TESTING COEFFICIENTS CONSTANCY & SPECIFICATION OF INTERACTIVE EFFECTS

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OUTLINE

1. Motivation.
2. Characterization of the null hypothesis.
3. Testing procedure.
4. Test statistic & critical values
5. Real data application.
6. Monte Carlo.

1. MOTIVATION

RANDOM COEFFICIENT MODEL (VARYING INTERCEPT)

Random Vector: $\left(Y, Z, X_{11}, ..., X_{1k_1}, X_{21}, ..., X_{2k_2}\right)$,

$$Y = \beta_0\,(Z) + \beta_1(Z) \cdot X_{11} + ... + \beta_{k_1}(Z) \cdot X_{1k_1}$$

$$+\, \delta_1 \cdot X_{21} + ... + \delta_{k_2} \cdot X_{2k_2} + \varepsilon,$$

$\beta_j : \mathbb{R} \to \mathbb{R}$ unknown functions, $j = 0, 1, ..., k_1$

$\beta_0(Z) \to$ Varying intercept

$\beta_j(Z) \to$ Varying marginal effects, $j = 1, ..., k_1$

$\delta_j \to$ Constant marginal effects, $j = 1, ..., k_2$

RANDOM COEFFICIENT MODEL (CONST. INTERCEPT)

Random Vector: $\left(Y, Z, X_{11}, ..., X_{1k_1}, X_{21}, ..., X_{2k_2}\right)$,

$$Y = \bar{\beta}_0 + \beta_1(Z) \cdot X_{11} + ... + \beta_{k_1}(Z) \cdot X_{1k_1}$$

$$+ \delta_1 \cdot X_{21} + ... + \delta_{k_2} \cdot X_{2k_2} + \varepsilon,$$

$\beta_j : \mathbb{R} \to \mathbb{R}$ unknown functions, $j = 0, 1, ..., k_1$

$\bar{\beta}_0 \to$ Constant intercept

$\beta_j(Z) \to$ Varying marginal effects, $j = 1, ..., k_1$

$\delta_j \to$ Constant marginal effects, $j = 1, ..., k_2$

REFERENCES ON VARYING COEFFICIENT MODELS

Partially Linear Model: Constant slopes (marginal effects)

$$k_2 = 0, \; Var\left(\beta_0(Z)\right) > 0 \;\; \& \;\; Var(\beta_j(Z)) = 0, \; j = 1, ..., k_1.$$

Shiller (1984, JASA), Wahba (1985, Ann. Stat.), Engle, Granger, Rice & Weiss (1986, JASA), N.E. Heckman (1986, JRSSB), Shick (1986), Speckman (1988, JRSSB), Chen (1988, Ann. Stat.), Robinson (1988, Eca.).

Varying Coefficient Model: (all coefficients varying)

$$k_2 \;=\; 0,\ Var(\beta_j(Z)) > 0,\ j = 1, ..., k_1.$$

Cleveland, Grosse and Shyu (1991, *Book*), Hastie & Tibshinari (1993, *JRSSB*), Chen & Tsay (1993, *JASA*), McCabe and Tremayne (1995, *Ann. Stat.*), Wu, Chiang & Hoover (1998, *JASA*), Fan & Zhang (1999, *Ann. Stat.)*, Chiang, Rice & Wu (2001, *JASA*), Hoover, Rice, Wu & Yang (1998, *JASA*), Fan & Zhang (2000, *JRSSB*), Cai, Fan & Yao (2000, *JASA*), Kim (2007, *Ann. Stat.)*, Hoderlain & Sherman (2015, *J. Econ.*), Feng, Gao, Peng & Zhang (2017, *J. Econ.*)

Semivarying Coefficient Model: (some coefficients varying)

$$Var(\beta_j(Z)) \quad > \quad 0, \; j = 0, 1, ..., k_1.$$

Zhang, Lee & Song (2002, *JMA*), Xia, Zhang & Tong (2004, *Biometrika*), Li, Xue & Lian (2011, *JMA*), Li, Chen & Lin (2011, *JSPI*), Hu & Xia (2012, *Stat. Sinica*), Hu (2014, *JSCC*), Shi-qin, Juan & Gang (2012, *Phys. Proc*), Li, Li, Liang & Hsiao (2017, *Econ. Rev.*).

TESTING CONSTANCY OF COEFFICIENTS

$$H_0 : Var\left(\beta_j(Z)\right) = 0 \text{ for all } j = 0, 1, ..., k_1$$

vs

$$H_1 : Var\left(\beta_j(Z)\right) > 0 \text{ for some } j = 0, 1, ..., k_1$$

Existing Proposals:

Look at the discrepancy between the restricted & unrestricted fits.

$\|$

Smooth estimates of $\beta_j(\cdot)$ needed for the unrestricted fit

$\Downarrow$

Kauermann & Tutz (1999, *Biometrika)*, Cai, Fan & Yao (2000, *JASA*), Fan & Zhang (2000, *JRSSB)*, Fan, Zhang & Zhang (2001, *Ann. Stat.*), or Qu & Li (2006, *Biometrics*)..

Our Proposal:

Mimicking classical stability tests in time-varying coefficient models

$$\Big\|$$

No smooth estimates of $\beta_j(\cdot)$ needed

$$\left(\beta_j(\cdot) \text{ possibly discontinuous, e.g. } \beta_j(Z) = \bar{\beta}_j \cdot 1_{\{Z \leq z_0\}}\right)$$

Based on CUSUM of residuals

$$\Downarrow$$

e.g. Hinkley (1970), Brown, Durbin & Evans (1975), Hawkins (1977, 1987), Nyblon (1989) or Andrews (1993).

Interpret (Y, X) sample sequentially observed according to Z

APPLICATION: INTERACTIVE EFFECTS MODEL CHECKING

$$Y = \bar{\beta}_0 + \beta_1(Z) \cdot X_{11} + ... + \beta_k(Z) \cdot X_{1k_1} + g_0\left(Z, \delta_0\right) + X_{11} \cdot g_1(Z, \delta_1) + ... + X_{1k_1} \cdot g_{k_1}(Z, \delta_{k_1}) + \varepsilon,$$

$p_j \times 1$ vector of parameters: $\delta_j \subseteq \mathbb{R}^{p_j}$, $j = 1, ..., k_1$

$g_j \longrightarrow$ Linear in parameters known function.

Example: $g_j(Z, \delta_j) = \delta_{j1} Z + \delta_{j2} Z^2 + ... + \delta_{jm_j} Z^{m_j}$, $j = 0, 1, .., k_1$

$$H_0 : Var\left(\beta_j(Z)\right) = 0 \text{ all } j = 1, ..., k_1$$

vs

$$H_1 Var\left(\beta_j(Z)\right) > 0, \text{ some } j = 1, ..., k_1$$

MOTIVATING EXAMPLE: ETHANOL DATA

Cleveland, Grosse & Shyu (1991), and many others, example:

88 observations on the exhaut from an engine fuelled by ethanol.

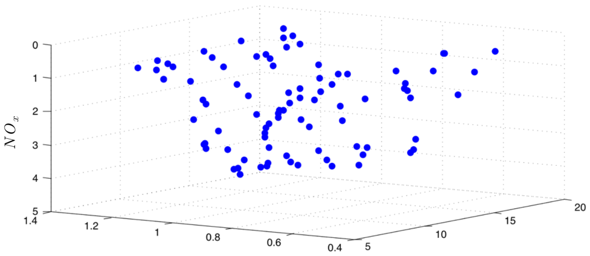
NO_x : Normalized concentration of nitric oxide & nitrogen dioxide

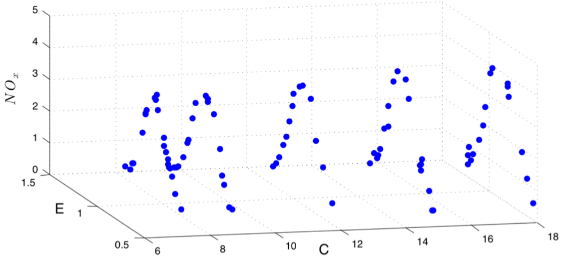
E : Equivalence ratio, measure of fuel-air mixture.

C : Compensation ratio of the engine.

