Identification of the Distribution of Valuations in an Incomplete Model of English Auctions*

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Abstract

An incomplete model of English auctions similar to the one studied in Haile and Tamer (2003) is studied, in which bidders are assumed to (i) bid no more than their valuations and (ii) never let an opponent win at a price they are willing to beat. The model is shown to fall in the class of Generalized Instrumental Variable Models introduced in Chesher and Rosen (2014), thereby extending the scope of set identification analysis using restrictions (i) and (ii) to models with information paradigms other than that of the indendent private values framework. The methods employed produce sharp bounds – that is *the identified set* – for model primitives. In the symmetric independent private values (IPV) setting that was the focus of Haile and Tamer (2003), the identified set for the distribution of bidder valuations is shown to refine the bounds available until now.

Keywords: English auctions, partial identification, sharp set identification, generalized instrumental variable models.

1 Introduction

The path breaking paper Haile and Tamer (2003) (HT) develops bounds on the common distribution of valuations in an incomplete model of an open outcry English ascending auction in a symmetric independent private values (IPV) setting.

One innovation in the paper was the use of an incomplete model based on weak plausible restrictions on bidder behavior, namely that a bidder never bids more than her valuation and never

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allows an opponent to win at a price she is willing to beat. An advantage of an incomplete model is that it does not require specification of the mechanism relating bids to valuations. Results obtained using the incomplete model are robust to misspecification of such a mechanism. The incomplete model may be a better basis for empirical work than the button auction model of Milgrom and Weber (1982) sometimes used to approximate the process delivering bids in an English open outcry auction.

On the down side the incomplete model is partially, not point, identifying for the primitive of interest, namely the common conditional probability distribution of valuations given auction characteristics. HT derive bounds on this distribution in an independent private values setting and show how to use these bounds to make inferences about the distribution and interesting functionals of the distribution such as the optimal reserve price.

The question of the sharpness of those bounds was left open in HT. In this paper we resolve this question. We consider a slightly simplified version of the model in HT and show that the model falls in the class of Generalized Instrumental Variable (GIV) models introduced in Chesher and Rosen (2014), (CR). We obtain a characterization of the identified set (sharp bounds) for the auction model by applying the general characterization for GIV models given in CR.[1] We show that there are observable implications additional to those given in HT and in numerical calculations demonstrate that they can be binding.

The characterization of sharp bounds on valuation distributions comprises a dense system of infinitely many inequalities restricting not just the value of the distribution function via pointwise bounds on its level but also restricting its shape as it passes between the pointwise bounds.

In this paper we not only resolve the question of sharpness in the original HT model in which valuations are (conditionally) independent and identically distributed, but we also expand the application of their intuitively appealing restrictions on bidder behavior to non-IPV settings. Theorems 1 and 2 in Section 3 provide general characterizations that do not require IPV. These results provide a framework for identification analysis incorporating further restrictions that appear in econometric models of auctions. Section 4 demonstrates how this can be done by specializing these results to the IPV case to obtain the sharp bounds on the valuation distribution referred to above. Section 5 illustrates how our analysis can be applied to models that feature unobservable auction specific heterogeneity, a special and important class of models in which the IPV restriction does not hold.

Partial identification has been usefully applied to address other issues in auction models since HT. Tang (2011) and Armstrong (2013) both study first-price sealed bid auctions. Tang (2011) assumes equilibrium behavior but allows for a general affiliated values model that nests private and common value models. Without parametric distributional assumptions model primitives are generally partially identified, and bounds on seller revenue under counterfactual reserve prices and

[1] In this paper we use the expression *identified set* to refer to *sharp bounds* throughout. Non-sharp bounds are referred to simply as bounds or *outer regions*.

auction format are derived. Armstrong (2013) studies a model in which bidders play equilibrium strategies but have symmetric independent private values *conditional* on unobservable heterogeneity, and derives bounds on the mean of the bid and valuation distribution, and other functionals thereof. Aradillas-Lopez, Gandhi, and Quint (2013) study second price auctions that allow for correlated private values. Theorem 4 of Athey and Haile (2002) previously showed non-identification of the valuation distribution in such models, even if bidder behavior follows the button auction model equilibrium. Aradillas-Lopez, Gandhi, and Quint (2013) impose a slight relaxation of the button auction equilibrium, assuming that transaction prices are determined by the second highest bidder valuation. They combine restrictions on the joint distribution of the number of bidders and the valuation distribution with variation in the number of bidders to bound seller profit and bidder surplus.

The restrictions of the auction models we study are set out in Section 2. In Section 3 GIV models are introduced and the auction model is placed in the GIV context, and the identified set for such models is characterized. The identified set for the auction model with independent private values is characterized in Section 4 and the inequalities that feature in this characterization are explored in Section 4.1. The identified set for an auction model with additive unobservable auction specific heterogeneity is characterized in Section 5. Some numerical examples are presented in Section 6. Section 7 concludes.

2 Model

We study open outcry English ascending auctions with a finite number of bidders, M, which may vary from auction to auction. The model imposes the slight simplifications that there is no reserve price and the minimum bid increment is zero. These conditions simplify the exposition and are easily relaxed.[2]

Auctions are characterized by a vector of observed final bids B, a vector of valuations V, the number of bidders M, and auction characteristics Z. B, V, M, Z are presumed to be realized on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with sigma algebra $\mathcal{A}$ endowed with the Borel sets on Ω. Valuations V are not observed. Observations of (B, Z, M) across auctions render the joint distribution of these variables identified. The goal of our identification analysis is to determine what this joint distribution reveals about $F_z(\cdot)$. In some applications the number of bidders, M, could be an element of Z, in which case $F_z(\cdot)$ could vary with m. In the remainder of the paper, inequalities involving random variables, such as those in Restrictions 1 and 2 below, and those stated in Lemma 1, are to be understood to mean these inequalities hold $\mathbb{P}$ almost surely. For any random vector $X = (X_1, \ldots, X_M)$, the notation $X_{m:M}$ denotes the m^{th} order statistic of X, so for example

[2] These conditions are also imposed in Appendix D of HT in which the sharpness of identified sets is discussed. As is the case in HT, with a reserve price r our analysis applies to the distribution of valuations truncated below at r.

$X_{M:M} = \max(X)$ and $X_{1:M} = \min(X)$.

Restriction 1. In an auction with M bidders, the final bids and valuations are realizations of random vectors $B = (B_1, \ldots, B_M)$ and $V = (V_1, \ldots, V_M)$ such that for all $m = 1, ..., M$, $B_m \leq V_m$ almost surely.

Restriction 2. In every auction the second highest valuation, $V_{M-1:M}$, is no larger than the highest final bid, $B_{M:M}$. That is, $V_{M-1:M} \leq B_{M:M}$ almost surely.

Restrictions 1 and 2 are the HT restrictions on bidder behavior. They admit the standard button auction equilibrium, but also allow for other bidder behavior consistent with that observed in ascending oral auctions, such as jump bids.

HT study the identifying power of Restrictions 1 and 2 when in addition bidder valuations are restricted to the conditional independent private values paradigm, stated here as Restriction CIPV.

Restriction CIPV (Conditional Independent Private Values)**.** There are independent private values conditional on auction characteristics $Z = z$ such that the valuations of bidders are identically and independently continuously distributed with conditional distribution function given $Z = z$ denoted by $F_z(\cdot)$.

The approach taken here applies identification analysis from CR, which automatically delivers sharp bounds for model primitives without need for a constructive proof of sharpness. Moreover, the analysis is applicable in the absence of the CIPV restriction, and thereby establishes how the intuitively appealing restrictions 1-2 of HT can be much more broadly applied. We additionally consider the following restriction on observed bids.

Restriction EX (Exchangeability)**.** Conditional on auction characteristics $Z = z$, observed final bids $(B_1, \ldots, B_M)$ are exchangeable.

Restriction EX imposes that bids are symmetric given auction characteristics z. Restrictions 1-2 and CIPV were imposed by HT, while Restriction EX was not. In Theorem 3 we present bounds on $F_z(\cdot)$ that refine those of HT using only Restrictions 1-2 and CIPV. The bounds are shown to be sharp if, additionally, Restriction EX holds, or if the researcher only has data on bid order statistics.

3 Generalized Instrumental Variable models

This auction model falls in the class of Generalized Instrumental Variable (GIV) models introduced in Chesher and Rosen (2014).

We consider M-bidder auctions for some particular value of M and use the results in CR to characterize the identified set (i.e. sharp bounds) for valuation distributions delivered by a joint distribution of M final bids. In cases where M is not included as an element of Z, so that $F_z(\cdot)$ is restricted to be invariant with respect to M, the intersection of the sets obtained with different values of M gives the identified set of valuation distributions in situations in which there is variation in the number of bidders across auctions.

A GIV model places restrictions on a process that generates values of observed endogenous variables, Y, given exogenous variables Z and U, where Z is observed and U is unobserved. The variables (Y, Z, U) take values on $\mathcal{R}_{YZU}$ which is a subset of a suitably dimensioned Euclidean space.

GIV models place restrictions on a structural function $h : \mathcal{R}_{YZU} \to \mathbb{R}$ which defines the admissible combinations of values of Y and U that can occur at each value z of Z which has support $\mathcal{R}_Z$. Admissible combinations of values of (Y, U) at $Z = z$ are zero level sets of this function, as follows.

$$\mathcal{L}(z; h) = \{(y, u) : h(y, z, u) = 0\}$$

For each value of U and Z we can define a Y-level set

$$\mathcal{Y}(u, z; h) \equiv \{y : h(y, z, u) = 0\}$$

which is singleton for all u and z in complete models, but not in incomplete models such as that studied here. Likewise, for each value of Y and Z we can define a U-level set

$$\mathcal{U}(u, z; h) \equiv \{u : h(y, z, u) = 0\}. \tag{3.1}$$

GIV models place restrictions on such structural functions and also on a collection of conditional distributions

$$\mathcal{G}_{U|Z} \equiv \{G_{U|Z}(\cdot|z) : z \in \mathcal{R}_Z\}$$

whose elements are conditional distributions of U given $Z = z$ obtained as z varies across the support of Z. $G_{U|Z}(\mathcal{S}|z)$ denotes the probability that $U \in \mathcal{S}$ conditional on $Z = z$ under the law $G_{U|Z}$.

3.1 Unordered Final Bids

In an auction of M bidders each bidder is characterized by an observed final bid B_m and private valuation V_m for the object at auction. The vector $(B_1, \ldots, B_M)$ denotes observed final bids for each of the M bidders in the auction, and the vector $V = (V_1, \ldots, V_M)$ denotes the corresponding unobserved private valuations. Neither vector B nor V need be ordered from smallest to largest or *vice-versa*. B_m and V_m correspond to the bid and valuation of the same bidder, but the order of the bids and valuation in B and V is otherwise arbitrary.

Valuations V_m are each restricted to have strictly increasing marginal cumulative distribution function $F_z(\cdot)$ on their support conditional on auction characteristics $Z = z$. The valuations need not be independent. The support of V_m is a subset of $[\underline{v}, \overline{v}] \subseteq \overline{\mathbb{R}}$, the extended real line.

For each $m = 1, ..., M$, define $W_m \equiv F_z(V_m)$, such that each variable W_m is marginally uniformly distributed on the unit interval. The vector $W \equiv (W_1, \ldots, W_M)$ plays the role of unobservable vector U for GIV analysis when considering unordered final bids. Statements predicated by $\forall m$ are to be understood to hold for all $m = 1, ..., M$.

The GIV level set of unobservable W corresponding to that in (3.1) given the HT assumptions from observed bids B is

$$\mathcal{W}(B, F_z) \equiv \left\{ w \in [0,1]^M : \forall m,\ F_z(B_m) \leq w_m \wedge F_z\left(B_{m^*(B)}\right) \geq \max_{m \neq m^*(B)} w_m \right\}, \tag{3.2}$$

where $m^*(B)$ denotes the index of the winning bidder.

A GIV structural function which expresses these restrictions, with bid vector B taking the role of Y, is

$$h(B, z, W) = \sum_{m=1}^{M} \max((F_z(B_m) - W_m), 0) + \max(\left(\max_{m \neq m^*(B)} W_m - F_z\left(B_{m^*(B)}\right) \right), 0). \tag{3.3}$$

Thus we have cast the auction model as a GIV model in which the structural function h is a known functional of the collection of conditional valuation distributions $\{F_z(\cdot) : z \in \mathcal{R}_Z\}$. We use the notation $\mathcal{F}$ to denote a collection of such conditional distribution functions, and F to denote those $\mathcal{F}$ permitted by the model, and which embody the researcher's prior information on the distribution functions $F_z(\cdot)$.[3] The restrictions of the auction model on the distribution of (W, Z) are: (i) W and Z are independently distributed and (ii) the conditional distribution of W given $Z = z$, denoted G_z, is the joint distribution of M marginally uniform variates.

If Restriction CIPV is imposed, then the components of W are mutually independent conditional on Z and, for any set $\mathcal{S} \subseteq \mathcal{R}_W$, $G_z(\mathcal{S}) = G_W(\mathcal{S})$ is the probability that a random M-vector of independent uniform$(0,1)$ variates takes a value in the set $\mathcal{S}$. In the absence of restriction CIPV, $G_{W|Z}(\cdot|z)$ must still be such that the *marginal* distribution of each component of W given Z is uniform$(0,1)$, but the components of W may be correlated. The joint distribution of W given $Z = z$ may also then vary with z. We use the notation $\mathcal{G}$ to denote a collection of distributions $\{G_z : z \in \mathcal{R}_Z\}$, and G to denote those collections $\mathcal{G}$ which are admitted by the model specification. For example, the conditional distributions G_z could each be left unrestricted across different values of z, their dependence on z could be parameterized through an index function, or they could be explicitly parameterized by an M-dimensional copula.

Applying Theorem 2 and Lemma 1 of CR, the identified set for the pair $F_z(\cdot)$ and $G_z(\cdot)$ are those such that for all sets $\mathcal{S}$ in a collection of test sets $\mathsf{Q}(h, z)$ the following inequality is satisfied

[3] For example, a model could restrict each $F_z(\cdot)$, $z \in \mathcal{R}_Z$ to a parametric family, or it could restrict $F_z(\cdot)$ to be invariant with respect to certain components of z.

almost surely

$$G_z(\mathcal{S}) \geq \mathbb{P}\left[\mathcal{U}(Y,Z;h) \subseteq \mathcal{S}|Z=z\right]. \tag{3.4}$$

The collection of test sets $\mathsf{Q}(h,z)$ is defined in Theorem 3 of CR. It comprises certain unions of the members of the collection of U-level sets $\mathcal{U}(y,z;h)$ obtained as y takes values in the conditional support of Y given $Z=z$.[4] The following theorem provides the formal result for the auction model where $\mathcal{W}(B,F_z)$ takes the place of $\mathcal{U}(Y,Z;h)$.

Theorem 1. *Let $\mathcal{F} \in \mathsf{F}$, $\mathcal{G} \in \mathsf{G}$, and Restrictions 1 and 2 hold. The identified set for*

$$\left(F_z\left(\cdot\right), G_z\left(\cdot\right) : z \in \mathcal{R}_Z\right)$$

are those collections of conditional distributions admitted by F and G such that for almost every $z \in \mathcal{R}_Z$, for all $\mathcal{S}$ that are unions of sets of the form $\mathcal{W}(b,F_z)$, $b \in \mathcal{R}_B$:

$$\mathbb{P}\left[\mathcal{W}\left(B,F_z\right) \subseteq \mathcal{S}|Z=z\right] \leq G_z\left(\mathcal{S}\right). \tag{3.5}$$

Theorem 1 directly applies results from CR to characterize the identified set for the marginal distribution of valuations $F_z(\cdot)$ and their copula $G_z(\cdot)$ across values of z. It imposes only Restrictions 1 and 2 on bidder behavior, without recourse to independence of bidder valuations or exchangeability of observed bids. A feature of the Theorem is thus its generality. Nonetheless, with so few restrictions in place, one may wonder just how informative this characterization may be in practice. Indeed, empirical auction models typically do impose additional restrictions, which will further refine the set given by (3.5), and which, thinking of implementation, may facilitate estimation and inference.

