Inference for the jump part of quadratic variation of Itô semimartingales

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Abstract

When asset prices are modelled by Itô semimartingales, their quadratic variation consists of a continuous and a jump component. This paper is about inference on the jump part of the quadratic variation, which we estimate by using the difference of realised variance and realised multipower variation.

The main contribution of this paper is that we provide a bivariate asymptotic limit theory for realised variance and realised multipower variation in the presence of jumps. From that result, we can then deduce the asymptotic distribution of the estimator of the jump component of quadratic variation and can make inference on it. Furthermore, we present consistent estimators for the asymptotic variances of the limit distributions which allows us to derive a feasible asymptotic theory. Monte Carlo studies reveal a good finite sample performance of the proposed feasible limit theory, and an empirical study shows the relevance of our result in practice.

1 Introduction

Inference on the variation of asset prices has been studied in great detail in the last decade. Due to the fact that high frequency asset price data have become widely available, one can now use nonparametric methods which exploit the specific structure of high frequency data to learn about the price variation over a given period of time. While logarithmic asset prices have often been modelled by Brownian semimartingales, the focus of research has recently shifted towards more general models which allow for jumps in the price process. This paper follows

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this recent stream of research by assuming that the logarithmic asset price is given by an Itô semimartingale of the form

$$dY_t = b_t dt + \sigma_t dW_t + dJ_t,$$

which consists of a Brownian semimartingale ($b_t dt + \sigma_t dW_t$) and a jump component (dJ_t) (the exact assumptions and regularity conditions will be defined more precisely below).

This paper is about inference on the jump part of the quadratic variation process of the price process. The quadratic variation (see e.g. Protter (2004)) of the price process is given by

$$[Y]_t = [Y]_t^c + [Y]_t^d,$$

where

$$[Y]_t^c = \int_0^t \sigma_s^2 ds \qquad \text{and} \qquad [Y]_t^d = \sum_{0 \le s \le t} (\Delta J_s)^2$$

denote the continuous and discontinuous (or jump) parts of the quadratic variation, respectively. While inference on the continuous part of the quadratic variation has been studied in detail in the literature (see e.g. Barndorff-Nielsen & Shephard (2002)), inference on the discontinuous part has not been studied explicitly yet. However, an indirect way to gain information on the jump part of quadratic variation is given by any test for the presence of jumps (e.g. Barndorff-Nielsen & Shephard (2006), Aït-Sahalia & Jacod (2006) and Jacod & Todorov (2007)). So Barndorff-Nielsen & Shephard (2006) have studied a related question and introduced several non–parametric tests for testing the null hypothesis of no jumps versus the alternative of jumps. In order to make inference on the jump part of quadratic variation, our first steps will follow the methodology of Barndorff-Nielsen & Shephard (2006), who exploited the fact that jumps in the asset price are reflected in a jump part of the quadratic variation and vice versa. So their main idea was to compare two measures of variance: one which is not robust to jumps, a quantity called *realised variance* (see e.g. Comte & Renault (1998), Barndorff-Nielsen & Shephard (2002), Andersen, Bollerslev, Diebold & Labys (2001)), that estimates the total variation of the price process, and one which is robust to jumps, called *realised bipower variation* (see e.g. Barndorff-Nielsen & Shephard (2004), Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006)), and only estimates the continuous part of the variance. By using the difference or the ratio of these two quantities, they have found a consistent estimator for the jump part of the total price variation. Furthermore, they have derived the asymptotic distribution of these test statistics under the null hypothesis (that there are no jumps). Huang & Tauchen (2005) have carried out an extensive simulation study based on these asymptotic results; this revealed a very good finite sample performance of the proposed test statistics.

However, the asymptotic distribution of these test statistics under the alternative hypothesis (that there are jumps) has not been known yet. So in order to be able to make inference on the jump part of quadratic variation or to derive the asymptotic distribution of the test statistics under the alternative hypothesis, we have to find the asymptotic distribution of a consistent estimator of the jump part of the quadratic variation. This is exactly the task we tackle in this paper. Recently, the concept of bipower variation has been extended to *multipower variation* (see e.g. Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), Barndorff-Nielsen, Shephard & Winkel (2006), Woerner (2006)). Jacod (2006) has derived a central limit theorem for realised multipower variation from realised tripower onwards (but not for realised bipower variation). We follow this line of research and derive the main result: If one uses multipower variation of higher powers than two (e.g. tripower or quadpower variation) for estimating the continuous part of the price variation, similar test statistics as the ones proposed by Barndorff-Nielsen & Shephard (2006) can be constructed whose distributions can be calculated when there are jumps. So our key result is that

$$\frac{1}{\sqrt{\Delta_n}}\Big(\text{ (Realised variance - realised multipower variation)} - [Y]^d\Big)$$
$$\to Mixed\ Normal\left(0, constant \int \sigma_s^4 ds + 2\sum_s \left(\sigma_{s-}^2 + \sigma_s^2\right)(\Delta J_s)^2\right),$$

where the powers in the multipower variation have to satisfy some constraints as explained later (see equation (13)).

So in a first stage, we derive infeasible limit results (as given above), which means that the asymptotic distribution (particularly the asymptotic variance) depends on components of the price process which we do not observe. In a next step, we replace the unobserved asymptotic variance by a consistent estimator, which we construct by using a similar framework as the one studied in Veraart (2007) for realised versions. So in the end, we are able to make inference on the jump part of the quadratic variation and not only on the integrated variance.

The remaining part of the paper is structured as follows. Section 4.2 introduces the notation and the main model assumptions. In Section 4.3, we review the most important facts about realised variance and realised multipower variation. Section 4.4 contains the main contribution of this paper. First, we sketch some of the important theoretical work by Jacod (2007, 2006) on univariate asymptotic results for realised variance and realised multipower variation. Then, we present our main result: the asymptotic distribution of a bivariate process of realised variance and realised multipower variation in the presence of jumps. From this, we can derive the asymptotic distribution of various jump–test statistics under both the null and alternative hypotheses. Furthermore, we show how the infeasible limit results can be converted into a feasible limit theory. In Section 4.5, we carry out a detailed simulation study, which we use for assessing the finite sample performance of our proposed test statistics. Furthermore, we compare test statistics based on different powers, and we investigate the trade–off of an efficiency gain for estimating the continuous part of the variance by multipower variation of low power (tripower, quadpower) versus a decrease in the finite sample bias by using multipower variation of high power (10–power, 20–power). An empirical study is then carried out in Section 4.6, where we study some high frequency equity data and identify jumps in the price process. Finally, Section 4.7 concludes the paper and gives some prospect on future research. The proof of our main theorem and the tables with the results from the simulation study are given in the Appendices (Section 4.8.1 and 4.8.2, respectively).

2 Setup

We assume that the logarithmic asset price is given by a real–valued Itô semimartingale $Y = (Y_t)_{t\geq 0}$, which is defined on a probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ as given below, where we use the same assumptions as in Jacod (2007).

Hypothesis (H) Let

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \kappa(\delta) \star (\underline{\mu} - \underline{\nu})_t + \kappa'(\delta) \star \underline{\mu}_t, \tag{1}$$

where

- W is one–dimensional $(\mathcal{F}_t)_{t\geq 0}$–Brownian motion and $\underline{\mu}$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$;
- κ a continuous truncation function which is bounded, has compact support and $\kappa(x) = x$ in a neighbourhood of 0 and $\kappa'(x) = x - \kappa(x)$;
- ν denotes the compensator of the jump measure μ of X and $\nu(dt, dx) = dtF_t(dx)$.
- Let δ denote a predictable map from $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ on $\mathbb{R}$. Then $F_t(\omega, dx)$ is the image of the Lebesgue measure on $\mathbb{R}$ by the map $x \mapsto \delta(\omega, t, x)$;
- $\underline{\nu}(ds, dx) = ds \otimes dx$ denotes the predictable compensator of $\underline{\mu}$.
- The processes (b_t) and $\int (1 \wedge x^2) F_t(dx)$ are locally bounded $(\mathcal{F}_t)_{t\geq 0}$–predictable, and the process (σ_t) is càdlàg adapted.

Note that every Itô semimartingale admits a representation as in (1) where $\underline{\mu}$ is a Poisson random measure and $\underline{\nu}(ds, dx) = ds \otimes dx$. Further note that σ and W can be dependent in this general model framework and, hence, our model accounts for the leverage effect.

Another assumption is concerned with the jump part of the semimartingale.

Hypothesis (K) We assume that (H) is satisfied and that the coefficient δ (see (1)) satisfies $|\delta(\omega, t, x)| \leq \gamma_k(x)$ for all $t \leq T_k(\omega)$, where γ_k denote some deterministic functions on $\mathbb{R}$ which satisfy $\int (1 \wedge x^2) \circ \gamma_k(x) dx < \infty$ and (T_k) are stopping times increasing to $+\infty$.

We need also an assumption for the volatility process. In this paper, we shall focus on volatility processes which satisfy the following conditions.

Hypothesis (L-s) (H) holds and the volatility process σ has the form

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_u du + \int_0^t \widetilde{\sigma}_u dW_u + \int_0^t \widetilde{\sigma}'_u dW'_u + \kappa(\widetilde{\delta}) \star (\underline{\mu} - \underline{\nu})_t + \kappa'(\widetilde{\delta}) \star \underline{\mu}_t,$$

and

- W' is another Brownian motion on the space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which is independent of W;
- the process $(\widetilde{b}_t)$ is optional and locally bounded;
- the processes (b_t), $(\widetilde{\sigma}_t)$, $(\widetilde{\sigma}'_t)$ are adapted left–continuous with right limits in t, and locally bounded;
- the functions $\delta(\omega, t, x)$ and $\tilde{\delta}(\omega, t, x)$ are predictable and left–continuous with right limits in t. Also, $|\delta(\omega, t, x)| \leq \gamma_k(x)$ and $|\tilde{\delta}(\omega, t, x)| \leq \tilde{\gamma}_k(x)$ for all $t \leq T_k(\omega)$, where $\gamma_k, \tilde{\gamma}_k$ are deterministic functions on $\mathbb{R}$ with $\int \phi_s \circ \gamma_k(x) dx < \infty$ (where we define $0^0 = 0$) — note that this is the condition where the s comes in — and $\int \phi_2 \circ \tilde{\gamma}_k(x) dx < \infty$. We define ϕ_s by $\phi_s(x) = 1 \wedge |x|^s$ if $0 < s < \infty$ and by $\phi_s(x) = \mathrm{I}_{\mathbb{R}\setminus\{0\}}(x)$ if $s = 0$. Furthermore, (T_k) denotes a sequence of stopping times increasing to $+\infty$.

So under (H) and (L-s) for any $s \in [0,2]$, we essentially consider a Brownian semimartingale with drift and jumps. Note that we assume in (L-s) that $s \in [0,2]$. If $s \leq s' \leq 2$, then (L-s)⇒ (L-s')⇒ (K) ⇒(H). Also note that (L-0) implies that X has locally finitely many jumps and if X is continuous, then all hypotheses (L-s) are identical for all $s \in [0,2]$ (see Jacod (2007, p.6)). Finally, we formulate a hypothesis which guarantees that the semimartingale has a Brownian semimartingale component which is nowhere degenerate.

Hypothesis (H') Hypothesis (H) holds and (σ_t^2) and (σ_{t-}^2) do not vanish.

For our asymptotic theory, we need some further notation, which follows Jacod (2007)'s framework. Let $(\Omega', \mathcal{A}', \mathbb{P}')$ denote an auxiliary space which supports two Brownian motions $\overline{W}$ and $\widetilde{W}$, two sequences of $\mathcal{N}(0,1)$ random variables, denoted by (U_p) and $(U_p)'$ and, further, a sequence of random variables (ξ_p) which are uniformly distributed on $[0,1]$. All these processes are assumed to be mutually independent. Now we extend our original probability space and we write

$$\widetilde{\Omega} = \Omega \times \Omega', \qquad \widetilde{\mathcal{A}} = \mathcal{A} \otimes \mathcal{A}', \qquad \widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'.$$

One can now extend, in the obvious way, the variables $Y_t,\ b_t, \ldots$ defined on Ω and $\overline{W},\ \widetilde{W},\ U_p, \ldots$ defined on Ω' to the product space (without change of notation). Let $\widetilde{\mathbb{E}}$ denote the expectation with respect to $\widetilde{\mathbb{P}}$. Further, let (T_p) denote stopping times which are an enumeration of the jump times of Y. Finally, we write $(\widetilde{\mathcal{F}}_t)$ for the smallest right–continuous filtration of $\widetilde{\mathcal{A}}$ which contains $(\mathcal{F}_t)$ and with respect to which $\overline{W}$ is adapted and, further, such that U_p, U_p' and ξ_p are $\widetilde{\mathcal{F}}_{T_p}$–measurable for all p.

Straightforwardly, $\overline{W}$ and $\widetilde{W}$ are $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$-Brownian motions under $\widetilde{\mathbb{P}}$, which also holds for W and W'. Further $\underline{\mu}$ is a Poisson measure with compensator $\underline{\nu}$ for the bigger filtration.

3 Review of Realised Variance and Realised Multipower Variation

After having introduced the admittedly quite tedious notation for the continuous–time price process, we now turn our attention to its discrete–time observations.

Let us assume that we observe the process Y over an interval $[0,t]$ at times $i\Delta_n$ for $\Delta_n > 0$ and $i = 0, \ldots, [t/\Delta_n]$. So for its discretely observed increments, we write

$$\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n} \quad \text{for } i = 1, \ldots [t/\Delta_n].$$

In practice, these increments are used to construct estimators for the variance or integrated variance. For example, it is well–known that the *realised variance*, which is the sum of the squared increments, estimates the quadratic variation of the underlying process consistently, i.e.

$$RV_t^n = \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n Y)^2 \xrightarrow{ucp} [Y]_t, \quad \text{as } n \to \infty,$$

where the convergence is uniformly on compacts in probability (ucp) (see Protter (2004, p. 57), Andersen, Bollerslev, Diebold & Ebens (2001) and Barndorff-Nielsen & Shephard (2002)).

