Evaluating Factor Pricing Models Using High Frequency Panels[1]

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Abstract

This paper develops a new framework and statistical tools to analyze stock returns using high frequency data. We consider a continuous-time multi-factor model via a continuous-time multivariate regression model incorporating realistic empirical features, such as persistent stochastic volatilities with leverage effects. We find that conventional regression approach often leads to misleading and inconsistent test results. We overcome this by using samples collected at random intervals, which are set by the clock running inversely proportional to the market volatility. We find that the size factor has difficulty in explaining the size-based portfolios, while the book-to-market factor is a valid pricing factor.

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1. Introduction

The empirical anomalies related to the Capital Asset Pricing Model (CAPM), typified by the size, value, and momentum effects, lie at the center of multi-factor asset pricing models. Especially, the model by Fama and French (1992, 1993) incorporates the excess returns on two portfolios capturing the size and value premiums as the additional factors. They estimated and tested this three-factor model using twenty-five equity returns from portfolios sorted by stocks' sizes and book-to-market ratios. The research in empirical finance has been focusing on multi-factor asset pricing models since then, and mainly geared toward identifying asset pricing anomalies, thereby new pricing factors. Finding a new factor typically begins with grouping stocks by a characteristic, such as size, book to market ratio, or past return performances. Then, econometric analyses follow, verifying if there exist significant, abnormal returns not explained by the incumbent pricing factors, and testing if a new model embedding an additional factor made from the anomaly variable is rejected. That is, empirical asset pricing involves the construction of panel data sets of returns, and the ensuing statistical investigation of those data series with some economic restrictions.

In the paper, we develop a new framework and a new set of statistical tools for high frequency panels and use them to reexamine Fama-French regressions.[5] Our approach utilizes some recent econometric research on models with high frequency observations. Fama-French regressions have still been analyzed largely within the classical regression framework. There are at least two dimensions that we may look into for a new opportunity using our approach. First, asset return data sets are available at several different frequencies, e.g., daily, monthly and yearly. However, very few attempts have been made to address the issue of how to use these data sets provided at multiple frequencies. In modern financial markets, information flows almost in real time and assets are traded at high frequencies. Thus, a valid asset pricing model under the premise of well-functioning markets must delineate relationships between asset returns and pricing factors at the (high) frequency of market clearing. This implies that a proper integration of higher frequency models is needed to accurately estimate and test asset pricing models at lower frequencies.

Second, financial asset returns used in Fama-French regressions are extremely volatile at high frequencies. This excessive volatility introduces too much noise to make it meaningful to run regressions at high frequencies. Moreover, virtually all asset returns show a strong evidence of time-varying and stochastic volatilities, and of leverage effects. The time-varying and stochastic volatilities would have only a second-order effect, if they are asymptotically stationary. Unfortunately, however, all empirical researches reported in the literature unanimously and unambiguously find that they are nonstationary, which is attributable to structural breaks, switching regimes and/or near-unit roots,[6] and endogenous

[5]The empirical asset pricing literature often uses the term, Fama-French regressions to refer to multi-factor pricing models containing size or firm distress factors. For instance, a model including a momentum factor in addition to the three Fama-French factors is called, four-factor Fama-French model. See Carhart (1997) for details. Following this convention, we regard Fama-French regressions as multi-factor models in contrast to the CAPM.

[6]The reader is referred to Jacquier, Polson and Ross (1994, 2004), So, Lam and Li (1998) and Kim, Lee, and Park (2009) for the evidence of nonstationarity in stock return volatilities.

due to leverage effects. As shown in Chung and Park (2007), the nonstationary volatilities generally affect the limit distributions and invalidate the standard tests.[7] The negligence or misspecification of the time-varying and stochastic volatilities would therefore have a first-order effect. The presence of leverage effects introduces endogeneity in volatilities, which makes it more complicated to deal with the nonstationarity of volatilities.[8] It would certainly be a challenging problem to statistically analyze regressions with endogenous nonstationary stochastic volatilities.

To analyze Fama-French regressions, we derive a continuous time multifactor pricing model and consider the corresponding panel regression. Our model is very general in the sense that it allows for time-varying and stochastic volatilities, which are both nonstationary and endogenous. The error term is just given as a general martingale differential, consisting of two components, namely, the common component and the idiosyncratic component, which are independent of each other. The common component is specified as having volatility driven by the market, but otherwise it is entirely unrestricted. We may of course permit the presence of endogenous nonstationarity in the volatility process of the common component. On the other hand, the idiosyncratic component is only assumed to be cross-sectionally independent and have an asymptotically stationary volatility process. Our specification for the idiosyncratic component is therefore also very flexible and unrestrictive. In fact, the only meaningful restriction imposed on our error component model is that its nonstationary volatility component is generated exclusively by the market. This implies in particular that only the market risk is non-diversifiable over time.[9] Our specification of the error components is justified both theoretically and empirically in the paper.

For the statistical analysis of our model, we develop a new methodology relying on the sampling at random intervals in lieu of fixed intervals, and using the realized variance measure at a higher frequency to estimate the variance of the resulting sample. Our approach exploits a well known theorem in the theory of stochastic processes, due to Dambis, Dubins and Schwarz, which is often referred to as the DDS theorem.[10] It implies that any realization from a continuous martingale can be regarded as a realization from Brownian motion if it is read using the clock running at a speed inversely proportional to its quadratic variation. At least on its continuous part, a martingale generated with an arbitrary volatility process can therefore be converted into a Brownian motion simply by a time change in sampling. Consequently, general martingale differentials now become Brownian differentials, which are independent and identically distributed normals.[11] The DDS theorem is not directly applicable if the error process is discountinuous and has jumps. However, our approach remains to be valid asymptotically also in this case, as long as jumps are exogenous and occurs

[7] They also show that the regressions may even become spurious in case that the nonstationary volatilities are excessive.

[8] For more details of the leverage effects, the reader is referred to Harvey and Shepard (1996), Jacquier, Polson and Ross (2004), Yu (2005) and Kim, Lee, and Park (2009).

[9] As shown in Park (2002), the usual law of large numbers does not hold in the presence of nonstationary volatility.

[10] Readers are referred to, e.g., Revuz and Yor (1994) for more details about the DDS theorem.

[11] For the application of this approach in the univariate setup, see Phillips and Yu (2005), and Andersen, Bollerslev and Dobrev (2007). It has been more systematically and rigorously developed recently by Park (2009).

intermittently. This is because our statistical theory only requires asymptotic normality, not normality in finite samples, of the regression errors after time change.

For our model, we may use the market volatility to set the required random sampling intervals. This is because only the market volatility drives the endogenous nonstationarity in our error component model. As long as the volatility in the common component is taken care of by sampling at proper random intervals, the errors become asymptotically normal. The variance of the errors collected at the random intervals is also determined by the idiosyncratic component, but its volatility is asymptotically stationary as in the standard regression model. Moreover, the error variance can be estimated readily by the realized variance obtained from higher frequency observations available at each random interval. Our methodology therefore utilizes observations at both high and low frequencies. We use observations at a high frequency to set the random sampling intervals, and to estimate the variance of collected samples. On the other hand, are used samples collected at a low frequency to analyze the main regressions. They are analyzed at a low frequency to avoid distortions caused by excessive volatilities existing at high frequency observations. At the same time, however, we do not discard the available observations at the higher frequency, i.e., we also use them to deal with time-varying and stochastic volatilities in observations collected at a low frequency.

With this new econometric methodology in hand, we revisit the classic issues in empirical asset pricing. We estimate and test the CAPM and various multi-factor Fama-French models on five data sets of daily equity returns, which consist of three sets of decile portfolios sorted respectively by size, book-to-market ratio (B/M), and past performances (momentum); a set of twenty five portfolios sorted by size and B/M; and a set of thirty portfolios from different industries. For our random time regressions, we select the sampling intervals using the realized variance series of the daily excess market returns by setting the realized variance over each random sampling interval at the level comparable to the average realized variance of the monthly excess market returns. In the paper, we compare the results from our random time regressions with those from fixed time regressions using monthly observations.

We find that conventional regressions on fixed time intervals yield confusing test results. Specifically, with the fixed time sampling, we cannot reject the CAPM on the size portfolios and the B/M portfolios even if the estimated market betas cannot explain the higher risk premium generated by small size or high book-to-market ratio.[12] This result is inconsistent with the vast amount of literature on the existence of size and value premia of stock returns. Furthermore, when we incorporate the B/M or the size factor into the CAPM regression on each of the corresponding data sets, the fixed-time monthly OLS regressions cannot reject the two-factor models with even higher p-values, stating that one cannot statistically reject neither CAPM nor respective two-factor models on those portfolios. However, when all three factors are included, i.e., when the Fama-French 3-factor model is used for estimation on 25 portfolios sorted both by size and B/M, we have a flat rejection of the model. That is, this conventional method ignoring time-varying volatilities gives logically inconsistent test results.

[12]For the industry portfolios, the CAPM is not rejected either. However, it is rejected in case of the momentum portfolios.

Meanwhile, our random sampling approach based on time change decisively rejects the CAPM, reproducing the asset pricing anomalies compatible with the previous literature. Then, we estimate the two-factor models on each corresponding portfolios to find that the B/M factor is indeed a valid pricing factor explaining variations in stock returns due to different book-to-market ratios. However, the test result shows that the size factor is not sufficient to capture the cross sections of stock returns. Consistent with these findings, the three-factor model on 25 portfolios is rejected, and it turns out to be closely related to the small firm effect. Therefore, the random sampling approach offers a reliable and correct statistical method to estimate and test multi-factor asset pricing models with high frequency data. Related, we find that the estimates of beta coefficients in most cases studied are not critically different across the two econometric procedures. Thus, their differences seem to come mainly from the estimates of constant terms and variance-covariance matrix of residual terms, implying the importance of properly treating highly persistent stochastic volatilities of residual terms.

Finally, when applied to the industry portfolios, we again obtain similar results: the conventional method cannot reject the CAPM, despite significant deviations of abnormal returns from zero, resulting in the rejection of the CAPM in the random sampling case. The main reason for the rejection turns out to be the portfolio returns from consumer product companies. This only prevails in the random sampling case. We find that the returns from the consumer goods industry help explain the size effect of the Fama-French portfolios, especially the returns of the micro cap, growth firms. In sum, our empirical results coherently show that by appropriately handling stochastic volatilities, our method provides an accurate statistical procedure for both estimation and testing, without losing the attractive features of OLS regression. Thus, the good news that we want to convey is that empirical researchers can run OLS regressions of multi-factor pricing models using data sets in any (especially high) frequencies and accurately test the adequacy of those models, provided that the persistent market stochastic volatilities are well treated using our time-change method. Another related point to be made from our empirical result is that we still need valid pricing factors to explain cross sectional behaviors of stock returns via factor models.

The rest of the paper is organized as follows. In Section 2, we develop a continuous time multi-factor model of asset returns with stochastic volatilities and propose a panel regression model based on our theoretical framework. Section 3 presents a statistical procedure to analyze our model and asymptotic theories. In so doing, we also provide a statistical toolkit necessary for our empirical analysis. Section 4 describes the data sets used in our empirical analysis and provide empirical evidence for various specifications of our model. We then employ our new methodology to reexamine the CAPM and Fama-French regressions in Section 5, where empirical results from our analysis of Fama-French regressions are summarized and compared with other results reported in the literature. Section 6 concludes the paper. Useful lemmas and their proofs and the proofs of the main theorems are collected in Mathematical Appendix.

2. The Model and Assumptions

2.1 Theoretical Background

In this section, we derive a continuous-time beta model of asset returns on which our study of Fama-French regressions will be based. For the derivation of our model, we let π be the state price density given by

$$\frac{d\pi_t}{\pi_t} = \upsilon_t dt + \sum_{j=1}^{J} \tau_{jt} dV_{jt}, \tag{1}$$

where υ and (τ_j) are respectively drift and volatility processes, and (V_j) are independent Brownian motions. Subsequently, we specify the price process (P_i) of security $i = 1, \ldots, I$ as

$$\frac{dP_{it}}{P_{it}} = \mu_{it} dt + \sigma_t \left(\sum_{j=1}^{J} \kappa_{ij} dV_{jt} + \sum_{k=1}^{K} \lambda_{ik} dW_k \right) + \omega_{it} dZ_{it}, \tag{2}$$

where (μ_i) and (σ, ω_i) are drift and volatility processes, $(\kappa_{ij}, \lambda_{ik})$ are nonrandom coefficients, and (Z_i) and (W_k) are independent Brownian motions. Throughout the paper, we assume that

Assumption 2.1 (Z_i), (V_j) and (W_k) are independent Brownian motions such that (ω_i, Z_i) and (σ, V_j, W_k) are independent of each other, and such that (W_k) are Brownian motions independent of (V_j) conditional on σ.

State price density π is a process which makes (πP_i) a martingale for all $i = 1, \ldots, I$. It is well known since Harrison and Kreps (1979) that the existence of a state price density implies no arbitrage in the asset market. Throughout this section, we regard the instantaneous returns of a risky asset (dP_i/P_i) as the total returns from trading gains and the dividends paid between t and $t + dt$.

The drift term (μ_i) in (2) measures the risk return trade off, which will be determined below. For the specification of the diffusion term in (2), we introduce the component with the common volatility σ, as well as the component representing the idiosyncratic volatility (ω_i) specific to asset i. The common volatility component is then further divided into two components, the one involving (V_j) and the other consisting only of (W_k) that are independent of (V_j). In total, we have three terms describing the stochastic evolution of (dP_i/P_i). The first term involving (V_j) results from the covariations with the state price density π, and is therefore closely related to the pricing factor. The coefficient (κ_{ij}) measures the proportionality of the risk of security i relative to that of (V_j). Meanwhile, the second and third terms including (W_k) and (Z_i) have no bearing on π, and are not used to pin down the conditional mean component (μ_i) in (2). Instead, (W_k) and (Z_i) are viewed as fluctuations related to the part of a firm's cash flows which makes volatile the dividend process, and in turn, the gross return process. Alluded is that the remainder of the firms' cash flows will matter for valuing the equity of these firms and these are already included in the first part $(\sigma\kappa_{ij}dV_j)$.[13] What (W_k) and (Z_i) capture, we believe, are the fluctuations of the dividends of these firms which do not affect investors' discount factors for pricing purposes. This is a sensible assumption based on the empirical evidence that realized sample paths of firms' dividends are much more volatile than those of aggregate consumption growth or other macroeconomic variables, which ought to be associated with the state price density process.[14] Our setup states that if individual assets' payoffs are not correlated with the state-price density π, there will be no risk-return trade-off, which will be reflected via the terms in (μ_i) despite the volatility of asset returns.

Now we introduce pricing factors (Q_j), which we specify as

$$\frac{dQ_{jt}}{Q_{jt}} = \nu_{jt}dt + \rho_j\sigma_t dV_{jt} \tag{3}$$

for $j = 1, 2, ..., J$, where in particular (ν_j) are drift processes and (ρ_j) are nonrandom coefficients. In our specification, (Q_j) can be understood as the price of a portfolio made out of individual assets so that only systematic diffusion part relevant for pricing will remain.[15] For instance, we can think of the price of a portfolio with a long position for small firms (or high book-to-market ratio) and a short position for large firms (or low book-to-market ratio) as a factor. In the similar context, the first factor Q_1 is set to be the market factor with the unit corresponding coefficient of ρ's, i.e., $\rho_1 = 1$.[16] The subsequent derivation of our model depends crucially on the existence of a common volatility movement σ, especially with *constant* proportionality of risk for all assets. When the assumption of constant proportion is relaxed, we obtain a conditional beta representation, leading to conditional factor models. With some additional assumptions on the structure of betas, we may also consider such models as in Ang and Kristensen (2009). However, given our emphasis on the Fama-French regressions on stock returns, we do not pursue this route in this paper.

We derive our main formula by invoking the definition of the state price density. Under no arbitrage condition, we may easily deduce from (1), (2) and (3) that

$$\begin{aligned} \mu_{it} &= -\upsilon_t - \sum_{j=1}^{J} \kappa_{ij}\sigma_t\tau_{jt} \\ \nu_{jt} &= -\upsilon_t - \rho_j\sigma_t\tau_{jt} \end{aligned} \tag{4}$$

holds for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Note that the left-hand-side of the first equation in (4) represents a conditional mean return for holding security i. If $\kappa_{ij} = 0$ for all j, i.e., there is no risk for this asset's payoffs, then $-\upsilon_t$ is the resultant return process, thereby it stands for the instantaneously riskless rate denoted as r_t^f. As mentioned earlier, this equation

[13] It is an important task to quantify the relative contributions to explaining systematic return variations between the discount factor risk and cash flow risk. However, it is beyond the scope of our paper.

[14] Alternatively, one may consider that the asset market is incomplete in the sense that a source of shock (W_k) is either not priced or priced with a significantly downward bias via the conditional mean component.

[15] It is possible to allow for the presence of non-pricing factors (W_k) we introduced in (2). This will, however, make the Fama-French OLS regressions invalid, as we will explain later.

[16] That is, we regard the market as the asset that includes only the systematic component for pricing with the reference value of 1 for the beta, which will be introduced later.

describes the important characteristics of risk-return trade-off via conditional covariation between an asset's return and the discount factor π. Upon setting $\upsilon_t = -r_t^f$, it follows immediately from (4) that

$$\mu_{it} - r_t^f = \sum_{j=1}^{J} \beta_{ij}(\nu_{jt} - r_t^f), \tag{5}$$

where $\beta_{ij} = \kappa_{ij}/\rho_j$ for $i = 1, \ldots, I$ and $j = 1, \ldots, J$.

Now we have from (2), (3) and (5) that

$$\frac{dY_{it}}{Y_{it}} = \alpha_i + \sum_{j=1}^{J} \beta_{ij} \frac{dX_{jt}}{X_{jt}} + dU_{it} \tag{6}$$

with $\alpha_i = 0$ and

$$dU_{it} = \sigma_t \sum_{k=1}^{K} \lambda_{ik} dW_{kt} + \omega_{it} dZ_{it}, \tag{7}$$

if we define

$$\frac{dY_{it}}{Y_{it}} = \frac{dP_{it}}{P_{it}} - r_t^f dt$$
$$\frac{dX_{jt}}{X_{jt}} = \frac{dQ_{jt}}{Q_{jt}} - r_t^f dt,$$

for $i = 1, \ldots, I$ and $j = 1, \ldots, J$.

Our subsequent empirical analysis will be based on the model given by (6) and (7). Imposing the loadings of all other factors than the market factor to zero gives us the conventional CAPM regression in continuous time. One important restriction given in this model is that the constant coefficient α_i is not present for all i in our theoretical models. Since we only use excess returns for both factors and test assets, $\alpha_i = 0$ must hold for all i. In this vein, we call α the pricing errors throughout the paper, where we write $\alpha = (\alpha_1, \ldots, \alpha_I)'$. Testing the hypothesis of $\alpha = 0$ has been a focal point of empirical asset pricing literature. Unlike the conventional discrete-time CAPM or multi-factor models, note that our continuous time model offers an error structure derived from the underlying asset pricing model. Therefore, the estimation of the model (6) and the related statistical inference require further elaboration. To tackle this, we develop our econometric method and procedure below.[17]

2.2 Regression Formulation

Our model (6) is formulated as an instantaneous regression, where both the regressand and regressors are measured over an infinitesimal time interval. The regressions for observations

[17] In our model, the error process U is assumed to be a continuous process. However, the assumption can be relaxed and we may allow for the presence of jumps. This will be discussed in the next section.

collected at any prescribed time intervals may easily be obtained from (6). If time series observations are collected over the intervals defined by

$$0 \equiv T_0 < T_1 < \cdots < T_N \equiv T \tag{8}$$

over the time interval $[0, T]$, then we have the corresponding regression model

$$\int_{T_{n-1}}^{T_n} \frac{dY_{it}}{Y_{it}} = \alpha_i (T_n - T_{n-1}) + \sum_{j=1}^{J} \beta_{ij} \int_{T_{n-1}}^{T_n} \frac{dX_{jt}}{X_{jt}} + (U_{iT_n} - U_{iT_{n-1}}) \tag{9}$$

with

$$U_{iT_n} - U_{iT_{n-1}} = \sum_{k=1}^{K} \lambda_{ik} \int_{T_{n-1}}^{T_n} \sigma_t dW_{kt} + \int_{T_{n-1}}^{T_n} \omega_{it} dZ_{it} \tag{10}$$

for $n = 1, \ldots, N$. In the paper, we consider both fixed and random sampling schemes. For the fixed sampling scheme, we set $T_n - T_{n-1}$ to be nonrandom and constant for all $n = 1, \ldots, N$, like a month or a year. Instead, we define (T_n) to be a non-decreasing sequence of stopping times, or a time change, for the random sampling scheme. In particular, we will use in the paper the time change given by the volatility process σ in the common volatility factor. As discussed, σ is the volatility of the market factor introduced earlier below (3).

For our random sampling scheme, we let

$$dS_t = \frac{dX_{1t}}{X_{1t}} = \frac{dQ_{1t}}{Q_{1t}} - r_t^f dt \tag{11}$$

be the instantaneous market excess return, and define the time change (T_n) as

$$[S]_{T_n} - [S]_{T_{n-1}} = \int_{T_{n-1}}^{T_n} \sigma_t^2 dt = \Delta \tag{12}$$

for $n = 1, \ldots, N$, where Δ is a fixed constant.[18] This compares with the corresponding fixed sampling scheme (T_n) given by

$$T_n = (n/N)T \tag{13}$$

for $n = 1, \ldots, N$. In particular, if we set

$$T_n = n\Delta = (n/N)[S]_T, \tag{14}$$

then the random sampling scheme in (12) yields the same number of observations as the fixed sampling scheme in (13) for regression (9). In what follows, we will often simply refer to the sampling schemes in (12) with (14) and (13) as the random and fixed sampling schemes, respectively.

[18] The choice of Δ is an important problem, and the reader is referred to Park (2009) for more discussions on this subject. For the empirical analysis in the paper, we simply set Δ so that the random sampling scheme has the same number of observations N as the fixed sampling scheme at monthly frequency.

The motivation for our random sampling scheme (12) is to effectively deal with the endogenous nonstationarity of market volatility σ in the common error component of (10). It is well known and clearly demonstrated in the literature that the market volatility has an autoregressive root that is very close to unity. Also, its leverage effect on the market excess return is quite strongly negative. The reader is referred to Jacquier, Polson and Rossi (1994, 2004) and Kim, Lee and Park (2009) for more discussions on the nonstationarity and leverage effect of market volatility. In this situation, the usual law of large numbers and central limit theory do not hold and hence the usual chi-square tests for inference in regression (9) are invalid as shown in e.g., Park (2002). This poses a serious problem in analyzing Fama-French regressions. Under the random sampling scheme, however, we have

$$\int_{T_{n-1}}^{T_n} \sigma_t dW_{kt} =_d \mathbb{N}(0, \Delta) \tag{15}$$

for all $n = 1, \ldots, N$ and $k = 1, \ldots, K$, and that they are independent of each other. Here and elsewhere in the paper, we use $\mathbb{N}$ to signify normal distribution. This is due to a theorem by Dambis, Dubins and Schwarz, which will be called the DDS theorem in the paper. Of course, the normality in (15) only applies to the random sampling scheme.