The remainder of the paper studies models with additional restrictions, and develops characterizations of the resulting identified sets that refine those obtained using Theorem 1. To do this we turn to bounds on valuation distributions derived from knowledge of only the distribution of *ordered* final bids, that is using order statistics of the bid distribution, as were also employed in HT.

We show in the next Section that when the Restriction EX holds there is no loss of information in having knowledge only of the distribution of ordered final bids, in the sense that the identified set obtained using the distribution of ordered final bids is sharp and is the same as that obtained using the distribution of unordered final bids. If Restriction EX does not hold, then the bounds derived from information in the distribution of ordered bids still apply, but they may not be sharp relative to the bounds obtained from the distribution of unordered final bids. Exchangeability is reasonable assumption in several contexts, for example when there is no observable information about specific

[4] In general the collection $\mathsf{Q}(h,z)$ contains all sets that can be constructed as unions of sets, (3.10), on the support of the random set $\mathcal{U}(Y,Z;h)$. In particular models some unions can be neglected because the inequalities they deliver are satisfied if inequalities associated with other unions are satisfied. There is more detail and discussion in CR14.

bidder identities. It is an implication of restrictions used in many applications, although it does rule out some interesting cases, such as when bidders have different observable types or when bidders form collusive bidding rings. In such settings Theorem 1 nonetheless applies and can still be taken as a starting point for identification analysis.

3.2 Ordered Final Bids

Let $Y = (Y_1, ..., Y_M)$ denote ordered final bids, so that

$$Y_m \equiv B_{m:M}, \qquad m \in \{1, \ldots, M\}.$$

It is convenient to write the ordered valuations as functions of uniform order statistics. To this end define

$$U \equiv (W_{1:M}, \ldots, W_{M:M}) = (F_z(V_{1:M}), \ldots, F_z(V_{M:M})).$$

The components of U order those of W from smallest to largest, so the m^{th} component U_m is the m^{th} order statistic of W. The distribution of U is therefore that of the order statistics of M uniform$(0,1)$ but possibly dependent random variables. Without placing restrictions on the joint distribution of valuations, many such distributions of U are possible, depending on the copula of V. The notation $G_z^o(\mathcal{S})$ is used to denote the probability placed on the event $U \in \mathcal{S}$ when U has one such distribution. The admissible collection of joint distributions for W conveyed by $G_z(\cdot)$ restricts the collection of admissible $G_z^o(\cdot)$.[5]

The following inequalities involving the order statistics of final bids B and latent valuations V set out in Lemma 1 are a consequence of Restrictions 1 and 2. The proof of the lemma, like all other proofs, is provided in Appendix A.

Lemma 1. *Let Restrictions 1-2 hold. Then for all m and M*

$$B_{m:M} \leq V_{m:M} \tag{3.6}$$

$$B_{M:M} \geq V_{M-1:M}. \tag{3.7}$$

In similar manner to HT, we can base identification analysis on the restrictions (3.6) and (3.7) on bid and valuation order statistics. We show that in fact when final bids are exchangeable, application of the GIV analysis to *ordered* bids and valuations delivers the same sharp bounds as are obtained using the information in the distribution of unordered final bids.

[5]The notation $G_z(\mathcal{S})$ and $G_z^o(\mathcal{S})$ distinguises between a joint distribution of M marginally uniform$(0,1)$ random variables, and the joint distribution of the *order statistics* of M marginally uniform$(0,1)$ random variables, respectively. The dependence structure among the M marginally uniform$(0,1)$ components need not be known.

The restrictions (3.6) and (3.7) of Lemma 1 can be written as

$$\forall m, \quad Y_m \leq V_{m:M} = F_z^{-1}(U_m) \text{ and } Y_M \geq V_{M-1:M} = F_z^{-1}(U_{M-1})$$

and, on applying the increasing function $F_z(\cdot)$, they are as follows:

$$\forall m, \quad F_z(Y_m) \leq U_m \text{ and } F_z(Y_M) \geq U_{M-1}. \tag{3.8}$$

A GIV structural function which expresses these restrictions is

$$h(Y, z, U) = \sum_{m=1}^{M} \max((F_z(Y_m) - U_m), 0) + \max((U_{M-1} - F_z(Y_M)), 0). \tag{3.9}$$

The corresponding U-level sets are:

$$\mathcal{U}(Y, z; h) = \left\{ u : \left(\bigwedge_{m=1}^{M} (u_m \geq F_z(Y_m)) \right) \wedge (F_z(Y_M) \geq u_{M-1}) \right\}, \tag{3.10}$$

it being understood that for all m, $u_m \geq u_{m-1}$.[6] These are not singleton sets. Figure 1 illustrates for the 2 bidder case. The U-level set $U((y_1', y_2'), z; h)$ is the blue rectangle below the 45° line.

The auction model has now been cast as a GIV model involving ordered bids Y and latent variables U. The structural function h is again a known functional of $\{F_z(\cdot) : z \in \mathcal{R}_Z\}$, and once again U and Z are independently distributed. Applying Theorem 4 of CR gives the following result.

Theorem 2. *Let $\mathcal{F} \in \mathsf{F}$, $\mathcal{G} \in \mathsf{G}$, and Restrictions 1-2 and EX hold. The identified set for $(F_z(\cdot), G_z^o(\cdot) : z \in \mathcal{R}_Z)$ are those collections of conditional distributions admitted by F and G such that for almost every $z \in \mathcal{R}_Z$, for all $\mathcal{S}$ that are unions of sets of the form $\mathcal{U}(Y, z; h)$:*

$$\mathbb{P}[\mathcal{U}(Y, z; h) \subseteq \mathcal{S} | z] \leq G_z^o(\mathcal{S}), \tag{3.11}$$

where $G_z^o(\mathcal{S})$ denotes the probability that $U \in \mathcal{S}$ conditional on $Z = z$.

Relative to Theorem 1, Theorem 2 simplifies characterization of the identified set in case where Restriction 3 holds. The simplification lies in that the collection of inequalities defining the identified set involves only probabilities featuring *ordered* bids. When considering test sets, one need only consider unions of sets (3.10) in which $u_1 \leq \cdots \leq u_M$. This is an $M!$ reduction in the number of sets whose unions must be considered when constructing test sets $\mathcal{S}$.

The following section builds on this result to characterize the identified set for $F_z(\cdot)$, each $z \in \mathcal{R}_Z$, when in addition Restriction CIPV holds. The resulting inequalities are closely examined

[6] If there was a minimum bid increment Δ, then (3.10) would have $F_z(y_M + \Delta)$ in place of $F_z(y_M)$.

for this important special case. Section 5 applies this result to a model with auction specific unobserved heterogeneity in which there are affiliated private values.

4 The Identified Set for $F_z(\cdot)$ With IPV

Under Restriction CIPV, bidder valuations in each auction are independent and identically distributed with conditional distribution function $F_z(\cdot)$ given observable auction characteristics $Z = z$. Thus the components of $W = (F_z(V_{1:M}), ..., F_z(V_{M:M}))$ are independent and each distributed uniform$(0,1)$. The order statistics of W, which are $U = (F_z(V_{1:M}), ..., F_z(V_{M:M}))$ have the distribution of the order statistics of M uniform$(0,1)$ random variables. Thus $G_z^o(\mathcal{S})$ is known; it is $M!$ times the volume of the set $\mathcal{S}$, here denoted $G_U(\mathcal{S})$.[7] Only $\mathcal{F} = \{F_z(\cdot) : z \in \mathcal{R}_Z\}$ is the object of identification analysis.

Theorem 3. *Let $\mathcal{F} \in \mathsf{F}$ and Restrictions 1-2 and CIPV hold. The set*

$$\mathcal{F}^* \equiv \{\mathcal{F} \in \mathsf{F} : \text{ for all closed } \mathcal{S} \subseteq \mathcal{R}_U,\ G_U(\mathcal{S}) \geq \mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{S}|z] \ \text{ a.e. } z \in \mathcal{R}_z\}, \tag{4.1}$$

comprises bounds on the collection of conditional distributions $\{F_z(\cdot) : z \in \mathcal{R}_Z\}$. If, in addition, either Restriction EX holds, or only the order statistics of the bids are observable rather than the bids themselves, then these these bounds are sharp.

The theorem characterizes bounds on the collection of possible valuation distributions conditional on auction characteristics z as those that satisfy (3.4) for all closed sets $\mathcal{S} \subseteq \mathcal{R}_U$. If, additionally, bids are exchangeable, or if only the distribution of bid order statistics conditional on $Z = z$ is identified from the data, rather than the distribution of bids, then these bounds are sharp and (4.1) delivers the identified set.

In CR we characterize a sub-family of closed sets on $\mathcal{R}_U$, denoted $\mathsf{Q}(h, z)$ such that if (3.4) holds for all $\mathcal{S} \in \mathsf{Q}(h, z)$, then it must also hold for all closed $\mathcal{S} \subseteq \mathcal{R}_U$. In Section 4.1 below we consider the form of inequalities generated by particular sets $\mathcal{S}$, and then consider their identifying power in the examples of Section 6.

The set $\mathcal{U}(Y, Z; h)$ in (3.4) is a random set (Molchanov (2005)) whose realizations are U-level sets as set out in (3.10). Its conditional probability distribution given $Z = z$ is determined by the probability distribution of Y given $Z = z$. In the auction setting this is the conditional distribution of ordered final bids in auctions with $Z = z$. The probability on the right hand side of (3.4) is a conditional containment functional. It is equal to the conditional probability given $Z = z$ that Y

[7] This is so because the joint distribution of the uniform order statistics is uniform on the part of the unit M-cube in which $u_1 \leq u_2 \leq \cdots \leq u_M$, with density equal to $M!$. See Section 2.2 in David and Nagaraja (2003).

lies in the set $\mathcal{A}(\mathcal{S}, z; h)$ where

$$\mathcal{A}(\mathcal{S}, z; h) \equiv \{y : \mathcal{U}(y, z; h) \subseteq \mathcal{S}\}$$

are those of values of y which can only arise when U takes a value in $\mathcal{S}$.

When $\mathcal{S}$ is a U-level set, say $\mathcal{U}(y', z; h)$ with $y' \equiv (y'_1, \dots, y'_M)$, there is

$$\mathcal{A}(\mathcal{U}(y', z; h), z; h) = \left\{ y : \left(y_M = y'_M\right) \wedge \left(\bigwedge_{m=1}^{M-1} \left(y_m \geq y'_m\right) \right) \right\} \tag{4.2}$$

it being understood that for all m, $y_m \geq y_{m-1}$.

Figures 2 and 3 show that the effect of changing the value y'_M (which produces the magenta colored rectangles in these Figures) is to produce a new level set that *is not* a subset of $\mathcal{U}(y', z; h)$, hence the equality in (4.2).

Figure 4 shows that the effect of increasing the value y'_{M-1} is to produce a new level set that *is* a subset of $\mathcal{U}(y', z; h)$, hence the weak inequalities in (4.2).

If Y is continuously distributed the equality in (4.2) causes the probability $\mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{U}(y', z; h) | Z = z]$ to be zero and the inequality (3.4) does not deliver an informative bound when $\mathcal{S} = \mathcal{U}(y', z; h)$. So, when bids are continuously distributed, amongst the unions of U-level sets in a collection $\mathsf{Q}(h, z)$, nontrivial bounds are only delivered by unions of a collection of level sets whose members have values of the maximum bid Y_M ranging over a set of values of nonzero measure.

We proceed to consider particular unions of this sort, as follows.

$$\mathcal{U}(y', y''_M, z; h) \equiv \bigcup_{y_M \in [y'_M, y''_M]} \mathcal{U}((y'_1, \dots, y'_{M-1}, y'_M), z; h), \qquad y''_M \geq y'_M$$

Such unions are termed *contiguous unions* of U-level sets.[8]

The region in $\mathcal{R}_U$ occupied by such a contiguous union is

$$\mathcal{U}(y', y''_M, z; h) = \left\{ u : \left(\bigwedge_{m=1}^{M} \left(u_m \geq F_z(y'_m)\right) \right) \wedge \left(F_z(y''_M) \geq u_{M-1}\right) \right\}$$

it being understood that, for all m, $u_m \geq u_{m-1}$. Figure 5 illustrates for the 2 bidder case. The contiguous union is the region under the 45° line outlined in blue - a rectangle with its top left hand corner removed.

The probability mass placed on this region by the distribution of the uniform order statistics

[8] A simple U-level set as in (3.10) is obtained on setting $y'_M = y''_M$. When Y_M is not continuously distributed this member of $\mathsf{Q}(h, z)$ may deliver nontrivial bounds.

is:

$$G_U\left(\mathcal{U}(y', y''_M, z; h)\right) = M! \int_{F_z(y'_M)}^{1} \int_{F_z(y'_{M-1})}^{\min(u_M, F_z(y''_M))} \int_{F_z(y'_{M-2})}^{u_{M-1}} \cdots \int_{F_z(y'_1)}^{u_2} du. \tag{4.3}$$

The set of values of Y that deliver U-level sets that are subsets of $\mathcal{U}(y', y''_M, z; h)$ is

$$\mathcal{A}(\mathcal{U}(y', y''_M, z; h), z; h) = \left\{ y : \left(y'_M \leq y_M \leq y''_M\right) \wedge \left(\bigwedge_{m=1}^{M-1} \left(y_m \geq y'_m\right) \right) \right\}$$

it being understood that $y_1 \leq \cdots \leq y_M$. This region is indicated by the shaded area in Figure 5 for the case in which $M = 2$.[9] The conditional containment functional on the right hand side of (3.4) is calculated as follows.

$$\mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{U}(y', y''_M, z; h)|z] = \mathbb{P}\left[\left(y'_M \leq Y_M \leq y''_M\right) \wedge \left(\bigwedge_{m=1}^{M-1} \left(Y_m \geq y'_m\right) \right) \middle| z \right] \tag{4.4}$$

This is a probability that can be estimated using data on values of ordered bids while the probability $G_U\left(\mathcal{U}(y', y''_M, z; h)\right)$ is determined entirely by the chosen values of y', y''_M, z and the distribution function of valuations, F_z, whose membership of the identified set is under consideration.

For any choice of F_z a list of values of (y', y''_M) delivers a list of inequalities on calculating (3.4) and if one or more of the inequalities is violated the candidate valuation distribution F_z is outside the identified set.

The inequalities that arise for particular choices of (y', y''_M) are now explored. The first choices to be considered deliver the inequalities in HT, then other choices are considered which deliver additional inequalities.

4.1 Inequalities defining the identified set

4.1.1 Valuations stochastically dominate bids

With $y''_M = +\infty$ and with M element y' as follows:

$$y' = (-\infty, -\infty, \ldots, -\infty, \underbrace{v}_{\text{position } n}, v, \ldots, v)$$

the containment functional probability (4.4) is:

$$\mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{U}(y', y''_M, z; h)|z] = \mathbb{P}[Y_n \geq v|z],$$

[9] The shaded area shows values of $F_z(y_1)$ and $F_z(y_2)$ that give rise to U-level sets that are subsets of $\mathcal{U}(y', y''_M, z; h)$ with $M = 2$.

the probability mass placed by the distribution G_U on the contiguous union, (4.3), is

$$G_U\left(\mathcal{U}(y', y''_M, z; h)\right) = \mathbb{P}[U_n \geq F_z(v)] = \mathbb{P}[V_n \geq v|z; F_z]$$

and so the condition (3.4) delivers the following inequalities.[10]

$$\forall n, \forall v: \qquad \mathbb{P}[V_n \leq v|z; F_z] \leq \mathbb{P}[Y_n \leq v|z]$$

These inequalities hold for a valuation distribution function F_z if and only if under that distribution there is the stochastic ordering of order statistics of bids and valuations required by the restriction (3.6).

