Econometrics of Share Auctions*

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Abstract

The purpose of this paper is to propose structural econometric methods for the empirical study of Wilson's (1979) share auction model. This is a common value model in which a single and perfectly divisible good is sold to a group of symmetric and risk-neutral buyers. The parameters in the distribution function of the value of the good and the signals received by the buyers are estimated using a two-step estimation procedure. The methods are applied to French Treasury securities auctions held in 1995. A counterfactual comparison shows that the Treasury's revenue in the discriminatory share auction (the mechanism adopted by the French Treasury) is higher than in the uniform share auction.

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1 Introduction

Until now the structural econometric auction literature has mainly focused its attention on the standard first-price auction of a single indivisible good.[1] Various structural estimation techniques have been developed for this auction model under different bidding environments. Donald and Paarsch (1993, 1996), Elyakime, Laffont, Loisel and Vuong (1994), Laffont, Ossard and Vuong (1995), and Guerre, Perrigne and Vuong (2000) consider the Independent Private Value (IPV) paradigm so that the structural element to be estimated is the distribution of private values of the buyers. Li, Perrigne and Vuong (2002) consider an affiliated private value framework so that their objective is to estimate the joint distribution of private values. Finally, Paarsch (1992), Li, Perrigne and Vuong (2000) and Hong and Shum (2003) consider the common value paradigm and propose methods to estimate the joint distribution function of the value of the good and the signals received by the buyers.

The purpose of this paper is to propose structural econometric methods for the empirical study of Wilson's (1979) share auction model. This is a common value model in which a single and perfectly *divisible* good is sold to a group of symmetric and risk-neutral buyers. As in the standard common value auction model for an indivisible good, the value of the object is unknown at the time of bidding, and prior to the auction the buyers independently receive signals that are informative about the value. Unlike the standard model, each potential buyer not only submits a price but also a share (or fraction) he requests at that price. Each bidder can submit as many price/share combinations as he wants, thereby constituting a demand curve for the good. By aggregating the individual demand curves, the auctioneer can determine the equilibrium price that clears the market. Wilson restricts his analysis primarily to two auction formats: the uniform price auction and the discriminatory auction. In both auction formats the allocation devise is the same, and consists in awarding to each bidder the fraction of the good he requested at the equilibrium price. The payment rule, however, differs in the two auction mechanisms. Roughly speaking, in the uniform auction, all winners (bidders with positive demand at the equilibrium price) pay the equilibrium price, whereas in the discriminatory auction, for each marginal

[1] Some recent papers, however, consider the estimation of multi-unit auctions. See for example Donald, Paarsch and Robert (1999), and Cantillon and Pesendorfer (2001).

share they receive, winners pay the price at which the bid was submitted.

Although basically our estimation procedure can be applied to both the discriminatory and uniform share auction model, we only present the methods for the former auction format. The structural elements of interest are the marginal distribution function of the value of the good, and the conditional distribution function of the signal given the value. The two distribution functions are specified parametrically so that the goal is to estimate a vector of unknown parameters. We do this via a two-step procedure which is similar in spirit as the two-step procedure proposed by Elyakime et al. (1994) and Guerre et al. (2000) (see Jofre-Bonet and Pesendorfer (2003) and Hortaçsu (2002) for other applications and extensions of this two-step procedure). As in their first step, we exploit the relationship between the distribution of observables (shares) and the structural distribution of the model, estimate the distribution of observables, and insert the estimate into the first-order condition. The second step of their method of statistical inference is however different in nature from our's. Their first-order condition allows them to simulate for each bidder a private value, which in turn allows them to estimate the distribution of private values using the generated pseudo sample. We shall instead exploit the fact that our Euler condition can be seen as a set of moment restrictions that depend on the unknown parameters of interest. Minimization of the empirical counterpart of the moment restrictions then leads to our estimator.

As Elyakime et al. (1994) and Guerre et al. (2002) have pointed out, an attractive feature of the two-step method is that it only relies on the first-order condition implied by the bidder's maximization problem. Although it must be assumed that there exists an equilibrium strategy and that all bidders behave according to this strategy, the explicit form of the equilibrium does not need to be known. In our context this is of particular importance since recent work has shown that, unlike the uniform share auction, it appears difficult to find explicit optimal strategies in the case of the discriminatory share auction (Viswanathan, Wang and Witelski (2000) and Hortaçsu (2001) have obtained explicit solutions for the very specific two-bidder case only).

This paper also studies the identification of the share auction model and the asymptotic properties of our two-step estimator. We show that the model is parametrically identified. That is, given our parametric choice of the distribution functions, the true value of the parameter is uniquely determined from the moment restrictions. The asymptotic properties are derived by drawing on Newey and McFadden (1994). Our estimator belongs to their class of semiparametric two-step estimators. By verifying the regularity conditions of Newey and McFadden for this class of models, we show that our estimator is consistent and asymptotically normally distributed, and indicate how the asymptotic variance matrix can be consistently estimated.

The methods are applied to data from Treasury securities auctions. Treasury securities auctions are often cited as a good example of a share auction (see for example the survey on Treasury auction theory by Das and Sundaram, 1996). Treasury securities auctions are auctions in which huge quantities of strictly identical goods—i.e. securities—are sold to a group of buyers, generally large investment institutions. Since these institutions typically require different amounts of securities at different prices, the Treasury allows them to submit price/quantity combinations, that is a demand curve for the securities. Normalizing the total amount of securities offered to one, and dividing the quantity bids by the total volume, a Treasury auction is indeed a good approximation of a share auction.

Our empirical analysis is based on very detailed bidder-level data from French Treasury securities auctions held in 1995. In France, the Treasury sells the securities via discriminatory auctions. In the first part of the empirical analysis our purpose is therefore to estimate the parameters of the discriminatory auction model using the two-step estimation procedure. The second part is more policy-oriented. Given an explicit optimal strategy in the uniform auction model (derived under the same distributional assumptions as in the discriminatory model), and using the parameter estimates of the discriminatory auction model, we can approximate the hypothetical equilibrium price that would emerge in the uniform auction, and thereby also the corresponding revenue. Comparing the observed revenue from the discriminatory auction with the hypothetical revenue generated by the uniform auction, we can evaluate if the discriminatory auction is revenue-superior to the uniform auction, or not. Our counterfactual comparison shows that the Treasury's revenue in the discriminatory auction is significantly higher than in the uniform auction.

We hereby contribute to a debate that has been going on at least since Friedman (1960). He supported the uniform auction format, claiming that collusion is less likely in this auction system and that for this reason the uniform auction would be better from the Treasury's viewpoint than a discriminatory auction. His article initiated a vast literature on the revenue generating properties of the uniform and discriminatory auction formats. Although most of the papers in this literature are theoretical, there are some empirical contributions. The empirical studies on the optimal revenue debate are either based on experimental data (see for example Smith (1967), and Miller and Plott (1985)), or on natural experiment-type data (see for example Umlauf (1993), who examines the revenue issue by exploiting the fact that the Mexican treasury switched from uniform pricing to discriminatory pricing, and Berg, Boukai and Landsberger (1998) who exploit the fact that the Norwegian central Bank simultaneously applied both auction rules to similar types of securities).

Hortaçsu (2002), Castellanos and Oviedo (2002), and Armantier and Sbaï (2003) are the only papers we are aware of that also address the revenue-issue by using structural econometric methods. Hortaçsu proposes a model of strategic bidding behavior in the discriminatory share auction. His model differs from Wilson's model because the IPV paradigm is adopted instead of the common value paradigm, and also because the prices and shares submitted by the bidders are discrete instead of continuous variables. His method is distribution-free, and the techniques are applied to Turkish Treasury auctions. Castellanos and Oviedo apply our method of statistical inference to Mexican Treasury auctions. Finally, Armantier and Sbaï extend Wang and Zender's (2002) common value model for the discriminatory share auction by taking into account both risk-aversion and asymmetry among bidders. They use a parametric estimation method developed by Armantier, Florens and Richard (2002), and their application is based on French Treasury auctions. We will discuss the results of these 3 contributions in the empirical part of our paper.

The paper proceeds as follows. Section 2 presents Wilson's (1979) share auction model, the identification of the model, the estimation method, and the asymptotic properties of the two-step estimator. Section 3 describes the institutional background of the French Treasury securities auctions, and contains a descriptive analysis of the auction data. Section 4 presents the results, and Section 5 concludes.

2 Share auction theory and the estimation method

2.1 The share auction model

This subsection presents the theory of share auctions developed by Wilson (1979). A divisible good is auctioned. There are $n \geq 2$ risk-neutral bidders. The marginal value of the good is constant and is the same for all bidders but unknown at the start of the auction. It is assumed that the value of the good is a realization of a random variable V.[2] The distribution function of V is denoted $F_V(v) = \Pr(V \leq v)$.[3] Prior to the auction, each bidder $i = 1, ..., n$ receives a private signal about the value of the good. The signal received by individual i is assumed to be a realization of the random variable S_i. As usual in common value-type auction models, the bidder's signals $S_1, ..., S_n$ are i.i.d. given V. The distribution function of S_i given V is thus the same for all bidders i, and is denoted $F_{S|V}(s|v) = \Pr(S_i \leq s|V = v)$. It is assumed that signal S_i is only observed by bidder i, and not by the seller or the other potential buyers. Furthermore, the number of bidders n, and the distribution functions $F_V(\cdot)$ and $F_{S|V}(\cdot|\cdot)$ are common knowledge.

Each bidder i is required to submit to the auctioneer a tender (sealed, and written) stating, for each price of the good, the desired share of the good. The price-share combinations submitted by bidder i constitute bidder's i demand function for the good. By aggregating the individual demand functions, the auctioneer can determine the equilibrium price that clears the market, i.e., the price for which aggregate demand equals one. Given the pre-announced allocation devise and pricing rule, the shares are then awarded to the winners, who pay the owner of the good. In the two auction formats mentioned in the introduction, the uniform price auction and the discriminatory auction, the allocation mechanism is the same, and consists in allocating to each bidder the fraction of the good he requested at the equilibrium price. The payment rule, however, differs in the two auction formats. In the uniform price auction each winner simply pays the equilibrium price multiplied by his requested share. In the discriminatory auction each winner has to pay the area under his inverse demand curve between zero and his requested share,

[2] Note that in this model both the marginal value and the value are equal to V.

[3] Throughout the paper random variables are distinguished from their realizations by denoting the former by upper case letters and the latter by lower case letters.

so that here the payment is bidder-specific.

Let $x_i(\cdot,\cdot)$ be a strategy of bidder i. A strategy is a function of the price of the good p and the signal s_i, such that when bidder i receives signal $S_i = s_i$, he submits a demand schedule specifying that at each price p he demands a share $x_i(p, s_i)$ of the good. In characterizing an optimal strategy, attention is restricted to symmetric strategies, so that $x_i(\cdot,\cdot) = x(\cdot,\cdot)$ for all i. By *optimal* strategy is meant a symmetric Bayesian Nash equilibrium of the auction game. A strategy is thus optimal if no player can deviate in a profitable way from equilibrium behavior if the other players adopt the optimal strategy.

The form of an optimal strategy depends, among other things, on the auction mechanism that is used to sell the good. Consider first the characterization of an optimal strategy in the case of a uniform auction. Let $x(\cdot,\cdot)$ now designate the optimal strategy. Suppose that all bidders except i use the strategy $x(\cdot,\cdot)$, and that i uses the strategy $y(\cdot,\cdot)$. Let p^0 denote the equilibrium price, i.e., p^0 is the price such that

$$\sum_{j\neq i} x(p^0, s_j) + y(p^0, s_i) = 1. \tag{1}$$

The above market clearing equation shows that the equilibrium price depends on the signals received by the competitors of bidder i. Since these signals are unknown, the equilibrium price is also unknown to bidder i. But since bidder i knows the distribution function from which the signals S_j, $j \neq i$, are drawn, and also the function $x(\cdot,\cdot)$, he can determine the (conditional) distribution function of the random variable P^0. That is, he can determine

$$\begin{aligned}
H(p; v, y) &\equiv \Pr\left(P^0 \leq p | V = v, y(p, s_i) = y, S_i = s_i\right) \\
&= \Pr\left(\sum_{j\neq i} x(p, S_j) \leq 1 - y | V = v, S_i = s_i\right) \\
&= \Pr\left(\sum_{j\neq i} x(p, S_j) \leq 1 - y | V = v\right).
\end{aligned}$$

When bidder i uses the strategy $y(\cdot,\cdot)$, and if the value of the good and equilibrium price are respectively v and p^0, his profit is $(v - p^0)y(p^0, s_i)$. Bidder's i expected profit in a uniform auction is therefore

$$E\left\{\int_0^\infty (V - p)y(p, s_i)dH(p; V, y(p, s_i))|S_i = s_i\right\} \tag{2}$$

where, as the notation suggests, the expectation is with respect to V given $S_i = s_i$. The strategy $x(\cdot,\cdot)$ is indeed optimal if the maximum of (2) is attained at $y(\cdot,\cdot) = x(\cdot,\cdot)$. A solution to this maximization problem can be characterized by applying the principles of calculus of variations. Wilson has shown that the necessary condition for optimization (the Euler condition) is that

$$0 = E\left\{(V-p)\partial H(p;V,y)/\partial p + x(p,s_i)\partial H(p;V,y)/\partial y | S_i = s_i\right\} \qquad (3)$$

where the partial derivatives of H with respect to p and y are evaluated at $y = x(p, s_i)$. This condition holds for all $p \in [p^{min}, p^{max}]$, where $[p^{min}, p^{max}]$ is the support, at the equilibrium, of the distribution function of the equilibrium price.

Next consider the discriminatory auction. Using the same notation as above, bidder's i profit in a discriminatory auction, when he adopts the strategy $y(\cdot,\cdot)$, and when the value of the good and equilibrium price are v and p^0, equals $(v-p^0)y(p^0, s_i) - \int_{p^0}^{\overline{p}} y(u, s_i)du$, where $\overline{p}$ is the largest price for which demand $y(\cdot, s_i)$ is non-negative. Bidder's i expected profit is therefore

$$E\left\{\int_0^{\infty}\left[(V-p)y(p,s_i) - \int_p^{\overline{p}} y(u,s_i)du\right] dH(p;V,y(p,s_i))|S_i = s_i\right\}. \qquad (4)$$

Wilson (1979) does not derive the Euler condition for the discriminatory share auction. We show in appendix A that the Euler condition in this case is

$$0 = E\left\{(V-p)\partial H(p;V,y)/\partial p - H(p;V,y)|S_i = s_i\right\} \qquad (5)$$

where the distribution H and the derivative of H are evaluated at $y = x(p, s_i)$, and the condition must hold for all $p \in [p^{min}, p^{max}]$. Note that this Euler condition differs from the uniform-auction Euler condition (3) only in the second term of the expectation.

Our method of statistical inference is highly facilitated by rewriting the above Euler equation. The following proposition states how (5) can conveniently be reformulated.

Proposition 1. *The Euler condition (5) can be rewritten as*

$$\begin{aligned} 0 \quad = \quad & E\left\{(n-1)\cdot\left[E\left\{V|S_1 = s_1, ..., S_n = s_n\right\} - p\right]\cdot 1(P^0 \le p)\right\} \\ & - E\left\{(p - P^0)\cdot 1(P^0 \le p)\right\}. \end{aligned} \qquad (6)$$

In the above equation $1(\cdot)$ represents the indicator function, the first expectation is with respect to $S_1, ..., S_n$ (the random variable P^0 only depends on $S_1, ..., S_n$), and, as is clear from the notation, the second expectation is with respect to V given $S_1, ..., S_n$, and the third is with respect to P^0. The condition (6) must hold for all $p \in [p^{min}, p^{max}]$. The proof of Proposition 1 is given in appendix B. We also show in appendix B that Proposition 1 remains valid in the case of asymmetric bidders. That is, even when the distribution of S_i given V is no longer the same for each bidder i, the restriction (6) still holds.

The Euler condition forms the basis of our estimation method for the discriminatory share auction model. As explained in the next subsection, our estimator is defined as the minimum of an empirical counterpart of the Euler restriction. The crucial advantage of the reformulated Euler condition (6) over the Euler condition (5) is that it is much easier to obtain an empirical counterpart of the former. This comes from the fact that (6) no longer depends on $H(\cdot; \cdot, \cdot)$, the distribution function of the market clearing price. Indeed this function (and its derivative with respect to p) is difficult to compute and evaluate because of its implicit dependence on the equilibrium strategy $x(\cdot, \cdot)$.

2.2 Estimation

In Section 2.2.1 we rewrite the Euler conditions (6) once more and show that they can be seen as moment conditions that depend on the unknown parameters of the distribution functions. Section 2.2.2 is devoted to the issue of identification of the model. Section 2.2.3 presents our two-step estimator and Section 2.2.4 is devoted to the asymptotic properties of the estimator.

2.2.1 Theoretical moments

The estimation procedure exploits the fact that the results from several different auctions are available. Suppose there are L auctions and let l index the l-th auction. In many applications, the goods sold in the different auctions are not completely identical. Also, the number of bidders typically varies from auction to auction. To capture this between-auction heterogeneity we introduce the commonly known vector of variables z_l characterizing the good sold at the l-th auction, and the number of bidders, n_l.

