A Common Value Auction with State Dependent Participation*

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We study a common value, first-price auction in which the number of bidders is endogenous: the seller (auctioneer) knows the value and solicits bidders at a cost. The number of bidders, which is unobservable, may thus depend on the true value. Therefore, being solicited already conveys information. This "solicitation effect" may soften competition and may impede information aggregation. Under certain conditions, there is an equilibrium in which the auctioneer solicits many bidders, yet the resulting price is not competitive and it fails to aggregate any information. This stands in contrast to the familiar outcomes of standard auctions.

This paper analyzes a common value, first-price auction with the novel feature that the number of bidders may depend on the value. Specifically, there is a single good and two states, ℓ and h, with the common value of the good, v_ω, $\omega = \ell, h$, satisfying $v_h > v_\ell$. In state ω there are n_ω bidders, who receive conditionally independent signals, but do not observe ω or n_ω.

We explore the bidding equilibria of this game. One of the main insights is that, owing to the state dependent participation, competition between bidders can be softened to the point where bidders with high signals are pooling on a common bid (atom) below the ex-ante expected value. This means that, when there are many bidders, the winning bid is essentially independent of the state and possibly non-competitive. This is interesting

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and somewhat surprising from the perspective of auction analysis. But, perhaps more importantly, this reduced competitiveness has implications for the extent of information aggregation by markets with adverse selection.

To understand this insight, recall that, in an ordinary common value auction, winning at a lower bid reflects more negatively on the value of the object than winning at a higher bid. Consequently, partially informed bidders try to evade adverse selection by bidding more aggressively and in the process inject their information into the price. For this reason, an ordinary auction with many bidders is both nearly competitive and may aggregate the information well. In contrast, with state dependent participation, more aggressive bidding might involve more severe adverse selection. If there are sufficiently more bidders in state ℓ than in state h, just being included in the auction already involves a "participation curse" that depresses the expected value estimate held by a bidder. A bidder who overbids everybody else, would bear the full strength of this "curse." But a winning bid from the "middle" of the winning bid distribution would give rise to a "middle winner's blessing" that partly offsets the "participation curse." The reason is that more bidders are likely to bid near or above the "middle" in state ℓ and therefore, conditional on winning, the probability of state h is higher than its interim probability. This induces bidders to escape the "participation curse" by aiming at a "middling" bid. They are thus driven away from overbidding and towards more pooling. Consequently, both the competition and the incorporation of information into prices are dampened.

Our main characterization result concerns the general form of the bidding equilibrium, when there are many bidders in each state. To explain it, let G_ω denote the c.d.f. of the (conditionally independent) signals received by bidders in state ω, with support on $[\underline{x}, \bar{x}]$ and density g_ω. Assume that the likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ is monotonically increasing so that $\bar{x}$ is the most favorable signal for h. When n_ℓ and n_h are sufficiently large, the general form of bidding equilibria is determined by the magnitude of $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$. If this ratio is below 1, then any bidding equilibrium is necessarily of the pooling type mentioned above. If this ratio is above 1, then any bidding equilibrium is necessarily of a separating type, resembling the equilibrium of an ordinary common value auction with no significant atoms in the winning bid distribution and with a higher expected winning bid in state h than in ℓ.

State dependent participation may arise for a number of reasons. This paper focuses on a straightforward reason—costly solicitation of bidders by an informed seller, who knows ω, and invites n_ω bidders to participate.[1] It is natural to wonder whether an equilibrium of the full game with a strategic solicitation decision rules out any of the patterns described

[1] In Murto and Välimäki (2015) state dependent participation arises because the bidders make costly entry decisions after receiving their signals; in Atakan and Ekmekci (2015) bidders' entry decisions differ across states due to differences in the value of outside options. A range of behavioral considerations might have a similar effect as well.

above. Indeed, another insight of this paper is that a non-competitive atom of the type discussed above may arise in an equilibrium of the full game.[2]

Let us point out this paper's main contributions. From the conceptual perspective, one contribution is the idea of considering an auction with state dependent participation. As mentioned before, this could arise for a variety of reasons and, while there might be some differences among different scenarios of this type, the main considerations should be similar. Therefore, shining the light on price formation in the presence of this consideration is interesting in it's own right, independently of the specifics of an underlying process that determines the participation.