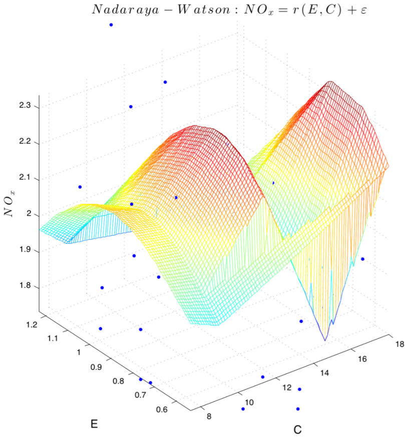
$$NO_X = \beta_0(E) + \beta_1(E) \cdot C + \varepsilon$$

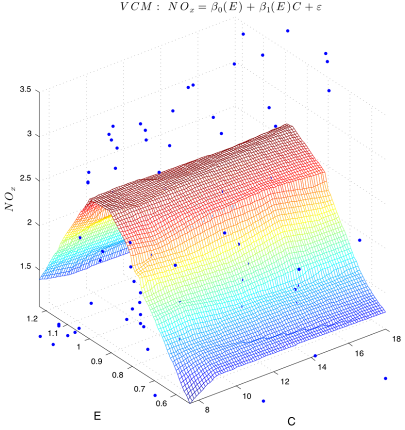
ETHANOL DATA





NON-PARAMETRIC & SEMIPARAMETIC FITS



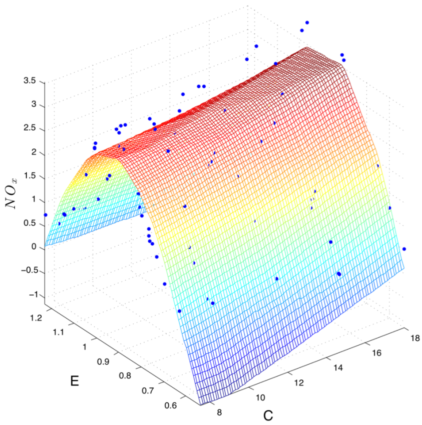


Plug-in bandwidths

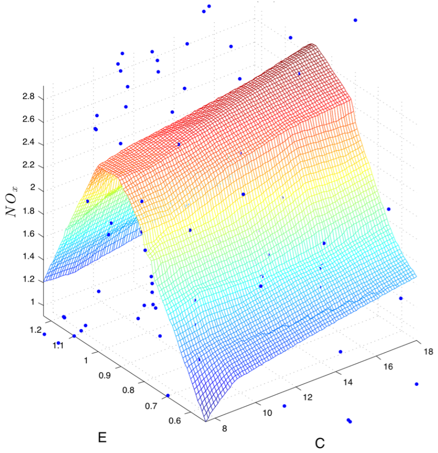
PARAMETRIC & SEMI-PARAMETRIC FITS

p-value=0.06

$$NO_x = \beta_0 + \beta_1 E + \beta_2 C + \beta_3 E^2 + \beta_4 C^2 + \beta_5 C \cdot E + \varepsilon$$

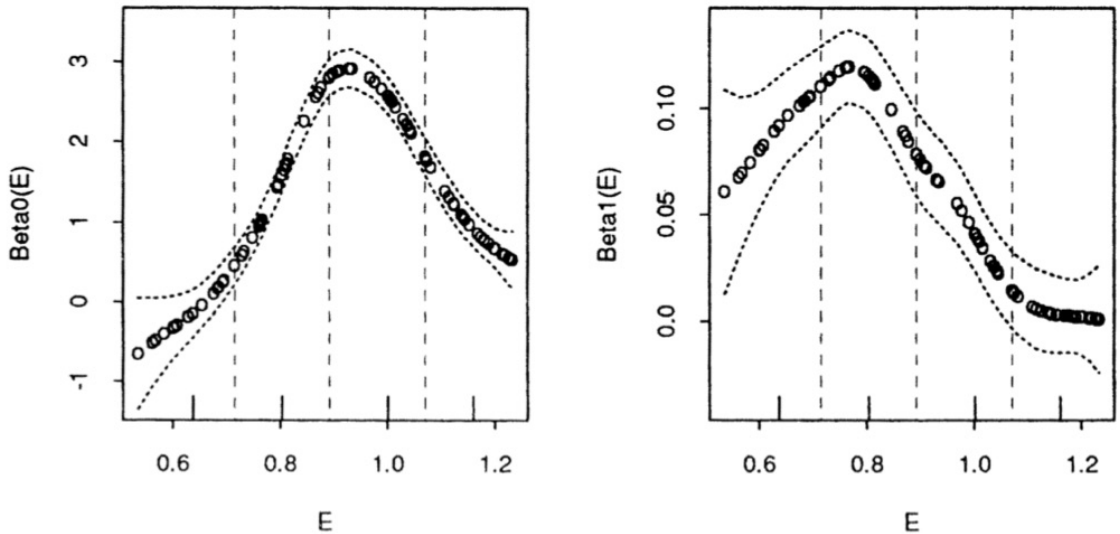


$$NO_x = \beta_0(E) + \beta_1 C + \varepsilon$$



Plug-in bandwidths

Varying coefficient estimates



Hastie & Tibshinari (1993, JRSSB)

MOTIVATING EXAMPLE: RETURNS OF EDUCATION

Blackbuern & Newmark (1992, QJE): Use *IQ* as proxy variable of ability in returns of education.

Source: Young Men's Cohort National Longitudinal Survey (663 obs.)

WAGE : USD monthly earnings $\longleftarrow$ EDUC : Years of education

$$\left.\begin{array}{l}
\mathit{IQ} : \text{Intelligence quotient (proxy of ability)} \\
\\
\mathit{EXPER} : \text{Years of work experience} \\
\mathit{TENURE} : \text{Years with current employer} \\
\mathit{BLACK} : \text{Dummy if black} \\
\mathit{SOUTH} : \text{Dummy 1 if live in south} \\
\mathit{URBAN} : \text{Dummy 1 if live in urban area (SMSA)} \\
\mathit{MARRIED} : \text{Dummy 1 if married.}
\end{array}\right\} \begin{array}{c}\text{Control} \\ \text{variables}\end{array}$$

VARYING COEFFICIENTS DEPENDING ON IQ

1^{st} IQ significance test

$$Log\,(WAGE) = \beta_0(IQ) + \beta_1(IQ) \cdot EDUC + \delta_1 \cdot EXPER + \delta_2 \cdot TENURE$$

$$+\delta_3 \cdot BLACK + \delta_4 \cdot SOUTH + \delta_5 \cdot URBAN + \delta_6 \cdot MARRIED + \varepsilon$$

$$H_0 : Var(\beta_j(IQ)) = 0,\ j = 0, 1 \quad \text{vs} \quad H_1: Var(\beta_0(IQ)) > 0 \text{ or } Var(\beta_1(IQ)) > 0$$

OLS fits under H_0

$$Log\,\widehat{(WAGE)} = \underset{(0.054)}{5.395} + \underset{(0.006)}{0.065}\,EDUC + ...$$

$$Log\,\widehat{(WAGE)} = \underset{(0.128)}{5.176} + \underset{(0.006)}{0.054}\,EDUC + \underset{(0.001)}{0.0036}\,IQ + ...$$

$$Log\,\widehat{(WAGE)} = \underset{(0.5)}{5.6} + \underset{(0.041)}{0.018}\,EDUC - \underset{(0.0052)}{0.009}\,IQ + \underset{(0.00038)}{0.00034}\,IQ \cdot EDUC + ...$$

2^{nd} Specification test

$$H_0 : Log\,(WAGE) = \bar{\beta}_1 \cdot EDUC + \bar{\beta}_0 + \delta_1 \cdot IQ + \delta_2 \cdot (IQ \cdot EDUC)$$

$$+\delta_3 \cdot EXPER + \delta_4 \cdot TENURE + \ \ldots\ + \varepsilon$$

$$H_1 : Log\,(WAGE) = [\beta_1(IQ) + \delta_2 \cdot IQ] \cdot EDUC + \bar{\beta}_0 + \delta_1 IQ +$$

$$+\delta_2 EXPER + \delta_3 TENURE + \ \ldots\ + \varepsilon$$

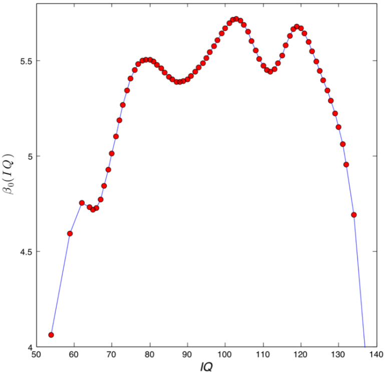
with $\quad Var\,(\beta_1(IQ)) > 0$

VARYING COEFFICIENTS ESTIMATES

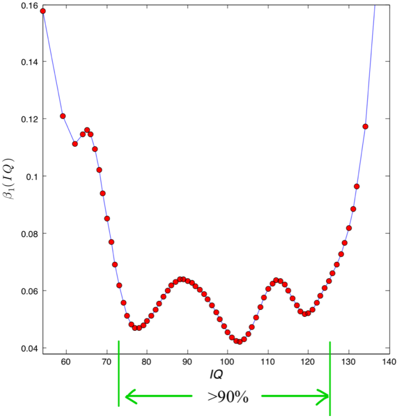
$$Log\,(WAGE) = \beta_0(IQ) + \beta_1(IQ)EDUC + \ldots$$

