Dynamic Models with Unobserved State Variables and Heterogeneity: Time Inconsistency in Drug Compliance*

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Feburary, 2013
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Abstract

Time preference plays an important role in understanding inter-temporal economic behavior. The frequently-used dynamic models suffer from under-identification, where the discount factors are often assumed based on market interest rate. The identification of these discount factors is only achieved through certain exclusion restrictions in revealed preference data. Moreover, behavior literature have shown strong evidence of time inconsistency, and illustrate that agents' discount factors could vary across individuals and across time for the same individual. Built upon the recent development in the measurement-error literature, we establish the identification of heterogeneous hyperbolic discounting preference with unobserved state variables in the stationary environment. We then develop an estimation strategy and apply it to consumers' prescription drug refilling. We find substantial time inconsistency and heterogeneity in agents' inter-temporal choice behavior. This provides novel policy opportunities to reduce patients' non-compliance behavior and improve their well-being.

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1 Introduction

Time preference plays an essential role in understanding inter-temporal economic behavior. Economic agents are often assumed to discount future streams of utility or profits exponentially over the time in these dynamic models. However, literature (e.g., Rust (1994); Magnac and Thesmar (2002)) illustrates that these dynamic models often suffer from under-identification problems. Consumers' preference parameters are only identified conditional on the assumption of the discount factors and the subjective beliefs about future market conditions. The exponential discount factors are only identified with certain exclusion restrictions (Magnac and Thesmar (2002)) (and rational self-fulfilling expectations). Such exclusion restrictions may be rare in the field data or difficult to validate.

Moreover, behavior literature challenges the validity of standard exponential discounting hypotheses and has shown strong evidence that hyperbolic discounting might explain agent behavior better in many settings (Thaler (1981), Loewenstein and Prelec (1992), Laibson (1997), O'Donoghue and Rabin (2001)). The commonly observed preference reversals such as that decision makers choose larger and later of two prizes when both are distant in time, but prefer the smaller but earlier one as both prizes draw nearer to the present, are not easily reconciled within standard exponential discounting setting, which could parsimoniously be explained by the hyperbolic discounting framework. Under hyperbolic discounting, agents discount the immediate future from the present more heavily than the same interval starting at a future date. This can be illustrated using the quasi-hyperbolic discount function $D_t \equiv \beta\delta^t$, where $\beta < 1$ is the short-run present bias factor and δ is the long-term discount factor. When $\beta = 1$, it reduces to the exponential discounting model. Hyperbolic discounting can lead to time inconsistent preferences and preference reversals. For example, an agent who exhibits hyperbolic discounting can prefer 10 dollar now ($t = 0$) to 12 dollar in a year ($t = 1$), yet prefer 12 dollar in two years ($t = 2$) to 10 dollar in one year ($t = 1$). Based on the insight above, experimentalists study the implication of time-inconsistent preference and the associated problems of self-control behaviors such as patient compliance and vice goods consumption.

Discount factors could also vary across individuals (heterogeneity) or differ substantially from the market interest rate (Frederick, Loewenstein, and O'donoghue (2002); Dubé, Hitsch, and Jindal (2009)). In a field-experimental setting, Yao, Mela, Chiang, and Chen (2011) identify heterogeneous discount factors in a finite mixture fashion by first identifying consumers' heterogeneous utilities and the distribution of random consumption shocks using data that have no dynamics involved, then recovering the discount factors when the tariff menu with dynamic decision process was exogenously imposed.

In this paper, we incorporate unobserved state variables and heterogeneous time preference into a dynamic discrete choice model, where agents exhibit potentially hyperbolic discounting preferences in the form of standard a exponential discounting factor (δ) and heterogeneous present-bias factors $\boldsymbol{\beta} \in \{\beta_1, ..., \beta_K\}, K \geq 1$. We show that $\beta_1, ..., \beta_K$, δ and agents' preference can be identified in four steps under mild conditions. First, built upon the recent development in the measurement-error literature (e.g. Hu and Shum (2010)), we establish the identification of conditional choice probability which depends on both observed and unobserved state variables, and the state transition process for both observed and unobserved state variables of each type (type is indicated by the present-bias factor) of the agents. Based on the nonparametrically identified CCP and state transition processes, we then prove in the second step that the current-period value function can also be nonparametrically identified conditional on given discounting factors ($\boldsymbol{\beta}$, δ). Next, we employ employ the restrictions introduced by heterogenous present-bias factor which affect the state transition process but not the agents' static payoff to achieve the identification of ($\boldsymbol{\beta}$, δ). Lastly, we show agents' instantaneous utility function is identified from value function and discounting factors. Our first step of identification is constructive, it provides us with a novel estimating strategy to nonparametrically estimate CCP and the transition processes with the unobserved state variables. These estimates then serve as the inputs for our estimation of ($\boldsymbol{\beta}$, δ) using maximum likelihood.

In the empirical application, we address patients' noncompliance behavior in drug refilling using a dynamic discrete choice with unobserved state variables and heterogeneous time preferences. The tendency of patients not to take their prescribed medications on a consistent and continuing basis (so called "noncompliance") has emerged as an issue of significant concern because of the resulting huge health care costs and lost producibility.[1] An increased understanding of the reasons for noncompliance with prescribed medication is an important step to reduce the cost and improve treatment effectiveness, and thus patient health. In this application, we model patients' decision of compliance/noncompliance as a dynamic discrete choice problem where patients are potentially time inconsistent and exhibit nonzero and heterogeneous present-bias factor $\boldsymbol{\beta}$. Patients' decision of compliance is affected by a set of observed state variables including insurance status, whether the prescription is new or refilling as well as the cost of the prescription. The unobserved "type" of patients, which is captured by the heterogeneous present-bias factor $\boldsymbol{\beta}$ and the unobserved health status also affect patients' compliance. By the empirical results, we provide better understanding on how patients' potentially time-inconsistency and their health status affect their decision of compliance, and the insight may of novel policy recommendation to reduce patients' non-compliance behavior and improve patients' health.

[1] This behavior has been reported to contribute to 125,000 premature deaths each year in the United States and an additional \$100 billion in costs to the economy in terms of increased health care costs and lost productivity.

The contributions of this paper are two-fold. First, we embed unobserved variables into agents' time-inconsistent behavior, and show that the heterogeneity of time preference ($\boldsymbol{\beta}$) provides identifying power for identification of hyperbolic discounting preference. Fang and Wang (2012) utilize exclusive restrictions argument (similar to Magnac and Thesmar (2002)) to study semi-parametric identification of hyperbolic discounting preference of dynamic discrete choice model. However they do not allow unobserved heterogeneity or unobserved state variables other than serially independent choice-specific errors. In a finite-horizon (non-stationary) case, Mahajan and Tarozzi (2010) demonstrate how the hyperbolic discounting parameters can be identified with additional data of elicited beliefs and response to time-preference questions. They allow for unobserved agent types who differ in both time-preference as well as per-period utility parameters, by using the exclusive variable to affect type probabilities. This is closely related with Kasahara and Shimotsu (2009), which requires the exclusive variable to affect the choice probabilities. Chung, Steenburgh, and Sudhir (2009) utilize the bonus-based compensation structure, and estimate a dynamic structural model of sales force compensation with hyperbolic discounting. None of these papers allow for the serially correlated sate variable even though early literature such as Miller (1984) and Pakes (1986) study the standard dynamic problem with an unobserved process partly determining the state variables. Our results explicitly show that the impacts of discounting factors and unobserved state variables on agents' dynamic discrete choices can be separately identified. For the purpose of identification, we employ the restrictions imposed by agents' heterogenous time inconsistency. This distinguish our work from existing ones where exclusion variables are observed and used for identification, which is difficult to justify in many applications.

Second, we provide an estimating strategy that is closely related to identification and can be easily implemented. Estimation of dynamic discrete choice models with unobserved heterogeneity or serially correlated unobservables has been addressed in a growing literature during the past a few years. Imai, Jain, and Ching (2009) and Norets (2009) consider a Bayesian approach of estimation. Gallant, Hong, and Khwaja (2008) and Blevins (2011) develops simulation estimators for dynamic games with serially-correlated unobservables, utilizing state-of-the-art recursive importance sampling ("particle filtering") techniques. Arcidiacono and Miller (2011) provide an expectation-maximization (EM) algorithm for estimating dynamic games where the unobservables are assumed to follow a discrete Markov process. Our estimation strategy is different from the existing ones in that condition choice probability and transition process of state variables can be nonparametrically estimated with the presence of unobserved state variables. Consequently, using these objectives as input, discounting factors and the parameters in agents' preference are estimated parametrically by maximum likelihood.

The remaining of this paper is organized as follows. In Section 2 we present a general dynamic choice model with hyperbolic discounting time preferences. In Section 3 we show the identification of the model and discusses the conditions required in our identification. Section 4 proposes a estimation method based on our identification result. Section 5 illustrates the performance of our estimators by a Monte Carlo experiment. Section 6 provide details about our empirical application. Finally, Section 7 concludes and discusses some issues in our analysis.

2 The Model

2.1 Basic Setup

Consider a dynamic setting where an agent $i \in \mathcal{I} = \{1, 2, ..., I\}$ exhibits hyperbolic discounting time preference characterized by (β, δ), where $\beta \in (0, 1]$ and $\delta \in (0, 1]$, and the agent's intertemporal preference is represented by

$$U_t(u_t, u_{t+1}, ...) \equiv u_t + \beta \sum_{s=t+1}^{T} \delta^{s-t} u_s. \tag{1}$$

In such a setting, the discount function for the agent is $\{1, \beta\delta, \beta\delta^2, ...\}$, where δ is the "standard discount factor" that captures the agent's long-run and time-consistent discounting and β is the "present-bias factor" which describes agent's short-term impatience. The discount function degenerates to the standard exponential one when $\beta = 1$.

The intertemporal utility is additively time separable at time $t = 1, 2, ...T$.[2] The agent chooses among a set of alternatives, $Y_t = j \in \mathcal{J} = \{0, 1, 2, ..., J\}$, based on a list of variables, $(X_t, H_t, \tau, \varepsilon)$, where X_t, H_t are vectors of observed and unobserved state variables, respectively, τ is the unobserved heterogeneity, i.e., different types of hyperbolic discount factor β_τ.[3] The unobserved variables are denoted by $\Omega_t = (H_t, \tau)$.

Assumption 1. *The instantaneous utility functions are described by:*

$$\forall j \in \mathcal{J}/\{0\}, u_{jt}(X_t, \Omega_t; \varepsilon) = \widetilde{u}_{jt}(X_t, H_t) + \varepsilon_{jt}, \forall t.$$

where $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t}, ..., \varepsilon_{Jt}) \in R^J$ *is mean independent of* (X_t, Ω_t)*:* $E(\varepsilon_{jt}|X_t, \Omega_t) = 0$ *and the joint distribution,* $G(\cdot)$*, is absolutely continuous with respect to the Lebesgue measure in* R^J*, which is assumed to be the extreme value distribution.*

[2]The time horizon could be infinite or finite. The identification we study here is the stationary process (infinite horizon, $T = \infty$).

[3]If the heterogeneity is in the standard discounting factor δ, the choices will be different across agents after controlling for all other variables but all the agents will exhibit the same level of time-consistency.

Note that the present-bias factor β does not enter the instantaneous utility. This unobserved heterogeneity affects agents choice only through the effects on transition process of state variables. The assumptions on random shock ε_t standard for dynamic discrete choice models. Especially, assuming $G(\cdot)$ to be distributed according to an extreme value distribution loses no generality for our identification methodology.