From the substantive perspective, two of the paper's main contributions are, first, the introduction of a new model that seems relevant for numerous economically interesting situations and, second, obtaining insights with obvious implications. The most distinct of those insights are the potential emergence of an atom in the winning bid distribution and the associated failures of competitiveness and information aggregation. Let us elaborate on these two contributions.

Viewed narrowly as a model of a single auction with bidder solicitation, it captures a hitherto neglected element of common scenarios where the sale of an asset or the contracting out of a project take the form of a bid collection process by some deadline. The bidders may not know how many other bidders would submit bids by the deadline, nor do they know how much effort the seller is making to interest potential bidders, but are aware that such efforts may be related to the seller's private information. Our model might be better suited for discussing such scenarios than standard auction models.[3]

Viewed more broadly, the "auction" is just a convenient abstraction of a free form price formation process that takes place in a decentralized market environment, rather than in a formal mechanism.[4] To have in mind a concrete scenario of this type, consider a stylized market for investment finance.[5] Entrepreneurs, each of whom is seeking to finance a single investment project, face investors looking to acquire such projects. Each entrepreneur (who is the counterpart of the "auctioneer" in our model) knows the value of her own project and contacts multiple investors, who in turn observe signals of the value of projects presented to them and respond with acquisition offers. Our model adds to this environment the recognition that the entrepreneur's efforts may depend on their private information and that this may have some real consequences.[6]

[2] We focus here on this more striking result. In Lauermann and Wolinsky (2013) we show that the separating pattern may also arise in a full equilibrium.

[3] Subramanian (2000) contains numerous examples illustrating this point.

[4] This is in the spirit of Wilson (1977) and Milgrom (1979, 1981).

[5] This is just a parable. Our model is not tailored to this application and by no means intends to offer a detailed discussion of financial markets.

[6] Since quite a few of the main points of interest can already be addressed in the auction-like interaction of a single entrepreneur and the investors she contacts, our formal model and analysis are couched in terms

Before discussing the significance of the insight concerning the emergence of a non-competitive atom, it is useful to get two points out of the way. One point is that the stark atom of our simple model need not be taken literally. As explained above, the atom is a robust consequence of the "middle winner's blessing" effect. In a version of the model with noise, it would translate to a "cloud" of bunched bids, as illustrated by the example of Section 7.1. Thus, in a richer world the implication would be that the winning bid distribution is quite flat rather than having an exact atom. The other point is that the phenomenon of nearly identical (or unnaturally similar) bid prices has been observed in formal auctions[7] and is familiar from markets as well. It is usually attributed to collusion, and this may well be the right explanation in many cases. Our point is that the bunching of bids or prices is not unheard of in what economists like to call the "real world," and our insight might provide a potential alternative way of thinking about some such scenarios.

One implication of the emergence of an atom is a disconnect between prices and values. This might have consequences for the efficiency of resource allocation. In the very basic common value environment that we are considering, trade is always desirable, and the price is just a transfer. But plausible extensions of this model, like adding some heterogeneity in private values to either side of the market, would introduce efficiency considerations. In such a case, the discrepancy between prices and values and the failure to aggregate information would translate to real economic costs.

This insight also points out adverse selection as another possible factor for explaining the phenomenon of "sticky prices"—the failure of prices to respond to some changes in the underlying parameters. Traditionally this is explained by institutional rigidities, menu costs, and efficiency wages. An atom that would arise in a market of the sort we have in mind would obviously remain an equilibrium in the face of some changes in the magnitude of the parameters.

