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Jaap H Abbring and Øystein Daljord

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Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
www.cepr.org

IDENTIFYING THE DISCOUNT FACTOR IN DYNAMIC DISCRETE CHOICE MODELS

Abstract

The identification of the discount factor in dynamic discrete models is important for counterfactual analysis, but hard. Existing approaches either take the discount factor to be known or rely on high level exclusion restrictions that are difficult to interpret and hard to satisfy in applications, in particular in industrial organization. We provide identification results under an exclusion restriction on primitive utility that is more directly useful to applied researchers. We also show that our and existing exclusion restrictions limit the choice and state transition probability data in different ways; that is, they give the model nontrivial and distinct empirical content.

JEL Classification: C14, C25, D91, D92

Keywords: discount factor, dynamic discrete choice, empirical content, identification

Jaap H Abbring - j.h.abbring@uvt.nl
CentER, Tilburg University and CEPR

Øystein Daljord - Oeystein.Daljord@chicagobooth.edu
Booth School of Business, University of Chicago

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1. INTRODUCTION

The identification of the discount factor in dynamic discrete choice models is crucial for their application to the evaluation of agents' responses to dynamic interventions. Rust (1994, Lemma 3.3) and Magnac and Thesmar (2002, Proposition 2) showed that the discount factor is not identified from data on choices and state variables without further restrictions. Magnac and Thesmar's Proposition 4 established identification based on such a restriction, the existence of a state variable that affects, in some specific way, expected discounted future utility but not the "current value," which is a difference in expected discounted utilities between two particular choice sequences. Nevertheless, later work on identification (e.g. Bajari et al., 2015; Norets and Tang, 2014) usually takes the discount factor to be known, perhaps because Magnac and Thesmar's exclusion restriction is difficult to interpret and hard to satisfy in applications, in particular in those applications in e.g. industrial organization that use stationary models with infinite horizon.

We discuss this limitation of Magnac and Thesmar's result and provide identification results under an exclusion restriction on primitive utility that is more directly useful to applied researchers. We also show that there is some scope for testing Magnac and Thesmar's exclusion restriction and ours, as these have nontrivial and different empirical contents. Specifically, for both restrictions, we give an example of data that are consistent with one restriction but not with the other. Finally, we show that identification of the discount factor does not follow as a special case of Fang and Wang's (2015) generic identification results for dynamic discrete choice models with hyperbolic discounting.

2. MODEL

Consider a stationary version of Magnac and Thesmar's model. Time is discrete with an infinite horizon. In each period, agents first observe state variables x and ε, where x takes values in $\tilde{X} = \{x_1, \dots, x_J\}$ and $\varepsilon = \{\varepsilon_1, \dots, \varepsilon_K\}$ is continuously distributed on $\mathbb{R}^K$; for $J, K \geq 2$. Then, they choose d from the set of alternatives $\tilde{D} = \{1, 2, \dots, K\}$ and collect utility $u_d(x, \varepsilon) = u^*_d(x) + \varepsilon_d$. Finally, they move to the next period with new state variables x' and ε' drawn from a Markov transition distribution controlled by d. Following Magnac and Thesmar, we assume that a version of Rust's (1987) conditional independence assumption holds. Specifically, x' is drawn independently of ε from the transition distribution $Q_i\left(\cdot|x\right)$ for any choice $i \in \tilde{D}$; and $\varepsilon_1, \dots, \varepsilon_K$ are independently drawn from mean zero type-1 extreme value distributions.[1] Agents maximize the rationally expected utility flow discounted with factor $\beta \in [0,1)$.

Each choice d equals the option i that maximizes the choice-specific expected discounted utility (or, simply, "value") $v_i(x,\varepsilon)$. The additive separability of $u_i(x,\varepsilon)$ and conditional independence imply that $v_i(x,\varepsilon) = v_i^*(x) + \varepsilon_i$, with v_i^* the unique solution to

$$\begin{aligned} v_i^*(x) &= u_i^*(x) + \beta\mathbb{E}\left[\max_{i'\in\tilde{D}}\{v_{i'}^*(x') + \varepsilon_{i'}'\}\,\middle|\, d = i, x\right] \\ &= u_i^*(x) + \beta\int\mathbb{E}\left[\max_{i'\in\tilde{D}}\{v_{i'}^*(x') + \varepsilon_{i'}'\}\right] dQ_i(x'|x) \end{aligned} \tag{1}$$

for all $i \in \tilde{D}$. Here, for given $\tilde{x} \in \tilde{X}$,

$$\mathbb{E}\left[\max_{i'\in\tilde{D}}\{v_{i'}^*(\tilde{x}) + \varepsilon_{i'}'\}\right] = \ln\left(\sum_{i'\in\tilde{D}}\exp\left(v_{i'}^*(\tilde{x})\right)\right) \tag{2}$$

is the McFadden surplus for the choice among $i' \in \tilde{D}$ with utilities $v_{i'}^*(\tilde{x}) + \varepsilon_{i'}'$.

Suppose we have data on choices d and state variables x that allow us to determine $Q_i(\cdot|\tilde{x})$ and the choice probabilities $p_i(\tilde{x}) = \Pr(d = i|x = \tilde{x})$ for all $i \in \tilde{D}$ and $\tilde{x} \in \tilde{X}$. The model is identified if and only if we can uniquely determine its primitives from these data. As we discuss in Section 5, there exist unique (up to a standard utility normalization) values of the primitives that rationalize the data for any given discount factor $\beta \in [0,1)$. Thus, we can and will focus our identification analysis on β.

The choice probabilities are fully determined by

$$\ln\left(p_i(\tilde{x})\right) - \ln\left(p_K(\tilde{x})\right) = v_i^*(\tilde{x}) - v_K^*(\tilde{x}), \qquad i \in \tilde{D}/\{K\},\ \tilde{x} \in \tilde{X}. \tag{3}$$

[1] Magnac and Thesmar (2002) show that the distribution of ε cannot be identified and take it to be known. Our type-1 extreme value assumption leads to the canonical multinomial logit case. Our results extend directly to any other known continuous distribution on $\mathbb{R}^K$.