The marginal distribution of the n^{th} order statistic of M identically and independently distributed uniform variates is $Beta(n, M+1-n)$.[11] Let $Q(p; n, M)$ denote the associated quantile function. The restrictions placed on valuation distributions by the inequality (3.4) and the test sets under consideration in this Section are, written in terms of uniform order statistics:

$$\forall n, \forall v: \qquad \mathbb{P}[U_n \leq F_z(v)] \leq \mathbb{P}[Y_n \leq v|z]$$

which can be written as follows.

$$\forall v: \qquad F_z(v) \leq \min_n Q(\mathbb{P}[Y_n \leq v|z]; n, M) \tag{4.5}$$

This continuum of pointwise upper bounds must hold for all valuation distribution functions in the identified set. This is the bound given in Theorem 1 of HT.

Figures 6 and 7 show the contiguous unions of U-level sets (the regions bordered in blue) delivering this inequality for 2 bidder auctions. The regions shaded blue (triangular in one case and trapezoidal in the other) indicate the values of $(F_z(y_2), F_z(y_1))$ that deliver U-level sets that are subsets of these contiguous unions.

4.1.2 The highest bid stochastically dominates the second highest valuation

With $y''_M = v$ and $y' = (-\infty, -\infty, \ldots, -\infty)$ the containment functional probability (4.4) is:

$$\mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{U}(y', y''_M, z; h)|z] = \mathbb{P}\left[Y_M \leq v|z\right],$$

[10] The notation $\mathbb{P}[V_n \leq s|z; F_z]$ serves to remind that V_n is an order statistic of valuations which are identically and independently distributed with conditional distribution function F_z.

[11] See Section 2.3 in David and Nagaraja (2003). The density function of this Beta random variable is proportional to $u^{n-1}(1-u)^{M-n}$.

the probability mass placed by the distribution G_U on the contiguous union, (4.3), is

$$G_U\left(\mathcal{U}(y', y''_M, z; h)\right) = \mathbb{P}[U_{M-1} \leq F_z(v)] = \mathbb{P}[V_{M-1} \leq v|z; F_z]$$

so, with this choice of (y', y''_M), the condition (3.4) delivers the following inequalities.

$$\forall v: \quad \mathbb{P}[V_{M-1} \leq v|z; F_z] \geq \mathbb{P}[Y_M \leq v|z] \tag{4.6}$$

These inequalities hold for a valuation distribution function F_z if and only if under that distribution the second highest valuation is stochastically dominated by the highest bid as required by the restriction (3.7).

All valuation distribution functions in the identified set must satisfy:

$$\forall v: \qquad \mathbb{P}[U_{M-1} \leq F_z(v)] \geq \mathbb{P}\left[Y_M \leq v|z\right]$$

which is (4.6) rewritten in terms of a uniform order statistic, equivalently

$$\forall v: \qquad F_z(v) \geq Q(\mathbb{P}\left[Y_M \leq v|z\right]; M-1, M). \tag{4.7}$$

This is the bound given in Theorem 2 of HT when the minimum bid increment considered there is set equal to zero.[12]

Figure 8 shows, outlined in blue, the contiguous union of U-level sets delivering this inequality. The triangular shaded region indicates the values of $(F_z(y_2), F_z(y_1))$ that deliver U-level sets that are subsets of this contiguous union.

4.1.3 Contiguous unions depending on a single value of y

In the two cases just considered contiguous unions of U-level sets are determined by (y', y''_M) in which only a single value, v, of Y appears. The inequalities they deliver place a continuum of pointwise upper and lower bounds on the value of the valuation distribution function, $F_z(v)$. When Y is continuously distributed these are the only contiguous unions determined by a single value of Y that deliver nontrivial inequalities. It is necessary to give separate consideration to cases in which final bids are continuously distributed and cases in which they are not.

Bids continuously distributed First suppose that y''_M takes some finite value v as in Section 4.1.2. We must have $y'_M < v$ otherwise the containment functional is zero if Y is continuously distributed. The only possible value for y'_M that does not introduce a second finite value is $-\infty$ and since $y'_m \leq y'_M$ for all M we arrive at the case considered in Section 4.1.2.

[12]With a positive minimum bid increment Δ, Y_M is replaced by $Y_M + \Delta$.

Now suppose a single finite value v determines the vector y'. The only feasible value for y''_M is $+\infty$ because we must have $y''_M > y'_M$ to obtain a nontrivial inequality with Y continuously distributed. Since the elements of y' must be ordered we arrive at the case considered in Section 4.1.1.

Bids not continuously distributed When Y is not continuously distributed the case with $y''_M = v$ and

$$y' = (-\infty, -\infty, \ldots, -\infty, \underbrace{v}_{\text{position } n}, v, \ldots, v)$$

may deliver a nontrivial inequality when $n = M$, but not when $n < M$.

With $n = M$ there is[13]

$$\begin{aligned} G_U\left(\mathcal{U}(y', y''_M, z; h)\right) &= \mathbb{P}[U_M \geq F_z(v) \geq U_{M-1}] \\ &= M F_z(v)^{M-1}(1 - F_z(v)) \end{aligned}$$

$$\mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{U}(y', y''_M, z; h)|z] = \mathbb{P}\left[Y_M = v|z\right]$$

and the condition (3.4) delivers the following inequalities.

$$\forall v: \quad M F_z(v)^{M-1}(1 - F_z(v)) \geq \mathbb{P}\left[Y_M = v|z\right]$$

With $n < M$, $G_U\left(\mathcal{U}(y', y''_M, z; h)\right)$ is zero because, in the set $\mathcal{U}(y', y''_M, z; h)$ under consideration, $U_{M-1} = F_z(v)$ and U_{M-1} is the second largest uniform order statistic which is continuously distributed. When Y is not continuously distributed the probability

$$\mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{U}(y', y''_M, z; h)|z] = \mathbb{P}\left[\left.\bigwedge_{m=n}^{M} Y_M = v\right| z\right] \tag{4.8}$$

may be positive in which case the inequality (3.4) is violated whatever candidate distribution F_z is considered.

This violation for any F_z arises because a nonzero probability (4.8) with $n < M$ cannot occur under the restrictions of the model. This is so because if for some v, $Y_{M-1} = Y_M = v$ with nonzero probability then under the restrictions of the model the second largest valuation, $V_{M-1:M}$, is equal

[13] The expression for $G_U\left(\mathcal{U}(y', y''_M, z; h)\right)$ is obtained as:

$$\begin{aligned} \mathbb{P}[U_M \geq F_z(v) \geq U_{M-1}] &= M! \int_{F_z(v)}^{1} \int_0^{F_z(v)} \int_0^{u_{M-1}} \cdots \int_0^{u_2} du \\ &= M! \int_{F_z(v)}^{1} \int_0^{F_z(v)} \frac{u_{M-1}^{M-2}}{(M-2)!} du_{M-1} du_M \end{aligned}$$

which delivers the result as stated.

to v with nonzero probability which violates the requirement that valuations are continuously distributed.

4.1.4 Contiguous unions depending on two values of y

The bounds (4.5) and (4.7) are the bounds developed in HT. We now show that valuation distribution functions in the identified set are subject to additional restrictions. To do this we turn to inequalities delivered by the containment functional inequality (3.4) applied to test sets $\mathcal{S}$ which are contiguous unions of U-level sets characterized by *two* values of Y.

There are just two types of contiguous union of U-level sets that are determined by two values of Y, v_1 and v_2.

Case 1 In this case: $y''_M = +\infty$ and, with $v_1 \geq v_2$,

$$y' = (-\infty, \ldots, -\infty, \underbrace{v_2}_{\text{position } n_2}, \ldots, v_2, \underbrace{v_1}_{\text{position } n_1}, \ldots, v_1).$$

The containment functional probability (4.4) is

$$\mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{U}(y', y''_M, z; h)|z] = \mathbb{P}\left[\bigwedge_{m=n_1}^{M} Y_m \geq v_1 \wedge \bigwedge_{m=n_2}^{n_1-1} Y_m \geq v_2 \middle| z\right]$$

while (4.3) delivers

$$G_U\left(\mathcal{U}(y', y''_M, z; h)\right) = \mathbb{P}\left[\bigwedge_{m=n_1}^{M} U_m \geq F_z(v_1) \wedge \bigwedge_{m=n_2}^{n_1-1} U_m \geq F_z(v_2)\right]$$

which, plugged into (3.4) delivers inequalities which must be satisfied by all valuation distribution functions in the identified set for all $n_2 < n_1 \leq M$ and all $v_1 \geq v_2$.

As an example, the inequalities obtained with $n_1 = M$ and $n_2 = M - 1$ for which

$$y' = (-\infty, \ldots, -\infty, v_2, v_1)$$

are as follows.

$$\forall v_1 \geq v_2: \qquad 1 - F_z(v_1)^M - MF_z(v_2)^{M-1} + MF_z(v_1)F_z(v_2)^{M-1} \geq \mathbb{P}\left[Y_M \geq v_1 \wedge Y_{M-1} \geq v_2 | z\right] \quad (4.9)$$

These inequalities must be satisfied by all valuation distribution functions F_z in the identified set.

Case 2 In this case: $y''_M = v_1$ and, with $v_1 > v_2$,

$$y' = (-\infty, \ldots, -\infty, \underbrace{v_2}_{\text{position } n}, \ldots, v_2).$$

The containment functional probability (4.4) is

$$\mathbb{P}[\mathcal{U}(Y,Z;h) \subseteq \mathcal{U}(y', y''_M, z; h)|z] =$$
$$\mathbb{P}\left[(v_2 \leq Y_M \leq v_1) \wedge \left(\bigwedge_{m=n}^{M-1} (Y_m \geq v_2) \right) \middle| z \right]$$

while (4.3) delivers

$$G_U\left(\mathcal{U}(y', y''_M, z; h)\right) = \mathbb{P}\left[\left(\bigwedge_{m=n}^{M} U_m \geq F_z(v_2) \right) \wedge (F_z(v_1) \geq U_{M-1}) \right]$$

which, plugged into (3.4) delivers inequalities which must be satisfied by all valuation distribution functions in the identified set for all $n \leq M$ and all $v_1 > v_2$. As an example, the inequalities obtained with $n = M - 1$ are as follows.

$$\begin{aligned} \forall v_1 > v_2 : \quad & F_z(v_1)^M - M F_z(v_1) F_z(v_2)^{M-1} + (M-1) F_z(v_2)^M \\ & + M(1 - F_z(v_1))(F_z(v_1)^{M-1} - F_z(v_2)^{M-1}) \geq \\ & \mathbb{P}\left[v_1 \geq Y_M \geq v_2 \wedge Y_{M-1} \geq v_2 | z \right] \end{aligned} \quad (4.10)$$

Discussion The inequalities presented in this Section, of which (4.9) and (4.10) are examples, may not be satisfied by all valuation distributions which satisfy the HT bounds (4.5) and (4.7). Section 6 presents numerical calculations for two examples in which one or both of the new inequalities are binding.

The new bounds place restrictions on *pairs* of coordinates that can be connected by distribution functions in the identified set.

Contiguous unions of U level sets that are determined by n values of Y place restrictions on n-tuples of coordinates that can be connected by valuation distribution functions in the identified set, and n can be as large as $M + 1$.

There are also test sets which are unions of contiguous unions which are not themselves contiguous unions, so there are potentially restrictions on collections of many more than $M+1$ coordinates that can be connected by valuation distribution functions in the identified set.

5 Auction Specific Unobservable Heterogeneity

In this section we consider a model in which auction specific unobserved heterogeneity affects bidders' valuations. Allowing for such heterogeneity is important when the good being auctioned has some features observed by bidders, but not observed by the researcher, that have a common effect on each bidders' value. The unobserved variable could for example be a measure of quality unavailable to the researcher.

Unobservable auction-specific heterogeneity has featured in a variety of auction models in the recent literature, with examples including Krasnokutskaya (2011), Armstrong (2013), Roberts (2013), and Quint (2015), where bidder behavior accords with equilibrium. This section examines the use of the restrictions 1-2, originally used by HT studying open outcry ascending auctions under the IPV paradigm, to study identification when auction specific unobservable heterogeneity is allowed.

Bidder valuations are now restricted to comprise the sum of a private value component $\overline{V}_m$ and an auction-specific component V^+ as follows.

Restriction UAH (Unobservable Auction Heterogeneity). Valuations in an M bidder auction are given by $V = (V_1, ..., V_M)$ where for each m, $V_m = \overline{V}_m + V^+$, where $\overline{V}_m \sim F_z(\cdot)$ and $V^+ \sim F_{z+}(\cdot)$. Both $F_z(\cdot)$ and $F_{z+}(\cdot)$ are strictly increasing on their supports.

Restriction UAH introduces auction-specific unobservable heterogeneity. Both distributions $F_z(\cdot)$ and $F_{z+}(\cdot)$ may vary with z. The joint distribution of $\overline{V}_1, \ldots, \overline{V}_M$ and V^+ is left unrestricted. The notation $\mathcal{F}_+$ is used to denote a particular collection $\{F_{z+}(\cdot) : z \in \mathcal{R}_Z\}$ and F_+ denotes those collections of $\mathcal{F}_+$ admitted by the model specification.[14]

Define the random vector $U \in [0,1]^{M+1}$ as

$$U \equiv \left(F_z\left(\overline{V}_{1:M}\right), ..., F_z\left(\overline{V}_{M:M}\right), F_{z+}\left(V^+\right)\right)$$

such that the first M components of U have a joint distribution which is the joint distribution of the order statistics of M (possibly dependent) marginally uniform$(0,1)$ variables. The last $(M+1)^{th}$ component of U is marginally uniform$(0,1)$. This is a generalization of the vector U defined in the previous section, which can be obtained by setting V^+ equal to zero and $V_m = \overline{V}_m$.

As in previous sections we continue to work with bid order statistics, with each m^{th} bid order statistic denoted $Y_m = B_{m:M}$, and with $G_z(\mathcal{S})$ denoting the probability that $U \in \mathcal{S}$ conditional on $Z = z$ for any $\mathcal{S} \in [0,1]^{M+1}$. $\mathcal{G}$ denotes a particular collection $\{G_z(\cdot) : z \in \mathcal{R}_Z\}$ and G denotes such collections which are admitted by the model specification. $G_z^-(\mathcal{S})$ is used to denote the probability that the first M components of U belong to any set $\mathcal{S} \subseteq [0,1]^M$.

[14] For example, F_+ could specify that $F_{z+}(\cdot)$ does not vary with z.

The set of feasible values of unobservable U as a function of Y is given by

$$\mathcal{U}\left(Y;F_z,F_{z+}\right)\equiv\left\{\left(u,u^+\right)\in\left[0,1\right]^{M+1}:\begin{array}{c} u_1\leq\cdots\leq u_M, \\ \wedge\quad \forall m,\,Y_m\leq F_z^{-1}\left(u_m\right)+F_{z+}^{-1}\left(u^+\right), \\ \wedge\quad Y_M\geq F_z^{-1}\left(u_{M-1}\right)+F_{z+}^{-1}\left(u^+\right)\end{array}\right\}, \tag{5.1}$$

equivalently

$$\mathcal{U}\left(Y;F_z,F_{z+}\right)\equiv\left\{\begin{array}{c}\left(u,u^+\right)\in\left[0,1\right]^{M+1}, \\ u_1\leq\cdots\leq u_M\end{array}:\max_{m=1,\ldots,M}Y_m-F_z^{-1}\left(u_m\right)\leq F_{z+}^{-1}\left(u^+\right)\leq Y_M-F_z^{-1}\left(u_{M-1}\right)\right\}.$$

With a slight abuse of notation, for any set $\mathcal{Y}\subseteq\mathcal{R}_Y$ let

$$\mathcal{U}\left(\mathcal{Y};F_z,F_+\right)\equiv\bigcup_{y\in\mathcal{Y}}\mathcal{U}\left(y;F_z,F_{z+}\right), \tag{5.2}$$

denote the union of level sets $\mathcal{U}\left(y;F_z,F_{z+}\right)$ across $y\in\mathcal{Y}\subseteq\mathcal{R}_Y$.