The agent makes a choice from the set $\mathcal{J}$ in each period to maximize her current utility $U_t(u_t, u_{t+1}, ...)$ given the state variables $(X_t,\ H_t)$, her type τ, and the choice-specific shock ε_{jt}, $\varepsilon_t = (\varepsilon_{1t},\ \varepsilon_{2t},...,\varepsilon_{Jt})$. A strategy profile $\boldsymbol{s}$ for this agent in an infinite horizon is defined as $\boldsymbol{s} = \{s_t\}_{t=1}^{\infty}$ where $s_t : \mathcal{X} \times \mathcal{S} \times R^J \to \mathcal{J}, \forall t$. Considering the time-inconsistent preferences of the agent, we first introduce a long-run expected continuation utility perceived by the agent at period t, $V_t(X_t, \Omega, \varepsilon_t; \boldsymbol{s}_t^+)$ where $\boldsymbol{s}_t^+$ is the continuation strategy profile from period t on, i.e., $\boldsymbol{s}_t^+ = \{s_l\}_{l=t}^{\infty}$. The utility $V_t(X_t, \Omega, \varepsilon_t; \boldsymbol{s}_t^+)$ satisfies a "Bellman like" equation as follows (Harris and Laibson (2004)).

$$V_t(X_t, \Omega_t, \varepsilon_t; \boldsymbol{s}_t^+) = u_{j^*t}(X_t, \Omega_t, \varepsilon_{j^*t}) + \delta E\Big[V_{t+1}(x_{t+1}, \Omega_{t+1}, \varepsilon_{t+1}; \boldsymbol{s}_{t+1}^+)|X_t, \Omega_t; j^*\Big],$$

where $j^* \equiv s_t(X_t, \Omega_t, \varepsilon_t) \in \mathcal{J}$ is the choice determined by the strategy $\boldsymbol{s_t}$, and the expectation is taken over X, Ω and ε at period $t+1$. Based on this long-run expected continuation utility, the "observed" strategy profile which generates our observation of agent's choice must satisfy:

$$\begin{aligned} s_t(X_t, \Omega_t, \varepsilon_t) &= \arg\max_{j \in \mathcal{J}} \Big\{ u_{jt} + \beta\delta E\Big[V_{t+1}(X_{t+1}, \Omega_{t+1}, \varepsilon_{t+1}; \boldsymbol{s}_{t+1}^+)|X_t, \Omega_t; j\Big] \Big\}, \\ & \forall X_t \in \mathcal{X}, \Omega_t \in \mathcal{S}, \varepsilon_t \in R^J, t = 1, 2, ... \end{aligned} \tag{2}$$

To further define and characterize the value function, we first specify how the vector of unobserved state variables Ω enters the state transition process (for ease of notation, we drop the subscript t and denote the variables of next period by prime).

Assumption 2. *Random preference shocks at two periods, denoted as ε and ε' are independent conditional on $(X, \Omega; Y = j)$. This implies that the transition governing (X, Ω, ε) is*

$$\begin{aligned} f(X', \Omega', \varepsilon'|X, \Omega, \varepsilon; Y = j) &= f(\varepsilon'|X', \Omega')f(X', \Omega'|X, \Omega, ; Y = j) \\ f(\varepsilon'|X', \Omega') &= f(\varepsilon) \\ f(X', \Omega'|X, \Omega; Y = j) &= f(X'|\Omega', X; Y = j)f(\Omega'|\Omega, X; Y = j) \end{aligned} \tag{3}$$

The first two assumptions state the conditional independence of the transition of the state variables and the distribution of the shocks, which are both standard in the literature, e.g, see Rust (1994). The third assumption is on the transition of the unobserved state variables and the dependence of transition of observed variables on the unobserved variables. The vector of unobserved variables Ω is serially correlated across time. In our empirical application, Ω is two-dimensional, $\Omega = (H_t, \tau) \in \mathcal{H} \times \Gamma$, with the first term, H_t being the serially correlated health status of agents, the unobserved heterogeneity τ indicates the unobserved "type" which captures the heterogenous present-bias factor of agents. Both variables are discrete with a cardinality $|\mathcal{H}|$ and $|\Gamma|$, respectively.[4]

2.2 Value functions

The characterization of agents' (optimal) strategy profile $\boldsymbol{s}$ enables us to express the deterministic component of the current choice-specific value function $Z_j(X, \Omega)$ as

$$Z_j(X, \Omega) = \widetilde{u}_j(X, \Omega) + \beta\delta \int V(X^{'}, \Omega^{'}) f(X^{'}, \Omega^{'}|X, \Omega; j = \boldsymbol{s}(X, \Omega, \varepsilon)) dX^{'} d\Omega^{'}, \tag{4}$$

where $\varepsilon = (\varepsilon_1, ...\varepsilon_J) \in R^J$ is a vector, $V(X^{'}, \Omega^{'})$ is the perceived long-run value function defined as $V(X, \Omega) = E_\varepsilon V(X, \Omega, \varepsilon; \boldsymbol{s})$ where $V(X, \Omega, \varepsilon; \boldsymbol{s})$ is the stationary value function and $\boldsymbol{s}$ is the strategy profile defined in the following. The agent at the current period chooses the alternative j if and only if j satisfies

$$j = \arg\max_{k \in \mathcal{J}} \{Z_k(X, \Omega) + \varepsilon_k\}.$$

Equivalently, the strategy profile $\boldsymbol{s}(X, \Omega, \varepsilon)$ we introduced can be expressed as

$$\boldsymbol{s}(X, \Omega, \varepsilon) = \arg\max_{k \in \mathcal{J}} \{Z_k(X, \Omega) + \varepsilon_k\}.$$

Consequently, the observed current-period choice probability (CCP) is

$$\Pr\Big(Z_j(X, \Omega) + \varepsilon_j > \max_{k \in \mathcal{J}, k \neq j} Z_k(X, \Omega) + \varepsilon_k\Big) = \frac{\exp\big(Z_j(X, \Omega)\big)}{\sum_{k \in \mathcal{J}} \exp\big(Z_k(X, \Omega)\big)}. \tag{5}$$

With the strategy profile $\boldsymbol{s}(X, \Omega, \varepsilon)$ and current period choice-specific value function $Z_j(X, \Omega)$ defined above, we characterize the perceived choice-specific long-run value function $V_j(X, \Omega)$ (net of the random utility ε_j) and its relationship with $Z_j(X, \Omega)$. First, $V_j(X, \Omega)$ can be naturally defined as,

$$V_j(X, \Omega) = \widetilde{u}_j(X, \Omega) + \delta \int V(X^{'}, \Omega^{'}) f(X^{'}, \Omega^{'}|X, \Omega; Y = j) dX^{'} d\Omega^{'}. \tag{6}$$

[4] If either of H_t and τ is continuous, the model can still be identified using the similar approach.

From Eq.(4) and Eq.(6), we obtain a relationship between $V(\cdot)$ and $Z(\cdot)$.

$$V_j(X,\Omega) = Z_j(X,\Omega) + (1-\beta)\delta \int V(X',\Omega')f(X',\Omega'|X,\Omega;Y=j)dX'd\Omega'. \tag{7}$$

This result, together with the definition $V(X,\Omega) \equiv E_\varepsilon V(X,\Omega,\varepsilon;\boldsymbol{s}) = E_\varepsilon[V_k(X,\Omega)+\varepsilon_k)$, where k is the alternative chosen according to the strategy profile $\boldsymbol{s}(X,\Omega,\varepsilon)$, allows us to build up an explicit and deeper connection between $V(X,\Omega)$ and $Z(X,\Omega)$, which is crucial for our identification of value function.[5]

$$\begin{aligned} V(X,\Omega) &= \ln\left(\sum_{j\in\mathcal{J}} \exp\left(Z_j(X,\Omega)\right)\right) \\ &\quad +(1-\beta)\delta \sum_{j\in\mathcal{J}} \underbrace{\frac{\exp\left(Z_j(X,\Omega)\right)}{\sum_{l\in\mathcal{J}} \exp\left(Z_l(X,\Omega)\right)}}_{\text{CCP}} \int V(X',\Omega')\underbrace{f(X',\Omega'|X,\Omega;Y=j)}_{\text{transition process}}dX'd\Omega' \end{aligned} \tag{8}$$

Eq.(8) links two value functions $V(\cdot)$ and $Z(\cdot)$ to the CCP and transition process of X and Ω. Note that the agent's choice is jointly determined by $V(\cdot)$ and $Z(\cdot)$, however, the one-to-one mapping between value functions and conditional choice probabilities in Hotz and Miller (1993) only exists between $Z(\cdot)$ and the CCP in hyperbolic discounting case. Actually, our model nests the models with standard discounting factor by setting $\beta = 1$ and consequently $V(\cdot) = Z(\cdot)$.

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[5]

$$\begin{aligned} V(X,\Omega) &\equiv E_\varepsilon V(X,\Omega,\varepsilon;\boldsymbol{s}) = \int \left\{V_k(X,\Omega)+\epsilon_k\right\}f(\varepsilon)d\varepsilon \\ &= \int \left\{Z_k(X,\Omega) + (1-\beta)\delta\int V(X',\Omega')f(X',\Omega'|X,\Omega;Y=k)dX'd\Omega' + \epsilon_k\right\}f(\varepsilon)d\varepsilon \\ &= \int \left\{Z_k(X,\Omega)+\epsilon_k\right\}f(\varepsilon)d\varepsilon + (1-\beta)\delta\int V(X',\Omega')f(X',\Omega'|X,\Omega;k)dX'd\Omega' f(\varepsilon)d\varepsilon \\ &= \int \max_{j\in\mathcal{J}}\{Z_j(X,\Omega)+\epsilon_j\}f(\varepsilon)d\varepsilon + (1-\beta)\delta\int V(X',\Omega')f(X',\Omega'|X,\Omega;Y=k)dX'd\Omega' f(\varepsilon)d\varepsilon \\ &= \ln\left(\sum_{j\in\mathcal{J}}\exp\left(Z_j(X,\Omega)\right)\right) \\ &\quad +(1-\beta)\delta\sum_{j\in\mathcal{J}}\Pr\left(k=\arg\max_{j\in\mathcal{J}}\{Z_j(X,\Omega)+\varepsilon_j\}\right)\int V(X',\Omega')f(X',\Omega'|X,\Omega;j)dX'd\Omega' \\ &= \ln\left(\sum_{j\in\mathcal{J}}\exp\left(Z_j(X,\Omega)\right)\right) \\ &\quad +(1-\beta)\delta\sum_{j\in\mathcal{J}} \underbrace{\frac{\exp\left(Z_j(X,\Omega)\right)}{\sum_{l\in\mathcal{J}}\exp\left(Z_l(X,\Omega)\right)}}_{\text{CCP}} \int V(X',\Omega')\underbrace{f(X',\Omega'|X,\Omega;j)}_{\text{transition process}}dX'd\Omega', \end{aligned}$$

where the 2nd line is due to Eq.(7), the 5th and 6th lines are derived using the extreme value distribution of ε.

3 Identification

In this section, we discuss identification of the dynamic model. The data we observe are $\{X_t, Y_t\}_{t=1}^{\infty}$, where $\{X_t, Y_t\} \in \mathcal{X} \times \mathcal{J}$. We assume that the structure of the model, denoted by $\boldsymbol{\theta}$, is defined by the following parameters:

$$\boldsymbol{\theta} = \left\{ (\boldsymbol{\beta}, \delta), \boldsymbol{\pi}, G(\cdot), \left(\widetilde{u}_j(x, \Omega), Z_j(x^{'}, \Omega^{'}), V_j(x^{'}, \Omega^{'}) : j \in \mathcal{J}; x, x^{'} \in \mathcal{X}; \Omega, \Omega^{'} \in \mathcal{S} \right) \right\},$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_m)$ with $\{\pi_s\}_{s=1}^{m}$ being the probability of type s, i.e., those agents with present bias β_s, $G(\cdot)$ is the known distribution of choice-specific payoff shocks $\boldsymbol{\varepsilon}$, which is assumed to be a Type-I extreme value distribution. Our objectives of identification is $\boldsymbol{\beta} = (\beta_1, ..., \beta_m)$, which denotes the present-bias factor for decision makers of m different types, the standard discounting factor δ, and the parameters of agents' utility function $\widetilde{u}_j(x, \Omega)$.

Remark. The structure $\boldsymbol{\theta}$ above is different from that in Fang and Wang (2012) where the structure is

$$\boldsymbol{\theta} = \left\{ (\beta, \widetilde{\beta}, \delta), G(\cdot), \left(\widetilde{u}_j(x), Z_j(x^{'}), V_j(x^{'}) : j \in \mathcal{J}; x, x^{'} \in \mathcal{X} \right) \right\},$$

where both utility function and value function only depend on observed variable X, hence the CCP and transition process of X are both observed from the data. In additional, there is no unobserved heterogeneity for decision makers in Fang and Wang (2012). They distinguish between naive and sophisticated agents by introducing an additional factor $\widetilde{\beta}$: agents are partially naive if $\widetilde{\beta} \in [\beta, 1]$, completely naive if $\widetilde{\beta} = 1$ and sophisticated if $\widetilde{\beta} = \beta$. Basically, $\widetilde{\beta}$ indicates the extent to which a decision maker anticipates her future present-bias. Nevertheless, all the agents are homogeneous in Fang and Wang (2012). Our setting is corresponding to the case of sophisticated agents, i.e., $\widetilde{\beta} = \beta$ while agents are of different types who have different CCP and different transition process.■

Our identification is achieved in four steps. First, conditional on the distribution of the random utility component $\{\varepsilon\}$, we employ the recently developed econometric methodologies (Hu (2008) and Hu and Shum (2010)) to identify the Markov kernel $f(Y_t, X_t, \Omega_t | Y_{t-1}, X_{t-1}, \Omega_{t-1})$ using four periods of data $\{X_t, Y_t\}_{t=1}^{4}$ under the assumption of stationarity. Since for Markovian dynamic models, the kernel $f(Y_t, X_t, \Omega_t | Y_{t-1}, X_{t-1}, \Omega_{t-1})$ factors into the conditional choice probabilities (CCP), $f(Y_t | X_t, \Omega_t)$ and state transition process,[6] $f(X_t, \Omega_t | X_{t-1}, \Omega_{t-1}; Y_{t-1})$, we further show that the CCP and state transition process conditional on all H_t and τ are identified from the kernel. Consequently, the type probability $\boldsymbol{\pi}$ is also identified. Second, after the CCP's and the law of motion for the observed and unobserved state variables are recovered, we then

[6] We assume perfect expectation of agents on state transition, as in Magnac and Thesmar (2002).

use them as input to identify the structural parameters: current and long-run choice-specific value functions, $Z_j(X_t, \Omega_t)$ and $V_j(X_t, \Omega_t)$. Third, we discuss how to utilize the implications of heterogenous present-bias to identify the discount factors $(\boldsymbol{\beta}, \delta)$ for each type of agents. Lastly, we recover the contemporaneous payoff $\widetilde{u}_{jt}(X_t, \Omega_t)$ for all states X and choice j.[7]

3.1 Identification of CCP and State Transitions

We show that the Markov transition kernel $f_{Y_t,X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}}$ is identified from the observed joint distribution of X and Y of four periods, i.e., $f_{\{X_s,Y_s\}_{s=t-2}^{t+1}}$. This step of identification is related to Hu and Shum (2012), but our identification focus on the model with discrete variables. One of the prominent advantages is that that the essential assumptions of identification are empirically testable.