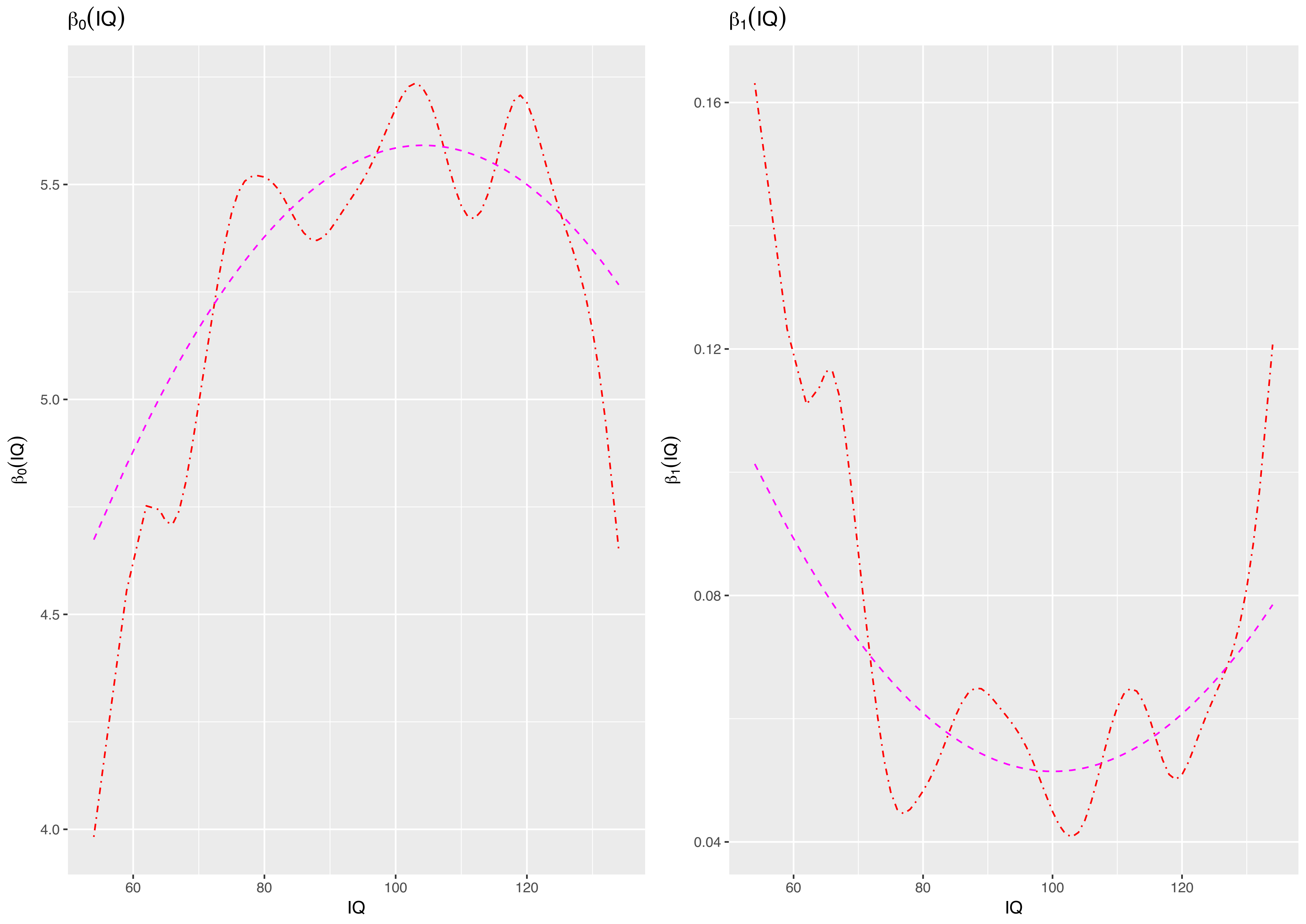
Plug-in bandwidth

$\beta_0(IQ)$



$\beta_1(IQ)$





$\beta_0(IQ)$
5.5
5.0
4.5
4.0
60
80
100
120
IQ
$\beta_1(IQ)$
0.16
0.12
0.08
0.04
60
80
100
120
IQ

NOTATION & BASIC ASSUMPTIONS

$$Y = \mathbf{X}_1' \boldsymbol{\beta}(Z) + \mathbf{X}_2' \, \delta + \varepsilon,$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ X_{11} \\ \vdots \\ X_{1k_1} \end{pmatrix}, \ \mathbf{X}_2 = \begin{pmatrix} X_{21} \\ X_{22} \\ \vdots \\ X_{2k_1} \end{pmatrix}, \ \boldsymbol{\beta}(\cdot) = \begin{pmatrix} \beta_0(\cdot) \\ \beta_1(\cdot) \\ \vdots \\ \beta_{k_1}(\cdot) \end{pmatrix}, \ \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{k_2} \end{pmatrix}$$

$$\mathbb{E}\left(\varepsilon \middle| Z, \mathbf{X}\right) = 0 \ a.s.$$

$$F_Z(z) = \mathbb{P}\left(Z \leq z\right) \ \underline{\text{continuous}} \rightarrow U = F_Z(Z) \sim U(0,1)$$

$$\boldsymbol{\Sigma}\left(u\right) = \mathbb{E}\left(\mathbf{X}\left(u\right)\mathbf{X}\left(u\right)'\right) \ \underline{\text{non-singular}}, \quad \underset{(2(k_1+1)+k_2)\times 1}{\mathbf{X}\left(u\right)} = \begin{pmatrix} \mathbf{X}_1 \cdot 1_{\{U \leq u\}} \\ \mathbf{X}_1 \cdot 1_{\{U > u\}} \\ \mathbf{X}_2 \end{pmatrix}$$

2. CHARACTERIZATION OF THE NULL HYPOTHESIS.

CHARACTERIZATION OF $H_0 : \beta(Z) = \bar{\beta}$ *a.s.*

$$\begin{pmatrix} \boldsymbol{\theta}^-(u) \\ \boldsymbol{\theta}^+(u) \\ \boldsymbol{\theta}^o(u) \end{pmatrix} = \underset{\mathbf{b}^+,\mathbf{b}^-,\mathbf{b}^o}{\arg\min} \Big\{ \mathbb{E}\left[\left(Y - \mathbf{X}_1'\mathbf{b}^- - \mathbf{X}_2'\mathbf{b}^o\right)^2 1_{\{U\le u\}}\right] + \mathbb{E}\left[\left(Y - \mathbf{X}'\mathbf{b}^+ - \mathbf{X}_2'\mathbf{b}^o\right)^2 1_{\{U>u\}}\right] \Big\}$$

$$= \underset{\mathbf{b}^+,\mathbf{b}^-,\mathbf{b}^o}{\arg\min} \mathbb{E}\left(Y - \mathbf{X}_1' 1_{\{U\le u\}}\mathbf{b}^- - \mathbf{X}' 1_{\{U>u\}}\mathbf{b}^+ - \mathbf{X}_2'\mathbf{b}^o\right)^2$$

$$= \mathbb{E}\left(\mathbf{X}(u)\mathbf{X}(u)'\right)^{-1}\mathbb{E}\left(\mathbf{X}(u)\,Y\right)$$

$$= \begin{pmatrix} \bar{\beta} \\ \bar{\beta} \\ \delta \end{pmatrix} \quad \text{under } H_0$$

TARGET FUNCTION

$$\boldsymbol{\eta}(u) = \boldsymbol{\theta}^{-}(u) - \boldsymbol{\theta}^{+}(u)$$

$$H_0 : Var\left(\beta_j(Z)\right) = 0 \text{ all } j = 0, \ldots, k_1$$

$$\not\Uparrow \quad \Downarrow$$

$$\boldsymbol{\eta}(u) = \mathbf{0} \text{ for all } u \in (0, 1)$$

We consider

$$\boxed{\begin{array}{c} H_0 : \boldsymbol{\eta}(u) = \mathbf{0} \text{ all } u \in (0,1) \\ \text{vs} \\ H_0 : \boldsymbol{\eta}(u) \neq \mathbf{0} \text{ some } u \in (0,1) \end{array}}$$