The idiosyncratic error component of (10) is expected to behave much more nicely. Under Assumption 2.1, $(\int_{T_{n-1}}^{T_n} \omega_{it} dZ_{it})$ becomes independent across i and has variance

$$\mathbb{E}\left(\int_{T_{n-1}}^{T_n} \omega_{it} dZ_{it}\right)^2 = \mathbb{E}\left(\int_{T_{n-1}}^{T_n} \omega_{it}^2 dt\right) \tag{16}$$

for each $i = 1, \ldots, I$. In what follows, we assume

Assumption 2.2 For all $i = 1, \ldots, I$, we have

$$\frac{1}{N}\sum_{n=1}^{N}\int_{T_{n-1}}^{T_n} \omega_{it}^2 dt \to_p \varpi_i^2$$

as $N \to \infty$, for some $\varpi_i^2 > 0$.

Assumption 2.2 is not stringent and should be satisfied widely. It holds under mild regularity conditions if, for instance, the volatilities generated by the idiosyncratic component over the random sampling intervals are asymptotically stationary. In particular, the presence of nonstationarity is not allowed in the idiosyncratic error component of our model. Note that we still permit endogeneity in (ω_i). In the special case where the idiosyncratic volatilities (ω_i) are independent of the driving Brownian motions (Z_i), we have $\int_{T_{n-1}}^{T_n} \omega_{it} dZ_{it} =_d \mathbb{MN}\left(0, \int_{T_{n-1}}^{T_n} \omega_{it}^2 dt\right)$, where $\mathbb{MN}$ denotes mixed normal distribution.[19]

[19] This will be the case, if there is no leverage effect on the asset return generated from the idiosyncratic error component.

For the statistical inference in our model, we need an estimate for the error covariance matrix for regression (9). It is easy to obtain the asymptotic error covariance matrix implied by our error component model (10). Under the random sampling scheme, note that we have

$$\mathbb{E}(U_{iT_n} - U_{iT_{n-1}})^2 = \Delta \sum_{k=1}^{K} \lambda_{ik}^2 + \mathbb{E}\left(\int_{T_{n-1}}^{T_n} \omega_{it}^2 dt\right)$$

$$\mathbb{E}(U_{iT_n} - U_{iT_{n-1}})(U_{jT_n} - U_{jT_{n-1}}) = \Delta \sum_{k=1}^{K} \lambda_{ik}\lambda_{jk}$$

for all $0 \leq i \leq I$ and $0 \leq i \neq j \leq I$. Therefore, if we define $U_{T_n} - U_{T_{n-1}} = (U_{1T_n} - U_{1T_{n-1}}, \ldots, U_{IT_n} - U_{IT_{n-1}})'$, then we would expect to have

$$\mathbb{E}(U_{T_n} - U_{T_{n-1}})(U_{T_n} - U_{T_{n-1}})' \approx \Sigma$$

asymptotically, where Σ is a matrix with the i-th diagonal entry $\Delta \sum_{k=1}^{K} \lambda_{ik}^2 + \varpi_i^2$ and (i, j)-th off-diagonal entry $\Delta \sum_{k=1}^{K} \lambda_{ik}\lambda_{jk}$. Subsequently, we call Σ the asymptotic error covariance matrix for our regression (9).[20]

The asymptotic error covariance matrix Σ can be estimated using two different approaches. As in the conventional approach, we may estimate Σ by

$$\hat{\Sigma} = \frac{1}{N} \sum_{n=1}^{N} (U_{T_n} - U_{T_{n-1}})(U_{T_n} - U_{T_{n-1}})'. \tag{17}$$

Clearly, we have $\hat{\Sigma} \to_p \Sigma$ as $N \to \infty$, if we assume some extra regularity conditions to ensure that

$$\frac{1}{N} \sum_{n=1}^{N} \left(\int_{T_{n-1}}^{T_n} \omega_{it} dZ_{it}\right)^2 \to_p \varpi_i^2, \ \frac{1}{N} \sum_{n=1}^{N} \left(\int_{T_{n-1}}^{T_n} \omega_{it} dZ_{it}\right)\left(\int_{T_{n-1}}^{T_n} \omega_{jt} dZ_{jt}\right) \to_p 0 \tag{18}$$

for all i and for all $i \neq j$. It is easy to see that (18) holds under appropriate assumptions, due in particular to (16) and Assumption 2.2, and the independence of $(\int_{T_{n-1}}^{T_n} \omega_{it} dZ_{it})$ across $i = 1, \ldots, I$.

Furthermore, we may estimate the asymptotic error covariance matrix Σ using

$$\tilde{\Sigma} = \frac{1}{N} \int_0^T [U, U']_t \, dt, \tag{19}$$

where $[U, U']$ is the matrix of quadratic variations and covariations of $U = (U_1, \ldots, U_I)'$.

[20]For K finite and known, our model imposes some restrictions on the asymptotic error covariance matrix Σ. However, these restrictions will not be exploited in the paper, since the inference on K is beyond the scope of this paper.

Note that

$$[U_i]_{T_n} - [U_i]_{T_{n-1}} = \Delta \sum_{k=1}^{K} \lambda_{ik}^2 + \int_{T_{n-1}}^{T_n} \omega_{it}^2 dt$$

$$[U_i, U_j]_{T_n} - [U_i, U_j]_{T_{n-1}} = \Delta \sum_{k=1}^{K} \lambda_{ik}\lambda_{jk}$$

for all $1 \leq i \leq I$ and $1 \leq i \neq j \leq I$. Therefore, we have $\tilde{\Sigma} \to_p \Sigma$ as $N \to \infty$, which holds without any extra regularity conditions.

Due to Assumption 2.1, the usual condition for exogeneity of the regressors in (9) holds and the OLS procedure is valid for regression (9) for our random sampling scheme as well as the fixed sampling scheme. To see this more clearly, we let

$$\mathcal{F}_n = \sigma\left(\Big(U_{it}, i = 1, \ldots, I,\, t \leq T_n\Big), \Big(X_{jt}, j = 1, \ldots, J,\, t \leq T_{n+1}\Big)\right),$$

$n = 1, \ldots, N$, for our fixed or random sampling scheme (T_n). Then we may easily see that the regressors $(\int_{T_{n-1}}^{T_n} dX_{jt}/X_{jt})$, $j = 1, \ldots, J$, are all $\mathcal{F}_{n-1}$-measurable, and the regression errors $(U_{iT_n} - U_{iT_{n-1}})$ satisfy the orthogonality condition

$$\mathbb{E}\left[U_{iT_n} - U_{iT_{n-1}} \Big| \mathcal{F}_{n-1}\right] = 0$$

for $i = 1, \ldots, I$, as required for the validity of the OLS regression in (9). Recall in particular that we assume in Assumption 2.1 (W_k) are Brownian motions independent of (V_j) conditional on σ.

3. Statistical Procedure and Asymptotic Theory

In this section, we introduce the actual statistical procedure to analyze our model, and develop their asymptotic theory. For our development, it will be convenient to rewrite our model (9) as a more conventional regression. Therefore, we rewrite our model (9) as

$$y_{ni} = \alpha_i c_n + \sum_{j=1}^{J} \beta_{ij} x_{nj} + u_{ni}, \tag{20}$$

where

$$y_{ni} = \int_{T_{n-1}}^{T_n} \frac{dY_{it}}{Y_{it}}, \qquad c_n = T_n - T_{n-1},$$

$$x_{nj} = \int_{T_{n-1}}^{T_n} \frac{dX_{jt}}{X_{jt}}, \qquad u_{ni} = U_{iT_n} - U_{iT_{n-1}} \tag{21}$$

for $n = 1, \ldots, N$ and $i = 1, \ldots, I$. Under Assumptions 2.1 and 2.2, our choice of random sampling time (T_n) yields a regression model with errors, which are devoid of endogenous nonstationarity in volatility and have asymptotically stationary volatilities. Note in particular that the regression errors (u_n), $u_n = (u_{n1}, \ldots, u_{nI})'$, are approximately multivariate normal with mild heterogeneity, even in the presence of very general form of stochastic volatility on the underlying error process. In what follows, we let $y_n = (y_{n1}, \ldots, y_{nI})'$ and $x_n = (x_{n1}, \ldots, x_{nJ})'$.

For the subsequent development of our procedure and theory, we assume that

Assumption 3.1 $N^{-1}\sum_{n=1}^N x_n x_n' \to_p \Lambda > 0$ and $N^{-1/2}\sum_{n=1}^N x_n u_n' \to_d \mathbb{N}(0, \Lambda \otimes \Sigma)$, as $N \to \infty$.

Assumption 3.1 is necessary for all our regression asymptotics, and holds under very general conditions. For the expositional convenience, we just present the necessary high-level assumptions instead of laying out the details of required technical conditions.

Of course, (y_n) and (x_n) are not directly observable, and have to be estimated. We assume throughout the section that a sample providing observations for

$$(Y_{i,m\delta}, X_{j,m\delta}) \tag{22}$$

is available for $m = 0, \ldots, M$ with δ-interval in time, for each of $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Moreover, from $(X_{1,m\delta})$ and $(r^f_{m\delta})$, we obtain the observations $(S_{m\delta})$ for the excess market return process S introduced in (11) as

$$S_{m\delta} = \frac{X_{1,m\delta} - X_{1,(m-1)\delta}}{X_{1,(m-1)\delta}} = \frac{Q_{1,m\delta} - Q_{1,(m-1)\delta}}{Q_{1,(m-1)\delta}} - r^f_{(m-1)\delta}\,\delta$$

for $m = 1, \ldots, M$. We let $M\delta = T$, so that T is the horizon of the sample with size M collected at δ-interval in time. Our subsequent procedure is based on the asymptotic theory requiring $\delta \to 0$ and $T \to \infty$. In particular, δ should be small relative to T.[21]

To implement our approach based on regression (20) under the random sampling scheme, we need to estimate the time change (T_n), which we discuss below. If δ is small relative to T, we may estimate the quadratic variation $[S]$ of the excess market return process S using $(S_{m\delta})$. Indeed, if we set

$$[S]^\delta_t = \sum_{m\delta \le t} (S_{m\delta} - S_{(m-1)\delta})^2,$$

then we may expect $[S]^\delta \approx [S]$ over $[0, T]$ if δ is small enough compared with T. Once, we obtain an estimate $[S]^\delta$ of $[S]$, the corresponding estimate of the time change (T_n) may easily be obtained, accordingly as in (12), for a prescribed value of Δ. We propose the estimate (T^δ_n) of (T_n), which is given by

$$T^\delta_n = \delta \operatorname*{argmin}_{1 \le \ell \le M} \left| \sum_{m=1}^{\ell} (S_{m\delta} - S_{(m-1)\delta})^2 - n\Delta \right|, \tag{23}$$

[21] We use daily observations over approximately forty-five years for the empirical analysis in the paper, for which we believe our asymptotics are highly suitable. Of course, our theory allows for observations collected at intraday ultra-high frequencies. However, they appear to introduce much more noise than signal to our inference procedure especially if used over a long sampling horizon.

and define $M_n = \delta^{-1}T_n^\delta$ for each $n = 1, \ldots, N$. For the fixed time sampling scheme, we may set $M_n = \delta^{-1}T_n$ with T_n defined in (13) in what follows.

Now we define

$$y_{ni}^\delta = \sum_{m=M_{n-1}+1}^{M_n} \frac{Y_{i,m\delta} - Y_{i,(m-1)\delta}}{Y_{i,(m-1)\delta}}, \qquad c_n^\delta = T_n^\delta - T_{n-1}^\delta,$$
$$x_{nj}^\delta = \sum_{m=M_{n-1}+1}^{M_n} \frac{X_{j,m\delta} - X_{j,(m-1)\delta}}{X_{j,(m-1)\delta}}, \tag{24}$$

correspondingly as (21). It is quite obvious that (y_{ni}^δ), (c_n^δ), (x_{nj}^δ) get close to (y_{ni}), (c_n), (x_{nj}) under appropriate conditions as $\delta \to 0$ and $T \to \infty$. As in (20), we consider

$$y_{ni}^\delta = \alpha_i c_n^\delta + \sum_{j=1}^J \beta_{ij} x_{nj}^\delta + u_{ni}^\delta \tag{25}$$

for $n = 1, \ldots, N$ and $i = 1, \ldots, I$, where $(y_{ni}^\delta), (c_n^\delta)$ and (x_{nj}^δ) are introduced in (24) and (u_{ni}^δ) is defined by $u_{ni}^\delta = U_{iT_n^\delta} - U_{iT_{n-1}^\delta}$. It is intuitively clear that regression (25) has the same statistical properties as regression (20) if δ becomes sufficiently small relative to T. Note that we have a sample of size N to fit regression (25), which is formulated using a sample of size M in (22) with $M > N$. We call the latter *the original sample*, and the former *the regression sample*.

We need to introduce some technical conditions to ensure that the regression (25) constructed from discrete samples is asymptotically equivalent to our original regression (20) in continuous time.

Assumption 3.2 We let (a) $a_T(t-s) \leq \int_s^t \sigma_u^2 du \leq b_T(t-s)$ for all $0 \leq s \leq t \leq T$ with (a_T) and (b_T) depending only upon T, (b) $\sup_{t\geq 0} |\nu_{jt} - r_t^f| = O_p(1)$ for all $j = 1, \ldots, J$, (c) $\inf_t X_{jt} > 0$ and $\sup_{0\leq t\leq T} X_{jt} = O_p(c_T)$ for all $j = 1, \ldots, J$ with (c_T) depending only upon T, and (d) $\sup_{t\geq 0} \omega_{it} = O_p(1)$ for all $i = 1, \ldots, I$. Furthermore, we set (e) $\delta = O(T^{-4-\varepsilon}(a_T^2/b_T^7 c_T^4))$ for some $\varepsilon > 0$.

Under Assumption 3.2, we have

Theorem 3.1 For all $i = 1, \ldots, I$ and $j = 1, \ldots, J$,

$$\max_{1\leq n\leq N} \left| c_n^\delta - c_n \right|, \ \max_{1\leq n\leq N} \left| x_{nj}^\delta - x_{nj} \right|, \ \max_{1\leq n\leq N} \left| u_{ni}^\delta - u_{ni} \right|, \ \max_{1\leq n\leq N} \left| y_{ni}^\delta - y_{ni} \right| = o_p(N^{-1/2})$$

as $N \to \infty$.

Our regression (25) can be analyzed exactly as the standard multivariate regression model. In particular, the single equation OLS estimators for (α_i) and (β_{ij}) are fully efficient asymptotically. Therefore, we may run the OLS regression on (25) for each $i = 1, \ldots, I$.

This does not require the estimation of the asymptotic error covariance matrix Σ introduced earlier in the previous section. However, we need to estimate Σ for the test of a joint hypothesis involving multiple regression coefficients across $i = 1, \ldots, I$. For the estimation of Σ in regression (25), we may follow the usual two step procedure: In the first step, we estimate (α_i) and (β_{ij}) for each i by the single equation method. Then we use the fitted residuals in the second step to estimate Σ, which can be obtained as usual by

$$\hat{\Sigma}^\delta = \frac{1}{N} \sum_{n=1}^{N} \hat{u}_n^\delta \hat{u}_n^{\delta\prime}, \tag{26}$$

where $\hat{u}_n^\delta = (\hat{u}_{n1}^\delta, \ldots, \hat{u}_{nI}^\delta)'$ with $(\hat{u}_{ni}^\delta)$ being the fitted residual from regression (25) for equation i.

The error variance estimate $\hat{\Sigma}^\delta$ is expected to behave well only when $N >> I$, i.e., the size of regression sample is substantially bigger than the number of cross sectional units.[22] In our approach, there is another way to estimate the asymptotic error variance Σ using the original sample. The estimator would be useful especially when N is small relative to I.[23] It is indeed well defined even if $N < I$, as long as the size M of the original sample is large enough. To introduce the estimator more explicitly, we let $(\hat{\alpha}_i)$ and $(\hat{\beta}_{ij})$ be the OLS estimators of (α_i) and (β_{ij}) obtained from regression (25). Moreover, we define

$$\hat{U}_{i,m\delta} - \hat{U}_{i,(m-1)\delta} = \frac{Y_{i,m\delta} - Y_{i,(m-1)\delta}}{Y_{i,(m-1)\delta}} - \hat{\alpha}_i \delta - \sum_{j=1}^{J} \hat{\beta}_{ij} \frac{X_{j,m\delta} - X_{j,(m-1)\delta}}{X_{j,(m-1)\delta}}$$

and

$$\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta} = \big(\hat{U}_{1,m\delta} - \hat{U}_{1,(m-1)\delta}, \ldots, \hat{U}_{I,m\delta} - \hat{U}_{I,(m-1)\delta}\big)'.$$

Then the asymptotic error variance of regression (25) can be estimated by

$$\tilde{\Sigma}^\delta = \frac{1}{N} \sum_{m=1}^{M} \big(\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta}\big)\big(\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta}\big)'. \tag{27}$$

From Theorem 3.1, it is well expected that

Corollary 3.2 We have

$$\hat{\Sigma}^\delta = \hat{\Sigma} + O_p(N^{-1/2}), \qquad \tilde{\Sigma}^\delta = \tilde{\Sigma} + O_p(N^{-1/2})$$

for all large N.

In the CAPM and Fama-French regressions, it is of one of the main interests to test for the hypothesis

$$\mathbb{H}_0 : \alpha_1 = \cdots = \alpha_I = 0. \tag{28}$$

[22] The estimator $\hat{\Sigma}^\delta$ defined in (26) even has a rank deficiency and becomes singular if $N < I$.

[23] Suppose, for instance, we run regressions at yearly frequency, when the data are available at daily frequency, for forty years on the panel consisting of twenty-five cross sectional units.

The rejection of the hypothesis implies that the proposed model is not a true model and presumably requires a new factor.

The Wald test for the hypothesis can be easily formulated in our model (25), which may simply be regarded as the classical multivariate regression. The test statistic $\tau(\alpha)$ is defined by

$$\tau(\alpha) = \left(c'c - c'X(X'X)^{-1}X'c\right)^{-1}\hat{\alpha}'\bar{\Sigma}^{-1}\hat{\alpha}, \tag{29}$$

where c is an N-dimensional vector with c_n^δ as its n-th component and X is an $N \times J$ matrix with x_{nj}^δ as its (n, j)-th element, and $\bar{\Sigma} = \hat{\Sigma}$ or $\tilde{\Sigma}$. The test statistic $\tau(\alpha)$ has chi-square limit distribution with I-degrees of freedom. As discussed, we need some extra technical conditions if we use $\hat{\Sigma}$. It is also possible to use F-distribution after an appropriate adjustment for the degrees of freedom, as in Gibbons, Ross and Shanken (1989).[24] We may similarly test the hypothesis $\mathbb{H}_0 : \beta_{1j} = \cdots = \beta_{Ij} = 0$ for some factor j, using the statistic

$$\tau(\beta_j) = \left(x_j'x_j - x_j'X_j^c(X_j^{c\prime}X_j^c)^{-1}X_j^{c\prime}x_j\right)\hat{\beta}_j'\bar{\Sigma}^{-1}\hat{\beta}_j \tag{30}$$

where x_j is an N-dimensional vector with x_{nj}^δ as its n-th component and X_j^c is an $N \times J$ matrix defined by deleting the j-th column from X and adding c as one of its columns.

In order are some discussions on how we deal with the presence of jumps. Our theoretical development thus far assumes that the error process is given by a process with a continuous sample path a.s. However, for the validity of our econometric methodology, we do not need to assume that the error process is continuous. Indeed, our random sampling scheme is well expected to remove endogenous and nonstationary volatilities even in the presence of jumps. In this case, the DDS theorem does not apply and the regression errors are not in general normally distributed. Nevertheless, this does not affect our statistical theory, since it does not rely on the normality of the regression errors. In fact, following Choi and Park (2010), it is rather straightforward to show that Assumption 2.1 continues to hold in general if the error processes (U_i) are discontinuous and have jumps, with Σ given by the probability limit of $\hat{\Sigma}$ and $\tilde{\Sigma}$ introduced respectively in (17) and (19). A variety of jumps may be allowed as long as they occur exogenously in discrete time intervals. Moreover, we can show as in Choi and Park (2010) that the presence of jumps in (U_i) does not affect the asymptotic validity of all our subsequent procedures to analyze continuous time processes using discrete observations.

To be more consistent with our theoretical model, however, we assume that jumps are generated independently from the continuous part of the model and do not include any information on the model parameters. Therefore, jumps are regarded as pure noise. Accordingly, we simply get rid of the observations that appear to be contaminated with jumps for our empirical analysis in the paper. We first use a test by Lee and Mykland (2008) to find the locations of jumps. Once we find their locations, we identify the sampling intervals $[T_{n-1}^\delta, T_n^\delta]$ to which they belong, and simply discard the corresponding regression

[24]Strictly speaking, their test, often referred to as the GRS test in the literature, is not applicable in our context, since we do not assume normality. Of course, the estimation samples would be closer to normal under the random sampling scheme, and it would be more appropriate to use the random sampling scheme for the GRS test. We do not report their test in the paper, however, since in our case the degrees of freedom adjustment is negligible and their tests always yield the same results qualitatively as the Wald tests.

samples.[25] It is also possible that we test for the presence of jumps in each of the time intervals $[T^\delta_{n-1}, T^\delta_n]$, $n = 1, \ldots, N$, using the test developed by, e.g., Barndorff-Nielson and Shepard (2004b), and delete the regression samples from any of the time intervals which are tested positive. This procedure, however, makes sense only when sufficiently large enough number of the original samples exist in all of the time intervals.

There are various methods developed in the literature that are comparable to our procedure in the paper. Andersen, Bollerslev, Diebold and Wu (2006), Barndorff-Nielsen and Shephard (2004a) and Todorov and Bollerslev (2007) all consider the inferential problem in continuous time regression model similar to ours. Indeed, we may directly apply their methods to estimate (β_i) in our regression model (6).[26] However, their approach is different from ours in that they fix T and let $\delta \to 0$. They focus more on the analysis of quadratic covariations of the regressands and regressors in continuous time over a fixed time interval. It would therefore be more appropriate to apply their methods for ultra-high frequency samples observed over a relatively short time horizon. In contrast, our methodology would be more useful to analyze continuous time regression model over longer time horizons, since we require $T \to \infty$ as well as $\delta \to 0$. For the inference on constant term (α_i) in regression (6), none of the aforementioned existing methods is applicable and it is absolutely necessary to utilize samples over long time horizons. In particular, all other existing methods are not applicable to test for the hypothesis (28).