From Theorem 2 it follows that under Restrictions 1-2 and UAH, for each z the identified set for $\left(F_z\left(\cdot\right),F_{z+}\left(\cdot\right),G_z\left(\cdot\right)\right)$ are those that satisfy

$$\forall\mathcal{Y}\subseteq\mathcal{R}_Y,\quad\mathbb{P}\left[Y\in\mathcal{Y}|z\right]\leq G_z\left(\mathcal{U}\left(\mathcal{Y};F_z,F_{z+}\right)\right).$$

For any $\mathcal{Y}\subseteq\mathcal{R}_Y$ the probability $\mathbb{P}\left[Y\in\mathcal{Y}|z\right]$ is identified from knowledge of the joint distribution of (Y,Z). The probability $G_z\left(\mathcal{U}\left(\mathcal{Y};F_z,F_{z+}\right)\right)$ is the probability of the event that $U\in\mathcal{U}\left(y;F_z,F_{z+}\right)$ for some $y\in\mathcal{Y}$. That is

$$\left\{U\in\mathcal{U}\left(\mathcal{Y};F_z,F_{z+}\right)\right\}\Leftrightarrow\left\{\exists y\in\mathcal{Y}:U\in\mathcal{U}\left(y;F_z,F_{z+}\right)\right\},$$

and consequently,

$$\begin{aligned} G_z\left(\mathcal{U}\left(\mathcal{Y};F_z,F_{z+}\right)\right) &= G_z\left(\exists y\in\mathcal{Y}:U\in\mathcal{U}\left(y;F_z,F_{z+}\right)\right) \\ &= G_z\left(\exists y\in\mathcal{Y}:\max_{1\leq m\leq M}y_m-F_z^{-1}\left(U_m\right)\leq F_{z+}^{-1}\left(U^+\right)\leq y_M-F_z^{-1}\left(U_{M-1}\right)\right) \end{aligned} \tag{5.3}$$

where with a slight abuse of notation, $G_z\left(\mathcal{E}\right)$ denotes the probability of event $\mathcal{E}$ determined by the realization of random vector U when U is distributed G_z.

For any given set $\mathcal{Y}\subseteq\mathcal{R}_Y$, if the event

$$\mathcal{E}\left(\mathcal{Y}\right)\equiv\left\{\exists y\in\mathcal{Y}:\max_{m=1,\ldots,M}y_m-F_z^{-1}\left(U_m\right)\leq F_{z+}^{-1}\left(U^+\right)\leq y_M-F_z^{-1}\left(U_{M-1}\right)\right\}$$

whose probability determines $G_z(\mathcal{U}(\mathcal{Y}; F_z, F_{z+}))$ in (5.3) occurs, then the event

$$\tilde{\mathcal{E}}(\mathcal{Y}) \equiv \left\{ \exists y \in \mathcal{Y} : \max_{m=1,\ldots,M} y_m - F_z^{-1}(U_m) \leq y_M - F_z^{-1}(U_{M-1}) \right\} \tag{5.4}$$

must occur. Furthermore, when $\tilde{\mathcal{E}}(\mathcal{Y})$ occurs, then without further restrictions imposed on the distribution of auction-specific heterogeneity, there necessarily exists *some* cumulative distribution function F_{z+} for V^+ such that $\mathcal{E}(\mathcal{Y})$ occurs.

The M inequalities appearing in $\tilde{\mathcal{E}}(\mathcal{Y})$ can alternatively be obtained by differencing the inequalities delivered by the HT restrictions 1-2 appearing in (5.1). By definition U is an element of the set defined in (5.1) if and only if

$$\forall m, \quad Y_m \leq F_z^{-1}(U_m) + F_{z+}^{-1}\left(U^+\right) \quad \wedge \quad Y_M \geq F_z^{-1}(U_{M-1}) + F_{z+}^{-1}\left(U^+\right),$$

since the combination of any such pair of inequalities for a given m implies

$$Y_m - Y_M \leq F_z^{-1}(U_m) - F_z^{-1}(U_{M-1}). \tag{5.5}$$

Looked at in this way the inequality that is crucial in determining the identified set under auction-specific heterogeneity is obtained in similar manner to the derivation of observable implications in which fixed effects do not appear in panel data models. Here the auction-specific unobservable is akin to a fixed effect that appears in each of the inequalities delivered by restrictions 1-2. Combining these inequalities appropriately produces further observable implications from which the common unobservable term $F_{z+}^{-1}(u^+)$ is absent. The development here produces collections of such inequalities.

Defining

$$D \equiv (Y_M - Y_1, ..., Y_M - Y_{M-2}), \tag{5.6}$$

the inequalities (5.5) taken over all m can be written

$$\forall m = 1, ..., M-2: \quad D_m \geq F_z^{-1}(u_{M-1}) - F_z^{-1}(u_m).$$

Note that D only has $M-2$ components because (5.5) holds trivially for $m \in \{M-1, M\}$.

Let $\mathcal{D}$ denote a set of vectors on the support of random vector D. It follows that for any set $\mathcal{D} \subseteq \mathcal{R}_D$,

$$D \in \mathcal{D} \Rightarrow \left\{ \exists d \in \mathcal{D} : \max_{m=1,\ldots,M-2} \left\{ F_z^{-1}(U_{M-1}) - F_z^{-1}(U_m) - d_m \right\} \leq 0 \right\}$$

and consequently

$$\mathbb{P}[D \in \mathcal{D}|Z] \leq G_z^- \left(\exists d \in \mathcal{D} : \max_{m=1,\ldots,M-2} \left\{ F_z^{-1}(U_{M-1}) - F_z^{-1}(U_m) - d_m \right\} \leq 0 \right),$$

where again with a slight abuse of notation, $G_z^-(\mathcal{E})$ denotes the probability of event $\mathcal{E}$ determined by the realization of $(U_1, ..., U_M)$ when this vector is distributed G_z^-.

The development using CR from which (5.3) was obtained allows us to establish that without additional restrictions placed on F_{z+}, these inequalities characterize sharp bounds on $(F_z(\cdot), G_z(\cdot))$. The following theorem collects the formal results.

Theorem 4. *Let $\mathcal{F} \in \mathsf{F}$, $\mathcal{F}_+ \in \mathsf{F}_+$, and $\mathcal{G} \in \mathsf{G}$, and let Restrictions 1-2, EX, and UAH hold. Then (i) The identified set for $\{F_z(\cdot), F_{z+}(\cdot), G_z(\cdot) : z \in \mathcal{R}_z\}$ are those admitted by $\mathsf{F}, \mathsf{F}_+, \mathsf{G}$ that satisfy, for almost every $z \in \mathcal{R}_z$:*

$$\forall \mathcal{Y} \subseteq \mathcal{R}_Y, \quad \mathbb{P}[Y \in \mathcal{Y}|z] \leq G_z \left(\exists y \in \mathcal{Y} : \max_{m=1,\ldots,M} y_m - F_z^{-1}(U_m) \leq F_{z+}^{-1}\left(U^+\right) \leq y_M - F_z^{-1}(U_{M-1}) \right).$$

(ii) With no restrictions placed on F_+, the identified set for $\{F_z(\cdot), G_z^-(\cdot) : z \in \mathcal{R}_z\}$ are those admitted by F, G that satisfy, for almost every $z \in \mathcal{R}_z$:

$$\forall \mathcal{D} \subseteq \mathcal{R}_D, \quad \mathbb{P}[D \in \mathcal{D}|Z] \leq G_z^- \left(\exists d \in \mathcal{D} : \max_{m=1,\ldots,M-2} \left\{ F_z^{-1}(U_{M-1}) - F_z^{-1}(U_m) - d_m \right\} \leq 0 \right). \tag{5.7}$$

(iii) With no restrictions placed on F_+, the identified set for $\{F_z(\cdot) : z \in R_z\}$ are those admitted by F such that for some $G_z^-(\cdot)$ admitted by G, (5.7) holds. If, in addition, Restriction CIPV holds, then $G_z^-(\cdot)$ corresponds to the joint distribution of the order statistics of $M-1$ independent uniform$(0,1)$ variables, so that for any $F_z(\cdot)$ the probability on the right of (5.7) is known.

In addition to characterizing identified sets under the stated restrictions, Theorem 4 provides a starting point for examining further simplifications of these characterizations that may be attainable under particular restrictions on F, F_+, and G. For example, the theorem allows for, but requires neither independence of observed characteristics Z and latent auction heterogeneity V^+, nor independence between V^+ and $\overline{V}$. Such restrictions may allow further simplification of these characterizations.

6 Bounds in numerical examples

This Section presents graphs of some bounds for two particular joint distributions of ordered bids in 2 bidder auctions. Details of the calculations and the valuation distributions employed are given in Section B of the Appendix. In both cases the bid and valuation distributions satisfy the conditions of the auction model. The Figures show *survivor* functions, $\bar{F}_z(v) = 1 - F_z(v)$, and bounds on survivor functions.

Figure 9 shows in blue the HT bounds (4.5) and (4.7) on the valuation survivor function for Example 1. The valuation survivor function employed in the example is drawn in red. It is the survivor function of a random variable which is a mixture of two lognormal distributions.

In Figure 10 two valuation values are selected, $v_1 = 14.5$ and $v_2 = 10$ and the upper and lower bounds on $\bar{F}_z(v)$ are marked by black circles at these two values. Figure 11 shows a unit square within which we can plot possible values of the pair $(\bar{F}_z(14.5), \bar{F}_z(10))$. In this Figure the HT bounds are shown as a blue rectangle. Since $\bar{F}_z(14.5) \leq \bar{F}_z(10)$ the pair of ordinates must lie above the 45° line, drawn in orange. The new inequality (4.9) requires ordinate pairs $(\bar{F}_z(14.5), \bar{F}_z(10))$ to lie above the magenta line which first falls and then increases. This line lies below the blue rectangle and the inequality (4.9) delivers no refinement in this case. The new inequality (4.10) requires ordinate pairs to lie above the red, increasing, line and this does deliver a refinement.

Figure 11 shows that valuation survivor functions in the identified set cannot take relatively low values at $v = 10$ and relatively high values at $v = 14.5$. Survivor functions satisfying the pointwise bounds on their ordinates (4.5) and (4.7) that are relatively flat over this range are excluded from the identified set.

In Example 2 the valuation distribution function is a mixture of normal distributions. This is drawn in red in Figure 12 with bounds drawn in blue. Figure 13 shows two selected valuation values, $v_1 = 12.5$ and $v_2 = 11.5$. The blue rectangle in Figure 14 shows the pointwise bounds on the ordinates $(\bar{F}_z(12.5), \bar{F}_z(11.5))$ which must lie above the orange 45° line since the survivor function is decreasing. The new bounds (4.9) and (4.10) deliver respectively the magenta and red lines in Figure 14. Ordinate pairs $(\bar{F}_z(12.5), \bar{F}_z(11.5))$ of survivor functions in the identified set must lie above both lines. In this example both of the new inequalities serve to refine the pointwise bounds (4.5) and (4.7).

Finally we consider a *parametric* model which restricts valuations to have a Beta distribution, $Beta(\theta_1, \theta_2)$, with support on the unit interval.

We consider three-coordinate inequalities obtained using test sets which are contiguous unions with $y''_M = v_1$ and, with $v_1 \geq v_2 \geq v_3$,

$$y' = (-\infty, \ldots, -\infty, v_3, v_2).$$

The containment functional probability (4.4) for these test sets is

$$\mathbb{P}[\mathcal{U}(Y, Z; h) \subseteq \mathcal{U}(y', y''_M, z; h)|z] = \mathbb{P}\left[v_1 \geq Y_M \geq v_2 \wedge Y_{M-1} \geq v_3 \middle| z\right]$$

and equation (4.3) delivers

$$G_U\left(\mathcal{U}(y', y''_M, z; h)\right) = \mathbb{P}\left[U_M \geq F_z(v_2) \wedge F_z(v_1) \geq U_{M-1} \geq F_z(v_3)\right]$$

which is as follows.

$$\begin{aligned} G_U\left(\mathcal{U}(y', y''_M, z; h)\right) = {} & M(1 - F_z(v_1))(F_z(v_1)^{M-1} - F_z(v_3)^{M-1}) \\ & - MF_z(v_3)^{M-1}(F_z(v_1) - F_z(v_2)) + (F_z(v_1)^M - F_z(v_2)^M) \end{aligned}$$

Calculations are done for the two bidder case ($M = 2$) and a particular probability distribution of ordered bids.[15] Values of (v_1, v_2, v_3) are chosen as all selections satisfying $v_3 \leq v_2 \leq v_1$ taken from the list $\mathcal{V} \equiv (0, 0.05, 0.10, \ldots, 0.95, 1)$. Calculations are done on a 100×100 rectangular grid of values of $\theta \equiv (\theta_1, \theta_2)$ with $\theta_1 \in [0.5, 2.5]$ and $\theta_2 \in [0.65, 4.50]$.

The blue-bordered region in Figure 15 shows the values of θ that satisfy the the point-wise bounds of HT. These are the values of θ that satisfy single coordinate inequalities with $(v_1, v_2, v_3) \in \{(0, v, 1), (0, 0, v), (v, v, 1)\}$ for $v \in \mathcal{V}$. The red-bordered region shows the values of θ that satisfy the two and three coordinate inequalities and the crosshatched region shows values of θ that satisfy all the inequalities. The probability distribution generating value $\theta = (1, 2)$ is marked in yellow. There is a significant reduction in the calculated identified set for θ on considering the new inequalities. Experience in practice will depend on the bid distributions that arise and on the models employed.

7 Concluding remarks

The incomplete model of English auctions studied in HT falls in the class of Generalized Instrumental Variable models introduced in CR. Results in CR characterize the identified set of structures delivered by a GIV model. Applying the results to the auction model delivers a characterization of the sharp identified set for that model.

The CR development of sharp identified sets uses results from random set theory. CR shows that a structure, comprising a structural function h and distribution of unobservable random variables, G_U, is in the identified set of structures delivered by a model and distribution of observable random variables if and only if G_U is the distribution of a selection[16] of the random U-level set delivered by the structural function h and the distribution of observable values under consideration. The identified set is then characterized using necessary and sufficient conditions for this selectionability

[15] This distribution is obtained as follows. Valuations are identically and distributed $Beta(1, 2)$ variates. Let $V \equiv (V_1, V_2)$ denoted ordered valuations with $V_1 \leq V_2$; let $R \equiv (R_1, R_2)$ be independently distributed $Beta(2, 2)$ variates; let $D \in \{0, 1\}$ be such that $P[D = 1] = 1/2$ with V, R and D are mutually independently distributed. The lowest and highest final bids are $B_1 = D \times R_1 \times V_1$ and $B_2 = R_2 \times V_1 + (1 - R_2)V_2$. Probabilities are obtained by simulation using 10^6 independent draws of (V, R, D).

[16] A *selection* of a random set is a random variable that lies in the random set with probability one. A probability distribution is *selectionable* with respect to a random set if there exists a selection of the random set which has that probability distribution.

property to hold.

The characterization given in CR and applied here to the auction model delivers a complete description of the identified set. This means that *all* structures admitted by the model that can deliver the distribution of ordered bids, and *only* such structures, satisfy the inequalities that comprise the characterization.

HT left the issue of the sharpness of their bounds as an open question.[17] The approach adopted in HT to determining sharpness is a constructive one, effectively searching for admissible bidding strategies which deliver the distribution of final bids used to calculate the bounds for every distribution of valuations in a proposed identified set. As noted in HT[18] this is difficult to carry through in the auction model. Constructive proofs of sharpness have the advantage that they deliver at least one of the many complete, observationally equivalent, specifications of the process under study. However they are frequently hard to obtain. The method set out in CR and applied here has the advantage that sharpness is guaranteed.