1.) Consider the case

$$\mathbb{E}\left(\mathbf{X}_i\mathbf{X}_j' \middle| Z\right) = \mathbb{E}\left(\mathbf{X}_i\mathbf{X}_j'\right), \quad i,j = 1,2 \quad a.s.$$

$$\Downarrow$$

$$\eta(u) = \frac{\mathbb{E}\left(\beta(Z)\cdot 1_{\{F_Z(Z)\leq u\}}\right) - u\mathbb{E}\left(\beta(Z)\right)}{u(1-u)} = \mathbf{0} \text{ all } u \in (0,1)$$

$$\Updownarrow$$

$$H_0 : Var\left(\beta_j(Z)\right) = 0 \text{ all } j = 0, ..., k_1$$

2.) Consider the case

$$\boldsymbol{\delta} = \mathbf{0}, \quad \text{i.e.} \quad Y = \mathbf{X}_1'\beta(Z) + \varepsilon,$$

$$\boldsymbol{\eta}(u) = \mathbf{0} \text{ for all } u \in (0,1)$$

$$\Updownarrow$$

$$\mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'\beta(Z)1_{\{U\leq u\}}\right)\mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'1_{\{U\leq u\}}\right)^{-1} = \mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'\beta(Z)\right)\mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'\right)^{-1}$$

$$\Updownarrow$$

$$H_0 : Var\left(\beta_j(Z)\right) = 0 \text{ all } j = 0, ..., k_1$$

3. TESTING PROCEDURE

INDUCED ORDER STATISTICS (IOS) OR CONCOMITANTS

Given a sample $\{Y_i, Z_i, \mathbf{X}_i\}_{i=1}^n$,

Order statistics of $\{Z_i\}_{i=1}^n$: $Z_{1:n} < Z_{2:n} < ... < Z_{n:n}$

Generic sample $\longrightarrow \{\xi_i\}_{i=1}^n$

$\xi -$ Induced order statistics (or $\xi -$ concomitants) of $\{Z_i\}_{i=1}^n$

$$\Downarrow$$

$\xi_{[1:n]}, ..., \xi_{[n:n]}$ such that $\xi_{[i:n]} = \xi_j \iff Z_{i:n} = Z_j$.

$\theta^{\pm}(\cdot)$ & $\eta(\cdot)$ ESTIMATES

$$\begin{pmatrix} \hat{\theta}^{-} \\ \hat{\theta}^{+} \\ \hat{\theta}^{o} \end{pmatrix}(u) = \arg\min_{\theta^{+},\theta^{-},\theta^{o}} \left\{ \sum_{i=1}^{\lfloor nu \rfloor} \Big(Y_{[i:n]} - \mathbf{X}'_{1[i:n]}\theta^{-} - \mathbf{X}'_{2[i:n]}\theta^{o} \Big)^2 \right.$$

$$+ \sum_{i=1+\lfloor nu \rfloor}^{n} \Big(Y_{[i:n]} - \mathbf{X}'_{1[i:n]}\theta^{+} - \mathbf{X}'_{2[i:n]}\theta^{o} \Big)^2$$

$$= \sum_{i=1}^{n} \Big(Y_i - \mathbf{X}'_{1i} 1_{\{Z_i \le Z_{\lfloor nu \rfloor : n}\}} \theta^{+} - \mathbf{X}'_{1i} 1_{\{Z_i > Z_{\lfloor nu \rfloor : n}\}} \theta^{-} - \mathbf{X}'_{2i}\theta^{o} \Big)^2$$

$$\boxed{\hat{\eta} = \hat{\theta}^{-} - \hat{\theta}^{+} \quad \text{estimates} \quad \eta = \theta^{-} - \theta^{+}}$$

$$\hat{\boldsymbol{\theta}}(u) = \begin{pmatrix} \hat{\boldsymbol{\theta}}^{-} \\ \hat{\boldsymbol{\theta}}^{+} \\ \hat{\boldsymbol{\theta}}^{o} \end{pmatrix}(u) = \hat{\boldsymbol{\Sigma}}^{-1}(u)\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i(u)Y_i,$$

$$\hat{\boldsymbol{\Sigma}}^{-1}(u) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i(u)\mathbf{X}_i'(u) \;\;\&\;\; \mathbf{X}_i(u) = \begin{pmatrix} \mathbf{X}_{1i} \cdot 1_{\{Z_i \le Z_{\lfloor nu \rfloor : n}\}} \\ \mathbf{X}_{1i} \cdot 1_{\{Z_i > Z_{\lfloor nu \rfloor : n}\}} \\ \mathbf{X}_{2i} \end{pmatrix}$$

Testing procedure:

Check whether $\hat{\boldsymbol{\eta}}(u) = \left(\hat{\boldsymbol{\theta}}^{-} - \hat{\boldsymbol{\theta}}^{+}\right)(u)$ is close to zero uniformly in u

CONSISTENCY

Assume:

A.1 $\mathbb{E}\left\|\mathbf{X}_j Y\right\| < \infty$, $j = 1, 2$, and $\mathbf{\Sigma}(u)$ exists and p.d. for all $u \in [\pi, 1-\pi]$, $\pi \in (0,1)$

A.2. F_Z is continuous.

Applying a Glivenko-Cantelli argument,

$$\lim_{n\to\infty} \sup_{u\in[0,1]} \left\|\left(\hat{\mathbf{\Sigma}} - \mathbf{\Sigma}\right)(u)\right\| = 0 \text{ a.s.}$$

$$\lim_{n\to\infty} \sup_{u\in[0,1]} \left\|\frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i(u) Y_i - \mathbb{E}\left(\mathbf{X}(u) Y\right)\right\| = 0 \text{ a.s.}$$

$$\Downarrow$$

$$\lim_{n\to\infty} \sup_{u\in[\pi,1-\pi]} \left\|\left(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}\right)(u)\right\| = 0 \text{ a.s.}, \quad \pi \in (0,1)$$

ASYMPTOTIC NORMALITY

Under $H_0 : \beta(Z) = \bar{\beta}$ a.s.,

$$\sqrt{n}\begin{pmatrix} \hat{\boldsymbol{\theta}}^{-}(u) - \bar{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\theta}}^{+}(u) - \bar{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\theta}}^{o}(u) - \delta \end{pmatrix} = \hat{\boldsymbol{\Sigma}}^{-1}(u)\hat{\mathbf{B}}(u), \quad u \in [\pi, 1-\pi]$$

$$\hat{\mathbf{B}}(u) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i(u)\varepsilon_i = \begin{pmatrix} \hat{\mathbf{B}}_1(u) \\ \hat{\mathbf{B}}_1(1) - \hat{\mathbf{B}}_1(u) \\ \hat{\mathbf{B}}_2(1) \end{pmatrix}$$

$$\hat{\mathbf{B}}_j(u) = \frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nu \rfloor}\mathbf{X}_{j[i:n]}\varepsilon_{[i:n]}, \; j = 1, 2,$$

CLT for $\hat{\mathbf{B}}$ based on an invariance principle for partial sums of IOS: Battacharya (1974, 76), Stute (1993,1997), Davidov & Egorov(2000, 01)

Davidov & Egorov (2001): uniformly in $u \in [0,1]$, $j = 1,2$,

$$\begin{aligned} \hat{\mathbf{B}}_j(u) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{X}_{ji} \varepsilon_i 1_{\{Z_i \leq Z_{\lfloor nu \rfloor : n}\}} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{X}_{ji} \varepsilon_i 1_{\{F_Z(Z_i) \leq u\}} + o_p(1) \end{aligned}$$

Assuming $A.1$, $A.2$ and $\mathbb{E} \|\mathbf{X}_j \varepsilon\|^2 < \infty$,

$$\left\{ \begin{pmatrix} \hat{\mathbf{B}}_1(u) \\ \hat{\mathbf{B}}_2(u) \end{pmatrix} \right\}_{u \in [0,1]} \to_d \left\{ \begin{pmatrix} \mathbf{B}_1(u) \\ \mathbf{B}_2(u) \end{pmatrix} \right\}_{u \in [0,1]} \text{ in } D[0,1],$$

where $\mathbf{B}_j$ are mean zero Gaussian processes and for $u, v \in [0,1]$, $j, \ell = 1,2$

$$\mathbb{E}\left(\mathbf{B}_j(u) \mathbf{B}_\ell(v)\right) = \mathbb{E}\left(\mathbf{X}_j \mathbf{X}_\ell' \varepsilon^2 1_{\{F_Z(Z) \leq \min(u,v)\}}\right) = \mathbb{E}\left(\mathbf{X}_j(u) \mathbf{X}_\ell'(v) \varepsilon^2\right).$$

A.3. $\mathbb{E}\left\|\mathbf{X}_j\varepsilon\right\|^2 < \infty$, $j = 1, 2$.