It is assumed that the random variables (N_l, Z_l), $l = 1, ..., L$, are independently and identically distributed. The value of the good in the l-th auction, V_l, is assumed to depend on Z_l but not on N_l. Similarly, the signal received by individual i in auction l, S_{il}, depends on Z_l (and on V_l) but not on N_l. Conditionally on $Z_1, ..., Z_L$, the values $V_1, ..., V_L$ are assumed to be independently and identically distributed. Furthermore, $S_{1l}, ..., S_{n_l l}$ are independent conditionally on (V_l, Z_l), and the signals S_{il} and $S_{i'l'}$ are independent conditionally on Z_l and $Z_{l'}$ for all $l \neq l'$. We adopt a parametric framework. The distribution functions are specified parametrically and are thus known up to a vector of parameters. The conditional distribution of V_l given $Z_l = z$ is denoted $F_{V|Z}(\cdot|z;\theta_1)$, where θ_1 is a vector of parameters. The conditional distribution function of S_{il} given $V_l = v$ and $Z_l = z$ is denoted $F_{S|V,Z}(\cdot|v, z;\theta_2)$, where θ_2 is a vector of parameters. Given the distribution functions $F_{V|Z}(\cdot|\cdot;\cdot)$ and $F_{S|V,Z}(\cdot|\cdot,\cdot;\cdot)$, we can determine the distribution function of S_{il} given $Z_l = z$, denoted $F_{S|Z}(\cdot|z;\theta)$, where $\theta = (\theta_1', \theta_2')'$. The true value of θ is denoted θ^0.

The objective of this subsection is to rewrite the Euler condition (6) once again and show how the reformulated theoretical moments depend on the true parameter value. To proceed, let us from now on explicitly write the optimal strategy as a function of not only the price and the signal, but also the number of bidders, the vector of characteristics of the good, and the parameter. That is, $x(p, s, n, z;\theta^0)$ is the optimal demand for the good at price p for an individual with signal s, when the auction is attended by n bidders and the good on sale has characteristics z, and when the true value of the parameter equals θ^0. Furthermore, for any given $p \in [p^{min}(n, z), p^{max}(n, z)]$, let $G(\cdot|n, z;p)$ denote the distribution function of $x(p, S_{il}, N_l, Z_l;\theta^0)$ conditionally on $N_l = n$ and $Z_l = z$. We have

$$\begin{aligned}
G(x|n, z;p) &= \Pr(x(p, S_{il}, N_l, Z_l;\theta^0) \leq x|N_l = n, Z_l = z) \\
&= \Pr(x(p, S_{il}, n, z;\theta^0) \leq x|N_l = n, Z_l = z) \\
&= \Pr(S_{il} \geq x^{-1}(p, x, n, z;\theta^0)|N_l = n, Z_l = z) \\
&= \Pr(S_{il} \geq x^{-1}(p, x, n, z;\theta^0)|Z_l = z) \\
&= 1 - F_{S|Z}(x^{-1}(p, x, n, z;\theta^0)|z;\theta^0)
\end{aligned} \quad (7)$$

where[4] the fourth equation follows from the assumption that S_{il} and N_l are conditionally independent. The third equation holds under the additional

[4]Note that there is the function $x(\cdot,\cdot,\cdot,\cdot;\cdot)$, the inverse function $x^{-1}(\cdot,\cdot,\cdot,\cdot;\cdot)$, and the scalar x.

hypothesis that the optimal strategy $x(\cdot, s, \cdot, \cdot; \cdot)$ is a strictly decreasing function in s.[5] Note that from (7) we immediately obtain the inverse demand function:

$$x^{-1}(p, x, n, z; \theta^0) = F_{S|Z}^{-1}(1 - G(x|n, z; p)|z; \theta^0). \tag{8}$$

Let us now rewrite the Euler condition (6), first by incorporating the auction-specific notation and variables. For auction l with characteristics z_l and with n_l bidders, the condition becomes

$$\begin{aligned} 0 = E\Big\{(n_l - 1) \cdot \Big[E\Big\{V_l|S_{1l} = s_{1l}, ..., S_{n_l l} = s_{n_l l}, N_l = n_l, Z_l = z_l\Big\} - p\Big] \\ \cdot 1(P_l^0 \leq p)|N_l = n_l, Z_l = z_l\Big\} - E\Big\{(p - P_l^0) \cdot 1(P_l^0 \leq p)|N_l = n_l, Z_l = z_l\Big\} \end{aligned} \tag{9}$$

where the random variable P_l^0 represents the equilibrium price in auction l, and the first expectation is with respect to $S_{1l}, ..., S_{n_l l}$ given $N_l = n_l, Z_l = z_l$. The condition must hold for all possible values of n_l and z_l, all $p \in [p^{min}(n_l, z_l), p^{max}(n_l, z_l))]$ and all $l = 1, ..., L$. Letting x_{ilp} represent the observed optimal demand by bidder i in auction l at the price p, i.e., $x_{ilp} = x(p, s_{il}, n_l, z_l; \theta^0)$, and since $s_{il} = x^{-1}(p, x_{ilp}, n_l, z_l; \theta^0)$, and using (8), the condition (9) can be rewritten as

$$\begin{aligned} 0 = E\Big\{(n_l - 1) \cdot \Big[E\Big\{V_l|S_{1l} = F_{S|Z}^{-1}(1 - G(x_{1lp}|n_l, z_l; p)|z_l; \theta^0), \\ ..., S_{n_l l} = F_{S|Z}^{-1}(1 - G(x_{n_l lp}|n_l, z_l; p)|z_l; \theta^0), N_l = n_l, Z_l = z_l\Big\} - p\Big] \\ \cdot 1(P_l^0 \leq p)|N_l = n_l, Z_l = z_l\Big\} - E\Big\{(p - P_l^0) \cdot 1(P_l^0 \leq p)|N_l = n_l, Z_l = z_l\Big\} \end{aligned} \tag{10}$$

where now the first expectation is with respect to the random variables $X_{1lp}, ..., X_{n_l lp}$, given $N_l = n_l, Z_l = z_l$.

Since the above moment condition is actually a conditional moment condition (the first and third expectations are conditional on N_l, Z_l), there is

[5] Given the specific parametric specifications of the distributions functions $F_{V|Z}(\cdot|\cdot; \cdot)$ and $F_{S|V,Z}(\cdot|\cdot, \cdot; \cdot)$ used in Section 4.1, we derive an explicit optimal strategy in the uniform auction model. This strategy (the unique equilibrium in a large class of demand functions) turns out to be strictly decreasing in s. It is quite natural therefore to assume that the optimal strategy in the discriminatory auction also satisfies this property.

an infinity of possible unconditional moments for each p. More precisely, the condition (10) implies that

$$0 = E\Big\{w(N_l, Z_l) \cdot (N_l - 1) \cdot \Big[E\Big\{V_l | S_{1l} = F^{-1}_{S|Z}(1 - G(x_{1lp}|n_l, z_l; p)|z_l; \theta^0),$$
$$..., S_{n_l l} = F^{-1}_{S|Z}(1 - G(x_{n_l lp}|n_l, z_l; p)|z_l; \theta^0), N_l = n_l, Z_l = z_l\Big\} - p\Big]$$
$$\cdot 1(P^0_l \le p)\Big\} - E\Big\{w(N_l, Z_l) \cdot (p - P^0_l) \cdot 1(P^0_l \le p)\Big\} \quad (11)$$

for any function $w(\cdot, \cdot)$.[6] The first expectation is an unconditional expectation with respect to $N_l, Z_l, X_{1lp}, ..., X_{n_l lp}$, the third is an unconditional expectation with respect to N_l, Z_l, P^0_l. For a given function $w(\cdot, \cdot)$, the above condition must hold for all $p \in \cap_{l/w(n_l, z_l) \neq 0}[p^{min}(n_l, z_l), p^{max}(n_l, z_l)]$. Note that if the weighting function is such that $w(n_l, z_l) = 0$ for "many" l, the set of values of p for which (11) should hold is in a sense "large"; inversely if $w(n_l, z_l) = 0$ for only "few" l, the set of price values that should satisfy the condition is "small". The moment condition (11) forms the basis of our estimation method. Section 2.2.3 proposes empirical counterparts for the theoretical moments and shows how the former can used in our method of statistical inference. But first we turn to the issue of identification.

2.2.2 Identification

An important issue in the structural estimation of auction models is whether the fundamental elements of the model are identified. In the share auction model the fundamental elements are the conditional distribution functions of $V|Z$ and $S|V, Z$. Although we have adopted a parametric framework in this paper (i.e. the distribution functions are assumed to be known up to a vector of parameters), we shall investigate both the parametric and nonparametric identification of the model.

In our setting, the nonparametric identification problem consists in establishing whether the distribution functions of $V|Z$ and $S|V, Z$ are uniquely determined from the Euler condition and knowledge of the distribution function of the observable variables. More precisely, the question that should be answered is: given the distribution function of share bids $G(\cdot|\cdot, \cdot; \cdot)$ defined in (7), and given the "nonparametric version" of the moment condition

[6] A function $w(\cdot, \cdot)$ is a scalar function of the "instrumental" variables n_l, z_l.

(11)[7] for all possible functions $w(\cdot,\cdot)$ and all possible p, are the nonparametric distribution functions $F_{V|Z}(\cdot|\cdot)$ and $F_{S|V,Z}(\cdot|\cdot,\cdot)$ uniquely determined? To answer this question, it is instructive to recall the identification issue in a closely related model, namely the common value first-price auction model for a single indivisible good. Laffont and Vuong (1996) and Athey and Haile (2002) have shown that the general common value model (where the signals are affiliated random variables) is not nonparametrically identified. There are no nonparametric identification results for the pure common value model (also known as the "mineral rights model"), but the intuition is that this less general model is also non-identified. Our pure common value discriminatory share auction model being closely linked to the mineral rights model, it is likely that the share auction is also not identified. Given a normalization of the signals similar to the normalization considered by Athey and Haile (2002, page 2125), it might be possible to identify the distribution of $S|V,Z$ but one can probably not recover the distribution of $V|Z$. It is of particular interest to understand the precise sources of nonidentification, and to study what are the possible identifying restrictions (as is done by Li, Perrigne and Vuong (2000) in the case of the common value first-price auction model). However, this topic is beyond the scope of this paper.

Instead we turn now to the parametric identifiability of our model. The question that has to be answered now is whether the true parameter θ^0 is uniquely determined given the distribution function of share bids, the theoretical moment conditions and the parametric specifications of the distribution functions. Of course the answer to this question is dependent on which specific parametric specifications are chosen. The next proposition establishes the parametric identification of the share auction model for the case where the distribution functions of $V|Z$ and $S|V,Z$ are given by (15) and (16) respectively. These specifications are used in the empirical part of the paper (Section 4.1 and Section 4.2), and can be seen as a generalization of the specifications chosen by Wilson (1979, example 1). We will refer to this parametric configuration as the gamma/exponential configuration.

Proposition 2. *Given i) the distribution function $G(\cdot|\cdot,\cdot;\cdot)$, ii) the moment condition (11) for all functions $w(\cdot,\cdot)$ and for all possible p, and iii) the parametric specifications of $F_{V|Z}(\cdot|z;\theta_1)$ and $F_{S|V,Z}(\cdot|v,z;\theta_2)$ defined in respectively (15) and (16) (the gamma/exponential configuration), the true*

[7]The same condition as condition (11) except that the parametric distribution functions are replaced by nonparametric ones.

value $\theta^0 = (\theta_1^{0\prime}, \theta_2^{0\prime})'$ *is uniquely determined.*

The proof of Proposition 2 is in appendix D. Parametric identification is obtained despite the fact that condition (6) is weaker than condition (5) (we have taken the expectation of (5) over S_i). As the proof shows, identifiability comes from variations in p and the number of bidders.

2.2.3 Definition of the estimator

Our estimator of the parameter of interest θ^0 is a two-step estimator. The first step consists in estimating the distribution function of bids $G(\cdot|\cdot,\cdot;\cdot)$. This distribution function is not parameterized in any way but is is instead left completely unspecified. The advantage of not imposing any a priori restrictions on the distribution function of bids is that potential misspecification problems are avoided.

For any given p, the distribution function $G(\cdot|n, z; p)$ can be estimated nonparametrically from the observed share bids x_{ilp}, $i = 1, ..., n_l$, $l = 1, ..., L$, and the variables n_l, z_l, $l = 1, ..., L$, using kernel estimation methods. The kernel estimate of $G(\cdot|\cdot,\cdot;p)$ is

$$\hat{G}(x|n, z; p) = \frac{\sum_{l=1}^{L} \frac{1}{n_l} \sum_{i=1}^{n_l} 1(x_{ilp} \leq x) K\left(\frac{n-n_l}{h_N}, \frac{z-z_l}{h_Z}\right)}{\sum_{l=1}^{L} K\left(\frac{n-n_l}{h_N}, \frac{z-z_l}{h_Z}\right)} \tag{12}$$

where $K(\cdot,\cdot)$ is a kernel and h_N and h_Z are bandwidth parameters (h_Z is actually a vector of bandwidth parameters with the same dimension as z).

The second step of the estimation method consists in minimizing, over θ, an appropriate criterion function involving the empirical counterparts of our theoretical moments. Since (11) must hold for any weighting function $w(\cdot,\cdot)$ and, given $w(\cdot,\cdot)$, for all $p \in \cap_{l/w(n_l,z_l)\neq 0}[p^{min}(n_l, z_l), p^{max}(n_l, z_l)]$, there is an infinity of moment conditions of the form (11), and for each of these theoretical moments an empirical counterpart is readily found (see below). The question now arises which of the theoretical moments should be exploited in the method of statistical inference. One option is to somehow use *all* possible moments, but, for a practical reason given in the next subsection, this is not the course that we shall follow. Instead, our estimation method exploits a fixed number of the theoretical moments. That is, we impose the restriction (11) only for a finite number of different prices and weighting functions.

Our estimation method uses $t = 1, ..., T$ moment conditions. The t-th moment imposes that condition (11) should hold for $w(\cdot,\cdot) = w_t(\cdot,\cdot)$ and $p = p_t$, compatible with our choice of $w(\cdot,\cdot)$. To find an empirical counterpart for this theoretical moment, consider again condition (11). Replacing θ^0 by an arbitrary value θ, and given the price p_t and the weighting function $w_t(\cdot,\cdot)$, the only unknown component in (11) is the distribution function $G(\cdot|\cdot,\cdot;p_t)$. Replacing this distribution function by its consistent estimate $\hat{G}(\cdot|\cdot,\cdot;p_t)$, a natural empirical counterpart of the righthand side of (11) is the sample average

$$
\begin{aligned}
& m_t(x_{11p_t}, ..., x_{n_L L p_t}, n_1, ..., n_L, p_1^0, ..., p_L^0, z_1, ..., z_L, p_t; \theta) \\
\equiv & \frac{1}{L} \sum_{l=1}^{L} w_t(n_l, z_l) \cdot (n_l - 1) \cdot 1(p_l^0 \le p_t) \cdot \Big[E\Big\{ V_l | S_{1l} = F_{S|Z}^{-1}(1 - \hat{G}(x_{1lp_t} | n_l, z_l; p_t) | z_l; \theta), \\
& \qquad ..., S_{n_l l} = F_{S|Z}^{-1}(1 - \hat{G}(x_{n_l l p_t} | n_l, z_l; p_t) | z_l; \theta), N_l = n_l, Z_l = z_l \Big\} - p_t \Big] \\
& \qquad - \frac{1}{L} \sum_{l=1}^{L} w_t(n_l, z_l) \cdot (p_t - p_l^0) \cdot 1(p_l^0 \le p_t) \quad (13)
\end{aligned}
$$

where the expectation is with respect to the distribution function of V_l given $S_{1l}, ..., S_{n_l l}, Z_l$, with the parameter being equal to θ.

The second step of our estimation procedure consists in minimizing over θ the sum of the T squared empirical moments. More precisely, the second-stage estimate of θ^0 is

$$
\hat{\theta} = Arg \min_{\theta} \sum_{t=1}^{T} m_t^2(x_{11p_t}, ..., x_{n_L L p_t}, n_1, ..., n_L, p_1^0, ..., p_L^0, z_1, ..., z_L, p_t; \theta). \quad (14)
$$

Before turning to the asymptotic properties of the two-step estimator, we want to make 2 remarks. The first remark concerns the choice of the moments. Unfortunately, we do not have results on how to choose the T prices and weighting functions optimally. In Section 4 we nevertheless recommend (on the basis of theoretical and practical arguments) to select them in a particular way. Our second remark concerns, once again, the identification of the model. In section 2.2.2 it was shown that the share auction is parametrically identified if (11) holds for any function $w(\cdot,\cdot)$ and all possible p. Since in our estimation method the restriction (11) is imposed only for a finite number of price values and weighting functions, it is necessary to assume that the identification is not lost by considering a subset of moments.