The new bounds for the auction model exploit information contained in the joint distribution of ordered final bids unlike the pointwise bounds on the levels of valuation distributions which depend only on marginal distributions of ordered bids. This new information is useful because the joint distribution of ordered final bids is informative about the spacing of ordered bids which in turn is informative about the shape of the valuation distribution.

The characterizations using the CR development additionally open the door to the use of Restrictions 1-2 in auction models that do not require independent private values. These restrictions on bidder behavior, introduced by HT, are intuitively appealing in open outcry auctions where the usual button auction equilibrium may not always seem an appropriate model of bidder behavior. HT showed that even though these restrictions may seem a strong relaxation of equilibrium and render the model incomplete, they can still be used to learn useful information about valuation distributions in a model with IPV. In many auctions studied in empirical research, the IPV paradigm may be questionable, and this has sometimes motivated the use of auction models allowing for other information paradigms, such as affiliated private values. Theorems 1, 2, and 4 do not require IPV. Theorem 4 in particular applies to models that allow for affiliated private values through auction-specific unobservable heterogeneity, which has been a focus of some recent papers in the literature, see for example Krasnokutskaya (2011), Armstrong (2013), Roberts (2013), and Quint (2015).

Another important area of study are auction models with selective entry. Gentry and Li (2014) take a constructive approach to proof of sharpness in such model. They produce pointwise bounds on the value of a distribution function of valuations at each value of its argument and prove pointwise sharpness. Taking the approach adopted here may lead to shape restrictions which lead

[17] See Section VIII of HT.
[18] See Appendix D of HT.

to refinement of the identified set of valuation distribution functions.

The characterization of the sharp identified set in the English auction model involves a dense system of inequalities. The inequalities restrict not only the level of the valuation distribution function at each point in its support but also the shape of the function as it passes between the pointwise bounds. In practice, with a finite amount of data and computational resource, some selection of inequalities will be required. How to make that selection is an open research question.

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A Proofs of results stated in the main text

Proof of Lemma 1. Consider a realization (b, v) of (B, V). Under Restriction 1 the number of elements of b with values greater than $v_{m:M}$ is at most $M - m$. Therefore in all realizations of (B, V), $b_{m:M} \leq v_{m:M}$ for all m and M, from which (3.6) follows immediately. The second result, (3.7), follows directly from Restriction 2. $\square$

Proof of Theorem 1. The result follows from application of Corollary 1 of Theorem 2 and Lemma 1 in CR with U-sets $\mathcal{W}(B, F_z)$ as defined in (3.2). $\square$

Proof of Theorem 2. From Theorem 1 we have that the identified set for $(F_z(\cdot), G_z(\cdot) : z \in \mathcal{R}_Z)$ are those such that

$$\mathbb{P}[\mathcal{W}(B, F_z) \subseteq \mathcal{S} | Z = z] \leq G_z(\mathcal{S}) \tag{A.1}$$

for all $\mathcal{S}$ comprising unions of sets on the support of $\mathcal{W}(B, F_z)$, a.e. $z \in \mathcal{R}_Z$. From CR Lemma 1 this is equivalent to

$$\forall \mathcal{S} \in \mathsf{C}(\mathcal{R}_W), \quad \mathbb{P}[\mathcal{W}(B, F_z) \subseteq \mathcal{S} | Z] \leq G_z(\mathcal{S}) \tag{A.2}$$

where $\mathsf{C}(\mathcal{R}_W)$ denotes the collection of closed sets $\mathcal{S}$ on $[0,1]^M$ the support of W.

For the purpose of the proof, fix $z \in \mathcal{R}_Z$ at an arbitrary value. All probability statements below are to be understood to be conditional on $Z = z$.

Let $\tilde{W}$ be a random M vector defined on $(\Omega, \mathcal{A}, \mathbb{P})$ distributed $G_z(\cdot)$ and independent of B. Then (A.2) is equivalent to

$$\forall \mathcal{S} \in \mathsf{C}(\mathcal{R}_W), \quad \mathbb{P}[\mathcal{W}(B, F_z) \subseteq \mathcal{S}] \leq \mathbb{P}\left(\tilde{W} \in \mathcal{S}\right). \tag{A.3}$$

Let $\tilde{B}$ denote that random M-vector whose components are precisely those of B, reordered in such a way that for each $m = 1, ..., M$, the m^{th} lowest component has the same index as that of the m^{th} lowest component of $\tilde{U}$. $\tilde{B}$ is therefore a random permutation of B, and since $\tilde{U}$ and B are independent, exchangeability of B implies that $\tilde{B}$ has the same distribution as B.

We consequently have that (A.3) holds if and only if

$$\forall \mathcal{S} \in \mathsf{C}(\mathcal{R}_U), \quad \mathbb{P}\left[\mathcal{W}\left(\tilde{B}, F_z\right) \subseteq \mathcal{S}\right] \leq \mathbb{P}\left[\tilde{W} \in \mathcal{S}\right]. \tag{A.4}$$

Equivalently, by Artstein's inequality we have that the distribution of $\tilde{W}$ is selectionable with respect to the distribution of $\mathcal{W}\left(\tilde{B}, F\right)$, such that there exist random vectors $\tilde{W}^* \stackrel{d}{=} \tilde{W}$ ($\stackrel{d}{=} W$) and $\tilde{B}^* \stackrel{d}{=} \tilde{B}$ ($\stackrel{d}{=} B$) such that $\mathbb{P}\left[\tilde{W}^* \in \mathcal{W}\left(\tilde{B}^*, F\right)\right] = 1$.[19] By definition of the set-valued mapping

[19] Artstein's inequality is from Artstein (1983), see also Norberg (1992) and Molchanov (2005, Sectdion 1.4.8).

$\mathcal{W}(\cdot, F)$, we therefore have that (A.4) is equivalent to the statement that with probability one

$$\forall m,\ F\left(\tilde{B}^*_m\right) \leq \tilde{W}^*_m \wedge F\left(\tilde{B}^*_{m\left(\tilde{B}^*\right)}\right) \geq \max_{m \neq m\left(\tilde{B}^*\right)} \tilde{W}^*_m.$$

Because the elements of $\tilde{B}^*$ and $\tilde{W}^*$ have the same ordering, this holds if and only if with probability one

$$\forall m,\ F\left(\tilde{B}^*_{m:M}\right) \leq \tilde{W}^*_{m:M} \wedge F\left(\tilde{B}^*_{M:M}\right) \geq \tilde{W}^*_{M-1:M},$$

or equivalently, since $\tilde{B}^* \overset{d}{=} B$, $Y = (B_{1:M}, ..., B_{M:M})$, and $\tilde{W}^* \overset{d}{=} W$, if the distribution of $U = (W_{1:M}, ..., W_{M:M})$ is selectionable with respect to the distribution of

$$\mathcal{U}(Y, z; h) \equiv \left\{u \in [0,1]^M : u_1 \leq \cdots \leq u_M \wedge \forall m,\ u_m \geq F(Y_m) \wedge u_{M-1} \leq F(Y_M)\right\}.$$

The distribution of U is that of the order statistics of W, such that for any set $\mathcal{S}$, $\mathbb{P}[U \in \mathcal{S}] = G^o_z(\mathcal{S})$, so that by application of Artstein's (1983) inequality, the above selectionability condition is equivalent to

$$\forall \mathcal{S} \in \mathsf{C}(\mathcal{R}_U), \quad \mathbb{P}[\mathcal{U}(Y, z; h) \subseteq \mathcal{S} | z] \leq G^o_z(\mathcal{S}).$$

Since the choice of z was arbitrary, this concludes the proof. □

Proof of Theorem 3. Under Restriction IPV, for each $z \in \mathcal{R}_Z$, the random variables $V_1, ..., V_M$ are independently distributed F_z conditional on $Z = z$. Since $U_m = F_z^{-1}(V_{m:M})$, it follows that the distribution of $U = (U_1, ..., U_M)$ is that of the order statistics of M independent and identically distributed uniform$(0,1)$ variates, which is uniform with constant density on that part of the unit M-cube where $U_1 \leq U_2 \leq \cdots \leq U_M$, see for example Section 2.2 of David and Nagaraja (2003). Thus the characterization of Theorem **2**, but with $G^o_z(\mathcal{S})$ replaced with $G_U(\mathcal{S})$, the probability placed on $\mathcal{S}$ by this known distribution. □

Proof of Theorem 4. Part (i) follows from application of Theorem 2 as described in the main text. Part (ii) follows from first noticing that CR Corollary 1 and Lemma 1 in conjunction with the definition of random vector D imply that the set of $(F_z(\cdot), G^-_z(\cdot))$ satisfying (5.7) are precisely those such that conditional on $Z = z$ there exist random vectors $\tilde{Y} \overset{d}{=} Y | Z = z$ and $\tilde{U} \sim G^-_z(\cdot)$ satisfying (5.5) with each Y_m replaced by $\tilde{Y}_m$ and U replaced by $\tilde{U}$. This guarantees that for all m,

$$Y_m - F_z^{-1}(U_m) \leq Y_M - F_z^{-1}(U_{M-1}),$$

and so there exists a random variable $\tilde{V}^+$ such that with probability one

$$Y_m - F_z^{-1}(U_m) \leq V^+ \leq Y_M - F_z^{-1}(U_{M-1}).$$

Thus we have established the existence of random vectors $\tilde{Y} \overset{d}{=} Y|Z = z$ and $\tilde{U} \sim G_z^-(\cdot)$ such that Restrictions 1-2 and UAH hold. Part(iii) follows from taking the implied set of feasible F_z from part (ii) and that with Restriction CIPV, $G_z^-(\cdot)$ is known. Part (iv) holds because there are no restrictions placed on any of the conditional distributions across different values of z. □

B Calculation of bid probabilities in Section 6

The probabilities used in the two examples in Section 6 were produced by simulation using 10^6 independent draws of identically distributed independent pairs of valuations from a valuation distribution and a fully specified stochastic mechanism that delivers final bids given valuations. Distribution functions of ordered bids and the various probabilities that appear in bounds are simply calculated as proportions of simulated ordered bids that meet the required conditions. In the two examples valuations have different distributions and final bids are obtained from valuations in different ways.

In Example 1 the valuation distribution is specified as a mixture of two log normal distributions, one $LN(0, 1)$ and the other $LN(2.5, 0.5^2)$ with mixture weights respectively 0.3 and 0.7.[20] In each of the 10^6 simulated auctions two independent realizations of this mixture distribution are sampled and sorted to deliver realizations of ordered valuations.

Independently distributed random variables, Θ_1 and Θ_2 with identical symmetric Beta distributions, expected value 0.5 and standard deviation 0.05, are assigned to respectively the low and high valuation players. A fair coin toss determines who bids first. At each round of the auction the current bidder bids their value Θ times their valuation plus $(1 - \Theta)$ times the bid on the table. At the first round the bid on the table is zero. The auction ends when the bid on the table exceeds the lower of the two valuations. A player who makes no bid has a final bid recorded as zero.

In Example 2 ordered valuations are produced as in Example 1 but with the distribution from which the valuations are sampled specified as a mixture of normal distributions, one $N(10, 1)$ the other $N(12.5, 0.5^2)$ with mixture weights 0.5 attached to each distribution. The final bid of a low valuation bidder is calculated as their valuation minus an amount which is the absolute value of a independent realization of a standard normal variable. The final bid of a high bidder is simulated as a weighted average of the low and high valuations with the weight given by a realization of a uniform variate with support on $[0, 1]$. The intention in example 2 is just to produce final bid distributions which respect the stochastic dominance conditions of the model. Example 1 by contrast obtains bid distributions by simulating one possible bidding behavior.

[20] $LN(\mu, \sigma^2)$ is a random variable whose logarithm is $N(\mu, \sigma^2)$.

Figure 1: The U-level set $\mathcal{U}((y_1', y_2'), z; h)$ containing values of uniform order statistics, $u_2 \geq u_1$, that can give rise to order statistics of bids, (y_1', y_2'). F_z is the distribution function of valuations. As labelled this is for the 2 bidder case. In the M bidder case this shows a projection of a level set with u_2 (u_1) denoting the largest (second largest) order statistic of M i.i.d. uniform variates.

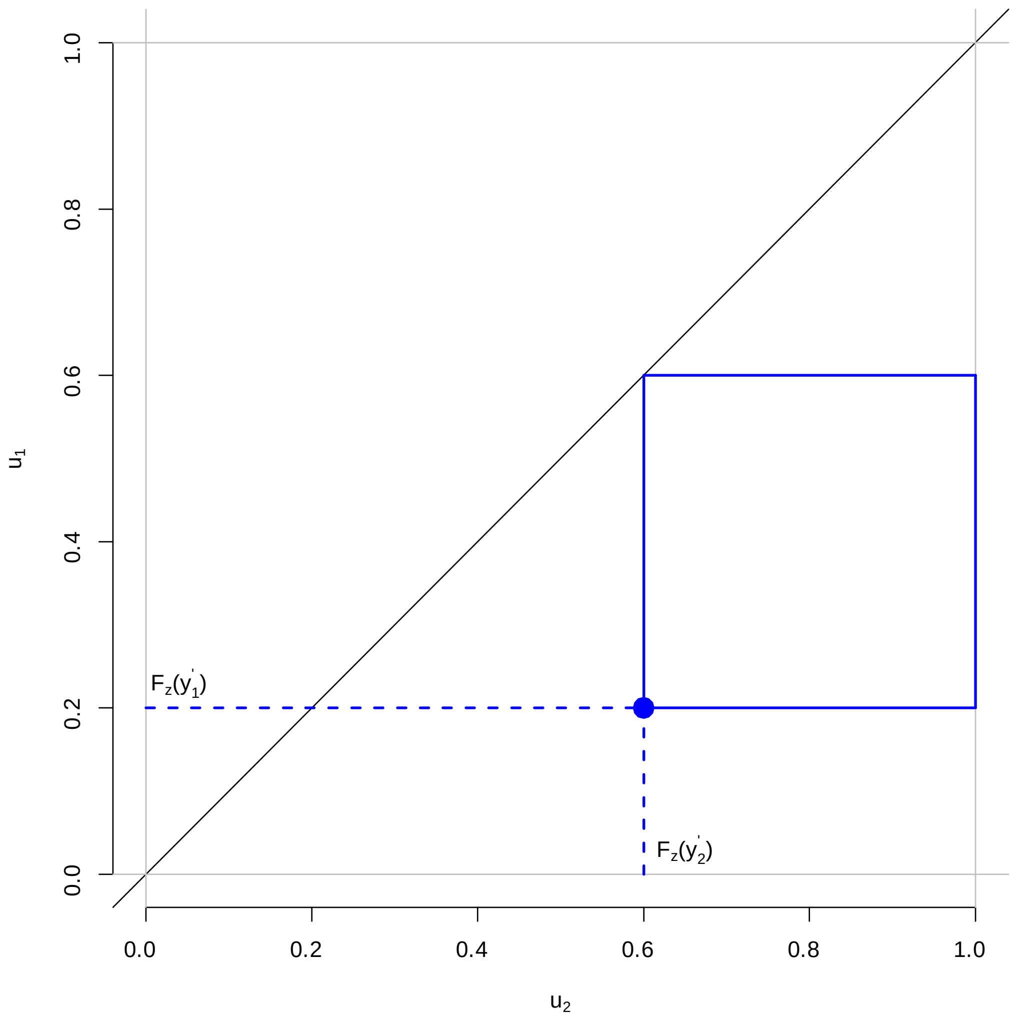

Figure 2: In blue the U-level set $\mathcal{U}((y_1', y_2'), z; h)$ containing values of uniform order statistics, $u_2 \geq u_1$, that can give rise to order statistics of bids, (y_1', y_2'). F_z is the distribution function of valuations. In magenta the U-level set obtained as y_2' is reduced as shown by the arrow. This is never a subset of the original U-level set outlined in blue. As labelled this is for the 2 bidder case.