1.1 Literature Connections

The question of information aggregation by prices is a fundamental question of economic theory. It was initially addressed in the context of competitive markets by the rational expectations literature. It was subsequently addressed in auction market models that account for strategic behavior. In the context of a common value auction, this information aggregation question translates to whether the winning bid is near the true value when there are many bidders (of course, there is no reason to expect it when only a few bidders participate). Translated to the two-state model considered here, Milgrom's (1979) result is that the winning bid in an ordinary common value auction approaches the true value as the number of bidders grows if and only if the likelihood ratio of the two states is

of an auction model.

[7] See, e.g., Mund (1960) and Comanor and Schankerman (1976).

unbounded over the support of the signal distribution (see also Wilson (1977)). Although Milgrom does not say it, his arguments seem to imply continuity in the sense that the availability of signals with large but bounded likelihood ratio would result in a significant degree of information aggregation. Kremer (2006) showed that ordinary common value auctions become competitive in the sense that the expected price approaches the expected value when the number of bidders grows. Our analysis shows that, with state dependent participation both of these results may fail and uncovers the conditions under which they are valid.

From the perspective of auction theory, the closest papers are the above mentioned papers, Murto and Valimaki (2015) and Atakan and Ekmekci (2015). They also have a common value auction with state dependent participation,[8] but they explore other mechanisms that generate it.

Broecker (1990) and Riordan (1993) model competition among incompletely informed banks over the business of potential borrowers as an ordinary auction—the borrowers contact all the banks for quotes. Our analysis implies that such competition may be significantly affected when borrowers choose how many banks to contact based on their private information.

Our model can also be thought of as adding adverse selection to Burdett and Judd's (1983) simultaneous ("batch-") search model. In that model, a buyer obtains a sample of prices from sellers of a homogenous product. Their buyer is the counterpart of our seller. Our model endows this buyer with private information that might affect the seller's cost. This might be relevant for markets of certain services, such as repair or the above mentioned credit markets. The presence of adverse selection gives rise to different analysis and results. In particular, in Burdett and Judd's model, the more convincing equilibrium becomes competitive when the sampling cost becomes negligible, while this is not necessarily the case in our model.

In markets of the sort we are interested in, the contacts made by agents do not always follow a rigid protocol—sometimes they are indeed simultaneous, sometimes sequential, and sometimes a combination of the two. We focus here on the simultaneous case; we explored the sequential scenario in Lauermann and Wolinsky (2016). While there are, of course, salient relations between these two papers, there are significant differences. First, the analysis of these two models is different and there are some meaningful differences in the results. But, perhaps more importantly, there are significant qualitative differences in the manner in which information is incorporated into prices and allocations. In the sequential-search-with-bargaining model there is no direct price competition. The search forces still drive the prices to proximity with the average value, so the outcome is nearly competitive when search frictions are small. But the price is not a direct instrument—

[8] Remark to the referees: These papers are subsequent to the earlier versions of our paper.

uninformed agents with promising signals cannot actively overbid—and therefore the extent of information aggregation is determined by the interaction of search and the signal technology. In contrast, the auction setting assigns a prominent role to price competition. The uninformed may try to evade adverse selection by bidding more aggressively, and in the process inject their information into the price. For this reason, a large ordinary auction is both nearly competitive and aggregates information well. In the non-competitive equilibria of our model, more aggressive bidding leads to even more severe adverse selection. In this sense, non-competitive equilibria are tied closely to the bidding, and these equilibria have no counterpart in the search model.

Finally, Lauermann and Wolinsky (2013) is a fuller and more technical version of the present paper. It contains some of the characterization results omitted from this paper, in particular the complete characterization of the separating equilibria and existence results. All proofs are relegated to an appendix, where a $*$ next to the result indicates that the proof is in the online appendix.

2 Model

Basics.—This is a single-good, common value, first-price auction environment with two underlying states, h and ℓ. There are N potential bidders (buyers). The common values of the good for all potential bidders in the two states are v_ℓ and v_h, respectively, with $0 \leq v_\ell < v_h$. The seller's cost is zero.