Besides, one can use the *realised bipower variation* (as defined by Barndorff-Nielsen & Shephard (2004, 2006)) for estimating the continuous part of the quadratic variation of Itô semimartingales (see Jacod (2006)). So for $\mu_1 = \sqrt{2/\pi}$, one obtains

$$\mu_1^{-2} \sum_{i=1}^{[t/\Delta_n]-1} |\Delta_i^n Y| \left|\Delta_{i+1}^n Y\right| \stackrel{ucp}{\longrightarrow} [Y^c]_t = \int_0^t \sigma_s^2 ds, \quad \text{as } n \to \infty.$$

This concept can be further generalised to realised multipower variation (see e.g. Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) for a treatment of realised multipower variation in the absence of jumps and Woerner (2006) and Jacod (2006) for the corresponding results in the presence of jumps). Let $\mathbf{r} = (r_1, \ldots, r_I)$ be a multi–index with $r_i > 0$. Further, we write $|\mathbf{r}| = r_1 + \cdots + r_I$ and $\mathbf{r}_+ = \max_{1 \le i \le I} r_i$ and $\mathbf{r}_- = \min_{1 \le i \le I} r_i$. Let

$$\mu_r = \mathbb{E}|U|^r, \quad \text{for } U \sim N(0,1).$$

Then

$$\Delta_n^{1-|\mathbf{r}|/2} \mu_{\mathbf{r}}^{-1} \sum_{i=1}^{[t/\Delta_n]-I} \prod_{j=1}^{I} |\Delta_{i+j-1}^n Y|^{r_j} \stackrel{ucp}{\to} \int_0^t |\sigma_u|^{|\mathbf{r}|} du, \text{ as } n \to \infty,$$

where $\mu_{\mathbf{r}} = \prod_{j=1}^{I} \mu_{r_j}$. Now we define

$$RMPV(\mathbf{r})_t^n = \frac{[t/\Delta_n]}{[t/\Delta_n] - I} \Delta_n^{1-|\mathbf{r}|/2} \mu_{\mathbf{r}}^{-1} \sum_{i=1}^{[t/\Delta_n]-I} \prod_{j=1}^{I} |\Delta_{i+j-1}^n Y|^{r_j},$$

where the factor $[t/\Delta_n]/([t/\Delta_n] - I)$ accounts for the fact that there are just $([t/\Delta_n] - I)$ terms in the sum rather than $[t/\Delta_n]$ summands as in the realised variance. By multiplying the realised multipower variation by the factor above, we make it more easily comparable to realised variance. In particular, when studying the difference of those two quantities, we do not end up with a finite sample bias which is just caused by the fact that we are comparing two similar sums with a different number of summands. Clearly,

$$RMPV(\mathbf{r})_t^n \stackrel{ucp}{\to} \int_0^t |\sigma_u|^{|\mathbf{r}|} du, \text{ as } n \to \infty.$$

Note that if $|\mathbf{r}| = 2$, then the factor $\Delta_n^{1-|\mathbf{r}|/2} = 1$ and, hence, it disappears. In particular, we are interested in realised multipower variations with equal power r_i. So we define for $k, I \in \mathbb{N}$:

$$RMPV(k;I)_t^n = \frac{[t/\Delta_n]}{[t/\Delta_n] - I} \Delta_n^{1-k/2} \mu_{k/I}^{-I} \sum_{i=1}^{[t/\Delta_n]-I} \prod_{j=1}^{I} |\Delta_{i+j-1}^n Y|^{k/I}.$$

Then

$$RMPV(k;I)_t^n \stackrel{ucp}{\longrightarrow} \int_0^t |\sigma_u|^k du, \quad \text{as } n \to \infty.$$

In particular we are interested in the case $k = 2$ when

$$RMPV(2;I)_t^n = \frac{[t/\Delta_n]}{[t/\Delta_n] - I} \mu_{2/I}^{-I} \sum_{i=1}^{[t/\Delta_n]-I} \prod_{j=1}^{I} |\Delta_{i+j-1}^n Y|^{2/I} \stackrel{ucp}{\longrightarrow} \int_0^t \sigma_u^2 du, \text{ as } n \to \infty.$$

Then, clearly,

$$RV_t^n - RMPV(2;I)_t^n \stackrel{ucp}{\longrightarrow} [Y]_t^d, \quad \text{as } n \to \infty.$$

So, the difference of realised variance and realised multipower variation is a consistent estimator for the jump part of the total variation.

Estimating the jump component of the total variation is hence a fairly easy task. However, things get significantly more complicated when we want to make inference on the jump component, which requires establishing an appropriate asymptotic theory. Let us first review some univariate asymptotic results for realised variance and realised multipower variation, which have been proven under the assumption that the price process has no jumps and, hence, is just given by a Brownian semimartingale.

From Barndorff-Nielsen & Shephard (2002, 2007*b*), we know that we obtain the following central limit result for realised variance in the absence of jumps. As $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}}\left(RV_t^n - \int_0^t \sigma_s^2 ds\right) \stackrel{stably\ in\ law}{\longrightarrow} \sqrt{2}\int_0^t \sigma_u^2 dB_u \sim MN\left(0, 2\int_0^t \sigma_s^4 ds\right)$$

stably in law as a process (for the definition of *stable convergence as a process* see e.g. Jacod & Shiryaev (2003)) where the two letters *MN* stand for *mixed normal distribution*. From Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), we get the following result for realised multipower variation in the absence of jumps. Under (L-s) for $s < 1$ and $\frac{s}{2-s} < \mathbf{r}_- < \mathbf{r}_+ < 1$ and as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}}\left(RMPV(\mathbf{r})_t^n - \int_0^t |\sigma_s|^{|\mathbf{r}|} ds\right) \stackrel{stably\ in\ law}{\longrightarrow} \mu_{\mathbf{r}}^{-1}\sqrt{A(\mathbf{r})}\int_0^t |\sigma_u|^{|\mathbf{r}|} d\widetilde{B}_u$$
$$\sim MN\left(0, \mu_{\mathbf{r}}^{-2}A(\mathbf{r})\int_0^t \sigma_s^{2|\mathbf{r}|} ds\right),$$

stably in law, where

$$A(\mathbf{r}) = \prod_{i=1}^{I}\mu_{2r_i} - (2I-1)\prod_{i=1}^{I}\mu_{r_i}^2 + 2\sum_{i=1}^{I-1}\prod_{j=1}^{i}\mu_{r_j}\prod_{j=I-i+1}^{I}\mu_{r_j}\prod_{j=1}^{I-i}\mu_{r_j+r_{j+i}},$$

where an empty product is set to 1. Note that an analogous result also holds in the presence of jumps as shown by Woerner (2006) and Jacod (2006). So in our special case of multipower variation, we get as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}}\left(RMPV(2;I)_t^n - \int_0^t \sigma_s^2 ds\right) \stackrel{\text{stably in law}}{\to} \sqrt{\omega_I^2 \mu_{I/2}^{-I}}\int_0^t \sigma_u^2 d\widetilde{B}_u$$
$$\sim MN\left(0, \omega_I^2 \mu_{I/2}^{-2I}\int_0^t \sigma_s^4 ds\right),$$

where

$$\omega_I^2 = \mu_{4/I}^I + (1-2I)\mu_{2/I}^{2I} + 2\sum_{j=1}^{I-1}\mu_{4/I}^{I-j}\,\mu_{2/I}^{2j}.$$

Note that both the Brownian motions B and $\widetilde{B}$, which appear in the limit processes, are independent of σ and W.

In the flavour of Barndorff-Nielsen & Shephard (2006), we can now construct test statistics based on the difference of realised variance and realised multipower variation. So we get

$$\frac{1}{\sqrt{\Delta_n}}(RV_t^n - RMPV(2;I)_t^n) \overset{\text{stably in law}}{\rightarrow} \sqrt{\theta_I}\int_0^t \sigma_s^2 d\bar{W}_s \sim N\left(0, \theta_I \int_0^t \sigma_s^4 ds\right), \tag{2}$$

where

$$\theta_I = \mu_{2/I}^{-2I}\omega_I^2 - 2.$$

This results builds the base for a *linear test statistic* to test for jumps in the price process.

From Slutsky's lemma one can derive a *ratio test statistic*:

$$\frac{1}{\sqrt{\Delta_n}}\left(\frac{RMPV(2;I)_t^n}{RV_t^n} - 1\right) \overset{\text{stably in law}}{\rightarrow} N\left(0, \frac{\theta_I \int_0^t \sigma_s^4 ds}{\left(\int_0^t \sigma_s^2 ds\right)^2}\right). \tag{3}$$

And, finally, we can apply the delta method and derive the corresponding result for the log–transformation:

$$\frac{1}{\sqrt{\Delta_n}}(\log(RV_t^n) - \log(RMPV(2;I)_t^n)) \xrightarrow{stably\ in\ law} N\left(0, \theta_I \frac{\int_0^t \sigma_s^4 ds}{\left(\int_0^t \sigma_s^2 ds\right)^2}\right), \tag{4}$$

which can be used for constructing a *log–linear test statistic*.

However, these limit results only hold under the null hypothesis that there are no jumps. The main contribution of this paper is that we derive the asymptotic distribution of these test statistic also under the alternative hypothesis that there are jumps. More generally, we study the asymptotic properties of the bivariate process of realised variance and realised multipower variation.

4 Central Limit Theorems in the Presence of Jumps

Let Y be our general real–valued semimartingale as defined above, which has both a Brownian semimartingale and a jump component. We are interested in studying the asymptotic properties of the bivariate process

$$\frac{1}{\sqrt{\Delta_n}}\begin{pmatrix} RV_t^n - [Y]_{\Delta_n[t/\Delta_n]} \\ RMPV(2;I)_t^n - [Y]_t^c \end{pmatrix}. \tag{5}$$

From Jacod (2007, 2006), we already know the univariate limit results for both components.

Realised variance: Assume that (L-2) is satisfied. Then as $n \to \infty$ we get from Jacod (2007, Theorem 2.11 (ii))

$$\frac{1}{\sqrt{\Delta_n}}\left(RV_t^n - [Y]_{\Delta_n[t/\Delta_n]}\right) \xrightarrow{stably\ in\ law} L_t^{(1)} + L_t^{(2)}, \tag{6}$$

where the convergence is stably in law as a process. The limiting process is given by $L_t^{(1)} + L_t^{(2)}$, where

$$L_t^{(1)} = \sqrt{2} \int_0^t \sigma_u^2 d\overline{W}_u, \tag{7}$$

$$L_t^{(2)} = 2 \sum_{p:\ T_p \leq t} \Delta Y_{T_p} \left(\sqrt{\xi_p} U_p \sigma_{T_p-} + \sqrt{1-\xi_p}\, U_p' \sigma_{T_p} \right). \tag{8}$$

Furthermore, Jacod (2007) makes the following remarks:

- Stable convergence in law only holds when the discretised process $[Y]_{\Delta_n[t/\Delta_n]}$ is used in (6). However, $\frac{1}{\sqrt{\Delta_n}} (RV_t^n - [Y]_t)$ converges *finite–dimensionally stably in law* to the limit described above (see Jacod (2007, Remark 2.14)). But the latter result will be sufficient for us since we are interested in making inference on the jump part of the quadratic variation at a fixed time t.
- The processes (7) and (8) define semimartingales on the extended space.
- Conditionally on $\mathcal{A}$, $L^{(1)}$ and $L^{(2)}$ are *independent* and $L^{(1)}$ is a *martingale* with *Gaussian law*, and if Y and σ do not jump together, $L^{(2)}$ is also a martingale with Gaussian law. Their variances are given by ((Jacod 2007, p. 8))

$$\widetilde{\mathbb{E}} \left(\left(L_t^{(1)} \right)^2 \middle| \mathcal{A} \right) = 2 \int_0^t \sigma_u^4 du, \tag{9}$$

$$\widetilde{\mathbb{E}} \left(\left(L_t^{(2)} \right)^2 \middle| \mathcal{A} \right) = 2 \sum_{p:\ T_p \leq t} \left(\Delta Y_{T_p} \right)^2 \left(\sigma_{T_p}^2 + \sigma_{T_p-}^2 \right). \tag{10}$$

  So conditionally on $\mathcal{A}$, the asymptotic variance of the bias between realised variance and quadratic variation is given by

$$2 \int_0^t \sigma_u^4 du + 2 \sum_{p:\ T_P \leq t} \left(\Delta Y_{T_p} \right)^2 \left(\sigma_{T_p}^2 + \sigma_{T_p-}^2 \right). \tag{11}$$

- When there are no jumps, the limit is given by (7), which is a well–known result, e.g. Jacod (1994), Jacod & Protter (1998) and Barndorff-Nielsen & Shephard (2002).

Realised multipower variation: The asymptotic distribution of multipower variations in the presence of jumps has first been derived by Woerner (2006). A later study by Jacod (2006, Theorem 6.2) contains the following result. Assume that (L-s) holds for some $s < 1$ and that we have (H'). Furthermore let $\mathbf{r}$ be a multi–index such that $\frac{s}{2-s} < \mathbf{r}_- \leq \mathbf{r}_+ < 1$. Then, as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}} \left(RMPV(\mathbf{r})_t^n - \int_0^t |\sigma_u|^{|\mathbf{r}|} du \right) \overset{stably\ in\ law}{\longrightarrow} \mu_{\mathbf{r}}^{-1} \sqrt{A(\mathbf{r})} \int_0^t |\sigma_u|^{|\mathbf{r}|} d\widetilde{W}_u,$$

stably in law as a process.

In the next section, we combine these two results and derive a bivariate limit result, which is the main contribution of this paper.

Remark We suppose that it is possible to derive a central limit theorem for realised bipower variation in the presence of jumps. However, the central limit theorem for realised bipower variation will differ from the ones for realised tripower, realised quadpower etc.. As mentioned in Barndorff-Nielsen, Shephard & Winkel (2006, Section 3.1), the limit process will exhibit a jump component in addition to the Brownian semimartingale component. So we expect to obtain a similar central limit result to that for realised variance. This aspect will be studied in more detail in future research.

4.1 Main Result

Let $(Y_t)_{t\geq 0}$ denote a one–dimensional semimartingale.

Theorem 4.1 *Assume (L-s) for some $s<1$, (H') and let $\mathbf{r}$ be a multi–index such $\frac{s}{2-s} < \mathbf{r}- \leq \mathbf{r}+ < 1$. Then*

$$\frac{1}{\sqrt{\Delta_n}}\begin{pmatrix} RV_t^n - [Y]_{\Delta_n[t/\Delta_n]} \\ \mu_{\mathbf{r}}^{-1} RMPV(\mathbf{r})_t^n - \int_0^t |\sigma_u|^{|\mathbf{r}|}du \end{pmatrix} \xrightarrow{stably\ in\ law} \begin{pmatrix} \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u + 2\sum_{p:\ T_p\leq t}\Delta Y_{T_p}\left(\sqrt{\xi_p}U_p\sigma_{T_{p-}} + \sqrt{1-\xi_p}\,U_p'\sigma_{T_p}\right) \\ \sqrt{2}\int_0^t \sigma_u^{|\mathbf{r}|} d\overline{W}_u + \sqrt{\theta_{\mathbf{r}}}\int_0^t |\sigma_u|^{|\mathbf{r}|} d\widetilde{W}_u \end{pmatrix},$$

where the convergence is stable in law as a process and $\theta_{\mathbf{r}} = (\mu_{\mathbf{r}}^{-1}\sqrt{A(\mathbf{r})})^2 - 2$.

If σ and Y do not jump together, the first component is the sum of two independent martingales which have, conditional on $\mathcal{A}$, Gaussian law. Note that in that case $\sigma_{T_{p-}} = \sigma_{T_p}$ since T_p are the jump times of Y.