Thus, with the transition probabilities $Q_i(\cdot|\tilde{x})$, the value contrasts $v_i^*(\tilde{x}) - v_K^*(\tilde{x})$ for $i \in \tilde{D}/\{K\}$ and $\tilde{x} \in \tilde{X}$ capture all the model's implications for the data. Hotz and Miller (1993) pointed out that (3) can be inverted to identify the value contrasts from the choice probabilities. To use this, we first rewrite (1) as

$$v_i^*(x) = u_i^*(x) + \beta \int (m(x') + v_K^*(x'))\, dQ_i(x'|x), \tag{4}$$

where, for given $\tilde{x} \in \tilde{X}$, $m(\tilde{x}) = \mathbb{E}\left[\max_{i' \in \tilde{D}}\{v_{i'}^*(\tilde{x}) - v_K^*(\tilde{x}) + \varepsilon'_{i'}\}\right]$ is the "excess surplus" (over $v_K^*(\tilde{x})$), the McFadden surplus for the choice among $i' \in \tilde{D}$ with utilities $v_{i'}^*(\tilde{x}) - v_K^*(\tilde{x}) + \varepsilon'_{i'}$. By (3), $m(\tilde{x}) = -\ln(p_K(\tilde{x}))$.

3. MAGNAC AND THESMAR'S IDENTIFICATION RESULT

Let $\mathbf{v}_i$, $\mathbf{p}_i$, $\mathbf{u}_i$, and $\mathbf{m}$ be $J \times 1$ vectors with j-th elements $v_i^*(x_j)$, $p_i(x_j)$, $u_i^*(x_j)$, and $m(x_j)$, respectively. Let $\mathbf{Q}_i$ be the $J \times J$ matrix with (j, j')-th entry $Q_i(x_{j'}|x_j)$ and $\mathbf{I}$ be a $J \times J$ identity matrix. Note that the $J \times 1$ vector $\mathbf{m} + \mathbf{v}_K$ stacks the McFadden surpluses in (2).

In this notation, the data are $\{\mathbf{p}_i, \mathbf{Q}_i; i \in \tilde{D}\}$ and directly identify $\mathbf{m} = -\ln \mathbf{p}_K$. We can rewrite (4) as $v_i^*(x) = u_i^*(x) + \beta \mathbf{Q}_i(x)\left[\mathbf{m} + \mathbf{v}_K\right]$, where $\mathbf{Q}_i(x_j)$ is the j-th row of $\mathbf{Q}_i$. Subtracting the same expression for $v_K^*(x)$, rearranging, and substituting (3), we get

$$\ln(p_i(x)) - \ln(p_K(x)) = \beta\left[\mathbf{Q}_i(x) - \mathbf{Q}_K(x)\right]\mathbf{m} + U_i(x), \tag{5}$$

where $U_i(x) = u_i^*(x) - u_K^*(x) + \beta\left[\mathbf{Q}_i(x) - \mathbf{Q}_K(x)\right]\mathbf{v}_K$ is Magnac and Thesmar's "current value" of choice i in state x. Their Proposition 4 assumes the existence of an option $i \in \tilde{D}/\{K\}$ and $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ such that $\tilde{x}_1 \neq \tilde{x}_2$ and $U_i(\tilde{x}_1) = U_i(\tilde{x}_2)$. Under this exclusion restriction, differencing (5) evaluated at $\tilde{x}_1$ and $\tilde{x}_2$ yields

$$\begin{aligned} &\ln(p_i(\tilde{x}_1)/p_K(\tilde{x}_1)) - \ln(p_i(\tilde{x}_2)/p_K(\tilde{x}_2)) \\ &\qquad = \beta\left[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)\right]\mathbf{m}. \end{aligned} \tag{6}$$

Provided that Magnac and Thesmar's rank condition

$$\left[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)\right]\mathbf{m} \neq 0 \tag{7}$$

holds, this linear (in β) equation uniquely determines β in terms of the data.

This identification argument can be interpreted in terms of an experiment that shifts the contrast $[\mathbf{Q}_i(x) - \mathbf{Q}_K(x)]\,\mathbf{m}$ between expected excess surpluses under choices i and K by changing the state x from $\tilde{x}_2$ to $\tilde{x}_1$, while keeping the current value $U_i(\tilde{x}_1) = U_i(\tilde{x}_2)$ constant. The discount factor is the per unit effect of that observed shift on the observed log choice probability ratio $\ln(p_i(x)/p_K(x))$.

A shift in the expectation contrast $\mathbf{Q}_i(x) - \mathbf{Q}_K(x)$ does not suffice for identification. For example, suppose that the exclusion restriction holds for some $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$, but that the excess surplus $m(x_1) = \cdots = m(x_J)$ is constant, so that the expected excess surplus contrast $[\mathbf{Q}_i(x) - \mathbf{Q}_K(x)]\,\mathbf{m} = 0$. Then, a shift in the expectation contrast does not shift the expected excess surplus contrast and thus does not change the decision problem. Consequently, this shift is not informative on β and Magnac and Thesmar's rank condition (7) fails.

Rank condition (7) has a meaningful interpretation and is verifiable in data. The exclusion restriction $U_i(\tilde{x}_1) = U_i(\tilde{x}_2)$, however, is more problematic, because it imposes opaque conditions on the primitives that are unlikely to be satisfied in applications. The current values depend on both current utilities and discounted expected future values. Specifically, they involve elements of $\mathbf{v}_K$, which by (4) equals

$$\mathbf{v}_K = [\mathbf{I} - \beta\mathbf{Q}_K]^{-1}\,[\mathbf{u}_K + \beta\mathbf{Q}_K\mathbf{m}]\,. \tag{8}$$

The current value is in fact a value contrast between two sequences of choices: choose i now, K in the next period, and choose optimally ever after, relative to choose K now, K in the next period, and choose optimally ever after. Because this particular value contrast does not correspond to common economic choice sequences, the applied value of Magnac and Thesmar's restriction is limited (Dubé et al., 2014). It is hard to think of naturally occurring experiments that shift the expected contrasts in excess surplus, i.e. satisfy the rank condition, without also shifting the current value and thus violating the exclusion restriction. Still, hard does not mean impossible. In Example 1 below, we show that due to the special structure of that particular problem, both conditions may plausibly be met.