The original Fama-French regressions and their variants have largely been analyzed in discrete time models using low-frequency observations spanning relatively long time horizons. It is possible to accommodate the presence of nonstationary stochastic volatilities in discrete time framework. In fact, various discrete-time regression models with nonstationary stochastic volatilities are suggested and studied by several authors including Hansen (1995), Chung and Park (2007) and Xu (2007). In particular, we may apply the methodologies developed in Hansen (1995) and Chung and Park (2007) to do inference in appropriate discrete-time models corresponding to our continuous-time model (6). However, the form of nonstationary stochastic volatility we may consider in discrete-time model is rather limited and somewhat unrealistic. The required statistical procedure to properly deal with the presence of nonstationary stochastic volatility is nevertheless quite complicated and difficult to implement. On the other hand, our continuous time approach permits truly general nonstationary stochastic volatility, and provides a very simple yet extremely powerful methodology to effectively deal with it.

[25] Of course, this pretesting on jumps would render the size of the subsequent test deviate from its nominal test. This is, however, ignored for simplicity.

[26] As shown in Barndorff-Nielsen and Shephard (2004a), (β_i) in (6) can be estimated consistently simply by the usual high-frequency regression without constant term, if $\delta \to 0$ with T fixed. It can be shown that the regression continues to yield a consistent estimate for (β_i) under our setup requiring $T \to \infty$. The inclusion of constant term (α_i) does not affect the consistency of the estimate for (β_i), as long as the integrated regressors $(\int_0^T dX_{jt}/X_{jt})$ are not exceedingly explosive.

4. Data and Preliminary Analysis

4.1 Data

This section describes the data sets used in our empirical analysis. We make use of decile portfolios stratified by sizes, book-to-market ratios (B/M), and past performances. We also use 25 portfolios sorted by sizes and B/M and 30 industry portfolios. All the data sets are available at Kenneth French's web page.[27] For pricing factors, we adopt the market (MKT), the size (SMB), and the B/M (HML), often referred to as the Fama-French factors, and the momentum factor (MMT). The data sets cover the period of July, 1963 and December, 2008, and all of the returns in the data sets are of the daily frequency and annualized. Table 1 presents summary statistics of the factors and the corresponding portfolio returns. Specifically, Panel A reports means and standard deviations of the factors, together with correlations across each other. High Sharpe ratios of HML and MMT state that buying and holding distressed firms or better performing firms would have been lucrative investment strategies during this period. In terms of correlations, both SMB and HML have moderately negative correlations with MKT, while MMT is weakly, negatively correlated with MKT. Correlations across SMB, HML, and MMT are small. Panel B of Table 1 reports means and standard deviations of annualized returns stratified into ten portfolios. The eleventh row in each group refers to portfolio strategies with long positions of high returns and short positions of low returns, often called the hedged portfolio returns.[28] The size strategy yields about 1.6% per annum, while the book-to-market strategy earns about 5.6% per annum. During this period, the momentum strategy of buying past winners and selling past losers produces 18% per year, which is quite substantial. These summary statistics suggest that they are good candidates for pricing factors, as discussed in the previous literature. How about the volatility structures of these portfolio returns? We delve into this issue in the next subsection.

4.2 Preliminary Analysis

Our factor pricing model specified in (6) and (7) imposes some special error structure in the Fama-French regressions, which motivated us to invent a new methodology. Before we reexamine the Fama-French regressions using our methodology, it is therefore necessary that we investigate whether various specifications of our model are empirically justifiable. For this purpose, we consider the conventional 3-factor Fama-French regression which uses 25 portfolio returns sorted by size and book-to-market ratio as regressands and Fama-French factors as regressors.

An important implication of the Assumption 2.1 is that the error processes (dU_i) in (6) are correlated cross-sectionally due to the presence of the common component (dW_k).[29] To

[27] http://mba.tuck.darthmouth.edu/pages/faculty/ken.french/index.html

[28] This, respectively, corresponds to (i) the returns from the smallest size (1st group) of market equity minus the returns from the largest size (10th group) for the size strategy, (ii) the returns from the highest B/M (10th group) minus the lowest B/M (1st group) for the book-to-market strategy, and (iii) the returns from the winner (10th group) minus the loser (1st group) in case of the momentum strategy.

[29] Independence of (ω_i, Z_i) and (ω_i) is not likely to be empirically testable by construction. However,

see how much cross-correlations exist among the errors in a typical factor pricing model, we test for diagonality of the covariance matrix of fitted residuals estimated in the usual way from the aforementioned conventional Fama-French regression. We use the residuals from both fixed time regression and our new random time regression, which was formally introduced in (25), and apply the LM test of diagonality suggested by Breusch and Pagan (1980). As can be seen in Table 2, the null of diagonality is rejected in both cases, indicating that there exist cross-correlations among the errors, which may be generated by the common error component (dW_k) or a common error component not captured by the factors already included.

Our specification of the pricing formula given in (6) and (7) presumes the presence of common volatility factor σ in the diffusion terms (dV_j) of all pricing factors (dQ_j/Q_j) specified in (3) and more importantly in the common component (dW_k) of the errors (dU_i). Especially, we assume that the common volatility factor σ is given by the volatility of the excess market return which is defined from the first pricing factor as in (11). To see if this assumption is empirically justified, we plot the quadratic variation series of daily excess market return along with those of the daily 25 fitted residuals from the Fama-French three factor regression in Figure 1.[30] It is clear that the 25 residual quadratic variation series follow closely that of the excess market return signified by the dark thicker line, thereby strongly supporting our assumption that the volatility of excess market return represents the common component of individual residual volatilities.

To more carefully investigate the appropriateness of our assumption, we also estimate the instantaneous variances of the 25 fitted residuals and compare them with those of the excess market returns. Note that the quadratic variation series presented in Figure 1 can be regarded as the estimates for the integrated variances of the fitted residuals, and that the instantaneous variances are the time derivatives of integrated variances. For the actual estimation, we apply the local linear smoothing method to the quadratic variation series we obtained earlier and compute their time derivatives. We also conduct the principal component analysis to extract the leading factor from the estimates of the instantaneous variances for the fitted residuals. The leading factor is expected to represent the nonstationary volatility factor in the fitted residuals. Our results are provided in Figure 2. The magnitudes of the estimated instantaneous variances of the fitted residuals are not exactly identical to those of the market, or those of the extracted leading factor. However, it is rather strongly suggested that they fluctuate together. In particular, their cycles are remarkably overlapped. For instance, the timings of peaks and troughs for the instantaneous variance series of the market and the extracted leading factor appear to coincide perfectly.

We also investigate whether the errors (u_{ni}) are orthogonal to the regressors (c_n) and (x_{nj}) in our regression (20), especially under random sampling scheme. Of course, this is crucial for the validity of OLS procedure. Indeed, they may be correlated with each other. It happens, for instance, if the pricing factors (dQ_j/Q_j) in (3) has nonpricing volatility components (dW_k), as well as the pricing volatility components (dV_j). To see whether the

whether there is a common component in error terms is a fair empirical question, which we illustrate in short.

[30]To obtain the residuals at the daily level, we use the coefficient estimates from the random time Fama-French three factor regression.

orthogonality between the regressors and the regression errors is a plausible tenet, we run the fixed time regression using monthly observations to estimate the regression coefficients, and use the estimates to obtain the fitted residuals at the daily frequency. Then we obtain the time change and compute the sample correlations between the regressors (c_n) and (x_{nj}), and the regression errors (u_{ni}), for the random time regression. If the assumed orthogonality does not hold, then we must have at least some evidence of nonzero correlation between the regressors and the regression errors under the random sampling scheme. The results are reported in Table 3. The values of the actual sample correlations are quite low for all regressors, supporting the validity of OLS in the random time regressions.

Lastly, based on all the results, we are ready to check if the volatility structure we impose on error terms is plausible and well treated by the random sampling scheme proposed in the paper. To empirically evaluate this issue, we consider a stochastic volatility model to measure the degree of persistency in the stochastic volatilities of regression errors in (20). Therefore, we specify $u_{ni} = \sqrt{f_i(v_{ni})}\varepsilon_{ni}$ for $n = 1, \ldots, N$ and $i = 1, \ldots, I$ with $\mathbb{E}\varepsilon_{ni}^2 = 1$, where (v_{ni}) is the latent volatility factor generated as $v_{ni} = \rho_i v_{n-1,i} + \eta_{ni}$ and (f_i) is the volatility function. We use the logistic function for the volatility function f_i, and allow for nonzero correlation between (ε_{ni}) and (η_{ni}) which represents the leverage effect.[31] The stochastic volatility model is fitted for each i using the fitted residuals from regression (25) based on the random sampling scheme, and the latent volatility factor is extracted using the conventional density-based Kalman filter method. For comparison, we also estimate the stochastic volatility model using the fitted residuals from the fixed time regression. The extracted volatility factors are given in Figures 3-5 and the estimated values of the AR coefficients (ρ_i) of the extracted volatility factors are presented in Table 4.

It seems evident that the extracted volatility factors of the residuals from random time regressions are not persistent. Note that the volatilities of the individual portfolios consist of both the nonstationary common trend and stationary idiosyncratic components, and only the nonstationary common component is corrected via our random sampling method. Thus, one may expect that the estimated AR coefficients in this case reflect only the stationary component of the extracted volatilities. Indeed, the average of the estimated AR coefficients is around 0.724 in the random time, which is stationary. This is in sharp contrast with the volatility factors extracted from the fixed time residuals, most of which have the estimated AR coefficient very close to unity. In addition, the observed high persistency of the volatility factors extracted from the fixed time residuals is quite similar to that of the fixed time excess market return, as can be seen in the first panel of Figure 6. The estimated AR coefficient of the fixed time market volatility factor is 0.9505.[32] On the other hand, the AR coefficient of the extracted volatility factor from the time changed market return is much smaller, indeed close to zero, and its sample path clearly shows no persistency as displayed in the second panel of Figure 6. Putting things together, the empirical results confirm that our volatility setup is realistic and properly handled with the random sampling scheme.

[31] The reader is referred to Kim, Lee and Park (2009) for more details about the stochastic volatility model and estimation methodology we use in the paper.

[32] This is also consistent with the well known fact that the stochastic volatility of market return is highly persistent. See aforementioned references for the nonstationarity in stock return volatilities.

5. Reexamination of Fama-French Regressions

5.1 Tests of the CAPM

In this section, we examine the CAPM regressions on five sets of daily portfolio returns. First three sets consist of eleven portfolios, ten of which are sorted out by a firm characteristic (sizes, B/M ratios or prior returns), and the eleventh one refers to the hedge portfolio explained in the previous section. The next set consists of thirty industry portfolios. Finally, the last data set comprises traditional 25 portfolios sorted by sizes and B/M ratios. As discussed in Section 2, we run regression (9) under the two sampling schemes, fixed time and random time. In case of the fixed time sampling, we construct monthly data by integrating portfolio returns over each month. For the random time sampling scheme, we follow (12) and set Δ at the level of quadratic variation comparable to the average, monthly excess market return. Tables 5 to 9 report estimates of alphas and betas with standard errors for each portfolio, followed by the Wald statistic defined in (29) to test if the model is rejected.

Table 5 reports results for the decile size portfolios, and the size strategy (1st−10th decile) portfolio. Beta estimates in both sampling schemes are close to each other and MKT mildly captures exposures to taking risks for small firms (i.e., beta is higher for small firms). However, comparing the alpha estimates, one can clearly see that there is a huge difference between the fixed time and the random time sampling schemes. There exists a significant risk component not captured by the market factor according to small firms' alpha estimates in the random sampling case, whereas the fixed sampling result is much weaker. As expected, the Wald test statistic states that the CAPM is not rejected in case of the fixed sampling regression, while p-value of the random sampling case is 0.0000, a clear rejection. Figure 7 displays this finding graphically. Fixed sampling results show a hump-shape of alpha estimates, which is somewhat confusing, if the size effect does matter. On the contrary, the random-sampling result with a proper treatment of stochastic volatilities, shows a nice emergence of monotonically decreasing size premium.

Table 6 reports basically the identical information for the book-to-market portfolios. However, this case shows another evidence that the conventional method fails in doing reliable statistical inferences. Unlike Table 5 with size-based portfolios, Table 6 report that both alphas and betas are similarly estimated, and the estimated amount of value premium is around 6% to 7% per annum. Figure 8 illustrates that the estimated alphas are similar across the two methods. However, when the Wald statistics are compared, the fixed sampling scheme cannot reject the CAPM, while the random sampling rejects the model with p-value of 0.0000. In addition, the estimated market betas dictate that the growth stocks are riskier than the value stocks, implying that the CAPM is probably not pricing these portfolios correctly, as shown by Fama and French (1993).

Based on the results, we suspect that the model is correctly rejected in the random time regressions, and the conventional fixed time regressions seem to have difficulty in doing this. However, at this stage, a natural question arises: Fama and French (1993) and many authors have used the conventional OLS with the Gibbons, Ross, and Shanken (GRS) tests to reject the CAPM and even various other Fama-French models. Why do our fixed time sampling results differ from the previous OLS results? Recall that the main difference between the conventional OLS and our fixed sampling OLS is the way that data series are constructed. The conventional monthly return data use two data points of asset prices between two consecutive months, while our data are constructed by integrating the daily data over a month in fixed sampling cases. The two methods would produce the same monthly data if the instantaneous returns were defined as the differentials of logarithm of prices, viz., $d\log(P_{i,t})$. However, our instantaneous returns are constructed as the ratios of the price differentials to previous prices, viz. $dP_{i,t}/P_{i,t-1}$, and under this definition the two data construction methods can produce substantially different monthly return data.

But, can we, then, achieve the same results by directly using the monthly return data with the conventional OLS machinery instead of using random sampling scheme on a higher frequency data, because both will take a look at the data at a frequency comparable to monthly frequency after all? Note that our model is written in continuous time, then aggregated over time to make the model testable in discrete time environment. As shown in Section 3, the asymptotics and resultant test statistics of the model are different from those of the discrete-time counterparts. If continuous time diffusion models better describe the actual market clearing processes, which we believe, then, these differences are critical in evaluating the empirical asset pricing models. To further investigate this point, we run the OLS regressions on the conventional monthly returns with the same decile portfolios. We find two interesting results. First, in both size and B/M based decile portfolios, the GRS statistics report p-values around 0.031 and 0.038 respectively. Therefore, the CAPM is not rejected at 3% despite the prevalent size or value effects. Second, as we vary the starting date of the data, p-values vary significantly between 0.002 and 0.208.[33] This result may be an indirect evidence of conditional factor models. But, even in conditional models, a final test on whether a model is rejected would be to look at whether or not the long-run average of alphas be zero.[34] Thus, the use of low frequency data does not necessarily give reliable and accurate test results. On the contrary, our random sampling results are quite robust to such variations. This is a subtle but an important point: Conventional methods may fail to reject a model too easily, and often produce puzzling results.

Table 7 and Figure 9 report the results on the momentum portfolios. The momentum strategy generates a huge average abnormal returns of 20% to 22% in both cases and the CAPM is decisively rejected in both sampling regressions. Thus, putting things together, unless deviations from the true model are really obvious, like in the case with the momentum factor, the conventional testing procedure is inoperable and fails to reject a proposed model too often. As emphasized in our earlier discussions, the failure of the conventional testing procedure is due to the fact that variance-covariance matrix of the error terms is very difficult to estimate in the presence of non-stationary stochastic volatilities. Indeed, existing empirical studies unequivocally show that they are nonstationary, though their sources may

[33] Although not monotonic, CAPM on size portfolios is more difficult to reject when the sample period gets longer, while the opposite is likely to be true for the CAPM regressions on value portfolios. We do not report the results as a separate table since similar exercises have been performed in other studies. Nevertheless, our argument here is germane and new in the context of testing factor pricing models with nonstationary volatilities in a high frequency setting.

[34] See Ang and Kristensen (2009) for more details. They test conditional factor pricing models using a non-parametric method.

differ. And the models with time-varying and stochastic volatilities would yield misleading results, as we discussed earlier. In addition, the presence of leverage effects, also prevalent in stock return data, brings about endogeneity in volatilities, which further complicates the treatment of the nonstationary volatilities. We also run a similar exercise for the 25 Fama-French portfolios sorted by sizes and B/M ratios and report the results in Table 8 and Figure 10. Now, even the fixed sampling scheme rejects the CAPM with p-value of 0.0000, which contradicts the test results with the decile portfolios. On the contrary, the random sampling method rejects the model, compatible with the results from the decile portfolios.

As a final exercise for the CAPM, we examine the unmanaged industry portfolios consisting of thirty groups and report the results in Table 9. The familiar story prevails again. Despite the similar estimates of alphas and betas on average, Wald test statistics say that the conventional approach cannot reject the CAPM at 4.53%, while our approach rejects the model with zero p-value. To analyze what makes the difference between the two approaches in this case, we plot the alphas connecting each industry that belongs to one of more broadly defined five groups of industries in Figure 11. Most conspicuous are the industries in consumer goods group featuring consistently positive alphas in our random time regression, while the fixed time regression produces a mixed bag of results. This suggests that consumption growth may be a valid pricing factor together with the financial market factor, which is reminiscent of consumption-based pricing models employing more flexible preferences such as Epstein and Zin (1989). Summing up, the random sampling method works reliably in a high-frequency environment, contrary to its fixed sampling counterpart. More importantly, all the test results for the CAPM based on the random sampling provide a strong case for multi-factor models.

5.2 Tests of the Fama-French Models

In this section, we investigate multi-factor models of asset returns. Continued from the previous section, we begin with two-factor models, incorporating the size, B/M or momentum factor into the CAPM on each of the corresponding decile portfolio data sets. In Tables 10 and 11, like the CAPM, the fixed time OLS regressions cannot reject the two-factor models with the factors (MKT,SMB) and (MKT,HML) at even higher p-values, stating that the size and B/M factors are relevant pricing factors, despite that the CAPM is not rejected on the same data sets. This is a contradicting result caused by the imprecise statistical method. Meanwhile, the random sampling result shows that the two-factor model with the B/M is not rejected at 8% of p-value for the 10 B/M-based portfolios, though the model with the size factor fails to explain the 10 size-based portfolios. That is, our method suggests that the B/M factor is indeed a valid pricing factor for explaining the variations of stock returns over the cross-section of B/M ratio groups, while the size factor may be insufficient to account for the spectrum of asset returns in light of the firm sizes. In addition, we want to note that this is consistent and plausible with the random-sampling CAPM results on size groups that are decisive rejections. Table 12 reports that the two-factor model with the momentum is rejected in both approaches.

One common feature in all three of the two factor models we consider here is that the abnormal returns of the hedged portfolios are not statistically different from zero, suggesting that pricing errors are small. What then drives the rejections of the model according to the Wald statistics in the random sampling case? A closer look at the Tables 10 and 12 reveals that other portfolios than those used in forming the hedged portfolios, such as Size 5 or Momentum 3, turn out to have significantly non-zero abnormal returns. Thus, the Wald test for all assets, compared to the test on a hedged portfolio alone, is a more stringent test for verifying if a factor model can explain all the returns considered than just the ones specifically aimed at matching certain characteristics. Therefore, if the results between the two tests in a factor pricing model clearly disagree, then the proposed model may need new factors because it is likely to have difficulty in fitting the returns sorted by other characteristics. Consistent with this view, Table 11 shows that the medium value stocks as well as the hedged portfolio do not have significantly non-zero abnormal returns, hence the model is not rejected.

Based on this observation and following the tradition, now we estimate and test the three-factor Fama-French model on the data set with 25 portfolios. Tables 13 and 14 show that both fixed and random time regressions reject the model. This result is somewhat anticipated from the random sampling results on the two factor model (MKT, SMB) in Table 10, where the size factor fails to explain the 10 size-based portfolios. Compared to the CAPM results on the 25 portfolios, the extent to which the model misbehaves appears to be smaller, yet the p-values based on Wald statistic imply a clear rejection of the Fama-French model, which requires careful scrutiny. The first panels for the fixed and random time regressions in Figure 12 display the deviations of alpha estimates from zero for the 25 portfolios. The first group consisting of the smallest stocks have the largest magnitude of deviations, which is a common feature in both the fixed sampling and random sampling cases. Related, the two graphs in second panels of Figure 12 connect the alpha estimates with either the same sizes or B/M groups. Compared to the corresponding graphs in Figure 10 for the CAPM, the lines are much closer to the horizontal axis of zero value and even the slope is obviously reversed in some cases. However, one observation, corresponding to (size, B/M) $= (1, 1)$ distinctively deviates from zero value, which seems to drive the rejection of the model. Note that this refers to small cap, growth stocks with low book-to-market ratios. To be more precise, we plot the average excess returns for each portfolio and the predicted returns from the Fama-French regressions in Figure 13, following Cochrane (2001, p441). It is easy to observe that the $(1, 1)$ portfolio is quite off from the 45 degree line compared to other portfolios and it displays a significant premium within the smallest B/M group.[35] This is a part of the size premium, yet we must note that the lower, left panels in Figures 12 and 13 display that the small growth stocks show the stark contrast to the typical pattern of the size premium, hinting that the conventional size factor may be not enough in capturing this behavior.[36] This suggests that either an additional factor or a replacing factor may be

[35] This effect also appears in Cochrane (2001) yet with a much weaker pattern. We suspect that the difference comes from the data period which is between 1947 to 1996 in his case.

[36] There may be a common economic fundamental that affects both size and B/M portfolio returns in a different fashion than the conventional size and B/M factors do. Fama and French (1995) report that both the size and B/M premiums are related to the earnings of the firms. They find that the small firms have persistently lower earnings and the growth stocks have persistently high earnings, though the former link is weak. If persistent high earnings imply low cash flow risk, and the small firm effect is dominated by the

needed to justify premiums related to buying large cap stocks and selling small cap stocks in the group of firms with small distress.[37]

We recall that the portfolio returns from the consumer goods industry feature significant abnormal returns when CAPM is used. Admittedly, there is no direct connection between the small growth stocks and the consumer goods industry. However, given the signifying role of consumption goods as a foundational link between a discount factor and asset prices, we believe that including the consumption sector returns as a factor is a worthy trial. Related, Lettau and Ludvigson (2001) found that a macroeconomic factor that captures the consumption wealth ratio can substantially improve the performance of the consumption CAPM. Motivated by those findings, we form a consumer goods industry factor, called the CMR factor as the excess returns on the portfolio of the firms producing consumer goods. Regarding the definition of the consumer goods sector, we simply use the returns from the consumer goods sector out of the data with 5-industry category available in the Kenneth French's data library. Then, we run regressions of multi-factor models incorporating the consumer goods industry factor (CMR) on the 10 size-based and B/M-based portfolios, and the 25 portfolios to see if a model with the CMR factor can help explain the behaviors of asset returns, especially the small growth stocks. We report the results from three-factor models, which include the market, the CMR and either the size or B/M factor.[38]

Table 15 displays the results from the three-factor models with the CMR factor on the portfolios sorted by the size and B/M ratio.[39] In comparison with the results in Tables 10 and 11, we observe that p-values increase in each case and the beta coefficients for the CMR are mostly significant. Thus, it is inferred that the CMR factor helps explain both the value-based and size-based portfolios. Especially, the model with the market, B/M and CMR factors is not rejected at 10%.[40] Figure 14 shows that the overall fit is good for both of the three-factor models. Based on this positive result, we select these two three-factor models to investigate the 25 portfolios. Unfortunately, both models are rejected at 0.0000 of p-values, but we find that in case of the model with the market, B/M and CMR factors, the pricing error for the (1,1) portfolio gets significantly reduced and the overall fit appears to be generally comparable to that of the Fama-French model. This is summarized in Table 16 and Figure 15. For the model with the market, size and CMR, overall fit is worse and the Wald statistic is higher, hence it is clearly inferior to the model with the market, B/M and CMR, as well as the traditional Fama-French model. The results suggest that the size factor is not entirely satisfactory in terms of capturing the cross sectional behaviors of asset

value effect, this story may justify why a small, growth stock is a good asset to short-sell. However, it still does not explain why the large cap within the smallest B/M ratio is a risky bet as illustrated in the lower left panel of the Figure 12.