Remark The one–dimensional limit result for the multipower variation holds as soon as (L-s) for some $s<1$, (H') hold and $\frac{s}{2-s} < \mathbf{r}- \leq \mathbf{r}+ < 1$. In order to obtain the limit result for the realised variance, we need the assumption (L-2) which is clearly implied by (L-s) for some $s<1$.

Corollary 4.2 *Assume (L-s) for some $s<1$, (H') and that Y and σ have no common jumps. For $I\in\mathbb{N}$ with $2<I<\frac{2}{s}(2-s)$, we obtain:*

$$\frac{1}{\sqrt{\Delta_n}}\begin{pmatrix} RV_t^n - [Y]_{\Delta_n[t/\Delta_n]} \\ RMVP(2;I)_t^n - \int_0^t \sigma_u^2 du \end{pmatrix} \xrightarrow{stably\ in\ law} \begin{pmatrix} \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u + \sqrt{2}\sum_{p:\ T_p\leq t}\Delta Y_{T_p}\sigma_{T_p}\left(\sqrt{\xi_p}U_p + \sqrt{1-\xi_p}\,U_p'\right) \\ \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u + \sqrt{\theta_I}\int_0^t \sigma_u^2 d\widetilde{W}_u \end{pmatrix}, \quad (12)$$

where (12) has, conditionally on $\mathcal{A}$, Gaussian law with zero mean and variance

$$\begin{pmatrix} 2\int_0^t \sigma_u^4 du + 4\sum_{p:\ T_p\leq t}\left(\Delta Y_{T_p}\right)^2\sigma_{T_p}^2 & 2\int_0^t \sigma_u^4 du \\ 2\int_0^t \sigma_u^4 du & (2+\theta_I)\int_0^t \sigma_u^4 du \end{pmatrix},$$

where $\theta_I = \mu_{2/I}^{-2I}\omega_I^2 - 2$.

The following corollary contains the result which is of most importance in applications and can be regarded as key result of this paper.

Corollary 4.3 *Assume (L-s) for some $s < 1$, (H') and that Y and σ have no common jumps. For $I \in \mathbb{N}$ with $2 < I < \frac{2}{s}(2-s)$ we obtain:*

$$\frac{1}{\sqrt{\Delta_n}}(RV_t^n - RMPV(2;I)_t^n - [Y]^d_{\Delta_n[t/\Delta_n]}) \overset{stably\ in\ law}{\longrightarrow} L_t, \tag{13}$$

where L_t has, conditionally on $\mathcal{A}$, Gaussian law with zero mean and variance given by

$$\theta_I \int_0^t \sigma_u^4 du + 4 \sum_{p:\ T_P \le t} \left(\Delta Y_{T_p}\right)^2 \sigma_{T_p}^2 .$$

Proof Let $c = (1, -1)'$. Then

$$c' \frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n - [Y]_{\Delta_n[t/\Delta_n]} \\ RMVP(2;I)_t^n - \int_0^t \sigma_u^2 du \end{pmatrix} = \frac{1}{\sqrt{\Delta_n}}((RV_t^n - RMVP(2;I)_t^n) - [Y]^d_{\Delta_n[t/\Delta_n]}).$$

The expression above converges stably in a law as a process to a process which, conditionally on $\mathcal{A}$, is $N\left(0, (1,\ -1)M_t\binom{1}{-1}\right)$–distributed, where

$$M_t = \begin{pmatrix} 2\int_0^t \sigma_u^4 du + 4\sum_{p:\ T_P \le t}\left(\Delta Y_{T_p}\right)^2 \sigma_{T_p}^2 & 2\int_0^t \sigma_u^4 du \\ 2\int_0^t \sigma_u^4 du & (2+\theta_I)\int_0^t \sigma_u^4 du \end{pmatrix}.$$

□

4.2 Distribution of the Test Statistics under the Alternative Hypothesis

Now we have all results for constructing feasible test statistics and for deriving their distributions under both the null and the alternative hypothesis (assuming that Y and σ have no common jumps and that the assumptions of Theorem 4.1 are satisfied).

- For the *linear test* (see (2)), we obtain the following limit result under the alternative hypothesis, i.e. in the presence of jumps:

  $$\frac{1}{\sqrt{\Delta_n}}(RV_t^n - RMPV(2;I)_t^n - [Y]^d_{\Delta_n[t/\Delta_n]})$$

  converges stably in law as a process to a process, which, conditionally on $\mathcal{A}$, has Gaussian law with zero mean and variance

  $$\theta_I \int_0^t \sigma_s^4 ds + 4\sum_{0 \le s \le t} \sigma_s^2 \left(\Delta Y_s\right)^2 .$$

- For the *ratio test* (see (3)), we obtain the following limit result under the alternative hypothesis:

$$\frac{1}{\sqrt{\Delta_n}}\left(\frac{RMPV(2;I)_t^n}{RV_t^n} - 1 + \frac{[Y]^d_{\Delta_n[t/\Delta_n]}}{RV_t^n}\right)$$

  converges stably in law as a process to a process, which, conditionally on $\mathcal{A}$, has Gaussian law with zero mean and variance

$$\frac{\theta_I \int_0^t \sigma_s^4 ds + 4\sum_{0\le s\le t} \sigma_s^2 \left(\Delta Y_s\right)^2}{\left(\int_0^t \sigma_s^2 ds + \sum_{0\le s\le t} \left(\Delta Y_s\right)^2\right)^2}.$$

- And from the bivariate delta method (for a bivariate function $g(x,y) = \log(x) - \log(y)$), we deduce the distribution of (4) under the alternative hypothesis:

$$\frac{1}{\sqrt{\Delta_n}}(\log\left(RV_t^n\right) - \log\left(RMPV(2;I)_t^n\right) - (\log\left([Y]_{\Delta_n[t/\Delta_n]}\right) - \log\left([Y]^c_{\Delta_n[t/\Delta_n]}\right))$$

  converges stably in law as a process to a process, which, conditionally on $\mathcal{A}$, has Gaussian law with zero mean and variance

$$\left(\frac{2}{[Y]_t^2} - \frac{4}{[Y]_t[Y]_t^c} + \frac{(2+\theta_I)}{([Y]_t^c)^2}\right)\int_0^t \sigma_u^4 du + \frac{4}{[Y]_t^2}\sum_{0\le s\le t} \sigma_s^2(\Delta Y_s)^2.$$

4.3 Feasible Standard Errors

In order to derive feasible test statistics, we need estimators for the asymptotic variances. From Barndorff-Nielsen & Shephard (2002) and Jacod (2006), we know that the continuous part of the asymptotic variance can be consistently estimated by special cases of the realised multipower variation — even in the presence of jumps. For $I \geq 3$

$$\frac{1}{\Delta_n} RMPV(4;I)_t^n \xrightarrow{ucp} \int_0^t \sigma_s^4 ds.$$

So, how can we estimate the jump part of the asymptotic variance? From Veraart (2007), we know that one can use an estimator which is based on the difference of a generalised version of realised variance and realised multipower variation. In particular, we write K_n for a sequence which satisfies

$$K_n \to \infty \qquad \text{and} \qquad \Delta_n K_n \to 0 \qquad \text{as } n \to \infty.$$

We can extend a result by Lee & Mykland (2006) and write

$$\hat{\sigma}^{2(-)}_{(i-1)\Delta_n} = \frac{\mu_1^{-2}}{(K_n-2)\,\Delta_n}\sum_{j=i-K_n+1}^{i-2} \left|\Delta_j^n X\right|\left|\Delta_{j+1}^n X\right|,$$

$$\hat{\sigma}^{2(+)}_{(i-1)\Delta_n} = \frac{\mu_1^{-2}}{(K_n-2)\,\Delta_n}\sum_{j=i+2}^{i+K_n-1} \left|\Delta_j^n X\right|\left|\Delta_{j+1}^n X\right|.$$

for the locally averaged realised bipower variation. From Veraart (2007), we know that

$$\sum_{i=1}^{t/\Delta_n} \left(\hat{\sigma}^{2(-)}_{(i-1)\Delta_n} + \hat{\sigma}^{2(+)}_{(i-1)\Delta_n}\right) (\Delta_i^n Y)^2 \xrightarrow{\mathbb{P}} 2\int_0^t \sigma_s^4 ds + \sum_{0\leq s\leq t} \left(\sigma_{s-}^2 + \sigma_s^2\right) (\Delta Y_s)^2,$$

as $n \to \infty$ and, therefore,

$$c_1 \sum_{i=1}^{t/\Delta_n} \left(\hat{\sigma}^{2(-)}_{(i-1)\Delta_n} + \hat{\sigma}^{2(+)}_{(i-1)\Delta_n}\right) (\Delta_i^n Y)^2 - c_2 \frac{1}{\Delta_n} RMPV(4;I)_t^n$$
$$\xrightarrow{\mathbb{P}} (2c_1 - c_2)\int_0^t \sigma_s^4 ds + c_1 \sum_{0\leq s\leq t} \left(\sigma_{s-}^2 + \sigma_s^2\right) (\Delta Y_s)^2,$$

for constants c_1, c_2 with $2c_1 \geq c_2$. And in order to make sure that the variance estimator is always positive, we use

$$\hat{A}_t^n(c_1,c_2) = \max\left\{ c_1 \sum_{i=1}^{t/\Delta_n} \left(\hat{\sigma}^{2(-)}_{(i-1)\Delta_n} + \hat{\sigma}^{2(+)}_{(i-1)\Delta_n}\right) (\Delta_i^n Y)^2 - c_2 \frac{1}{\Delta_n} RMPV(4;I)_t^n, \right.$$
$$\left. (2c_1 - c_2)\frac{1}{\Delta_n} RMPV(4;I)_t^n \right\}.$$

So e.g. we obtain for $n \to \infty$:

$$\hat{A}_t^n(2, 4-\theta_I) \to \theta_I \int_0^t \sigma_s^4 ds + 2\sum_{0\leq s\leq t} \left(\sigma_{s-}^2 + \sigma_s^2\right) (\Delta Y_s)^2 .$$

Clearly, in the absence of common jumps of Y and σ, we could also use the slightly simpler estimator of the asymptotic variance given by

$$\max\left\{ 2c_1 \sum_{i=1}^{t/\Delta_n} \hat{\sigma}^{2(-)}_{(i-1)\Delta_n} (\Delta_i^n Y)^2 - c_2 \frac{1}{\Delta_n} RMPV(4;I)_t^n, (2c_1 - c_2)\frac{1}{\Delta_n} RMPV(4;I)_t^n \right\}.$$

Now we can define feasible standard errors for the linear test statistic, the ratio test statistic and the log–linear test statistic under the alternative hypothesis, that there are jumps. Let $I, \tilde{I}$ denote positive integers which are greater or equal to 3.

Feasible standard error for linear test:

$$\frac{1}{\sqrt{\Delta_n}} \frac{(RV_t^n - RMPV(2;\tilde{I})_t^n)}{\sqrt{\hat{A}_t^n(4, 4-\theta_I)}}.$$

Feasible standard error for ratio test:

$$\frac{1}{\sqrt{\Delta_n}} \left(\frac{RMPV(2;\tilde{I})_t^n}{RV_t^n} - 1\right) \frac{RV_t^n}{\sqrt{\hat{A}_t^n(4, 4-\theta_I)}}.$$

Feasible standard error for log–linear test:

$$\frac{1}{\sqrt{\Delta_n}}(\log(RV_t^n) - \log\left(RMPV(2;\tilde{I})_t^n\right))\frac{RV_t^n RMPV(2;\tilde{I})_t^n}{\sqrt{\hat{A}_t^n(c_1,c_2)}},$$

where

$$c_1 = 4(RMPV(2;I)_t^n)^2,$$
$$c_2 = 4(RMPV(2;I)_t^n)^2 - (2(RMPV(2;I)_t^n)^2 - 4RV_t^n RMPV(2;I)_t^n + (2+\theta_I)(RV_t^n)^2).$$

5 Simulation Study

In this section, we will study the finite sample performance of our test statistics by carrying out a detailed simulation study.

So far, we have seen that we can use any realised multipower variation $RMPV(2;I)_t^n$ with $I \geq 3$ for constructing a test for jumps because from the tripower onwards all multipower variations (which satisfy $\frac{s}{2-s} < 2/I < 1$ for sufficiently small s) are robust toward jumps. So, which multipower variation shall we choose to construct our test statistic?

Basically, we are confronted with the following trade–off: We know that in the absence of jumps, realised variance is the most efficient consistent estimator of integrated variance. Using higher multipower variation in such a model setting results in an efficiency loss. Recall that for $U \sim N(0,1)$ we have:

$$\mu_r = \mathbb{E}|U|^r = \frac{\sqrt{2^r}\Gamma\left(\frac{1}{2}(r+1)\right)}{\sqrt{\pi}},$$

$$\omega_I^2 = \mu_{4/I}^I + (1-2I)\mu_{2/I}^{2I} + 2\sum_{j=1}^{I-1}\mu_{4/I}^{I-j}\,\mu_{2/I}^{2j},$$

$$\theta_I = \left(\mu_{2/I}^{-I}\sqrt{\omega_I^2}\right)^2 - 2 = \mu_{2/I}^{-2I}\,\omega_I^2 - 2.$$

So when we look at the values of θ_I for various I in Table 1, we see that θ_I increases for increasing I, which describes the loss in efficiency.

I	1	2	3	4	5	6	∞
θ_I	0	0.608	1.061	1.377	1.605	1.776	2.934

Table 1: Different values for θ_I

It is interesting to see that θ_I actually converges to a finite number

$$\lim_{I\to\infty}\theta_I = \frac{\pi^2}{2} - 2,$$

so the loss in efficiency is bounded, which can also be seen in Figure 1.