We restrict our analysis to first-order Markov models by making the following assumption.

Assumption 3. *First-order Markov:* $f_{Y_t,X_t,\Omega_t|\{Y_s,X_s,\Omega_s\}_{s=1}^{t-1}} = f_{Y_t,X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}}$.

This first-order Markov assumption is standard in the literature on dynamic discrete choice. The joint distribution of patients' choices and state variables (observed and unobserved) at period t is only dependent on the information at $t-1$.

Assumption 4. *Limited feedback: The vector of unobserved state variables at period* $t-1$, Ω_{t-1} *does not directly impact* (Y_t, X_t).

$$\begin{aligned} f_{Y_t,X_t|Y_{t-1},X_{t-1},\Omega_t,\Omega_{t-1}} &= f_{Y_t,X_t|Y_{t-1},X_{t-1},\Omega_t} = f_{Y_t|X_t,Y_{t-1},X_{t-1},\Omega_t} \cdot f_{X_t|Y_{t-1},X_{t-1},\Omega_t} \\ &= \underbrace{f_{Y_t|X_t,\Omega_t}}_{\text{CCP: } f_{Y_t|X_t,\Omega_t}} \cdot \underbrace{f_{X_t|Y_{t-1},X_{t-1},\Omega_t}}_{X_t \text{ law of motion}}. \end{aligned} \tag{9}$$

The limited feedback assumption eliminates Ω_{t-1} as a conditioning variable in both terms of the last line of the above equation. In Markovian dynamic model, the first term further simplifies to $f(Y_t|X_t, \Omega_t)$ (CCP), because the Markovian laws of motion for (X_t, Ω_t) imply that the optimal policy function depends only on the current state variables. Thus, this assumption imposes weaker restriction on the first term than the Markovian dynamic models. In the second term $f(X_t|Y_{t-1}, X_{t-1}, \Omega_t)$, the limited feedback rules out direct feedback of Ω_{t-1} on X_t. However, it allows the indirect effect via Ω_{t-1}'s influence on Y_{t-1} or X_{t-1}. This implicitly assumes that the unobserved state variables Ω_t are realized before X_t, so that X_t depends on Ω_t. While this is less restrictive than the assumption that X_t evolves independently of both Ω_{t-1} and Ω_t, which

[7] A question: whether the heterogeneity of β and δ across individuals are distinguishable.

has been made in many empirical applications to estimate the X_t law of motion directly from the data, it does rule out the situation where Ω_t is realized after X_t. In our empirical application of drug refilling, $\Omega_t = (H_t, \tau)$ where H_t is observed health status of patients, it realizes before the realization of observed variables X_t. Previous work with unobserved state variables, such as Erdem, Imai, and Keane (2003), Crawford and Shum (2005), and Hendel and Nevo (2006), also satisfy the Markov and limited feedback assumptions. In the case of unobserved heterogeneity, where $\Omega_{t-1} = \Omega_t$, $\forall t$, the limited feedback assumption is trivially satisfied. Importantly, the limited feedback assumption does not restrict the law of motion for Ω_t and Ω_t is still allowed to depend stochastically on $\Omega_{t-1}, Y_{t-1}, X_{t-1}$. This assumption may be relaxed by either imposing additional restrictions on CCP or identifying a higher-order Markov.

Under assumptions 3 and 4, we have the following result on the decomposition of the Markov kernel $f_{Y_t,X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}}$.

Lemma 1. *Decomposition of the Markov kernel: the Markov law of motion can be decomposed as follows.*

$$\begin{aligned} f_{Y_t,X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}} &= f_{Y_t|X_t,\Omega_t} f_{X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}} \\ &= f_{Y_t|X_t,\Omega_t} f_{X_t|\Omega_t,Y_{t-1},X_{t-1}} f_{\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}} \\ &= \underbrace{f_{Y_t|X_t,\Omega_t}}_{CCP} \underbrace{f_{X_t|Y_{t-1},X_{t-1},\Omega_t}}_{X_t \text{ law of motion}} \underbrace{f_{\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}}}_{\Omega_t \text{ law of motion}}. \end{aligned} \tag{10}$$

Proof The second equality follows directly from assumption 3 and 4. ■

It is clear that from lemma 1, identification of the Markov kernel $f_{Y_t,X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}}$ together with any two objectives on the R.H.S. guarantees the identification of CCP and laws of motion. Our strategy of identification is to first identify the Markov kernel then identify the CCP and law of motion for X_t. Consequently, the law of motion for Ω_t is identified.

Our identification procedure will focus on the model with discrete unobserved variable Ω even though it can be easily extended to the continuous scenario.[8] More specifically, $\forall\, t$, $\Omega_t = (H_t, \tau) \in \mathcal{H} \times \Gamma$ has $|\mathcal{H}| \times |\Gamma|$ possible values. Without loss of generality, we define $|\mathcal{H}| \times |\Gamma| \equiv m \geq 2$, which is an unknown integer. The choice variable $Y_t \in \mathcal{J}$ is also discrete as we defined earlier, while the observed variable $X_t \in \mathcal{X}$ is a vector with its components being either continuous or discrete. For ease of notation, we combine all the observed variables together by grouping Y_t, X_t into $W_t = (Y_t, X_t)$ and denote its realization $w_t \in \mathcal{X} \times \mathcal{J}$.

[8]Please see Hu and Shum (2010) for details of continuous variables.

To accommodate the discreteness of the unobserved Ω, we discretize W_t to $M \geq 2$ segments if it is continuous, otherwise M is just the cardinality of $\mathcal{X} \times \mathcal{J}$.[9] For given w_t, $w_{t-1} \in \mathcal{X} \times \mathcal{J}$, we define matrices $L_{W_{t+1},w_t,w_{t-1},W_{t-2}}$, $L_{W_{t+1}|w_t,\Omega_t}$, $L_{\Omega_{t-1}|w_{t-1},W_{t-2}}$, and $D_{w_t|w_{t-1},\Omega_{t-1}}$ with dimension $M \times M, M \times m, m \times M$, and $m \times m$, respectively,

$$\begin{aligned} L_{W_{t+1},w_t,w_{t-1},W_{t-2}} &= \left[f_{W_{t+1},W_t,W_{t-1},W_{t-2}}(i, w_t, w_{t-1}, j)\right]_{i,j} \\ L_{W_{t+1}|w_t,\Omega_t} &= \left[f_{W_{t+1}|W_t,\Omega_t}(i|w_t, j)\right]_{i,j} \\ L_{\Omega_{t-1}|w_{t-1},W_{t-2}} &= \left[f_{\Omega_t|W_{t-1},W_{t-2}}(i|w_{t-1}, j)\right]_{i,j} \\ D_{w_t|w_{t-1},\Omega_{t-1}} &= \begin{bmatrix} f_{W_t|W_{t-1},\Omega_{t-1}}(w_t|w_{t-1}, 1) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & f_{W_t|W_{t-1},\Omega_{t-1}}(w_t|w_{t-1}, m) \end{bmatrix} \end{aligned}$$

Other matrices in the paper are all similarly defined. Given the matrix notation above, we first show the inference of the cardinality of Ω which is the product of the number of types of agents and the possible

Assumption 5. *For all the methods of discretization, the rank of both $L_{W_{t+1},w_t,w_{t-1},W_{t-2}}$ and $L_{W_t,w_{t-1},W_{t-2}}$ for fixed $(w_t, w_{t-1}) \in \mathcal{X} \times \mathcal{J}$ is less than or equal to the cardinality of the unobserved variable Ω_t, i.e. $Rank(L_{W_{t+1},w_t,w_{t-1},W_{t-2}}) = Rank(L_{W_t,w_{t-1},W_{t-2}}) \leq |\mathcal{H}| \times |\Gamma|$. Moreover, there exists a method of discretization such that $Rank(L_{W_{t+1},w_t,w_{t-1},W_{t-2}}) = Rank(L_{W_t,w_{t-1},W_{t-2}}) = |\mathcal{H}| \times |\Gamma|$.*

The existence of a method of discretization under which

$$Rank(L_{W_{t+1},w_t,w_{t-1},W_{t-2}}) = Rank(L_{W_t,w_{t-1},W_{t-2}}) = |\mathcal{H}| \times |\Gamma|$$

holds permits us to choose $M = m$, then all the matrices defined above are of dimension $m \times m$. Consequently, the restrictions this assumption imposes to the model can be analyzed from the following relationship,

$$\begin{aligned} L_{W_{t+1},w_t,w_{t-1},W_{t-2}} &= L_{W_{t+1}|w_t,\Omega_t} D_{w_t|w_{t-1},\Omega_t} L_{\Omega_t,w_{t-1},W_{t-2}} \\ L_{W_t,w_{t-1},W_{t-2}} &= L_{W_t|w_{t-1},\Omega_{t-1}} L_{\Omega_{t-1},w_{t-1},W_{t-2}}. \end{aligned}$$

According to the assumption that the cardinality of $\mathcal{H} \times \Gamma$ is equal to m, $\text{Rank}(D_{w_t|w_{t-1},\Omega_t}) = m$ if $f(w_t|w_{t-1}, \Omega_t) \neq 0$ for all Ω_t. The full rank condition of $L_{W_{t+1}|w_t,\Omega_t}$ states how an unobserved state $\omega_t \in \mathcal{H} \times \Gamma$ affects the distribution of observables $w_{t+1} \in \mathcal{X} \times \mathcal{J}$ cannot be (linearly) explained by the way other $\tilde{\omega}_t \in (\mathcal{H} \times \Gamma)/\omega_t$ affect w_{t+1}. Similar full-rank condition is generally

[9] Need to discuss carefully that discretization doesn't lose any information. Basically, it implies that the model is over-identified under the assumptions.

assumed in the literature on identification. Another restriction implicitly imposed by the assumption is $|\mathcal{X}| \times |\mathcal{J}| \geq |\mathcal{H}| \times |\Gamma|$, otherwise there will be no methods of discretization such that $Rank(L_{W_{t+1},w_t,w_{t-1},W_{t-2}}) = Rank(L_{W_t,w_{t-1},W_{t-2}}) = |\mathcal{H}| \times |\Gamma|$.The condition $|\mathcal{X}| \times |\mathcal{J}| \geq |\mathcal{H}| \times |\Gamma|$ is satisfied in the following main cases: first, any component of X_t is continuous, thus W_t is continuous, and it can be arbitrarily discretized into M segments such that $|\mathcal{X}| \times |\mathcal{J}| \geq |\mathcal{H}| \times |\Gamma|$ hold.[10] Second, both X_t and Y_t are discrete. Then $|\mathcal{H}| \times |\Gamma|$ is at most $|\mathcal{X}| \times |\mathcal{J}|$ in order for the nonparametric identification to be achieved. Nevertheless, $|\mathcal{H}| \times |\Gamma|$ may always be identified when $|\mathcal{X}| \times |\mathcal{J}| < |\mathcal{H}| \times |\Gamma|$ if we parameterize the matrices discussed above.

Remark 1. The unobserved state variable Ω_t is two dimensional and assumption 5 only allows us to identify the distinct combinations of H_t and τ which affect the transition process and CCP differently. For instance, H_t and τ are both binary variables indicating a patient's health status is good or bad, and the patient is less or more consistent, respectively, in our empirical application, therefore $m = 4$. However, assumption 5 only identifies $m = 4$ while it is still uncertain which of the following three cases is true: $|\mathcal{H}| = 1, |\Gamma| = 4$, or $|\mathcal{H}| = 2, |\Gamma| = 2$, or $|\mathcal{H}| = 4, |\Gamma| = 1$. Hence, additional conditions are required to further identify $|\mathcal{H}|$ and $|\Gamma|$.