[37] As one of the usual suspects, we try the momentum factor, making a four-factor model. However, we find that the momentum factor is orthogonal to the size effect within low book-to-market ratios.

[38] We also tried the two-factor model consisting only of the market and the CMR. To conserve the space we do not report the results here but it appears that the CMR factor captures some of the size premiums, but not the value premiums.

[39] From now on, we only report the random sampling results, because the fixed sampling results on the industry portfolio do not pick up the CMR factor as shown in Table 9 and Figure 11.

[40] When we estimated a four-factor model, i.e., the Fama-French 3-factor model with the CMR factor, we find that the p-values get lower to 0.0000 and 0.0003 for the B/M and size portfolios, respectively.

returns and the returns from the consumer goods industry are useful in complementing this deficiency. However, further investigation is necessary to incorporate both the size and consumer factors into the factor pricing model since the four factor model has a much lower p-value in spite of the additional factor.

6. Conclusion

This paper develops a new econometric framework and tools to analyze multi-factor asset pricing models. We consider a continuous-time factor model with a specific error component structure consistent with an underlying asset pricing theory. We show that our error structure is empirically supported as well. It is well known that asset returns have highly persistent, time-varying and stochastic volatilities which can substantially harm the reliability of estimation and testing of asset pricing models, especially when a higher sampling frequency is chosen. We overcome this difficulty by using samples collected at random intervals, instead of those sampled at calendar time. Specifically, the clock is running inversely proportional to the market volatility. That is, a time interval is short when volatilities are high, and vice versa. Under our random sampling scheme, Fama-French regressions may simply be regarded as the classical regressions having normal errors with variance given by the averaged quadratic variation of the martingale differential errors. Our method is quite simple: We run the usual OLS regressions on the time-changed data so that potential complexities from handling high-frequency data and nonstationary volatilities do not arise.

We apply our methods to various portfolios sorted by certain characteristics used to identify pricing factors. We find that the tests based on conventional regression models on fixed time intervals often produce invalid and contradicting test results. These issues do not prevail in the random time regressions. Even in comparison with the conventional regressions on lower frequency data, our test appear to yield more reliable results. Our additional empirical findings can be highlighted as follows. First, size premium is still an important part of cross sectional return variations. According to the fixed sampling scheme, size strategy produces around 0.6% annually, while our random sampling regression states around 5.6% per annum. In addition, even after including the size factor, the size-based portfolios are not fully explained. This is not the case for the value-based portfolios. Second, we also find that the three-factor or four-factor Fama-French models cannot fully account for the size, value, and momentum premia, and the rejection of the three-factor model appears to mainly come from the small firms with low book-to-market ratios in case of the 25 portfolios sorted by the size and book-to-market ratios. Of course, this is well documented in Fama and French (1993). However, we want to point out that although Fama and French argue that their model still explains cross sectional variations very well despite this puzzling behavior, this effect does not only survive over time but appears to get even stronger according to our empirical results. Third, our CAPM and multi-factor results on industry portfolios suggest some potential role to be played by an additional factor based on consumer goods industry sector. It is noteworthy that this anomaly does not prevail in the fixed sampling case.

In an attempt to find out a better factor pricing model, we form a consumer factor using the returns from the consumer goods industry sector and test the model on all the portfolios we consider. Interestingly, we find that this consumer factor has some explanatory power on the returns of the small growth stocks. This suggests that factors motivated by economic theories can shed light on the issue of explaining the cross sections of stock returns, because these factors are likely to be robust to alternative sets of portfolios to be explained. Related, a recent work by Fama and French (2008) shows that there are many other asset pricing anomalies related to net stock issues, accruals, asset growth, and profitability. Some of them are even robust across all size groups, and the conventional Fama-French model is not able to deliver satisfying performance. In this vein, a quest for valid pricing factors thereby a new and better asset pricing model is still an important task to sharpen our understanding on how financial markets reward taking systematic risks and uncertainties. We hope that our newly developed tool is a useful addition to this enterprise.

Mathematical Appendix

Useful Lemmas and Their Proofs

Let (A_t) be an Ito process given by

$$\frac{dA_t}{A_t} = f_t dt + g_t dB_t,$$

where (B_t) is a Brownian motion with respect to a filtration $(\mathcal{F}_t)$, to which (f_t) and (g_t) are adapted. We assume

Assumption A1 For all $0 \leq s \leq t \leq T$, $a_T(t-s) \leq \int_s^t g_u^2 du \leq b_T(t-s)$, where a_T and b_T are some constants depending only upon T.

Assumption A2 $\sup_{t\geq 0} |f_t| = O_p(1)$.

Assumption A3 $\inf_{t\geq 0} A_t > 0$ and $\sup_{0\leq t\leq T} A_t = O_p(c_T)$, with (c_T) depending only on T.

In the subsequent development of our theory, we assume that the Ito process A satisfies Assumptions A1-A3.

Lemma A1 We have

$$\sup_{|s-t|\leq\delta} |A_t - A_s| = O_p\left(\delta^{1/2-\varepsilon} b_T^{1/2} c_T\right)$$

for any $\varepsilon > 0$, uniformly in $0 \leq s, t \leq T$.

Proof of Lemma A1 Write

$$A_t - A_s = \int_s^t f_u A_u du + \int_s^t g_u A_u dB_u$$

for $0 \leq s \leq t \leq T$. We may easily deduce that

$$\left| \int_s^t f_u A_u du \right| \leq \left(\sup_{0 \leq t \leq T} A_t \right) \int_s^t |f_u| du = O_p(\delta c_T) \tag{31}$$

uniformly in $0 \leq s, t \leq T$, due to Assumptions A2 and A3. Moreover, if we let $C_t = \int_0^t g_s A_s dB_s$, then C is a continuous martingale with

$$\begin{aligned} [C]_t - [C]_s &= \int_s^t g_u^2 A_u^2 du \\ &\leq \left(\sup_{0 \leq t \leq T} A_t^2 \right) \int_s^t g_u^2 du = O_p(\delta b_T c_T^2) \end{aligned} \tag{32}$$

uniformly in $0 \leq s, t \leq T$. Since C is a continuous martingale, we may represent it as

$$C_t = (D \circ [C])_t \tag{33}$$

with the DDS Brownian motion D of C, due to the celebrated theorem by Dambis, Dubins and Schwarz in e.g., Revuz and Yor (1993, Theorem 5.1.6, p173). Now we may deduce from (33), together with the modulus of continuity of Brownian motion and (32), that

$$\begin{aligned} \sup_{|t-s| \leq \delta} |C_t - C_s| &\leq \sup_{|t-s| \leq \delta} \left| (D \circ [C])_t - (D \circ [C])_s \right| \\ &\leq \sup_{|t-s| \leq \delta} \left| [C]_t - [C]_s \right|^{1/2-\varepsilon} = O_p \left(\delta^{1/2-\varepsilon} b_T^{1/2} c_T \right) \end{aligned} \tag{34}$$

for any $\varepsilon > 0$, uniformly in $0 \leq s, t \leq T$. Upon noticing that $c_T \delta = o\left(\delta^{1/2-\varepsilon} b_T c_T\right)$ for any $\varepsilon > 0$, the stated result follows immediately from (31) and (34). The proof is therefore complete. □

Lemma A2 We have

$$\max_{1 \leq m \leq M} \left| \int_{(m-1)\delta}^{m\delta} \frac{dA_t}{A_t} - \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}} \right| = O_p(\delta^{1-\varepsilon} b_T c_T)$$

for any $\varepsilon > 0$.

Proof of Lemma A2 Define

$$\begin{aligned} R_m &= \int_{(m-1)\delta}^{m\delta} \frac{dA_t}{A_t} - \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}} \\ &= \int_{(m-1)\delta}^{m\delta} \left(1 - \frac{A_t}{A_{(m-1)\delta}} \right) \frac{dA_t}{A_t} = \int_{(m-1)\delta}^{m\delta} \frac{A_t - A_{(m-1)\delta}}{A_{(m-1)\delta}} \left(f_t dt + g_t dB_t \right). \end{aligned} \tag{35}$$

We have

$$\int_{(m-1)\delta}^{m\delta} \frac{A_t - A_{(m-1)\delta}}{A_{(m-1)\delta}} f_t dt \leq \frac{1}{\inf_t A_t} \left(\sup_{(m-1)\delta \leq t \leq m\delta} |A_t - A_{(m-1)\delta}| \right) \int_{(m-1)\delta}^{m\delta} f_t dt$$
$$= O_p\left(\delta^{3/2-\varepsilon} b_T^{1/2} c_T\right) = O_p\left(\delta^{1-\varepsilon} b_T c_T\right) \tag{36}$$

uniformly in $m = 1, \cdots, M$, due in particular to Lemma A1. Moreover,

$$\int_{(m-1)\delta}^{t} \frac{A_s - A_{(m-1)\delta}}{A_{(m-1)\delta}} g_s dB_s$$

is a continuous martingale, whose increment in quadratic variation over interval $[(m-1)\delta, m\delta]$ is bounded by

$$\int_{(m-1)\delta}^{m\delta} \left(\frac{A_t - A_{(m-1)\delta}}{A_{(m-1)\delta}} \right) g_t^2 dt \leq \frac{1}{\inf_t A_t^2} \left(\sup_{(m-1)\delta \leq t \leq m\delta} |A_t - A_{(m-1)\delta}|^2 \right) \int_{(m-1)\delta}^{m\delta} g_t^2 dt$$
$$= O_p\left(\delta^{2-\varepsilon} b_T^2 c_T^2\right).$$

Consequently, we may show that

$$\int_{(m-1)\delta}^{m\delta} \frac{A_t - A_{(m-1)\delta}}{A_{(m-1)\delta}} g_t dB_t = O_p\left(\delta^{1-\varepsilon} b_T c_T\right) \tag{37}$$

uniformly in $m = 1, \cdots, M$, using the same argument as in the proof of Lemma A2. The stated result now follows immediately from (35), (36) and (37). □

Subsequently, we let

$$dF_t = \frac{dA_t}{A_t} \qquad \text{and} \qquad dG_t = g_t dB_t,$$

and define

$$[F]_t^\delta = \sum_{m\delta \leq t} \left(\frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}} \right)^2$$
$$[G]_t^\delta = \sum_{m\delta \leq t} (G_{m\delta} - G_{(m-1)\delta})^2.$$

Lemma A3 We have

$$\sup_{0 \leq t \leq T} \left| [G]_t^\delta - [G]_t \right| = O_p\left((\delta T)^{1/2} b_T \right).$$

Proof of Lemma A3 Under Assumption A1, the stated result follows immediately from Lemma A3.1 of Park (2009).

Lemma A4 We have

$$\sup_{0\le t\le T}\left|[F]_t^\delta - [G]_t\right| = O_p\left(\delta^{1/2-\varepsilon}Tb_T^{3/2}c_T^2\right)$$

for any $\varepsilon > 0$.

Proof of Lemma A4 Define

$$[F^\delta]_t = \sum_{m\delta\le t}(F_{m\delta}-F_{(m-1)\delta})^2 = \sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}\frac{dA_t}{A_t}\right)^2,$$

and note that

$$\left|[F]_t^\delta - [G]_t^\delta\right| \le \left|[F]_t^\delta - [F^\delta]_t\right| + \left|[F^\delta]_t - [G]_t^\delta\right|. \tag{38}$$

We may readily deduce from Lemmas A1 and A2 that

$$\begin{aligned}
\left|[F]_t^\delta - [F^\delta]_t\right| =& \sum_{m\delta\le t}\left[\left(\frac{A_{m\delta}-A_{(m-1)\delta}}{A_{(m-1)\delta}}\right)^2 - \left(\int_{(m-1)\delta}^{m\delta}\frac{dA_t}{A_t}\right)^2\right] \\
\le& \frac{2}{\inf_t A_t}\left(\max_{1\le m\le M}|A_{m\delta}-A_{(m-1)\delta}|\right) \\
& M\left(\max_{1\le m\le M}\left|\int_{(m-1)\delta}^{m\delta}\frac{dA_t}{A_t} - \frac{A_{m\delta}-A_{(m-1)\delta}}{A_{(m-1)\delta}}\right|\right) \\
=&(T/\delta)O_p\left(\delta^{1/2-\varepsilon}b_T^{1/2}c_T\right)O_p\left(\delta^{1-\varepsilon}b_Tc_T\right) = O_p\left(\delta^{1/2-\varepsilon}Tb_T^{3/2}c_T^2\right)
\end{aligned} \tag{39}$$

for all $0 \le t \le T$.

Moreover, it follows that

$$[F^\delta]_t = [G]_t^\delta + 2\sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}f_tdt\right)(G_{m\delta}-G_{(m-1)\delta}) + \sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}f_tdt\right)^2,$$

where we have

$$\sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}f_tdt\right)^2 \le MO_p(\delta^2) = O_p(\delta T)$$

and

$$\begin{aligned}
&\left|\sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}f_tdt\right)(G_{m\delta}-G_{(m-1)\delta})\right| \\
&\le \left[\sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}f_tdt\right)^2\right]^{1/2}\left[\sum_{m\delta\le t}(G_{m\delta}-G_{(m-1)\delta})\right]^{1/2} \\
&= O_p\left((\delta T)^{1/2}\right)O_p\left((Tb_T)^{1/2}\right) = O_p\left(\delta^{1/2}Tb_T^{1/2}\right)
\end{aligned}$$

uniformly in $0 \le t \le T$. Note that $\delta T = o\left(\delta^{1/2} T b_T^{1/2}\right)$. Consequently, we have

$$\left|[F^\delta]_t - [G]_t^\delta\right| = O_p\left(\delta^{1/2} T b_T^{1/2}\right) \tag{40}$$

uniformly in $0 \le t \le T$. The stated result follows from Lemma A3, and (38), (39) and (40). Note that

$$\delta^{1/2} T b_T^{1/2}, (\delta T)^{1/2} b_T = o\left(\delta^{1/2-\varepsilon} T b_T^{3/2} c_T^2\right),$$

and therefore, the terms we consider in Lemma A3 and (40) become negligible. □

In what follows, we let

$$H_t = \inf_{s>0}\left\{[G]_s > t\right\}$$

and analogously define

$$H_t^\delta = \inf_{s>0}\left\{[F]_s^\delta > t\right\}$$

for $0 \le t \le [G]_T$.

Lemma A5 We have

$$\sup_{0\le t\le [G]_T}\left|H_t^\delta - H_t\right| = O_p\left(\delta^{1/2-\varepsilon} T a_T^{-1} b_T^{3/2} c_T^2\right)$$

for any $\varepsilon > 0$.

Proof of Lemma A5 The proof is virtually identical to that of Corollary 3.3 of Park (2009), and therefore, it is omitted.

In the following lemma, we define M_n by $\delta M_n = H_{n\Delta}^\delta$ for $n = 1, \ldots N$.

Lemma A6 We have

$$\max_{1\le n\le N}\left|\int_{H_{(n-1)\Delta}}^{H_{n\Delta}} \frac{dA_t}{A_t} - \sum_{m=M_{n-1}+1}^{M_n} \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}}\right| = O_p\left(\delta^{1/4-\varepsilon} T^{1/2} a_T^{-1/2} b_T^{5/4} c_T\right)$$

for any $\varepsilon > 0$.

Proof of Lemma A6 We let

$$R_n = \int_{H_{(n-1)\Delta}}^{H_{n\Delta}} \frac{dA_t}{A_t} - \sum_{m=M_{n-1}+1}^{M_n} \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}},$$

and write

$$|R_n| \le |R_n^a| + \left|R_n^b\right|, \tag{41}$$

where

$$R_n^a = \int_{H_{(n-1)\Delta}}^{H_{n\Delta}} \frac{dA_t}{A_t} - \int_{H_{(n-1)\Delta}^\delta}^{H_{n\Delta}^\delta} \frac{dA_t}{A_t}$$

$$R_n^b = \int_{H_{(n-1)\Delta}^\delta}^{H_{n\Delta}^\delta} \frac{dA_t}{A_t} - \sum_{m=M_{n-1}+1}^{M_n} \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}}.$$

Moreover, we define

$$I_n = \min\left(H_{n\Delta}, H_{n\Delta}^\delta\right) \qquad \text{and} \qquad J_n = \max\left(H_{n\Delta}, H_{n\Delta}^\delta\right)$$

for $n = 1, \ldots, N$.

We have

$$|R_n^a| \leq \left| \int_{H_{(n-1)\Delta}}^{H_{n\Delta}} f_t dt - \int_{H_{(n-1)\Delta}^\delta}^{H_{n\Delta}^\delta} f_t dt \right| + \left| \int_{H_{(n-1)\Delta}}^{H_{n\Delta}} g_t dB_t - \int_{H_{(n-1)\Delta}^\delta}^{H_{n\Delta}^\delta} g_t dB_t \right|.$$

The first term is bounded by

$$2 \max_{1\leq n\leq N} \int_{I_n}^{J_n} f_t dt \leq 2 \left(\sup_{0\leq t\leq T} |f_t| \right) \max_{1\leq n\leq N} \left| H_{n\Delta} - H_{n\Delta}^\delta \right|$$

for all $n = 1, \ldots, N$, and the quadratic variation of the second term is bounded by

$$2 \max_{1\leq n\leq N} \int_{I_n}^{J_n} g_t^2 dt \leq 2b_T \max_{1\leq n\leq N} \left| H_{n\Delta} - H_{n\Delta}^\delta \right|$$

for all $n = 1, \ldots, N$. Clearly, the first term is of order smaller than that of the second term. Therefore, it follows from Lemma A5 that

$$R_n^a = O_p\left(\delta^{1/4-\varepsilon} T^{1/2} a_T^{-1/2} b_T^{5/4} c_T\right), \tag{42}$$

uniformly in $n = 1, \ldots, N$.

Furthermore, we have

$$\begin{aligned} \left| R_n^b \right| &\leq \sum_{m=M_{n-1}+1}^{M_n} \left| \int_{(m-1)\delta}^{m\delta} \frac{dA_t}{A_t} - \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}} \right| \\ &\leq \max_{1\leq n\leq N} \left| H_{n\Delta}^\delta - H_{(n-1)\Delta}^\delta \right| \max_{1\leq m\leq M} \left| \int_{(m-1)\delta}^{m\delta} \frac{dA_t}{A_t} - \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}} \right| \end{aligned}$$

for all $n = 1, \ldots, N$. However, we may readily deduce that

$$\max_{1\leq n\leq N} \left| H_{n\Delta}^\delta - H_{(n-1)\Delta}^\delta \right| \leq \max_{1\leq n\leq N} \left| H_{n\Delta} - H_{(n-1)\Delta} \right| + 2 \max_{1\leq n\leq N} \left| H_{n\Delta} - H_{n\Delta}^\delta \right|,$$

and

$$\max_{1\leq n\leq N}\left|H_{n\Delta}-H_{(n-1)\Delta}\right|\leq\frac{\Delta}{a_T}=O_p\left(a_T^{-1}\right).$$

Consequently, it follows from Lemma A2 that

$$R_n^b=O_p\left(\delta^{1-\varepsilon}a_T^{-1}b_Tc_T\right)\tag{43}$$

for any $\varepsilon>0$, uniformly in $n=1,\ldots,N$. Note that

$$\max_{1\leq n\leq N}\left|H_{n\Delta}-H_{n\Delta}^{\delta}\right|=O_p\left(\delta^{1/2-\varepsilon}Ta_T^{-1}b_T^{3/2}c_T^2\right)=o_p\left(a_T^{-1}\right),$$

due to Lemma A5. The stated result now follows immediately from (41), (42) and (43). Note that R_n^b is of order smaller than that of the first term of R_n^a. □

The Proofs of Theorems

Proof of Theorem 3.1 Throughout the proof, we set $T_n=H_{n\Delta}$, where H is introduced above Lemma A5. Note that $(Tb_T)^{-1/2}=O(N^{-1/2})$, since $N\Delta\leq Tb_T$ and Δ is constant. The result for (c_n) may easily be obtained if we let $X_1=A$ and apply Lemma A5. It follows that

$$\begin{aligned}\max_{1\leq n\leq N}\left|c_n^{\delta}-c_n\right|&=\max_{1\leq n\leq N}\left|(T_n^{\delta}-T_{n-1}^{\delta})-(T_n-T_{n-1})\right|\\&\leq 2\max_{1\leq n\leq N}\left|H_{n\Delta}-H_{n\Delta}^{\delta}\right|\\&=O_p\left(\delta^{1/2-\varepsilon}Ta_T^{-1}b_T^{3/2}c_T^2\right)=o_p((Tb_T)^{-1/2})=o_p(N^{-1/2}).\end{aligned}$$

Similarly, we may simply apply Lemma A6 with $X_j=A$, and note that

$$\frac{\delta^{1/4-\varepsilon}T^{1/2}b_T^{5/4}c_T}{a_T^{1/2}}=o\left(\frac{1}{T^{1/2}b_T^{1/2}}\right)=o(N^{-1/2}).$$

to deduce the stated result for (x_{nj}).

The proof for (u_{in}) is slightly more involved. Note that

$$\max_{1\leq n\leq N}\left|u_{ni}^{\delta}-u_{ni}\right|\leq 2\max_{1\leq n\leq N}\left|U_{iT_n^{\delta}}-U_{iT_n}\right|.\tag{44}$$

However, we have

$$U_{iT_n^{\delta}}-U_{iT_n}=\int_0^{T_n^{\delta}}\omega_{it}dZ_{it}-\int_0^{T_n}\omega_{it}dZ_{it},$$

whose quadratic variation is bounded by

$$\max_{1\leq n\leq N}\left|T_n^{\delta}-T_n\right|=\max_{1\leq n\leq N}\left|H_{n\Delta}^{\delta}-H_{n\Delta}\right|=O_p\left(\delta^{1/2-\varepsilon}Ta_T^{-1}b_T^{3/2}c_T^2\right)$$

uniformly in $n = 1, \ldots, N$, due to Lemma A5. It follows that

$$\max_{1\leq n\leq N} \left| U_{iT_n^\delta} - U_{iT_n} \right| = O_p\left((\delta^{1/2-\varepsilon} T a_T^{-1} b_T^{3/2} c_T^2)^{1/2} \right), \tag{45}$$

and therefore,

$$\max_{1\leq n\leq N} \left| u_{ni}^\delta - u_{ni} \right| = o(N^{-1/2}),$$

due to (44) and (45), and $(\delta^{1/2-\varepsilon} T a_T^{-1} b_T^{3/2} c_T^2)^{1/2} = o((Tb_T)^{-1/2}) = o(N^{-1/2})$.