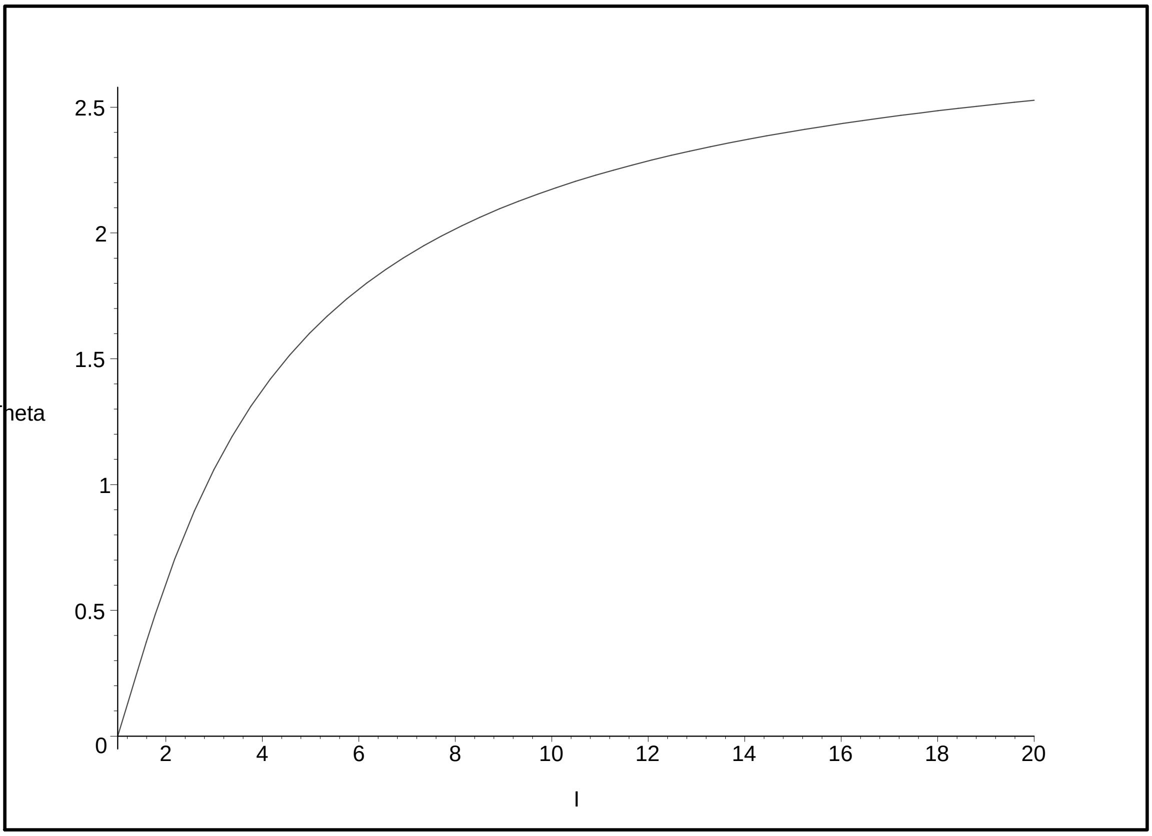

Figure 1: Different values for θ_I

Given these results, we might be tempted to focus on tripower variation only for estimating integrated variance in the presence of jumps since it is the multipower variation of lowest power (hence the most efficient one) which is robust towards jumps when we study its asymptotic distribution.

However, there is also another issue which is worth studying a bit further. That is the problem of the finite sample bias when we consider the difference of realised variance and realised multipower variation.

5.1 Finite Sample Bias — A Jump–Diffusion Model

So we know that in theory one can use the difference between the realised variance and any multipower variation of power $2/I$ for $I = 3, 4, \ldots$. From a statistical point of view, we might want to use tripower variation since it is the most efficient statistic. However, in this section, we study the finite sample bias between realised variance and multipower variation and show that it seems to get smaller when I increases, which means that using tripower variation would result in the biggest possible finite sample bias one can find.

Before we focus on more advanced models in our simulation study, we study a very basic model in this section: a Brownian motion with jumps. Let

$$Y_t = W_t + \sum_{i=1}^{N_t} c_i,$$

where W is a Brownian motion, N_t is a Poisson process with intensity 1 and i.i.d. $c_i \sim N(0, \sigma_p^2)$ and $t \in [0,1]$. In order to simplify the exposition, we set $t = 1$. Let $I_i^n = ((i-1)\Delta_n, i\Delta_n]$ for $i = 1, \ldots, M$, where $M = [1/\Delta_n]$. Hence, for $j = 1, 2, \ldots$ we have

$$\mathbb{P}(\text{There are } j \text{ jumps in } I_i^n) = P(N_{i\Delta_n} - N_{(i-1)\Delta_n} = j) = \frac{e^{-\lambda\Delta_n}(\lambda\Delta_n)^j}{j!}.$$

So the mean of realised variance, realised multipower variation and the sum of the squared jumps can be easily derived.

Realised variance: For the realised variance, we get

$$\mathbb{E}\left(RV_1^n\right) = \mathbb{E}\left(\sum_{i=1}^{M} (\Delta_i^n Y)^2\right) = M\mathbb{E}\left(\Delta_i^n Y\right)^2 = \left(1 + \lambda\sigma_p^2\right)\Delta_n \left[\frac{1}{\Delta_n}\right],$$

since

$$\begin{aligned}\mathbb{E}\left(\Delta_i^n Y\right)^2 &= \sum_{j=0}^{J} \mathbb{P}(N_{i\Delta_n} - N_{(i-1)\Delta_n} = j)\, \mathbb{E}\left(\Delta_i^n W + \sum_{i=1}^{j} c_i\right)^2 \\ &= \sum_{j=0}^{J} \frac{e^{-\lambda\Delta_n}(\lambda\Delta_n)^j}{j!}(\Delta_n + j\sigma_p^2) = \left(1 + \lambda\sigma_p^2\right)\Delta_n.\end{aligned}$$

Realised multipower variation: Let $I = 3, 4, \ldots$. From the independence and stationarity of the increments, we get for an $i \in \{1, \ldots, M - I\}$:

$$\begin{aligned}\mathbb{E}\left(RMPV(2; I)_1^n\right) &= \mathbb{E}\left(\frac{M}{M-I}\mu_{2/I}^{-I} \sum_{i=1}^{M-I} \prod_{i'=0}^{I-1} |\Delta_{i+i'}^n Y|^{2/I}\right) \\ &= M\mu_{2/I}^{-I}\left(\mathbb{E}\left(|\Delta_i^n Y|^{2/I}\right)\right)^I \\ &= \left[\frac{1}{\Delta_n}\right]\left(\sum_{j=0}^{\infty} \frac{e^{-\lambda\Delta_n}(\lambda\Delta_n)^j}{j!}\left(\Delta_n + j\sigma_p^2\right)^{1/I}\right)^I,\end{aligned}$$

since

$$\mathbb{E}\left(|\Delta_i^n Y|^{2/I}\right) = \sum_{j=0}^{\infty} \mathbb{E}\left(|\Delta_i^n Y|^{2/I} \,\middle|\, N_{i\Delta_n} - N_{(i-1)\Delta_n} = j\right) \mathbb{P}(N_{i\Delta_n} - N_{(i-1)\Delta_n} = j)$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\lambda\Delta_n}(\lambda\Delta_n)^j}{j!}\mathbb{E}\left(\left|\Delta_i^n W + \sum_{j'=1}^{j} c_{j'}\right|^{2/I}\right) = \sum_{j=0}^{\infty} \frac{e^{-\lambda\Delta_n}(\lambda\Delta_n)^j}{j!}\mu_{2/I}\left(\Delta_n + j\sigma_p^2\right)^{1/I}.$$

Sum of squared jumps:

$$\mathbb{E}\left(\sum_{i=1}^{N_1} c_i^2\right) = \sum_{J=0}^{\infty} j\sigma_p^2 \frac{e^{-\lambda}\lambda^j}{j!} = \sigma_p^2\lambda.$$

Bias: So we obtain

$$\mathbb{E}\left(\frac{1}{\sqrt{\Delta_n}}\left(RV_1^n - RMPV(2;I)_1^n - \sum_{i=1}^{N_1} c_i^2\right)\right)$$

$$= \frac{1}{\sqrt{\Delta_n}}\left(\left(1+\lambda\sigma_p^2\right)\Delta_n\left[\frac{1}{\Delta_n}\right] - \left[\frac{1}{\Delta_n}\right]\left(\sum_{j=0}^{\infty}\frac{e^{-\lambda\Delta_n}(\lambda\Delta_n)^j}{j!}\left(\Delta_n + j\sigma_p^2\right)^{1/I}\right)^I - \lambda\sigma_p^2\right)$$

$$\frac{1}{\sqrt{\Delta_n}}\left(\left(1+\lambda\sigma_p^2\right)\Delta_n\left[\frac{1}{\Delta_n}\right] - \left[\frac{1}{\Delta_n}\right]\Delta_n\left(\sum_{j=0}^{\infty}\frac{e^{-\lambda\Delta_n}(\lambda\Delta_n)^j}{j!}\left(1+\frac{j\sigma_p^2}{\Delta_n}\right)^{1/I}\right)^I - \lambda\sigma_p^2\right).$$

Clearly, the bias converges to 0 when $\Delta_n \to 0$. However, if we fix Δ_n, then the bias converges to the following expression

$$\frac{1}{\sqrt{\Delta_n}}\left(\left(1+\lambda\sigma_p^2\right)\Delta_n\left[\frac{1}{\Delta_n}\right] - \left[\frac{1}{\Delta_n}\right]\Delta_n \exp\left(\frac{\sum_{j=0}^{\infty}\frac{(\lambda\Delta_n)^j \log\left(1+j\sigma_p^2/\Delta_n\right)}{j!}}{\exp(\lambda\Delta_n)}\right) - \lambda\sigma_p^2\right),$$

when $I \to \infty$. In the following, we provide two tables and plots of the finite sample bias of the linear test under the hypothesis that there are jumps for various values of I and for different jump sizes. For the jump intensity we have chosen $\lambda = 0.6$, which results in one jump on $[0,1]$ and $\lambda = 1.6$, which results in two jumps in $[0,1]$.

	$\lambda = 0.6$				$\lambda = 1.6$			
M \ I	3	4	5	6	3	4	5	6
39	-0.33 (-0.20)	-0.3 (-0.18)	-0.28 (-0.18)	-0.27 (-0.17)	-0.7 (-0.54)	-0.65 (-0.50)	-0.62 (-0.48)	-0.59 (-0.47)
78	-0.35 (-0.21)	-0.31 (-0.19)	-0.29 (-0.18)	-0.27 (-0.17)	-0.71 (-0.58)	-0.64 (-0.53)	-0.6 (-0.50)	-0.58 (-0.48)
390	-0.38 (-0.22)	-0.33 (-0.18)	-0.3 (-0.16)	-0.28 (-0.15)	-0.71 (-0.59)	-0.57 (-0.49)	-0.51 (-0.44)	-0.47 (-0.41)
1560	-0.33 (-0.20)	-0.25 (-0.15)	-0.22 (-0.13)	-0.2 (-0.12)	-0.66 (-0.53)	-0.51 (-0.41)	-0.44 (-0.35)	-0.4 (-0.32)
23400	-0.21 (-0.14)	-0.12 (-0.09)	-0.09 (-0.07)	-0.07 (-0.06)	-0.47 (-0.38)	-0.3 (-0.24)	-0.24 (-0.19)	-0.2 (-0.16)

Table 2: Estimated bias (theoretical bias) for $\sigma_p^2 = 0.1$. The estimation results are based on 5000 replications. Recall that $M = 1/\Delta_n$ and that I denotes the power in $RMPV(2;I)_1^n$.

In Table 2 and Figure 2, we study the finite sample bias when the jump size distribution is drawn from a N(0, 0.1) distribution, whereas, in Table 3 and Figure 3, we allow for bigger

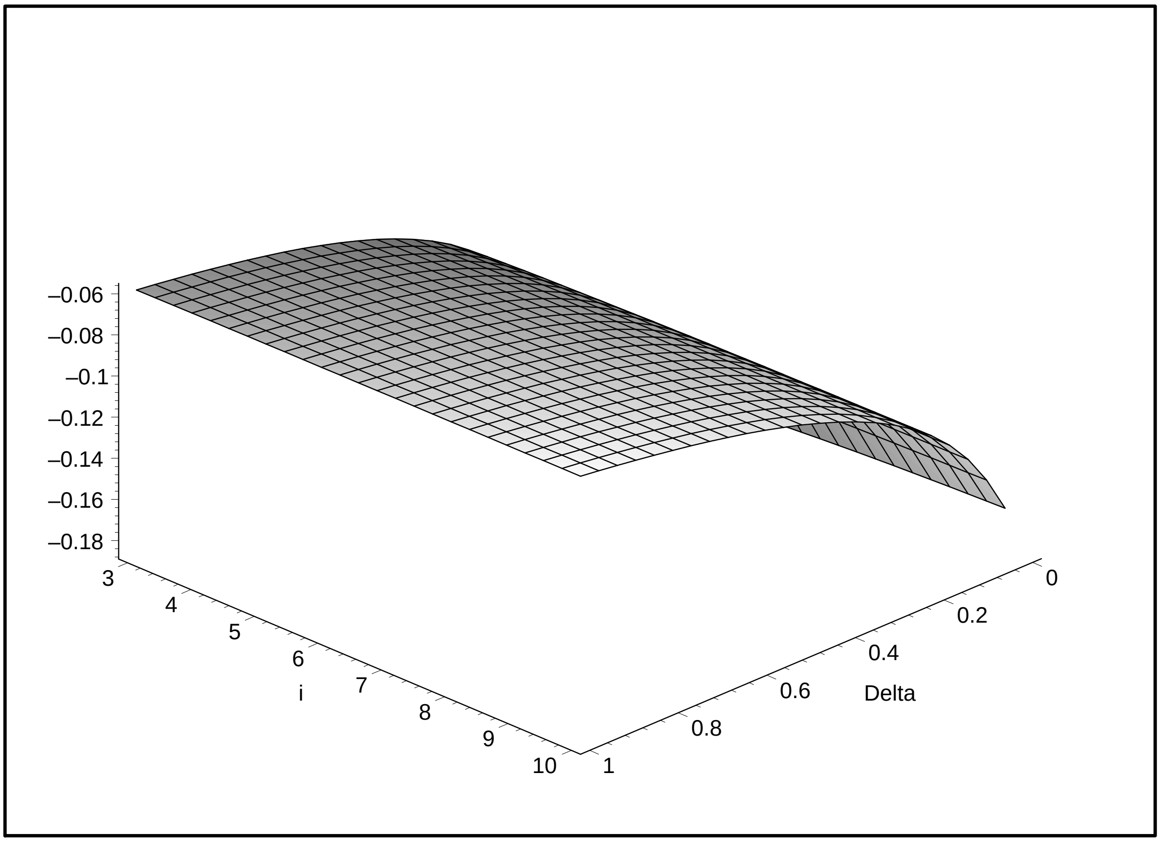

Figure 2: Finite sample bias for $\lambda = 0.6$ and $\sigma_p^2 = 0.1$.

jumps and simulate the jump size from a N(0, 1) distribution. It is striking that the finite sample bias is quite big — particularly when we allow for many/big jumps and when M is still quite small. However, we also observe that by using multipower variation of higher powers, the bias seems to shrink significantly.

5.2 Assessing the Finite Sample Performance of Various Test Statistics

Now we turn our attention to slightly more advanced models and assess the finite sample performance of the linear, ratio and log–linear test statistic both in the absence and in the presence of jumps.