Unclear economic intuition for the exclusion restrictions becomes an issue when, as in typical applications, the exclusion restrictions are assumptions that are not themselves tested in the analysis. Without a clear understanding of the economic substance of the current value concept, it is both hard to assess how plausible identifying assumptions based on that concept are, and to find variation that satisfies the restrictions. By defining the exclusion restrictions directly on the primitives, which we do in the next section, we can generally improve the intuition for them.[2] That is helpful when evaluating the quality of the identifying assumptions, and aids the search for sources of identifying variation.

4. A NEW IDENTIFICATION RESULT

Like Magnac and Thesmar, we start with (5) or, equivalently,

$$\ln \mathbf{p}_i - \ln \mathbf{p}_K = \beta \left[\mathbf{Q}_i - \mathbf{Q}_K\right] \left[\mathbf{m} + \mathbf{v}_K\right] + \mathbf{u}_i - \mathbf{u}_K. \tag{9}$$

Instead of controlling the contribution of $\mathbf{v}_K$ to the right hand side with an exclusion restriction on the current value, we exploit that, in the stationary case, it can be expressed in terms of the model primitives. Substituting (8) in (9) and rearranging gives

$$\begin{aligned} \ln \mathbf{p}_i - \ln \mathbf{p}_K = \beta \left[\mathbf{Q}_i - \mathbf{Q}_K\right] \left[\mathbf{I} - \beta \mathbf{Q}_K\right]^{-1} \mathbf{m} + \mathbf{u}_i \\ - \left[\mathbf{I} - \beta \mathbf{Q}_i\right] \left[\mathbf{I} - \beta \mathbf{Q}_K\right]^{-1} \mathbf{u}_K. \end{aligned} \tag{10}$$

Intuition from static discrete choice analysis and Magnac and Thesmar's results for dynamic models suggest that, for identification, we need to fix utility in one reference alternative, say $\mathbf{u}_K$. Intuitively, choices only depend on, and thus inform about, utility contrasts. Thus, following e.g. Fang and Wang (2015) and Bajari et al. (2015), we set $\mathbf{u}_K = \mathbf{0}$.[3] This normalization cannot be refuted

[2] Alternatively, the restrictions could be defined on functions of primitives that correspond to economically meaningful concepts.

[3] Our identification analysis of β applies without change to the case in which $u_K^*(x)$ is constant, but not necessarily zero. It can be straightforwardly extended to the case in which $u_K^*(x)$ is known but not necessarily constant, which would require a different rank condition. Note that none of these normalizations collapses Magnac and Thesmar's exclusion restriction on current values to an easily interpretable restriction on primitives.

by data without further restrictions.[4] Despite this lack of empirical content, it is not completely innocuous, as it may affect the model's counterfactual predictions (see e.g. Norets and Tang, 2014, Lemma 2). It is however standard and allows us to focus on the identification of the discount factor.[5]

Now, assume that there exist a choice $i \in \tilde{D}/\{K\}$ and a pair of states $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ such that $\tilde{x}_1 \neq \tilde{x}_2$ and $u_i^*(\tilde{x}_1) = u_i^*(\tilde{x}_2)$. This exclusion restriction has the advantage over Magnac and Thesmar's that it is a simple restriction on primitive utility. Under the primitive utility restriction, (10) implies

$$\begin{aligned}\ln\left(p_i(\tilde{x}_1)/p_K(\tilde{x}_1)\right) - \ln\left(p_i(\tilde{x}_2)/p_K(\tilde{x}_2)\right)& \\ = \beta\left[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)\right]&\left[\mathbf{I} - \beta\mathbf{Q}_K\right]^{-1}\mathbf{m}.\end{aligned} \tag{11}$$

Unlike the right hand side of (6), the right hand side of (11) is not linear in β. Nevertheless, given data on transition and choice probabilities, it is a smooth, known function of β. It is therefore very easy to verify, for example graphically, whether there is a unique value of β that equates it to the known left hand side of (11). Before we illustrate this with examples, we first discuss (11) in some more detail.

Like (6), (11) is constructive in the sense that it defines a moment condition that can be used for estimation purposes. We have effectively moved the terms that frustrated the interpretation of the exclusion restriction from the realm of untestable assumptions to the moment condition, which is verifiable in data.[6] Specifically, the right hand side of (11) equals β times the sum of two terms,

$$\left[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)\right]\mathbf{m}$$

[4]In Section 5, we note that the normalized model can rationalize any choice and state transition probability data.

[5]Chou (2015) recently provided identification results for dynamic discrete choice models without this normalization. Chou's Propositions 3, 7, and 8 provide conditions for identification of the discount factors in a nonstationary model; his results for the stationary model studied here take the discount factor to be known. Another difference is that we emphasize the economic interpretation of the identifying conditions and that we provide results on their empirical content.

[6]Section 5 shows that the two exclusion restrictions have nontrivial and distinct empirical contents. That is, the data carry some information on them. However, we also show that data are often consistent with both. Hence, no uniformly consistent test on either restriction exists.

and

$$[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\,\mathbf{v}_K =$$
$$[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\left[\beta\mathbf{Q}_K + \beta^2\mathbf{Q}_K^2 + \cdots\right]\mathbf{m}.$$

The first term is the known shift in the expected excess surplus contrast from changing the state from $\tilde{x}_2$ to $\tilde{x}_1$ that is used in Magnac and Thesmar's identification argument. The second term is the corresponding shift in choice K's expected value contrasts, which is not directly known because it depends on β. The two terms add up to a shift in the expected surplus contrast that depends on β and therefore does not directly identify β from its effect on the log choice probability ratio. In particular, one obvious candidate to replace Magnac and Thesmar's rank condition (7) here, a nonzero shift in expected surplus contrasts for all $\beta \in (0, 1)$, does not suffice for identification, except in special cases.

One such special case arises if $\mathbf{Q}_K$ is such that our version of Magnac and Thesmar's rank condition is equivalent to theirs.