Under A.1, A.2 & A.3

$$\left\{\hat{\mathbf{B}}\left(u\right)\right\}_{u\in[0,1]} \to_d \left\{\mathbf{B}\left(u\right)\right\}_{u\in[0,1]} \quad \text{in } D\left[0,1\right]$$

$$\hat{\mathbf{B}}\left(u\right) = \begin{pmatrix} \hat{\mathbf{B}}_1\left(u\right) \\ \hat{\mathbf{B}}_1\left(1\right) - \hat{\mathbf{B}}_1\left(u\right) \\ \hat{\mathbf{B}}_2\left(1\right) \end{pmatrix} \quad \& \quad \mathbf{B}\left(u\right) = \begin{pmatrix} \mathbf{B}_1\left(u\right) \\ \mathbf{B}_1\left(1\right) - \mathbf{B}_1\left(u\right) \\ \mathbf{B}_2\left(1\right) \end{pmatrix}$$

$$\mathbb{E}\left(\mathbf{B}\left(u\right)\mathbf{B}'\left(v\right)\right) = \mathbb{E}\left(\mathbf{X}(u)\mathbf{X}'(v)\varepsilon^2\right) = \mathbf{\Omega}\left(u,v\right),$$

Therefore under H_0, A.1, A2 & A.3, for $\pi \in (0,1)$,

$$\left\{ \sqrt{n} \begin{pmatrix} \hat{\boldsymbol{\theta}}^{-}(u) - \bar{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\theta}}^{+}(u) - \bar{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\theta}}^{o}(u) - \boldsymbol{\delta} \end{pmatrix} \right\}_{u \in [\pi, 1-\pi]} \to_d \left\{ \boldsymbol{\Sigma}^{-1}(u) \, \mathbf{B}(u) \right\}_{u \in [\pi, 1-\pi]}$$

$$\Downarrow$$

$$\left\{ \sqrt{n} \hat{\boldsymbol{\eta}}(u) \right\}_{u \in [\pi, 1-\pi]} \to_d \left\{ \hat{\boldsymbol{\eta}}_{\infty}(u) \right\}_{u \in [\pi, 1-\pi]} \text{ in } D[\pi, 1-\pi]$$

$$\left\{ \hat{\boldsymbol{\eta}}_{\infty}(u) \right\}_{u \in [\pi, 1-\pi]} \overset{d}{=} \left\{ \left[\mathbf{I}_{k_1} \ \vdots - \mathbf{I}_{k_1} \ \vdots \mathbf{0}_{k_2} \right] \boldsymbol{\Sigma}^{-1}(u) \, \mathbf{B}(u) \right\}_{[\pi, 1-\pi]}$$

Particular Case: $\pi \in [0,1]$,

$$\mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'\middle| Z\right) = \mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'\right) \quad \text{a.s.} \quad \text{and} \quad \mathbb{E}\left(\varepsilon^2\middle| Z\right) = \mathbb{E}\left(\varepsilon^2\right) = \sigma^2 \; a.s.$$

$$\boldsymbol{\Sigma}(u) = \begin{bmatrix} u\mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'\right) & 0 \\ 0 & (1-u)\mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'\right) \end{bmatrix},$$

$$\mathbf{B}_1(u) = \sigma \cdot \mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'\right)^{1/2} \cdot \mathbf{W}_0(u)$$

$$\left\{\hat{\boldsymbol{\eta}}_\infty(u)\right\}_{u\in[\pi,1-\pi]} \stackrel{d}{=} \left\{\sigma \cdot \mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1'\right)^{-1/2} \frac{\mathbf{W}_0(u) - u\mathbf{W}_0(1)}{u(1-u)}\right\}_{u\in[\pi,1-\pi]}$$

$\mathbf{W}_0 \rightarrow (k_1+1)\times 1$ vector of independent standard Wiener's processes

4. TEST STATISTIC & CRITICAL VALUES

STANDARDIZATION

Estimate

$$\begin{aligned}\boldsymbol{\Xi}(u) &= AVar\left(\sqrt{n}\hat{\boldsymbol{\eta}}\,(u)\right)\\ &= \left[\mathbf{I}_{k_{1+1}} \vdots - \mathbf{I}_{k_{1+1}} \vdots \mathbf{0}_{k_2}\right] \boldsymbol{\Sigma}^{-1}(u)\,\boldsymbol{\Omega}(u,u)\,\boldsymbol{\Sigma}^{-1}(u) \begin{bmatrix}\mathbf{I}_{k_{1+1}}\\ -\mathbf{I}_{k_{1+1}}\\ \mathbf{0}_{k_2}\end{bmatrix}\end{aligned}$$

by

$$\hat{\boldsymbol{\Xi}}(u) = \left[\mathbf{I}_{k_{1+1}} \vdots - \mathbf{I}_{k_{1+1}} \vdots \mathbf{0}_{k_2}\right] \hat{\boldsymbol{\Sigma}}^{-1}(u)\,\hat{\boldsymbol{\Omega}}(u,u)\,\hat{\boldsymbol{\Sigma}}^{-1}(u) \begin{bmatrix}\mathbf{I}_{k_{1+1}}\\ -\mathbf{I}_{k_{1+1}}\\ \mathbf{0}_{k_2}\end{bmatrix}$$

$$\hat{\boldsymbol{\Omega}}(u,u) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i(u)\mathbf{X}_i'(u)\hat{\varepsilon}_i^2, \quad \hat{\varepsilon}_i \rightarrow \text{OLS residuals under } H_0.$$

TEST STATISTICS

Cramér-v. Mises type,

$$\hat{\varphi}_{\pi}^{(1)} = \sum_{i=K+\lfloor n\pi \rfloor}^{n-K-\lfloor n\pi \rfloor} \hat{\boldsymbol{\eta}}\left(\frac{i}{n}\right)' \hat{\boldsymbol{\Xi}}^{-1}\left(\frac{i}{n}\right) \hat{\boldsymbol{\eta}}\left(\frac{i}{n}\right),$$

Kolmogorov-Smirnov type (as suggested in Csörgo & Horvath 1997)

$$\hat{\varphi}_{\pi}^{(2)} = n \max_{K+\lfloor n\pi \rfloor \leq i \leq n-K-\lfloor n\pi \rfloor} \left(\frac{i\,(n-i)}{n}\right) \hat{\boldsymbol{\eta}}\left(\frac{i}{n}\right)' \hat{\boldsymbol{\Xi}}^{-1}\left(\frac{i}{n}\right) \hat{\boldsymbol{\eta}}\left(\frac{i}{n}\right),$$

$K = 1 + k_1 + k_2 \rightarrow \hat{\boldsymbol{\Xi}}$ degrees of freedom

TESTS UNDER MEAN INDEPENDENCE

Assume $k_2 = 0$ (no $\mathbf{X}_2$) and

$\mathbb{E}\left(\mathbf{X}_1\mathbf{X}_1 \middle| Z\right) = \mathbb{E}\left(\mathbf{X}_1\mathbf{X}_j\right), \; j = 1, 2, \;$ a.s. and $\mathbb{E}\left(\varepsilon^2 \middle| Z\right) = \mathbb{E}\left(\varepsilon^2\right)$ *a.s.*

Applying results in Csörgo and Horvath (1997), under H_1,

$$\lim_{n\to\infty} \hat{\varphi}_0^{(j)} = \infty \text{ a.s. } j = 1, 2$$

and under H_0,

$$\hat{\varphi}_0^{(1)} \to_d \int_0^1 \frac{\left[\mathbf{W}_0(u) - u\mathbf{W}_0(u)\right]' \left[\mathbf{W}_0(u) - u\mathbf{W}_0(u)\right]}{u(1-u)} du$$

Tabulated in Scholz and Stephens (1997)

$$\hat{\varphi}_0^{(2)} \to_d \sup_{0\leq u\leq 1} \left[\mathbf{W}_0(u) - u\mathbf{W}_0(u)\right]' \left[\mathbf{W}_0(u) - u\mathbf{W}_0(u)\right]$$

Tabulated in Kiefer (1959)

$\mathbf{W}_0 \to (k_1 + 1) \times 1$ vector of independent Wiener's processes

TEST IN THE GENERAL CASE

Under H_1, $\pi \in (0,1)$

$$\lim_{n\to\infty} \hat{\varphi}_{\pi}^{(j)} = \infty \text{ a.s. } j = 1,2$$

and under H_0, $\pi \in (0,1)$,

$$\hat{\varphi}_{\pi}^{(1)} \to_d \int_{\pi}^{1-\pi} \hat{\boldsymbol{\eta}}_{\infty}'(u)\, \Xi^{-1}(u) \hat{\boldsymbol{\eta}}_{\infty}(u)\, du,$$

$$\hat{\varphi}_{\pi}^{(2)} \to_d \sup_{\pi \le u \le 1-\pi} u(1-u) \hat{\boldsymbol{\eta}}_{\infty}'(u)\, \Xi^{-1}(u) \hat{\boldsymbol{\eta}}_{\infty}(u)$$

Critical values estimated using wild bootstrap.