Nature draws a state $\omega \in \{\ell, h\}$ with prior probabilities $\rho_\ell > 0$ and $\rho_h > 0$, $\rho_\ell + \rho_h = 1$. The seller learns the realization of the state ω and invites n_ω bidders, $1 \leq n_\omega \leq N$. If $n_\omega < N$, the seller selects the invitees randomly with equal probability. We use $\mathbf{n}$ to denote the vector (n_ℓ, n_h).

The seller incurs a solicitation cost $s > 0$ for each invited bidder. We assume that $N \geq \frac{v_h}{s}$. Therefore, N does not constrain the seller.

Each invited bidder observes a private signal $x \in [\underline{x}, \bar{x}]$ and submits a bid $b \in [0, v_h]$. Conditional on the state $\omega \in \{\ell, h\}$, signals are independently and identically distributed according to a cumulative distribution function (c.d.f.) G_ω. A bidder neither observes ω nor n_ω.

The invited bidders bid simultaneously: The highest bid wins and ties are broken randomly with equal probabilities.

If in state $\omega \in \{h, \ell\}$ the winning bid is p, then the payoffs are $v_\omega - p$ for the winning bidder and zero for all others. The seller's payoff is $p - n_\omega s$.

Further Details.— The signal distributions G_ω, $\omega \in \{\ell, h\}$, have identical supports, $[\underline{x}, \bar{x}] \subset \mathbb{R}$, no atoms, and strictly positive densities g_ω. The likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ is non-decreasing and right-continuous, with $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \lim_{x \to \bar{x}} \frac{g_h(x)}{g_\ell(x)}$. This is the weak monotone likelihood ratio property (MLRP): larger values of x indicate a (weakly) higher likelihood of the higher value[9]. The signals are not trivial and boundedly informative,

$$0 < \frac{g_h(\underline{x})}{g_\ell(\underline{x})} < 1 < \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \infty.$$

Expected Payoffs and Equilibrium.— Recall that the state $\omega \in \{\ell, h\}$ and the number of bidders n_ω are unobservable to bidders. A bidder's posterior probability of ω, conditional on being solicited and receiving signal x, is

$$\Pr[\omega|x; \mathbf{n}] \triangleq \frac{\rho_\omega g_\omega(x) \frac{n_\omega}{N}}{\rho_\ell g_\ell(x) \frac{n_\ell}{N} + \rho_h g_h(x) \frac{n_h}{N}} = \frac{\rho_\omega g_\omega(x) n_\omega}{\rho_\ell g_\ell(x) n_\ell + \rho_h g_h(x) n_h}.$$

where ρ_ω, $g_\omega(x)$, and $\frac{n_\omega}{N}$, respectively, reflect the information contained in the prior belief, in the signal x, and in the bidder being invited. Notice that N cancels out and hence does not play any role in the analysis.

A bidding strategy β prescribes a bid as a function of the signal realization,

$$\beta : [\underline{x}, \bar{x}] \to [0, v_h].$$

We study symmetric, pure, and non-decreasing bidding strategies. Our companion paper Lauermann and Wolinsky (2013) establishes that equilibrium bidding strategies are necessarily non-decreasing when $n_\omega \geq 2$, $\omega = \ell, h$, which are the only cases considered in the present paper.