2.2.4 Asymptotic properties of the estimator

Newey and McFadden (1994, section 8) consider a class of estimators that are defined as the solution of a set of equations involving the observables, an unknown vector of parameters of interest, and a "first-step" estimate of a function. Since the first-step estimator is a nonparametric estimator of a function rather than an estimator of a finite-dimensional parameter, this class is referred to as the class of semiparametric two-step estimators. Newey and McFadden show that these estimators can converge at a rate equal to the root of the number of observations, even though the first-step estimator converges at a slower rate. They also give regularity conditions for asymptotic normality of the second-step estimator.

By rewriting the criterion function (14) we show in Février, Préget and Visser (2002) that our estimator of θ^0 belongs to the class of semiparametric two-step estimators (the nonparametric first-step estimate being $\hat{G}$). By verifying that the regularity conditions of Newey and McFadden hold, Février et al. show that our estimator is $\sqrt{L}$-consistent, and that it is asymptotically normally distributed. For the form of the asymptotic variance of the estimator we refer to appendix C, as it involves a lot of (too much) new notation.

We restrict ourselves to a finite number of moments T to fit into the framework of Newey and McFadden. The extension to a setting where an infinity of moment conditions is exploited is beyond the scope of this paper (see however Carrasco and Florens (1999) for an extension of the GMM to a continuum of moments).

3 Data

3.1 The institutional setting of the French government securities auctions

Since 1985 the French government securities are sold through auctions.[8] The Treasury securities are auctioned via discriminatory auctions.[9] The auctions are organized by the Bank of France, but all decisions regarding the dates the auctions are held, the characteristics of the securities, the quantities offered, etc., are taken by the Treasury. The three main types of French Treasury securities are:

- The *Bons du Trésor à taux Fixe et à intérêts précomptés* (the BTFs); these are tradable fixed-rate short-term discount Treasury bills with maturities of up to one year.
- The *Bons du Trésor à taux Fixe et à intérêts ANnuels* (the BTANs); these are tradable fixed-rate medium-term Treasury notes with interest paid annually and with maturities of two or five years.
- The *Obligations Assimilables du Trésor* (the OATs); these are fungible Treasury bonds with maturities ranging between 7 and 30 years.

Since our empirical analysis is based on the auctions for OATs and BTANs that were held in 1995, we describe the auction environment for these particular securities only, and present the auction rules as they prevailed at that time, even though they may have slightly changed by now.

The Treasury auctions for OATs and BTANs are held once a month—OATs on the first Thursday of the month and BTANs on the third Thursday. The scheduling of the auctions is as follows:

[8] This section draws on the French Government Securities annual report (we use the 1995 version; see also the web site of the French Treasury: http://www.oat.finances.gouv.fr), and on personal discussions with Sébastien Moynot and Benoît Cœuré of the French Treasury.

[9] Bartolini and Cottarelli (1997) find that 42 out of 77 countries in their sample (including all G-7 countries) used auctions in 1993 to sell government bills. More than 90% of the 42 countries that relied on auctions adopted the discriminatory auction format.

- Two business days prior to the auction, the Treasury announces the line that is to be auctioned, i.e., the Treasury describes the characteristics of the securities (the nominal yield, the maturity, etc.), and also determines the amount of securities it plans to sell.
- A bid consists of a price/quantity pair. The price is expressed as a percentage (of the nominal value of the security), and the quantity is the amount (in FFr) of the security the bidder wants at the corresponding price.[10] The minimal amount that bidders may submit is FFr1 million for the BTANs and FFr50 million for the OATs. Bidders are allowed to submit as many bids as they wish. Bids can be submitted until 10 minutes before the start of the auction. Most bids are submitted via TELSAT, a computerized bidding system, but bidders can also submit their proposals in sealed envelops directly to the Bank of France. The start of the auction is at 11 am.
- The Bank of France ranks bids in descending order of price, and immediately transmits the list of ranked bids to the Treasury without revealing the identity of the bidders.
- The Treasury calculates the stop-out price (the equilibrium price), i.e., the price for which aggregate demand equals the amount of securities offered at the auction. The highest prices are served first, and the allocation of securities stops when the amount (sold at auction) is reached. Any ties are settled on a pro rata basis.
- The auction results are announced via TELSAT (and also via some other information networks). The delay between the deadline for submission and the release of the results was less than 20 minutes in 1995.
- Delivery and settlement of the OATs take place on the 25th of the month the auction is held. In the case of 2-year (resp. 5-year) BTANs this occurs on the 5th (resp. 12th) of the month following the auction.

The majority of bidders are so-called *Spécialistes en Valeurs du Trésor* (SVTs).[11] The SVTs are primary dealers selected by the Treasury among

[10]For example, a bidder submitting (90%;FFr100 million) states that at the price of 90% he requests FFr100 million worth of securities. Re-formulated in a more familiar way, assuming the nominal value equals FFr2000, this bidder demands 50000 securities (FFr100 million divided by FFr2000) when the price per security is FFr1800 (90% of FFr2000).

[11]On 1 April 1995 there were 19 SVTs.

the most active players on the government securities market. They belong to large French or international banks. Generally operating within these banks as independent entities, the SVTs are traders who buy and sell the securities and try to maximize profits. The SVTs sell the securities not only on the secondary market, but also directly to their clients (pension funds and insurance companies) and their own bank. On average the SVTs account for 90% of the securities bought at auctions, the remaining 10% being purchased by other banks or financial institutions.

France belongs to a small group of countries[12] in which bidders have the possibility to submit *Offres Non Compétitives* (ONCs). These non-competitive bids (the competitive bids being the auction bids described until now) can actually only be submitted by the SVTs. There are two kinds of non-competitive bids: the ONC1s, which must be submitted at the same time as the competitive bids, and the ONC2s, which may be submitted once the auction is over and until one day after the auction. The SVTs are not obliged to engage in non-competitive bidding. Each SVT can not submit more than one ONC1 bid and one ONC2 bid. Unlike a competitive bid, a non-competitive bid consists only in an amount (in FFr) of the security the bidder wants. The amount submitted by a bidder is sealed and may not exceed a certain bidder-specific limit, but except for this restriction each bidder is guaranteed the quantity he bids for. The limit may differ for the two types of ONCs, and its height is determined by the participation of the bidder in the three last auctions. The price at which the non-competitive bids must be paid is identical for all bidders and corresponds to the quantity-weighted average price of the awarded competitive bids. Since ONC1s are submitted *before* the main auction and ONC2s *after* the auction results are revealed, the former are submitted under *price-uncertainty* and the latter under *price-certainty.*

French Treasury securities can not only be bought at auction but also on two other markets: the so-called when-issued market and the secondary market (see Bikhchandani and Huang (1993) for a detailed description of the US-version of these two market forms). The when-issued market for a security is a forward market that starts the day when the Treasury announces an auction for that security, and ends on the settlement date. Once the announcement has been made by the Treasury, dealers can trade forward

[12]According to Bartolini and Cottarelli (1997), only 38% of 40 countries that relied on auctions accepted non-competitive bids.

contracts on the Treasury securities that are to be auctioned. Contract sellers and contract buyers commit themselves to respectively deliver and take delivery of certain specified amounts of securities at the forward prices. The forward contracts are delivered on the issue date (the settlement date) of the security, which explains why the market is called a when-issued market. The BTANs and OATs are also traded on the secondary market. The secondary market is a permanently active market where the smaller financial institutions and individual investors can trade in securities and where the competitive bidders can resale their securities obtained at auction.

3.2 The link between theory and practice

Although Wilson (1979) did not mention Treasury auctions in his article (may be because at that time the practice of selling securities at auction was much less widespread than nowadays; see Bartolini and Cottarelli (1997)), his model and its underlying assumptions are often regarded as well adapted to the context of Treasury auctions.[13] It is clear however from our description of the French institutional setting that there are some deviations between theory on the one hand and real-life auctions on the other. The purpose of this subsection is to comment on these deviations. We will also motivate some of the assumptions underlying the share auction model.

Perhaps the most crucial assumption underlying the share auction model is that bidding behavior can be adequately modelled within the common value paradigm. An argument in favor of the common value assumption is that in France the bidders' objective is to resell the securities purchased at auction. Indeed, according to officials at the French Treasury, the SVTs eventually resell all the securities they have won. Since the SVTs resell their securities on essentially the same competitive market, there is a common value component in the bidders' valuation. Most of the securities are resold well after the day of the auction.[14] At the time of bidding, the future common

[13]For instance, in their discussion about the theoretical literature on divisible-good auctions, Back and Zender (1993) refer to Wilson's share auction model as the most relevant model for Treasury securities. Bikhchandani and Huang (1993) state that in Treasury securities auctions the common value assumption is appropriate because the value for each bidder is a common and unknown resale price. Each investor is likely to have some private information about the resale price of the security, on which to base a bid function. Finally, Das and Sundaram (1996) argue that bidders in Treasury securities auctions can be considered as symmetric and risk-neutral.

[14]Although some SVTs resell part of their securities immediately, i.e., during the after-

resale value of the securities is therefore unknown to the SVTs. An argument in favor of the assumption that the SVTs receive different signals about the future value is that they typically have different forecasts on interest movements and different anticipations of the market demand for securities.

So there are arguments for using a common value environment. However, there may also be private value aspects to the Treasury auctions we are analyzing. As mentioned in section (3.1), the SVTs resell part of the securities directly to their clients. It is likely that these clients place orders (stipulating the amount of securities they wish to buy from the SVTs at given transaction prices) *before* the opening of the auction. This means that the SVTs know, in advance, that certain quantities of T-bills can be disposed of at the predetermined transaction prices. This in turn means that the SVTs may not only participate in the Treasury auctions for speculative purposes (as in the common value paradigm), but also to fill customer orders. It may thus be possible that the valuation of each SVT is determined not only by a common value component but also by a private value component, the latter corresponding to the transaction price proposed by the client of the SVT (which is unknown to other SVTs).[15]

Two other model assumptions that need to be motivated are the risk-neutrality assumption and the symmetry assumption, i.e., the assumption that the conditional distribution from which the signals are drawn is the same for all agents. The risk-neutrality hypothesis seems natural given that the bidders in our sample are large financial institutions. These institutions are wealthy agents and are therefore unlikely to be averse to risk.

The symmetry assumption may seem less natural as the SVTs that participated in the auctions of 1995 differed considerably in size and financial importance. In 1995 there were for example financial heavyweights such as *Crédit Lyonnais* and *Société Générale*, but also smaller institutions such as

noon of the auction day (in which case it can be argued that the resale value at the time of bidding is known from the secondary market), many T-bills are resold days or even weeks thereafter. The fact that resale is not immediate is not surprising since there is an important time lag between the moment the auction is organized and the moment the securities are actually delivered and paid (see section (3.1)).

[15] Hortaçsu (2002) argues that there are private value aspects in his data because bidders in Turkey (mostly banks) primarily buy securities to fulfill their liquid asset reserve requirements. The securities won at auction are therefore for the bidder's personal use only, and the private values are the privately known liquidity needs of the banks. Hortaçsu's argument does not apply in our context as bidders in France do not enter the auctions to meet liquid asset reserve requirements.

Banque d'Escompte and *Jouis Dreyfuss Finance*.[16] This seems like evidence against our hypothesis of symmetry across bidders. However, the auxiliary data that we have on the transaction behavior of the SVTs (kindly given to us by the French Treasury) are rather in support of the symmetry assumption. These data are summarized in Table 1. The table reports for each SVT the total volume of T-bill transactions in 1995 (quantity of T-bills bought by the SVT in 1995 plus the quantity sold in that year).[17] The transaction data are given separately for the BTANs and OATs.

Table 1. Annual volume (FFr millions) of T-bill transactions

SVT	BTAN	OAT
1	526 541 (11%)	564 430 (10%)
2	343 722 (7%)	515 588 (9%)
3	359 467 (8%)	465 576 (8%)
4	363 048 (8%)	432 786 (8%)
5	305 332 (6%)	433 722 (8%)
6	352 273 (7%)	351 596 (6%)
7	309 322 (6%)	318 588 (5%)
8	218 745 (5%)	260 500 (4%)
9	270 869 (6%)	379 563 (7%)
10	241 348 (5%)	318 876 (6%)
11	258 056 (5%)	324 900 (6%)
12	238 761 (5%)	247 326 (4%)
13	218 296 (5%)	261 707 (5%)
14	225 613 (5%)	195 862 (3%)
15	200 944 (4%)	208 932 (4%)
16	190 780 (4%)	221 927 (4%)
17	158 296 (3%)	194 929 (3%)

Although the traded quantities of T-bills are certainly not identical, the disparity between the SVTs is not that important. Most SVTs have transaction volumes varying between 4% and 8% of the total trade activity. The SVTs

[16] The above mentioned web site of the French Treasury can be consulted for the list of SVTs that are currently active in the French Treasury auctions. There is much overlap between this list (2003 version) and the one of 1995.

[17] The original dataset contains information on traded volumes for each quarter of 1995 and a total of 20 different SVTs. We aggregated the quarterly data into annual data, and only kept the SVTs whose transaction data are available in all quarters (17 out of 20 SVTs).

are thus quite similar in terms of their T-bill transaction activities. Admittedly, this is by no means a definite or overwhelming proof of the symmetry assumption. But unfortunately we do not have other relevant information on the SVTs that would allow us to perform additional tests of the symmetry hypothesis. Also, given the strictly anonymous nature of our bidding data, it would anyhow be hard if not impossible to estimate a generalized asymmetric auction model.[18] For these reasons we shall maintain the symmetry assumption, acknowledging that it is potentially restrictive.

Next we comment on the deviations between theory and practice. The first deviation is that, unlike the share auction model, part of the bidders in France—the SVTs—have the possibility to submit ONC1s and ONC2s. The fact that bidders have this opportunity implies that their maximization problem differs from the maximization problem (4). If non-competitive bidding is allowed then agents maximize expected earnings derived from competitive *and* non-competitive bidding, and consequently the actual optimal bidding strategies may differ from those derived in Section 2.1. We have not attempted to extend Wilson's model to account for non-competitive bidding because i) it is very difficult to actually model the phenomenon, and ii) our data would not allow us to estimate an extended model since we do not observe the demand for ONCs at the individual bidder level (only the aggregate demand for both types of ONCs are recorded in the data).

The second deviation is that unlike most Treasury auctions in the world (including the BTF auctions in France), the French Treasury does not announce the precise amount of OATs or BTANs it plans to sell. Instead it announces an issue interval, wherein the total quantity of securities eventually sold necessarily lies. Ex-ante this induces some uncertainty about the total amount of securities sold at auction. Although in practice the intervals announced by the Treasury are quite tight, we will address this uncertainty issue in Section 4.3.

The third deviation is that Wilson's model describes the main auction as

[18]It should be stressed however that our estimation method can easily be adapted to the case of asymmetric bidders. This is because, as mentioned just after Proposition 1, the Euler condition (6) is still valid in the asymmetric setup, and (11) remains in this case an appropriate basis for statistical inference. The distribution function $G(\cdot|\cdot,\cdot;\cdot)$ would have to be estimated separately for each subgroup of identical bidders (or, in the extreme asymmetric case where all bidders are different, separately for each agent). Thus, when the data have a richer panel data type structure, the less restrictive asymmetric share auction model can be estimated.

an isolated market, unaffected by what happens on the when-issued market and the secondary market. The interdependencies that exist between the three market forms might affect bidder's behavior at the auction. This has been shown by Bikhchandani and Huang (1989) in a model where the Treasury bills obtained at a discriminatory common value auction can be resold on a secondary market. They show that the information linkage between the primary auction and the secondary market affects bidding behavior at the auction. Similarly, Haile (2001) shows that resale opportunities influence bidding behavior at an English single-unit auction with private values. Although it would be interesting to extend Wilson's share auction model in these directions, we feel that this is beyond the scope of this paper, and we thus implicitly assume that the type of signalling (caused by the presence of the resale market) studied by Bikhchandani et al. (1989) and Haile (2001) is absent in our data.

The fourth and final deviation we want to discuss it that Wilson's model assumes that bidders submit continuous downward sloping bid functions, whereas in our data the SVTs only submit a certain number of price/quantity bids (the number of bids varies among bidders, but on average it equals 2.86; see Table 4). We assume that the optimal continuous bid function does exist but that bidders only submit some points that *belong* to this function. Furthermore, in calculating the hypothetical revenue under the uniform auction (Section 4), we shall explicitly take into account that agents submit a limited number of price/quantity combinations.

3.3 Descriptive analysis

Our empirical analysis is based on all French-franc denominated OAT and BTAN auctions that were held in 1995.[19] Table A1 in the appendix gives the auction dates, the lines auctioned (this column in the table gives for each auction the nominal yield of the security and the year of maturity), the settlement dates, and the exact maturity dates. As mentioned in subsection (3.1), OATs and BTANs were auctioned once per month—OATs on the first Thursday of the month and BTANs on the third Thursday of the month.

[19]Three other types of French government securities were issued through auctions by the Treasury in 1995: BTFs, ECU-denominated OATs/BTANs, and OATs for private investors. We have excluded the data from these auctions for homogeneity reasons. Indeed, the auction rules under which they were conducted differed from the rules under which the auctions in our sample were held.

There are therefore 24 different auction dates. As table A1 indicates, the Treasury did not necessarily sell just one line on a given auction date, but often sold two or even three lines on the same day. If several lines were offered on a given day, they were sold simultaneously but via strictly separate auctions.[20] Six different lines of OATs and five different lines of BTANs were issued in 1995.