Remark 2. Assumption 5 permits us to chose $M = m$ such that all the matrices defined above are square with full rank, i.e., the matrix $L_{W_{t+1},w_t,w_{t-1},W_{t-2}}$ is invertible for all $(w_t, w_{t-1}) \in \mathcal{X} \times \mathcal{X}$. If it is not applicable to choose $M = m$ (e.g., $|\mathcal{X}| \times |\mathcal{J}| = 5, |\mathcal{H}| \times |\Gamma| = 4$), then we can reduce the dimension of the observed matrices such that they are all $m \times m$. We discuss the implementation in Monte Carlo experiment.■

The next assumption of identification is imposed to the conditional probability $f_{W_t|W_{t-1},\Omega_t}$.

Assumption 6. *There exist two pairs* $(w_t, \widetilde{w}_t)$ *and* $(w_{t-1}, \widetilde{w}_{t-1})$ *in* $\mathcal{X} \times \mathcal{J}$*, such that*

$$\frac{f_{W_t|W_{t-1},\Omega_t}(w_t|w_{t-1},k) f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|\widetilde{w}_{t-1},k)}{f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|w_{t-1},k) f_{W_t|W_{t-1},\Omega_t}(w_t|\widetilde{w}_{t-1},k)} \neq \frac{f_{W_t|W_{t-1},\Omega_t}(w_t|w_{t-1},l) f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|\widetilde{w}_{t-1},l)}{f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|w_{t-1},l) f_{W_t|W_{t-1},\Omega_t}(w_t|\widetilde{w}_{t-1},l)},$$
$$\forall k, l \in \mathcal{H} \times \Gamma.$$

Taking log to the expression above, we obtain an equivalent condition to the assumption

[10] In practice, we can always start from a large enough value M and test the full rank condition by varying M to verify $M = m$.

above,

$$\begin{aligned}
& \log f_{W_t|W_{t-1},\Omega_t}(w_t|w_{t-1},k) - \log f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|\widetilde{w}_{t-1},k) \\
& -[\log f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|w_{t-1},k) - \log f_{W_t|W_{t-1},\Omega_t}(w_t|\widetilde{w}_{t-1},k)] \\
\neq\ & \log f_{W_t|W_{t-1},\Omega_t}(w_t|w_{t-1},l) - \log f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|\widetilde{w}_{t-1},l) \\
& -[\log f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|w_{t-1},l) - \log f_{W_t|W_{t-1},\Omega_t}(w_t|\widetilde{w}_{t-1},l)] \\
& \forall k,l \in \mathcal{H} \times \Gamma.
\end{aligned}$$

The restrictions imposed by the assumption 6 can be seen clearly from the inequality above: when $\widetilde{w}_t \to w_t$, $\widetilde{w}_{t-1} \to w_{t-1}$ the L.H.S. of the inequality represents a second derivative of $\log f_{W_t|W_{t-1},\Omega_t}$ with respect to W_t and W_{t-1}. The inequality implies that such a derivative has to be different for any two realization of Ω_t. Moreover, an implicit restriction imposed by this assumption is that there are at least two distinct realizations for W_t and W_{t-1}, which can be naturally satisfied since there exists at least two choice Y_t. This assumption only requires for *any* two realizations in $\mathcal{X} \times \mathcal{J}$ for each of the two consecutive periods to hold, which makes it less restrictive.

Assumption 7. *(a) There exists a functional $\mathcal{F}(\cdot)$ such that $\mathcal{F}\left(f_{W_{t+1}|w_t,\Omega_t}(\cdot|w_t,\omega_t)\right)$ is monotonic in ω_t for any $w_t \in \mathcal{X} \times \mathcal{J}$.*
(b) Furthermore, there exists a functional $\mathcal{G}(\cdot)$ such that $\mathcal{G}\left(f_{Y_{t+1},X_{t+1}|w_t,H_t,\tau}(\cdot|w_t,h_t,\tau)\right)$ is monotonic in h_t for any $y_t \in \mathcal{J}$ or $x_t \in \mathcal{X}$.[11]

The monotonicity imposed above is not as restrictive as it looks for the following two reasons: first, the functional $\mathcal{F}(\cdot)$ could take any reasonable form. For example, if the conditional probability matrix $L_{W_t|w_{t-1},\Omega_t}$ is strictly diagonally dominant, hence the columns can be ordered according to the position of the maximal entry of each column. Alternatively, the ordering condition may be derived from some known properties of the distribution $f_{W_t|w_{t-1},\Omega_t}$. For instance, if it is known that $E(W_{t+1}|w_{t-1},\Omega_t = \omega_t)$ is monotonic in ω_t, we are able to correctly order all the columns of $L_{W_t|w_{t-1},\Omega_t}$. Second, in empirical applications, this assumption is model specific and oftentimes it is implied by the model in analysis naturally. For instance, in our drug refill example $\Omega_t = \{H_t, \tau\}$ where H_t and τ indicate a patient's health status (Good or Not), and the patient's level of time-inconsistency (Patience or Impatience), respectively. $W_t = (Y_t, X_t)$ where Y_t a binary choice variable of compliance or noncompliance, X_t is cost of the prescription, being

[11] This assumption needs to be specified carefully, i.e., which part of $H_t \times \tau$ affects Y_t or/and X_t. It enables us to identify not only the transition process but also $|\mathcal{H}|$ and $|\Gamma|$.

low or high. Then the monotonicity can be natural:

$$\begin{aligned} & P(\text{Compliance, Lowcost}|\text{Good,Patience}) \\ > & P(\text{Compliance,Lowcost}|\text{Good,Impatience}) \\ > & P(\text{Compliance,Lowcost}|\text{Not,Patience}) \\ > & P(\text{Compliance,Lowcost}|\text{Not,Impatience}). \end{aligned}$$

Assumption 5 identifies the cardinality $|\mathcal{H}| \times |\Gamma|$, while still allowing different combinations of H_t and τ. For instance, suppose we identify $|\mathcal{H}| \times |\Gamma| = 4$, which implies that there are three possible scenarios: four categories of health status and homogeneous present bias ($|\mathcal{H}| = 4, |\Gamma| = 1$), homogeneous health status but four different present bias levels ($|\mathcal{H}| = 1, |\Gamma| = 4$), and the specification above about the health status and present bias ($|\mathcal{H}| = 2, |\Gamma| = 2$). To further identify $|\mathcal{H}|$ and $|\Gamma|$, we need part (b) of the assumption above. Without loss of generality, suppose $f_{X_t|H_t}$ is monotonic in H_t but does not depend on τ, then $|\mathcal{H}|$ and $|\Gamma|$ are both identified.

Having presented the necessary assumptions for our identification, we show the identification of CCP and transition process by first providing a decomposition of the Markov kernel $L_{W_t,\Omega_t|W_{t-1},\Omega_{t-1}}$.

Lemma 2. *The Markov law of motion $L_{w_t,\Omega_t|w_{t-1},\Omega_{t-1}}$ can be decomposed as follows.*[12]

$$L_{w_t,\Omega_t|w_{t-1},\Omega_{t-1}} = L^{-1}_{W_{t+1}|w_t,\Omega_t} L_{W_{t+1},w_t,w_{t-1},W_{t-2}} L^{-1}_{W_t,w_{t-1},W_{t-2}} L_{W_t|w_{t-1},\Omega_{t-1}}.$$

Proof See Appendix. ■

This lemma implies that the identification of the Markov law of motion $L_{w_t,\Omega_t|w_{t-1},\Omega_{t-1}}$ boils down to that of $L_{W_{t+1}|w_t,\Omega_t}$ and $L_{W_t|w_{t-1},\Omega_{t-1}}$. Under the assumption of stationarity, we have $L_{W_{t+1}|w_t,\Omega_t} = L_{W_t|w_{t-1},\Omega_{t-1}}$. Thus it is sufficient to identify $L_{W_{t+1}|w_t,\Omega_t}$ in order to achieve identification of $L_{W_t,\Omega_t|W_{t-1},\Omega_{t-1}}$. We present the identifying result of $L_{W_{t+1}|w_t,\Omega_t}$ in the following lemma.

Lemma 3. *The transition process $L_{W_{t+1}|w_t,\Omega_t}$ is identified from the joint distribution $f_{W_{t+1},w_t,w_{t-1},W_{t-2}}$. In the stationary case, $L_{W_t|w_{t-1},\Omega_{t-1}}$ is identified using the same information.*

Proof See Appendix. ■

[12] In this lemma, the invertibility of the matrix $L_{W_t,w_{t-1},W_{t-2}}$ is assumed. $L_{W_t,w_{t-1},W_{t-2}} = L_{W_t|w_{t-1},\Omega_{t-1}} L_{\Omega_{t-1},w_{t-1},W_{t-2}}$. Assumption 5 implies that $L_{W_t|w_{t-1},\Omega_{t-1}}$ is of full rank.

The results of Lemma 2 and 3 identify the Markov kernel $f_{Y_t,X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}}$. As a consequence, the transition process of Ω, $f_{\Omega_t|X_{t-1},Y_{t-1},\Omega_{t-1}}$ can be obtained by summing out observables X_t, Y_t,

$$f_{\Omega_t|X_{t-1},Y_{t-1},\Omega_{t-1}} = \sum_{Y_t}\sum_{X_t} f_{Y_t,X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}}$$

On the other hand, summing out Y_t leads to joint transition process of (X_t, Ω_t):

$$\begin{aligned} f_{X_t,\Omega_t|X_{t-1},Y_{t-1},\Omega_{t-1}} &= \sum_{Y_t} f_{Y_t,X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}} \\ &= f_{X_t|X_{t-1},Y_{t-1},\Omega_t} f_{\Omega_t|X_{t-1},Y_{t-1},\Omega_{t-1}}. \end{aligned}$$

Hence the transition process of X_t, $f_{X_t|X_{t-1},Y_{t-1},\Omega_t}$ is also identified. At last, the conditional choice probability $f_{Y_t|X_t,\Omega_t}$ is determined by the identification of $f_{Y_t,X_t,\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}}$, $f_{\Omega_t|X_{t-1},Y_{t-1},\Omega_{t-1}}$, $f_{X_t|X_{t-1},Y_{t-1},\Omega_{t-1}}$ and the result of decomposition in Lemma 1. The results of identification are summarized in the following proposition.

Proposition 1. *Under assumptions 1, 2 3, 4, 5, 6, and 7, the choice probability $f(Y_t|X_t, \Omega_t)$, and the transition process of state variables $f(X_t|\Omega_t, Y_{t-1}, X_{t-1})$, and $f(\Omega_t|Y_{t-1}, X_{t-1}, \Omega_{t-1})$ are all identified from the four periods of data, $\{X_s, Y_s\}_{s=t-2}^{t+1}$.*

Since CCP-based estimation (Hotz, Miller, Sanders, and Smith (1994); Bajari, Benkard, and Levin (2007)) requires forward simulation of the Markov process $\{Y_s, X_s, \Omega_s\}$, $s = t-2, t-1, ...$ starting from some initial conditional $f_{Y_{t-2},X_{t-2},\Omega_{t-2}}$. When there are unobserved state variables, this becomes difficult because Ω_{t-2} is not observed. Fortunately, our identification results in proposition 1, allow us to identify the marginal densities $f_{Y_{t-2},X_{t-2},\Omega_{t-2}}$.

Corollary 1. *Under the assumptions 1-7, the observed joint density $f_{Y_{t+1},X_{t+1},X_t,Y_t,Y_{t-1},X_{t-1},Y_{t-2},X_{t-2}}$ uniquely determines $f_{Y_{t-2},X_{t-2},\Omega_{t-2}}$ for $t \in \{3, 4, ..., T-1\}$.*

Proof See Appendix. ■

In the first step of identification we discussed above, we are able to identify the conditional choice probabilities, state transition probabilities for both observables and unobservable, and the initial state. All these results will serve as the inputs for the next step's identification.

Remark 1. For each type of agents, we obtain their type-specific CCP, $\Pr(Y_t = j|X_t, H_t; \tau)$ and transition process $\Pr(X_t|H_t, X_{t-1}; j, \tau)$, $\Pr(H_t|X_{t-1}, H_{t-1}; j, \tau)$ which depend on the structural parameters of the model: type specific utility functions $u_j(X, H; \tau)$ and the discounting factors β_τ and δ. Given the identified type-specific CCP $\Pr(Y_t = j|X_t, H_t; \tau)$, type probability $\boldsymbol{\pi}$ can be directly identified from the relationship between observed $\Pr(Y_t|X_t)$ and $\Pr(Y_t|X_t, H_t; \tau)$.