Let $\pi_\omega(b|\beta, n)$ be the probability of winning with bid b, given state ω, bidding strategy β employed by the other bidders, and n bidders. The expected payoff to a bidder who bids b, conditional on participating and observing the signal x, given the bidding strategy β and the participation $\mathbf{n} = (n_\ell, n_h)$, is

$$U(b|x; \beta, \mathbf{n}) = \frac{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b|\beta, n_\ell)(v_\ell - b) + \rho_h g_h(x) n_h \pi_h(b|\beta, n_h)(v_h - b)}{\rho_\ell g_\ell(x) n_\ell + \rho_h g_h(x) n_h}. \quad (1)$$

Alternatively, we can write

$$U(b|x; \beta, \mathbf{n}) = \Pr[\text{ win at } b \mid x; \beta, \mathbf{n}] (\mathbb{E}[\, v \mid x, \text{win at } b; \beta, \mathbf{n}] - b), \quad (2)$$

where

$$\mathbb{E}[\, v \mid x, \text{ win at } b; \beta, \mathbf{n}] = \frac{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b|\beta, n_\ell) v_\ell + \rho_h g_h(x) n_h \pi_h(b|\beta, n_h) v_h}{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b|\beta, n_\ell) + \rho_h g_h(x) n_h \pi_h(b|\beta, n_h)}, \quad (3)$$

and

$$\Pr[\text{ win at } b \mid x; \beta, \mathbf{n}] = \frac{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b|\beta, n_\ell) + \rho_h g_h(x) n_h \pi_h(b|\beta, n_h)}{\rho_\ell g_\ell(x) n_\ell + \rho_h g_h(x) n_h}. \quad (4)$$

[9] Weak MLRP means that discrete signals are a special case of our model.

The numerator and denominator of (1), (3), and (4) can be divided by $\rho_\ell g_\ell(x) n_\ell \pi_\ell(b|\beta, n_\ell)$ or $\rho_\ell g_\ell(x) n_\ell$ to express them in terms of the compound likelihood ratios $\frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{n_h}{n_\ell}\frac{\pi_h(p|\beta,n_h)}{\pi_\ell(p|\beta,n_\ell)}$ or $\frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{n_h}{n_\ell}$. When convenient, we sometimes use those transformations.

Let $\mathbb{E}[v]$, without any conditioning, denote the expected ex-ante value of the good

$$\mathbb{E}(v) \triangleq \rho_\ell v_\ell + \rho_h v_h.$$

From here on, the profile $(\beta, \mathbf{n})$ will typically be suppressed from the arguments and we write $U(b|x)$, $\mathbb{E}[\, v \mid x, \text{ win at } b]$, and so forth to simplify the notation.

3 Bidding Game and Bidding Equilibrium

This part focuses on the bidding behavior for a given pattern of state dependent participation. The understanding of this situation is both of interest in its own right (as discussed in the introduction) and used as a building block for the analysis of a full model that includes endogenous solicitation.

A *bidding game* $\Gamma_0(N, \mathbf{n})$ is the game among the bidders given state dependent participation $\mathbf{n} = (n_\ell, n_h)$. The ordinary common value auction is a special case of the bidding game with $n_\ell = n_h$.

Recall that the state $\omega \in \{\ell, h\}$ and the number of bidders n_ω are unobservable to bidders.

A *bidding equilibrium* of $\Gamma_0(N, \mathbf{n})$ is a non-decreasing bidding strategy β such that $b = \beta(x)$ maximizes $U(\cdot|x; \beta, \mathbf{n})$ for all x.

One significant consequence of the state dependent participation is the emergence of *atoms* in the bidding equilibrium. The strategy β has an atom at p if

$$x_-(p) \triangleq \inf\{x \in [\underline{x}, \bar{x}] \,|\, \beta(x) \geq p\} < \sup\{x \in [\underline{x}, \bar{x}] \,|\, \beta(x) \leq p\} \triangleq x_+(p), \qquad (5)$$

where $\sup \emptyset = \underline{x}$ and $\inf \emptyset = \bar{x}$. In auctions with private values, a standard argument involving slight overbidding or undercutting precludes atoms in which bidders get positive payoffs. This argument does not apply directly to common value auctions, since overbidding the atom may have different consequences in different underlying states owing to possibly different frequencies of bids that are tied in the atom in the different states. Still, as is shown below, a somewhat more subtle argument still precludes atoms in an ordinary common value auction ($n_\ell = n_h = n$), except at the lowest equilibrium bid. However, when $n_\ell > n_h$, atoms may arise in a bidding equilibrium.