Apart from the information already given in table A1, we observe for each auction all competitive bids submitted by all bidders (the non-competitive bids are not observed in our data), the stop-out price, the amount of securities sold at auction, and the total amounts of ONC1s and ONC2s awarded by the Treasury. Our data are completely anonymous, i.e., the auction participants are unidentified in our data. We cannot therefore tell from the data which bidder in a given auction corresponds to which bidder in some other auction.

Table 2 gives some overall information about the auctions. In 1995 a total of 45 auctions took place: 25 OAT auctions and 20 BTAN auctions. The total quantity issued by the Treasury in the 45 auctions is FFr464.579 billion, so the mean amount of securities sold is FFr10.324 billion per auction. The total quantity can be split up into the total amount of awarded competitive bids (FFr 423.72 billion), total amount of awarded ONC1s (FFr4.831 billion), and total of awarded ONC2s (FFr36.028 billion). A total of 937 "different" bidders have participated in the 45 auctions, and the total number of competitive bids submitted by these bidders is 2677. About 38% of these 2677 bids were served by the Treasury, 16% were only partially served (because they were at the stop-out price), and almost half of the bids (46%) were loosing bids.

[20] For example, the investors interested in both lines issued on May 1995—"7.75% April 2005" OATs and "8.5% October 2008" OATs—had to submit their bids separately for the two auctions, according to the procedures described in subsection (3.1), and at 11 am the two lines were auctioned independently from each other.

Table 2. Overall information about the auctions

Number of auctions	45
OAT	25 (56%)
BTAN	20 (44%)
Number of bidders	937
Number of bids	2677
Totally served	1 016 (38%)
Partially served	423 (16%)
Not served	1 238 (46%)
Total amount issued by the Treasury (FFr millions)	464 579
competitive bids (FFr millions)	423 720 (91%)
ONC1 (FFr millions)	4 831 (1%)
ONC2 (FFr millions)	36 028 (8%)

Table 3 presents summary statistics per auction. The average number of bidders per auction is 20.82. The number of bidders is quite stable across auctions and is close to the total number of SVTs in 1995. The number of bids per auction ranges between 28 and 102 bids, and the mean is about 60 bids. The mean of the auction coverage—which is the ratio of the sum of all submitted competitive bids and ONC1s to the total amount served (the awarded competitive bids and ONC1s)—is equal to 2.25.

Table 3. Summary statistics per auction

Variable	Mean	Std. dev.	Min	Max	Obs
Number of bidders	20.82	1.71	15	23	45
Number of bids	59.49	17.41	28	102	45
Amount issued by Treasury (FFr millions)	10 324	5 922	2 052	21 849	45
Winning competitive bids (FFr millions)	9 416	5 335	1 800	19 125	45
ONC1 (FFr millions)	107	121	0	496	45
ONC2 (FFr millions)	801	820	0	2 553	45
Auction coverage	2.25	0.75	1.29	5.18	45
Maturity of security (in days)	3 749	3 227	586	11 231	45
Nominal yield (%)	7.31	0.80	5.75	8.50	45
Secondary market price	98.07	9.29	71.33	108.50	45
Stop-out price	97.94	9.40	70.88	108.16	45
Highest price bid - lowest price bid	0.32	0.13	0.10	0.68	45
Auction scatter (average price - stop-out price)	0.03	0.02	0.00	0.16	45

The mean of the maturity of the security—defined as the number of days between the settlement date and the date of maturity—is 3 749 days. The average nominal yield in the sample is 7.31%. The average of the secondary market price—defined as the opening secondary market price of the security on the day of the auction—equals 98.07. The stop-out price varies between 70.88 and 108.16, and has a mean equal to 97.94. The dispersion of the submitted prices—defined as the highest price minus the lowest price— is on average 0.32. Finally, the mean of the auction scatter—still another measure of the dispersion of auction prices, and defined as the difference between the weighted average price of the winning bids and the stop-out price—equals 0.03.

Table 4 gives summary statistics per bidder or per bid. The number of submissions per bidder ranges from 1 to 9, and the mean equals 2.86. This is slightly lower than the mean of 3.2 bids found by Gordy (1999) (based on data from Portuguese Treasury auctions), but substantially lower than the mean of 6.9 bids found by Hortaçsu (2002) (Turkish Treasury auctions).

Table 4. Summary statistics per bidder or per bid

Variable	Mean	Std. dev.	Min	Max	Obs
Number of bids	2.86	1.58	1	9	937
Demanded quantity per bid (FFr millions)	326	328	10	2500	2677
Price bid	98.54	7.93	70.54	108.26	2677
Highest price bid - lowest price bid	0.07	0.07	0	0.54	937

In our sample the submitted quantities range from FFr10 million to FFr2.5 billion, and prices from 70.54 to 108.26. The price dispersion per bidder ranges between 0 (if the bidder has submitted only one bid) and 0.54.

4 Results

In Section 4.1 we present the two-step estimates for the gamma/exponential configuration. Section 4.2 reports the results of the revenue-comparison between the discriminatory and uniform auction formats. Finally, Section 4.3 is devoted to the robustness of our results. We will estimate another set of parametric specifications (the normal configuration), and study the sensitivity of the results that are obtained under the gamma/exponential configuration.

4.1 Estimation of the parameters of the discriminatory model

The secondary market price, the nominal yield and the maturity of the security (divided by 1000) sold at the l-th auction are the variables included in the vector z_l. The dimension of z_l is thus equal to 3, and we denote $z_l = (z_{1l}, z_{2l}, z_{3l})$. It is assumed that the total quantity of securities sold at auction is known ex-ante to the bidders, i.e., there is no supply uncertainty. In order to render the data coherent with the share auction framework, all observed bids are divided by the total quantity sold at the l-th auction, and the total quantity is normalized to 1. In the first step of our estimation procedure we nonparametrically estimate the distribution function $G(x|n, z; p)$ using the Epanechnikov kernel. To avoid any confusion with the vector z_l, we denote the 3-dimensional vector of explanatory variables at which $G(\cdot|\cdot, \cdot; \cdot)$ is evaluated, as $z = (z^1, z^2, z^3)$. In expression (12) we thus have $K\left(\frac{n-n_l}{h_N}, \frac{z-z_l}{h_Z}\right) = K\left(\frac{n-n_l}{h_N}\right) K\left(\frac{z^1-z_{1l}}{h_{1Z}}\right) K\left(\frac{z^2-z_{2l}}{h_{2Z}}\right) K\left(\frac{z^3-z_{3l}}{h_{3Z}}\right)$ where $K(u) = 0.75(1-u^2)1\left(|u| \leq 1\right)$, and h_N, h_{1Z}, h_{2Z} and h_{3Z} are the bandwidth parameters. The choice of all bandwidth parameters were chosen according to the rule of thumb defining each bandwidth as 2.214 multiplied by the standard error of the variable multiplied by the number of observations (L) to the power $-\frac{1}{7}$.[21] We find $h_N = 2.2$, $h_{1Z} = 11.9$ (bandwidth of the secondary market price), $h_{2Z} = 1.0$ (nominal yield), and $h_{3Z} = 4.1$ (maturity of the security divided by 1000).

To proceed with the second step, we have to choose parametric specifications for the distribution functions of the signal and the value. For the revenue comparison in the next subsection, the specifications should be chosen such that explicit optimal strategies can be obtained in the uniform share auction model. Bearing this in mind, we assume that the value V_l given $Z_l = z_l$ has the distribution function

$$F_{V|Z}(v|z_l; \theta_1) = \int_0^v \gamma u^{\gamma-1} \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} u^{\gamma(\alpha_l-1)} \exp\left[-\beta_l u^\gamma\right] du \qquad (15)$$

[21]Newey and McFadden (1994, pp. 2203-2210) impose conditions on the choice of the kernel and the convergence rate of the bandwidth parameters. To satisfy these conditions, we chose the Epanechnikov kernel ($m = 2$ in the notation of Newey and McFadden's lemma 8.10, p. 2206), and the convergence rate $L^{-\frac{1}{7}}$. Furthermore in this formula, the factor 2.214 is the constant associated with the Epanechnikov kernel.

where

$$\alpha_l = (1, z_l) \cdot \alpha$$
$$\beta_l = (1, z_l) \cdot \beta$$

and $\Gamma(\cdot)$ is the gamma function, α and β are vectors (of dimension 4 by 1) of parameters, and γ is a scalar parameter. Note that if $\gamma = 1$ then the above distribution function corresponds to the gamma distribution function with parameters α_l and β_l, i.e., in this case V_l follows a gamma distribution with conditional mean α_l/β_l and conditional variance α_l/β_l^2; if $\gamma \neq 1$ then V_l^γ is distributed as a gamma distribution with parameters α_l and β_l; if $\alpha_l = 1$ then V_l follows a weibull distribution with parameters γ and β_l. Note also that $\theta_1' = (\alpha', \beta', \gamma)$.

We furthermore assume that the signal S_{il} given $V_l = v_l$ and $Z_l = z_l$ follows an exponential distribution:

$$F_{S|V,Z}(s|v_l, z_l; \theta_2) = 1 - \exp\left[-s v_l^\gamma\right] \tag{16}$$

where γ is the scalar parameter that also appears in the conditional distribution function of V_l. Note that the conditional expectation and variance of S_{il} are assumed to be independent of z_l. Note also that $\theta_2 = \gamma$, so the complete vector of parameters is therefore $\theta' = (\alpha', \beta', \gamma)$. As mentioned already, we refer to the set of specifications (15) and (16) as the gamma/exponential configuration.

In the second step of the estimation procedure we estimate θ^0, the true value of θ. The estimate of this parameter is defined by (14), where, given the specifications (15) and (16), the conditional expectation of V_l appearing in the empirical moment $m(.)$ is:

$$\begin{aligned} E\Big\{ V_l | S_{1l} &= F_{S|Z}^{-1}(1 - \hat{G}(x_{1lp}|n_l, z_l; p)|z_l; \theta), \\ &\quad ..., S_{n_l l} = F_{S|Z}^{-1}(1 - \hat{G}(x_{n_l lp}|n_l, z_l; p)|z_l; \theta), N_l = n_l, Z_l = z_l \Big\} \\ &= \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{\left(\beta_l + \sum_{i=1}^{n_l} \beta_l \left[\frac{1}{\hat{G}^{1/\alpha_l}(x_{ilp}|n_l, z_l; p)} - 1\right]\right)^{1/\gamma}}. \end{aligned}$$

The choice of the weighting functions is made in a pragmatic way. Recall that for a given function $w(\cdot, \cdot)$, the first order condition (11) is verified only for prices in the interval $\cap_{l/w(n_l, z_l) \neq 0}[p^{min}(n_l, z_l), p^{max}(n_l, z_l)]$. This constrains the choice of the weighting functions as one can not select a function that assigns positive weights to auctions whose equilibrium prices are very different.[22] The natural choice that consists in choosing the weighting functions as polynomials in the variables n_l, z_l leads for example to unsatisfactory results (the parameter values took unexpected signs and the estimated expectations of V_l were unreasonably high compared to the stop-out prices and secondary market prices).

This difficulty with polynomials led us to choose $T = 45$ (i.e. the number of moments equals the number of auctions in the sample) and kernel functions for the weighting functions: $w_t(n_l, z_l) = K\left(\frac{n_t - n_l}{h_N}\right) K\left(\frac{z_{1t} - z_{1l}}{h_{1Z}}\right) K\left(\frac{z_{2t} - z_{2l}}{h_{2Z}}\right) K\left(\frac{z_{3t} - z_{3l}}{h_{3Z}}\right)$ where the kernel $K(\cdot)$ and the bandwidths are the same as above. The values of n_t and $z_t = (z_{1t}, z_{2t}, z_{3t})$, $t = 1, ..., T$, are set equal to the observed values taken by the number of bidders and good-characteristics observed in the L auctions. In coherence with the constraint induced by the weighting functions, the prices $p_1, ..., p_T$ are set equal to the observed stop-out prices.[23]

There are two other reasons to choose the prices $p_1, ..., p_T$ close to the equilibrium prices observed in the sample. The first reason is a practical one. From a practical point of view it seems unwise to select values that are too "small". Indeed, in practice the demand functions are probably filled in with relatively less precision for low values of p, as bidders may anticipate that demands expressed far below the expected equilibrium price are not executed and thus without real consequence. On the other hand, it may also be unwise to select price values that are too "large". Indeed, as the equilibrium demand functions may be relatively flat for large values of p, the variation in the estimates $G(\cdot|\cdot, \cdot; \cdot)$ may be too small, which might in turn lead to numerical problems in the second step of the estimation procedure. The more theoretical reason for selecting prices close to the equilibrium values

[22] Suppose, for example, that 2 auctions l' and l'' have equilibrium prices that are very different. It is then likely that the supports $[p^{min}(n_{l'}, z_{l'}), p^{max}(n_{l'}, z_{l'})]$ and $[p^{min}(n_{l''}, z_{l''}), p^{max}(n_{l''}, z_{l''})]$ do not overlap. If the weighting function assigns positive weights to both auctions, i.e., $w(n_{l'}, z_{l'}) > 0$ and $w(n_{l''}, z_{l''}) > 0$, then $\cap_{l/w(n_l, z_l) \neq 0}[p^{min}(n_l, z_l), p^{max}(n_l, z_l)] = \emptyset$, that is there is no price p for which the condition (11) holds.

[23] The price p_t belongs, by definition, to $[p^{min}(n_t, z_t), p^{max}(n_t, z_t)]$. Furthermore, $w_t(n_l, z_l)$ is positive if and only if n_l and z_l are close to n_t and z_t, which in turn means that the support of the equilibrium price $[p^{min}(n_l, z_l), p^{max}(n_l, z_l)]$ should be close to the support $[p^{min}(n_t, z_t), p^{max}(n_t, z_t)]$. It thus seems reasonable to assume that p_t belongs to $\cap_{l/w_t(n_l, z_l) \neq 0}[p^{min}(n_l, z_l), p^{max}(n_l, z_l)]$ as requested when using the first order condition.

is that the identifiability comes from imposing the Euler condition for p in an interval around the maximal equilibrium price (see the proof of Proposition 2 in appendix D).

The second-step estimates are presented in Table 5, together with the estimated standard errors (using the estimated asymptotic variance matrix of $\hat{\theta}$ given in appendix C).

Table 5. Estimation of θ: the gamma/exponential configuration

Estimate of α :	
Constant	-15848.93 (320.72)**
Secondary market price	66.67 (3.48)**
Nominal yield	1617.29 (8.76)**
Maturity of security (in days/1000)	142.24 (6.96)**
Estimate of β :	
Constant	8596.67 (114.87)**
Secondary market price	-104.30 (1.24)**
Nominal yield	340.41 (6.77)**
Maturity of security (in days/1000)	-2.04 (1.57)
γ	12.28 (0.0085)**

Note: Actual estimates and standard errors (in parentheses) in β are very small, and those reported in the table are multiplied by 10^{24}. **=significant at the 5% level.

All parameters are significantly different from zero at the 5% level, except the parameter corresponding to the maturity of the security in β. Given the estimate $\hat{\theta}$ and using (15), we can calculate $E(V_l|Z_l = z_l)$, and the derivative of this expectation with respect to each variable in z_l. Evaluated at the empirical mean of the explanatory variables (i.e. we replace z_l by the averages reported in Table 3), we find that the conditional mean of the value equals 99.79, which is above the average secondary market price (98.07) and the average stop-out-price (97.94). The derivative with respect to the nominal yield, the secondary market price and the maturity are 1.06, 1.18 and 0.40 respectively. All these variables thus have a positive effect on the expected value of the security. Table A2 in the appendix gives for all auctions $l = 1, ..., 45$, the estimated expectations $E\{V_l|S_{1l} = \hat{s}_{1l}, ..., S_{n_l l} = \hat{s}_{n_l l}, Z_l = z_l\}$ and $E\{V_l|Z_l = z_l\}$ (where $\hat{s}_{il} = F_{S|Z}^{-1}(1 - \hat{G}(x_{ilp_l^0}|n_l, z_l; p_l^0)|z_l; \hat{\theta})$), the stop-out-price p_l^0, and the secondary market price. As one may expect when looking at the first-order condition (11), the stop-out-price is practically always less than the expected value.[24] This finding is in accordance with the standard result in a first-price auction, where players bid less than the expected value conditional on their signal and the highest signal of the competitors. Table A2 also shows that the expected value is generally greater than the secondary market price. This result may explain why many bidders do not resale their securities immediately.

4.2 Revenue comparison

In this subsection we compare the actual income of the French Treasury with the hypothetical income the Treasury would have earned had it adopted the uniform share auction mechanism. The actual revenue from a given discriminatory auction simply equals the sum, over all winning bids in the auction, of awarded quantities multiplied by the associated prices. The total actual income earned by the Treasury is then the sum, over the 45 auctions held in 1995, of these actual revenues. In 1995 the total actual income thus calculated amounts to FFr421.543 billion.