Example 1 If

$$\mathbf{Q}_K = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

the distribution of x' is concentrated on the same point, x_J, if K is chosen, independently of x. For example, in Rust's (1987) bus engine renewal problem, the choice to renew always returns the observed state variable, mileage since last renewal, to zero, independently from its mileage just before renewal. In this case, $v_K^*(\tilde{x}) = \beta\,(m(x_J) + v_K^*(x_J))$ for all $\tilde{x} \in \tilde{X}$, so that the second (value of choice K) term in the right hand side of (11) is zero. Consequently, a nonzero shift in expected surplus contrasts is equivalent to Magnac and Thesmar's rank condition (7) and suffices for identification. In fact, because in this case $\beta[\mathbf{Q}_i - \mathbf{Q}_K]\mathbf{v}_K = 0$, our primitive utility restriction is equivalent to Magnac and Thesmar's current value restriction and their identification result applies. As an aside, note that (7) simplifies to

$$[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2))]\,\mathbf{m} \neq 0.$$

in this example. That is, it simply requires that the expected excess surplus differs between states $\tilde{x}_1$ and $\tilde{x}_2$ under choice $i \neq K$.

Even if our version of Magnac and Thesmar's rank condition is not equivalent to theirs, it may suffice for identification.

Example 2 If the left hand side of (11) equals 0, i.e. if the choice probability ratio does not change between states $\tilde{x}_2$ and $\tilde{x}_1$, then (11) requires that either $\beta \in (0, 1)$ such that this shift in surplus contrasts is zero or $\beta = 0$. Consequently, in this special case, it is sufficient and necessary for identification that the shift in expected surplus contrasts is nonzero for all $\beta \in (0, 1)$. Intuitively, this alternative to Magnac and Thesmar's rank condition ensures that there is sufficient variation in expected future payoffs, so that a lack of response in choices can only be explained by myopic behavior, $\beta = 0$.

In general, however, neither condition suffices for identification.

Example 3 Figure 1 plots the left hand side of (6) and (11) (solid black line) and the right hand sides of (6) (dashed red line) and (11) (solid blue curve) for a specific example with $K = 2$ choices and $J = 3$ states. The example's data satisfy Magnac and Thesmar's rank condition and our alternative to it (note that the right hand side of (11) is positive on $(0, 1)$).

The choice probabilities imply a relatively low excess surplus $m(x_3)$ in state x_3. Because the experiment underlying Magnac and Thesmar's rank condition moves probability mass away from state x_3, the right hand side of (6), and the first (excess surplus) term in the right hand side of (11), slope upward and equal the left hand side for only one value of β. Under the current value restriction, this is the only discount factor consistent with the data.

Under the primitive utility restriction, we also need to take account of the second (value of choice K) term in the right hand side of (11). In contrast to the excess surplus $m(x_3)$, the value $v_K(x_3)$ is relatively high, because $\mathbf{Q}_K(x_3)$ puts a relatively low (zero) probability on ending up in the low excess surplus state x_3. Consequently, the move of probability mass away from state x_3 renders the second term in the right hand side of (11) negative, and increasingly so with increasing β. It follows that the right hand side of (11) first equals its left hand side at a slightly higher discount factor than the one identified under Magnac and Thesmar's condition. Moreover, the negative contribution of the second term eventually grows so large that the right hand side of (11) again equals the left hand side at a discount factor closer to one. Thus, two distinct discount factors are consistent with the data under the primitive utility restriction.

Magnac and Thesmar's rank condition is not necessary for identification either.

EXAMPLE 4 Figure 2 presents an example in which the shift in expected excess surplus is zero, so that the right hand side of (6) and the first (excess surplus) term in the right hand side of (11) are zero, but the second (value of choice K) term in the right hand side of (11) is positive and increasing with β. There exists exactly one $\beta \in [0, 1)$ that solves (11), despite the violation of Magnac and Thesmar's rank condition.

Also note that there is no value of β that satisfies (6). Thus, even though the data can be rationalized by some specification of the model, they are not consistent with the current value restriction. In other words, this restriction has empirical content. We return to this point in Section 5.

More generally, strict monotonicity of the right hand side of (11), as in Example 4, suffices for identification. It is easy to derive conditions that imply such strict monotonicity, and thus identification, and that do not involve β. Without loss of generality— we can freely interchange $\tilde{x}_1$ and $\tilde{x}_2$— we focus on conditions under which it is strictly increasing or, equivalently, its derivative with respect to β is positive:

$$[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)] [\mathbf{I} - \beta \mathbf{Q}_K]^{-2} \mathbf{m} > 0.$$

For this, it suffices that

$$[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)] \mathbf{Q}_K^l \mathbf{m} \geq 0 \text{ for all } l \in \{0, 1, 2, \ldots\}, \tag{12}$$

with the inequality strict for at least one l. Like Magnac and Thesmar's rank condition (7), these conditions do not depend on β and require that specific expected excess surplus contrasts differ between states $\tilde{x}_1$ and $\tilde{x}_2$. It is easy to verify that they hold in Example 4.

We end this section with two further examples in which monotonicity is easy to establish. In the first, Magnac and Thesmar's rank condition is key, even though their exclusion restriction may not hold.

EXAMPLE 5 If $\mathbf{Q}_K = \mathbf{I}$, Magnac and Thesmar's current value restriction is *not* necessarily satisfied under our primitive utility restriction. Consequently, we need to identify β from (11). The left hand side of (11) reduces to

$$\frac{\beta}{1-\beta}\left[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)\right]\mathbf{m},$$

which is strictly monotonic in β if and only if Magnac and Thesmar's rank condition (7) holds.

The final example relies on a type of payoff monotonicity that is common in models of firm and industry dynamics.

EXAMPLE 6 Suppose that the states in $\tilde{X}$ are ordered so that the excess surplus is increasing in the state: $m_1 < m_2 < \cdots < m_J$. Moreover, let $\mathbf{Q}_K$ be stochastically increasing; that is, the distribution of x' given x first-order stochastically increases with x if K is chosen. Such structures, with monotone payoffs and monotone state transitions, naturally arise in problems of nonstrategic firm entry and exit, in which profits increase in an observed demand state variable and higher current demand implies stochastically higher future demand.