5.2.1 Simulation Design

Our simulation design follows the one described in Huang & Tauchen (2005). Note that in this simulation framework, all the parameters are chosen in such a way that they correspond

	$\lambda = 0.6$				$\lambda = 1.6$			
M \ I	3	4	5	6	3	4	5	6
39	-1.27 (-0.72)	-1.08 (-0.59)	-0.98 (-0.53)	-0.92 (-0.50)	-2.59 (-2.02)	-2.19 (-1.67)	-1.98 (-1.50)	-1.85 (-1.40)
78	-1.17 (-0.68)	-0.94 (-0.54)	-0.82 (-0.48)	-0.76 (-0.44)	-2.44 (-1.89)	-1.97 (-1.51)	-1.75 (-1.32)	-1.62 (-1.22)
390	-0.99 (-0.58)	-0.75 (-0.42)	-0.64 (-0.35)	-0.57 (-0.31)	-1.96 (-1.57)	-1.44 (-1.13)	-1.2 (-0.94)	-1.07 (-0.84)
1560	-0.83 (-0.48)	-0.56 (-0.32)	-0.45 (-0.25)	-0.39 (-0.21)	-1.59 (-1.30)	-1.04 (-0.86)	-0.82 (-0.68)	-0.71 (-0.58)
23400	-0.51 (-0.32)	-0.26 (-0.17)	-0.18 (-0.12)	-0.13 (-0.10)	-1.04 (-0.86)	-0.55 (-0.47)	-0.38 (-0.33)	-0.3 (-0.27)

Table 3: Estimated bias (theoretical bias) for $\sigma_p^2 = 1$. The estimation results are based on 5000 replications. Recall that $M = 1/\Delta_n$ and that I denotes the power in $RMPV(2; I)_1^n$.

to parameter values which one can find in real data (see the references in Huang & Tauchen (2005)). We simulate asset price data from three different models:

Constant volatility jump diffusion

$$dY_t = \mu dt + \exp(\beta_0 + \beta_1 v)dW_t^Y + dL_t^J,$$

Stochastic volatility jump diffusion

$$\begin{aligned} dY_t &= \mu dt + \exp(\beta_0 + \beta_1 v_t)dW_t^Y + dL_t^J, \\ dv_t &= \alpha_v v_t dt + dW_t^v, \end{aligned}$$

where W^Y, W^v are standard Brownian motions with $Corr(dW^Y, dW^v) = \rho$, v_t is the stochastic volatility factor, L_t^J compound Poisson process with constant jump intensity λ and jump size distribution $N(0,\ \sigma_{jmp}^2)$.

Two–factor stochastic volatility model

$$\begin{aligned} dY_t &= \mu dt + \text{s--}\exp(\beta_0 + \beta_1 v_t^{(1)} + \beta_2 v_t^{(2)})dW_t^Y, \\ dv_t^{(1)} &= \alpha_{v^{(1)}} v_t^{(1)} dt + dW_t^{v^{(1)}}, \\ dv_t^{(2)} &= \alpha_{v^{(2)}} v_t^{(2)} dt + (1 + \beta_{v^{(2)}} v_t^{(2)})dW_t^{v^{(2)}}, \end{aligned}$$

where $v_t^{(1)}$ is a Gaussian process, $v_t^{(2)}$ exhibits a feedback term in the diffusion function, s–$\exp$ is the usual exponential function with a polynomial function splined in at very high values of its argument to ensure that for $\beta_{v^{(2)}} \neq 0$ the growth conditions (for a solution to exist and the Euler scheme to work) are satisfied and $Corr\left(dW^Y, dW^{v^{(1)}}\right) = \rho_1$ and $Corr\left(dW^Y, dW^{v^{(2)}}\right) = \rho_2$.

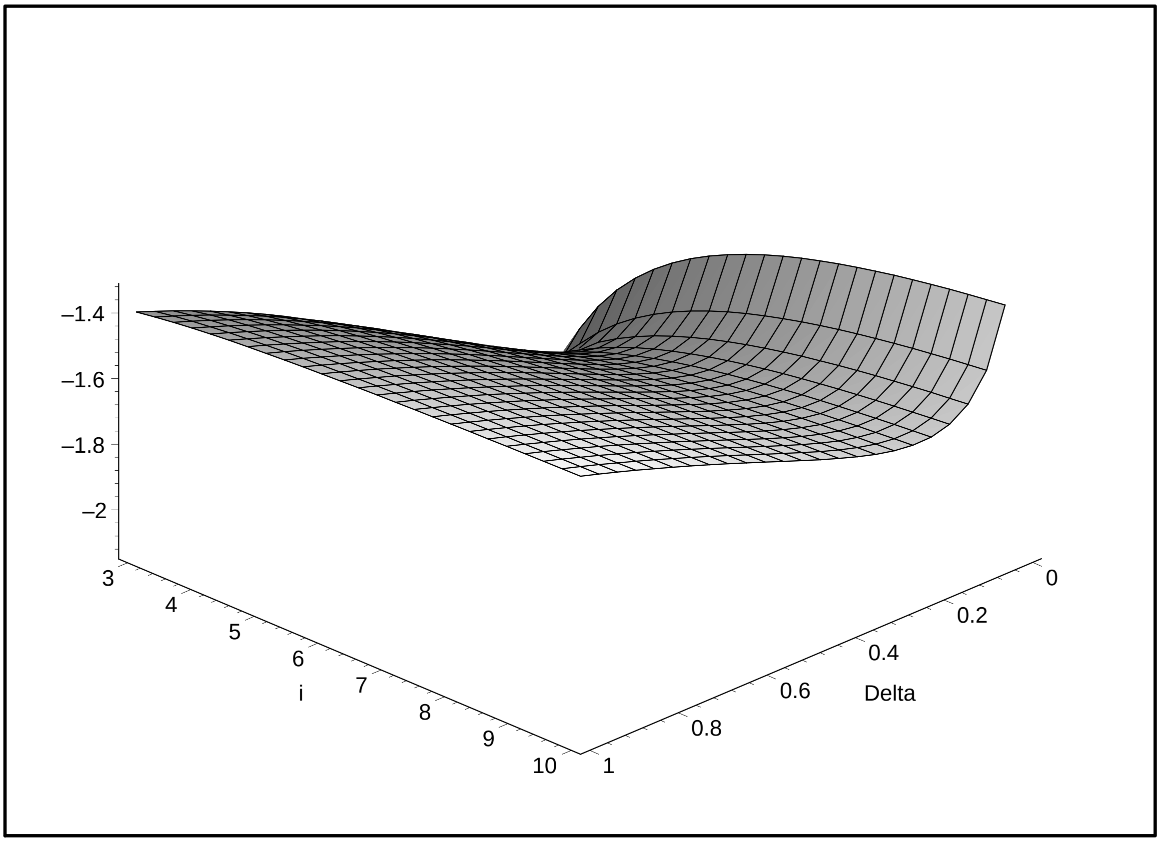

Figure 3: Finite sample bias for $\lambda = 1.6$ and $\sigma_p^2 = 1$.

Following Huang & Tauchen (2005), we choose as basic unit of time for our simulation one day. We then simulate the diffusion parts in the stochastic volatility models based on the Euler scheme, where we choose the increment of one second per tick on the Euler clock. For simulating the jump component, we follow exactly Huang & Tauchen (2005)'s approach by simulating the exponentially distributed jump times and then the normally distributed jump sizes (from $N(0,\ \sigma_{jmp}^2)$).

Our parameter choices are given in Table 4 and are identical to the choices in Huang & Tauchen (2005).

5.2.2 Simulation Results

Now we turn our attention to the simulation results based on the simulation design described in Huang & Tauchen (2005) where jumps occur randomly and their arrival times are exponentially distributed with parameter λ. We have simulated jumps at two different arrival rates. For the low value of λ we have not observed more than two jumps at any day in our 5000–day sample and for the high value of λ there were never more than three jumps a day, and in both cases

Parameter	Constant volatility jump diffusion	Stochastic volatility jump diffusion	Two–factor stochastic volatility model
μ	0.03		
β_0	0		-1.2
β_1	0.125		0.04
β_2	—		1.5
$\beta_{v^{(2)}}$	—		0.25
v	1	—	
ρ , ρ_1	—	-0.62	- 0.3
ρ_2	—		- 0.3
a_v , $a_{v^{(1)}}$	—	-0.1	- 0.00137
$a_{v^{(2)}}$	—		-1.386
p	$\{0.1, 0.2, 0.5, 0.7\}$		—
λ	$\{0.0114, 0.118\}$		—

Table 4: Choice of parameters for the constant volatility model with finite activity jumps, for the one–factor stochastic volatility model with finite activity jumps and for the two–factor stochastic volatility model.

there were many days where no jumps occurred at all. The simulation results are given in the Tables 5 – 9 in the Appendix (Section 4.8.2). For both the constant and the one–factor stochastic volatility model, the finite sample performance is really good. Also when we look at low values for M, e.g. the case for $M = 39$ (which corresponds to computing the realised variance and realised multipower variation based on 10–minute returns) the results are not too bad and the log–linear test seems to perform particularly well already at that low frequency. The performance of the linear and ratio test statistic occurs then to be a bit poorer when we look at the two–factor stochastic volatility model. But again, the log–linear test leads to good results. And clearly, when M increases, we see that the finite sample bias decreases and that our estimates for the asymptotic variance get even better.

So given these results, which test statistic do we want to use when we study real data? When we analyse data of relatively low frequency, it might be worth using the log–test and probably quadpower rather than tripower variation. However, if we want to test for jumps in data of a very high frequency, the performance of all three test statistics based on various powers in the multipower variation gets quite similar.

6 Empirical Study

In our empirical study, we focus on equity high frequency quote data. Here we have chosen IBM, General Motors (GM) and Shell (RDSA) intra–day TAQ data, available at WRDS, from 1 September 2005 to 31 August 2006. Figure 4 provides the plots of the three time series of logarithmic daily prices and the corresponding daily realised variances.

Before analysing the data, we have cleaned the data. Following methods used by Hansen & Lunde (2006*b*), we concentrate on quote data from one stock exchange only. Here we have chosen the NYSE. We only consider quotes where both the bid–size and the ask–size are greater

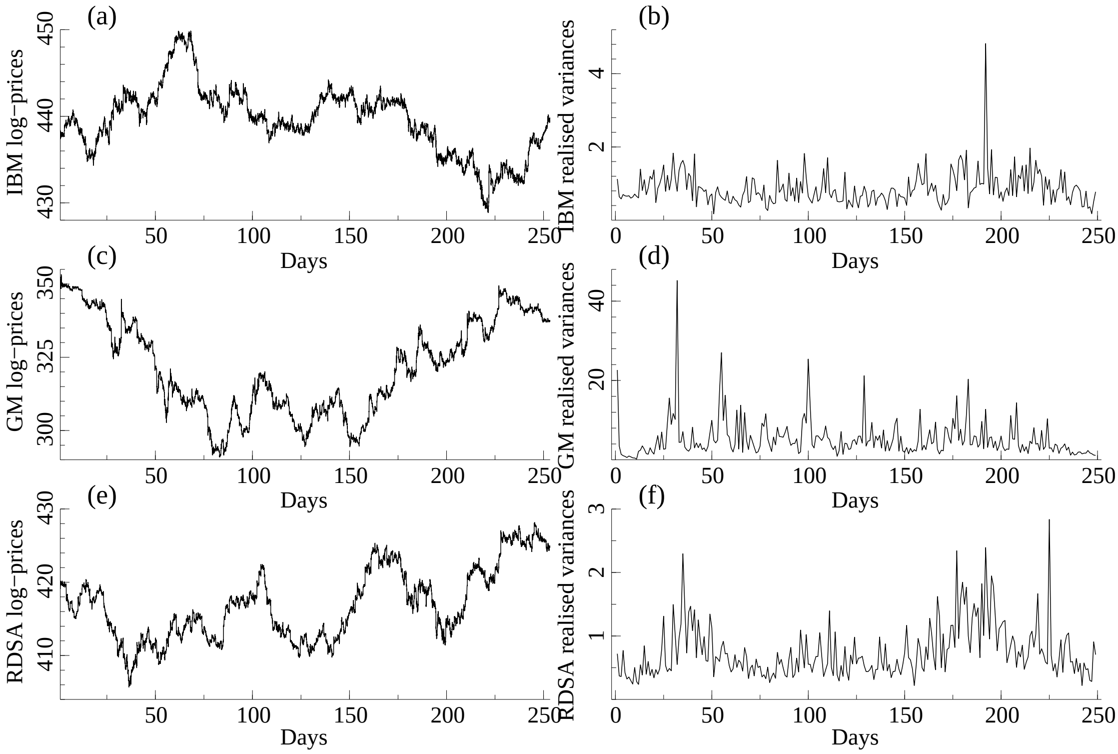

Figure 4: (a)Log–prices for IBM for September 2005 to August 2006; (b) daily realised variances for IBM; (c) log–prices for GM for September 2005 to August 2006; (d) daily realised variances for GM; (e) log–prices for RDSA for September 2005 to August 2006; (f) daily realised variances for RDSA.

than 0 and which are quoted in a normal trading environment (quote condition = 12 in the TAQ database). Since data at the beginning and at the end of a trading day differ quite a lot from the quotes during the day, we concentrate on data from 9.35 am until 15.55pm only. Furthermore, we have focused on the bid–prices only. In order to construct a time series of five minute returns of the log–bid–prices we use the previous tick sampling method. After we have cleaned the data, we have a data set consisting of 249 business days with 76 five minute returns per day, hence 18,924 returns.

We have computed the difference of realised variance and realised quadpower for each day in order to estimate the jump part of the quadratic variation. Besides we have calculated its upper and lower 95% confidence bounds. The corresponding plot is given in Figure 5.

We observe the following. For the IBM data, we find that on 18 days (out of 249 business days in the sample) the 0 is not in the confidence intervals, which indicates that there might have been a jump or even several jumps on these days. This corresponds to 7.2% of the days. For GM, we observe 41 days and, for RDSA, 36 days, where the 0 is not within the confidence bounds, which corresponds to 16.4% and 14.4% of the days, respectively.