The calculation of the hypothetical total income under the uniform auction format is less straightforward. First we need to determine an explicit optimal bidding strategy in the uniform auction format. Given our parametric specifications of the distribution functions (15) and (16), Février et al. (2002) show that an optimal strategy in the uniform auction is (i.e. a solution of (3))[25]

$$x(p, s_{il}, n_l, z_l; \theta) = \left[1 - \left\{\frac{\beta_l}{n_l} + s_{il}\right\}\left\{\frac{\Gamma(n_l + \alpha_l)}{\Gamma(n_l + \alpha_l + 1/\gamma)}\frac{1+\gamma}{\gamma}p\right\}^{\gamma}\right] / \left(n_l - 1\right). \tag{17}$$

Février et al. (2002) furthermore show that (17) is actually the unique equilibrium strategy in the class of demand functions that are linear in the signals. Admittedly, this does not exclude that there may be other types of equilibria in this game. The revenue comparisons that we report below are therefore conditional on the hypothesis that players will select the unique equilibrium with linear strategies.

[24] Intuitively, in this equation, we have $E\{V_l | S_{1l} = \hat{s}_{1l}, ..., S_{n_l l} = \hat{s}_{n_l l}, Z_l = z_l\} > p > p_l^0$.

[25] Of course, as with the Euler condition of the discriminatory model, the auction-specific notation and variables should be incorporated in the uniform Euler condition (3).

Note that, as mentioned in Section 2.2.1, the optimal bidding function (17) is decreasing in the signal s_{il} (it is also decreasing in p).

The strategy (17) is a generalization of the equilibrium strategy derived in Wilson (1979, example 1). Wilson assumes that V_l follows a gamma distribution with parameters α_l and β_l, and that S_{il} is exponentially distributed with parameter v_l. His setup thus corresponds to the special case where the parameter γ appearing in the distribution functions (15) and (16) is equal to 1. When $\gamma = 1$, (17) reduces to

$$x(p, s_{il}, n_l, z_l; \theta) = \left[1 - 2p\frac{\beta_l + n_l s_{il}}{n_l(n_l + \alpha_l)}\right] / \left(n_l - 1\right) \tag{18}$$

which is the optimal bidding function given by Wilson.[26]

Now that an optimal bidding strategy has been determined, a possible way to calculate the hypothetical income under the uniform auction format is as follows. Under the assumption that agents bid according to the optimal demand function (17), it can be shown that the equilibrium price in the l-th uniform auction satisfies

$$\begin{aligned} p_l^0 &= \frac{1}{1+1/\gamma^0} E\{V_l | S_{1l} = s_{1l}, ..., S_{n_l l} = s_{n_l l}, Z_l = z_l\} \\ &= \frac{1}{1+1/\gamma^0} \frac{\Gamma(n_l + \alpha_l^0 + 1/\gamma^0)}{\Gamma(n_l + \alpha_l^0)} \frac{1}{(\beta_l^0 + \sum_{i=1}^{n_l} s_{il})^{1/\gamma^0}} \end{aligned} \tag{19}$$

where $\alpha_l^0 = (1, z_l) \cdot \alpha^0$, $\beta_l^0 = (1, z_l) \cdot \beta^0$, and $\alpha^0, \beta^0, \gamma^0$ are the true parameter values. The equation for the equilibrium price given by Wilson (1979, page 682) corresponds to (19) with $\gamma^0 = 1$.[27]

Replacing, in (19), θ^0 by $\hat{\theta}$, and, for all i, s_{il} by the previously defined estimated signal $\hat{s}_{il}$, gives the estimated equilibrium price in the l-th uniform auction. The hypothetical revenue from auction l is then the product of total supply (i.e. the amount of securities sold at the l-th discriminatory auction) and the estimated stop-out price, and the total hypothetical income under the uniform auction follows from summation over all 45 auctions in the sample.

[26] In Wilson (1979) the numerator is $\beta_l - n_l s_{il}$ instead of $\beta_l + n_l s_{il}$. But this simply reflects a difference in the choice of the conditional distribution function of the signal. In Wilson the conditional distribution function is $F_{S|V,Z}(s_{il}|v_l, z_l; \theta_2) = e^{v_l s_{il}}$ for $s_{il} \leq 0$.

[27] Except that the denominator in the last term equals $\beta_l^0 - \sum_{i=1}^{n_l} s_{il}$ instead of $\beta_l^0 + \sum_{i=1}^{n_l} s_{il}$.

The previous method to calculate the hypothetical revenue relies on the assumption that bidders, in uniform auctions, submit continuous bid functions. We have seen in our data description, however, that on average bidders have submitted around 3 bids and that most of these bids are close to the equilibrium price. It seems therefore more appropriate in our context to calculate the hypothetical uniform revenue by explicitly taking into account the discrete nature of bidding. The alternative method is as follows. First we model the price/share pairs submitted by the bidders.[28] We assume that each bidder i with signal $\widehat{s}_{il}$ in auction l submits a vector of price/share pairs $(p_1, x(p_1, \widehat{s}_{il}, n_l, z_l; \widehat{\theta}), ..., (p_{D_i}, x(p_{D_i}, \widehat{s}_{il}, n_l, z_l; \widehat{\theta}))$ where $x(p_{d_i}, \widehat{s}_{il}, n_l, z_l; \widehat{\theta})$ is given by equation (17). D_i, the dimension of the vector, is a random variable. We define $D_i = 1 + Round(\widetilde{D_i})$ and assume that $\widetilde{D}$ follows a gamma distribution $\Gamma(a_D, b_D)$. Using the data from the discriminatory auctions (on the number of bids submitted by the agents), we find that $\widetilde{D} \sim \Gamma(1.39, 0.75)$.[29] We furthermore assume that the variables $p_{d_i} - p_l^0$ are i.i.d. random variables that follow a normal distribution $N(\mu_p, \sigma_p^2)$. Here p_l^0 is defined by (19). Using data from the discriminatory auctions (on the differences between the submitted prices and stop-out prices), we find $(p_d - p_l^0) \sim N(-1.53 \times 10^{-2}, 4.3 \times 10^{-4})$.

In order to calculate the hypothetical uniform revenue in auction l, we perform 100 simulations. In each simulation, we draw for each bidder i, the dimension D_i and the price/share pairs. We construct the bid functions, find the equilibrium price and derive the corresponding revenue. The estimated hypothetical revenue under the l-th uniform auction is then calculated as the average over the 100 simulations. The total hypothetical revenue under the uniform auction format follows then, as before, from summation over all 45 auctions.

We find that both the "continuous" and the "discrete" method lead to similar hypothetical revenues for the Treasury. Using the "continuous" method, we find that the total hypothetical uniform revenue equals FFr400.421 billion.[30] Therefore, had the French Treasury adopted the uniform auction format instead of the discriminatory auction format, it would have earned FFr21.122 billion less (5% of total income in the discriminatory auctions).

[28] We have tried several models but they all give similar results.

[29] The mean of D is therefore 2.86 and its standard deviation 1.58, as presented in Table 4.

[30] The estimation obtained with the "discrete" method is FFr400.061 billion.

In calculating the variance of the above estimate, we use the fact that the hypothetical revenue is simply the sum, over all auctions l, of p_l^0 times the amount of securities sold in the l-th auction. Hence, the estimated hypothetical revenue is some function of $\hat{\theta}$ and (via the estimated signals $\hat{s}_{il}$) $\hat{G}(\cdot)$, so its variance can be calculated by applying the delta method. In applying the delta method we ignore the variance in $\hat{G}(\cdot)$, i.e., we consider it as fixed. The estimated standard error thus calculated is FFr1.11 billion, and the 95% confidence interval for the hypothetical revenue is [FFr398.210 billion; FFr402.632 billion], implying that the difference in revenue between the discriminatory and uniform auction is significant at the 5% level.

When we restrict $\gamma = 1$, the results are very different: the estimated revenue under the uniform auction drops to FFr 215.462 billion, implying a substantial loss in income of FFr206.081 billion (almost 49%). However, since the hypothesis $\gamma = 1$ is rejected by the data (see Table 5), these last figures have no statistical justification, and cannot therefore be treated without much suspicion. They merely show that Wilson's model is too restrictive for the analysis of our data, and leads to overly negative conclusions regarding the performance of the uniform auction format.

Let us now compare our results with the 3 related econometric papers mentioned in the introduction. Hortaçsu (2002), using a sample of Turkish treasury auctions, also finds that the discriminatory auction is revenue-superior to the uniform auction. The revenue loss obtained by Hortaçsu is however larger than our estimated revenue loss of 5%. He reports counterfactual revenue comparisons for each of the 25 auctions in his sample. His *ex-ante* revenue differences vary between 0.12% and 27%, and on average the uniform auction generated 14% less than the discriminatory auction. Using treasury auctions held in Mexico, Castellanos and Oviedo (2002) find that the uniform auction produces more revenue than the discriminatory auction, which, interestingly, is in line with the conclusion that Umlauf (1993) draws from his analysis of the Mexican natural experiment. Since these authors adopt exactly the same parametric specifications as we do, their result is also an indirect proof of the fact that the gamma/exponential configuration is sufficiently flexible, i.e., depending on the data, the model can lead to any conclusion regarding the revenue comparison between the 2 auction formats. Finally, using another sample of French Treasury auctions, Armantier and Sbaï (2003) find that the revenue generated by uniform auctions is greater than the revenue in discriminatory auctions. There may be 2 explanations for the fact that their result is opposite to our's. One is that their model is different from our model as it allows for asymmetry among bidders and risk-aversion. Another and perhaps more plausible explanation is that their method of statistical inference, unlike the one we use, is based on an approximation of the equilibrium bid function.

4.3 Robustness analysis

In Section 2.2.2 we showed that the share auction model is probably not identified nonparametrically. This means that the economic question being answered in this paper may be sensitive to the parametric specification being utilized. It is therefore important to estimate alternative specifications and to check if our previous empirical results are robust.

The set of possible alternative models is unfortunately quite small since we need to confine ourselves to specifications that imply an analytical solution of the equilibrium strategy in the corresponding uniform auction. One possible alternative model that satisfies this criterion is the model considered by Kyle (1989).[31] In this model it is assumed that both the value of the good and the signal follow normal distributions: $V_l|Z_l = z_l \sim N(\mu_l^V, (\sigma^V)^2)$, and $S_{il}|V_l = v_l, Z_l = z_l \sim N(v_l, (\sigma^S)^2)$. The conditional mean of the value equals $\mu_l^V = (1, z_l)\cdot\alpha$, where α is again a vector of dimension 4 by 1. We have $\theta_1' = (\alpha', \sigma^V)$, $\theta_2 = \sigma^S$, and the complete parameter vector is therefore $\theta' = (\alpha', \sigma^V, \sigma^S)$. We refer to this parametric configuration as the normal configuration.

In Kyle's model it is necessary to consider supply-uncertainty (there does not exist an equilibrium in the uniform auction in the absence of supply uncertainty). In this version of the share auction model with supply uncertainty, the analogue of the market clearing equation (1) takes the form $\sum_{i=1}^{N_l} x(P_l^0, S_{il}) = 1 - \Omega_l$, where Ω_l represents the supply-uncertainty in auction l. As mentioned in Section 3.2, there is some degree of supply-uncertainty in the data as the Treasury only announces an issue interval for each auction.

It is assumed that the noise Ω_l follows a normal distribution: $\Omega_l \sim N(\mu^\Omega, (\sigma^\Omega)^2)$. The mean and variance can be estimated using the observations on the issue intervals announced by the Treasury. We find $\hat{\mu}^\Omega = 0.01$ and $\hat{\sigma}^\Omega = 0.04$. The fundamental parameter of interest θ can be estimated using our two-step approach. All methods described in Section 2.2 directly apply to the share auction model with supply uncertainty (the only thing

[31] We thank a referee for pointing out this model to us.

that one should keep in mind is that all relevant expectations are now also with respect to the noise Ω_l). The only difference is that in equation (7) (and in what follows) we have $G(\cdot|\cdot,\cdot;\cdot) = F_{S|Z}(\cdot|\cdot;\cdot)$. This is because in the normal configuration the equilibrium distribution under the uniform auction is increasing in the signal (see below). So it is natural to impose this restriction also in the estimation of θ under the discriminatory auction. In appendix D we show that in the case of the normal configuration the parameter θ^0 is identified. The results of the estimation are given in Table 6.

Table 6. Estimation of θ: the normal configuration

Estimate of α :	
Constant	-15.98 (7.14)**
Secondary market price	1.21 (0.12)**
Nominal yield	-0.54 (0.87)
Maturity of security (in days/1000)	0.48 (0.22)**
σ^V	0.74 (0.78)

Note: Actual estimates and standard errors (in parentheses) in α are very small, and those reported in the table are multiplied by 10^{24}. **=significant at the 5% level.

All parameters in α are significant except the parameter corresponding to the nominal yield of the security. We did not manage to estimate separately σ^S and σ^V. This numerical identification problem comes from the fact that there is too little variation in the number of bidders n_l in the data.[32] We therefore set the ratio $(\sigma^S/\sigma^V)^2$ arbitrarily equal to 0.01, and estimate σ^V. This parameter is strictly positive but not significant. The number of auctions in the sample is probably too small to estimate this parameter with sufficient precision.[33] Using the estimated parameters, we can again calculate $E(V_l|Z_l = z_l)$, and the derivative of this expectation with respect to each

[32] As the identification proof for the normal configuration shows (Appendix D), if n_l is the same for all l, the only thing that can be identified is $\sqrt{(\sigma^V)^2 + (\sigma^S)^2/(n_l + (\sigma^S)^2/(\sigma^V)^2)}$. Identification of σ^S and σ^V comes from the variation in n_l. If this variation is too small in the data, as in our case, there are numerical identification problems with these 2 parameters.

[33] This comes from the fact that the term $\frac{\sqrt{(\sigma^V)^2+(\sigma^S)^2}}{\tau_l^2(\sigma^S)^2}(n_l - 1)E\left\{\sum_{i=1}^{n_l}\Phi^{-1}(G(x_{ilp}|n_l;p))1(P_l^0 \leq p)|N_l = n_l\right\}$ from which σ^V is identified and estimated is very small compared to the other terms in the moment condition (see Appendix D).

variable in z_l. Evaluated at the empirical mean of the explanatory variables, we find that the conditional mean of the value equals 100.55, which is quite close to the value reported for the gamma/exponential configuration. The derivative with respect to the nominal yield, the secondary market price and the maturity are given by α and equal -0.54, 1.21 and 0.48, which again is quite similar to the results reported in Section 4.1 (except for the derivative with respect to the nominal yield, but the parameter associated with this variable is not significant). Table A2 in the appendix gives for all auctions the estimated expectations in the case of the normal configuration, and as the table shows, the patterns are similar for both configurations. Thus we may state that so far both configurations lead to similar conclusions.

Next we turn to the revenue comparison in the case of the normal configuration. It can be shown (see Kyle (1989)) that an optimal strategy in the uniform auction is

$$
\begin{aligned}
x(p, s_{il}, n_l, z_l; \theta, \mu^{\Omega}, \sigma^{\Omega}) &= \frac{n_l - 2}{n_l(n_l - 1)}(1 - \mu^{\Omega}) + \frac{2n_l - 2}{n_l(n_l - 1)} \\
&\quad \cdot \sqrt{\frac{n_l - 2}{n_l(n_l - 1)}} \frac{\sigma^{\Omega}\sigma^{S}}{(\sigma^{V})^2}\mu_l^{V} + \sqrt{\frac{n_l - 2}{n_l(n_l - 1)}} \frac{\sigma^{\Omega}}{\sigma^{S}} s_{il} \\
&\quad - \sqrt{\frac{n_l - 2}{n_l(n_l - 1)}} \Big[\frac{2n_l - 2}{n_l(n_l - 1)} \frac{\sigma^{\Omega}\sigma^{S}}{(\sigma^{V})^2} + \frac{\sigma^{\Omega}}{\sigma^{S}}\Big] p. \qquad (20)
\end{aligned}
$$

Note that the equilibrium strategy explicitly depends on the mean μ^{Ω} and the standard deviation σ^{Ω}. Using this strategy, we can in principle calculate the hypothetical total revenue under the uniform auction as in Section 4.2. However, the hypothetical total revenue turns out to be highly sensitive to small modifications in the value of σ^{Ω}. Indeed, any conclusion can be obtained regarding the revenue-comparison when the hypothetical uniform revenue is calculated for a set of values of σ^{Ω} closely around its estimate 0.04. This can be explained by the fact that in Kyle's model there only exists an equilibrium under the uniform auction when there is supply-uncertainty, i.e., when the variance of Ω_l is large. The degree of supply-uncertainty in our data is probably too small for the equilibrium strategy (20) to hold, which in turn implies that the hypothetical revenue under the uniform auction cannot be estimated in a reliable way.

5 Conclusion

This paper has proposed structural econometric methods for the empirical study of Wilson's share auction model. We have shown how the parameters of this model, i.e., the joint distribution function of the value of the good and the signals received by the bidders, can be estimated via a two-step estimation procedure. We have shown that the model is parametrically identified, and, using the estimation theory for semiparametric two-step estimators developed by Newey and McFadden (1994), we have established the asymptotic properties of our estimator. The methods have been applied to Treasury auctions held in France. Our results suggest that the Treasury's revenue in the discriminatory share auction is 5% higher than in the uniform share auction, which is a relatively high figure given the enormous amounts of money at stake. This result can be seen as an ex-post justification for the fact that the majority of countries rely on the discriminatory auctions to sell their Treasury securities.