There again is no reason for Magnac and Thesmar's current value restriction to hold under our primitive utility restriction. One approach to identification is to ignore this problem's special structure and simply check (11) graphically. The structure however allows us to formulate easy-to-handle sufficient conditions.

First, suppose that the transitions under choice K are not affected by a change of state from $\tilde{x}_2$ to $\tilde{x}_1$: $\mathbf{Q}_K(\tilde{x}_1) = \mathbf{Q}_K(\tilde{x}_2)$. Note that the expected excess surplus after l state transitions under choice K, $\mathbf{Q}_K^l$m, are increasing in the initial state. Thus, sufficient condition (12) is satisfied for all j if $\mathbf{Q}_i(\tilde{x}_1)$ first-order stochastically dominates $\mathbf{Q}_i(\tilde{x}_2)$; that is, if a move of state from $\tilde{x}_2$ to $\tilde{x}_1$ under choice i first-order stochastically increases the next period's state.

Next, if $\mathbf{Q}_K(\tilde{x}_1) = \mathbf{Q}_K(\tilde{x}_2)$ does not hold, condition (12) is satisfied for all l if the effect of a change of state from $\tilde{x}_2$ to $\tilde{x}_1$ on next period's state under choice i first-order dominates that same effect under choice K.

5. EMPIRICAL CONTENT

The previous two sections focused on identification. They give exclusion restrictions under which data generated by the model uniquely determine a set of model primitives. In applications, we need to entertain the possibility that the model is misspecified and did not generate the data to begin with.

First note that a version of Magnac and Thesmar's (2002) Proposition 2 holds: For any given data $\{\mathbf{p}_i, \mathbf{Q}_i; i \in \tilde{D}\}$, $\mathbf{u}_K = 0$, and $\beta \in [0, 1)$, there exists a unique set of primitive utilities $\{\mathbf{u}_i, i \in \tilde{D}/\{K\}\}$ that rationalizes the data. Specifically, $\mathbf{m} = -\ln \mathbf{p}_K$. Then, $\mathbf{v}_K$ follows from $\mathbf{u}_K = 0$ and (8). Next, by (3), $\mathbf{v}_i = \mathbf{v}_K + \ln \mathbf{p}_i - \ln \mathbf{p}_K$ for $i \in \tilde{D}/\{K\}$ ensures that the value functions are compatible with the choice probability data. In turn, by (4), these value functions are uniquely generated by the primitive utilities $\mathbf{u}_i = \mathbf{v}_i - \beta \mathbf{Q}_i [\mathbf{m} + \mathbf{v}_K]$ for $i \in \tilde{D}/\{K\}$ (note that $\mathbf{v}_K$ was already set to be consistent with $\mathbf{u}_K = 0$).

This result justifies our focus on the identification of the discount factor β in the previous two sections: Once the discount factor has been identified, we can find unique primitive utilities that rationalize the data. It also implies that the data cannot tell us whether the model without exclusion restrictions is false or not; that is, the unrestricted model has no empirical content. Therefore, we now turn to the empirical consequences of a violation of the assumed exclusion restriction. Such a violation can manifest itself in two distinct ways.

First, in some cases, it may be possible to find primitives that both satisfy the false exclusion restriction and are compatible with the data. If so, these primitives will in general not equal the true primitives. In Example 3, falsely assuming Magnac and Thesmar's current value restriction when the primitive utility restriction is true identifies a discount factor strictly below the true one. Because we can find primitive utilities that rationalize the data for any discount factor, the data can be of no help to determine the right restriction in this case. Instead, we need to argue in favor of one exclusion restriction or the other on other grounds. In Section 4, we presented a novel identification analysis for exactly this reason: The primitive utility restriction is comparatively easy to motivate and justify in applications.

Second, the data may be incompatible with the assumed exclusion restriction. For example, the data in Example 4 cannot be rationalized under the current value restriction, even though they are compatible with some specification of the model. Thus, the current value restriction gives the model empirical content: It nontrivially limits the data that can be observed. In that example, the data *are* consistent with an exclusion restriction on primitive utility. Conversely, there exist data that are inconsistent with the primitive utility restriction, but that can be rationalized by primitives that satisfy the current value restriction.

EXAMPLE 7 Figure 3 displays the left and right hand sides of (6) and (11) for a variant of Example 3's data in which the shift in the log choice probability ratio when moving the state from $\tilde{x}_2$ to $\tilde{x}_1$, and therefore the left hand side of (6) and (11), is twice as large. At the same time, the right hand sides are similar to those in Example 3 (as is easily verified by comparing Figure 3 to Figure 1). There is still a $\beta \in [0, 1)$ that solves (6), but (11) can no longer be met. Intuitively, the increasingly negative contribution of the second (value of choice K) term in the right hand side of (11) limits the possible log choice probability ratio response to the change in states to a level below the observed response.

Examples 3 and 7 establish that the two exclusion restrictions have nontrivial and distinct empirical contents, so that, to some extent, data can distinguish them. In practice, we can easily establish whether given data are consistent with one exclusion restriction or the other by verifying whether the corresponding moment condition, (6) or (11), or its empirical analog has a solution $\beta \in [0, 1)$.

The remainder of this section provides some further intuition for the empirical content that follows from the restrictions.

As we have already noted, without restrictions, the primitives are free to generate value contrasts $\mathbf{v}_i - \mathbf{v}_K$ that are compatible with any given choice data (that is, satisfy (3)). By extension, any observed shift in the log choice probability ratio from $\tilde{x}_2$ to $\tilde{x}_1$ can be rationalized by setting $u_i^*(\tilde{x}_1)$, $u_i^*(\tilde{x}_2)$, and β such that

$$\begin{aligned} (13) \qquad & v_i^*(\tilde{x}_1) - v_K^*(\tilde{x}_1) - v_i^*(\tilde{x}_2) + v_K^*(\tilde{x}_2) = \\ & u_i^*(\tilde{x}_1) - u_i^*(\tilde{x}_2) + \beta[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\,[\mathbf{m} + \mathbf{v}_K] \end{aligned}$$

matches this observed shift.