To finish the proof, we note that

$$\left| y_{ni}^\delta - y_{ni} \right| \leq |\alpha_i| \left| c_n^\delta - c_n \right| + \sum_{j=1}^J |\beta_{ij}| \left| x_{nj}^\delta - x_{nj} \right| + \left| u_{ni}^\delta - u_{ni} \right|,$$

uniformly in $i = 1, \ldots, I$, from which and our previous results we may easily deduce the stated result for $(y_{ni}$. □

Proof of Corollary 3.2 We may readily deduce the stated result for $\hat{\Sigma}$ from

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N \hat{u}_n^\delta \hat{u}_n^{\delta\prime} &= \frac{1}{N} \sum_{n=1}^N u_n^\delta u_n^{\delta\prime} + O_p(N^{-1/2}) \\ &= \frac{1}{N} \sum_{n=1}^N u_n u_n' + O_p(N^{-1/2}), \end{aligned}$$

due to the well known regression asymptotics and Theorem 3.1.

For the proof of our result for $\tilde{\Sigma}$, we assume that $I = J = 1$ and suppress the subscript i and j for notational simplicity. The proof for the general case is essentially the same and can easily be established as in the simple case we consider here. We write

$$\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta} = (U_{m\delta} - U_{(m-1)\delta}) - R_{m\delta}$$

with

$$R_{m\delta} = (\hat{\alpha} - \alpha)\delta + (\hat{\beta} - \beta) \frac{X_{m\delta} - X_{(m-1)\delta}}{X_{(m-1)\delta}},$$

so that

$$\left(\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta} \right)^2 = (U_{m\delta} - U_{(m-1)\delta})^2 - 2(U_{m\delta} - U_{(m-1)\delta}) R_{m\delta} \tag{46}$$

for $m = 1, \ldots, M$. However, we have

$$\begin{aligned} \frac{1}{N} \sum_{m=1}^M R_{m\delta}^2 &\leq 2(\hat{\alpha} - \alpha)^2 \frac{\delta^2 M}{N} + 2(\hat{\beta} - \beta)^2 \frac{1}{N} \sum_{m=1}^M \left(\frac{X_{m\delta} - X_{(m-1)\delta}}{X_{(m-1)\delta}} \right)^2 \\ &= o(N^{-2}) + O(N^{-1}) = O(N^{-1}), \end{aligned} \tag{47}$$

and

$$\left|\frac{1}{N}\sum_{m=1}^{M}(U_{m\delta}-U_{(m-1)\delta})R_{m\delta}\right|$$
$$\leq \left[\frac{1}{N}\sum_{m=1}^{M}(U_{m\delta}-U_{(m-1)\delta})^2\right]^{1/2}\left[\frac{1}{N}\sum_{m=1}^{M}R_{m\delta}^2\right]^{1/2} = O(N^{-1/2}). \tag{48}$$

Now it follows immediately from (46), (47) and (48) that $\tilde{\Sigma}^{\delta} = \tilde{\Sigma} + O_p(N^{-1/2})$, and the proof is complete. □

References

Andersen, T.G., T. Bollerslev, F. Diebold and G. Wu (2006). "Realized beta: Persistence and predictability," in T. Fomby and D. Terrell (eds.), *Advances in Econometrics: Econometric Analysis of Economic and Financial Time Series*, Volume 20.

Andersen, T.G., T. Bollerslev and D. Dobrev (2007). "No-arbitrage semi-martingale restrictions for continuous-time volatility models subject to leverage effects, jumps and i.i.d. noise: Theory and testable distributional implications," *Journal of Econometrics*, 138, 125-180.

Ang, A. and D. Kristensen (2009). "Testing conditional factor models," memeographed.

Barndorff-Nielsen, O.E. and N. Shephard (2004a). "Econometric analysis of realized covariation: High-frequency covariance, regression and correlation in financial economics," *Econometrica*, 2, 1-48.

Barndorff-Nielsen, O.E. and N. Shephard (2004b). "Power and bipower variation with stochastic volatility and jumps," *Journal of Financial Econometrics*, 2, 1-48.

Carhart, M.M. (1997). "On persistence in mutual fund performance," *Journal of Finance*, 52, 57-82.

Choi, Y. and J.Y. Park (2010). "Generalized methods of moments for regressions in continuous time," mimeographed.

Chung, H. and J.Y. Park (2007). "Nonlinear nonstationary heterosckedasticity in regression," *Journal of Econometrics*, 137, 230-259.

Cochrane, J. (2001). *Asset Pricing*, Princeton University Press: Princeton, New Jersey.

Epstein, L., and S. Zin (1989). "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework," *Econometrica*, 57, 937-968.

Fama, E.F. and K.R. French (1992). "The cross-section of expected stock returns," *Journal of Finance*, 47, 427-465.

Fama, E.F. and K.R. French (1993). "Common risk factors in the returns on stock and bonds," *Journal of Financial Economics*, 33, 3-56.

Fama, E.F. and K.R. French (1995). "Size and book-to-market factors in earnings and returns," *Journal of Finance*, 50, 131-155.

Fama, E.F. and K.R. French (1996). "Multifactor explanations of asset pricing anomalies," *Journal of Finance*, 51, 55-84.

Fama, E.F. and K.R. French (2008). "Dissecting anomalies," *Journal of Finance*, 73, 1653-1678.

Fan, J. and I. Gijbels (1996). *Local Polynomial Modelling and Its Applications*, Chapman & Hall: London, UK.

Gibbons, M.R., S.A. Ross, J. Shanken (1989). "A test of the efficiency of a given portfolio," *Econometrica*, 57, 1121-1152.

Hansen, B.E. (1995). "Regression with nonstationary volatility," *Econometrica*, 63, 1113-1132.

Harvey, A.C. and N. Shephard (1996). "The estimation of an asymmetric stochastic volatility model for asset returns," *Journal of Business and Economic Statistics*, 14, 429-434.

Jacquier, E., N.G. Polson and P.E. Rossi (1994). "Bayesian analysis of stochastic volatility model," *Journal of Business and Economic statistics*, 12, 371-389.

Jacquier, E., N.G. Polson and P.E. Rossi (2004). "Bayesian analysis of stochastic volatility models with fat-tails and correlated errors," *Journal of Econometrics*, 122, 185-212.

Kim, H., H. Lee and J.Y. Park (2009). "A general approach to extract stochastic volatilities with an empirical analysis of volatility premium," memeographed.

Lee, S. and P.A. Mykland (2008). "Jumps in financial markets: A new nonparametric test and jump dynamics," *Review of Financial Studies*, 21, 2535-2563.

Lettau, M. and S. Ludvigson (2001). "Resurrecting the (C)CAPM: A cross sectional test when risk premia are time-varying," *Journal of Political Economy*, 109, 1238-1287.

Park, J.Y. (2002). "Nonlinear nonstationary heteroskedasticity," *Journal of Econometrics*, 110, 383-415.

Park, J.Y. (2009). "Martingale regressions for conditional mean models in continuous time," mimeographed.

Phillips, P.C.B. and J. Yu (2005). "Jackknifing bond option prices," *Review of Financial Studies*, 18, 707-742.

Revuz, D. and M. Yor (1994). *Continuous Martingale and Brownian Motion*, 2nd ed., Springer-Verlag: New York, New York.

So, M.K.P., K. Lam and W.K. Li (1998). "A stochastic volatility model with Markov switching," *Journal of Business and Economic Statistics*, 16, 244-253.

Todorov, V. and T. Bollerslev (2007). "Jumps and betas: A new framework for disentangling and estimating systematic risks," forthcoming in *Journal of Econometrics.*

Xu, K.-L. (2007). "Bootstrapping autoregression under nonstationary volatility," *Econometrics Journal*, 10, 1-27.

Yu, J. (2005). "On leverage in a stochastic volatility model," *Journal of Econometrics*, 127, 165-178

Tables and Figures

Table 1: Summary Statistics of Factors and Portfolio Returns

Panel A: Factors

			Correlations			
	Mean	Stdev	MKT	SMB	HML	MMT
MKT	0.0446	0.1515	1.0000	-0.2200	-0.4302	-0.0644
SMB	0.0152	0.0798	-0.2200	1.0000	-0.0623	0.0667
HML	0.0511	0.0746	-0.4302	-0.0623	1.0000	-0.1215
MMT	0.0985	0.1025	-0.0644	0.0667	-0.1215	1.0000

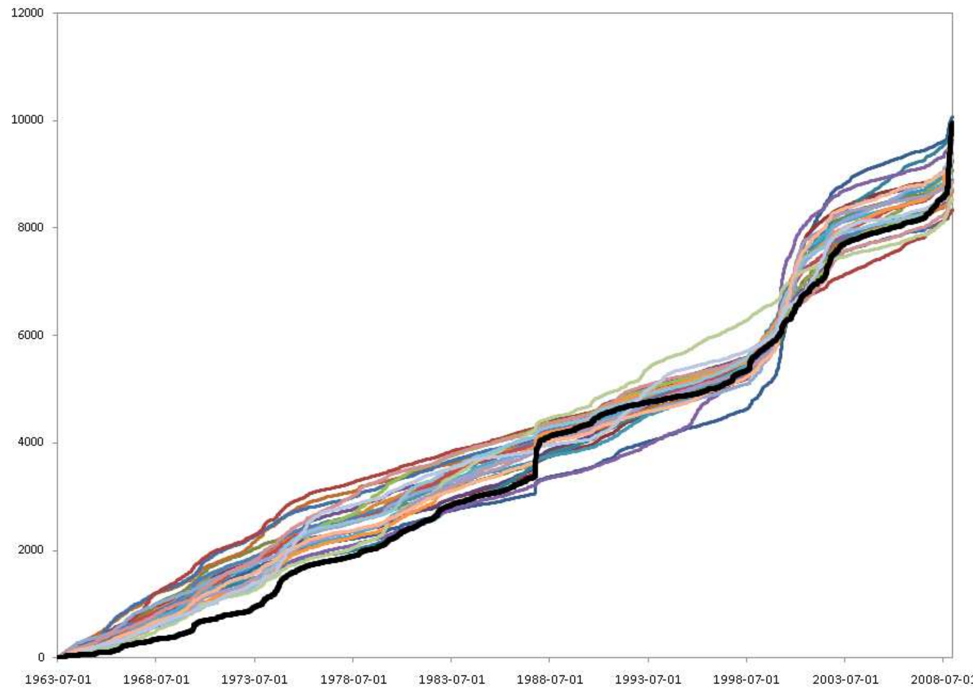
Panel B: Decile Portfolio Returns

Size	Mean	Stdev	B/M	Mean	Stdev	Momentum	Mean	Stdev
1 Small	0.0562	0.1267	1 Growth	0.0316	0.1779	1 Losers	-0.0584	0.2321
2	0.0559	0.1544	2	0.0449	0.1619	2	0.0068	0.1913
3	0.0646	0.1558	3	0.0499	0.1545	3	0.0382	0.1669
4	0.0619	0.1551	4	0.0499	0.1542	4	0.0389	0.1594
5	0.0656	0.1546	5	0.0477	0.1521	5	0.0295	0.1529
6	0.0581	0.1478	6	0.0572	0.1447	6	0.0448	0.1493
7	0.0595	0.1493	7	0.0665	0.1436	7	0.0450	0.1493
8	0.0561	0.1528	8	0.0724	0.1486	8	0.0757	0.1529
9	0.0507	0.1510	9	0.0831	0.1512	9	0.0685	0.1631
10 Big	0.0406	0.1608	10 Value	0.0878	0.1631	10 Winners	0.1240	0.2044
1-10	0.0156	0.1230	10-1	0.0562	0.1220	10-1	0.1824	0.1906

Table 2: Test of Diagonality of Variance-Covariance Matrix

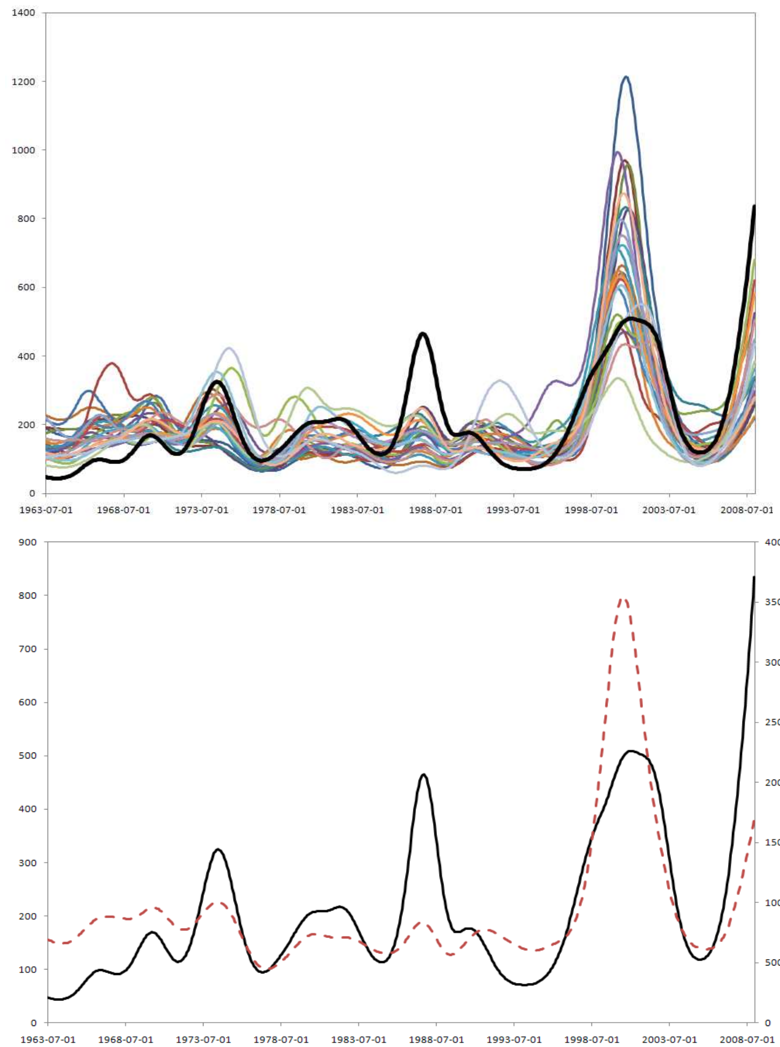
Model	LM Test	p-Value
Fixed Time Regression	15185.9766	0.0000
Random Time Regression	6974.5703	0.0000

Figure 1: Quadratic Variations of Market Return and Fitted Residuals



Notes: There are 26 lines in the figure. The thick darker line represents the quadratic variation series, estimated by the realized variance series, of the daily excess market returns, and the remaining 25 lines represent the quadratic variation series of the 25 fitted residuals from the Fama-French three factor regressions on 25 portfolios sorted by 5 size and 5 B/M ratio groups.

Figure 2: Instantaneous Variances of Market Return and Fitted Residuals



Notes: Top panel presents 26 lines, of which the thick darker line signifies the estimated instantaneous variance series of the daily excess market returns, and the remaining 25 lines represent those of the 25 fitted residuals from the three factor Fama-French regressions on 25 portfolios sorted by 5 size and 5 B/M ratio groups. The instantaneous variances are estimated by the derivatives of the quadratic variations that we obtain by using the local linear smoothing method with the rule of thumb bandwidth selection provided by Fan and Gijbels (1996). The bottom panel compares the estimated instantaneous variance series of the excess market returns (solid line) with that of the leading factor of the 25 fitted residuals (dashed line).

Table 3: Correlations between Residuals and Regressors in Random Time Regressions

	Correlation Coefficients				|Correlation Coefficients|			
(Size,B/M)	Alpha	MKT	SMB	HML	Alpha	MKT	SMB	HML
(1,1)	0.0677	-0.0563	0.0936	0.0390	0.0677	0.0563	0.0936	0.0390
(1,2)	-0.0955	-0.0810	-0.1141	0.0573	0.0955	0.0810	0.1141	0.0573
(1,3)	-0.0385	-0.0428	0.0797	-0.0165	0.0385	0.0428	0.0797	0.0165
(1,4)	-0.0630	0.0292	0.0973	-0.1158	0.0630	0.0292	0.0973	0.1158
(1,5)	0.0696	-0.0139	0.0743	-0.0308	0.0696	0.0139	0.0743	0.0308
(2,1)	-0.0893	-0.0380	-0.0182	0.0402	0.0893	0.0380	0.0182	0.0402
(2,2)	-0.1230	0.0412	0.0457	-0.0747	0.1230	0.0412	0.0457	0.0747
(2,3)	0.1077	0.0954	0.0239	-0.0651	0.1077	0.0954	0.0239	0.0651
(2,4)	-0.0034	0.0113	0.0102	-0.0831	0.0034	0.0113	0.0102	0.0831
(2,5)	-0.1078	0.1063	-0.0941	-0.0681	0.1078	0.1063	0.0941	0.0681
(3,1)	-0.0425	-0.0311	-0.0377	0.0160	0.0425	0.0311	0.0377	0.0160
(3,2)	0.0440	0.0756	0.2175	-0.1414	0.0440	0.0756	0.2175	0.1414
(3,3)	0.0453	0.0172	0.0805	-0.0606	0.0453	0.0172	0.0805	0.0606
(3,4)	0.0631	0.0846	0.1305	-0.1588	0.0631	0.0846	0.1305	0.1588
(3,5)	-0.1057	0.0792	0.0341	-0.0475	0.1057	0.0792	0.0341	0.0475
(4,1)	-0.1584	0.0163	-0.0567	-0.0279	0.1584	0.0163	0.0567	0.0279
(4,2)	-0.0719	-0.0367	0.0478	-0.0661	0.0719	0.0367	0.0478	0.0661
(4,3)	0.0291	-0.1139	0.0681	-0.0658	0.0291	0.1139	0.0681	0.0658
(4,4)	0.0495	-0.0255	0.0694	-0.0601	0.0495	0.0255	0.0694	0.0601
(4,5)	-0.0027	0.0335	0.0369	0.0179	0.0027	0.0335	0.0369	0.0179
(5,1)	-0.0180	0.0762	0.0225	-0.1200	0.0180	0.0762	0.0225	0.1200
(5,2)	-0.0452	0.0341	0.0017	-0.0080	0.0452	0.0341	0.0017	0.0080
(5,3)	0.0411	0.0481	0.0427	0.0396	0.0411	0.0481	0.0427	0.0396
(5,4)	-0.0371	-0.0800	-0.0014	-0.0591	0.0371	0.0800	0.0014	0.0591
(5,5)	0.0623	0.0531	0.0705	-0.0208	0.0623	0.0531	0.0705	0.0208
Mean	-0.0169	0.0113	0.0370	-0.0432	0.0632	0.0528	0.0628	0.0600
Stdev	0.0727	0.0607	0.0701	0.0579	0.0377	0.0301	0.0473	0.0394

Figure 3: Extracted Volatility Factors for Fitted Residuals, (1,1)-(2,5)

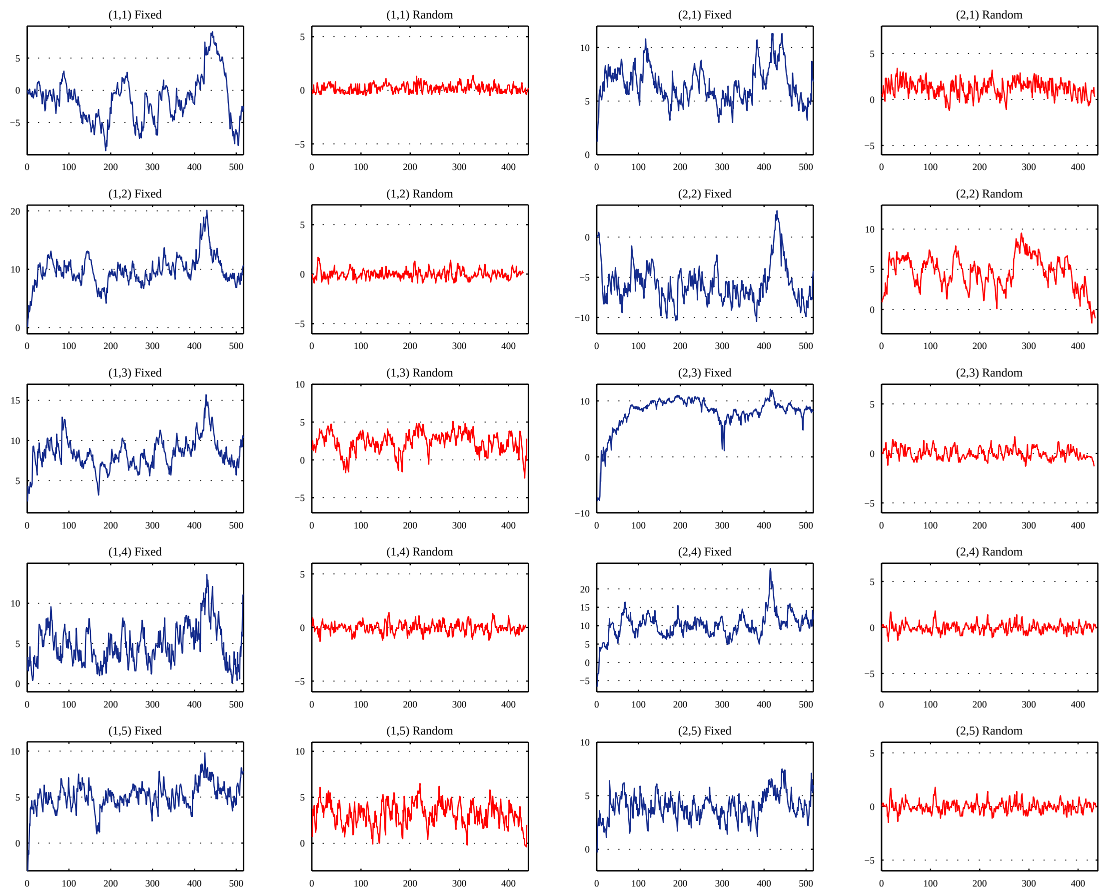


Figure 4: Extracted Volatility Factors for Fitted Residuals, (3,1)-(4,5)