References

[1] Armantier, O., J.-P. Florens and J.-F. Richard (2002), "Game theoretic models of incomplete information: constrained equilibrium and simulation," Working Paper, SUNY Stony Brook.

[2] Armantier, O. and E. Sbaï (2003), "Estimation and comparison of Treasury auction formats when bidders are asymmetric," Working Paper, SUNY Stony Brook.

[3] Athey, S. and P.A. Haile (2002), "Identification of standard auction models," *Econometrica*, 70, 2107-2140.

[4] Back, K. and J.F. Zender (1993), "Auctions of divisible goods: on the rationale for the Treasury experiment," *Review of Financial Studies,* 6, 733-764.

[5] Bartolini, L. and C. Cottarelli (1997), "Treasury bill auctions: issues and uses," in M. Blejer and T. Ter-Mminassian, eds., *Macroeconomic Dimensions of Public Finance,* 267-336, London: Routledge.

[6] Berg, S.A., B. Boukai and M. Landsberger (1998), "Bidding for Treasury securities under different auction rules: the Norwegian experience," Working Paper, Haifa University.

[7] Bikhchandani, S. and C. Huang (1989), "Auctions with resale markets: an exploratory model of Treasury bill markets," *Review of Financial Studies*, 2, 311-339.

[8] Bikhchandani, S. and C. Huang (1993), "The economics of Treasury securities markets," *Journal of Economic Perspectives*, 7, 117-134.

[9] Cantillon, E. and M. Pesendorfer (2001), "Combination bidding in multi-unit auctions," Working Paper, Yale University.

[10] Carrasco, M. and J.-P. Florens (1999), "Generalization of GMM to a continuum of moment conditions," *Econometric Theory*, 16, 797-834.

[11] Castellanos, S.G. and M.A. Oviedo (2002), "Optimal bidding in the Mexican Treasury securities auctions: a structural econometrics approach," Working paper, Yale University.

[12] Chiang, A.C. (1992), *Elements of Dynamic Optimization.* New York: McGraw-Hill International Editions.

[13] Das, S.R. and R.K. Sundaram (1996), "Auction theory: a survey with applications to Treasury Markets," *Financial Markets, Institutions and Instruments*, 5, 1-36.

[14] Donald, S. and H. Paarsch (1993), "Piecewise pseudo-maximum likelihood estimation in empirical models of auctions," *International Economic Review*, 34, 121-148.

[15] Donald, S. and H. Paarsch (1996), "Identification, estimation, and testing in parametric models of auctions within the independent private values paradigm," *Econometric Theory*, 12, 517-567.

[16] Donald, S., H. Paarsch and J. Robert (1999), "Identification, estimation, and testing in empirical models of sequential, ascending-price auctions with multi-unit demand: an application to Siberian timber-exports," Working Paper, University of Iowa.

[17] Elyakime, B., J.J. Laffont, P. Loisel and Q. Vuong (1994), "First-price sealed-bid auctions with secret reservation prices," *Annales d'Economie et Statistique*, 34, 115-141.

[18] Février, P., R. Préget and M. Visser (2002), "Econometrics of share auctions," Working Paper, CREST.

[19] French Government Securities, 1995 Annual Report, published by the Ministry of the Economy, Finance and Industry.

[20] Friedman, M. (1960), *A Program for Monetary Stability.* New York: Fordham University Press.

[21] Guerre, E., I. Perrigne and Q. Vuong (2000), "Optimal nonparametric estimation of first-price auctions," *Econometrica*, 68, 525-574.

[22] Gordy, M.B. (1999), "Hedging winner's curse with multiple bids: evidence from the Portuguese Treasury bill auction," *Review of Economics and Statistics*, 81, 448-465.

[23] Haile, P.A. (2001), "Auctions with resale markets: an application to U.S. forest service timber sales," *American Economic Review*, 91, 399-427.

[24] Hong, H. and M. Shum (2003), "Econometric models of asymmetric ascending auctions," *Journal of Econometrics*, 112, 327-358.

[25] Hortaçsu, A. (2002), "Mechanism choice and strategic bidding in divisible good auctions: an empirical analysis of the Turkish Treasury auction market," Working Paper, University of Chicago.

[26] Hortaçsu, A. (2001), "Bidding behavior in divisible good auctions: theory and evidence from the Turkish Treasury auction market," Working Paper, Stanford University.

[27] Jofre-Bonet, M. and M. Pesendorfer (2003), "Estimation of a dynamic auction game," *Econometrica*, 71, 1443-1489.

[28] Kyle, A. (1989), "Informed speculation with imperfect competetion," *Review of Economic Studies*, 56, 317-355.

[29] Laffont, J.J., H. Ossard and Q. Vuong (1995), "Econometrics of first-price auctions," *Econometrica*, 63, 953-980.

[30] Laffont, J.J. and Q. Vuong (1996), "Structural analysis of auction data," *American Economic Review, Papers and Proceedings*, 86, 414-420.

[31] Li, T., I. Perrigne and Q. Vuong (2000), "Conditionally independent private information in OCS wildcat auctions," *Journal of Econometrics*, 98, 129-161.

[32] Li, T., I. Perrigne and Q. Vuong (2002), "Structural estimation of the affiliated private value auction model," *RAND Journal of Economics*, 33, 171-193.

[33] Miller, G. and C. Plott (1985), "Revenue generating properties of sealed-bid auctions: an experimental analysis of one-price and discriminative auctions," in V.L. Smith, ed., *Research in Experimental Economics*, Volume 3, 159-181.

[34] Newey, W.K. and D.L. McFadden (1994), "Large sample estimation and hypothesis testing," in R.F. Engle and D.L. McFadden, eds., *Handbook of Econometrics,* Volume 4, 2113-2245.

[35] Paarsch, H.J. (1992), "Deciding between the common and private value paradigms in empirical models of auctions," *Journal of Econometrics*, 51, 191-215.

[36] Smith, V. (1967), "Experimental studies of discrimination versus competition in sealed bid auction markets," *Journal of Business*, 40, 56-84.

[37] Umlauf, S.R. (1993), "An empirical study of the Mexican Treasury bill auction," *Journal of Financial Economics,* 33, 313-340.

[38] Viswanathan, S., J.D. Wang and T. Witelski (2000), "Optimal bidding in multi-unit discriminatory auctions: two bidders," Working Paper, Duke University.

[39] Wang, J.D. and J.F. Zender (2002), "Auctioning divisible goods," *Economic Theory*, 19, 673-705.

[40] Wilson, R. (1979), "Auctions of shares," *Quarterly Journal of Economics,* 93*,* 675-689.

APPENDIX

A The Euler condition for the discriminatory auction model

First we rewrite the expected profit (4). We have

$$\begin{aligned} & E\left\{\int_0^\infty \left[(V-p)y(p,s_i) - \int_p^{\overline{p}} y(u,s_i)du\right] dH(p;V,y(p,s_i))|S_i = s_i\right\} \\ = \; & E\left\{-\int_0^\infty \left[-y(p,s_i) + (V-p)y_p(p,s_i) + y(p,s_i)\right] H(p;V,y(p,s_i))dp|S_i = s_i\right\} \\ = \; & E\left\{\int_0^\infty \left[-(V-p)y_p(p,s_i)H(p;V,y(p,s_i))\right] dp|S_i = s_i\right\} \end{aligned}$$

where the first equality follows from an integration by parts, and y_p is the derivative of y with respect to p. The expression in brackets can be written as a function $g(p, y(p, s_i), y_p(p, s_i), V)$. We want to stress that there is no connection at all between this function g and the distribution function G defined in the main text. Using this notation, the expected profit can be rewritten as

$$E\left\{\int_0^\infty \left[g(p,y(p,s_i),y_p(p,s_i),V)\right] dp|S_i = s_i\right\}. \tag{21}$$

The necessary condition for $y(\cdot, s_i)$ to maximize (21) is that for all p

$$0 = E\left\{\frac{\partial g}{\partial y} - \frac{d}{dp}\frac{\partial g}{\partial y_p}|S_i = s_i\right\} \tag{22}$$

(see Chiang, 1992, p. 46). Given the specific form of the function g, we have

$$\frac{\partial g}{\partial y} = -(V-p)y_p(p,s_i)\frac{\partial H(p;V,y(p,s_i))}{\partial y}$$

and

$$\frac{d}{dp}\frac{\partial g}{\partial y_p} = H(p;V,y(p,s_i)) - (V-p)\left\{\frac{\partial H(p;V,y(p,s_i))}{\partial p} + y_p(p,s_i)\frac{\partial H(p;V,y(p,s_i))}{\partial y}\right\}$$

so that (22) can be rewritten as

$$0 = E\left\{-H(p;V,y(p,s_i)) + (V-p)\frac{\partial H(p;V,y(p,s_i))}{\partial p}|S_i = s_i\right\}.$$

The strategy $x(\cdot,\cdot)$ is optimal if the above condition is satisfied for $y(\cdot,s_i) = x(\cdot,s_i)$, which gives the Euler condition (5).

B Proof of Proposition 1

Here we show that the Euler condition (5) can be rewritten as (6). We have

$$H(p;v,y) = \int \ldots \int_{s_j;j\neq i} 1\left(\sum_{j\neq i} x(p,s_j) \leq 1-y\right)\prod_{j\neq i} f_{S|V}(s_j|v)ds_j$$

where $f_{S|V}(\cdot|\cdot)$ is the density associated with $F_{S|V}(\cdot|\cdot)$. Defining $x_{jp} \equiv x(p,s_j)$, the above expression can be written as

$$H(p;v,y) = \int \ldots \int_{x_{jp};j\neq i} 1\left(\sum_{j\neq i} x_{jp} \leq 1-y\right)\prod_{j\neq i}\frac{\partial x^{-1}(p,x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p,x_{jp})|v)dx_{jp}$$

so that (because the integrand is symmetric in all the x_{jp})

$$\begin{aligned}\frac{\partial H(p;v,y)}{\partial p} &= (n-1)\int \ldots \int_{x_{jp};j\neq i} 1\left(\sum_{j\neq i} x_{jp} \leq 1-y\right)\\ &\quad \cdot\frac{\partial}{\partial p}\left[\frac{\partial x^{-1}(p,x_{1p})}{\partial x_{1p}} f_{S|V}(x^{-1}(p,x_{1p})|v)\right]dx_{1p}\\ &\quad \cdot\prod_{\substack{j\neq i\\ j\neq 1}}\frac{\partial x^{-1}(p,x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p,x_{jp})|v)dx_{jp}.\end{aligned}$$

Now define $H(p;v) = \Pr(P^0 \leq p|V=v)$ and $H(p) = \Pr(P^0 \leq p)$. We have

$$\begin{aligned}H(p;v) &= \int \ldots \int 1\left(\sum_{j=1}^{n} x(p,s_j) \leq 1\right)\prod_{j=1}^{n} f_{S|V}(s_j|v)ds_j\\ &= \int \ldots \int 1\left(\sum_{j=1}^{n} x_{jp} \leq 1\right)\prod_{j=1}^{n}\frac{\partial x^{-1}(p,x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p,x_{jp})|v)dx_{jp}.\end{aligned}$$

So, again by symmetry, we have

$$
\begin{aligned}
\frac{dH(p;v)}{dp} &= n \int ... \int 1 \left(\sum_{j=1}^{n} x_{jp} \leq 1 \right) \frac{\partial}{\partial p} \left[\frac{\partial x^{-1}(p, x_{1p})}{\partial x_{1p}} f_{S|V}(x^{-1}(p, x_{1p})|v) \right] dx_{1p} \\
& \quad \cdot \prod_{j=2}^{n} \frac{\partial x^{-1}(p, x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p, x_{jp})|v) dx_{jp}.
\end{aligned}
$$

After some straightforward calculations it follows that

$$
\begin{aligned}
E\left\{ \frac{\partial H(p; v, x(p, S_i))}{\partial p} |V = v \right\} &= \int \frac{\partial H(p; v, x(p, s_i))}{\partial p} f_{S|V}(s_i|v) ds_i \\
&= \frac{(n-1)}{n} \frac{dH(p;v)}{dp}
\end{aligned}
$$

and therefore

$$
E\left\{ \frac{\partial H(p; V, x(p, S_i))}{\partial p} \right\} = \frac{(n-1)}{n} \frac{dH(p)}{dp}.
$$

We also have

$$
\begin{aligned}
E\left\{ V \frac{\partial H(p; V, x(p, S_i))}{\partial p} \right\} &= \frac{(n-1)}{n} \int v f_V(v) \frac{dH(p;v)}{dp} dv \\
&= \frac{(n-1)}{n} \frac{d}{dp} \left[\int v f_V(v) \int ... \int 1 \left(P^0 \leq p \right) \prod_{j=1}^{n} f_{S|V}(s_j|v) ds_j dv \right] \\
&= \frac{(n-1)}{n} \frac{d}{dp} \left[E\left\{ E\left\{ V|S_1 = s_1, ..., S_n = s_n \right\} \cdot 1 \left(P^0 \leq p \right) \right\} \right]
\end{aligned}
$$

where $f_V(\cdot)$ is the density associated with $F_V(\cdot)$. Finally we have $E\left\{H(p; V, x(p, S_i))\right\} = H(p)$.

Therefore, taking the expectation with respect to V, S_i, the Euler condition (5) can be rewritten as

$$
\begin{aligned}
0 &= E\left\{ V \frac{\partial H(p; V, x(p, S_i))}{\partial p} \right\} - pE\left\{ \frac{\partial H(p; V, x(p, S_i))}{\partial p} \right\} - E\left\{ H(p; V, x(p, S_i)) \right\} \\
&= \frac{(n-1)}{n} \frac{d}{dp} \left[E\left\{ E\left\{ V|S_1 = s_1, ..., S_n = s_n \right\} \cdot 1 \left(P^0 \leq p \right) \right\} \right] \\
& \quad - p \frac{(n-1)}{n} \frac{dH(p)}{dp} - H(p).
\end{aligned}
$$

Integrating over p gives

$$\begin{aligned} C &= E\left\{(n-1)\cdot(E\left\{V|S_1=s_1,...,S_n=s_n\right\}-p)\cdot 1\left(P^0\leq p\right)\right\} \\ &\quad -E\left\{(p-P^0)\cdot 1\left(P^0\leq p\right)\right\} \end{aligned}$$

where C is the integration constant. By definition $\Pr(P^0 \leq p^{min}) = 0$. Replacing $p = p^{min}$ in the above expression, it follows then that $C = 0$, which gives the condition (6).

The above proof can easily be adapted to the case of asymmetric bidders. We just give a sketch of the proof of this more general case. In the asymmetric setup all players are assumed to be different, i.e., for each player i there is a specific distribution function of S_i conditionally on the value V. Using obvious notations, we introduce, for player i:

$$H^i(p;v,y) = \int ... \int_{s_j;j\neq i} 1\left(\sum_{j\neq i} x^j(p,s_j)\leq 1-y\right)\prod_{j\neq i} f^j_{S|V}(s_j|v)ds_j.$$

For player i the Euler condition (5) is

$$0 = E\left\{(V-p)\partial H^i(p;V,y)/\partial p - H^i(p;V,y)|S_i=s_i\right\}$$

where the distribution H^i and the derivative of H^i are evaluated at $y = x^i(p,s_i)$. Taking the expectation with respect to S_i, and summing over all i we have

$$0 = E\left\{(V-p)\sum_{i=1}^n \frac{\partial H^i(p;V,x^i(p,S_i))}{\partial p} - \sum_{i=1}^n H^i(p;V,x^i(p,S_i))\right\}.$$

Applying the same technique as above leads to

$$E\left\{\sum_{i=1}^n H^i(p;V,x^i(p,S_i))\right\} = nH(p)$$

and

$$E\left\{\sum_{i=1}^n \frac{\partial H^i(p;V,x^i(p,S_i))}{\partial p}\right\} = (n-1)\frac{dH(p)}{dp}$$

and

$$E\left\{\sum_{i=1}^{n} V \frac{\partial H^i(p; V, x^i(p, S_i))}{\partial p}\right\} = (n-1)\frac{d}{dp}\left[E\left\{E\left\{V|S_1 = s_1, ..., S_n = s_n\right\} \cdot 1\left(P^0 \leq p\right)\right\}\right].$$

This can be used, as previously, to recover equation (6).

C Asymptotic properties of the estimator

Our estimator belongs to the class of semiparametric two-steps estimators, the first-step being a kernel estimator. Newey and McFadden (1994, section 8.3) derive the asymptotic properties of such estimators. To apply their theorems, we need to introduce five additional functions.