Remark 2. When state variables are continuous, we can employ a similar procedure to conduct the identification. Not surprisingly, we require $W_t = (Y_t, X_t)$ to be continuous to achieve the identification. In the identification procedure above, all the matrices need to be substituted by the corresponding linear operators, and the decomposition of matrices will also be mimicked as spectrum decomposition of linear operators. The details can be found in Hu and Shum (2010).

Remark 3. The first step of identification we proposed above can also be applied to the case of finite horizon, where everything up till Lemma 3 keeps the same. Even though Lemma 3 does not hold, we can mimic the proof there and show that $L_{W_t|w_{t-1},\Omega_{t-1}}$ can be recovered from the joint distribution $f_{W_t,W_{t-1},W_{t-2},W_{t-3}}$. Thus one more period of observables is required for the identification of the Markov kernel, i.e., $f_{W_{t+1},w_t,w_{t-1},W_{t-2},W_{t-3}}$ determines $L_{W_{t+1}|w_t,\Omega_t}$, $L_{W_t|w_{t-1},\Omega_{t-1}}$ then $L_{W_t,\Omega_t|W_{t-1},\Omega_{t-1}}$. Consequently, the CCP and transition processes are similarly identified without additional assumptions. ■

3.2 Identification of value functions $Z_j(X,\Omega)$ and $V_j(X,\Omega)$

In the previous section, we show the identification of the conditional choice probabilities, $f_{Y_t|X_t,\Omega_t}$, and the state transition probabilities, $f_{X_t|\Omega_t,Y_{t-1},X_{t-1}}$ and $f_{\Omega_t|Y_{t-1},X_{t-1},\Omega_{t-1}}$. Consider that the unobserved variable is two dimensional, i.e., $\Omega_t = (H_t, \tau)$, the results of identification imply that CCP and transition process for each type of decision makers are recovered from their choice and observed variables. In this step of identification, we show that the choice-specific value function (current-period and long-run) and the utility for each type can also be identified for given β_τ and δ. Since the identification procedure across types is the same we neglect the type for ease of notation in the following identification.

We first denote the choice probability vector $\boldsymbol{p}(X,\Omega) = (p_1(X,\Omega), ..., p_J(X,\Omega))$, where

$$\begin{aligned} p_j(X_t,\Omega_t) &= \Pr\Big\{ j = \text{argmax}_{k\in\mathcal{J}}\big(Z_k(X_t,\Omega_t) + \varepsilon_{kt}\big)\Big\} \\ &= \frac{\exp\big(Z_j(X,\Omega)\big)}{\sum\limits_{k\in\mathcal{J}} \exp\big(Z_k(X,\Omega)\big)}. \end{aligned}$$

Since the agents observe both X and Ω, the results of one-to-one mapping between CCP and value functions in Hotz and Miller (1993) still hold in our setting, i.e., there is a one-to-one mapping Q which maps the choice probability vector to the J vector $\boldsymbol{Z}(X,\Omega) = \big(\Delta Z_1(X,\Omega), ..., \Delta Z_J(X,\Omega)\big)$, where $\Delta Z_j(X,\Omega) = Z_j(X,\Omega) - Z_0(X,\Omega)$ and $Z_0(X,\Omega)$ is the choice-specific value function for the reference alternative $j = 0$. Because the choice probability vector is identified in the first step and the mapping Q is determined by the known joint distribution $G(\cdot)$ and can be inverted, the value function differences $\Delta Z_1(X,\Omega), ..., \Delta Z_J(X,\Omega)$ are identified, i.e.,

$$\Delta Z_j(X,\Omega) = Z_j(X,\Omega) - Z_0(X,\Omega) = Q_j\big(p(X,\Omega); G(\cdot)\big). \tag{11}$$

Different from the dynamic discrete models where agents is time-consistent, the choice probabilities $p(X,\Omega)$ in our model are jointly determined by value functions $V(\cdot)$ and $Z(\cdot)$. Therefore, to recover $Z_0(\cdot)$ we have to identify the perceived long-run value functions $V(\cdot)$. For this purpose, we employ the relationship between $V(\cdot)$ and $Z(\cdot)$ described in equation (8), i.e., for all X and Ω,

$$\begin{aligned} V(X,\Omega) &= Z_0(X,\Omega) + \ln\left(\sum_{j\in\mathcal{J}} \exp\left(\Delta Z_j(X,\Omega)\right)\right) \\ &\quad +(1-\beta)\delta \sum_{j\in\mathcal{J}} p_j(X,\Omega) \sum_{X',\Omega'} V(X',\Omega') f(X',\Omega'|X,\Omega;j) \end{aligned}$$

where $\Delta Z_j(X,\Omega), p_j(X,\Omega)$,and $f(X',\Omega'|X,\Omega;j)$ are all identified. The unknowns in the above equation are $V(X,\Omega)$ and $Z_0(X,\Omega)$ for given β and δ. The equation above allows us to express $V(X,\Omega)$ which contains $|\mathcal{X}| * |\mathcal{H}| * |\Gamma|$ unknowns as a function of knowns and $Z_0(X,\Omega)$. These $|\mathcal{X}| * |\mathcal{H}| * |\Gamma|$ unknowns can be classified into $|\Gamma|$ (number of "types") "groups" according to the value of $\beta \in \{\beta_1, \beta_2, ..., \beta_{|\Gamma|}\}$. For ease of notation, all the equations in this subsection are β specific, i.e., for a given β in $\{\beta_1, \beta_2, ..., \beta_{|\Gamma|}\}$ and we neglect the corresponding subscript of β.

$$\boldsymbol{V} = \boldsymbol{Z}_0 + \boldsymbol{A} + (1-\beta)\delta \boldsymbol{P}\boldsymbol{F}\boldsymbol{V},$$

where $\boldsymbol{V}$, $\boldsymbol{Z}_0$ and $\boldsymbol{A}$ are column vectors with dimension $(|\mathcal{X}| * |\mathcal{H}|) \times 1$ for any given $\beta \in \{\beta_1, \beta_2, ..., \beta_{|\Gamma|}\}$, which are all defined similarly as follows.

$$\boldsymbol{V} = \begin{bmatrix} V(1,1) & ... & V(1,H) & V(2,1) & ... & V(2,H) & ... & V(X,H) \end{bmatrix}^T.$$

$\boldsymbol{P}$ and $\boldsymbol{F}$ are matrices that describe CCP, transition process of Ω and X, with dimension being $(|\mathcal{X}| * |\mathcal{H}|) \times (J+1) * (|\mathcal{X}| * |\mathcal{H}|)$, $(J+1) * (|\mathcal{X}| * |\mathcal{H}|) \times (|\mathcal{X}| * |\mathcal{H}|)$, respectively.

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_0 & \boldsymbol{P}_1 & ... & \boldsymbol{P}_J \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_0 & \boldsymbol{F}_1 & ... & \boldsymbol{F}_J \end{bmatrix}^T,$$

where

$$\boldsymbol{P}_j = \begin{bmatrix} P_j(1,1) & 0 & ... & 0 \\ 0 & P_j(1,2) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & P_j(X,H) \end{bmatrix},$$

is a diagonal matrix with dimension $(|\mathcal{X}| * |\mathcal{H}|) \times (|\mathcal{X}| * |\mathcal{H})$ and

$$\boldsymbol{F}_j = \begin{bmatrix} f(1,1|1,1;j) & ... & f(1,H|1,1;j) & f(2,1|1,1;j) & ... & f(2,H|1,1;j) & ... & f(X,H|1,1;j) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f(1,1|1,H;j) & ... & f(1,H|1,H;j) & f(2,1|1,H;j) & ... & f(2,H|1,H;j) & ... & f(X,H|1,H;j) \\ f(1,1|2,1;j) & ... & f(1,H|2,1;j) & f(2,1|2,1;j) & ... & f(2,H|2,1;j) & ... & f(X,H|2,1;j) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f(1,1|X,H;j) & ... & f(1,H|X,H;j) & f(2,1|X,H;j) & ... & f(2,H|X,H;j) & ... & f(X,H|X,H;j) \end{bmatrix}$$

is of dimension $(|\mathcal{X}| * |\mathcal{H}|) \times (|\mathcal{X}| * |\mathcal{H}|)$. Consequently, $\boldsymbol{V}$ can be solved as a function of knowns and $\boldsymbol{Z}_0$,

$$\boldsymbol{V} = \Big[\boldsymbol{I} - (1-\beta)\delta\boldsymbol{P}\boldsymbol{F}\Big]^{-1}(\boldsymbol{Z}_0 + \boldsymbol{A}). \tag{12}$$

Further recall that Eq.(4) can also be written as

$$Z_j(X,H,\beta) = \widetilde{u}_j(X,H) + \beta\delta \sum_{X',H'} V(X^{'},H^{'}) f(X^{'},H^{'}|X,H;Y=j).$$

For the reference alternative $j = 0$, we have

$$\begin{aligned} Z_0(X,H,\beta) &= \widetilde{u}_0(X,H) + \beta\delta \sum_{X',H'} V(X^{'},H^{'}) f(X^{'},H^{'}|X,H;Y=0) \\ &= \beta\delta \sum_{X',H'} V(X^{'},H^{'}) f(X^{'},H^{'}|X,H;Y=0), \end{aligned} \tag{13}$$

where the utility for the reference alternative $u_0(X,H)$ is normalized to zero for any X and H. In matrix form

$$\boldsymbol{Z}_0 = \beta\delta\boldsymbol{F}_0\,\boldsymbol{V} = \beta\delta\boldsymbol{F}_0\Big[\boldsymbol{I} - (1-\beta)\delta\boldsymbol{P}\boldsymbol{F}\Big]^{-1}(\boldsymbol{Z}_0 + \boldsymbol{A}).$$

The equation above implies a solution to $\boldsymbol{Z}_0$,

$$\boldsymbol{Z}_0 = \Big\{I - \beta\delta\boldsymbol{F}_0\Big[\boldsymbol{I} - (1-\beta)\delta\boldsymbol{P}\boldsymbol{F}\Big]^{-1}\Big\}^{-1}\beta\delta\boldsymbol{F}_0\Big[\boldsymbol{I} - (1-\beta)\delta\boldsymbol{P}\boldsymbol{F}\Big]^{-1}\boldsymbol{A}. \tag{14}$$

As a result, $Z_j(X,H,\beta) = Z_0(X,H,\beta) + \Delta Z_j(X,H,\beta)$ is also identified by combining equation (11) and equation (14). We summarize the identification results in this step in the following proposition:

Proposition 2. *Suppose assumptions 1-7 hold, then for given discount factors* $(\boldsymbol{\beta},\delta)$*, each type of agents' value functions* $V(\cdot)$ *and* $Z(\cdot)$ *are identified.*

3.3 Identification of discounting factors $(\boldsymbol{\beta}_\tau, \delta)$

Now we discuss conditions under which $(\boldsymbol{\beta}, \delta)$ can be identified. Magnac and Thesmar (2002) proposed identification of the exponential discounting factor δ (there is no present-bias factor β in their setting) using exclusion variables that affect both the current period utility and the state transition but have no impacts on the "current value function" which is not a model primitive. The existence of such exclusion variables is difficult to argue since it requires knowledge of the solution of the dynamic discrete choice model. Following a similar idea, Fang and Wang (2012) employ exclusion variables, which can be observed, that do not affect current utilities but affect future payoffs through its impact on the state transitions to show the identification of $\boldsymbol{\beta}$ and δ. Acknowledging the difficulty to find an exclusive variable, we explore the feature of incorporating the heterogenous present-bias β instead of looking for some observed variables to serve as restrictions for identification. Specifically, from the specification of the current period utility function in assumption 1, agents of different types (captured by different present-bias β) have the same instantaneous utility for given state variables X and H but they discount future differently, which may lead to their distinct choices. Recall that for given $\boldsymbol{\beta}$ and δ the per-period utility functions $\widetilde{u}_j(X, H), j \in \mathcal{J}, j \neq 0$ can be recovered from the current-period choice specific value functions $Z_j(X, H, \beta)$ as

$$\widetilde{u}_j(X, H) = Z_j(X, H, \beta) - \beta\delta \sum_{X'} \sum_{H'} V(X', H') f(X', H'|X, H; Y = j). \tag{15}$$