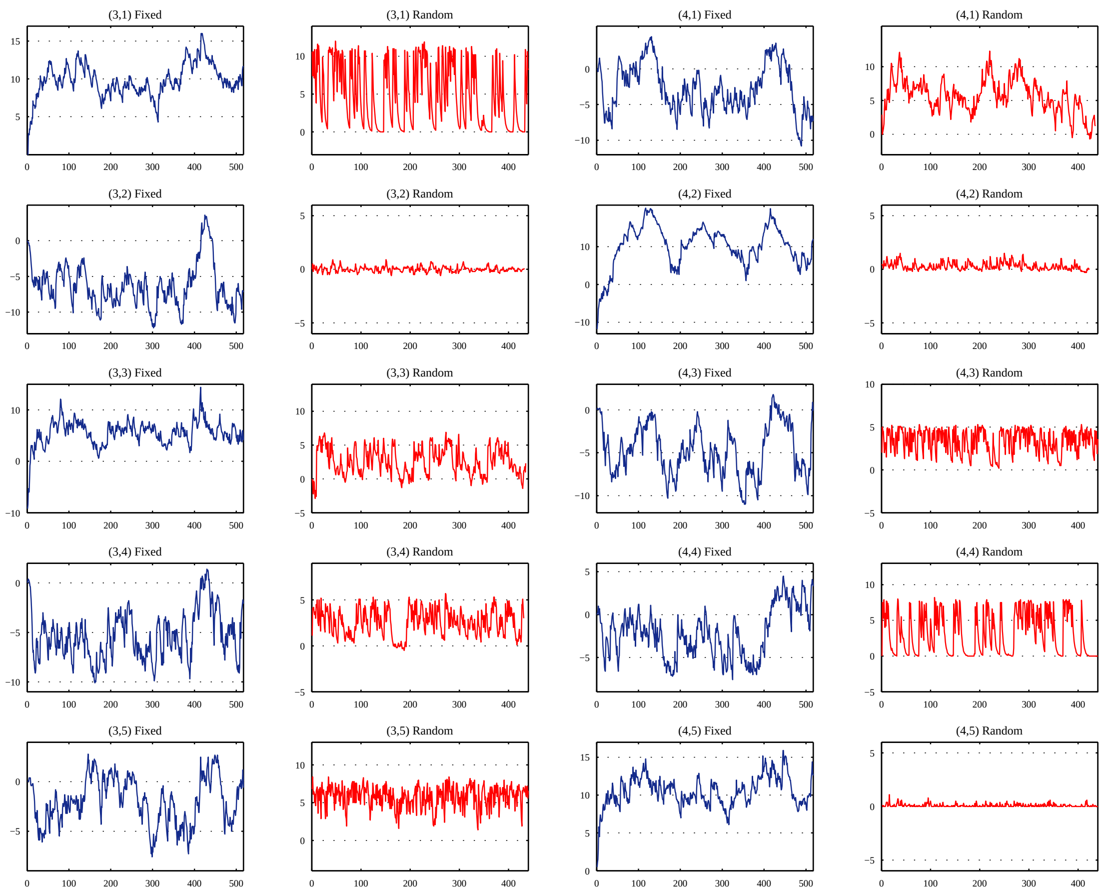


Figure 5: Extracted Volatility Factors for Fitted Residuals, (5,1)-(5,5)

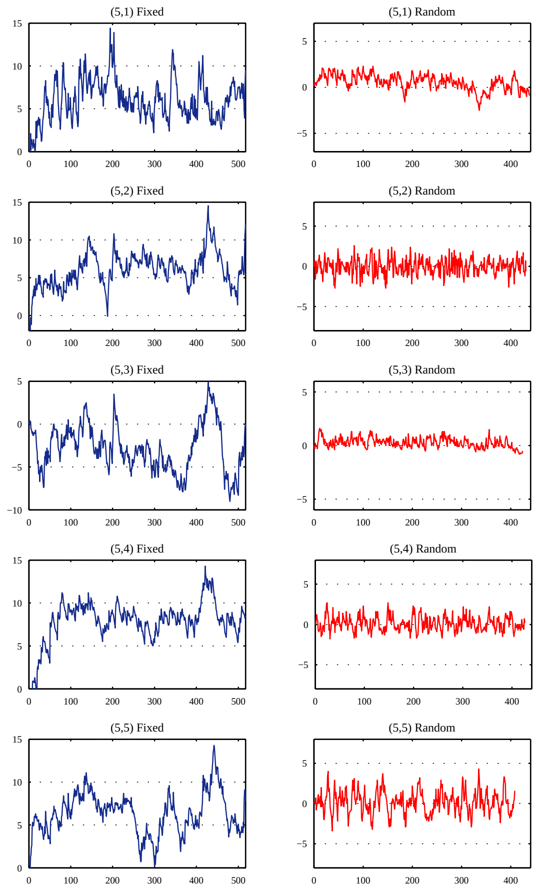
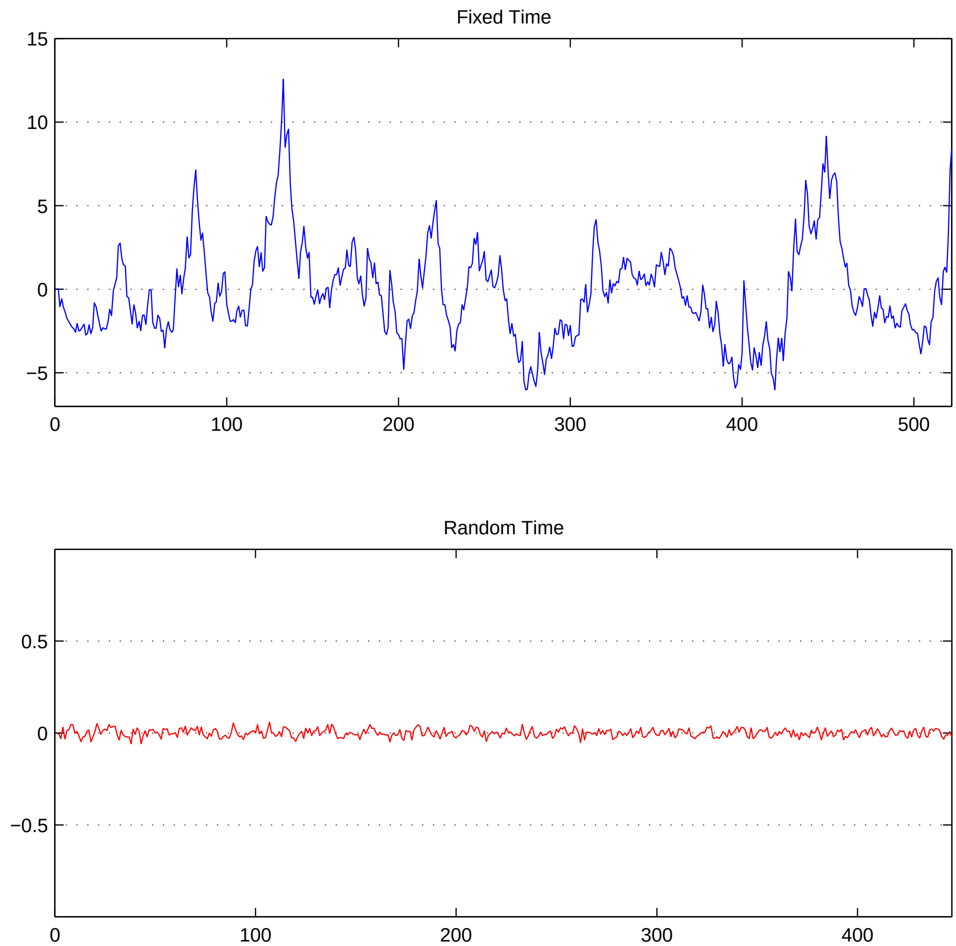


Table 4: AR Coefficients and Leverage Effects of Extracted Volatility Factors

	Fixed Time		Random Time	
(Size,B/M)	Alpha	Rho	Alpha	Rho
(1,1)	0.980 (0.020)	-0.338 (0.194)	0.430 (0.202)	-0.015 (0.147)
(1,2)	0.970 (0.024)	0.232 (0.280)	0.518 (0.159)	0.600 (0.166)
(1,3)	0.961 (0.029)	0.115 (0.194)	0.892 (0.007)	-0.108 (0.021)
(1,4)	0.994 (0.007)	-0.421 (0.191)	0.491 (0.315)	0.122 (0.160)
(1,5)	0.928 (0.054)	-0.082 (0.174)	0.858 (0.008)	0.175 (0.020)
(2,1)	0.954 (0.028)	-0.133 (0.204)	0.701 (0.029)	-0.277 (0.038)
(2,2)	0.980 (0.012)	0.491 (0.098)	0.957 (0.003)	-0.134 (0.028)
(2,3)	0.992 (0.009)	-0.174 (0.325)	0.847 (0.002)	-0.295 (0.015)
(2,4)	0.972 (0.039)	-0.426 (0.340)	0.491 (0.373)	-0.199 (0.100)
(2,5)	0.970 (0.001)	-0.100 (0.228)	0.518 (0.199)	0.161 (0.119)
(3,1)	0.969 (0.024)	-0.062 (0.261)	0.884 (0.008)	-0.070 (0.252)
(3,2)	0.989 (0.010)	-0.154 (0.227)	0.739 (0.008)	-0.167 (0.016)
(3,3)	0.949 (0.035)	-0.302 (0.200)	0.857 (0.001)	-0.448 (0.023)
(3,4)	0.981 (0.012)	-0.167 (0.180)	0.802 (0.011)	-0.208 (0.022)
(3,5)	0.969 (0.017)	-0.177 (0.195)	0.584 (0.125)	-0.254 (0.182)
(4,1)	0.989 (0.015)	-0.400 (0.202)	0.959 (0.020)	0.539 (0.023)
(4,2)	0.992 (0.023)	-0.258 (0.166)	0.641 (0.118)	-0.122 (0.161)
(4,3)	0.974 (0.013)	-0.403 (0.092)	0.788 (0.010)	-0.112 (0.145)
(4,4)	0.986 (0.016)	-0.460 (0.187)	0.872 (0.006)	-0.049 (0.157)
(4,5)	0.970 (0.020)	-0.189 (0.237)	0.828 (0.010)	0.392 (0.214)
(5,1)	0.944 (0.034)	-0.496 (0.094)	0.808 (0.008)	0.041 (0.199)
(5,2)	0.956 (0.029)	0.244 (0.175)	0.411 (0.144)	0.165 (0.155)
(5,3)	0.980 (0.012)	0.448 (0.124)	0.661 (0.382)	-0.129 (0.244)
(5,4)	0.962 (0.027)	-0.446 (0.166)	0.730 (0.214)	-0.269 (0.178)
(5,5)	0.964 (0.021)	-0.181 (0.189)	0.842 (0.144)	-0.219 (0.181)
Average	0.971 (0.021)	-0.154 (0.197)	0.724 (0.100)	-0.035 (0.119)

Figure 6: Extracted Volatility Factor from Excess Market Return



Notes: The excess market return series is first demeaned using non-parametric local-linear regression before we estimate stochastic volatility model with logistic volatility generating function. The AR coefficients of the extracted volatility factor is 0.9505 under the fixed time and 0.2476 under the random time scheme.

Table 5: Test of CAPM on Size Portfolios

	Fixed Time		Random Time	
Size	Alpha	Beta	Alpha	Beta
1 Small	0.0045	1.0772	0.0489	1.0215
	(0.0205)	(0.0394)	(0.0173)	(0.0419)
2	0.0047	1.1564	0.0262	1.1400
	(0.0170)	(0.0326)	(0.0142)	(0.0344)
3	0.0135	1.1536	0.0311	1.1364
	(0.0145)	(0.0279)	(0.0120)	(0.0292)
4	0.0141	1.1231	0.0277	1.1221
	(0.0135)	(0.0259)	(0.0108)	(0.0262)
5	0.0185	1.1048	0.0327	1.1047
	(0.0113)	(0.0218)	(0.0095)	(0.0232)
6	0.0118	1.0789	0.0198	1.0793
	(0.0097)	(0.0185)	(0.0082)	(0.0198)
7	0.0118	1.0798	0.0158	1.0619
	(0.0081)	(0.0156)	(0.0066)	(0.0161)
8	0.0098	1.0695	0.0096	1.0516
	(0.0070)	(0.0134)	(0.0059)	(0.0142)
9	0.0065	0.993	0.0051	0.9851
	(0.0057)	(0.0109)	(0.0046)	(0.0112)
10 Big	-0.0010	0.9217	-0.0073	0.9352
	(0.0054)	(0.0104)	(0.0045)	(0.0108)
1-10 Size Strategy	0.0055	0.1555	0.0562	0.0863
	(0.0250)	(0.0480)	(0.0211)	(0.0512)
Wald	12.2747 (0.2671)		54.6970 (0.0000)	

Figure 7: Alphas of Size Portfolios

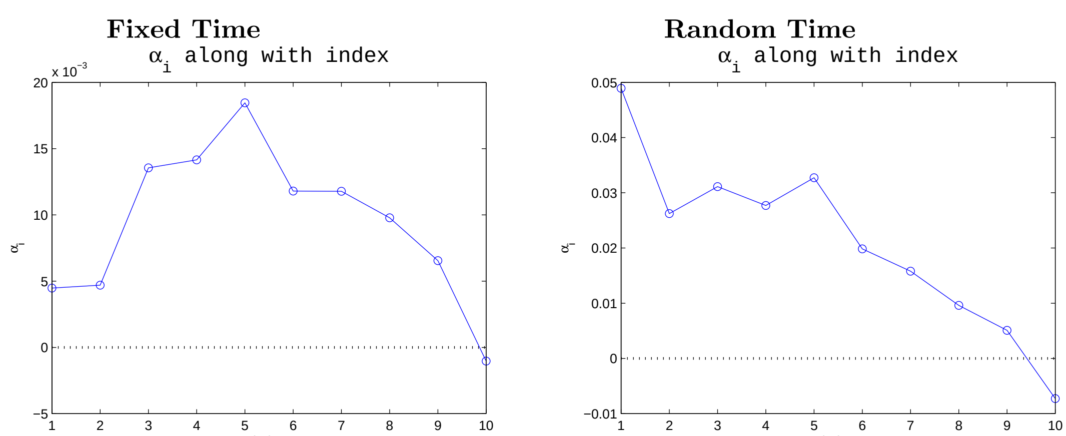
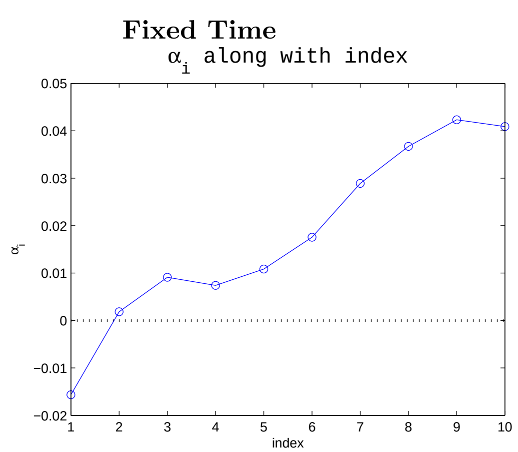


Table 6: Test of CAPM on Book-to-Market Portfolios

	Fixed Time		Random Time	
Book-to-Market	Alpha	Beta	Alpha	Beta
1 Growth	-0.0157	1.0872	-0.0188	1.1208
	(0.0101)	(0.0194)	(0.0089)	(0.0217)
2	0.0019	1.0101	-0.0101	1.0188
	(0.0075)	(0.0143)	(0.0061)	(0.0147)
3	0.0091	0.9677	0.0010	0.9916
	(0.0079)	(0.0152)	(0.0065)	(0.0159)
4	0.0074	0.9606	0.0048	0.9544
	(0.0092)	(0.0177)	(0.0077)	(0.0187)
5	0.0108	0.8679	0.0057	0.8817
	(0.0097)	(0.0186)	(0.0081)	(0.0197)
6	0.0176	0.8873	0.0242	0.8779
	(0.0094)	(0.0181)	(0.0079)	(0.0191)
7	0.0289	0.8379	0.0296	0.8086
	(0.0112)	(0.0214)	(0.0089)	(0.0215)
8	0.0367	0.8368	0.0298	0.8136
	(0.0116)	(0.0224)	(0.0094)	(0.0228)
9	0.0423	0.8850	0.0496	0.8638
	(0.0122)	(0.0235)	(0.0105)	(0.0256)
10 Value	0.0409	0.9898	0.0436	0.9914
	(0.0165)	(0.0316)	(0.0138)	(0.0336)
10-1 Book-to-Market Strategy	0.0566	-0.0974	0.0625	-0.1294
	(0.0232)	(0.0446)	(0.0203)	(0.0491)
Wald	15.7134 (0.1081)		46.2893 (0.0000)	

Figure 8: Alphas of Book-to-Market Portfolios



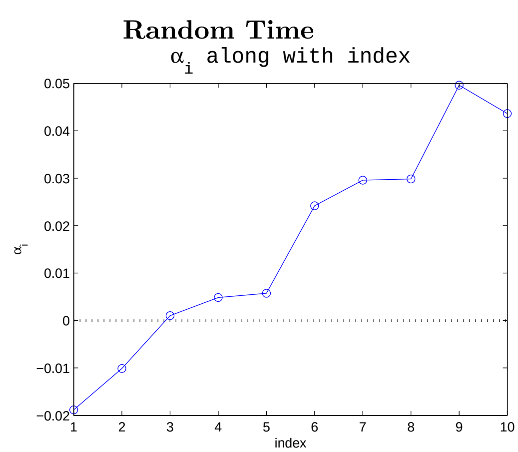


Table 7: Test of CAPM on Momentum Portfolios

	Fixed Time		Time Change	
Momentum	Alpha	Beta	Alpha	Beta
1 Losers	-0.1307	1.4675	-0.1220	1.4962
	(0.0205)	(0.0394)	(0.0198)	(0.0481)
2	-0.0483	1.1983	-0.0586	1.2041
	(0.0170)	(0.0326)	(0.0142)	(0.0344)
3	-0.0105	1.0247	-0.0134	1.0299
	(0.0145)	(0.0279)	(0.0113)	(0.0274)
4	-0.0037	0.9665	-0.0136	1.0014
	(0.0135)	(0.0259)	(0.0095)	(0.0231)
5	-0.0143	0.9158	-0.0194	0.9236
	(0.0113)	(0.0218)	(0.0083)	(0.0202)
6	0.0045	0.8758	-0.0010	0.9008
	(0.0097)	(0.0185)	(0.0074)	(0.0179)
7	0.0026	0.9129	0.0075	0.9111
	(0.0081)	(0.0156)	(0.0077)	(0.0186)
8	0.0346	0.8969	0.0349	0.8997
	(0.0070)	(0.0134)	(0.0082)	(0.0198)
9	0.0275	0.9855	0.0343	0.9675
	(0.0057)	(0.0109)	(0.0088)	(0.0212)
10 Winners	0.0739	1.1602	0.0785	1.1750
	(0.0054)	(0.0104)	(0.0139)	(0.0338)
10-1 Momentum Strategy	0.2046	-0.3073	0.2005	-0.3212
	(0.0348)	(0.0669)	(0.0291)	(0.0706)
Wald	69.1309 (0.0000)		104.1411 (0.0000)	

Figure 9: Alphas of Momentum Portfolios

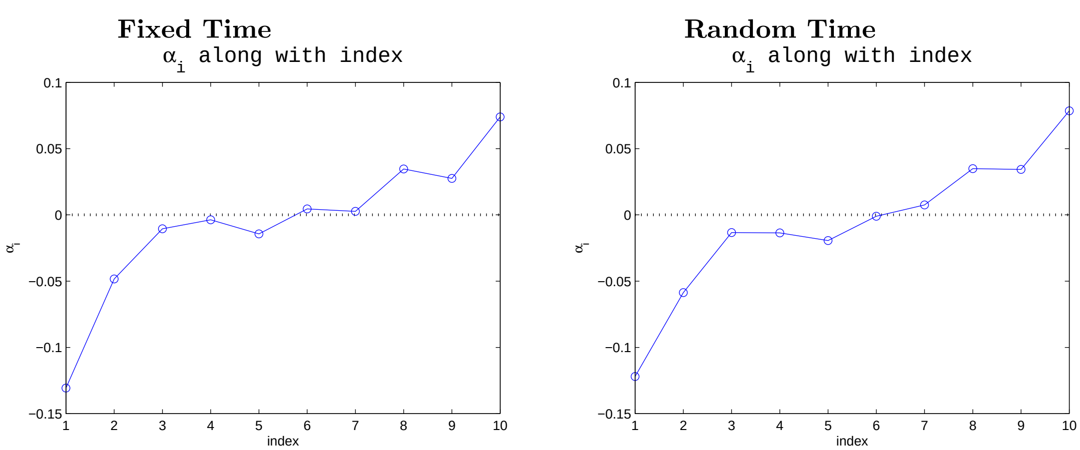
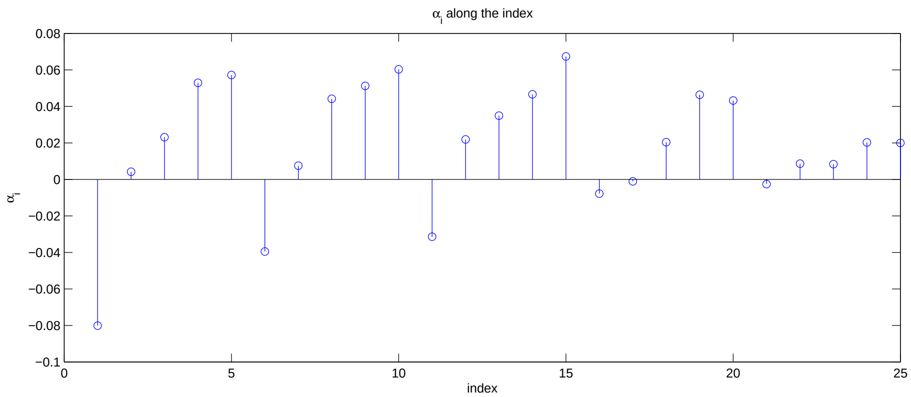


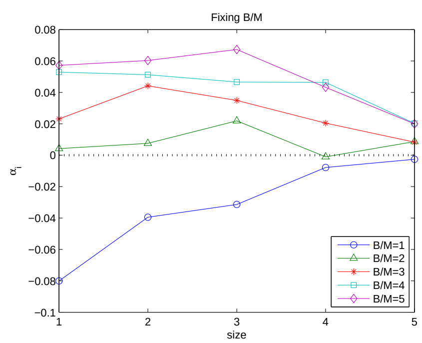
Table 8: Test of CAPM on (Size,B/M) Portfolios