- First of all, $r(x_{1lp_1}, ..., x_{n_l l p_T}, n_l, p_l^0, z_l, p_1, ..., p_T; \theta)$, a vector of dimension T, with components:

$$\begin{aligned} &r_t(x_{1lp_t}, ..., x_{n_l l p_t}, n_l, p_l^0, z_l, p_t; \theta) = \\ &w_t(n_l, z_l) \cdot (n_l - 1) \cdot 1(p_l^0 \leq p_t) \cdot \Big[E\Big\{V_l | S_{1l} = F_{S|Z}^{-1}(1 - G(x_{1lp_t}|n_l, z_l; p_t)|z_l; \theta), ... \\ &\quad , S_{n_l l} = F_{S|Z}^{-1}(1 - G(x_{n_l l p_t}|n_l, z_l; p_t)|z_l; \theta), N_l = n_l, Z_l = z_l\Big\} - p_t\Big] \\ &\quad - w_t(n_l, z_l) \cdot (p_t - p_l^0) \cdot 1(p_l^0 \leq p_t). \end{aligned}$$

- Second, $\nu(x_{1lp_1}, ..., x_{n_l l p_T}, n_l, p_l^0, z_l, p_1, ..., p_T; \theta^0)$, a vector of dimension $((n_{max} + 1) \times T)$ where $n_{max} = \max n_l$. The components for $i = 1, .., n_l$ are given by:

$$\begin{aligned} &\nu_{it}(x_{1lp_t}, ..., x_{n_l l p_t}, n_l, p_l^0, z_l, p_t; \theta^0) = \\ &-\frac{\partial E}{\partial S_{il}}\Big\{V_l | S_{1l} = F_{S|Z}^{-1}(1 - G(x_{1lp_t}|n_l, z_l; p_t)|z_l; \theta^0), ..., S_{n_l l} = F_{S|Z}^{-1}(1 - G(x_{n_l l p_t}|n_l, z_l; p_t)|z_l; \theta^0), \\ &\quad N_l = n_l, Z_l = z_l\Big\} \cdot (n_l - 1) w_t(n_l, z_l) F_{S|Z}^{-1\prime}(1 - G(x_{ilp_t}|n_l, z_l; p_t)|z_l; \theta^0) \\ &\quad \cdot \frac{1(p_l^0 \leq p_t)}{Pr(N_l = n_l, Z_l = z_l)} g(x_{1lp_t}, ..., x_{n_l l p_t}, n_l, z_l; p_t) \end{aligned}$$

where $g(\cdot;\cdot)$ is a joint density. The components for $i = n_l + 1$ are given by:

$$\nu_{(n_l+1)t}(x_{1lp_t}, ..., x_{n_l lp_t}, n_l, p_l^0, z_l, p_t; \theta^0) =$$
$$\sum_{i=1}^{n_l} \frac{\partial E}{\partial S_{il}} \Big\{ V_l | S_{1l} = F_{S|Z}^{-1}(1 - G(x_{1lp_t}|n_l, z_l; p_t)|z_l; \theta^0), ..., S_{n_l l} = F_{S|Z}^{-1}(1 - G(x_{n_l lp_t}|n_l, z_l; p_t)|z_l; \theta^0),$$
$$N_l = n_l, Z_l = z_l \Big\} \cdot (n_l - 1) w_t(n_l, z_l) F_{S|Z}^{-1'}(1 - G(x_{ilp_t}|n_l, z_l; p_t)|z_l; \theta^0)$$
$$\cdot \frac{1(p_l^0 \leq p_t) G(x_{ilp_t}|n_l, z_l; p_t)}{Pr(N_l = n_l, Z_l = z_l)} g(x_{1lp_t}, ..., x_{n_l lp_t}, n_l, z_l; p_t).$$

The components for $i > n_l + 1$ are equal to zero.

- Third, $\mu(x_{1lp_1}, ..., x_{n_l lp_T}, n_l, z_l, p_1, ..., p_T; \theta^0)$, a vector of dimension $(n_{max} \times T)$, with components:

$$\mu_{it}(x_{ilp_t}, n_l, z_l, p_t; \theta^0) = \int ... \int \nu_{it}(y_1, ..., y_{i-1}, x_{ilp_t}, y_{i+1}, ..., y_{n_l}, n_l, p_l^0, z_l, p_t; \theta^0) dy_1 ... dy_{n_l}.$$

- Fourth, $\lambda(n_l, z_l, p_t; \theta^0)$, a vector of dimension T, with components:

$$\lambda_t(n_l, z_l, p_t; \theta^0) = \int ... \int \nu_{(n_l+1)t}(y_1, ..., y_{n_l}, n_l, p_l^0, z_l, p_t; \theta^0) dy_1 ... dy_{n_l}.$$

- Fifth, $\delta(x_{1lp_1}, ..., x_{n_l lp_T}, n_l, z_l, p_1, ..., p_T; \theta^0)$, a vector of dimension T, with components:

$$\delta_t(x_{1lp_t}, ..., x_{n_l lp_t}, n_l, z_l, p_t; \theta^0) = \frac{1}{n_l} \sum_{i=1}^{n_l} \sum_{i'=1}^{n_l} \int_{x_{ilp_t}}^{+\infty} \mu_{i't}(y, n_l, z_l, p_t; \theta^0) dy +$$
$$\lambda_t(n_l, z_l, p_t; \theta^0).$$

With our notations, Theorem 8.12 of Newey and McFadden can be written as:

Theorem 1.

$$\sqrt{L}(\widehat{\theta} - \theta^0) \rightarrow N(0, (R'_\theta R_\theta)^{-1} R'_\theta \Omega R_\theta (R'_\theta R_\theta)^{-1})$$

where

$$R_\theta = E[\nabla_\theta r(X_{1lp_1}, ..., X_{n_l l p_T}, N_l, P_l^0, Z_l, p_1, ..., p_T; \theta^0)]$$

$$\Omega = Var[r(X_{1lp_1}, ..., X_{n_l l p_T}, N_l, P_l^0, Z_l, p_1, ..., p_T; \theta^0) + \delta(X_{1lp_1}, ..., X_{n_l l p_T}, N_l, Z_l, p_1, ..., p_T; \theta^0)].$$

Proof. Newey and McFadden have introduced the general function δ in their theorems 8.11 and 8.12. To construct δ in our specific context, we follow, step by step, the reasoning of Newey and McFadden (pp. 2207-2208).[34] $\square$

To study the estimation of the asymptotic variance-covariance matrix, we introduce the following additional functions.

- An estimation for $r_t(x_{1lp_t}, ..., x_{n_l l p_t}, n_l, p_l^0, z_l, p_t; \theta)$

$$\begin{aligned}
&\widehat{r}_t(x_{1lp_t}, ..., x_{n_l l p_t}, n_l, p_l^0, z_l, p_t; \theta) = \\
&w_t(n_l, z_l) \cdot (n_l - 1) \cdot 1(p_l^0 \leq p_t) \cdot \Big[E\Big\{ V_l | S_{1l} = F_{S|Z}^{-1}(1 - \widehat{G}(x_{1lp_t} | n_l, z_l; p_t) | z_l; \theta), ... \\
&\quad , S_{n_l l} = F_{S|Z}^{-1}(1 - \widehat{G}(x_{n_l l p_t} | n_l, z_l; p_t) | z_l; \theta), N_l = n_l, Z_l = z_l \Big\} - p_t \Big] \\
&\qquad - w_t(n_l, z_l) \cdot (p_t - p_l^0) \cdot 1(p_l^0 \leq p_t).
\end{aligned}$$

- An estimation for $\rho_{i't}(x_{ilp}, n_l, z_l, p_t; \theta^0) \equiv \int_{x_{ilp}}^{+\infty} \mu_{i't}(y, n_l, z_l, p_t; \theta^0) dy$

[34] The proof can found in Février, Préget and Visser (2002).

$$
\begin{aligned}
&\widehat{\rho}_{i't}(x_{ilp_t}, n_l, z_l, p_t; \widehat{\theta}) = \\
&\qquad -\frac{1}{L}\sum_{l'=1}^{L} \frac{\partial E}{\partial S_{jl'}}\Big\{ V_{l'} | S_{1l'} = F_{S|Z}^{-1}(1-\widehat{G}(x_{1l'p_t}|n_{l'}, z_{l'}; p_t)|z_{l'}; \widehat{\theta}), \\
&\quad ..., S_{n_{l'}l'} = F_{S|Z}^{-1}(1-\widehat{G}(x_{n_{l'}l'p_t}|n_{l'}, z_{l'}; p_t)|z_{l'}; \widehat{\theta}), N_{l'} = n_{l'}, Z_{l'} = z_{l'} \Big\} \\
&\qquad\quad \cdot (n_{l'}-1) w_t(n_{l'}, z_{l'}) F_{S|Z}^{-1\prime}(1-\widehat{G}(x_{i'l'p_t}|n_{l'}, z_{l'}; p_t)|z_{l'}; \widehat{\theta}) \\
&\qquad\qquad\qquad \cdot \frac{1(p^0_{l'} \leq p_t) 1(x_{ilp_t} \leq x_{i'l'p_t}) K\left(\frac{n_{l'}-n_l}{h_N}, \frac{z_{l'}-z_l}{h_Z}\right)}{\frac{1}{L}\sum_{l''} K\left(\frac{n_{l''}-n_{l'}}{h_N}, \frac{z_{l''}-z_{l'}}{h_Z}\right)}.
\end{aligned}
$$

- An estimation for $\lambda_t(n_l, z_l, p_t; \theta^0)$

$$
\begin{aligned}
&\widehat{\lambda}_t(n_l, z_l, p_t; \widehat{\theta}) = \\
&\qquad \frac{1}{L}\sum_{l'=1}^{L}\sum_{i=1}^{n_{l'}} \frac{\partial E}{\partial S_{il'}}\Big\{ V_{l'} | S_{1l'} = F_{S|Z}^{-1}(1-\widehat{G}(x_{1l'p_t}|n_{l'}, z_{l'}; p_t)|z_{l'}; \widehat{\theta}), \\
&\quad ..., S_{n_{l'}l'} = F_{S|Z}^{-1}(1-\widehat{G}(x_{n_{l'}l'p_t}|n_{l'}, z_{l'}; p_t)|z_{l'}; \widehat{\theta}), N_{l'} = n_{l'}, Z_{l'} = z_{l'} \Big\} \\
&\qquad\quad \cdot (n_{l'}-1) w_t(n_{l'}, z_{l'}) F_{S|Z}^{-1\prime}(1-\widehat{G}(x_{il'p_t}|n_{l'}, z_{l'}; p_t)|z_{l'}; \widehat{\theta}) \\
&\qquad\qquad\qquad \frac{1(p^0_{l'} \leq p_t) \widehat{G}(x_{il'p_t}|n_{l'}, z_{l'}; p_t) K\left(\frac{n_{l'}-n_l}{h_N}, \frac{z_{l'}-z_l}{h_Z}\right)}{\frac{1}{L}\sum_{l''} K\left(\frac{n_{l''}-n_{l'}}{h_N}, \frac{z_{l''}-z_{l'}}{h_Z}\right)}.
\end{aligned}
$$

- An estimation for $\delta_t(x_{1lp_t}, ..., x_{n_llp_t}, n_l, z_l, p_t; \theta^0)$

$$
\widehat{\delta}_t(x_{1lp_t}, ..., x_{n_llp_t}, n_l, z_l, p_t; \widehat{\theta}) = \frac{1}{n_l}\sum_{i=1}^{n_l}\sum_{i'=1}^{n_l} \widehat{\rho}_{i't}(x_{ilp_t}, n_l, z_l, p_t; \widehat{\theta}) + \widehat{\lambda}_t(n_l, z_l, p_t; \widehat{\theta}).
$$

We then obtain, under the hypotheses of Newey and McFadden, the analogue of their theorem 8.13:

Theorem 2.

$$(\widehat{R}'_\theta \widehat{R}_\theta)^{-1} \widehat{R}'_\theta \widehat{\Omega} \widehat{R}_\theta (\widehat{R}'_\theta \widehat{R}_\theta)^{-1} \to (R'_\theta R_\theta)^{-1} R'_\theta \Omega R_\theta (R'_\theta R_\theta)^{-1}$$

where

$$\widehat{R}_\theta = \frac{1}{L} \sum_{l=1}^{L} \nabla_\theta \widehat{r}(x_{1lp_1}, ..., x_{n_l l p_T}, n_l, p_l^0, z_l, p_1, ..., p_T; \widehat{\theta})$$

$$\widehat{\Omega} = \frac{1}{L} \sum_{l=1}^{L} [\widehat{r}(x_{1lp_1}, ..., x_{n_l l p_T}, n_l, p_l^0, z_l, p_1, ..., p_T; \widehat{\theta}) + \widehat{\delta}(x_{1lp_1}, ..., x_{n_l l p_T}, n_l, z_l, p_1, ..., p_T; \widehat{\theta})]$$
$$\cdot [\widehat{r}(x_{1lp_1}, ..., x_{n_l l p_T}, n_l, p_l^0, z_l, p_1, ..., p_T; \widehat{\theta}) + \widehat{\delta}(x_{1lp_1}, ..., x_{n_l l p_T}, n_l, z_l, p_1, ..., p_T; \widehat{\theta})]'.$$

D Proof of Proposition 2

This appendix gives the proof of Proposition 2. To simplify the proof, we consider the case where there are no auction characteristics Z_l. The proof of the general case is similar (except that the notations are more involved). First we proof the identification for the model with gamma/exponential specifications, and then for the model with normal specifications.

D.1 Gamma/exponential configuration

In the absence of auction characteristics, the distribution functions (15) and (16) become

$$F_V(v|\theta_1) = \int_0^v \gamma u^{\gamma-1} \frac{\beta}{\Gamma(\alpha)} u^{\gamma(\alpha-1)} \exp[-\beta u^\gamma] \, du \tag{23}$$

and

$$F_{S|V}(s|v;\theta_2) = 1 - \exp[-sv^\gamma]. \tag{24}$$

The parameters α, β, and γ are scalar parameters, $\theta_1 = (\alpha, \beta, \gamma)'$, $\theta_2 = \gamma$, and the complete vector of parameters is $\theta = (\alpha, \beta, \gamma)'$. Furthermore, after omitting the auction characteristics, and rearranging terms, the moment condition (11) becomes

$$
\begin{aligned}
&E\Big\{w(N_l)\cdot(N_l p - P_l^0)\cdot 1(P_l^0 \le p)\Big\} \\
&\quad = E\Big\{w(N_l)\cdot(N_l-1)\cdot\Big[E\Big\{V_l|S_{1l} = F_S^{-1}(1-G(x_{1lp}|n_l;p);\theta^0), \\
&\qquad ..., S_{n_l l} = F_S^{-1}(1-G(x_{n_l lp}|n_l;p);\theta^0), N_l = n_l\Big\}\Big]\cdot 1(P_l^0 \le p)\Big\}. \quad (25)
\end{aligned}
$$

where $F_S(\cdot;\cdot)$ is the marginal distribution function of S_{il}, and, analogously to (7), $G(x|n;p) = 1 - F_S(x^{-1}(x,p,n;\theta^0);\theta^0)$. Proving Proposition 2 now amounts to proving that θ^0 is uniquely determined given the distribution function $G(\cdot|\cdot;\cdot)$, and given that (25) must hold for any function $w(\cdot)$ and all possible p. This is equivalent to imposing

$$
\begin{aligned}
&E\Big\{(n_l p - P_l^0)\cdot 1(P_l^0 \le p)|N_l = n_l\Big\} \\
&\quad = E\Big\{(n_l-1)\cdot\Big[E\Big\{V_l|S_{1l} = F_S^{-1}(1-G(x_{1lp}|n_l;p);\theta^0), \\
&\qquad ..., S_{n_l l} = F_S^{-1}(1-G(x_{n_l lp}|n_l;p);\theta^0), N_l = n_l\Big\}\Big]\cdot 1(P_l^0 \le p)|N_l = n_l\Big\}. \quad (26)
\end{aligned}
$$

for all $n_l \ge 2$ and all $p \in [p^{min}(n_l), p^{max}(n_l)]$. Also note that the lefthand side of (26) does not explicitly depend on the parameter θ^0 (it only depends on n_l and p). The righthand side does explicitly depend on θ^0 (and n_l and p), and will be denoted $R(p,n_l;\theta^0)$. Proposition 2 can now be proved by showing that if there is a θ such that $R(p,n_l;\theta) = R(p,n_l;\theta^0)$ for all $n_l \ge 2$ and all $p \in [p^{min}(n_l), p^{max}(n_l)]$, then necessarily $\theta = \theta^0$.