BOOTSTRAP IMPLEMENTATION IN THE GENERAL CASE

1. Generate $\{\xi_i\}_{i=1}^n$ *iid* with $\mathbb{E}(\xi_1) = 0$ & $\mathbb{E}\left(\xi_1^2\right) = 1$.

2. Generate

$$Y_i^* = \mathbf{X}_{1i}'\hat{\boldsymbol{\beta}}^{LS} + \mathbf{X}_{2i}'\hat{\delta}^{LS} + (\hat{\varepsilon}_i \cdot \xi_i),$$

$$\hat{\varepsilon}_i = Y_i - \mathbf{X}_{1i}'\hat{\boldsymbol{\beta}}^{LS} - \mathbf{X}_{2i}'\hat{\delta}^{LS}, i = 1, .., n.$$

3. Compute bootstrap critical values and p-values using resamples

$$\left\{ Y_i^{*(b)}, X_i \right\}_{i=1}^n, b = 1, ..., B, \quad B \text{ large.}$$

The test is justified as in Stute, González-Manteiga & Quindimil (1998, *JASA)*.

$$Y = \bar{\beta}_0 + \mathbf{X}_1'\beta(Z) + \mathbf{X}_2'\delta + \varepsilon$$

$$\mathbf{X}_1 = \begin{pmatrix} X_{11} \\ \vdots \\ X_{1k_1} \end{pmatrix}, \quad \mathbf{X}_2 = vec\left(\mathbf{X}_1\mathbf{V}'\right), \ k_2 = p \cdot k_1,$$

$$\mathbf{V} = \begin{pmatrix} \varphi_1(Z) \\ \varphi_2(Z) \\ \vdots \\ \varphi_p(Z) \end{pmatrix} \ \& \ \ \varphi_j \rightarrow \text{known functions.}$$

FINITE SAMPLE PROPERTIES

Compare with CUSUM type test Stute (1997) based on

$$\hat{\varphi}(z, x_1, \ldots, x_k) = \frac{1}{n} \sum_{i=1}^{n} \hat{\varepsilon}_i \; 1_{\{Z_i \leq z\}} \prod_{j=1}^{k} 1_{\{X_{ij} \leq x_j\}}.$$

$\uparrow$

Resulting test is omnibus (poor performance as $k \uparrow$)

Also we compare with Cao, Fan & Yao (2000, *JASA*) test (LR)

$\uparrow$

$$\text{Compare} \begin{cases} \text{restricted } SSR \text{ (under } H_0\text{) (using smooth estimates)} \\ \& \\ \text{unrestricted } SSR \text{ (using OLS with the whole sample)} \end{cases}$$

5. REAL DATA APPLICATIONS

ETHANOL DATA

$$ON_x = \beta_0(E) + \beta_1(E)C + \varepsilon$$

$$H_0 : Var(\beta_0(E)) = Var(\beta_1(E)) = 0$$

vs

$$H_1 : Var(\beta_0(E)) > 0 \quad \text{or} \quad Var(\beta_1(E)) > 0$$

&

$$H_1 : Var(\beta_0(E)) > 0 \quad \& \quad Var(\beta_1(E)) = 0$$

$ON_x = \beta_0(E) + \beta_1(E)C + \varepsilon$		
$P - VALUE$		
$Trimming, \;\; \pi = 0, 01, 0.05, 0.10$		
	CvM	$K - S$
$H_0 : Var\left(\beta_0(E)\right) = Var\left(\beta_1(E)\right) = 0$ vs $H_1 : Var\left(\beta_0(E)\right) > 0 \quad \text{or} \quad Var\left(\beta_1(E)\right) > 0$	0.00	0.00
$CUSUM$	0.00	0.00
$SMOOTH$	0.00	

ETHANOL DATA: INTERACTIVE EFFECTS CHECKING

<table>
<tr><td colspan="3">$ON_x = \bar{\beta}_0 + \beta_1(E)C + \delta_1 E + \delta_2 (E \cdot C) + \varepsilon$

$P - VALUE$</td></tr>
<tr><td></td><td>CvM</td><td>$K - S$</td></tr>
<tr><td>$H_0 : Var(\beta_1(E)) = 0$

vs

$H_1 : Var(\beta_1(E)) > 0$</td><td>0.00</td><td>0.00</td></tr>
<tr><td>$CUSUM$</td><td>0.00</td><td>0.00</td></tr>
</table>

RETURNS OF EDUCATION

$$Log\left(WAGE\right) = \beta_0(IQ) + \beta_1(IQ)EDUC + \delta_1 EXPER + \delta_2 TENURE$$
$$+\delta_3 BLACK + \delta_4 SOUTH + \delta_5 URBAN + \delta_6 MARRIED + \varepsilon$$

$P - VALUE, \quad \pi = 0.01$		
	CvM	$K - S$
$H_0 : Var\left(\beta_0(E)\right) = Var\left(\beta_1(E)\right) = 0$ vs $H_1 : Var\left(\beta_0(E)\right) > 0$ or $Var\left(\beta_1(E)\right) > 0$	0.02	0.12
$CUSUM$	0.46	0.69
$SMOOTH$	0.23	

RETURNS OF EDUCATION: INTERACTIVE EFFECTS CHECK

$$Log\,(WAGE) = \bar{\beta}_0 + [\beta_1(IQ) - \delta_2 \cdot IQ] \cdot EDUC + \delta_1 \cdot IQ +$$

$$+\delta_2 \cdot EXPER + \delta_3 \cdot TENURE + ... + \varepsilon$$

$P - VALUE$		
	CvM	$K - S$
$H_0 : Var\,(\beta_1(IQ)) = 0$ vs		
$H_1 : Var\,(\beta_1(IQ)) > 0$	0.58	0.75
$CUSUM$	0.66	0.66

6. MONTE CARLO

MONTE CARLO

$$Y_i = \beta_0(Z_i) + \sum_{j=1}^{k} \beta_j(Z_i) X_{ji} + \underbrace{\left(\frac{e^{-\tau Z_i}}{\sqrt{Var(e^{-\tau Z_i})}} \right)}_{\sigma(Z)} U_i, \ \ \tau = 0, 1$$

$$Z_i \sim iid \ \ U(0,1) \perp U_i \sim iid \ \ N(0,1)$$

$$X_{ji} = Z_i + V_{ij}, \ \ V_{ij} \sim iid \ \ U(0,1) \perp U_i, Z_i$$

$$\beta_\ell(z) = \lambda \frac{\varphi(z)}{Var(\varphi(Z))^{1/2}}, \ \ \varphi(z) = \begin{cases} a) \ z \\ b) \ \sin(2\pi z) \\ c) \ [1 + e^{-z}]^{-1} \\ d) \ 1 + 2 \cdot 1_{\{Z_i \leq 0.4\}} \end{cases}$$

$$\downarrow$$

$$Var(\beta_\ell(Z)) = \lambda$$

Wild bootstrap: $\begin{cases} \mathbb{P}_\xi\left(\xi_i = -\frac{\sqrt{5}-1}{2}\right) = \frac{\sqrt{5}+1}{2\sqrt{5}} \\ \\ \mathbb{P}_\xi\left(\xi_i = \frac{\sqrt{5}+1}{2}\right) = \frac{\sqrt{5}-1}{2\sqrt{5}} \end{cases}$

Simulations: 1,000

Bootstrap replications: 1,000

Trimming: $\pi = 0.01$ (little effect in $\pi \in [0.01, 0.1]$)

MAIN CONCLUSIONS

- Little effect of trimming π.
- Little effect of heteroskedasticity.
- Excellent size accuracy of bootstrap and asymptotic (when applicable) critical values.
- Underlying $\beta_j(\cdot)$ model no important.
- Rejections depend mainly of magnitude of $Var(\beta_j(Z)) = \lambda$.

$$Y = \beta_0(Z) + X_{21} + ... + X_{1k_2} + \varepsilon, \text{ (Heter. } \tau = 1, \ \pi = 0.01)$$

% Rejections under $H_0 : Var(\beta_0(Z)) = 0$,

	CvM			K-S		
	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
	Our test					
50	5.2	5.5	5.4	5.4	5.4	4.9
100	4.7	4.9	4.7	5.0	5.4	4.9
200	5.2	6.0	4.6	5.9	6.2	5.5
	CUSUM test					
50	4.9	4.7	3.9	4.6	4.2	4.4
100	4.2	5.0	4.2	4.3	6.4	4.9
200	5.1	6.0	5.3	5.2	4.5	6.7

$$Y = \beta_0(Z) + X_{11}\beta_1(Z) + ... + X_{1k_1}\beta_{k_1}(Z) + X_{21}\delta_1 + \varepsilon,$$