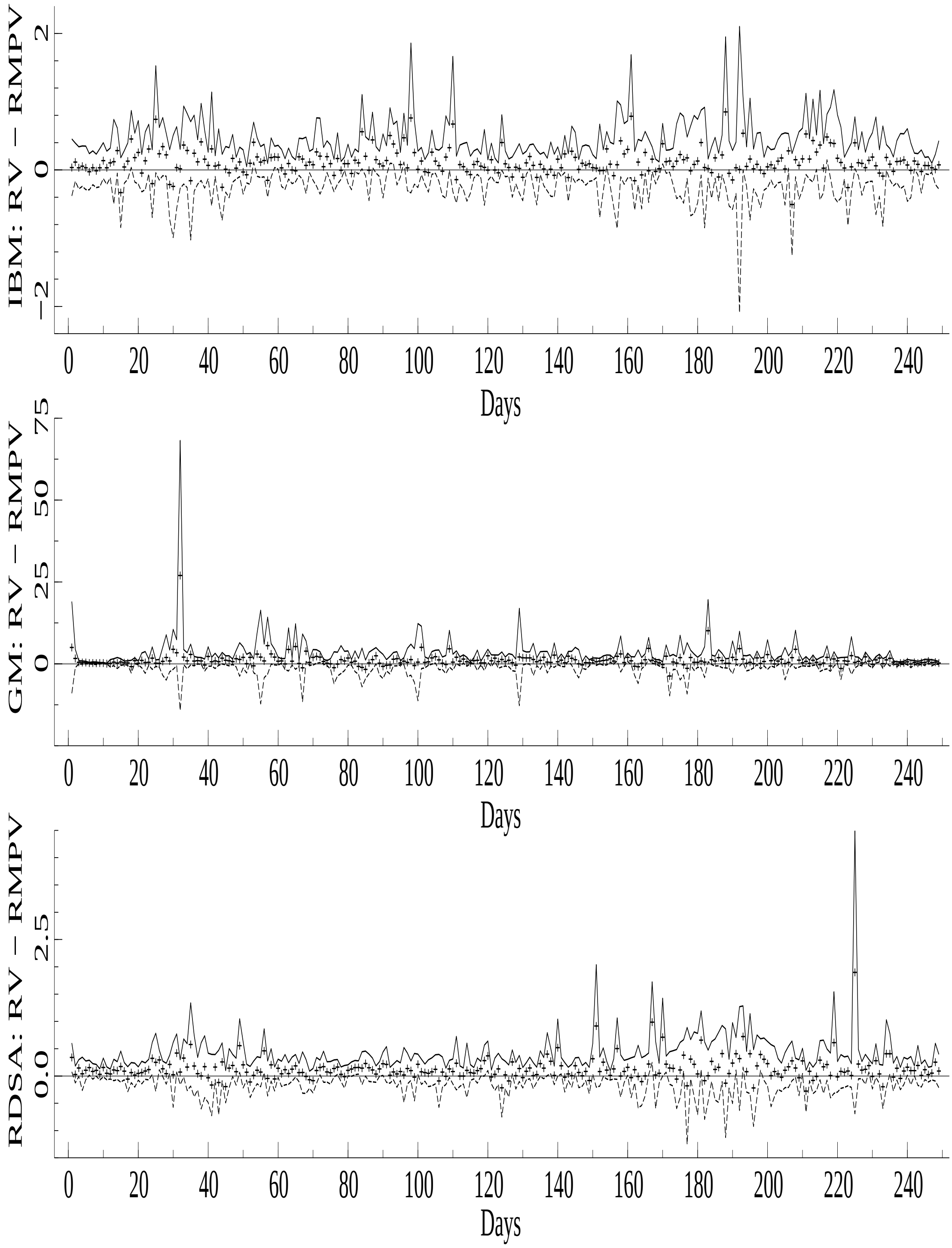

Figure 5: Estimated jump part of the quadratic variation with confidence bounds for IBM, GM and RDSA data.

7 Conclusion

In this paper, we have derived the joint distribution of realised variance and realised multipower in the presence of jumps. Hence we were able to derive the asymptotic distribution of various test statistics both under the null hypothesis of no jumps and under the alternative hypothesis. In particular, we can now make inference on the jump part of quadratic variation using our key result given in equation (13), which says that under some regularity assumptions and if Y and σ have no common jumps then

$$\frac{1}{\sqrt{\Delta_n}}(RV_t^n - RMPV(2;I)_t^n - [Y]^d_{\Delta_n[t/\Delta_n]}) \xrightarrow{stably\ in\ law} MN\left(0, \theta_I \int_0^t \sigma_u^4 du + 2 \sum_{p:\ T_P \leq t} \left(\Delta Y_{T_p}\right)^2 \left(\sigma^2_{T_p-} + \sigma^2_{T_p}\right)\right),$$

where $I \in \mathbb{N}$ with $2 < I < \frac{2}{s}(2-s)$ and $s < 1$.

Furthermore, we have shown how these (infeasible) limit results can be converted into feasible limit theorem, which can be used in practice.

We have carried out a detailed simulation study where we have chosen the parameters in such a way that the resulting data look very much like data we might find in practice. In our simulation study we have compared the final sample performance of the linear, ratio and log–linear test statistic for various multipower variations. We found that the finite sample performance is good.

We have applied our theoretical results to some high frequency equity data and have been able to identify days where the jump component of the quadratic variation seems to be significantly bigger than 0, which indicates that there might have been one or more jumps on these days.

In future work, it will be interesting to study particularly two questions in more detail. First, how do the results change when we allow for market microstructure noise in the model? How robust are our test statistics and how does the asymptotic distribution change? Second, how do these results extend to a multivariate framework? Very recent work by Barndorff-Nielsen & Shephard (2007*a*) and Jacod & Todorov (2007) has already addressed the question of testing for common and disjoint jumps of multivariate price processes. So it would be very interesting to see whether it would be possible to extend the results from this paper to a multivariate model setting.

A Proofs

Proof of Theorem 4.1 The univariate results follow from Jacod (2007, Theorem 2.11 (ii)) and Jacod (2006, Theorem 6.2). In order to derive the multivariate central limit result, we use a modified version of Jacod (2007, Theorem 2.12) which can account for multipower variation rather than power variation only. For the proof of the theorem, we essentially have to prove three lemmas (Lemma A.1 – Lemma A.3), which we will do in the following. But first of all, let us state one further stronger assumption which can be relaxed afterwards.

Hypothesis (SH) The hypothesis (H) holds and the processes (b_t), (c_t) and $(F_t(\phi_2))$ are bounded by a non–random constant and the jumps of Y are also bounded by a constant.

We refer to Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) and Jacod (2007, Section 4 and 5) for more details about how the validity of the corresponding limit results under stronger hypothesis leads to their validity under (H).

Remark Barndorff-Nielsen, Graversen, Jacod & Shephard (2006, Theorem 2 (in particular, Example 7)) contains the following bivariate limit theorem for realised variance and realised bipower variation for a Brownian semimartingale Y, i.e. in the absence of jumps, as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n - [Y]_t \\ RMPV((1,1))_t^n - [Y]_t \end{pmatrix} \xrightarrow{stably\ in\ law} \begin{pmatrix} \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u \\ \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u + \sqrt{\theta_2}\int_0^t \sigma_u^2 d\widetilde{W}_u \end{pmatrix},$$

stably in law for independent Brownian motions $\overline{W}$ and $\widetilde{W}$.

By using exactly the same reasoning, we can show that in the continuous semimartingale framework we obtain, as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n - [Y]_t \\ RMPV(2;I)_t^n - [Y]_t \end{pmatrix} \xrightarrow{stably\ in\ law} \begin{pmatrix} \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u \\ \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u + \sqrt{\theta_I}\int_0^t \sigma_u^2 d\widetilde{W}_u \end{pmatrix},$$

stably in law for independent Brownian motions $\overline{W}$ and $\widetilde{W}$.

In order to prove our main theorem, we need some further notation, which we will introduce in the following. Let Y_t^n denote a one–dimensional semimartingale. We consider functions $g_j : \mathbb{R} \to \mathcal{M}_{d_j,d_{j+1}}$ for $j = 1, \dots, I$, where $\mathcal{M}_{d_j,d_{j+1}}$ denotes a $d_j \times d_{j+1}$–dimensional matrix with real–valued entries. Note that we are in particular interested in the following choice of functions g_j for $j = 1, \dots, I$ and $I \geq 3$. Let $d_1 = \dots = d_I = 2$, $d_{I+1} = 1$ and

$$g_1(y) = \begin{pmatrix} y^2 & 0 \\ 0 & \mu_{2/I}^{-1}|y|^{2/I} \end{pmatrix}, \quad g_i(y) = \begin{pmatrix} 1 & 0 \\ 0 & \mu_{2/I}^{-1}|y|^{2/I} \end{pmatrix}, \quad g_I(y) = \begin{pmatrix} 1 \\ \mu_{2/I}^{-1}|y|^{2/I} \end{pmatrix}, \qquad (14)$$

for $i = 2, \dots I-1$. In the following, we will always set the second component $\left(\prod_{i'=1}^I g_{i'}\left(\frac{\Delta_{i+i'-1}^n Y}{\sqrt{\Delta_n}}\right)\right)^{(2)} = 0$ whenever $i > [t/\Delta_n] - I + 1$. Then,

$$\frac{1}{\sqrt{\Delta_n}}\left(\Delta_n \sum_{i=1}^{[t/\Delta_n]} \prod_{i'=1}^{I} g_{i'}\left(\frac{\Delta_{i+i'-1}^n Y}{\sqrt{\Delta_n}}\right)\right) = \frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n Y)^2 \\ \mu_{2/I}^{-I} \sum_{i=1}^{[t/\Delta_n]-I+1} \prod_{i'=1}^I \left|\Delta_{i+i'-1}^n Y\right|^{2/I} \end{pmatrix}$$
$$= \frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n \\ RMPV(2;I)_t^n \end{pmatrix}.$$

Further, we define $\beta_i^{i'} = \frac{1}{\sqrt{\Delta_n}}\sigma_{(i-1)\Delta_n}\Delta_{i+i'-1}^n W$ for $i' = 1, \dots, I$ and

$$\rho_i^n(g_{i'}) = \int g_{i'}(x) f_{\sigma_{(i-1)\Delta_n}}(x)dx,$$

(componentwise for the diagonal matrices and vectors defined above), where $f_{\sigma_{(i-1)\Delta_n}}$ is the density of a $N(0,\ \sigma^2_{(i-1)\Delta_n})$–distributed random variable. So, finally, we define the following random vector:

$$\overline{U}^n_t = \overline{U}^n(g_1,\ldots,g_I)_t = \sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}\left\{\prod_{i'=1}^{I} g_{i'}\left(\beta_i^{i'}\right) - \prod_{i'=1}^{I}\rho_i^n\left(g_{i'}\right)\right\}, \tag{15}$$

which is in $\mathcal{M}_{d_1,d_{I+1}}$. Note here that we will set the second component $\left(\prod_{i'=1}^{I} g_{i'}\left(\beta_i^{i'}\right)\right)^{(2)} - \left(\prod_{i'=1}^{I}\rho_i^n\left(g_{i'}\right)\right)^{(2)} = 0$ whenever $i > [t/\Delta_n] - I + 1$. The following Lemma states a central limit theorem for the random vector (15).

Lemma A.1 *Assume that (SH) holds and let $g_1,\ldots,g_I$ denote continuous even functions of at most polynomial growth with $g_i : \mathbb{R}^d \to \mathcal{M}_{d_i,d_{i+1}}$ for $i = 1,\ldots I$ as defined in (14). So, in particular, we have $d_1 = 2$ and $d_{I+1} = 1$. Let $\overline{U}^n = \overline{U}^n(g_1,\ldots g_I)$ denote the stochastic process defined in (15) with components*

$$\overline{U}^n_t\left(g_1,\ldots,g_I\right)^{(j)} = \sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]}\left\{\left(\prod_{i'=1}^{I} g_{i'}\left(\beta_i^{i'}\right)\right)^{(j)} - \left(\prod_{i'=1}^{I}\rho_i^n\left(g_{i'}\right)\right)^{(j)}\right\},$$

for $j = 1, 2$. Then the d_1–dimensional process $\overline{U}^n$ converges stably in law to a limit process $\overline{U}$ with components

$$\overline{U}^{(j)}_t = \sum_{k=1}^{2}\int_0^t \Sigma_u^{j,k} d\overline{W}^k_u, \quad j = 1, 2,$$

where the 2×2–dimensional process Σ, defined by

$$\Sigma_u = \begin{pmatrix} \sqrt{2}\sigma_u^2 & 0 \\ \sqrt{2}\sigma_u^2 & \sqrt{\theta_I}\sigma_u^2 \end{pmatrix} \tag{16}$$

is $(\mathcal{F}_t)$–optional.

Proof Since we are only dealing with Brownian semimartingales in this lemma, the result follows directly along the lines of Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, Proposition 5.2) or by extending the proof of Jacod (2007, Lemma 5.7), which we will sketch in the following.

One can easily show by induction on I that

$$\prod_{i'=1}^{I} g_{i'}\left(\beta_i^{i'}\right) - \prod_{i'=1}^{I}\rho_i^n(g_{i'}) = \sum_{j=1}^{I}\left(\prod_{i'=1}^{j-1} g_{i'}\left(\beta_i^{i'}\right)\right)\left[g_j\left(\beta_i^j\right) - \rho_i^n(g_j)\right]\left(\prod_{i'=j+1}^{I}\rho_i^n(g_{i'})\right),$$

where an empty product is set to 1. This term is not measurable with respect to $\mathcal{F}_{i\Delta_n}$, which we need in order to be able to apply Jacod & Shiryaev (2003, Theorem IX.7.19 and Theorem IX.7.28). So we use the same methods which have been applied in the proof of Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, Proposition 5.2). I.e. we shift the terms back in time to make them measurable w.r.t. $\mathcal{F}_{i\Delta_n}$. We do not shift the first term in the sum, but we shift the second term by one, the third term by two etc. and, finally, the Ith term by $I-1$. By doing that we get a new random variable

$$\zeta_i^n = \sqrt{\Delta_n} \sum_{j=1}^{I} \left(\prod_{i'=1}^{j-1} g_{i'} \left(\beta_i^{i'-(j-1)} \right) \right) \left[g_j \left(\beta_i^1 \right) - \rho_{i-(j-1)}^n (g_j) \right] \left(\prod_{i'=j+1}^{I} \rho_{i-(j-1)}^n (g_{i'}) \right), \quad (17)$$

which is clearly measurable with respect to $\mathcal{F}_{i\Delta_n}$. As in the proof of Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, Proposition 5.2) one can easily show that

$$\overline{U}_t^n(g_1, \ldots, g_I) - \sum_{i=I}^{[t/\Delta_n]-I+1} \zeta_i^n \xrightarrow{ucp} 0, \quad \text{as } n \to \infty.$$

Let $\mathbb{E}_{i-1}^n(\cdot) = \mathbb{E}\left(\cdot | \mathcal{F}_{(i-1)\Delta_n}\right)$. Trivially, we get $\mathbb{E}_{i-1}^n\left(\zeta_i^n\right) = 0$ and $\mathbb{E}_{i-1}^n\left(||\zeta_i^n||^4\right) \leq K\Delta_n^2$ (for a constant $K > 0$). Analogously to the proof of Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, Proposition 5.2), we obtain in particular that

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n \left(\zeta_i^{j;n} \zeta_i^{k;n} \right) \xrightarrow{ucp} \int_0^t \left(\Sigma_u \Sigma_u^* \right)^{j,k} du, \quad (18)$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n \left(\zeta_i^{j;n} \Delta_i^n N \right) \xrightarrow{ucp} 0, \quad \text{if } N = W \text{ or } N \in \mathcal{N}, \quad (19)$$

as $n \to \infty$, where $\mathcal{N}$ is the set of all bounded $(\mathcal{F}_t)$–martingales which are orthogonal to W. Now the result follows from Jacod & Shiryaev (2003, Theorem IX.7.19 and Theorem IX.7.28). □

Now we study the more general case where we allow for jumps in the price process Y. We start by introducing some notation (which is the same as in Jacod (2007)) and some more assumptions.