Under the primitive utility restriction, the range of the right hand side of (13) is limited by forcing $u_i^*(\tilde{x}_1) - u_i^*(\tilde{x}_2) = 0$, while under the current value restriction, the range is limited by requiring that $u_i^*(\tilde{x}_1) - u_i^*(\tilde{x}_2)$ is exactly offset by $[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\mathbf{v}_K$. The set of choice data that fall outside the range of the model under either restriction can be substantial.

For $K = J = 2$, the restriction $u_1^*(\tilde{x}_1) = u_1^*(\tilde{x}_2)$ and the normalization $u_2^*(\tilde{x}_1) = u_2^*(\tilde{x}_2)$ together imply that $v_1^*(\tilde{x}_1) - v_2^*(\tilde{x}_1) = v_1^*(\tilde{x}_2) - v_2^*(\tilde{x}_2)$. Thus, by (3), the model cannot rationalize data with state dependent choice probabilities. It can be compatible with state independent choice probabilities, but then β is not identified: Since $v_1^*(\tilde{x}_1) - v_2^*(\tilde{x}_1) = v_1^*(\tilde{x}_2) - v_2^*(\tilde{x}_2)$, it follows that $m(\tilde{x}_1) = m(\tilde{x}_2)$ and the experiment is uninformative, as seen in Example 2. A third choice with state dependent utility for the added choice or, alternatively, a third state with $u_1^*(x_3) \neq u_1^*(\tilde{x}_1) = u_1^*(\tilde{x}_2)$ is necessary to generate state dependence of the value contrasts.

For $K = J = 2$ and under the current value restriction, the model cannot only rationalize state independent choice probabilities but also, under some conditions on the transition probabilities, state dependent $p_1(\tilde{x}_1)$ and $p_2(\tilde{x}_2)$ that are sufficiently large. To see this, rewrite the moment condition in (6) as

$$(14) \qquad \ln\left(\frac{p_1(\tilde{x}_1)}{p_1(\tilde{x}_2)}\right) + (1 - \beta\Delta)\ln\left(\frac{1 - p_1(\tilde{x}_2)}{1 - p_1(\tilde{x}_1)}\right) = 0,$$

where $\Delta \in [-2, 2]$ is such that $[\mathbf{Q}_1(\tilde{x}_1) - \mathbf{Q}_2(\tilde{x}_1) - \mathbf{Q}_1(\tilde{x}_2) + \mathbf{Q}_2(\tilde{x}_2)] = [\Delta \ \ -\Delta]$. Evidently, $\Delta > 1$ is necessary to rationalize any state dependent choice data, but does not suffice. For any $\Delta \in (1, 2]$, (14) imposes further cross-restrictions on the choice probabilities that can be rationalized. If $\Delta = 2$, state dependent $p_1(\tilde{x}_1)$ and $p_1(\tilde{x}_2)$ are compatible with some discount factor $\beta \in [0, 1)$ if and only if $p_1(\tilde{x}_1) + p_1(\tilde{x}_2) > 1$. With lower Δ, only values of $p_1(\tilde{x}_1)$ and $p_1(\tilde{x}_2)$ closer to one are consistent with some $\beta \in [0, 1)$. Interestingly, with state dependent choice data, any such β necessarily takes values in $(\Delta^{-1}, 1) \subseteq (\frac{1}{2}, 1)$.

If either the number of choices $K \geq 3$ or the number of states $J \geq 3$, then both exclusion restrictions are compatible with nontrivial sets of choice and state transition probabilities. Their empirical contents are more subtle in this case and hard to characterize in general. We limit our discussion to one example.

Example 8 Let $\mathbf{Q}_K = \mathbf{I}$, as in Example 5. Suppose that the data imply a positive shift from $\tilde{x}_2$ to $\tilde{x}_1$ in the expected excess surplus contrast; i.e. $[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\mathbf{m} > 0$. Then, the moment condition in (11) can be rewritten as

$$(15) \qquad \frac{\beta}{1-\beta} = \frac{\ln\left(p_i(\tilde{x}_1)/p_K(\tilde{x}_1)\right) - \ln\left(p_i(\tilde{x}_2)/p_K(\tilde{x}_2)\right)}{[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\mathbf{m}}.$$

Because the left hand side of (15) takes all values in $[0, \infty)$ when β takes values in $[0, 1)$, any choice probabilities such that $\ln\left(p_i(\tilde{x}_1)/p_K(\tilde{x}_1)\right) - \ln\left(p_i(\tilde{x}_2)/p_K(\tilde{x}_2)\right) \geq 0$ can be rationalized with some β in this case. The intuition for this is straightforward. Because $\mathbf{Q}_K = \mathbf{I}$, not only the shift in the expected excess surplus contrast, but also the corresponding shift in the expected surplus contrast is positive. Moreover, the exclusion restriction does not allow the current period utility contrast to change between $\tilde{x}_2$ and $\tilde{x}_1$. Consequently, the value contrasts $v_i^*(\tilde{x}_1) - v_K^*(\tilde{x}_1) \geq v_i^*(\tilde{x}_2) - v_K^*(\tilde{x}_2)$, so that the model is only compatible with nonnegative shifts in the log choice probability ratio.

Under the current value restriction, Magnac and Thesmar's moment condition is

$$(16) \qquad \beta = \frac{\ln\left(p_i(\tilde{x}_1)/p_K(\tilde{x}_1)\right) - \ln\left(p_i(\tilde{x}_2)/p_K(\tilde{x}_2)\right)}{[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\mathbf{m}}.$$

The range of the left hand side of (16) is $[0, 1)$, which is a strict subset of the range $[0, \infty)$ of the left hand side of (15). At the same time, the right hand sides are the same.[7] Therefore, like (15), (16) cannot rationalize negative shifts in the log choice probability ratio. In addition, it is not compatible with too large positive shifts. Thus, the current value restriction has more empirical content than the primitive utility restriction in this case but, as Example 7 shows, this is not generally true.