% Rejections under $H_0 : Var(\beta_j(Z)) = 0, \;\; j = 0, 1, ..., k,$

	CvM				K-S			
	$k_1 = 0$	$k_1 = 1$	$k_1 = 2$	$k_1 = 3$	$k_1 = 0$	$k_1 = 1$	$k_1 = 2$	$k_1 = 3$
	Our test							
50	5.4	4.7	4.7	5.2	5.2	4.1	4.1	4.3
100	4.7	5.2	6.8	4.8	5.0	4.8	4.7	4.4
200	5.4	5.9	5.4	4.2	5.9	5.6	5.1	5.6
	CUSUM test							
50	4.9	4.7	3.9	3.9	4.6	4.2	4.4	5.4
100	4.2	5.0	4.2	4.1	4.3	6.4	4.9	5.2
200	5.1	6.0	5.3	4.9	5.2	4.5	6.7	5.5

$$Y = \beta_0(Z) + X_{11}\beta_1(Z) + ... + X_{1k_1}\beta_{k_1}(Z) + \varepsilon$$

$$H_0, \ X_1 \perp Z \ \& \ \varepsilon \perp Z$$

α	0.1				0.05				0.01			
k_1	0	1	2	3	0	1	2	3	0	1	2	3
Cramér v. Mises												
50	10.1	9.8	8.0	8.6	4.7	4.2	3.7	2.9	1.0	0.5	0.4	0.3
100	9.4	9.1	11.2	8.9	4.9	4.9	5.5	3.8	1.1	1.0	0.5	0.3
200	9.9	9.2	9.8	9.0	5.7	5.0	5.5	4.3	1.4	1.2	0.7	0.6
Kolmogorov-Smirnov												
50	10.0	11.1	10.5	10.3	5.3	4.7	4.7	4.5	0.8	0.4	0.5	0.4
100	11.0	11.2	12.2	9.8	5.0	5.4	5.9	4.6	1.3	1.0	0.7	0.6
200	10.9	10.0	11.3	10.7	5.2	4.9	5.3	4.8	1.2	1.1	1.0	0.8
Cramér v. Mises (Asymptotic)												
50	7.9	6.6	4.7	3.5	3.5	3.3	2.1	1.6	0.3	0.5	0.9	1.1
100	9.0	8.9	7.4	6.1	4.9	4.0	3.0	3.5	1.5	1.4	0.8	0.9
200	8.1	7.5	9.0	8.5	3.7	4.0	4.0	4.5	0.9	1.0	0.7	0.6
300	10.4	8.3	8.8	6.1	5.7	5.0	3.9	3.3	1.1	1.2	0.7	0.6
500	9.0	10.5	9.4	8.5	5.4	5.8	5.2	3.7	1.2	0.8	1.2	0.9
Kolmogorov-Smirnov (Asymptotic)												
50	4.7	7.3	9.1	11.4	1.7	4.4	6.3	8.7	0.3	2.0	4.6	7.0
100	7.9	8.5	9.8	9.0	3.4	5.3	5.9	6.7	0.6	2.5	3.2	4.2
200	7.2	9.6	8.9	10.8	3.1	4.9	4.9	7.5	1.2	1.7	2.5	3.9
300	9.5	9.0	8.6	8.1	4.7	4.8	4.5	4.9	1.1	1.5	1.9	2.4
500	9.3	9.9	9.0	9.3	5.2	5.5	5.9	5.5	1.2	1.9	2.0	2.5

$$Y = \beta_0(Z) + X_{21}\delta_1 + ... + X_{2k_2}\delta_{k_2} + \varepsilon$$

$$\beta_0(z) \propto z, j = 0, 1, ..., k,$$

% Rejections under $H_1 : Var(\beta_0(Z)) > 0.$

	CvM				K-S			
	k=1		k=3		k=1		k=3	
n\\λ	0.25	0.5	0.25	0.5	0.25	0.5	0.25	0.5
	Our Test							
50	19.3	57.4	10.7	26.8	16.8	49.3	9.8	20.2
100	37.7	89.9	16.3	52.5	31.4	82.2	15.5	40.7
200	69.0	99.7	30.2	88.2	58.5	98.7	25.0	76.1
	CUSUM test							
50	14.1	47.4	4.8	7.8	15.8	41.8	5.1	7.7
100	28.3	81.4	5.3	10.3	26.3	76.1	7.7	13.8
200	57.9	98.2	9.0	25.7	51.1	97.8	10.5	35.0

$$Y = \beta_0(Z) + X_{21}\delta_1 + ... + X_{2k_2}\delta_{k_2} + \varepsilon$$

$$\beta_0(z) \propto \sin(2\pi z)$$

% Rejections under $H_1 : Var(\beta_0(Z)) > 0$.

	CvM				K-S			
	k=1		k=3		k=1		k=3	
n\\λ	0.25	0.5	0.25	0.5	0.25	0.5	0.25	0.5
	Our Test							
50	20.9	57.4	15.4	26.8	21.3	49.3	16.8	20.2
100	36.1	89.9	27.4	52.5	38.5	82.2	30.1	40.7
200	61.6	99.7	48.4	88.2	65.7	98.7	50.7	76.1
	CUSUM test							
50	14.2	47.4	4.9	7.8	15.6	41.8	7.4	7.7
100	28.2	81.4	6.9	10.3	31.4	76.1	9.5	13.8
200	52.4	98.2	13.9	25.7	56.9	97.8	17.0	35.0

$$Y = \beta_0(Z) + X_{21}\delta_1 + ... + X_{2k_2}\delta_{k_2} + \varepsilon$$

$$\beta_0(z) \propto 1 + 2 \cdot 1_{\{z \le 0.4\}}$$

% Rejections under $H_1 : Var(\beta_0(Z)) > 0$.

	CvM				K-S			
	k=1		k=3		k=1		k=3	
n\\λ	0.25	0.5	0.25	0.5	0.25	0.5	0.25	0.5
	Our Test							
50	21.4	60.9	14.0	42.8	20.6	65.6	13.4	44.8
100	37.8	90.9	26.1	73.4	37.7	93.5	26.5	81.2
200	62.6	99.9	41.5	96.4	66.8	99.9	47.8	98.9
	CUSUM test							
50	14.8	46.2	5.4	9.7	13.7	53.2	7.3	13.3
100	27.5	80.8	7.0	20.0	27.9	87.0	8.9	27.2
200	49.6	99.0	12.1	41.7	55.1	99.7	14.1	58.7

$$Y = \beta_0(Z) + X_{11}\beta_1(Z) + ... + X_{1k_1}\beta_{k_1}(Z) + X_{21}\delta_1 + \varepsilon$$

$$\beta_j(z) \propto \sin(2\pi z),\ j = 1, .., k_1,$$

% Rejections under $H_1 : Var(\beta_j(Z)) = 0.25^2$, some $j = 0, ..., k$

	CvM			K-S		
	k1=0	k1=1	k1=3	k1=0	k1=1	k1=3
Our test						
50	20.9	33.2	55.4	21.3	42.4	68.2
100	36.1	72.1	97.2	38.5	81.7	99.5
200	61.6	97.4	100	65.7	98.6	100
CUSUM test						
50	14.2	21.1	23.3	15.6	25.0	21.8
100	28.2	47.2	61.3	31.4	57.8	59.2
200	52.4	86.8	96.0	56.9	93.9	96.5

SPECIFICATION TEST FOR INTERACTIVE EFECTS

$$H_0 : \mathbb{E}\left(Y \middle| X, Z \right) = \bar{\beta}_0 + Z\delta_1 + X\left[\bar{\beta}_1 + (Z \cdot X)\,\delta_2\right] \;\; a.s.$$

$$H_1 : \mathbb{E}\left(Y \middle| X, Z \right) = \bar{\beta}_0 + X\beta_1(Z) + Z\delta_1 + (Z \cdot X)\,\delta_2 \;\; a.s.$$

with $Var\left(\beta_1(Z)\right) > 0.$

Same designs for X, Z , ε & $\beta_0(Z)$

$n = 200$

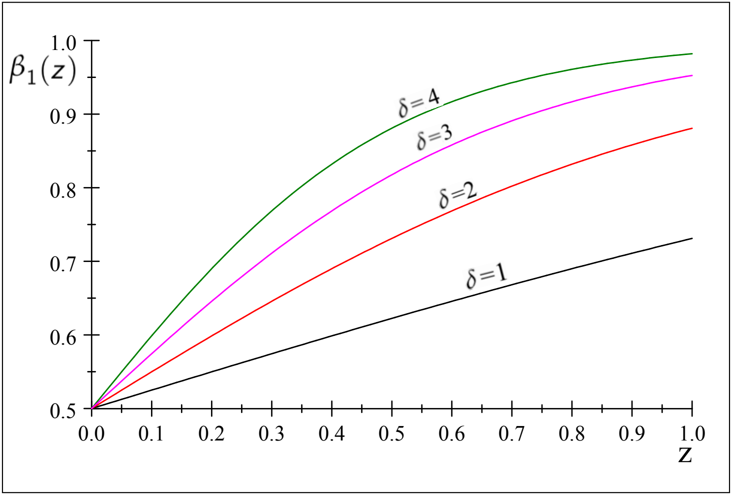
O $\longrightarrow$ $H_0 : \beta_1(Z) = 0$ vs $H_1 : \beta_1(Z) > 0$