Suppose we investigate two types with present-bias factor β_k and β_l, respectively. Then for given state variables X, H and choice $Y = j$, we have

$$\begin{aligned} & Z_j^k(X, H) - \beta_k\delta \sum_{X'} \sum_{H'} V^k(X', H') f^k(X', H'|X, H; Y = j) \\ = \ & Z_j^l(X, H) - \beta_l\delta \sum_{X'} \sum_{H'} V^l(X', H') f^l(X', H'|X, H; Y = j), \end{aligned}$$

where Z_j^m, V^m, and f^m are for the agents with present-bias factor $\beta_m, m = k, l$. Rewriting the relationship above in matrix form and plugging the identified Z, ΔZ and P, F, A, we have

$$\begin{aligned} & \boldsymbol{Z}_0^k + \boldsymbol{\Delta Z}_j^k + \beta_k\delta \boldsymbol{F}_j^k \Big[\boldsymbol{I} - (1 - \beta_k)\delta \boldsymbol{P^k F^k}\Big]^{-1} (\boldsymbol{Z}_0^k + \boldsymbol{A}^k) \\ = \ & \boldsymbol{Z}_0^l + \boldsymbol{\Delta Z}_j^l + \beta_l\delta \boldsymbol{F}_j^l \Big[\boldsymbol{I} - (1 - \beta_l)\delta \boldsymbol{P^l F^l}\Big]^{-1} (\boldsymbol{Z}_0^l + \boldsymbol{A}^l), \end{aligned} \tag{16}$$

where $k, l \in 1, 2, ..., |\Gamma|, k \neq l$, all the objectives with superscript k and l are for $\tau = \beta_k$ and $\tau = \beta_l$, respectively, and Z_0^k and Z_0^l are determined by (14). In the equation above, $\boldsymbol{\Delta Z}, \boldsymbol{P}, \boldsymbol{F}, \boldsymbol{A}$ are directly identified from data, which permits us to treat them as given in this step of identification even though they are type-specific. Therefore, the $1+|\Gamma|$ unknowns $\beta_1, \beta_2, ..., \beta_{|\Gamma|}$ and δ enter the equation above explicitly and there are $J \times |\mathcal{X}| * |\mathcal{H}| * |\Gamma| * (|\Gamma|-1)/2$ nonlinear equations in total. By a counting exercise, the identification of $\beta_1, \beta_2, ..., \beta_{|\Gamma|}$ and δ requires

$$J * |\mathcal{X}| * |\mathcal{H}| * |\Gamma| * (|\Gamma|-1)/2 \geq |\Gamma| + 1. \tag{17}$$

Satisfying such a restriction does not suffice the identification. We need the following assumption to guarantee the existence of solution to Eq.(16)

Assumption 8. *The Jacobian matrix of Eq.(16) is full rank at the true value of* $\beta_1, \beta_2, ..., \beta_{|\Gamma|}$ *and* δ*.*

Eq.(17) and assumption 8 ensure existence and local uniqueness of a solution $(\beta_1, \beta_2, ..., \beta_{|\Gamma|}, \delta)$.[13]

Remark. Employing the restrictions on the model imposed by the existence of unobserved heterogeneity overcomes the difficulty to find exclusive variables required as in Magnac and Thesmar (2002) and Fang and Wang (2012). Moreover, when the exclusion variables are observed and they are an explicit part of the model in analysis, a different subset of exclusion variables imply different models and the results are not directly comparable. However, our framework does not suffer this problem. ■

3.4 Identification of instantaneous utility function $u_j(X, H)$

It is straightforward to show the identification of instantaneous utility function according to (15), which can be expressed in matrix form as

$$\begin{aligned} \widetilde{\boldsymbol{u}}_j &= \boldsymbol{Z}_j - \beta\delta\boldsymbol{F}_j\boldsymbol{V} \\ &= \boldsymbol{Z}_0 + \boldsymbol{\Delta Z}_j + \beta\delta\boldsymbol{F}_j\Big[\boldsymbol{I} - (1-\beta)\delta\boldsymbol{PF}\Big]^{-1}(\boldsymbol{Z}_0 + \boldsymbol{A}), \end{aligned} \tag{18}$$

where everything on the right-hand side are identified in previous steps and

$$\widetilde{\boldsymbol{u}}_j = \Big[\ \widetilde{u}_j(1,1) \quad ... \quad \widetilde{u}_j(1,H) \quad \widetilde{u}_j(2,1) \quad ... \quad \widetilde{u}_j(2,H) \quad ... \quad \widetilde{u}_j(X,H)\ \Big]^T.$$

Therefore, the instantaneous utility function is identified for any $x \in |\mathcal{X}|, h \in |\mathcal{H}|$.

Remark. Equation (18) shows explicitly how introducing unobserved variables Ω may complicate the computation of utility functions (nonparametrically): the computation in equation (18)

[13] Instead of verifying assumption 8, we show it can hold by a numerical example where $|\mathcal{Y}| = |\mathcal{X}| = |\mathcal{H}| = |\Gamma| = 2$.

involves the inverse of square matrices with dimension $(|\mathcal{X}| * |\mathcal{H}|) \times (|\mathcal{X}| * |\mathcal{H}|)$. However, the inverse only needs to taken for square matrices with dimension $\mathcal{X} \times \mathcal{X}$ if there are only observed state variables. To simplify the estimation and obtain more accurate results, we will consider a semi-parametric estimation procedure to recover the parameters of agents' preference.■

The final identification results of the model is summarized in the following theorem

Theorem 1. *Under assumptions 1-8, all the model parameters* $(\widetilde{\boldsymbol{u}}, \boldsymbol{\beta}, \delta)$ *are identified.*

Remark. We need additional assumptions to extend our procedure of identification to dynamic discrete models with finite horizon and non-stationary state transitions where agents are potentially time-inconsistent. This can be seen from the first step of identification on CCP and state transition which requires data from five periods. For example, if we have five periods' of data $\{Y_t, X_t\}_{t=1}^{5}$, then we are only able to identify the CCP and state transition from period 3 to period 4. However, the backward induction requires the information for all the periods. Nevertheless, if we have additional information on the unobserved variables, e.g., a proxy, then we may still be able to identify the model using the results in measurement errors.

4 Estimation

Estimation of dynamic discrete choice models with unobserved heterogeneity or serially correlated unobservables has been addressed in a growing literature during the past a few years. Imai, Jain, and Ching (2009) and Norets (2009) consider a Bayesian approach of estimation. Gallant, Hong, and Khwaja (2008) and Blevins (2011) develop simulation estimators for dynamic games with serially-correlated unobservables, utilizing state-of-the-art recursive importance sampling ("particle filtering") techniques. Arcidiacono and Miller (2011) provide an expectation-maximization (EM) algorithm for estimating dynamic games with unobserved heterogeneity. Our estimating strategy distinguishes from the existing ones in that it depends on the nonparametric identification results we proposed in previous sections. More specifically, we estimate the model in two stages: first, we nonparmetrically estimate the CCP and transition process following the identification procedure. Second, using the results from the first stage as input, we recover the discount factors and parameters in agents' preference by maximum likelihood.

The first stage. This step of estimation follows directly from the constructive identification procedure. The Markov kernel $f_{w_t,\Omega_t|w_{t-1},\Omega_{t-1}}$ is identified as

$$L_{w_t,\Omega_t|w_{t-1},\Omega_{t-1}} = L^{-1}_{W_{t+1}|w_t,\Omega_t} L_{W_{t+1},w_t,w_{t-1},W_{t-2}} L^{-1}_{W_t,w_{t-1},W_{t-2}} L_{W_t|w_{t-1},\Omega_{t-1}},$$

where the matrix $L_{W_{t+1}|w_t,\Omega_t}$ can be recovered from a non-stochastic mapping $\phi(\cdot)$ from a square matrix to its eigenvectors

$$L_{W_{t+1}|w_t,\Omega_t} = \phi(\mathbf{A}\mathbf{B}^{-1}),$$

with

$$\begin{aligned}
\mathbf{A} &\equiv L_{W_{t+1},w_t,w_{t-1},W_{t-2}} L^{-1}_{W_{t+1},\widetilde{w}_t,w_{t-1},W_{t-2}}, \\
\mathbf{B} &\equiv L_{W_{t+1},w_t,\widetilde{w}_{t-1},W_{t-2}} L^{-1}_{W_{t+1},\widetilde{w}_t,\widetilde{w}_{t-1},W_{t-2}}.
\end{aligned}$$

Next, the transition process of Ω_t is obtained from $L_{w_t,\Omega_t|w_{t-1},\Omega_{t-1}}$ as

$$L_{\Omega_t|x_{t-1},y_{t-1},\Omega_{t-1}} = \sum_{x_t\in\mathcal{X}}\sum_{y_t\in\mathcal{J}} L_{y_t,x_t,\Omega_t|y_{t-1},x_{t-1},\Omega_{t-1}}. \tag{19}$$

On the other hand, integrating out Y_t leads to joint transition process of (X_t, Ω_t):

$$\begin{aligned}
L_{x_t,\Omega_t|x_{t-1},y_{t-1},\Omega_{t-1}} &= \sum_{y_t\in\mathcal{J}} L_{y_t,x_t,\Omega_t|y_{t-1},x_{t-1},\Omega_{t-1}} \\
&= L_{x_t|x_{t-1},y_{t-1},\Omega_t} L_{\Omega_t|x_{t-1},y_{t-1},\Omega_{t-1}}.
\end{aligned} \tag{20}$$

Combining this relationship with the identified transition process of Ω_t, we recover $L_{x_t|x_{t-1},y_{t-1},\Omega_t}$. Further recall that

$$f_{y_t,x_t,\Omega_t|y_{t-1},x_{t-1},\Omega_{t-1}} = \underbrace{f_{y_t|x_t,\Omega_t}}_{\text{CCP}} \underbrace{f_{x_t|y_{t-1},x_{t-1},\Omega_t}}_{X_t \text{ law of motion}} \underbrace{f_{\Omega_t|y_{t-1},x_{t-1},\Omega_{t-1}}}_{\Omega_t \text{ law of motion}}. \tag{21}$$

Finally, the conditional choice probability $f_{Y_t|X_t,\Omega_t}$ is identified.

Follow the identification procedure above, the objectives can be straightforwardly estimated from the data $\{Y_t, X_t\}$. First, the estimate for the Markov kernel, $\widehat{L}_{W_t,\Omega_t|W_{t-1},\Omega_{t-1}}$ is

$$\widehat{L}_{w_t,\Omega_t|w_{t-1},\Omega_{t-1}} = \widehat{L}^{-1}_{W_{t+1}|w_t,\Omega_t} \widehat{L}_{W_{t+1},w_t,w_{t-1},W_{t-2}} \widehat{L}^{-1}_{W_t,w_{t-1},W_{t-2}} \widehat{L}_{W_{t+1}|w_t,\Omega_t}, \tag{22}$$

where

$$\begin{aligned}
\left(\widehat{L}_{W_{t+1},w_t,w_{t-1},W_{t-2}}\right)_{i,j} &= \left(\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}(W_{t+1}=i, w_t, w_{t-1}, W_{t-2}=j)\right)_{i,j}. \\
\widehat{L}_{W_{t+1}|w_t,\Omega_t} &= \phi(\widehat{\mathbf{A}}\widehat{\mathbf{B}}^{-1}),
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\mathbf{A}} &= \widehat{L}_{W_{t+1},w_t,w_{t-1},W_{t-2}} \widehat{L}^{-1}_{W_{t+1},\widetilde{w}_t,w_{t-1},W_{t-2}}, \\
\widehat{\mathbf{B}} &= \widehat{L}_{W_{t+1},\widetilde{w}_t,\widetilde{w}_{t-1},W_{t-2}} \widehat{L}^{-1}_{W_{t+1},w_t,\widetilde{w}_{t-1},W_{t-2}}.
\end{aligned}$$

Consequently, the transition process of Ω is estimated as

$$\widehat{L}_{\Omega_t|x_{t-1},y_{t-1},\Omega_{t-1}} = \sum_{x_t\in\mathcal{X}}\sum_{y_t\in\mathcal{J}} \widehat{L}_{y_t,x_t,\Omega_t|y_{t-1},x_{t-1},\Omega_{t-1}}. \tag{23}$$

Similarly, the transition process of X_t can be identified from $\widehat{L}_{y_t,x_t,\Omega_t|y_{t-1},x_{t-1},\Omega_{t-1}}$ based on the decomposition equation (20),

$$\widehat{L}_{x_t|x_{t-1},y_{t-1},\Omega_{t-1}} = \widehat{L}^{-1}_{\Omega_t|x_{t-1},y_{t-1},\Omega_{t-1}} \sum_{y_t\in\mathcal{J}} \widehat{L}_{y_t,x_t,\Omega_t|y_{t-1},x_{t-1},\Omega_{t-1}}. \tag{24}$$

Finally, the CCP $\widehat{L}_{y_t|x_t,\Omega_t}$ is estimated as:

$$\widehat{L}_{y_t|x_t,\Omega_t} = \widehat{L}_{y_t,x_t,\Omega_t|y_{t-1},x_{t-1},\Omega_{t-1}} \widehat{L}^{-1}_{\Omega_t|x_{t-1},y_{t-1},\Omega_{t-1}} \widehat{L}^{-1}_{x_t|x_{t-1},y_{t-1},\Omega_{t-1}}. \tag{25}$$

The second stage. In this step of estimation, the parameters $\Theta = (\boldsymbol{\theta};\boldsymbol{\beta},\delta;\boldsymbol{\pi})$ are estimated using MLE, where $\boldsymbol{\theta}$ includes the parameters in the utility function, $\boldsymbol{\beta}$ and δ are discounting factors, $\boldsymbol{\pi} = (\pi_1,\pi_2,...,\pi_m)$ are type probabilities.