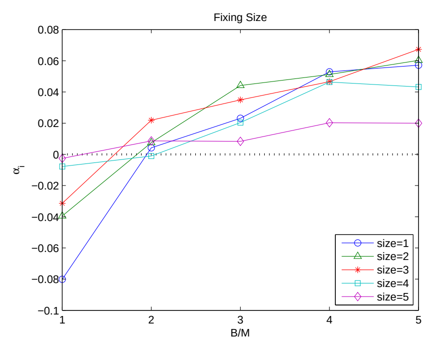
	Fixed Time		Random Time	
(Size,B/M)	Alpha	Beta	Alpha	Beta
(1,1)	-0.0801 (0.0246)	1.4533 (0.0472)	-0.0304 (0.0212)	1.4014 (0.0515)
(1,2)	0.0042 (0.0211)	1.2044 (0.0406)	0.0285 (0.0167)	1.1609 (0.0406)
(1,3)	0.0231 (0.0178)	1.0469 (0.0341)	0.0487 (0.0151)	1.0133 (0.0366)
(1,4)	0.0529 (0.0172)	0.9567 (0.0331)	0.0708 (0.0143)	0.9444 (0.0347)
(1,5)	0.0572 (0.0191)	1.0020 (0.0367)	0.0862 (0.0159)	0.9675 (0.0385)
(2,1)	-0.0395 (0.0187)	1.4118 (0.0359)	-0.0216 (0.0153)	1.3927 (0.0372)
(2,2)	0.0075 (0.0151)	1.1287 (0.0290)	0.0153 (0.0124)	1.1304 (0.0301)
(2,3)	0.0442 (0.0143)	1.0017 (0.0274)	0.0672 (0.0115)	1.0066 (0.0280)
(2,4)	0.0512 (0.0142)	0.9533 (0.0273)	0.0662 (0.0113)	0.9362 (0.0275)
(2,5)	0.0603 (0.0172)	1.0087 (0.0330)	0.0605 (0.0138)	1.0255 (0.0335)
(3,1)	-0.0314 (0.0156)	1.3526 (0.0300)	-0.0152 (0.0129)	1.3385 (0.0314)
(3,2)	0.0219 (0.0117)	1.0735 (0.0225)	0.0336 (0.0105)	1.0875 (0.0254)
(3,3)	0.0349 (0.0121)	0.9439 (0.0232)	0.0449 (0.0099)	0.9337 (0.0240)
(3,4)	0.0466 (0.0127)	0.8833 (0.0244)	0.0549 (0.0103)	0.8936 (0.0249)
(3,5)	0.0673 (0.0156)	0.9410 (0.0300)	0.0604 (0.0128)	0.9576 (0.0312)
(4,1)	-0.0078 (0.0115)	1.2331 (0.0221)	-0.0129 (0.0094)	1.2470 (0.0228)
(4,2)	-0.0010 (0.0100)	1.0550 (0.0192)	-0.0051 (0.0082)	1.0388 (0.0199)
(4,3)	0.0204 (0.0114)	0.9844 (0.0220)	0.0287 (0.0089)	0.9296 (0.0216)
(4,4)	0.0464 (0.0119)	0.9110 (0.0229)	0.0537 (0.0095)	0.8874 (0.0231)
(4,5)	0.0432 (0.0150)	0.9658 (0.0288)	0.0407 (0.0128)	0.9658 (0.0311)
(5,1)	-0.0026 (0.0088)	0.9998 (0.0169)	-0.0117 (0.0077)	1.0317 (0.0186)
(5,2)	0.0087 (0.0087)	0.9177 (0.0168)	-0.0010 (0.0072)	0.9329 (0.0174)
(5,3)	0.0083 (0.0106)	0.8352 (0.0204)	0.0040 (0.0089)	0.8506 (0.0216)
(5,4)	0.0203 (0.0126)	0.7907 (0.0241)	0.0107 (0.0101)	0.7602 (0.0246)
(5,5)	0.0200 (0.0158)	0.8337 (0.0303)	0.0223 (0.0129)	0.8432 (0.0313)
Wald	121.7332 (0.0000)		317.2629 (0.0000)	

Figure 10: Alphas of (Size,B/M) Portfolios

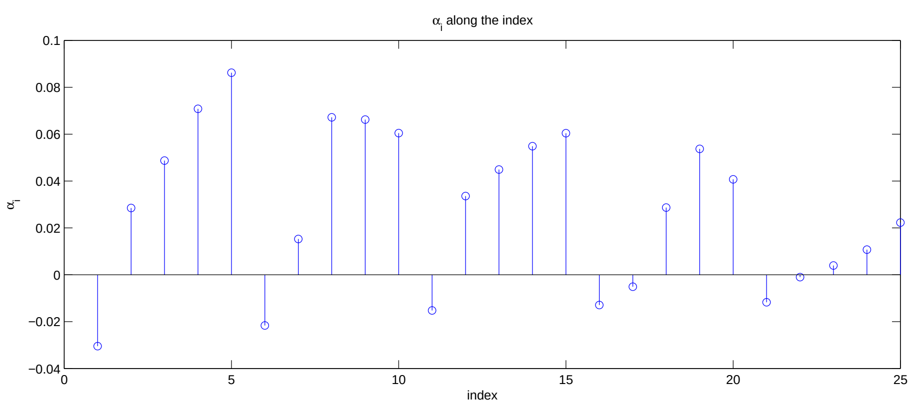
Fixed Time

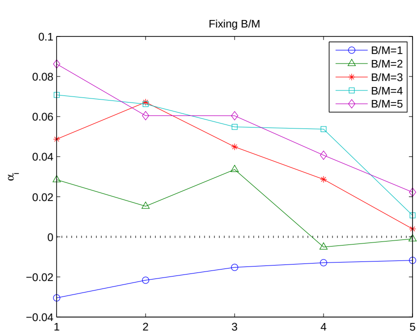






Random Time





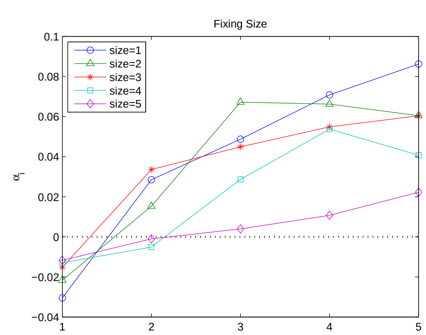
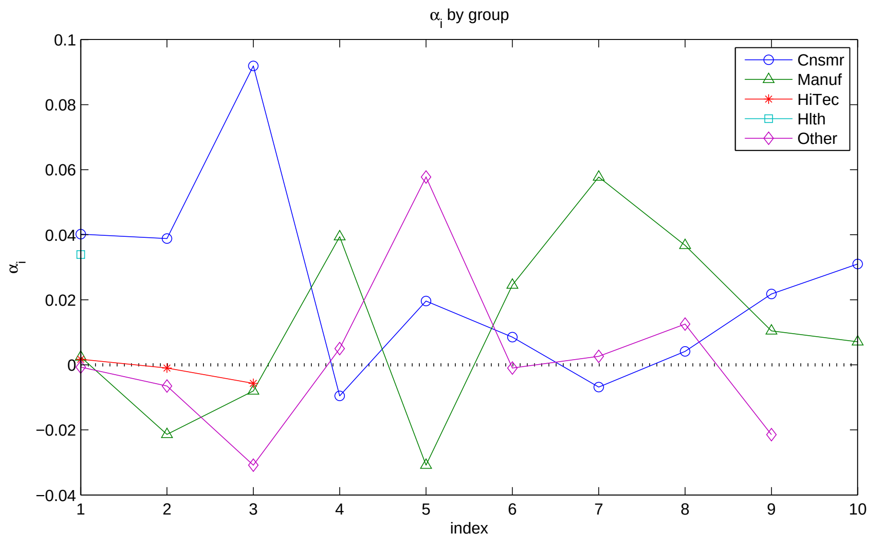


Table 9: Test of CAPM on Industry Portfolios

	Fixed Time		Random Time	
Industry	Alpha	Beta	Alpha	Beta
Food	0.0402 (0.0161)	0.7140 (0.0309)	0.0514 (0.0132)	0.6860 (0.0321)
Beer	0.0388 (0.0219)	0.8004 (0.0420)	0.0491 (0.0180)	0.7811 (0.0438)
Smoke	0.0919 (0.0294)	0.6287 (0.0566)	0.0571 (0.0243)	0.6583 (0.0590)
Games	-0.0007 (0.0223)	1.3058 (0.0428)	0.0044 (0.0174)	1.3323 (0.0421)
Books	-0.0095 (0.0166)	1.0430 (0.0319)	0.0100 (0.0139)	1.0281 (0.0337)
Hshld	0.0196 (0.0162)	0.8189 (0.0312)	0.0212 (0.0141)	0.8387 (0.0341)
Clths	0.0085 (0.0221)	1.0917 (0.0425)	0.0280 (0.0180)	1.1390 (0.0437)
Hlth	0.0339 (0.0168)	0.8485 (0.0322)	0.0326 (0.0144)	0.8941 (0.0351)
Chems	0.0024 (0.0160)	0.9740 (0.0308)	-0.0054 (0.0142)	0.9846 (0.0346)
Txtls	-0.0068 (0.0242)	0.9961 (0.0465)	0.0240 (0.0194)	1.0537 (0.0471)
Cnstr	-0.0065 (0.0154)	1.1475 (0.0295)	-0.0120 (0.0134)	1.1410 (0.0325)
Steel	-0.0214 (0.0234)	1.2390 (0.0449)	-0.0352 (0.0192)	1.2135 (0.0466)
FabPr	-0.0080 (0.0155)	1.1848 (0.0298)	-0.0120 (0.0127)	1.1839 (0.0307)
ElcEq	0.0394 (0.0168)	1.1628 (0.0323)	0.0368 (0.0140)	1.1811 (0.0339)
Autos	-0.0308 (0.0232)	1.0665 (0.0446)	-0.0274 (0.0181)	1.0683 (0.0439)
Carry	0.0245 (0.0216)	1.0758 (0.0415)	0.0215 (0.0175)	1.1023 (0.0424)
Mines	0.0050 (0.0329)	0.9203 (0.0631)	0.0067 (0.0272)	0.9220 (0.0659)
Coal	0.0578 (0.0431)	1.1051 (0.0827)	0.0127 (0.0340)	1.1016 (0.0826)
Oil	0.0368 (0.0216)	0.7588 (0.0414)	0.0311 (0.0191)	0.7483 (0.0463)
Util	0.0104 (0.0174)	0.5469 (0.0334)	0.0180 (0.0142)	0.4965 (0.0344)
Telcm	0.0017 (0.0170)	0.7654 (0.0327)	0.0264 (0.0137)	0.7500 (0.0332)
Servs	-0.0010 (0.0167)	1.3796 (0.0320)	0.0026 (0.0149)	1.3322 (0.0361)
BusEq	-0.0057 (0.0208)	1.3192 (0.0399)	-0.0219 (0.0169)	1.3429 (0.0411)
Paper	0.0071 (0.0157)	0.9096 (0.0301)	-0.0006 (0.0126)	0.9144 (0.0307)
Trans	0.0026 (0.0185)	1.0520 (0.0355)	-0.0114 (0.0150)	1.0343 (0.0363)
Whlsl	0.0041 (0.0162)	1.1093 (0.0312)	0.0259 (0.0137)	1.0832 (0.0332)
Rtail	0.0218 (0.0169)	0.9991 (0.0325)	0.0155 (0.0143)	1.0223 (0.0348)
Meals	0.0310 (0.0227)	1.1117 (0.0436)	0.0566 (0.0205)	1.1370 (0.0496)
Fin	0.0125 (0.0144)	1.0293 (0.0276)	0.0038 (0.0125)	1.0528 (0.0303)
Other	-0.0215 (0.0171)	1.0719 (0.0328)	-0.0021 (0.0130)	1.0960 (0.0316)
Wald	44.2420 (0.0453)		80.3369 (0.0000)	

Figure 11: Alphas of Industry Portfolios

Fixed Time



Random Time

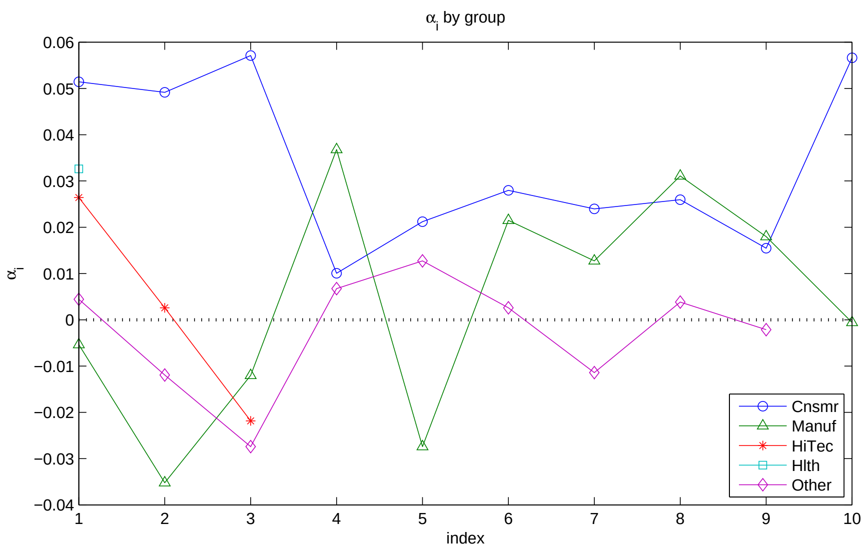


Table 10: Test of Two-Factor Model on Size Portfolios

	Fixed Time			Random Time		
Size	Alpha	Beta_MKT	Beta_SMB	Alpha	Beta_MKT	Beta_SMB
1 Small	-0.0038	0.8466	1.1876	0.0109	0.8010	1.2398
	(0.0095)	(0.0189)	(0.0272)	(0.0073)	(0.0184)	(0.0276)
2	-0.0026	0.9551	1.0362	-0.006	0.9534	1.049
	(0.0060)	(0.0121)	(0.0174)	(0.0051)	(0.0127)	(0.0191)
3	0.0074	0.9832	0.8771	0.0039	0.9790	0.8847
	(0.0056)	(0.0112)	(0.0161)	(0.0045)	(0.0111)	(0.0167)
4	0.0086	0.9682	0.7975	0.0036	0.9828	0.7831
	(0.0057)	(0.0114)	(0.0164)	(0.0043)	(0.0109)	(0.0163)
5	0.0139	0.9799	0.6434	0.0123	0.9863	0.6659
	(0.0056)	(0.0112)	(0.0162)	(0.0045)	(0.0113)	(0.0170)
6	0.0085	0.9872	0.4718	0.0046	0.9908	0.4978
	(0.0064)	(0.0128)	(0.0184)	(0.0052)	(0.0131)	(0.0197)
7	0.0093	1.0119	0.3494	0.0047	0.9976	0.3617
	(0.0061)	(0.0122)	(0.0175)	(0.0048)	(0.0121)	(0.0181)
8	0.0082	1.0249	0.2296	0.0026	1.0112	0.2271
	(0.0060)	(0.0120)	(0.0173)	(0.0052)	(0.0129)	(0.0194)
9	0.0063	0.9852	0.0402	0.0043	0.9808	0.0240
	(0.0057)	(0.0113)	(0.0163)	(0.0046)	(0.0116)	(0.0174)
10 Big	0.0010	0.9788	-0.2942	0.0016	0.9867	-0.2891
	(0.0030)	(0.0060)	(0.0087)	(0.0026)	(0.0064)	(0.0096)
1-10 Size Strategy	-0.0049	-0.1323	-1.4818	0.0092	-0.1857	1.5290
	(0.0250)	(0.0500)	(0.0719)	(0.0212)	(0.0531)	(0.0797)
Wald	12.0613 (0.2810)			31.3234 (0.0005)		

Table 11: Test of Two-Factor Model on Book-to-Market Portfolios

	Fixed Time			Random Time		
Book-to-Market	Alpha	Beta_MKT	Beta_HML	Alpha	Beta_MKT	Beta_HML
1 Growth	0.0157	0.9528	-0.5052	0.0136	0.9626	-0.5646
	(0.0075)	(0.0156)	(0.0239)	(0.0062)	(0.0161)	(0.0245)
2	0.0069	0.9885	-0.0811	-0.0045	0.9914	-0.0976
	(0.0075)	(0.0155)	(0.0239)	(0.0061)	(0.0160)	(0.0244)
3	0.0052	0.9845	0.0631	-0.0017	1.0050	0.0479
	(0.0080)	(0.0165)	(0.0254)	(0.0067)	(0.0175)	(0.0266)
4	-0.0084	1.0281	0.2538	-0.0114	1.0336	0.2824
	(0.0087)	(0.0180)	(0.0276)	(0.0072)	(0.0187)	(0.0285)
5	-0.0092	0.9537	0.3224	-0.014	0.9782	0.344
	(0.0088)	(0.0181)	(0.0279)	(0.0072)	(0.0189)	(0.0288)
6	-0.0063	0.9894	0.3837	0.0026	0.9832	0.3759
	(0.0080)	(0.0165)	(0.0254)	(0.0067)	(0.0175)	(0.0266)
7	-0.0049	0.9827	0.5443	0.0009	0.9486	0.4996
	(0.0085)	(0.0176)	(0.0271)	(0.0068)	(0.0179)	(0.0272)
8	-0.0061	1.0201	0.6890	-0.0075	0.9957	0.6497
	(0.0070)	(0.0145)	(0.0223)	(0.0057)	(0.0148)	(0.0225)
9	-0.0001	1.0665	0.6825	0.0093	1.0605	0.7017
	(0.0081)	(0.0168)	(0.0258)	(0.0068)	(0.0178)	(0.0271)
10 Value	-0.0126	1.2189	0.8612	-0.0050	1.2283	0.8454
	(0.0118)	(0.0243)	(0.0374)	(0.0100)	(0.0260)	(0.0396)
10-1 Book-to-Market	-0.0284	0.2661	1.3663	-0.0186	0.2658	1.4099
Strategy	(0.0237)	(0.0489)	(0.0751)	(0.0208)	(0.0543)	(0.0827)
Wald	9.8776 (0.4513)			16.7380 (0.0804)		

Table 12: Test of Two-Factor Model on Momentum Portfolios

	Fixed Time			Random Time		
Momentum	Alpha	Beta_MKT	Beta_MMT	Alpha	Beta_MKT	Beta_MMT
1 Losers	-0.0396	1.3428	-0.8501	-0.0238	1.3614	-0.9089
	(0.0147)	(0.0279)	(0.0296)	(0.0133)	(0.0314)	(0.0364)
2	0.0287	1.0928	-0.7188	0.0203	1.0957	-0.7310
	(0.0094)	(0.0178)	(0.0188)	(0.0075)	(0.0177)	(0.0205)
3	0.0454	0.9482	-0.5216	0.0444	0.9506	-0.5348
	(0.0090)	(0.0171)	(0.0181)	(0.0072)	(0.0170)	(0.0197)
4	0.0337	0.9152	-0.3498	0.0241	0.9495	-0.3494
	(0.0094)	(0.0178)	(0.0189)	(0.0079)	(0.0185)	(0.0215)
5	0.0111	0.8810	-0.2369	0.0035	0.8922	-0.2118
	(0.0098)	(0.0186)	(0.0197)	(0.0079)	(0.0186)	(0.0215)
6	0.0115	0.8662	-0.0653	0.0059	0.8913	-0.0638
	(0.0096)	(0.0183)	(0.0194)	(0.0077)	(0.0180)	(0.0209)
7	-0.0043	0.9224	0.0645	-0.0017	0.9236	0.0848
	(0.0089)	(0.0168)	(0.0178)	(0.0079)	(0.0186)	(0.0215)
8	0.0094	0.9315	0.2358	0.0100	0.9339	0.2309
	(0.0095)	(0.0180)	(0.0191)	(0.0075)	(0.0178)	(0.0206)
9	-0.0080	1.0341	0.3312	-0.0030	1.0186	0.3448
	(0.0084)	(0.0159)	(0.0169)	(0.0069)	(0.0162)	(0.0188)
10 Winners	0.0070	1.2518	0.6244	0.0116	1.2669	0.6195
	(0.0121)	(0.0230)	(0.0244)	(0.0097)	(0.0229)	(0.0266)
10-1 Momentum Strategy	0.0465	-0.0910	1.4745	0.0354	-0.0945	1.5283
	(0.0357)	(0.0677)	(0.0718)	(0.0305)	(0.0717)	(0.0831)
Wald	53.4570 (0.0000)			69.3201 (0.0000)		

Table 13: Test of Fama-French Three-Factor Model (Fixed Time)