6. RELATED LITERATURE

Fang and Wang (2015) recently investigated the identification of a dynamic discrete choice model with hyperbolic discounting, with geometric discounting as a special case. At first glance, their Proposition 2 seems to accomplish this note's improvements on Magnac and Thesmar and more. However, this is not true. Fang and Wang's Proposition 2 only establishes generic identification, where "generic" is defined over the space of possible values of the data data $\{\mathbf{p}_i, \mathbf{Q}_i; i \in \tilde{D}\}$. This severely limits its applicability to the present note's problem, in two ways.

First, geometric discounting is a singular case in the parameter space, in which the so called "present bias" and "potential naivety" parameters equal one. Thus, if at least one of these parameters is identified, then geometric discounting is a singular case in the data space as well, and Fang and Wang's Proposition 2 has no bearing on its identification.

Second, their rank condition, which only requires that $\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) \neq 0$, excludes a singular case in the data space, and therefore cannot be necessary for their generic identification result. Moreover, it does not suffice to get full identification in the geometric discounting case under either Magnac and Thesmar's or our exclusion restriction. In particular, it does not preclude that $\mathbf{Q}_i - \mathbf{Q}_K$ is a zero vector, in which case the right hand sides of both (6) and (11) are zero, and therefore not informative about β. Intuitively, because only value contrasts can be identified, the excluded variable should (at least) have different impacts on transitions under choice i and choice K. Specifically, under an exclusion re-

[7] The range of the right hand side can be shown to be $\mathbb{R}$ under the stated assumptions.

striction on primitive utility only, as in Fang and Wang, Section 4's alternative conditions are needed. This does not imply that Fang and Wang's generic identification result is false; after all, our alternative result proves identification for a singular case from Fang and Wang's perspective. However, it shows that Fang and Wang's generic identification result sheds no light on the conditions needed for identification of the geometric discount factor.

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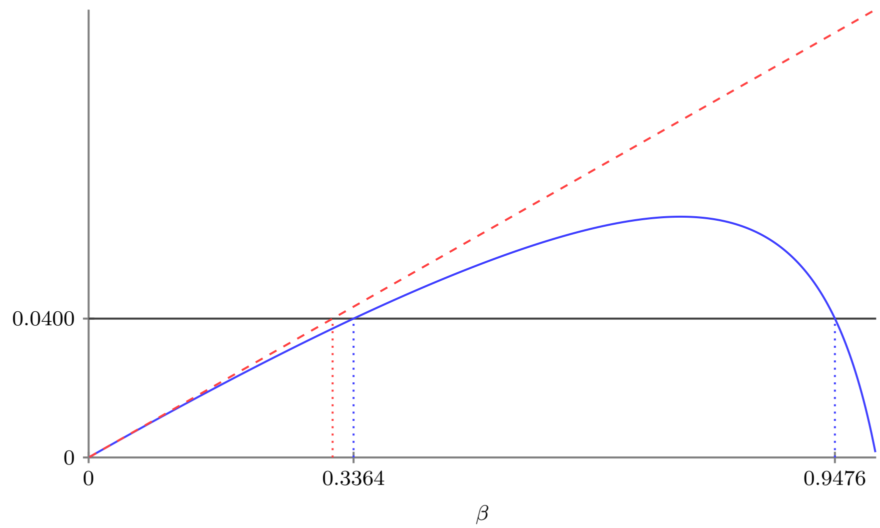
FIGURE 1.— Example in Which an Exclusion Restriction on Current Values Suffices for Identification but One on Primitive Utility Does Not

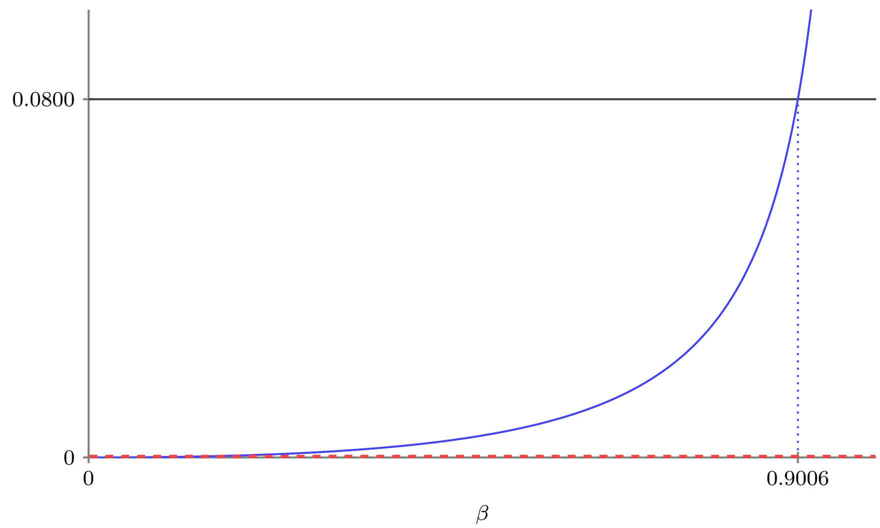


Note: For $J = 3$ states, $K = 2$ choices, $i = 1$, $\tilde{x}_1 = x_1$, and $\tilde{x}_2 = x_2$, this graph plots the left hand side of (6) and (11) (solid black horizontal line) and the right hand sides of (6) (dashed red line), and (11) (solid blue curve) as functions of β. The data are $\mathbf{Q}_i(\tilde{x}_1) = \begin{bmatrix} 0.25 & 0.25 & 0.50 \end{bmatrix}$, $\mathbf{Q}_i(\tilde{x}_2) = \begin{bmatrix} 0.00 & 0.25 & 0.75 \end{bmatrix}$,

$$\mathbf{Q}_K = \begin{bmatrix} 0.90 & 0.00 & 0.10 \\ 0.00 & 0.90 & 0.10 \\ 0.00 & 1.00 & 0.00 \end{bmatrix}, \ \mathbf{p}_i = \begin{bmatrix} 0.50 \\ 0.49 \\ 0.10 \end{bmatrix}, \text{ and } \mathbf{p}_K = \begin{bmatrix} 0.50 \\ 0.51 \\ 0.90 \end{bmatrix}.$$

Consequently, the left hand side of (6) and (11) equals $\ln(p_i(\tilde{x}_1)/p_K(\tilde{x}_1)) - \ln(p_i(\tilde{x}_2)/p_K(\tilde{x}_2)) = 0.0400$. Moreover, $\mathbf{m}' = \begin{bmatrix} 0.6931 & 0.6733 & 0.1054 \end{bmatrix}$ and $\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2) = \begin{bmatrix} -0.65 & 0.90 & -0.25 \end{bmatrix}$, so that the slope of the red dashed line equals $[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\,\mathbf{m} = 0.1291$. A unique value of β, 0.3098, solves (6), but two values of β solve (11): 0.3364 and 0.9476.