Hypothesis (SK) Assumptions (K) and (SH) are satisfied and the functions $\gamma_k = \gamma$ are bounded and do not depend on k.

Let $\epsilon > 0$ fixed. We define a process N by $N = \mathbb{1}_E * \underline{\mu}$, where $E = \{x : \gamma(x) > \epsilon\}$. Hence N is a Poisson process with parameter the Lebesgue measure of E, say λ.

Remark Note that under (SK) we have $\int_\mathbb{R} (1 \wedge \gamma^2(x))dx < \infty$ and $\sup_x \gamma(x) \leq K$ for a $K \geq 0$. Therefore we get $\int_\mathbb{R} \gamma^2(x) \mathbb{1}_{\{\gamma^2(x) \leq 1\}}(x)dx < \infty$ and $\int_\mathbb{R} \mathbb{1}_{\{\gamma^2(x) > 1\}}(x)dx < \infty$. So altogether, we obtain $\int_\mathbb{R} \gamma^2(x)dx < \infty$, since

$$\begin{aligned} \int_\mathbb{R} \gamma^2(x)dx &= \int_\mathbb{R} \gamma^2(x) \mathbb{1}_{\{\gamma^2(x) \leq 1\}}(x)dx + \int_\mathbb{R} \gamma^2(x) \mathbb{1}_{\{\gamma^2(x) > 1\}}(x)dx \\ &\leq \int_\mathbb{R} \gamma^2(x) \mathbb{1}_{\{\gamma^2(x) \leq 1\}}(x)dx + K \int_\mathbb{R} \mathbb{1}_{\{\gamma^2(x) > 1\}}(x)dx < \infty. \end{aligned}$$

Therefore, we can deduce that λ is indeed finite:

$$\lambda = \int_{\mathbb{R}} \mathbb{1}_{\{x:\gamma(x)>\epsilon\}}(x)dx \le \int_{\mathbb{R}} \frac{\gamma(x)^2}{\epsilon^2}dx < \infty.$$

Depending on ϵ, we define the following quantities:

- $S_1, S_2, \dots$ are the successive jump times of N,
- $I(n,p) = i,\ S_-(n,p) = (i-1)\Delta_n,\ S_+(n,p) = i\Delta_n$ on $\{(i-1)\Delta_n < S_p \le i\Delta_n\}$,
- $\alpha_-(n,p) = \frac{1}{\sqrt{\Delta_n}}\left(W_{S_p} - W_{S_-(n,p)}\right),\ \alpha_+(n,p) = \frac{1}{\sqrt{\Delta_n}}\left(W_{S_+(n,p)} - W_{S_p}\right)$,
- $R_p = \Delta Y_{S_p}$,
- $Y(\epsilon)_t = Y_t - \sum_{p:S_p\le t} R_p$,
- $R_p'^n = \Delta_i^n Y(\epsilon)$ on the set $\{(i-1)\Delta_n < S_p \le i\Delta_n\}$,
- $R_p' = \sqrt{\xi_p}U_p\sigma_{S_{p-}} + \sqrt{1-\xi_p}U_p'\sigma_{S_p}$,
- $\Omega_n(T,\epsilon) = \{\omega :$ each interval $[0,T] \cap ((i-1)\Delta_n, i\Delta_n]$ contains at most one $S_p(\omega)$; $|\Delta_i^n Y(\epsilon)(\omega)| \le 2\epsilon, \forall i \le T/\Delta_n\}$.

Lemma A.2 *Under (SK), the sequences* $\left(\overline{U}^n, (\alpha_-(n,p), \alpha_+(n,p))_{p\ge1}\right)$ *converge stably in law to* $\left(\overline{U}, \left(\sqrt{\xi_p}U_p, \sqrt{1-\xi_p}U_p'\right)_{p\ge1}\right)$ *as* $n \to \infty$.

Proof Most parts of this proof are identical to the corresponding proof by Jacod (2007, p. 31–32). However, in Step 2, we have adjusted the proof to allow for multipower variation.

Step 1: We have to prove that for all bounded $\mathcal{A}$–measurable random variables Ψ and all bounded Lipschitz functions Φ on the Skorohod space of d–dimensional functions on $\mathbb{R}_+$ endowed with a distance for the Skorohod topology, and all $q \ge 1$ and all continuous bounded functions f_p on $\mathbb{R}^2$, and with $A_n = \prod_{p=1}^q f_p(\alpha_-(n,p), \alpha_+(n,p))$ then

$$\mathbb{E}\left(\Psi\Phi\left(\bar{U}^n\right)A_n\right) \to \tilde{\mathbb{E}}\left(\Psi\Phi(\bar{U})\right)\prod_{p=1}^q \tilde{\mathbb{E}}\left(f_p(\sqrt{\xi_p}U_p, \sqrt{1-\xi_p}U_p')\right), \quad \text{as } n \to \infty. \quad (20)$$

Replacing Ψ by $\mathbb{E}\left(\Psi|\,\mathcal{G}\right)$ on both sides, it is sufficient to prove the limit result (20) for a Ψ which is measurable with respect to the separable σ–field $\mathcal{G}$ generated by both the measure $\underline{\mu}$ and the processes b, σ, W and Y.

Step 2: Let $\underline{\mu}'$ and $\underline{\mu}''$ ($\underline{\nu}'$ and $\underline{\nu}''$, respectively) denote the restrictions of $\underline{\mu}$ (and $\underline{\nu}$, respectively) to $\mathbb{R}_+ \times E^c$ and to $\mathbb{R}_+ \times E$. Further, let $(\mathcal{F}')$ denote the smallest filtration containing $(\mathcal{F}_t)$ such that $\underline{\mu}''$ is $\mathcal{F}_0'$-measurable. Clearly W is a Wiener process and $\underline{\mu}'$ is a Poisson random measure with compensator $\underline{\nu}'$ relative to $(\mathcal{F}_t)$, but also relative to $(\mathcal{F}_t')$.

Now we define a set of intervals surrounding the jump times of the Poisson process N. Let $m \in \mathbb{N}$ be any positive integer, then we define $S_p^{m-} = (S_p - 1/m)^+$, $S_p^{m+} = S_p + 1/m$ and

$B_m = \cup_{p\geq 1}(S_p^{m-}, S_p^{m+}]$. Since the indicator function $\mathbb{1}_{B_m}(\omega,t)$ is $\mathcal{F}_0' \otimes \mathbb{R}_+$–measurable, we can define the stochastic integral $W(m)_t = \int_0^t \mathbb{1}_{B_m}(u)dW_u$. Now let $(\mathcal{F}_t'^m)$ denote the smallest filtration containing $(\mathcal{F}_t')$ such that $W(m)$ is $(\mathcal{F}_0'^m)$–measurable. Further, we define the set $\Gamma_n(m,t) = \{i \in \mathbb{N} : i \leq [t/\Delta_n] \text{ and } B_m \cap ((i-1)\Delta_n, i\Delta_n] = \emptyset\}$. Similarly to Jacod (2007), we define two bivariate processes $\overline{U}'^n(m)$, where we just sum over the integers which are not "close" to the jump times, and $\overline{U}(m)$, with components:

$$\overline{U}'^n(m)_t^j = \sqrt{\Delta_n} \sum_{i\in\Gamma_n(m,t)} \left(\left(\prod_{i'=1}^I g_{i'}\left(\beta_i^{i'}\right)\right)^j - \left(\prod_{i'=1}^I \rho_i^n\left(g_{i'}\right)\right)^j \right),$$

$$\overline{U}(m)_t^j = \sum_{j'=1}^2 \int_0^t \Sigma_u^{j,j'} \mathbb{1}_{B_m^c(u)} d\overline{W}_u^{j'},$$

where Σ is defined by (16) and $j = 1,2$. Once again, note that both integrals are well–defined since $\overline{W}$ is a Brownian motion w.r.t. the smallest filtration containing $(\mathcal{F}_t')$ and $\mathcal{F}_0'^m$ at time 0. Clearly, $B_m \to \cup_p\{S_p\}$ for $m \to \infty$ and, hence, $\overline{U}(m) \xrightarrow{ucp} \overline{U}$ as $m \to \infty$. Note that

$$\begin{aligned}\Gamma^n(m,t)^c &= \{i : i \leq [t/\Delta_n], B_m \cap ((i-1)\Delta_n, i\Delta_n] \neq \emptyset\} \\ &\subseteq \left\{ i : i \leq [t/\Delta_n], \exists p : |i\Delta_n - S_p| \leq \frac{2}{m} \right\}.\end{aligned}$$

Note that in the following, the constant K can change from line to line, but will not depend on n, t and m (but will depend on ϵ).

Since the conditional expectation of $\zeta_i^{j;n}$ is zero, if we condition on the past before $(i-1)\Delta_n$ and the sequence of stopping times S_p, which are independent of W, i.e. $\mathbb{E}\left(\zeta_i^{j;n} \middle| \mathcal{F}'_{(i-1)\Delta_n}\right) = 0$, we reach that $\bar{U}^n(g_1,\ldots,g_I)_s^j - \bar{U}'^n(m)_s^j$ is indeed a martingale with respect to $(\mathcal{F}_t'^m)$. By applying Doob's inequality, we obtain the following:

$$\mathbb{E}\left(\sup_{s\leq t}\left|\bar{U}^n(g_1,\ldots,g_I)_s^j - \bar{U}'^n(m)_s^j\right|^2\right) \leq 4\Delta_n\, \mathbb{E}\left(\sum_{p=1}^\infty \sum_{i=1}^{[t/\Delta_n]} |\zeta_i^{j;n}|^2 \mathbb{1}_{\{|i\Delta_n - S_p| \leq 2/m\}}\right).$$

Since all functions g_i (for $i = 1,\ldots I$) are of at most polynomial growth, there exist constants $\tilde{p}_1,\ldots,\tilde{p}_I$ such that (by induction on I)

$$\begin{aligned}&\mathbb{E}\left(\sup_{s\leq t}\left|\bar{U}^n(g_1,\ldots,g_I)_s^j - \bar{U}'^n(m)_s^j\right|^2\right) \\ &\qquad \leq K\Delta_n\mathbb{E}\left(\sum_{p=1}^\infty \sum_{1\leq i\leq[t/\Delta_n]:\exists p:|i\Delta_n - S_p|\leq 2/m}\prod_{i'=1}^I (1+|\beta_i^{i',n}|^{\tilde{p}_{i'}})\right).\end{aligned}$$

Since σ is bounded and for fixed p, we get

$$\#\left\{i : i \leq [t/\Delta_n], |i\Delta_n - S_p| \leq \frac{2}{m}\right\} \leq \frac{4}{m\Delta_n},$$

we obtain from (SH) that

$$\mathbb{E}\left(\sup_{s\leq t}\left|\bar{U}^n(g_1,\ldots,g_I)^j_s - \bar{U}'^n(m)^j_s\right|^2\right) \leq \frac{K}{m}\mathbb{E}\left(\sum_{p=1}^{\infty}\mathbb{1}_{\{S_p\leq t+1\}}\right) = \frac{K}{m}\sum_{p=1}^{\infty}\mathbb{P}\left(S_p\leq t+1\right) = \frac{K}{m}\sum_{p=1}^{\infty}\mathbb{P}\left(N_{t+1}\geq p\right) = \frac{K}{m}\lambda(t+1).$$

It is now sufficient to prove that for each m and for each $\mathcal{G}$–measurable and bounded Ψ for fixed m, as $n\to\infty$:

$$\mathbb{E}\left(\Psi\Phi\left(\bar{U}'^n(m)\right)A_n\right) \to \tilde{\mathbb{E}}\left(\Psi\Phi(\bar{U}(m))\right)\prod_{p=1}^{q}\tilde{\mathbb{E}}\left(f_p(\sqrt{\xi_p}U_p,\sqrt{1-\xi_p}U'_p)\right), \tag{21}$$

since Ψ is Lipschitz and bounded.

Step 3: This part of the proof follows directly from Jacod (2007, page 32). Now we fix m and define a regular version $\mathcal{Q}=\mathcal{Q}_\omega(\cdot)$ of the probability measure $\mathbb{P}$ on $(\Omega,\mathcal{G})$, conditional on $\mathcal{F}_0'^m$, and similarly $\tilde{\mathcal{Q}}=\mathcal{Q}\times\mathbb{P}'$.

When $i\in\Gamma_n(m,t)$, then $\Delta_i^n W$ is independent of $\mathcal{F}_0'^m$ and hence also standard normally distributed under each $\mathcal{Q}_\omega$. So we can state (18) and (19), where we replace $\mathbb{E}^n_{i-1}$ by $\mathbb{E}_{\mathcal{Q}_\omega}\left(\cdot\middle|\mathcal{F}'^m_{(i-1)\Delta_n}\right)$. Since B_m^c is a locally finite unit of intervals, we get

$$\sum_{i\in\Gamma_n(m,t)}\mathbb{E}_{\mathcal{Q}_\omega}\left(\zeta_i^{j;n}\zeta_i^{j';n}\middle|\mathcal{F}'^m_{(i-1)\Delta_n}\right)\to\int_0^t\left(\Sigma_u\Sigma_u^*\right)^{j,j'}\mathbb{I}_{B_m^c}(u)du,$$

for $j,j'=1,2$ as $n\to\infty$. Therefore, we obtain for $n\to\infty$:

$$\mathbb{E}_{\mathcal{Q}_\omega}\left(\Psi\Phi\left(\overline{U}'^n(m)\right)\right)\to\tilde{\mathbb{E}}_{\tilde{\mathcal{Q}}_\omega}\left(\Psi\Phi\left(\overline{U}(m)\right)\right), \tag{22}$$

so $\overline{U}'^n(m)\overset{stably\ in\ law}{\longrightarrow}\overline{U}(m)$ under the measure $\mathcal{Q}_\omega$.

Step 4: Clearly A_n is $\mathcal{F}_0'^m$–measurable. Therefore, we can express the left hand side of (21)

$$\begin{aligned}\mathbb{E}\left(\Psi\Phi\left(\overline{U}'^n(m)\right)A_n\right) &= \mathbb{E}\left(A_n\mathbb{E}_{\mathcal{Q}.}\left(\Psi\Phi\left(\overline{U}'^n(m)\right)\right)\right)\\ &= \tilde{\mathbb{E}}\left(A_n\tilde{\mathbb{E}}_{\tilde{\mathcal{Q}}.}\left(\Psi\Phi\left(\overline{U}(m)\right)\right)\right)+\tilde{\mathbb{E}}\left(A_n\left(\mathbb{E}_{\mathcal{Q}.}\left(\Psi\Phi\left(\overline{U}'^n(m)\right)\right)\right.\right.\\ &\qquad\left.\left.-\tilde{\mathbb{E}}_{\tilde{\mathcal{Q}}.}\left(\Psi\Phi\left(\overline{U}(m)\right)\right)\right)\right)\end{aligned}$$

Note that all quantities in the formula above are bounded, so by (22) the second summand on the right hand side converges to 0 and $\Psi'=\tilde{\mathbb{E}}_{\tilde{\mathcal{Q}}.}\left(\Psi\Phi(\overline{U}(m)\right)$ is also a bounded and $\mathcal{F}_0'^m$–measurable variable. So, proving (21) is equivalent to proving

$$\tilde{\mathbb{E}}\left(\Psi'A_n\right)\to\tilde{\mathbb{E}}(\Psi')\prod_{p=1}^{q}\tilde{\mathbb{E}}\left(f_p\left(\sqrt{\xi_p}U_p,\sqrt{1-\xi_p}U'_p\right)\right),$$

for all Ψ' bounded and $\mathcal{F}_0^{'m}$–measurable, which is implied by

$$(\alpha_-(n,p), \alpha_+(n,p))_{p\geq 1} \stackrel{stably\ in\ law}{\longrightarrow} \left(\sqrt{\xi_p}U_p, \sqrt{1-\xi_p}U_p^{'}\right)_{p\geq 1}, \quad \text{as } n \to \infty.$$

However, this convergence result follows directly from Jacod & Protter (1998, Lemma 6.2).