Using the specifications (23) and (24), the righthand side of (26), evaluated at θ, is equal to

$$
\begin{aligned}
&R(p,n_l;\theta) \\
&= \int_0^{+\infty}\cdots\int_0^{+\infty}(n_l-1)\frac{\Gamma(n_l+\alpha+1/\gamma)}{\Gamma(n_l+\alpha)}\frac{g(x_{1lp},\ldots,x_{n_l lp}|n_l;p)\cdot 1(\sum_{i=1}^{n_l} x_{ilp} \le 1)dx_{1lp}\ldots dx_{n_l lp}}{\left(\beta + \sum_{i=1}^{n_l}\beta\left[\frac{1}{G^{1/\alpha}(x_{ilp}|n_l;p)} - 1\right]\right)^{1/\gamma}}
\end{aligned}
$$

where $g(x_{1lp},\dots,x_{n_l lp}|n_l;p)$ is the joint density of the share bids given $N_l = n_l$. Since $G(x_{ilp}|n_l;p) = 1 - F_S(s_{il};\theta^0) = \left(\frac{\beta^0}{\beta^0+s_{il}}\right)^{\alpha^0}$, we can rewrite the previous expression by changing the variables in the integrals and by using the fact that the joint density of the signals is $\frac{\Gamma(n_l+\alpha^0)}{\Gamma(\alpha^0)}\frac{(\beta^0)^{\alpha^0}}{(\beta^0+\sum_{i=1}^{n_l} s_{il})^{n_l+\alpha^0}}$. We get

$$R(p,n_l;\theta) = (n_l-1)\frac{\Gamma(n_l+\alpha+1/\gamma)}{\Gamma(n_l+\alpha)}\frac{\Gamma(n_l+\alpha^0)}{\Gamma(\alpha^0)}\frac{(\beta^0)^{\alpha^0}}{\beta^{1/\gamma}}$$
$$\cdot\int_0^{+\infty}\cdots\int_0^{+\infty}\frac{1\left(\sum_{i=1}^{n_l} x(p,s_{il},n_l;\theta^0)\le 1\right)ds_{1l}\dots ds_{n_l l}}{(\beta^0+\sum_{i=1}^{n_l} s_{il})^{n_l+\alpha^0}\left(1+\sum_{i=1}^{n_l}\left[\left(\frac{\beta^0+s_{il}}{\beta^0}\right)^{\alpha^0/\alpha}-1\right]\right)^{1/\gamma}}.$$

Finally we rewrite $R(p,n_l;\theta)$ as

$$R(p,n_l;\theta) = (n_l-1)\frac{\Gamma(n_l+\alpha+1/\gamma)}{\Gamma(n_l+\alpha)}\frac{\Gamma(n_l+\alpha^0)}{\Gamma(\alpha^0)}\frac{(\beta^0)^{\alpha^0}}{\beta^{1/\gamma}}$$
$$\cdot\int_0^{+\infty}\cdots\int_0^{+\infty} h(s_{1l},...,s_{n_l l})\cdot 1\left(\sum_{i=1}^{n_l} x(p,s_{il},n_l;\theta^0)\le 1\right)ds_{1l}\dots ds_{n_l l}$$

where

$$h(s_{1l},...,s_{n_l l}) = \frac{1}{(\beta^0+\sum_{i=1}^{n_l} s_{il})^{n_l+\alpha^0}\left(1+\sum_{i=1}^{n_l}\left[\left(\frac{\beta^0+s_{il}}{\beta^0}\right)^{\alpha^0/\alpha}-1\right]\right)^{1/\gamma}}.$$

We now proceed the proof by studying the n_l-th derivative of $R(p,n_l;\theta)$ with respect to p. Remark that $R(p,n_l;\theta)$ depends on p only via the multiple integral. We evaluate the n_l-th derivative of $R(p,n_l;\theta)$ at $p = p_l^{max} = p^{max}(n_l)$, the highest possible equilibrium price. Note that it is defined as the solution of $x(p_l^{max},0,n_l;\theta^0) = \frac{1}{n_l}$, i.e., p_l^{max} corresponds to the situation where all n_l bidders receive a signal equal to 0.[35] The reason for doing this

[35]For instance, in the case of the uniform auction there is an explicit solution for the maximal equilibrium price: $p_l^{max} = \frac{1}{1+1/\gamma^0}\frac{\Gamma(n_l+\alpha^0+1/\gamma^0)}{\Gamma(n_l+\alpha^0)}\frac{1}{(\beta^0)^{1/\gamma^0}}$.

is that the n_l-th derivative of the multiple integral, evaluated at $p = p_l^{max}$, turns out to be independent of θ. This property is crucial as it practically completes our proof of Proposition 2.

It can be shown that the n_l-th derivative of $R(p, n_l; \theta)$ at $p = p_l^{max}$ is

$$\begin{aligned}\frac{d^{n_l} R(p_l^{max}, n_l; \theta)}{dp^{n_l}} &= (n_l - 1)\frac{\Gamma(n_l + \alpha + 1/\gamma)}{\Gamma(n_l + \alpha)}\frac{\Gamma(n_l + \alpha^0)}{\Gamma(\alpha^0)}\frac{(\beta^0)^{\alpha^0}}{\beta^{1/\gamma}} \\ &\cdot\Big(- n_l \frac{\partial x^{-1}}{\partial p}(p_l^{max}, x(p_l^{max}, 0, n_l; \theta^0), n_l; \theta^0)\Big)^{n_l} h(0, \ldots, 0).\end{aligned}$$

The proof of this claim can be obtained from the authors. The main idea is to use the fact that for $p = p_l^{max}$, the function $1(\sum_{i=1}^{n_l} x(p, s_{il}, n_l; \theta^0) \leq 1)$ is equal to 1 for all s_{il}, and to study the multiple integral and its derivatives for p close to p_l^{max}. Note that the partial derivative in the above expression has no reason to be zero.[36] Also note that, as mentioned above, the n_l-the derivative of the multiple integral at $p = p_l^{max}$ (which corresponds to the last 2 terms in the above equation) is independent of θ.

This makes the proof of Proposition 2 almost complete. Indeed, if there is a θ such that $R(p, n_l; \theta) = R(p, n_l; \theta^0)$ for all $n_l \geq 2$ and all $p \in [p^{min}(n_l), p^{max}(n_l)]$, then necessarily $\frac{d^{n_l} R(p,n_l;\theta)}{dp^{n_l}} = \frac{d^{n_l} R(p,n_l;\theta^0)}{dp^{n_l}}$ for all $n_l \geq 2$ and all $p \in [p^{min}(n_l), p^{max}(n_l)]$. We must have in particular that $\frac{d^{n_l} R(p_l^{max},n_l;\theta)}{dp^{n_l}} = \frac{d^{n_l} R(p_l^{max},n_l;\theta^0)}{dp^{n_l}}$, which, using the above equation, implies that we must have

$$\frac{\Gamma(n_l + \alpha + 1/\gamma)}{\Gamma(n_l + \alpha)}\frac{1}{\beta^{1/\gamma}} = \frac{\Gamma(n_l + \alpha^0 + 1/\gamma^0)}{\Gamma(n_l + \alpha^0)}\frac{1}{(\beta^0)^{1/\gamma^0}} \tag{27}$$

for all $n_l \geq 2$. By taking the ratio of equation (27) for n_l and $n_l + 1$, we get:

$$\frac{n_l + \alpha + 1/\gamma}{n_l + \alpha} = \frac{n_l + \alpha^0 + 1/\gamma^0}{n_l + \alpha}$$

which implies that

$$n_l(1/\gamma - 1/\gamma^0) + \alpha^0/\gamma - \alpha/\gamma^0 = 0$$

must hold for all n_l. This in turn implies that γ^0 and α^0 are identified. From equation (27) it follows that β^0 is also identified. Thus we have shown that θ^0 is identified.

[36] In the case of the uniform auction mechanism, we have $\frac{\partial x^{-1}}{\partial p}(p_l^{max}, x(p_l^{max}, 0, n_l; \theta^0), n_l; \theta^0) = -\frac{n_l+\alpha^0}{2n_l(p_l^{max})^2}$.

D.2 Normal configuration

In the absence of auction characteristics, the distribution of the value is $V_l \sim N(\mu_V, \sigma_V^2)$, and the conditional distribution of the signal is $S_{il}|V_l = v \sim N(v, \sigma_S^2)$. The parameters μ_V, σ_V, and σ_S are scalar parameters, $\theta_1 = (\mu_V, \sigma_V)'$, $\theta_2 = \sigma_S$, and $\theta = (\mu_V, \sigma_V, \sigma_S)'$. Equation (26) is again used in the identification proof. The expression of $R(p, n_l; \theta)$, i.e., the righthand side of (26) evaluated at θ, is now $(1 - G(\cdot|\cdot;\cdot)$ is replaced by $G(\cdot|\cdot;\cdot))$:

$$R(p, n_l; \theta) = \quad (n_l - 1)E\left[\frac{1}{\tau_l^2}\left(\frac{\mu_V}{(\sigma_V)^2} + \frac{1}{\sigma_S^2}\sum_{i=1}^{n_l} F_S^{-1}(G(x_{ilp}|n_l; p); \theta)\right) 1(P_l^0 \le p)|N_l = n_l\right]$$

where

$$\tau_l^2 = \frac{1}{(\sigma_V)^2} + \frac{n_l}{\sigma_S^2}.$$

Using

$$F_S^{-1}(\cdot; \theta) = \sqrt{\sigma_V^2 + \sigma_S^2}\ \ \Phi^{-1}(\cdot) + \mu_V$$

where $\Phi(\cdot)$ is the distribution function of a normal distribution with mean 0 and variance 1, we obtain

$$\begin{aligned} R(p, n_l; \theta) \quad &= \quad \mu_V(n_l - 1)\Pr(P_l^0 \le p) + \frac{\sqrt{\sigma_V^2 + \sigma_S^2}}{\tau_l^2 \sigma_S^2}(n_l - 1) \\ &\quad \cdot E\left\{\sum_{i=1}^{n_l} \Phi^{-1}(G(x_{ilp}|n_l; p)) 1(P_l^0 \le p)|N_l = n_l\right\}. \end{aligned}$$

If there is a θ such that $R(p, n_l; \theta) = R(p, n_l; \theta^0)$ for all $n_l \ge 2$ and all $p \in [p^{min}(n_l), p^{max}(n_l)]$, then necessarily (we use the definition of τ_l)

$$\mu_V = \mu_V^0$$

and

$$\frac{\sqrt{\sigma_V^2 + \sigma_S^2}}{n_l + \sigma_S^2/(\sigma_V)^2} = \frac{\sqrt{(\sigma_V^0)^2 + (\sigma_S^0)^2}}{n_l + (\sigma_S^0)^2/(\sigma_V^0)^2}$$

for all $n_l \ge 2$ and all $p \in [p^{min}(n_l), p^{max}(n_l)]$. Thus μ_V^0 is identified. Variation in n_l then implies the identification of σ_V^0 and σ_S^0, which completes the proof.

Table A1. The auctions

Type of auction	Auction date	Line auctioned	Settlement date	Maturity date
OAT	5 Jan 95	7.5% 2005	25 Jan 95	25 Apr 05
OAT	5 Jan 95	6% 2025	25 Jan 95	25 Oct 25
OAT	2 Feb 95	7.5% 2005	27 Feb 95	25 Apr 05
OAT	2 Feb 95	8.5% 2008	27 Feb 95	25 Oct 08
OAT	2 Feb 95	6% 2025	27 Feb 95	25 Oct 25
OAT	2 Mar 95	7.5% 2005	27 Mar 95	25 Apr 95
OAT	2 Mar 95	8.5% 2008	27 Mar 95	25 Oct 08
OAT	6 Apr 95	7.75% 2005	25 Apr 95	25 Apr 05
OAT	4 May 95	7.75% 2005	26 May 95	25 Apr 05
OAT	4 May 95	8.5% 2008	26 May 95	25 Oct 08
OAT	1 Jun 95	8.5% 2002	26 Jun 95	25 Nov 02
OAT	1 Jun 95	7.75% 2005	26 Jun 95	25 Apr 05
OAT	1 Jun 95	6% 2025	26 Jun 95	25 Apr 25
OAT	6 Jul 95	7.75% 2005	25 Jul 95	25 Oct 05
OAT	6 Jul 95	8.5% 2008	25 Jul 95	25 Oct 08
OAT	6 Jul 95	6% 2025	25 Jul 95	25 Oct 25
OAT	3 Aug 95	8.5% 2002	25 Aug 95	25 Nov 02
OAT	3 Aug 95	7.75% 2005	25 Aug 95	25 Oct 05
OAT	3 Aug 95	8.5%2008	25 Aug 95	25 Oct 08
OAT	7 Sep 95	7.75% 2005	25 Sep 95	25 Oct 05
OAT	7 Sep 95	6% 2025	25 Sep 95	25 Oct 25
OAT	5 Oct 95	7.25% 2006	25 Oct 95	25 Apr 06
OAT	2 Nov 95	7.25% 2006	27 Nov 95	25 Apr 06
OAT	2 Nov 95	6% 2025	27 Nov 95	25 Oct 25
OAT	7 Dec 95	7.25% 2006	26 Dec 95	25 Apr 06
BTAN	19 Jan 95	7.75% 2000	13 Feb 95	12 Apr 00
BTAN	16 Feb 95	6.5% 1996	6 Mar 95	12 Oct 96
BTAN	16 Feb 95	7.75% 2000	13 Mar 95	12 Apr 00
BTAN	16 Mar 95	7.25% 1997	5 Apr 95	12 Aug 97
BTAN	20 Apr 95	7.25% 1997	5 May 95	12 Aug 97
BTAN	20 Apr 95	7.75% 2000	12 May 95	12 Apr 00
BTAN	18 May 95	7.25% 1997	6 Jun 95	12 Aug 97
BTAN	18 May 95	7.75% 2000	12 Jun 95	12 Apr 00
BTAN	15 Jun 95	7.25% 1997	5 Jul 95	12 Aug 97
BTAN	15 Jun 95	7.75% 2000	12 Jul 95	12 Apr 00
BTAN	20 Jul 95	7% 2000	11 Aug 95	12 Oct 00
BTAN	17 Aug 95	7.25% 1997	5 Sep 95	12 Aug 97
BTAN	17 Aug 95	7% 2000	12 Sep 95	12 Oct 00
BTAN	21 Sep 95	7.25% 1997	5 Oct 95	12 Aug 97
BTAN	21 Sep 95	7% 2000	12 Oct 95	12 Oct 00
BTAN	19 Oct 95	7.25% 1997	6 Nov 95	12 Aug 97
BTAN	19 Oct 95	7% 2000	13 Nov 95	12 Oct 00
BTAN	16 Nov 95	5.75% 1998	5 Dec 95	12 Mar 98
BTAN	21 Dec 95	5.75% 1998	5 Jan 96	12 Mar 98
BTAN	21 Dec 95	7% 2000	12 Jan 96	12 Oct 00

Table A2. Expected value of V, stop-out price, and secondary market price

$E[V_l \mid S_{1l}, ..., S_{n_l l}, Z_l]$	Stop-out price	$E[V_l \mid Z_l]$	Secondary market Price
96.28 94.68	93.96	96.40 95.88	94.17
71.75 71.99	70.88	72.09 72.60	71.33
97.75 97.12	95.72	97.80 97.91	95.86
104.37 105.56	102.26	104.40 105.90	102.39
75.15 74.30	72.94	75.41 75.03	73.35
98.72 98.37	96.56	98.77 99.17	96.91
105.65 107.16	103.28	105.67 107.17	103.45
101.98 102.21	99.68	101.99 102.52	99.65
102.36 102.36	99.74	102.39 103.06	99.75
107.63 109.01	104.70	107.65 108.78	104.80
111.79 110.78	106.74	111.80 110.15	106.80
105.84 105.84	101.76	105.82 105.70	102.37
80.70 81.46	77.46	80.55 80.70	78.07
103.97 104.60	101.20	103.94 104.28	101.21
110.49 110.77	106.46	110.51 110.50	106.25
79.27 79.05	76.18	79.22 79.14	76.80
115.56 111.79	107.96	115.57 111.47	107.91
107.03 107.48	102.80	106.99 106.43	103.00
118.80 114.34	108.16	118.80 113.21	108.50
106.71 106.84	102.72	106.68 106.23	102.85
79.92 80.07	77.22	79.84 79.96	77.50
99.87 100.45	97.78	99.88 100.89	98.16
102.01 102.43	99.62	102.02 102.81	99.75
80.20 81.22	77.80	80.05 80.29	77.80
108.32 107.15	102.60	108.27 106.48	102.80
101.07 100.69	99.41	101.10 101.32	99.50
97.69 99.98	99.21	97.81 100.77	99.01
101.87 101.89	100.19	101.86 102.08	100.14
99.90 99.75	99.21	99.99 100.78	99.24
101.02 101.31	99.97	101.02 101.62	99.95
102.83 102.96	100.90	102.81 102.94	100.88
104.03 103.66	101.47	104.02 103.70	101.54
106.77 106.37	103.27	106.74 105.75	103.21
103.25 103.17	101.30	103.23 103.16	101.24
106.08 106.21	102.80	106.04 105.33	102.87
101.59 101.96	99.90	101.61 102.40	100.06
104.90 104.67	101.96	104.85 104.07	102.02
102.98 103.30	100.78	102.95 103.22	100.75
105.29 104.90	102.13	105.24 104.26	102.19
104.76 104.34	101.47	104.72 104.15	101.53
103.94 103.85	101.61	103.90 103.53	101.60
102.56 102.93	100.60	102.54 102.97	100.57
99.58 103.19	100.36	99.31 103.00	100.42
105.56 103.78	100.73	105.29 103.42	100.78
115.78 108.48	103.91	115.72 107.19	104.08

Note: First column in $E[V_l \mid S_{1l}, ..., S_{n_l l}, Z_l]$ corresponds to the estimate obtained under the gamma/exponential configuration, and the second column to the one obtained under the normal configuration; similar definitions for the 2 columns in $E[V_l \mid Z_l]$.