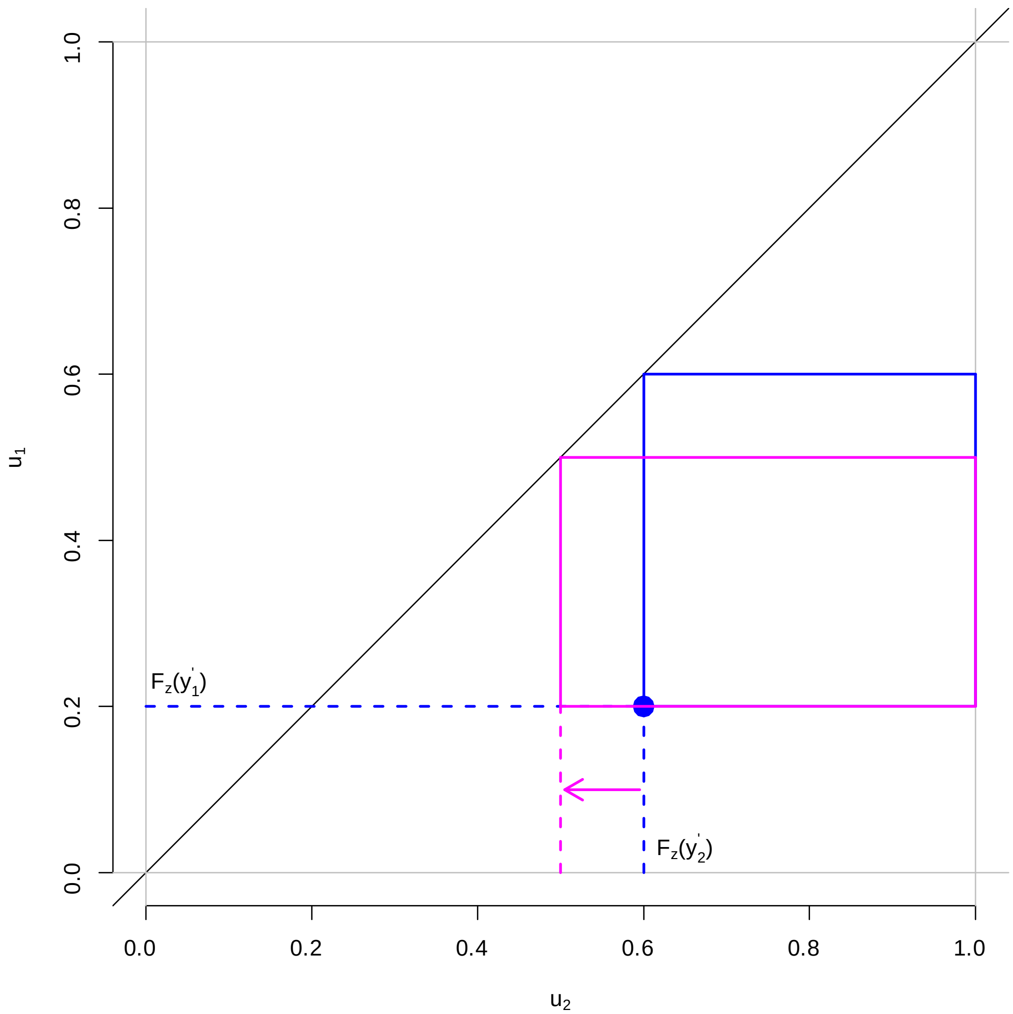

Figure 3: In blue the U-level set $\mathcal{U}((y'_1, y'_2), z; h)$ containing values of uniform order statistics, $u_2 \geq u_1$, that can give rise to order statistics of bids, (y'_1, y'_2). F_z is the distribution function of valuations. In magenta the U-level set obtained as y'_2 is increased as shown by the arrow. This is never a subset of the original U-level set outlined in blue. As labelled this is for the 2 bidder case.

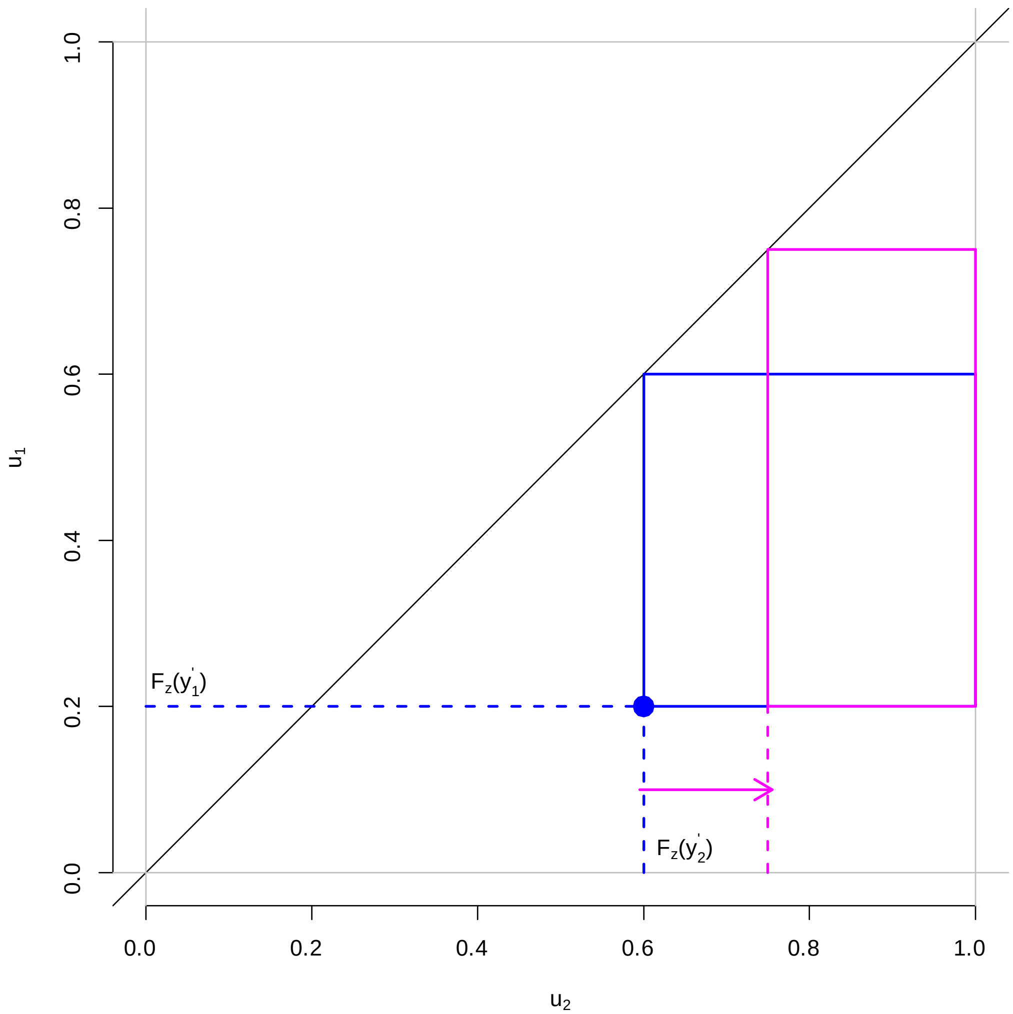

Figure 4: In blue the U-level set $\mathcal{U}((y_1', y_2'), z; h)$ containing values of uniform order statistics, $u_2 \geq u_1$, that can give rise to order statistics of bids, (y_1', y_2'). F_z is the distribution function of valuations. In magenta the U-level set obtained as y_1' is increased as shown by the arrow. This is always a subset of the original U-level set. As labelled this is for the 2 bidder case.

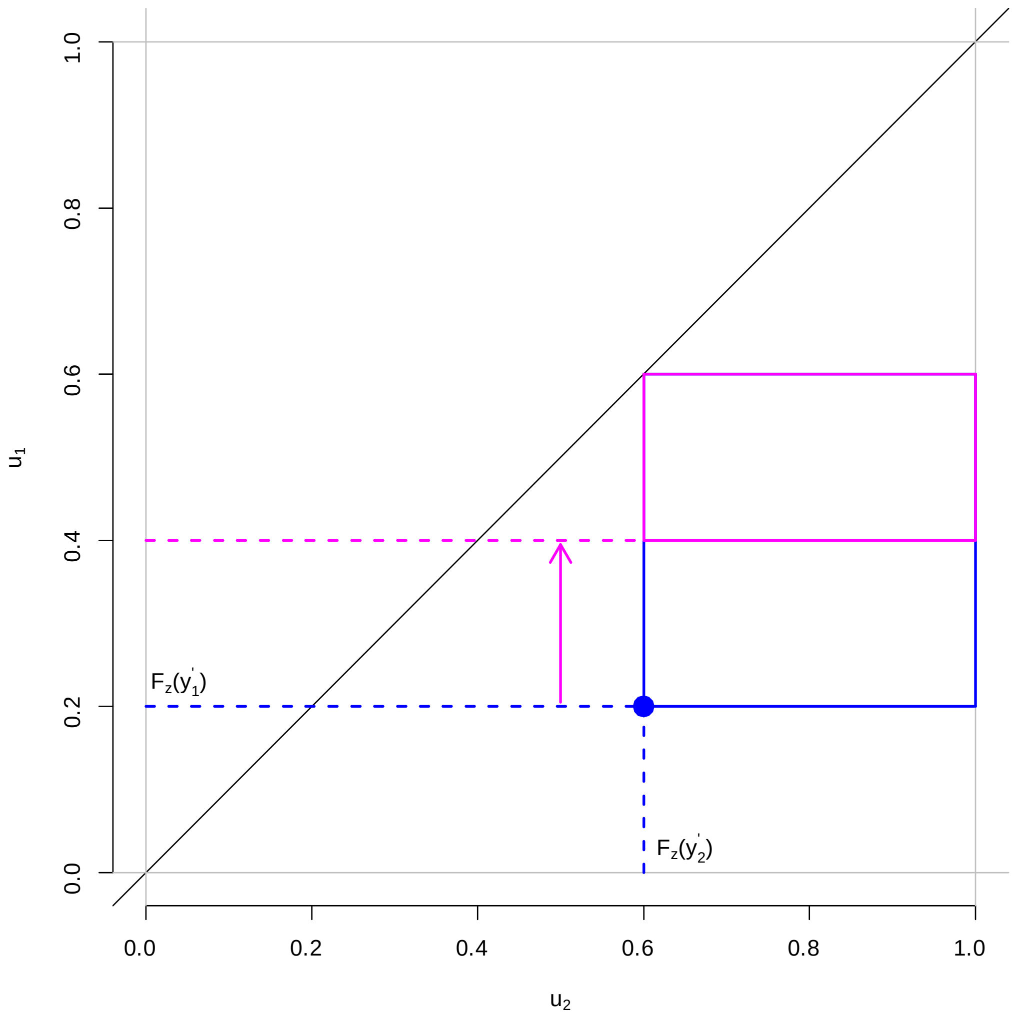

Figure 5: 2 bidder case. The contiguous union of level sets: $\mathcal{U}(y', y_2'', z; h)$ where $y' = (y_1', y_2')$, y_2' and y_2'' are values taken by the maximal order statistic of bids, Y_2, and y_1' is a value taken by the second largest order statistic of bids, Y_1. In the labels, F_z is the distribution function of valuations. The shaded area indicates the values of Y that give a U-level set which is a subset of the contiguous union.

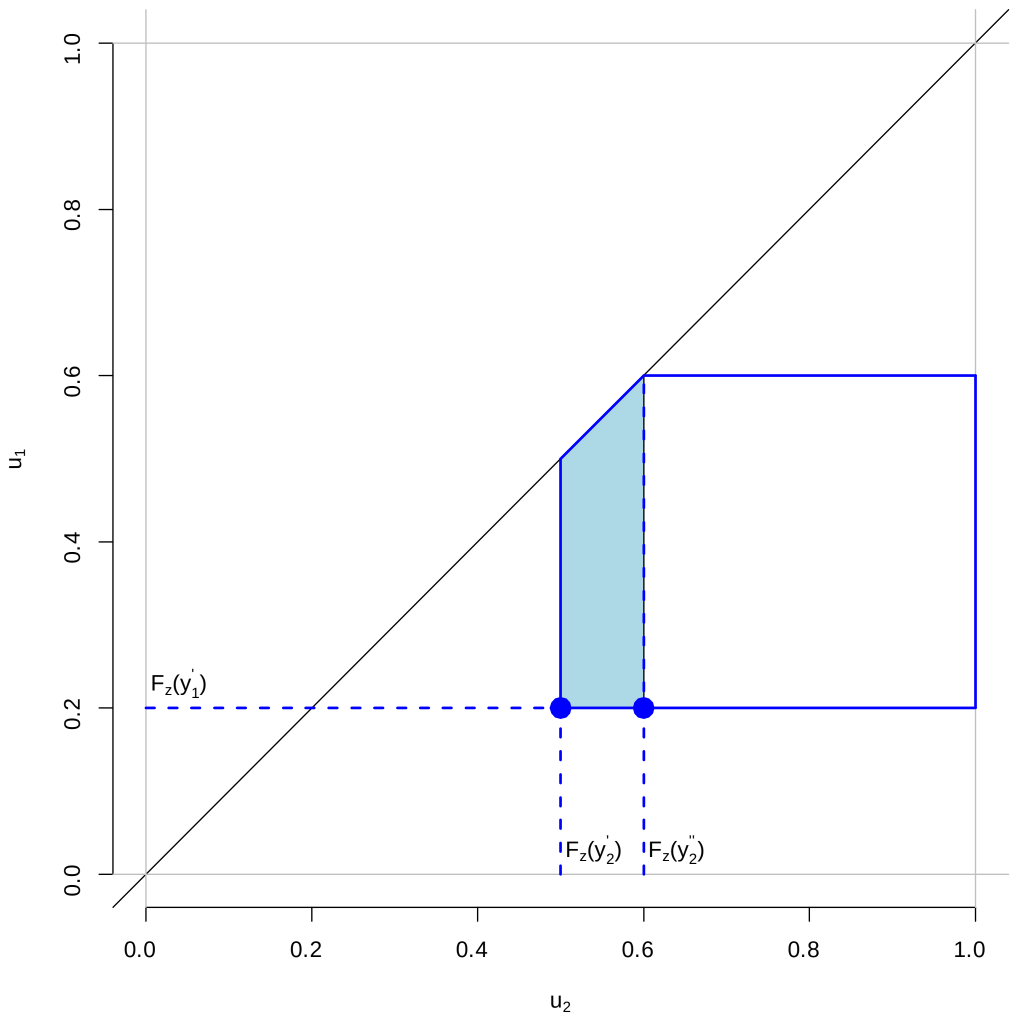

Figure 6: 2 bidder case. The triangular region outlined in blue is the contiguous union of level sets: $\mathcal{U}(y', y_2'', z; h)$ where $y_1' = y_2' = v$ and $y_2'' = \infty$. F_z is the distribution function of valuations. This choice of y' and y_2'' delivers the inequality requiring the second highest valuation to stochastically dominate the second highest bid. The shaded area indicates the values of Y that give a U-level set which is a subset of the contiguous union.