The type specific likelihood for an agent with discounting factors (β_m,δ) to choose Y_t conditional on state variables X_t, H_t, type $\tau = m$ is

$$\begin{aligned} & l_t(Y_t|X_t,H_t,\tau,\widehat{p};\Theta) \\ = & \frac{Y_{0t} + Y_{1t}\exp\Big[u_1(X_t,\tau,\Theta_1) - \beta_\tau\delta\sum_{H'}\sum_{X'}\ln[\widehat{p}_0(X',H',\tau)][\widehat{f}_1(X',H'|X,H,\tau) - \widehat{f}_0(X',H'|X,H,\tau)]\Big]}{1+\exp\Big[u_1(X_t,\tau,\Theta_1) - \beta_\tau\delta\sum_{H'}\sum_{X'}\ln[\widehat{p}_0(X',H',\tau)][\widehat{f}_1(X',H'|X,H,\tau) - \widehat{f}_0(X',H'|X,H,\tau)]\Big]}, \end{aligned}$$

where $Y_{0t} = 0, Y_{1t} = 1, \widehat{p}_0(X',H',\tau) \equiv f(Y' = 0|X',H',\tau)$ is the CCP estimated in the first step. $\widehat{f}_j(X',H'|X,H,\tau) = \widehat{f}(X'|X,H',\tau;Y=j) \times \widehat{f}(H'|X,H,\tau;Y=j), j = 0,1$, are state transition estimated in the first step, too.

Maximizing the log likelihood of the observed data to solve both Θ and π:

$$(\widehat{\Theta},\widehat{\pi}) = \arg\max_{\Theta,\pi}\sum_{i=1}^{N}\ln\Big[\sum_{\tau}\pi(\tau)\prod_{t=1}^{T} l_{nt}(Y_{nt}|X_{nt},H_{nt},\tau,\widehat{p};\Theta)\Big]$$

5 Monte Carlo Experiment

In this section, we provide Monte Carlo evidence of the performance for the proposed estimation in the last section. In order to focus on the estimation of the time preference parameters, we provide simple parameterizations for per-period utilities, imposing the restriction that they are common across types and linear in the unknown parameters.

We consider agents with a linear utility function who live for infinite horizon make a binary choice decision, $Y_t \in \mathcal{J} = \{0,1\}$. The observed state variable X, unobserved state variable H and heterogeneity τ are all binary, $\Omega = \{(0,0),(0,1),(1,0),(1,1)\} \equiv \{1,2,3,4\}$. The instantaneous payoff functions under two choice are parameterized as follows.

$$u_{jt} = \begin{cases} \gamma_0 + \gamma_1 x_t + \gamma_2 h_t + \varepsilon_{1t} & \text{if } j = 1, \\ \varepsilon_{0t} & \text{if } j = 0, \end{cases}$$

where the unobserved variable H_t is time-variant, affecting the transition process of X but not the instantaneous payoff, where ε_{0t}, and ε_{1t} are i.i.d. extreme value distributed with unit variance.

Agents are of two types with probability π_1 and π_2 and present bias factors $\beta_1 = 1$, $\beta_2 = 0.7$ respectively, and the standard discount factor $\delta = 0.9$. The transition process of X and H are

$$\begin{aligned} &\Pr(X' = 1|Y, X, H', \tau) = \frac{\exp(\varphi(X,Y,H',\tau))}{1+\exp(\varphi(X,Y,H',\tau))}, \\ &\Pr(H' = 1|Y, X, H, \tau) = \frac{\exp(\psi(X,H,Y,\tau))}{1+\exp(\psi(X,H,Y,\tau))}, \\ &\varphi(X, H', \tau) = \alpha_0 + \alpha_1 X + \alpha_2 Y + \alpha_3 H' + \alpha_4 \tau, \\ &\psi(X, H, \tau) = \eta_0 + \eta_1 X + \eta_2 Y + \eta_3 H + \eta_4 \tau. \end{aligned} \tag{26}$$

6 An Empirical Application: Medication Compliance

In this section, we apply our proposed identification and estimation strategy to consumers' medication compliance problem during prescription drug refilling process. We show how the seemingly inconsistent time preference and the resultant preference reversal cause some patients' noncompliance behavior often observed. By providing corresponding policy intervention we could potentially improve consumer welfare and their wellbeing.

6.1 Background

Consumers' non-compliance behavior has been a huge concern for both physicians and policy-makers. Existing research has focused on what drug or patient characteristics influence patients' compliance behavior while unable to explicitly pin down the underlying mechanism. Our study contributes to this literature by explicitly modeling consumers' inter-temporal behavior of their drug refilling choices while controlling drug or patient characteristics. This provides the potentially important mechanism for physicians, policymakers and even consumers themselves to understand the explicit tradeoff both within individuals and across individuals in the context of patient compliance.

According to the World Health Organization (2005), more than half of all patients do not take their medications as prescribed. The behavior of noncompliance and lack of persistence is at the cost of possible lost of health benefits and increased personal and social economic burden. Various interventions have been implemented to improve compliance and persistence for the purpose of reducing the public health burden of chronic disease. However, the interventions are not effective due to the complicated mechanism of compliance behavior. In this application, we model patients' as agents with heterogeneous discounting factors β and δ who make binary choice on drug refilling and analyze how patients' time-inconsistent preference, together with unobserved attitude of patients toward their physicians help us obtain better understanding of patients' noncompliance behavior.

6.2 Data and descriptive statistics

The data used in our application is compiled from several sources: (1) Catalina Health Resource blinded longitudinal prescription data warehouse (prescription and patient factors), (2) Redbook (price of drugs), First DataBank (brand or generic of drugs), Pharmacy experts (drug performance factors such as side effects, etc.), Verispan PSA (Direct to Consumer, DTC), and the Physician Drug & Diagnosis Audit (drug usage by ICD-9 disease code).

The panel data set contains Diovan (also referred as Valsartan) drug prescription and refill information for 9780 patients. The drug was initially approved in 1996 in the U.S. It is an angiotensin II receptor blocker (ARB) for the following three types of usage: (1) Treatment of hypertension, to lower blood pressure in order to reduce the risk of cardiovascular events, primarily strokes and myocardial infarctions; (2) Treatment of heart failure (NYHA class II-IV);[14] (3) Reduction of cardiovascular mortality in clinically stable patients with left ventricular failure or left ventricular dysfunction following myocardial infarction.[15] Different from some studies (e.g., Chan and Hamilton (2006)) where they focus on new drug experimentation, we study the scenario of a very well-established drug. Both consumers and physicians are well informed about the drugs' effectiveness and side effects (if any).[16]

[14] Diovan has been proved to significantly reduce hospitalization for heart failure.

[15] Recent large-scale medical research also has demonstrated Diovan can significantly reduce the incidence and progression of Alzheimer's disease and dementia besides cardiac-related problem.

[16] Typical side effects include headache, dizziness, and occasional fatigue, which are considered trivial for patients

Table 1: Summary Statistics

Variable	Mean	Standard Deviation	Minimum	Maximum
# of patients	5195	-	-	-
Age	62.268	13.854	18	90
New prescription (=1) or refill	0.331	0.470	0	1
Days supply	34.829	16.064	0	180
# of fills	9.143	2.812	4	27
Compliance	-5.239	18.918	-271	99
Cost	61.737	31.510	3.192	343.240
Cash pay (=1)	.161	.368	0	1
Average wholesale price (unit)	1.669	.128	1.335	2.171
Total days supply/observation period	.859	.229	.150	2.118
Lost days	1.670	.128	1.335	2.171
Compliant frequency (measure of % days compliant)	.880	.185	.146	3.333

In our data, a "fill" refers to a transaction where a patient either refilled an existing prescription or filled a new prescription, for a given drug. The patients are from three cohorts. All patients in a given cohort started therapy in the same month for a drug they had not taken before. Each patient is observed for one year. Cohort 1 begins in June, 2002; Cohort 2 begins in May, 2002; Cohort 3 begins in April, 2002. There are 895 drugs in the database. 5195 of the patients in our sample have more than four (re)fills and these patients are used for our estimation. [17] Table 1 presents summary statistics of our sample in analysis.

6.3 Empirical specifications

Each patient chooses to refill or not to refill a drug for each period. For the purpose of normalization, we set the patient's instantaneous utility to be zero if she chooses to refill a drug on time (compliance). The observed state variables include whether the prescription is new or refilled (new prescription), cost of prescription (cost), insurance status, age, and gender. We specify the instantaneous payoff from non-compliance relative to that of compliance is determined by whether the prescription is new or refill, the cost of the prescription, and whether they are insured. The cost is a continuous variable but a simple analysis shows that it can be naturally discretized into two or three segments. Thus $\boldsymbol{X} = \{Insurance, New\,prescription, Cost\}$. The unobserved

who take Diovan.

[17] Diovan's patent expired in September 2012 which is beyond of our study period.

Table 2: Summary of Variables

Variables	Notation	Explanation	Discrete/Continuous
choice variable	Y	compliance or not	binary (compliance=1)
observed variables	X_1	insurance status	binary (no=1)
	X_2	new prescription	binary (new=1)
	X_3	cost	continuous
unobserved variables	H	health status	discrete
	τ	present-bias	discrete

variables in our analysis are patients' health status, and the level of patients' present-bias, i.e., $\Omega = \{Health\ status,\ Present\ bias\}$. Table 2 summarizes all the state variables in analysis. We employ patients' health status as the exclusion variable because it does not affect the current period utility of refilling.[18] However, it affects transition of $\boldsymbol{X}$ through several possible channels: the patients' health status may directly affect their insurance status, as well as the cost of the prescription and whether they need new prescriptions. We also take into account the unobserved heterogeneity, which is captured by the different present bias β.

Y_t: Binary choice of compliance or not

X_t: New prescription versus refill (X_{1t}, binary variable, 1 indicates new prescription and 0 indicates refill), cost of prescription (X_{2t}, continuous variable), insurance status (X_{3t},binary variable, 1 pay by cash, 0 pay by 3rd part)

H_t: health status, discrete variable

τ: time-inconsistency of patients

$$u_1(\boldsymbol{X},\tau) - u_0(\boldsymbol{X},\tau) = \gamma_0 + \gamma_1 New\ prescription + \gamma_2 Insurance + \gamma_3 Cost + \gamma_4 Health\ status. \quad (27)$$

Justification of assumptions. The assumptions 4-8 imposed for identification are justified for our empirical application of drug refilling. The first-order Markov property implies that the history of all variables is sufficiently summarized by the period $t-1$'s states in terms of decision making for period t. The assumption of limited feedback essentially imposes the restriction that patients' health status determines the characteristics of the prescription and insurance status. The health status realizes after the patient's decision of refill at period $t-1$, therefor it may affect the communication between her and her physician. Consequently, the characteristics of the

[18]Health status may have the first order effects, i.e., affects the current period utility. However, it does not have the second order effects, i.e., health status doesn't enter the difference utility.

prescription for period t may be affected, too. The full rank assumption states that conditional on different realizations of Ω_t, the joint distribution of X and Y are not linearly correlated. To be specific, in our empirical application, $\Omega_t = \{Health\ status, Present\ bias\}$ where both the health status and present bias take two values, Good or Bad, Patience or Impatience, respectively. Then patients must be in one of the four categories $\{Good, Patient\}$, $\{Good, Impatience\}$, $\{Bad, patience\}$ and $\{Bad, Impatience\}$. The joint probability of Y $(Compliance, Not\ compliance)$ and X $(different\ combination\ of\ prescription, insurance, \cos t)$ for any of the four categories cannot be linearly expressed by that of other three categories. To illustrate the assumption of monotonicity, suppose $W_t = (Y_t, X_t)$ where Y_t a binary choice variable of compliance or noncompliance, X_t is cost of the prescription for simplicity, being low or high. Then the monotonicity can be natural:

$$
\begin{aligned}
& P(\text{Compliance, Lowcost}|\text{Good,Patience}) \\
> \; & P(\text{Compliance,Lowcost}|\text{Good,Impatience}) \\
> \; & P(\text{Compliance,Lowcost}|\text{Bad,Patience}) \\
> \; & P(\text{Compliance,Lowcost}|\text{Bad,Impatience}).
\end{aligned}
$$

6.4 Results and Discussion (pending)

We first nonparametrically estimate the number of types from the rank of $L_{W_{t+1},w_t,w_{t-1},W_{t-2}}$, where $W = (Y, X_1, X_2, X_3)$ which takes $2 \times 2 \times 2 \times K$ values after we discretize cost into K possible segments.