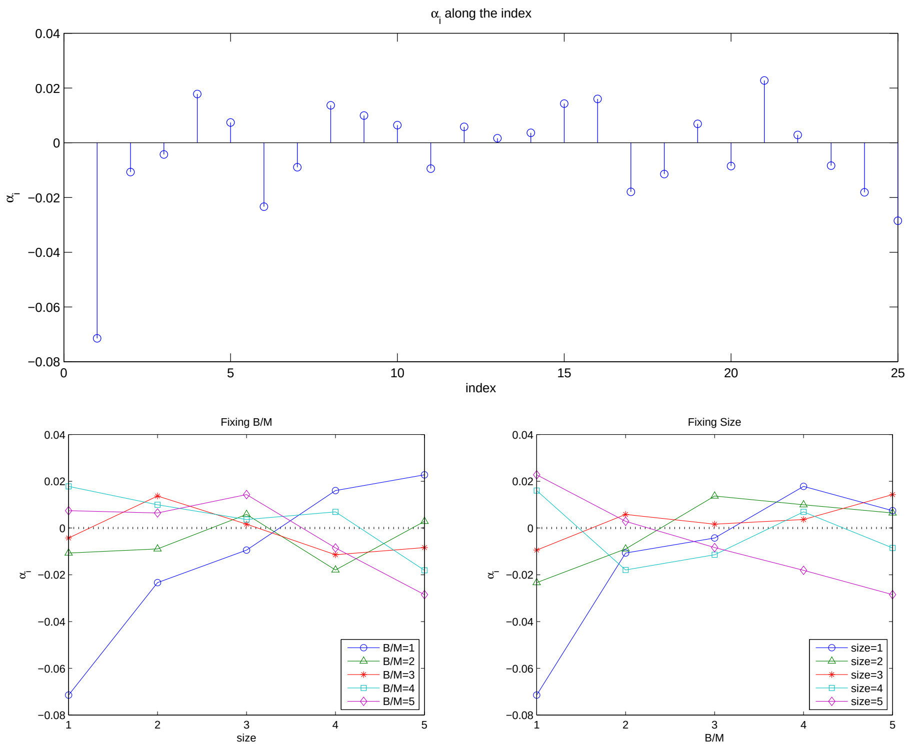
	Fixed Time			
(Size,B/M)	Alpha	Beta_MKT	Beta_SMB	Beta_HML
(1,1)	-0.0714 (0.0115)	1.1092 (0.0242)	1.3694 (0.0327)	-0.2936 (0.0367)
(1,2)	-0.0107 (0.0084)	0.9804 (0.0176)	1.2834 (0.0238)	0.0949 (0.0267)
(1,3)	-0.0043 (0.0070)	0.9231 (0.0147)	1.0747 (0.0199)	0.3192 (0.0224)
(1,4)	0.0178 (0.0070)	0.8809 (0.0148)	1.0080 (0.0200)	0.4508 (0.0225)
(1,5)	0.0074 (0.0073)	0.9737 (0.0155)	1.0771 (0.0209)	0.6798 (0.0235)
(2,1)	-0.0234 (0.0087)	1.1208 (0.0184)	0.9896 (0.0249)	-0.3714 (0.0280)
(2,2)	-0.0089 (0.0076)	1.0048 (0.0161)	0.8668 (0.0217)	0.1671 (0.0244)
(2,3)	0.0137 (0.0073)	0.9580 (0.0154)	0.7772 (0.0208)	0.4028 (0.0234)
(2,4)	0.0100 (0.0069)	0.9708 (0.0145)	0.7102 (0.0196)	0.5842 (0.0221)
(2,5)	0.0065 (0.0074)	1.0467 (0.0155)	0.8581 (0.0210)	0.7691 (0.0236)
(3,1)	-0.0095 (0.0082)	1.0944 (0.0172)	0.7330 (0.0232)	-0.4356 (0.0261)
(3,2)	0.0058 (0.0087)	1.0242 (0.0182)	0.5264 (0.0246)	0.1990 (0.0277)
(3,3)	0.0017 (0.0085)	0.9869 (0.0179)	0.4431 (0.0242)	0.4850 (0.0272)
(3,4)	0.0037 (0.0082)	0.9800 (0.0172)	0.3885 (0.0232)	0.6469 (0.0261)
(3,5)	0.0143 (0.0095)	1.0495 (0.0200)	0.5287 (0.0270)	0.7936 (0.0304)
(4,1)	0.0160 (0.0079)	1.0492 (0.0166)	0.3656 (0.0224)	-0.4244 (0.0252)
(4,2)	-0.0180 (0.0092)	1.0811 (0.0194)	0.2064 (0.0262)	0.2487 (0.0295)
(4,3)	-0.0114 (0.0094)	1.0844 (0.0198)	0.1621 (0.0267)	0.4940 (0.0301)
(4,4)	0.0069 (0.0086)	1.0330 (0.0181)	0.2090 (0.0245)	0.6110 (0.0275)
(4,5)	-0.0085 (0.0104)	1.1331 (0.0219)	0.2414 (0.0296)	0.8050 (0.0333)
(5,1)	0.0228 (0.0065)	0.9505 (0.0137)	-0.2645 (0.0185)	-0.3786 (0.0208)
(5,2)	0.0029 (0.0079)	0.9935 (0.0167)	-0.2270 (0.0226)	0.1191 (0.0254)
(5,3)	-0.0083 (0.0091)	0.9611 (0.0191)	-0.2432 (0.0258)	0.2957 (0.0290)
(5,4)	-0.0181 (0.0080)	1.0051 (0.0168)	-0.2246 (0.0228)	0.6419 (0.0256)
(5,5)	-0.0285 (0.0115)	1.0631 (0.0243)	-0.0982 (0.0328)	0.7906 (0.0369)
Wald	95.5585 (0.0000)			

Table 14: Test of Fama-French Three-Factor Model (Random Time)

	Random Time			
(Size,B/M)	Alpha	Beta_MKT	Beta_SMB	beta_HML
(1,1)	-0.0591 (0.0096)	1.0659 (0.0253)	1.4512 (0.0355)	-0.2758 (0.0384)
(1,2)	-0.0159 (0.0064)	0.9706 (0.0168)	1.2413 (0.0236)	0.1091 (0.0255)
(1,3)	-0.0034 (0.0058)	0.9019 (0.0153)	1.1160 (0.0214)	0.3107 (0.0231)
(1,4)	0.0152 (0.0055)	0.8736 (0.0145)	1.0433 (0.0203)	0.4096 (0.0219)
(1,5)	0.0143 (0.0057)	0.9556 (0.0149)	1.1072 (0.0209)	0.6604 (0.0226)
(2,1)	-0.0321 (0.0070)	1.1188 (0.0184)	0.9924 (0.0258)	-0.3473 (0.0279)
(2,2)	-0.0205 (0.0059)	1.0148 (0.0154)	0.8854 (0.0216)	0.1498 (0.0233)
(2,3)	0.0217 (0.0059)	0.9762 (0.0156)	0.7690 (0.0219)	0.3799 (0.0237)
(2,4)	0.0131 (0.0052)	0.9636 (0.0137)	0.7073 (0.0192)	0.5468 (0.0208)
(2,5)	-0.0074 (0.0065)	1.0955 (0.0171)	0.7967 (0.0239)	0.7553 (0.0259)
(3,1)	-0.0124 (0.0070)	1.0894 (0.0183)	0.7190 (0.0256)	-0.4325 (0.0277)
(3,2)	0.0061 (0.0071)	1.0144 (0.0187)	0.6323 (0.0262)	0.1404 (0.0283)
(3,3)	0.0041 (0.0068)	0.9756 (0.0179)	0.4790 (0.0251)	0.4538 (0.0271)
(3,4)	0.0086 (0.0068)	0.9768 (0.0177)	0.4348 (0.0249)	0.5728 (0.0269)
(3,5)	-0.0021 (0.0074)	1.0847 (0.0195)	0.5426 (0.0274)	0.7979 (0.0296)
(4,1)	0.0012 (0.0065)	1.0669 (0.0170)	0.3400 (0.0238)	-0.4269 (0.0257)
(4,2)	-0.0242 (0.0077)	1.0533 (0.0202)	0.2397 (0.0284)	0.2040 (0.0307)
(4,3)	-0.0019 (0.0075)	1.0095 (0.0197)	0.2106 (0.0276)	0.4189 (0.0299)
(4,4)	0.0136 (0.0069)	1.0032 (0.0181)	0.2430 (0.0254)	0.5673 (0.0274)
(4,5)	-0.0155 (0.0086)	1.1524 (0.0226)	0.2662 (0.0317)	0.8349 (0.0342)
(5,1)	0.0208 (0.0055)	0.9614 (0.0145)	-0.2688 (0.0203)	-0.4216 (0.0219)
(5,2)	-0.0013 (0.0065)	1.0091 (0.0172)	-0.2286 (0.0241)	0.1270 (0.0260)
(5,3)	-0.0086 (0.0076)	0.9843 (0.0199)	-0.2211 (0.0279)	0.3369 (0.0301)
(5,4)	-0.0163 (0.0067)	0.9642 (0.0177)	-0.2197 (0.0248)	0.5882 (0.0268)
(5,5)	-0.0213 (0.0092)	1.0758 (0.0242)	-0.0612 (0.0340)	0.7911 (0.0367)
Wald	166.0789 (0.0000)			

Figure 12: Fama-French Alphas of (Size,B/M) Portfolios

Fixed Time



Random Time

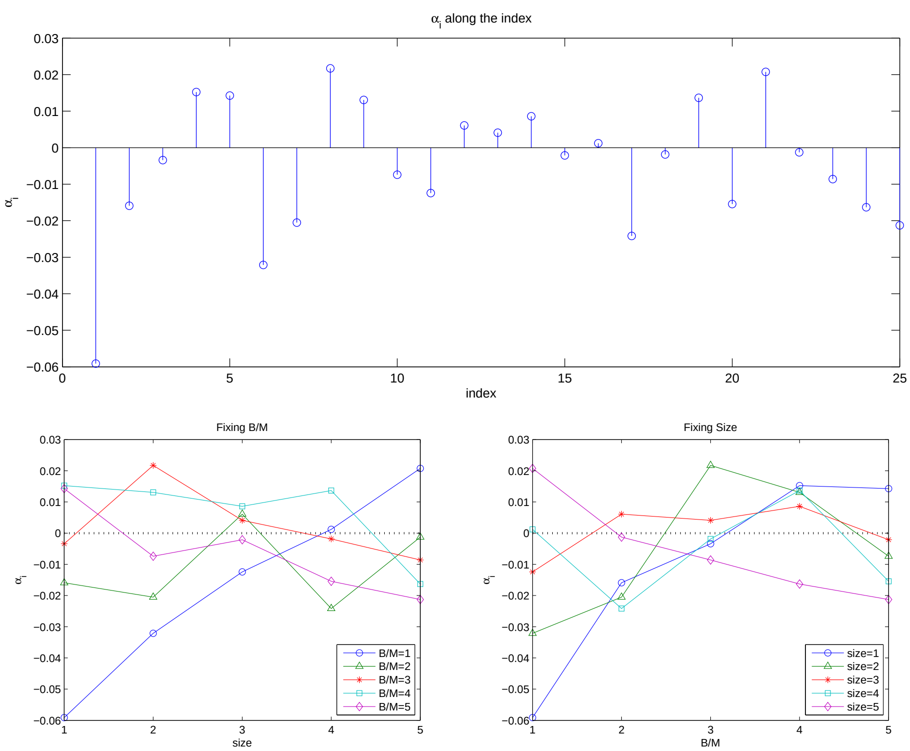
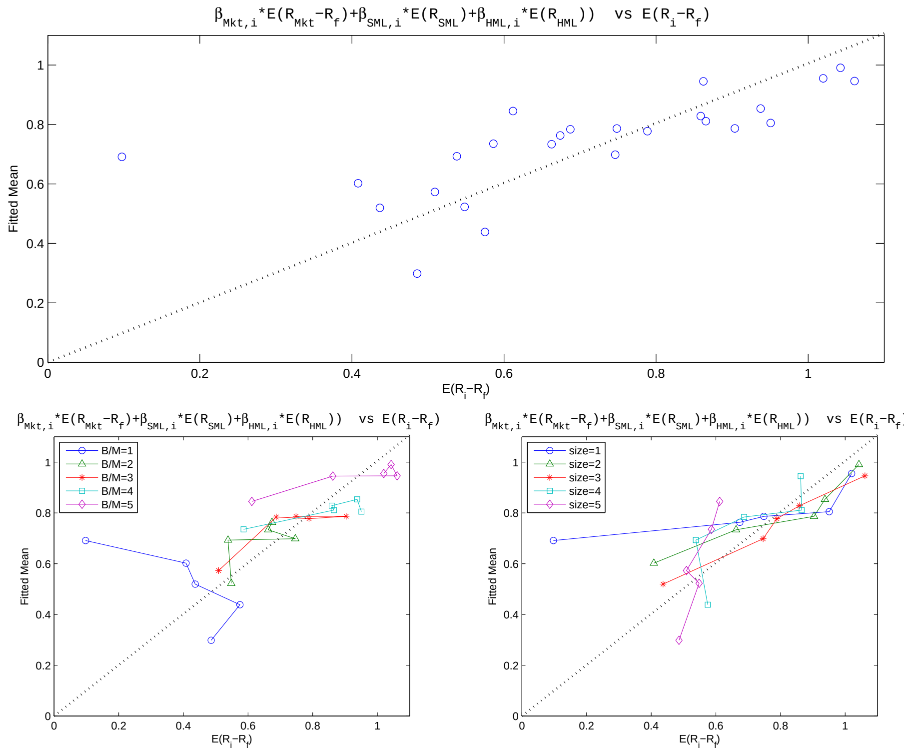


Figure 13: Fama-French Betas of (Size,B/M) Portfolios

Fixed Time



Random Time

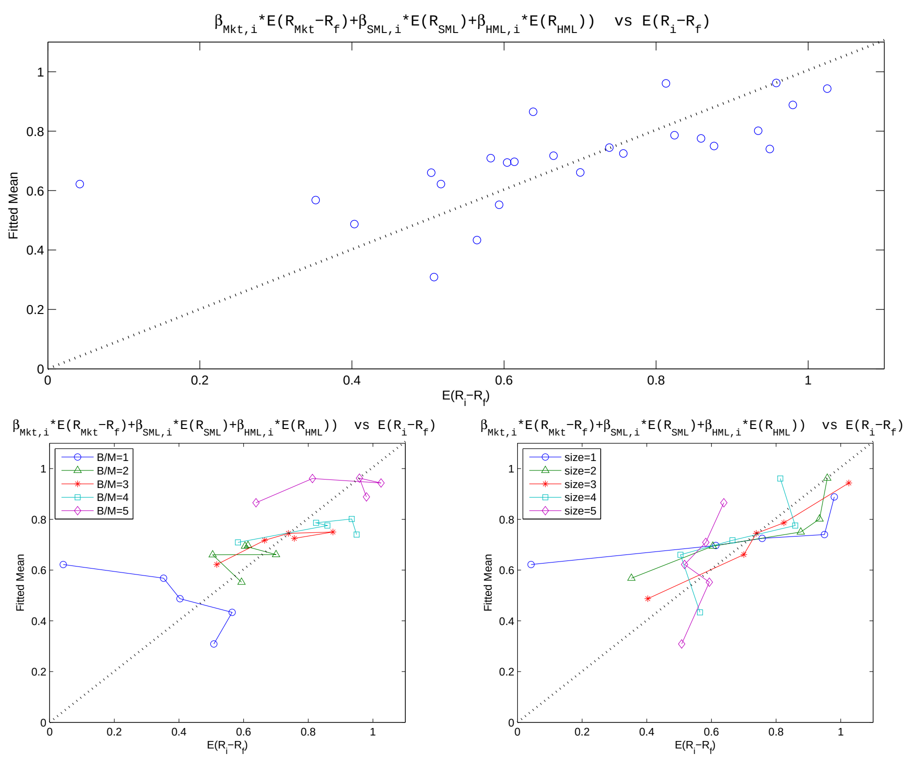


Table 15: Tests of Three-Factor Models with Consumer Industry Factor

Panel A: On B/M Portfolios (Random Time)

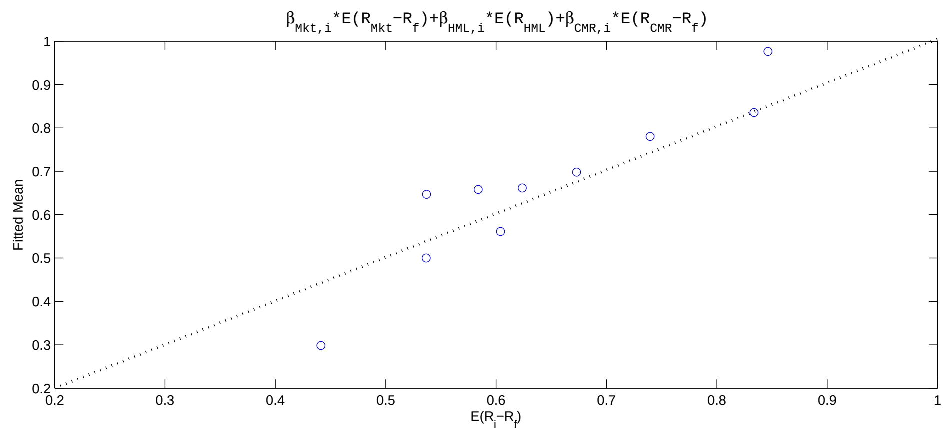
	Alpha	Beta_MKT	Beta_HML	Beta_CMR
1 Growth	0.0122 (-0.0058)	0.7432 (0.0328)	-0.5940 (0.0234)	0.2276 (0.0301)
2	-0.0064 (-0.0055)	0.6999 (0.0308)	-0.1367 (0.0221)	0.3026 (0.0284)
3	-0.0034 (0.0063)	0.7548 (0.0353)	0.0143 (0.0252)	0.2597 (0.0325)
4	-0.0120 (0.0071)	0.9383 (0.0402)	0.2696 (0.0287)	0.0989 (0.0369)
5	-0.0140 (0.0073)	0.9807 (0.0409)	0.3444 (0.0292)	-0.0027 (0.0376)
6	0.0027 (0.0067)	0.9919 (0.0378)	0.3770 (0.0270)	-0.0091 (0.0347)
7	0.0002 (0.0068)	0.8518 (0.0383)	0.4867 (0.0274)	0.1004 (0.0352)
8	-0.0080 (0.0056)	0.9151 (0.0317)	0.6389 (0.0227)	0.0837 (0.0292)
9	0.0087 (0.0068)	0.9646 (0.0381)	0.6889 (0.0272)	0.0995 (0.0350)
10 Value	-0.0052 (0.0100)	1.1857 (0.0562)	0.8397 (0.0402)	0.0442 (0.0517)
10-1 Book-to-Market Strategy	-0.0174 (0.0208)	0.4425 (0.1173)	1.4337 (0.0839)	-0.1834 (0.1079)
Wald		15.8061 (0.1053)		

Panel B: On Size Portfolios (Random Time)

	Alpha	Beta_MKT	Beta_SMB	Beta_CMR
1 Small	0.0103 (0.0074)	0.7624 (0.0401)	1.2405 (0.0276)	0.0415 (0.0383)
2	-0.0072 (0.0051)	0.8768 (0.0275)	1.0502 (0.0189)	0.0824 (0.0263)
3	0.0024 (0.0044)	0.8754 (0.0237)	0.8864 (0.0163)	0.1114 (0.0227)
4	0.0016 (0.0042)	0.8509 (0.0227)	0.7853 (0.0156)	0.1418 (0.0217)
5	0.0108 (0.0044)	0.8892 (0.0242)	0.6675 (0.0167)	0.1044 (0.0231)
6	0.0026 (0.0051)	0.8625 (0.0278)	0.5000 (0.0191)	0.1379 (0.0266)
7	0.0032 (0.0047)	0.8978 (0.0258)	0.3633 (0.0178)	0.1073 (0.0247)
8	0.0016 (0.0051)	0.9471 (0.0280)	0.2282 (0.0192)	0.0690 (0.0267)
9	0.0025 (0.0045)	0.8611 (0.0245)	0.0260 (0.0168)	0.1287 (0.0234)
10 Big	0.0013 (0.0026)	0.9685 (0.0140)	-0.2888 (0.0096)	0.0195 (0.0134)
1-10 Size Strategy	0.0089 (0.0213)	-0.2061 (0.1160)	1.5293 (0.0798)	0.0219 (0.1108)
Wald		27.2450 (0.0024)		

Figure 14: Betas of Three Factor Models with Consumer Industry Factor

On B/M Portfolios - Random Time



On Size Portfolios - Random Time

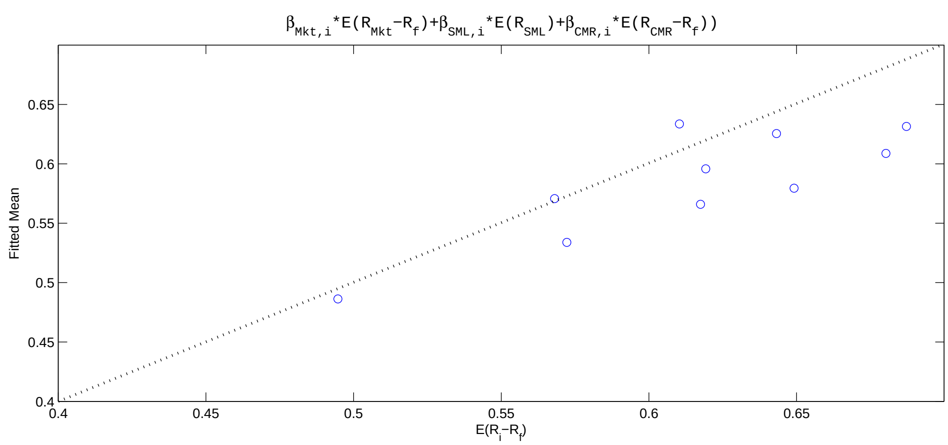


Table 16: Tests of Three-Factor Models with Consumer Industry and B/M Factors on (Size,B/M) Portfolios - Random Time

(Size, B/M)	Alpha	Beta_MKT	Beta_HML	Beta_CMR
(1,1)	0.0005 (0.0209)	1.4048 (0.1178)	-0.5015 (0.0843)	-0.1552 (0.1084)
(1,2)	0.0343 (0.0172)	1.1327 (0.0968)	-0.1011 (0.0692)	-0.0001 (0.0890)
(1,3)	0.0408 (0.0154)	0.9009 (0.0867)	0.1021 (0.0620)	0.1522 (0.0798)
(1,4)	0.0565 (0.0144)	0.8747 (0.0813)	0.2149 (0.0581)	0.1402 (0.0747)
(1,5)	0.0582 (0.0153)	0.9690 (0.0859)	0.4554 (0.0615)	0.1360 (0.0790)
(2,1)	0.0076 (0.0145)	1.1832 (0.0818)	-0.5241 (0.0585)	0.0675 (0.0752)
(2,2)	0.0140 (0.0126)	0.9340 (0.0712)	-0.0265 (0.0509)	0.2038 (0.0655)
(2,3)	0.0518 (0.0114)	0.9256 (0.0643)	0.2295 (0.0460)	0.1566 (0.0592)
(2,4)	0.0405 (0.0103)	0.8850 (0.0582)	0.4041 (0.0416)	0.1773 (0.0535)
(2,5)	0.0238 (0.0120)	1.0446 (0.0676)	0.5996 (0.0484)	0.1607 (0.0622)
(3,1)	0.0165 (0.0115)	1.1652 (0.0649)	-0.5566 (0.0464)	0.0187 (0.0597)
(3,2)	0.0300 (0.0105)	0.8426 (0.0590)	-0.0008 (0.0422)	0.2639 (0.0543)
(3,3)	0.0221 (0.0089)	0.8329 (0.0503)	0.3451 (0.0360)	0.2130 (0.0462)
(3,4)	0.0249 (0.0085)	0.8401 (0.0480)	0.4732 (0.0343)‘	0.2007 (0.0441)
(3,5)	0.0186 (0.0100)	0.9682 (0.0562)	0.6809 (0.0402)	0.1943 (0.0517)
(4,1)	0.0149 (0.0077)	1.0997 (0.0436)	-0.4860 (0.0312)	0.0119 (0.0401)
(4,2)	-0.0160 (0.0079)	0.8547 (0.0446)	0.1326 (0.0319)	0.2386 (0.0410)
(4,3)	0.0052 (0.0076)	0.8214 (0.0430)	0.3543 (0.0307)	0.2237 (0.0395)
(4,4)	0.0226 (0.0074)	0.8996 (0.0416)	0.5080 (0.0297)	0.1404 (0.0383)
(4,5)	-0.0057 (0.0091)	1.0395 (0.0510)	0.7700 (0.0365)	0.1533 (0.0469)
(5,1)	0.0081 (0.0059)	0.6528 (0.0332)	-0.4128 (0.0238)	0.2839 (0.0305)
(5,2)	-0.0116 (0.0069)	0.8168 (0.0391)	0.1439 (0.0279)	0.1687 (0.0359)
(5,3)	-0.0170 (0.0080)	1.0384 (0.0449)	0.3854 (0.0321)	-0.0860 (0.0413)
(5,4)	-0.0259 (0.0072)	0.8291 (0.0404)	0.6111 (0.0289)	0.1104 (0.0371)
(5,5)	-0.0242 (0.0091)	1.0052 (0.0515)	0.7930 (0.0368)	0.0650 (0.0473)
Wald	181.2525 (0.0000)			

Figure 15: Betas of Three Factor Models with Consumer Industry and B/M Factors