FIGURE 2.— Example in Which Magnac and Thesmar's Rank Condition Fails, but an Exclusion Restriction on Primitive Utility Suffices for Identification

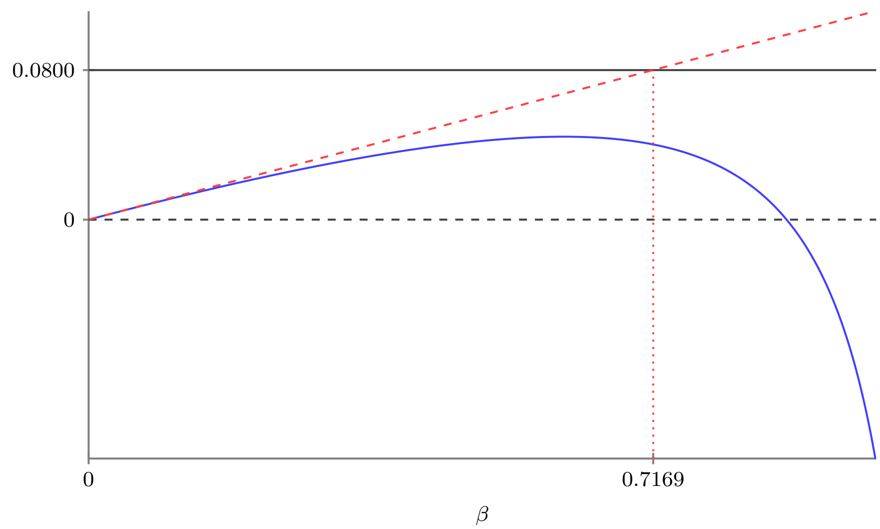


Note: For $J = 3$ states, $K = 2$ choices, $i = 1$, $\tilde{x}_1 = x_1$, and $\tilde{x}_2 = x_2$, this graph plots the left hand side of (6) and (11) (solid black horizontal line) and the right hand sides of (6) (dashed red line) and (11) (solid blue curve) as functions of β. The data are $\mathbf{Q}_i(\tilde{x}_1) = \begin{bmatrix} 0.00 & 0.25 & 0.75 \end{bmatrix}$, $\mathbf{Q}_i(\tilde{x}_2) = \begin{bmatrix} 0.25 & 0.25 & 0.50 \end{bmatrix}$,

$$\mathbf{Q}_K = \begin{bmatrix} 0.00 & 1.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}, \mathbf{p}_i = \begin{bmatrix} 0.50 \\ 0.48 \\ 0.50 \end{bmatrix}, \text{ and } \mathbf{p}_K = \begin{bmatrix} 0.50 \\ 0.52 \\ 0.50 \end{bmatrix}.$$

Consequently, the left hand side of (6) and (11) equals $\ln(p_i(\tilde{x}_1)/p_K(\tilde{x}_1)) - \ln(p_i(\tilde{x}_2)/p_K(\tilde{x}_2)) = 0.0800$. Moreover, $\mathbf{m}' = \begin{bmatrix} 0.6931 & 0.6539 & 0.6931 \end{bmatrix}$ and $\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2) = \begin{bmatrix} -0.25 & 0.00 & 0.25 \end{bmatrix}$, so that the slope of the red dashed line equals $[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\,\mathbf{m} = 0.0000$. A unique value of β, 0.9006, solves (11), but (6) has no solution.

FIGURE 3.— Example of Data that are Consistent with an Exclusion Restriction on Current Values but Not with One on Primitive Utility



Note: For $J = 3$ states, $K = 2$ choices, $i = 1$, $\tilde{x}_1 = x_1$, and $\tilde{x}_2 = x_2$, this graph plots the left hand side of (6) and (11) (solid black horizontal line) and the right hand sides of (6) (dashed red line) and (11) (solid blue curve) as functions of β. The data are $\mathbf{Q}_i(\tilde{x}_1) = \begin{bmatrix} 0.25 & 0.25 & 0.50 \end{bmatrix}$, $\mathbf{Q}_i(\tilde{x}_2) = \begin{bmatrix} 0.00 & 0.25 & 0.75 \end{bmatrix}$,

$$\mathbf{Q}_K = \begin{bmatrix} 0.90 & 0.00 & 0.10 \\ 0.00 & 0.90 & 0.10 \\ 0.00 & 1.00 & 0.00 \end{bmatrix}, \mathbf{p}_i = \begin{bmatrix} 0.50 \\ 0.48 \\ 0.10 \end{bmatrix}, \text{ and } \mathbf{p}_K = \begin{bmatrix} 0.50 \\ 0.52 \\ 0.90 \end{bmatrix}.$$

Consequently, the left hand side of (6) and (11) equals $\ln(p_i(\tilde{x}_1)/p_K(\tilde{x}_1)) - \ln(p_i(\tilde{x}_2)/p_K(\tilde{x}_2)) = 0.0800$. Moreover, $\mathbf{m}' = \begin{bmatrix} 0.6931 & 0.6539 & 0.1054 \end{bmatrix}$ and $\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2) = \begin{bmatrix} -0.65 & 0.90 & -0.25 \end{bmatrix}$, so that the slope of the red dashed line equals $[\mathbf{Q}_i(\tilde{x}_1) - \mathbf{Q}_K(\tilde{x}_1) - \mathbf{Q}_i(\tilde{x}_2) + \mathbf{Q}_K(\tilde{x}_2)]\,\mathbf{m} = 0.1116$. A unique value of β, 0.7169, solves (6), but (11) has no solution.