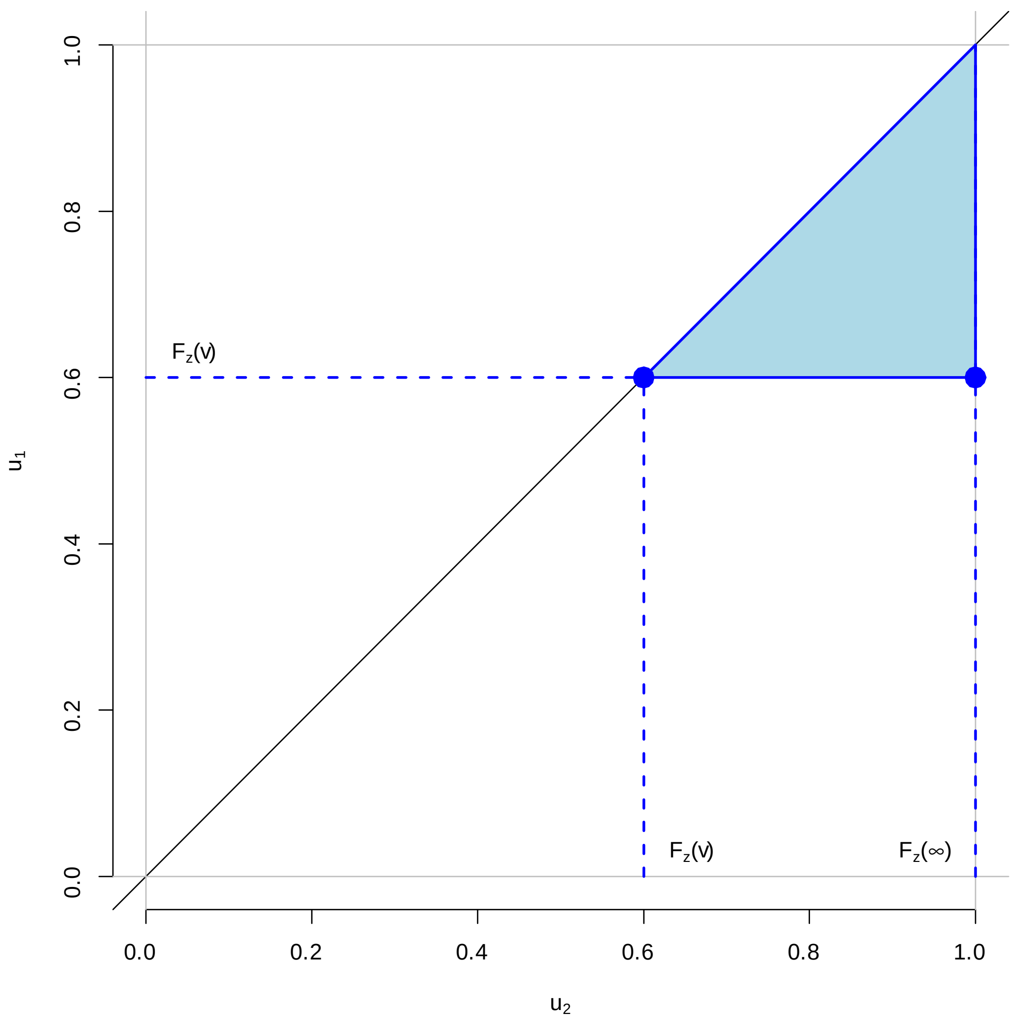

Figure 7: 2 bidder case. The trapezoidal region outlined in blue is the contiguous union of level sets: $\mathcal{U}(y', y_2'', z; h)$ where $y' = (-\infty, v)$ and $y_2'' = \infty$. F_z is the distribution function of valuations. This choice of y' and y_2'' delivers the inequality requiring the highest valuation stochastically dominates the highest bid. The shaded area indicates the values of Y that give a U-level set which is a subset of the contiguous union.

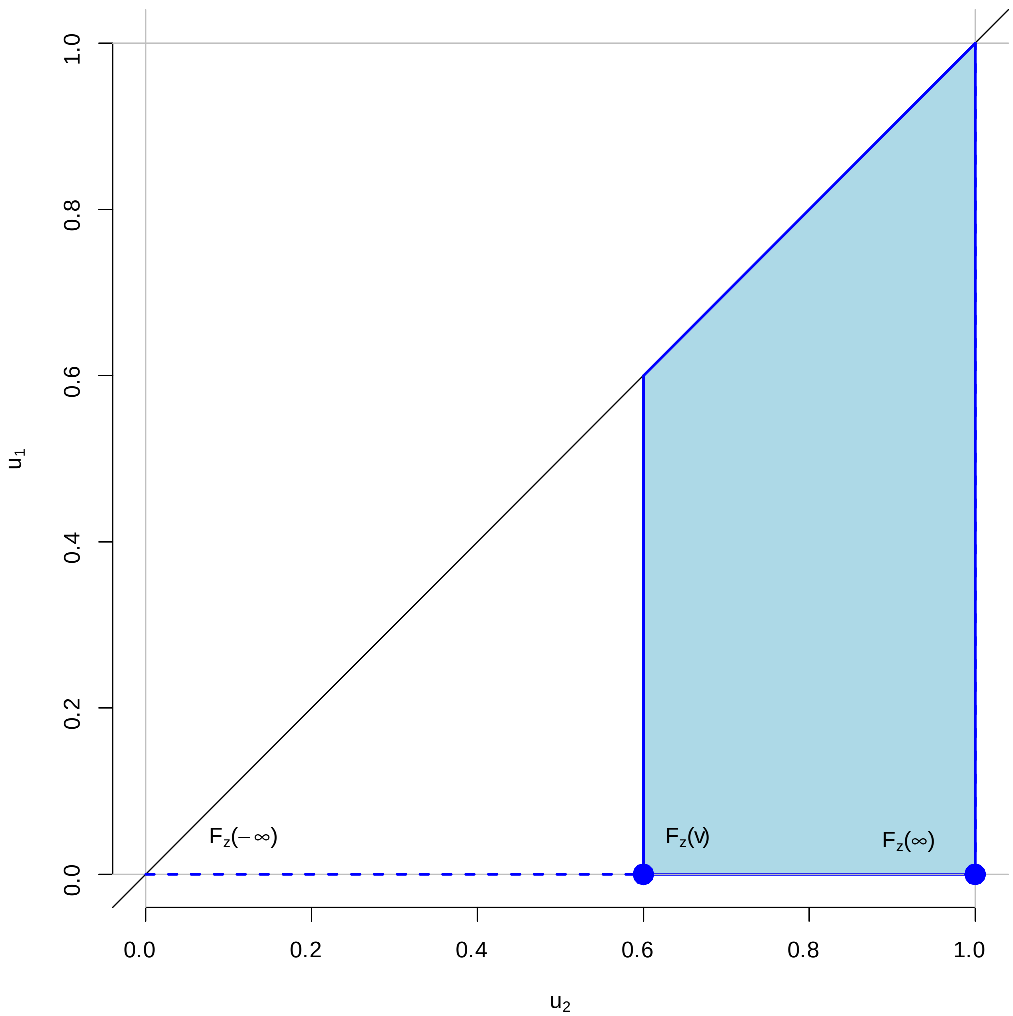

Figure 8: 2 bidder case. The trapezoidal region outlined in blue is the contiguous union of level sets: $\mathcal{U}(y', y_2'', z; h)$ where $y' = (-\infty, -\infty)$ and $y_2'' = v$. F_z is the distribution function of valuations. This choice of y' and y_2'' delivers the inequality requiring the highest bid stochastically dominates the second highest valuation. The shaded area indicates the values of Y that give a U-level set which is a subset of the contiguous union.

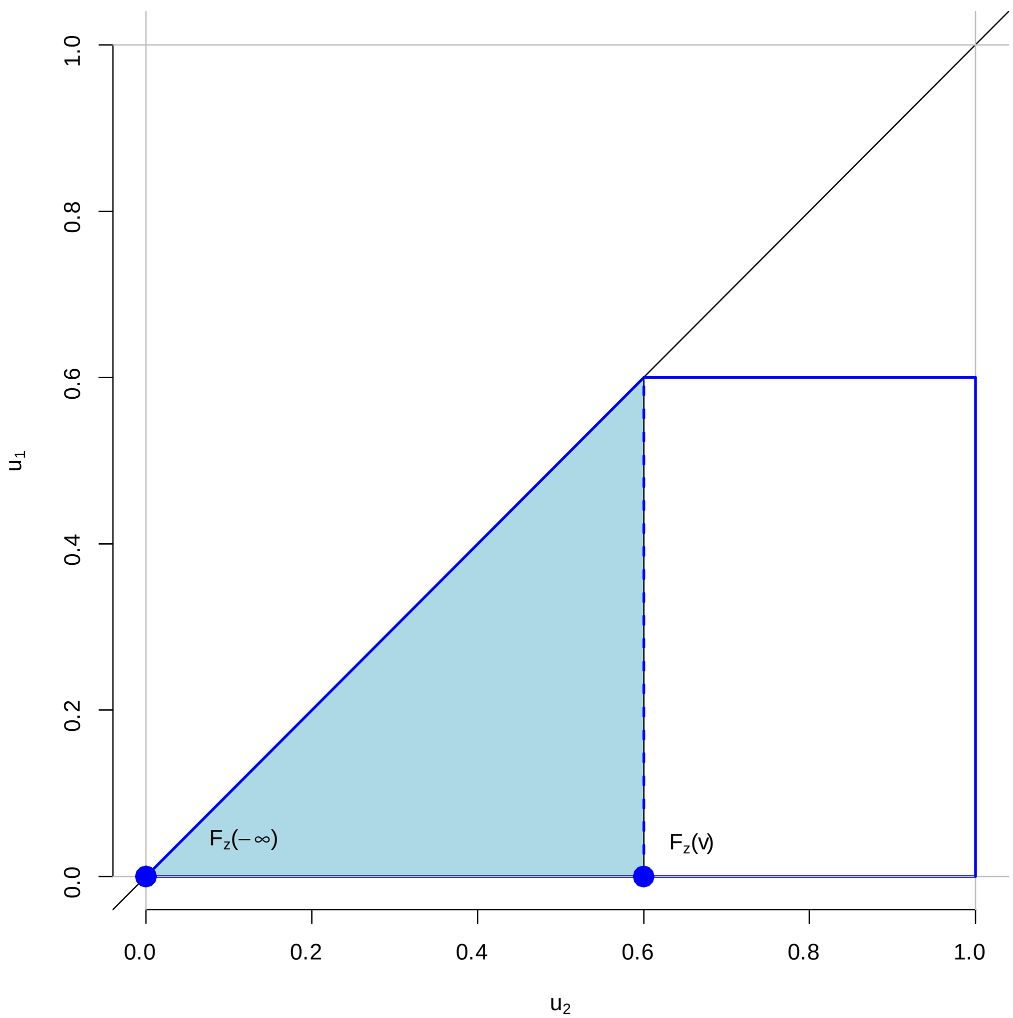

Figure 9: Example 1. Upper and lower bounds (blue) on the valuation survivor function (red).

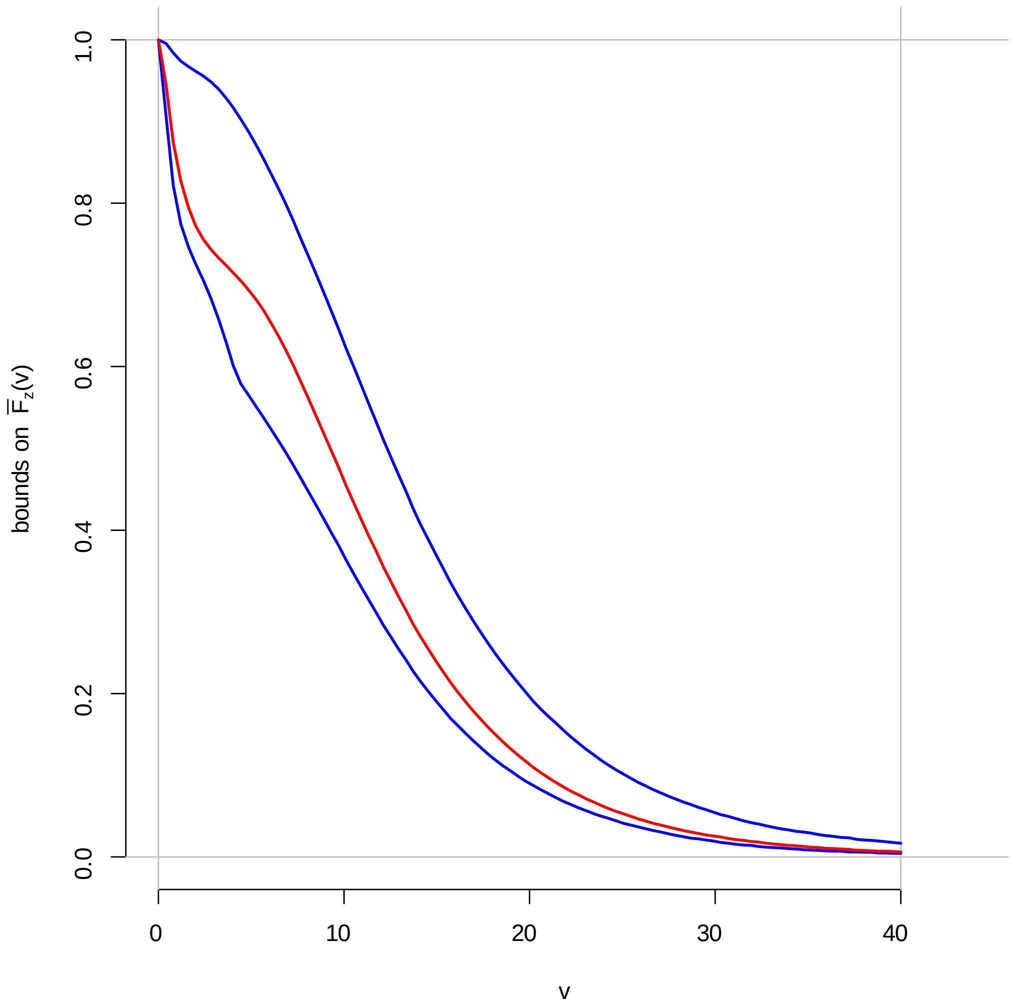

Figure 10: Example 1. Upper and lower bounds (blue) on the valuation survivor function (red). Two values of v, $v_1 = 14.5$ and $v_2 = 10$ are identified.

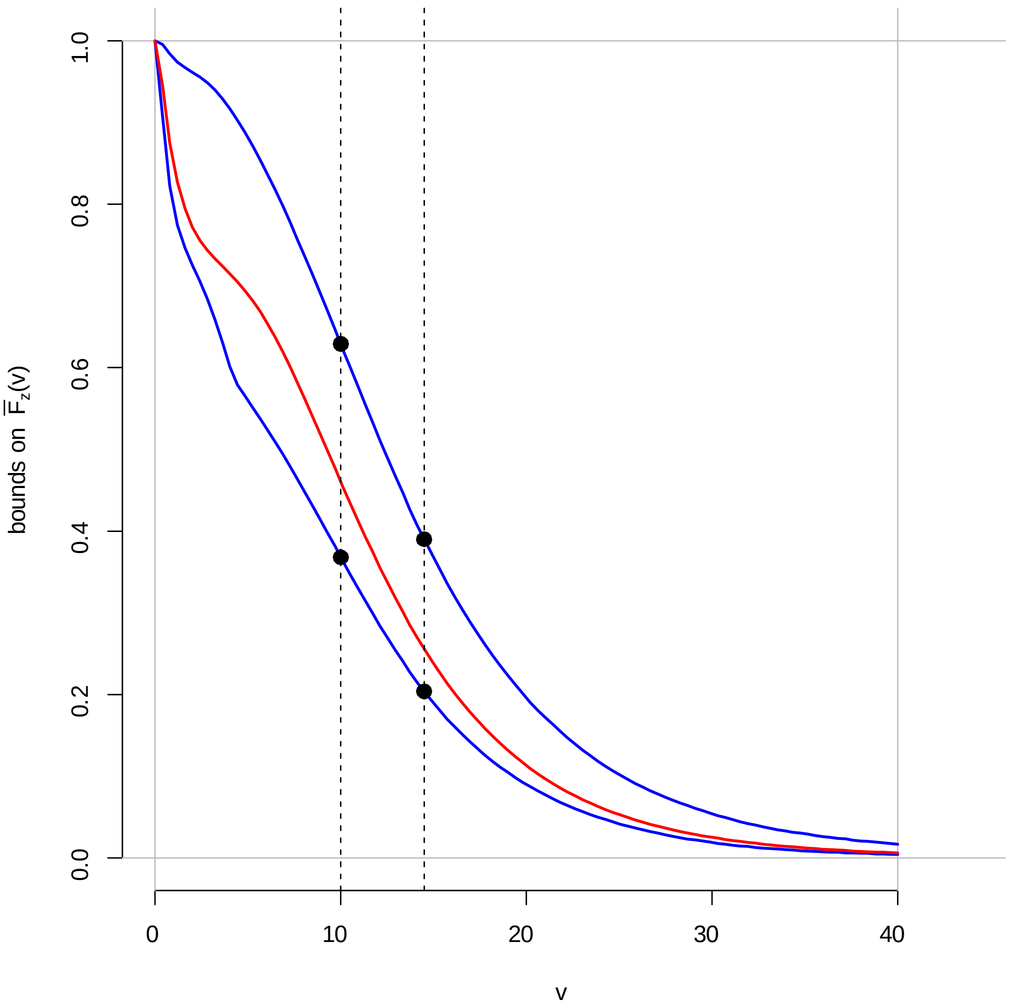

Figure 11: Example 1. The blue rectangle shows upper and lower bounds on $\bar{F}_z(v_1)$ and $\bar{F}_z(v_2)$ at $v_1 = 14.5$ and $v_2 = 10$. These ordinates of the valuation survivor function must lie above the 45° line (orange). The new bounds require they lie above the magenta and red lines as well. Only the red line delivered by inequality 4.10 is binding here.

