure 3 illustrates this.

B.5 Other Sources of Dynamic Inconsistency

B.5.1 Imperfect Altruism and Inter-generational Bargaining

Suppose there are two communities with access to a productive resource. They decide over how to share it by means of bargaining over usage rights. As long as these have not been settled, each period some surplus, normalized to one, is forgone. Upon failure to agree, both communities nominate a new delegate to engage in the bargaining on their behalf. Once they agree, however, all future generations enjoy the agreed surplus every period. I now sketch a simple version of this general problem with imperfect altruism of community members towards future ones.

Denote the two communities $i \in I$, each of which has a population of two members in any period $n \in \mathbb{N}$: a young member (i, y) and an old member (i, o). Each member lives for two periods, where in the first half of her life a member is called young, and in the second half it is called old; reproduction is therefore such that the old member at time t is replaced by a young one at time $t+1$. All young members of a community are identical, and so are all young ones (though they live at different times).

Each community member discounts payoffs exponentially (discount factor δ_i) and is altruistic towards all future generations, but with an extra discount (factor γ_i) for payoffs beyond her own lifetime: at any point in time, for any agreement with a delay of $t \in T$ periods, where community i's share is equal to q,

$$U_{i,o}\left(q,t\right)=\begin{cases} q+\gamma_i\delta_i\frac{1}{1-\delta_i}q & t=0 \\ \gamma_i\delta_i^t\frac{1}{1-\delta_i}q & t>0 \end{cases},\quad U_{i,y}\left(q,t\right)=\begin{cases} q+\delta_i U_{i,o}\left(q,0\right) & t=0 \\ \delta_i U_{i,o}\left(q,t-1\right) & t>1 \end{cases},\quad \left(\delta_i,\gamma_i\right)\in\left(0,1\right)^2.$$

It is straightforward to show that these preferences can be represented as $U_{i,g}\left(q,t\right)=d_{i,g}\left(t\right)q$, $g \in \{y, o\}$, such that

$$d_{i,o}\left(q,t\right)=\begin{cases} 1 & t=0 \\ \alpha_i\delta_i^t & t>0 \end{cases},\quad d_{i,y}\left(q,t\right)=\begin{cases} 1 & t=0 \\ \beta_i\delta_i & t=1, \\ \alpha_i\beta_i\delta_i^t & t>1 \end{cases}\quad \left\{\begin{array}{c} \alpha_i\equiv\frac{\gamma_i}{(1-\delta_i)+\gamma_i\delta_i} \\ \beta_i\equiv\frac{(1-\delta_i)+\gamma_i\delta_i}{\left(1-\delta_i^2\right)+\gamma_i\delta_i^2} \end{array}\right\}.$$

Notice that $0<\delta_i,\gamma_i<1$ implies $0<\alpha_i<\beta_i<1$. Since $\alpha_i<1$, old members' preferences exhibit a weak present bias, discounting the first period of delay with factor $\alpha_i\delta_i$ and thus more heavily than any other, which are discounted with constant factor δ_i. However, young members discount the second period of delay more than the first, namely with a factor $\alpha_i\delta_i$ less than that for the first one, which equals $\beta_i\delta_i$; all periods further in the future are discounted less, with constant factor δ_i.

Suppose that at the beginning of each period n, a round of bargaining takes place, where if n is odd, community 1's delegate gets to propose, and otherwise it is community 2's delegate. Since this means alternating offers in terms of communities, any bargaining protocol where each community either sends only young or only old members as delegates to the bargaining table results in a stationary game of the type analyzed in this paper, and the results of this paper, in particular characterization theorem 1, apply in a straightforward manner.

B.5.2 Non-linear Probability Weighting and Bargaining Under the Shadow of Breakdown Risk

One motive for impatience in the sense of discounting future payoffs is uncertainty, such as mortality risk. Halevy (2008) and Saito (2015) show how dynamically inconsistent discounting can be related to non-linear probability weighting.

Suppose that two parties bargain over a surplus of normalized size one, with alternating offers, where, after each round without agreement, there is a constant probability of $1-p \in (0,1)$ that bargaining (exogenously) breaks down before the next round, leaving players without any surplus. Applying the aforementioned authors' results in this context, in any round n a player i's preferences over shares q agree upon with delay $t \in T$ have the following representation, which—for the sake of simplicity—involves breakdown risk as the sole source of discounting:

$$U_i(q,t) = g_i\left(p^t\right) u_i(q), \tag{22}$$

where $g_i : [0,1] \to [0,1]$ is a so-called probability-weighting function, assumed continuous and increasing from $g_i(0) = 0$ to $g_i(1) = 1$, and $u_i : [0,1] \to \mathbb{R}_+$ is an instantaneous utility function, assumed continuous and increasing from $u_i(0) = 0$. These preferences are dynamically consistent if and only if g_i is the identity, in which case i maximizes expected utility.

Redefining, for a given "survival rate" p, $g_i(p^t) \equiv d_i(t)$, all results of this paper can be applied in a straightforward manner. Thus the players' probability weighting, which determines their dynamic inconsistency, can be related to the set of bargaining outcomes.

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