So the result follows. □

Finally, we generalise the results from Lemma A.2 and obtain the final auxiliary limit result which we need for the proof of our main theorem.

Lemma A.3 *Under the assumptions of Lemma A.2, the sequences* $\left(\overline{U}^n, \left(R_p^{'n}/\sqrt{\Delta_n}\right)_{p\geq 1}\right)$ *converge stably in law to* $\left(\overline{U}, \left(R_p^{'}\right)_{p\geq 1}\right)$*, as* $n \to \infty$.

Proof This proof goes along the lines of the proof of Jacod (2007, Lemma 5.9). However, for completeness we sketch Jacod (2007)'s proof here.

Since σ is càdlàg and by construction of $R_p^{'}$, we can deduce from Lemma A.2 that it is sufficient to prove

$$w_p^n = R_p^{'n}/\sqrt{\Delta_n} - \sigma_{S_-(n,p)}\alpha_-(n,p) - \sigma_{S_p}\alpha_+(n,p) \stackrel{\mathbb{P}}{\to} 0. \tag{23}$$

for any $p \geq 1$. Let $\underline{\mu}'$ and $\mathcal{F}_t^{'}$ be defined as in the previous proof. From

$$Y_t = Y_0 + \int_0^t b_s^{'} ds + \int_0^t \sigma_s dW_s + \delta \star (\underline{\mu} - \underline{\nu})_t,$$

where $b_t^{'} = b_t + \int \kappa'(\delta(t,x))$, we obtain:

$$Y_t(\epsilon) = Y_0 + \int_0^t b'(\epsilon)_s ds + \int_0^t \sigma_s dW_s + \delta \star (\underline{\mu}' - \underline{\nu}')_t,$$

where $b'(\epsilon)_t = b_t' + \int_E \delta(t,x)dx$ and the stochastic integral can be taken relative to both $(\mathcal{F}_t)$ and $(\mathcal{F}'_t)$. Further, $Y(\epsilon)$ satisfies (SH) for the filtration $(\mathcal{F}_t^{'})$. Based on the definition $Y' = Y - Y^c - Y_0$, we have $Y'(\epsilon) = Y(\epsilon) - Y^c - Y_0$. Then

$$w_p^n = \frac{1}{\sqrt{\Delta_n}}\left(\Delta_{I(n,p)}^n Y'(\epsilon) + \int_{S_-(n,p)}^{S_p} (\sigma_u - \sigma_{S_-(n,p)})dW_s + \int_{S_p}^{S_+(n,p)} (\sigma_u - \sigma_{S_p})dW_s\right).$$

Then it can be shown that for $\epsilon = \Delta^{1/4}$:

$$\begin{aligned}\mathbb{E}\left((w_p^n)^2\right) \leq K\sqrt{\Delta_n} + K\mathbb{E}\left(\frac{1}{\Delta_n}\int_{S_-(n,p)}^{S_+(n,p)} du \int_{E^c \cap \{x:|\delta(u,x)|\leq \Delta_n^{1/4}\}} \delta(u,x)^2 dx\right) \\ + K\mathbb{E}\left(\frac{1}{\Delta_n}\int_{S_-(n,p)}^{S_p} (\sigma_u - \sigma_{S_-(n,p)})^2 du + \frac{1}{\Delta_n}\int_{S_p}^{S_+(n,p)} (\sigma_u - \sigma_{S_p})^2\right) du.\end{aligned}$$

Finally, since $|\delta| \leq \gamma$ and $\int \gamma(x)^2 dx < \infty$ and since σ is càdlàg and bounded, the expression above converges to 0 as $n \to \infty$ (by Lebesgue's theorem). So $w_p^n \stackrel{\mathbb{P}}{\to} 0$. □

Now we can combine the results from the three Lemmas above to deduce the result of Theorem 4.1 analogously to the proof of Jacod (2007, Theorem 2.12). I.e. note that Lemma A.1 is multidimensional. The one–dimensional results have been deduced from the corresponding components of Lemma A.1 by Jacod (2007, Theorem 2.11 (ii)) for the realised variance and for the realised multipower variation by Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, p. 10–11) in the absence of jumps and Jacod (2006, Theorem 6.2) in the presence of jumps. So the way how these results are deduced from Lemma A.1 can be carried over separately for each component in the multidimensional case. So Theorem 4.1 holds. □

B Tables

		Linear test			Ratio test			Log-linear test		
M (K)	**I**	**Mean**	**S.D.**	**Cove.**	**Mean**	**S.D.**	**Cove.**	**Mean**	**S.D.**	**Cove.**
39	3	0	0.92	0.975	0	0.92	0.975	-0.02	0.92	0.975
(7)	4	0.02	0.95	0.967	-0.02	0.95	0.967	0	0.95	0.966
	10	0.07	1.11	0.924	-0.07	1.11	0.924	-0.01	1.05	0.939
78	3	-0.02	0.9	0.977	0.02	0.9	0.977	-0.04	0.91	0.973
(9)	4	0	0.93	0.971	0	0.93	0.971	-0.03	0.94	0.967
	10	0.02	1.02	0.946	-0.02	1.02	0.946	-0.03	1.01	0.951
390	3	0	0.91	0.97	0	0.91	0.97	-0.01	0.92	0.967
(20)	4	0.01	0.93	0.967	-0.01	0.93	0.967	0	0.93	0.966
	10	0.03	0.97	0.955	-0.03	0.97	0.955	0	0.97	0.958
1560	3	0	0.95	0.961	0	0.95	0.961	0	0.96	0.959
(40)	4	0.01	0.97	0.958	-0.01	0.97	0.958	0	0.97	0.957
	10	0.01	0.99	0.952	-0.01	0.99	0.951	0	0.99	0.951
23400	3	0.02	0.97	0.955	-0.02	0.97	0.955	0.01	0.97	0.955
(153)	4	0.02	0.98	0.953	-0.02	0.98	0.953	0.02	0.98	0.953
	10	0.02	0.99	0.947	-0.02	0.99	0.947	0.02	0.99	0.947

Table 5: Constant volatility model with $\lambda = 0.014$, $\sigma_p = 1.5$. We simulate data for 5000 days and compute the mean, standard deviation and coverage of the feasible linear test statistic, the feasible ratio test statistic and the feasible log-linear test statistic.

M (K)	**I**	Linear test **Mean**	**S.D.**	**Cove.**	Ratio test **Mean**	**S.D.**	**Cove.**	Log-linear test **Mean**	**S.D.**	**Cove.**
39	3	-0.06	0.96	0.965	0.06	0.96	0.965	-0.1	0.98	0.961
(7)	4	-0.02	0.98	0.959	0.02	0.98	0.959	-0.06	0.98	0.962
	10	0.03	1.1	0.925	-0.03	1.1	0.925	-0.05	1.06	0.94
78	3	-0.05	0.95	0.971	0.05	0.95	0.971	-0.09	0.95	0.965
(9)	4	-0.01	0.95	0.969	0.01	0.95	0.969	-0.05	0.94	0.97
	10	0.03	1.01	0.948	-0.03	1.01	0.948	-0.02	0.99	0.955
390	3	-0.06	0.95	0.961	0.06	0.95	0.961	-0.1	0.96	0.953
(20)	4	-0.03	0.95	0.961	0.03	0.95	0.961	-0.06	0.95	0.958
	10	0	0.97	0.951	0	0.97	0.951	-0.03	0.97	0.953
1560	3	-0.06	0.97	0.96	0.06	0.97	0.96	-0.09	0.99	0.953
(40)	4	-0.04	0.97	0.957	0.04	0.97	0.957	-0.06	0.98	0.953
	10	-0.02	0.98	0.955	0.02	0.98	0.955	-0.04	0.98	0.957
23400	3	-0.01	0.98	0.953	0.01	0.98	0.953	-0.04	0.98	0.953
(153)	4	-0.01	0.98	0.956	0.01	0.98	0.956	-0.02	0.98	0.957
	10	0	0.99	0.952	0	0.99	0.952	-0.01	0.99	0.953

Table 6: Constant volatility model with $\lambda = 0.118$, $\sigma_p = 1.5$. We simulate data for 5000 days and compute the mean, standard deviation and coverage of the feasible linear test statistic, the feasible ratio test statistic and the feasible log-linear test statistic.

M (K)	**I**	Linear test **Mean**	**S.D.**	**Cove.**	Ratio test **Mean**	**S.D.**	**Cove.**	Log-linear test **Mean**	**S.D.**	**Cove.**
39	3	0.01	0.93	0.973	-0.01	0.93	0.973	-0.01	0.93	0.972
(7)	4	0.04	0.96	0.965	-0.04	0.96	0.965	0	0.96	0.966
	10	0.07	1.09	0.925	-0.07	1.09	0.925	0	1.04	0.942
78	3	-0.01	0.88	0.978	0.01	0.88	0.978	-0.03	0.89	0.976
(9)	4	0	0.92	0.972	0	0.92	0.972	-0.02	0.92	0.97
	10	0.03	1.01	0.949	-0.03	1.01	0.949	-0.02	0.99	0.949
390	3	-0.02	0.92	0.972	0.02	0.92	0.972	-0.04	0.92	0.967
(20)	4	-0.01	0.94	0.964	0.01	0.94	0.964	-0.03	0.94	0.964
	10	0	0.97	0.955	0	0.97	0.955	-0.02	0.97	0.957
1560	3	0	0.96	0.959	0	0.96	0.959	-0.02	0.96	0.958
(40)	4	0	0.97	0.957	0	0.97	0.957	-0.02	0.97	0.957
	10	0	1	0.947	0	1	0.947	-0.02	1	0.946
23400	3	0	0.96	0.958	0	0.96	0.958	0	0.97	0.957
(153)	4	0	0.97	0.955	0	0.97	0.955	0	0.97	0.955
	10	0	0.99	0.95	0	0.99	0.95	0	0.99	0.949

Table 7: Stochastic volatility model with $\lambda = 0.014$, $\sigma_p = 1.5$. We simulate data for 5000 days and compute the mean, standard deviation and coverage of the feasible linear test statistic, the feasible ratio test statistic and the feasible log-linear test statistic.

M (K)	I	Linear test Mean	S.D.	Cove.	Ratio test Mean	S.D.	Cove.	Log-linear test Mean	S.D.	Cove.
39	3	-0.09	0.96	0.966	0.09	0.96	0.966	-0.14	0.99	0.957
(7)	4	-0.05	0.98	0.966	0.05	0.98	0.966	-0.1	0.99	0.961
	10	0	1.11	0.927	0	1.11	0.927	-0.08	1.06	0.938
78	3	-0.08	0.93	0.969	0.08	0.93	0.969	-0.12	0.96	0.962
(9)	4	-0.05	0.95	0.966	0.05	0.95	0.966	-0.09	0.96	0.963
	10	0	1.02	0.945	0	1.02	0.945	-0.07	1	0.946
390	3	-0.05	0.96	0.962	0.05	0.96	0.962	-0.1	0.98	0.955
(20)	4	-0.02	0.96	0.963	0.02	0.96	0.963	-0.05	0.96	0.959
	10	0.01	0.98	0.958	-0.01	0.98	0.958	-0.02	0.96	0.959
1560	3	-0.05	0.96	0.957	0.05	0.96	0.957	-0.08	0.98	0.952
(40)	4	-0.02	0.96	0.958	0.02	0.96	0.958	-0.04	0.96	0.957
	10	-0.01	0.98	0.952	0.01	0.98	0.952	-0.02	0.98	0.955
23400	3	-0.02	0.99	0.953	0.02	0.99	0.953	-0.04	1	0.949
(153)	4	0	0.99	0.953	0	0.99	0.953	-0.01	0.99	0.954
	10	0	0.98	0.954	0	0.98	0.954	0	0.98	0.954

Table 8: Stochastic volatility model with $\lambda = 0.118$, $\sigma_p = 1.5$. We simulate data for 5000 days and compute the mean, standard deviation and coverage of the feasible linear test statistic, the feasible ratio test statistic and the feasible log-linear test statistic.

M (K)	I	Linear test Mean	S.D.	Cove.	Ratio test Mean	S.D.	Cove.	Log-linear test Mean	S.D.	Cove.
39	3	0.26	1.05	0.943	-0.26	1.05	0.943	0.21	0.98	0.95
(7)	4	0.35	1.09	0.933	-0.35	1.09	0.933	0.28	0.99	0.945
	10	0.63	1.18	0.878	-0.63	1.18	0.878	0.43	0.97	0.947
78	3	0.16	0.96	0.963	-0.16	0.96	0.963	0.13	0.94	0.966
(9)	4	0.23	0.99	0.955	-0.23	0.99	0.955	0.19	0.96	0.962
	10	0.47	1.06	0.912	-0.47	1.06	0.912	0.36	0.95	0.956
390	3	0.06	0.91	0.975	-0.06	0.91	0.975	0.04	0.92	0.975
(20)	4	0.09	0.94	0.969	-0.09	0.94	0.969	0.07	0.94	0.97
	10	0.23	0.98	0.948	-0.23	0.98	0.948	0.18	0.96	0.961
1560	3	0.04	0.94	0.968	-0.04	0.94	0.968	0.03	0.94	0.967
(40)	4	0.06	0.95	0.961	-0.06	0.95	0.961	0.04	0.95	0.961
	10	0.12	0.99	0.95	-0.12	0.99	0.95	0.1	0.99	0.953
23400	3	0.02	0.98	0.955	-0.02	0.98	0.955	0.02	0.98	0.954
(153)	4	0.03	0.99	0.953	-0.03	0.99	0.953	0.03	0.99	0.952
	10	0.05	0.99	0.948	-0.05	0.99	0.948	0.04	0.99	0.949

Table 9: Two–factor stochastic volatility model. We simulate data for 5000 days and compute the mean, standard deviation and coverage of the feasible linear test statistic, the feasible ratio test statistic and the feasible log-linear test statistic.

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