7 Conclusions

Build on recent development in measurement-error and dynamic models literature, we establish the identification of heterogeneous hyperbolic discounting factors with unobserved state variables. Our proposed identification procedure employs the independence of current-period utility on heterogeneous present-bias of agents to identify the discount factors. In contrast to the existing identification results on hyperbolic discounting, our method requires less data (comparing with the approach that uses additional belief data) and overcomes the difficulty to find suitable observed variables as exclusion restriction (comparing with the approach using exclusion restrictions). Moreover, the present method allows us to explicitly take into account the impacts of unobserved serially correlated variables on agents' dynamic choices. Based on the identification strategy, we propose a simple two-step estimating algorithm. In the first step, we utilize the constructive identification procedure to recover the CCP and state transition process for both observed and unobserved variables. Then, in the second step, we utilize MLE to estimate agents' heterogeneous present-bias factors, the standard exponential discount factor and other model primitives.

Our current work restricts the analysis to stationary process with infinite horizon which makes sense for the application of the patient compliance issues. Extending our results to non-stationary problem with finite horizon could be interesting. In principle, if we have additional information on the unobserved variables, e.g., some proxy variables like elicited consumer beliefs discussed in Manski (2004), our identification and estimation strategy can be readily used to those setting. Nevertheless, combining both revealed and stated preference data could be one fruitful avenue for the future research.

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Appendix

Proof of Lemma 2 Using matrix notation, for any (w_t, w_{t-1}) we have

$$\begin{aligned} L_{W_{t+1},w_t,w_{t-1},W_{t-2}} &= L_{W_{t+1}|w_t,\Omega_t} L_{w_t,\Omega_t|w_{t-1},W_{t-2}} \\ &= L_{W_{t+1}|w_t,\Omega_t} L_{w_t,\Omega_t|w_{t-1},\Omega_{t-1}} L_{\Omega_{t-1}|w_{t-1},W_{t-2}} \end{aligned}$$

Similarly: $L_{W_t|w_{t-1},W_{t-2}} = L_{W_t|w_{t-1},\Omega_{t-1}} L_{\Omega_{t-1}|w_{t-1},W_{t-2}}$

Manipulating above two equations, we obtain an expression of the Markov law of motion

$$\begin{aligned} &= L^{-1}_{W_{t+1}|w_t,\Omega_t} L_{W_{t+1},w_t,w_{t-1},W_{t-2}} L^{-1}_{\Omega_{t-1}|w_{t-1},W_{t-2}} \\ &= L^{-1}_{W_{t+1}|w_t,\Omega_t} L_{W_{t+1},w_t|w_{t-1},W_{t-2}} L^{-1}_{W_t,w_{t-1},W_{t-2}} L_{W_t|w_{t-1},\Omega_{t-1}} \end{aligned}$$

Proof of Lemma 3 We first express the observed joint distribution $f_{W_{t+1},W_t,W_{t-1},W_{t-2}}$ as our identification objectives as follows:

$$\begin{aligned} f_{W_{t+1},W_t,W_{t-1},W_{t-2}} &= \sum_{\Omega_t} \sum_{\Omega_{t-1}} f_{W_{t+1},W_t,W_{t-1},W_{t-2},\Omega_t,\Omega_{t-1}} \\ &= \sum_{\Omega_t} \sum_{\Omega_{t-1}} f_{W_{t+1}|W_t,\Omega_t} \cdot f_{W_t,\Omega_t|W_{t-1},\Omega_{t-1}} \cdot f_{W_{t-1},W_{t-2},\Omega_{t-1}} \\ &= \sum_{\Omega_t} \sum_{\Omega_{t-1}} f_{W_{t+1}|W_t,\Omega_t} \cdot f_{W_t|W_{t-1},\Omega_t,\Omega_{t-1}} \cdot f_{\Omega_t,\Omega_{t-1},W_{t-1},W_{t-2}} \\ &= \sum_{\Omega_t} f_{W_{t+1}|W_t,\Omega_t} f_{W_t|W_{t-1},\Omega_t} \cdot f_{\Omega_t,W_{t-1},W_{t-2}} \end{aligned}$$

In matrix notation (for any fixed w_t, w_{t-1}) the relationship above is

$$L_{W_{t+1},w_t,w_{t-1},W_{t-2}} = L_{W_{t+1}|w_t,\Omega_{t-1}} D_{w_t|w_{t-1},\Omega_t} L_{\Omega_t,w_{t-1},W_{t-2}}$$

An important fact for the observed joint distribution $L_{W_{t+1},w_t,w_{t-1},W_{t-2}}$ is

$$L_{W_{t+1},w_t,w_{t-1},W_{t-2}} = \underbrace{L_{W_{t+1}|w_t,\Omega_t}}_{\text{no } w_{t-1}} \underbrace{D_{w_t|w_{t-1},\Omega_t}}_{\text{only } J \text{ unkwns.}} \underbrace{L_{\Omega_t,w_{t-1},W_{t-2}}}_{\text{no } w_t}$$

To proceed, we pick another realization of W_t, $\widetilde{w}_t \neq w_t$, then for (w_t, w_{t-1}), $(\widetilde{w}_t, w_{t-1})$, $(\widetilde{w}_t, \widetilde{w}_{t-1})$

$(w_t, \widetilde{w}_{t-1})$,

$$
\begin{aligned}
L_{W_{t+1},w_t,w_{t-1},W_{t-2}} &= L_{W_{t+1}|w_t,\Omega_t} \quad D_{w_t,w_{t-1},\Omega_t} \quad \underbrace{L_{\Omega_t,w_{t-1},W_{t-2}}}_{\parallel} \\
L_{W_{t+1},\overline{w}_t,w_{t-1},W_{t-2}} &= \underbrace{L_{W_{t+1}|\widetilde{w}_t,\Omega_t}}_{\parallel} \quad D_{\overline{w}_t,w_{t-1},\Omega_t} \quad \overbrace{L_{\Omega_t,w_{t-1},W_{t-2}}} \\
L_{W_{t+1},\widetilde{w}_t,\widetilde{w}_{t-1},W_{t-2}} &= \overbrace{L_{W_{t+1}|\widetilde{w}_t,\Omega_t}} \quad D_{\overline{w}_t|\widetilde{w}_{t-1},\Omega_t} \quad \underbrace{L_{\Omega_t,\widetilde{w}_{t-1},W_{t-2}}}_{\parallel} \\
L_{W_{t+1},w_t,\widetilde{w}_{t-1},W_{t-2}} &= L_{W_{t+1}|w_t,\Omega_t} \quad D_{w_t|\widetilde{w}_{t-1},\Omega_t} \quad \overbrace{L_{\Omega_t,\widetilde{w}_{t-1},W_{t-2}}}
\end{aligned}
$$

Under the assumption 5, LHS is invertible, and this fact enables us to eliminate $L_{\Omega_t,w_{t-1},W_{t-2}}$ using first two equations

$$
\begin{aligned}
\mathbf{A} &\equiv L_{W_{t+1},w_t,w_{t-1},W_{t-2}} L^{-1}_{W_{t+1},\widetilde{w}_t,w_{t-1},W_{t-2}} \\
&= L_{W_{t+1}|w_t,\Omega_t} D_{w_t|w_{t-1},\Omega_t} D^{-1}_{\widetilde{w}_t|w_{t-1},\Omega_t} L^{-1}_{W_{t+1}|\widetilde{w}_t,\Omega_t}
\end{aligned}
$$

Similarly, we can eliminate $L_{\Omega_t|\widetilde{w}_{t-1},W_{t-2}}$ using last two equations

$$
\begin{aligned}
\mathbf{B} &\equiv L_{W_{t+1},w_t,\widetilde{w}_{t-1},W_{t-2}} L^{-1}_{W_{t+1},\widetilde{w}_t,\widetilde{w}_{t-1},W_{t-2}} \\
&= L_{W_{t+1}|w_t,\Omega_t} D_{w_t|\widetilde{w}_{t-1},\Omega_t} D^{-1}_{\widetilde{w}_t|\widetilde{w}_{t-1},\Omega_t} L^{-1}_{W_{t+1}|\widetilde{w}_t,\Omega_t}
\end{aligned}
$$

The last step is to eliminate $L^{-1}_{W_{t+1}|\widetilde{w}_t,\Omega_t}$

$$
\mathbf{AB}^{-1} = L_{W_{t+1}|w_t,\Omega_t} D_{w_t,\widetilde{w}_t,w_{t-1},\widetilde{w}_{t-1},\Omega_t} L^{-1}_{W_{t+1}|w_t,\Omega_t}
$$

with diagonal matrix

$$
D_{w_t,\widetilde{w}_t,w_{t-1},\widetilde{w}_{t-1},\Omega_t} = D_{w_t|w_{t-1},X_t^*} D^{-1}_{\widetilde{w}_t|w_{t-1},\Omega_t} D_{\widetilde{w}_t|\widetilde{w}_{t-1},\Omega_t} D^{-1}_{w_t|\widetilde{w}_{t-1},\Omega_t}
$$

Consequently, the eigenvalue-eigenvector decomposition of observed $\mathbf{AB}^{-1}$

$$
\mathbf{AB}^{-1} = L_{W_{t+1}|w_t,\Omega_t} D_{w_t,\widetilde{w}_t,w_{t-1},\widetilde{w}_{t-1},\Omega_t} L^{-1}_{W_{t+1}|w_t,\Omega_t},
$$

with eigenvalues being the diagonal entry in $D_{w_t,\widetilde{w}_t,w_{t-1},\widetilde{w}_{t-1},\Omega_t}$, and

$$
\left(D_{w_t,\widetilde{w}_t,w_{t-1},\widetilde{w}_{t-1},\Omega_t}\right)_{j,j} = \frac{f_{W_t|W_{t-1},\Omega_t}(w_t|w_{t-1},j) f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|\widetilde{w}_{t-1},j)}{f_{W_t|W_{t-1},\Omega_t}(\widetilde{w}_t|w_{t-1},j) f_{W_t|W_{t-1},\Omega_t}(w_t|\widetilde{w}_{t-1},j)}
$$

To uniquely determine the uniqueness of the decomposition, the eigenvalues need to be distinct and this is guaranteed by assumption 6. Uniqueness of decomposition also requires normalization of eigenvectors, this can be done naturally because column in $L_{W_{t+1}|w_t,\Omega_t}$, sums to 1.

To pin-down the value of ω_t: need to "order" eigenvectors not necessary in the time-invariant case, $\Omega_t = \Omega_{t-1}$ useful in time-varying case: show how agents change types w/ time. $f_{W_{t+1}|W_t,\Omega_t(\cdot|w_t,\omega_t)}$ for any w_t is identified up to value of ω_t. To pin-down the value of ω_t : Assume there is *known* functional

$$h(w_t,\omega_t) \equiv G\left[f_{W_{t+1}|W_t,\Omega_t}\left(\cdot|w_t,\cdot\right)\right] \text{ is monotonic in } \omega_t.$$

Then set $\omega_t = G\left[f_{W_{t+1}|W_t,\Omega_t}\left(\cdot|w_t,\cdot\right)\right]$ $G\left[f\right]$ may be mean, mode, median, other quantile of f. Note: in unobserved heterogeneity case ($\Omega_t = \Omega,\ \forall t$), it is enough to identify $f_{W_{t+1}|W_t,\Omega_t}$.

Proof of Corollary 1 We first express the joint distribution of W_{t-1}, W_{t-2} as follows.

$$f_{W_{t-1},W_{t-2}} = \sum_{\Omega_{t-1}} f_{W_{t-1},W_{t-2},\Omega_{t-2}} = \sum_{\Omega_{t-1}} f_{W_{t-1}|W_{t-2},\Omega_{t-2}} f_{W_{t-2},\Omega_{t-2}}.$$

In matrix form, the equation above is expressed as

$$L_{W_{t-1},W_{t-2}} = L_{W_{t-1}|W_{t-2},\Omega_{t-2}} L_{W_{t-2},\Omega_{t-2}}.$$

Hence $L_{W_{t-2},\Omega_{t-2}}$ is identified as

$$L_{W_{t-2},\Omega_{t-2}} = L^{-1}_{W_{t-1}|W_{t-2},\Omega_{t-2}} L_{W_{t-1},W_{t-2}},$$

where $L_{W_{t-1}|W_{t-2},\Omega_{t-2}} = L_{W_t|W_{t-1},\Omega_{t-1}}$, which is due to stationarity, is identified in Lemma 3. Consequently, the initial condition $f_{Y_{t-2},X_{t-2},\Omega_{t-2}}$ is identified for $t \in \{3, 4, ..., T-1\}$