C $\longrightarrow$ Omnibus CUSUM test

$$H_1 : \begin{cases} a)\ \beta_1(Z) \propto [1+\exp(-Z)]^{-1} \\ b)\ \beta_1(Z) \propto \sin(2\pi Z) \\ c)\ \beta_1(Z) \propto 1 + 2 \cdot 1_{\{Z \leq 0.4\}} \end{cases}$$

	H_0		H_1: a)		H_1: b)		H_1: c)	
	O	C	O	C	O	C	O	C
$Var(\beta_1(Z)) = 0.25^2$								
50	5.0	4.5	5.0	4.6	25.5	16.3	12.6	8.8
100	5.0	5.8	5.0	5.7	53.5	42.7	28.7	19.4
200	5.8	5.4	5.8	6.1	88.1	80.2	56.2	40.0
$Var(\beta_1(Z)) = 0.5^2$								
50	5.0	5.4	5.4	4.5	74.1	54.7	40.7	25.0
100	5.0	5.8	5.2	6.1	98.8	94.8	82.9	66.0
200	5.8	5.4	7.4	6.2	100	100	99.5	96.6

$\beta_1(z) \;=\; [1+\exp(-z)]^{-1}$ is almost a straight line in $[0,1]$



%Rejections under H_1 with $\beta_1(Z) \propto [1+\exp(-\rho z)]^{-1}$

	$\delta = 1$		$\delta = 2$		$\delta = 3$		$\delta = 4$	
	O	C	O	C	O	C	O	C
$Var\left(\beta_0(Z)\right) = 0.5^2$								
50	5.4	4.5	6.6	5.8	12.4	9.4	20.4	13.6
100	5.2	6.1	10.2	7.5	21.4	16.1	41.7	27.0
200	7.4	6.2	16.10	11.7	40.4	27.1	73.3	49.6

THANK YOU!

TESTING COEFFICIENTS CONSTANCY OF A SUBSET OF COEFFICIENTS

$$Y = \mathbf{X}_1' \underset{k_1 \times 1}{\boldsymbol{\beta}_1}(Z) + \mathbf{X}_2' \underset{k_2 \times 1}{\boldsymbol{\beta}_2}(Z) + \varepsilon,$$

$$H_0 : Var\left(\boldsymbol{\beta}_{2j}(Z)\right) = 0,\ j = 1, .., k_2$$

vs

$$H_1 : Var\left(\boldsymbol{\beta}_{2j}(Z)\right) > 0,\ \text{some } j = 1, .., k_2$$

We apply our method estimating first $\boldsymbol{\beta}_1(\cdot)$ in the restricted model, i.e.

$$Y = \mathbf{X}_1'\boldsymbol{\beta}_1(Z) + \mathbf{X}_2'\bar{\boldsymbol{\beta}}_2 + \varepsilon,$$

e.g. Fan & Huang (2005, *Bernoulli*). Let $\hat{\boldsymbol{\beta}}_1(\cdot)$ be the estimator of $\boldsymbol{\beta}_1(\cdot)$ in the restricted model.

$$\begin{pmatrix} \bar{\boldsymbol{\theta}}^-(u) \\ \\ \bar{\boldsymbol{\theta}}^+(u) \end{pmatrix} = \underset{\boldsymbol{\theta}^+,\boldsymbol{\theta}^-}{\arg\min} \left\{ \sum_{i=1}^{\lfloor nu \rfloor} \left[\left(Y_{[i:n]} - \mathbf{X}_{1[i:n]}'\hat{\boldsymbol{\beta}}(Z_{i:n}) - \mathbf{X}_{2[i:n]}'\boldsymbol{\theta}^- \right)^2 \right] \right.$$

$$\left. + \sum_{i=1+\lfloor nu \rfloor}^{n} \left[\left(Y_{[i:n]} - \mathbf{X}_{1[i:n]}'\hat{\boldsymbol{\beta}}(Z_{i:n}) - \mathbf{X}_{2[i:n]}'\boldsymbol{\theta}^+ \right)^2 \right] \right\}$$

$$\bar{\boldsymbol{\eta}}(u) = \left(\bar{\boldsymbol{\theta}}^-(u) - \bar{\boldsymbol{\theta}}^+(u) \right) \rightarrow U - \mathit{process}$$

COMPARISON BETWEEN CvM, K-S, AND SMOOTHING TEST

$$Y = \beta_0(Z) + \beta_1 X_1 + ... + \beta_k X_k + \varepsilon$$

% Rejections under $H_0 : Var(\beta_0(Z)) = 0$.

	k=1			k=2			k=3			k=4		
	50	100	200	50	100	200	50	100	200	50	100	200
UNRESTRICTED												
CvM	4.9	5.0	4.4	3.6	5.3	5.0	4.0	4.8	4.7	1.6	4.9	5.0
KS	2.1	3.0	4.7	3.5	2.9	4.9	2.7	3.0	5.6	1.9	3.2	3.8
RESTRICTED												
CvM	6.2	5.0	4.2	5.8	4.8	6.4	5.8	4.8	6.4	6.2	4.8	6.0
KS	4.8	5.6	4.9	4.5	4.3	7.6	4.5	4.3	7.6	5.5	4.3	4.8
CUSUM												
CvM	5.4	4.3	4.1	6.5	4.2	5.7	5.6	4.6	4.9	4.5	3.8	5.3
KS	6.8	5.4	5.4	8.3	4.3	6.9	6.8	5.6	4.8	5.5	5.2	5.4
SMOOTH bandwidth=Cross Validation												
S	5.9	6.6	5.2	7.9	5.6	6.4	7.7	5.8	5.1	6.6	6.5	5.6

% Rejections under $H_1 : Var(\beta_0(Z)) > 0$,

$\beta_0(V) \propto 1 + 2 \cdot 1_{\{Z \leq 0.4\}}$, $Var(\beta_0(Z)) = 0.5^2$

	k=1			k=2			k=3			k=4		
	50	100	200	50	100	200	50	100	200	50	100	200
UNRESTRICTED												
CvM	40.4	77.3	98.9	16.5	51.2	89.9	12.0	34.2	74.4	7.4	22.5	60.8
KS	40.8	83.6	99.4	18.0	63.5	96.7	13.3	46.0	90.3	7.2	33.7	83.0
RESTRICTED												
CvM	60.7	90.2	99.7	43.3	80.5	98.4	37.9	70.8	95.2	35.1	61.5	92.2
KS	30.6	90.2	99.8	38.6	81.0	99.1	33.0	73.3	97.8	32.3	65.8	94.9
CUSUM												
CvM	48.9	80.2	98.8	18.9	42.1	78.6	10.3	22.7	43.5	7.8	13.6	28.9
KS	53.3	84.2	99.5	22.2	55.5	91.5	12.3	25.7	61.7	8.9	13.9	36.2
SMOOTH bandwidth=Cross Validation												
S	42.1	79.1	98.4	28.8	55.6	89.6	19.4	46.7	75.6	15.6	32.8	62.5

% Rejections under $H_1 : Var(\beta_0(Z)) > 0$,

$\beta_0(z) \propto \sin(2\pi z)$

	k=1			k=2			k=3			k=4		
	50	100	200	50	100	200	50	100	200	50	100	200
UNRESTRICTED												
CvM	40.4	79.0	98.4	19.6	57.9	93.5	13.5	40.5	87.0	9.0	31.2	80.0
KS	36.0	79.2	99.3	18.0	62.0	95.7	13.0	46.6	90.0	1.5	36.2	84.5
RESTRICTED												
CvM	56.8	90.5	99.9	45.9	84.6	99.1	41.8	79.7	99.0	39.8	73.7	97.6
KS	47.9	87.6	99.6	37.7	79.7	98.6	35.2	74.6	97.7	34.4	67.3	96.7
CUSUM												
CvM	46.0	81.5	98.8	18.5	48.4	83.1	12.0	24.1	56.3	8.3	16.3	37.6
KS	51.1	86.1	99.3	21.9	59.0	92.9	14.2	30.7	71.6	9.8	18.6	48.0
SMOOTH bandwidth=Cross Validation												
S	46.3	79.1	98.4	28.8	55.6	89.6	18.7	39.5	86.3	15.6	32.8	62.5