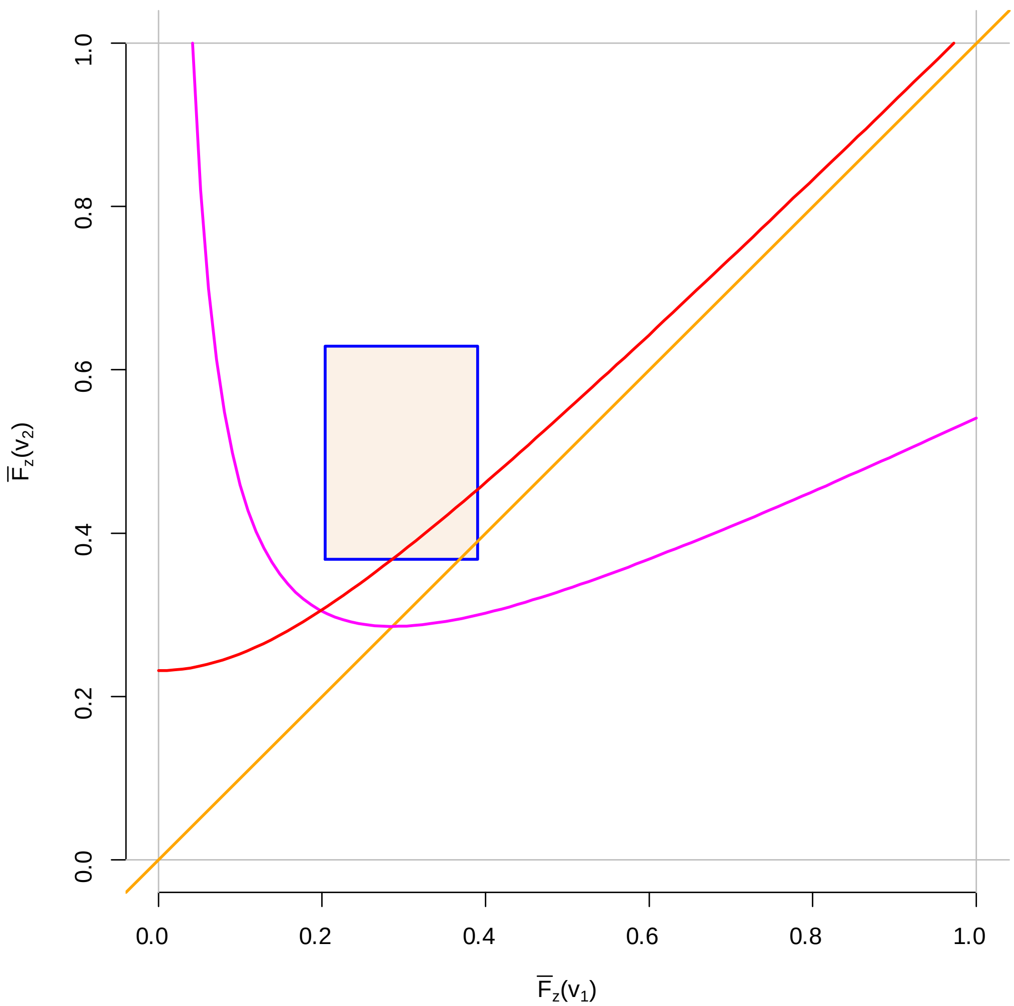

Figure 12: Example 2. Upper and lower bounds (blue) on the valuation survivor function (red).

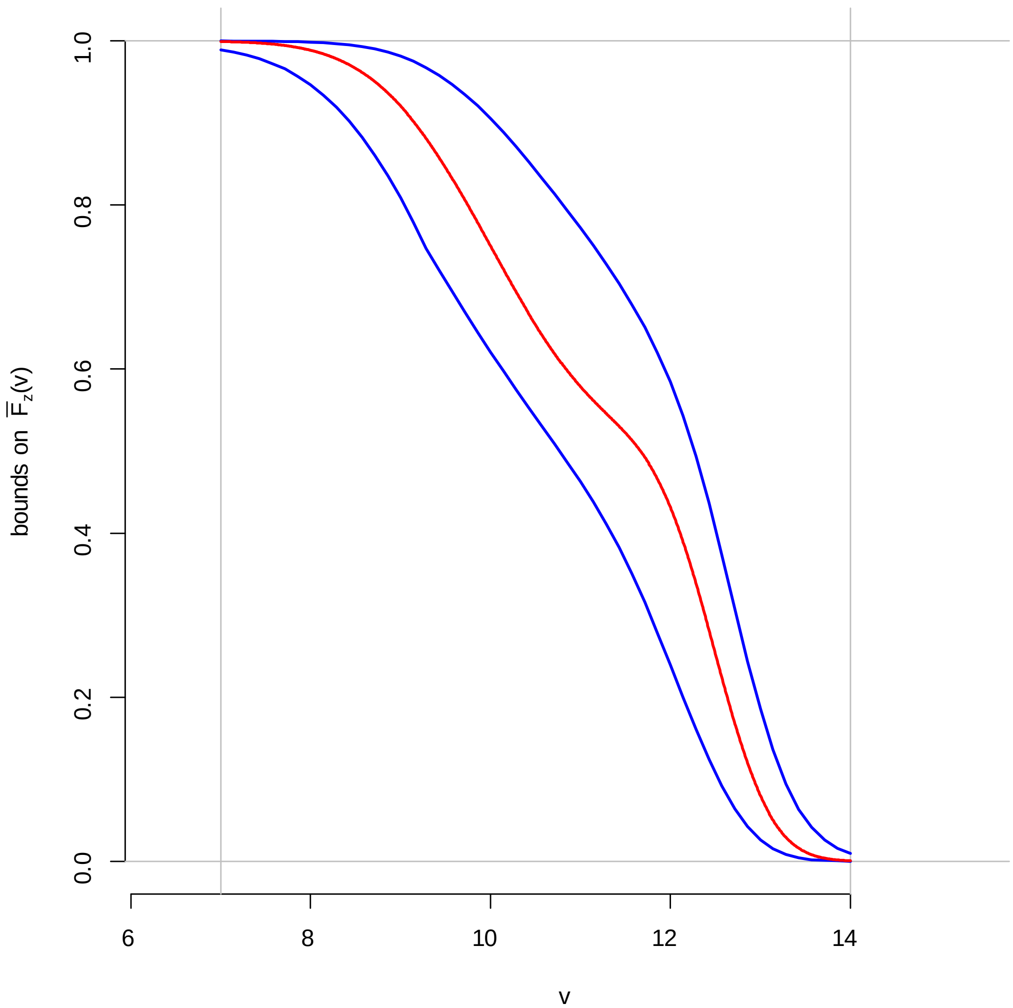

Figure 13: Example 2. Upper and lower bounds (blue) on the valuation survivor function (red). Two values of v are identified: $v_1 = 12.5$ and $v_2 = 11.5$.

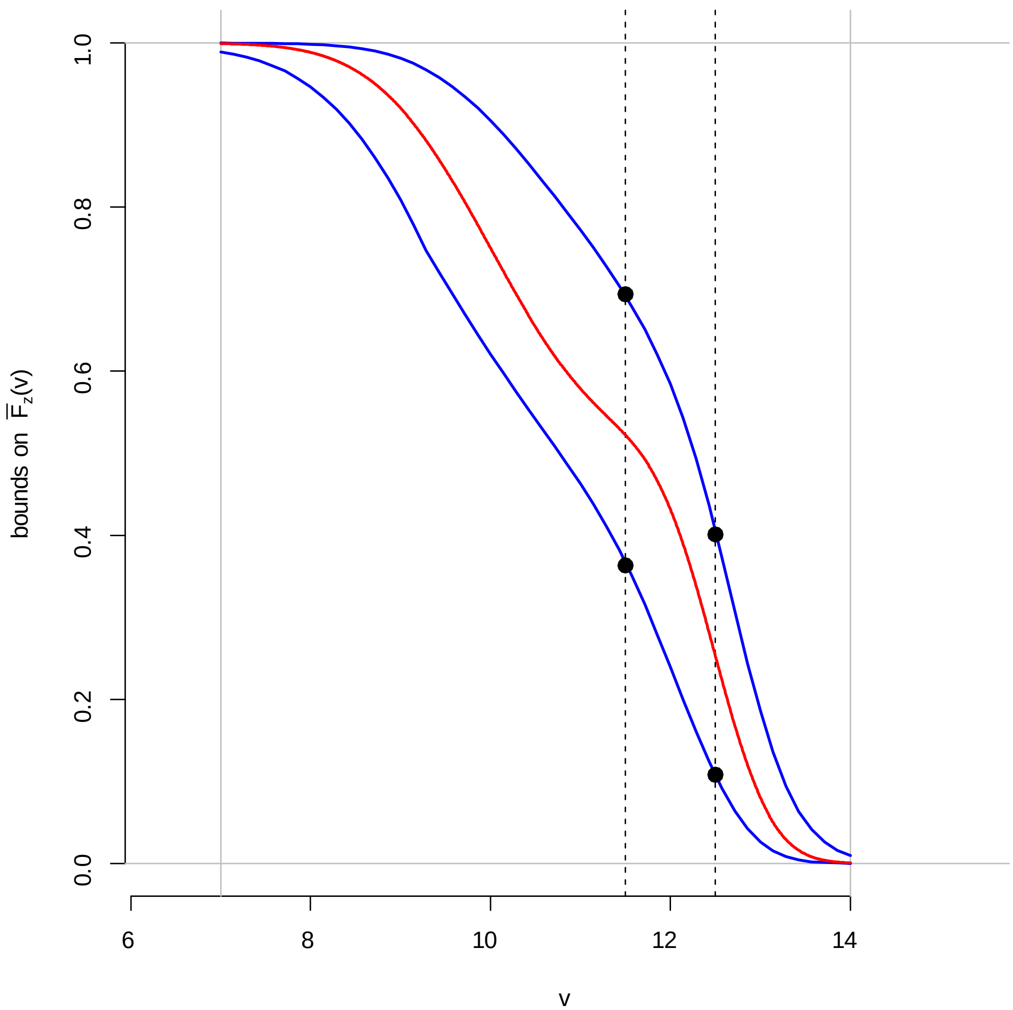

Figure 14: Example 2. The blue rectangle shows upper and lower bounds on $\bar{F}_z(v_1)$ and $\bar{F}_z(v_2)$ at $v_1 = 12.5$ and $v_2 = 11.5$. These ordinates of the valuation survivor function must lie above the 45° line (orange). The new bounds (4.9) and (4.10) require they lie above the magenta and red lines as well.

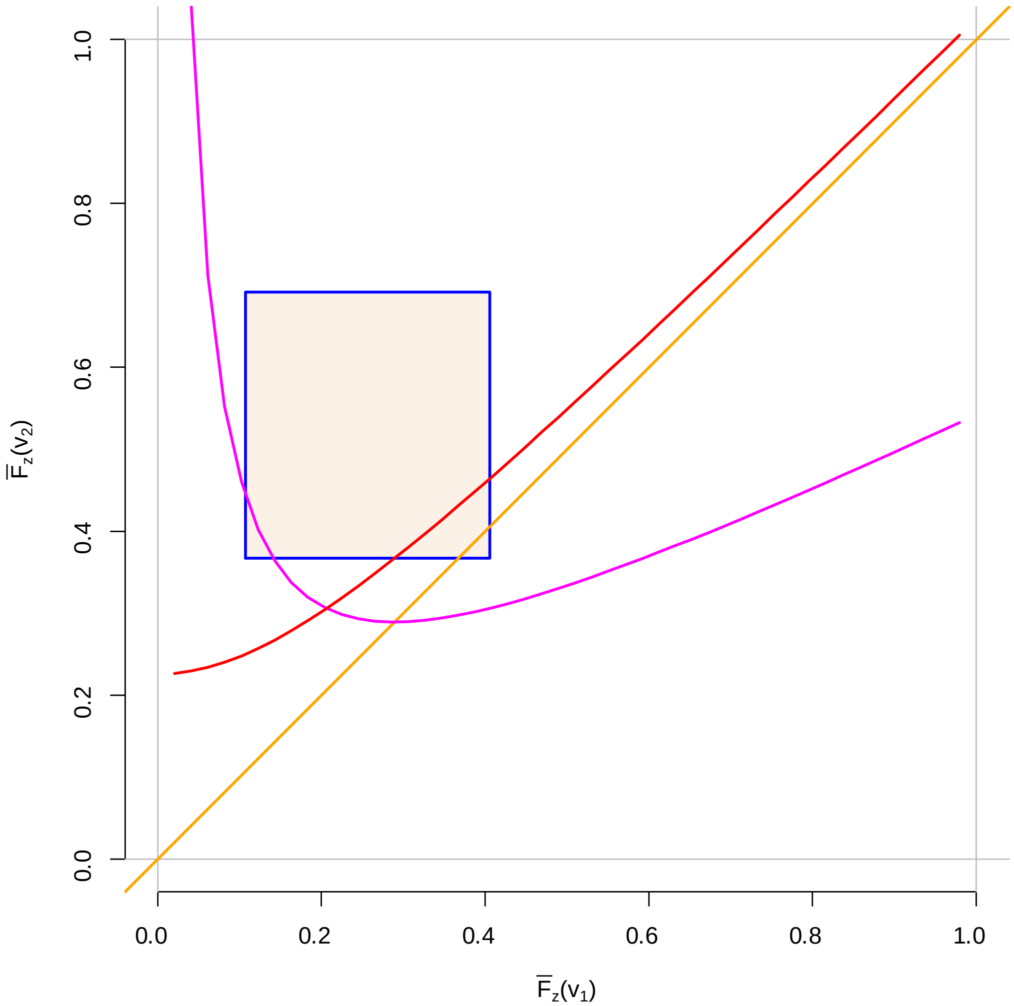

Figure 15: Outer regions for Beta valuation distribution parameters $\theta = (\theta_1, \theta_2)$ in a particular numerical example. The blue-bordered region contains values of θ satisfying the HT pointwise upper and lower bounds. The red-bordered region contains values of θ that satisfy the new inequalities. The magenta hatched region contains values that satisfy both sets of inequalities. The yellow dot marks the value of θ used to generate the probability distribution of valuations, and thus final bids, employed in the example.