3.1 Example of an Atom in a Bidding Equilibrium

Suppose that $v_\ell = 0$ and $v_h = 1$, with uniform prior $\rho_h = \rho_\ell = \frac{1}{2}$. Let $[\underline{x}, \bar{x}] = [0, 1]$, with densities $g_h(x) = 0.8 + 0.4x$, and $g_\ell(x) = 1.2 - 0.4x$. Thus, $\frac{g_h(x)}{g_\ell(x)}$ is increasing as required.

Claim 1 *Suppose $n_\ell = 6$ and $n_h = 2$. Let $\bar{b}$ be any number in $[\frac{1}{3}, \frac{4}{10}]$. There is a bidding equilibrium in which*

$$\beta(x) = \bar{b} \qquad \forall x \in [\underline{x}, \bar{x}].$$

Proof. Substituting $\rho_\ell = \rho_h = 0.5$, $v_\ell = 0$, $v_h = 1$, $n_\ell = 6$, and $n_h = 2$ into (3) and then dividing both the numerator and the denominator by $\rho_\ell g_\ell(x) \pi_\ell(b)$,

$$\mathbb{E}[v|x, \text{win at } b] = \frac{\frac{g_h(x)}{g_\ell(x)} \frac{2}{6} \frac{\pi_h(b)}{\pi_\ell(b)}}{1 + \frac{g_h(x)}{g_\ell(x)} \frac{2}{6} \frac{\pi_h(b)}{\pi_\ell(b)}}.$$

Since ties at the atom are broken randomly, $\pi_h(\bar{b}) = \frac{1}{n_h} = \frac{1}{2}$, $\pi_\ell(\bar{b}) = \frac{1}{n_\ell} = \frac{1}{6}$ and

$$\mathbb{E}[v|x, \text{win at } \bar{b}] = \frac{\frac{g_h(x)}{g_\ell(x)} \frac{2}{6} \frac{\frac{1}{2}}{\frac{1}{6}}}{1 + \frac{g_h(x)}{g_\ell(x)} \frac{2}{6} \frac{\frac{1}{2}}{\frac{1}{6}}} = \frac{\frac{g_h(x)}{g_\ell(x)}}{1 + \frac{g_h(x)}{g_\ell(x)}} \geq \frac{\frac{g_h(0)}{g_\ell(0)}}{1 + \frac{g_h(0)}{g_\ell(0)}} = \frac{4}{10} \geq \bar{b}.$$

Therefore, the expected payoff of bidding $\bar{b}$ is nonnegative.

A deviation to $b < \bar{b}$ yields zero payoff since $\pi_\omega(b) = 0$ for $\omega = \ell, h$. A deviation to $b > \bar{b}$ yields negative payoff since $\pi_\omega(b) = 1$, for $\omega = \ell, h$, and hence

$$\mathbb{E}[v|x, \text{win at } b > \bar{b}] = \frac{\frac{g_h(x)}{g_\ell(x)} \frac{2}{6} \frac{1}{1}}{1 + \frac{g_h(x)}{g_\ell(x)} \frac{2}{6} \frac{1}{1}} \leq \frac{\frac{g_h(1)}{g_\ell(1)} \frac{2}{6}}{1 + \frac{g_h(1)}{g_\ell(1)} \frac{2}{6}} = \frac{1}{3} \leq \bar{b} < b.$$

Therefore, there is no bid $b \neq \bar{b}$ that yields a higher expected payoff than $\bar{b}$. □

The key to the atom's immunity to deviations is $n_h/n_\ell < 1$. Slightly overbidding the atom would result in a discontinuous increase in payoff in state h, but an even more significant decrease in state ℓ. In other words, given the uniform tie-breaking rule, bidding in an atom provides insurance against winning too frequently in the negative payoff state ℓ ("hiding in the crowd"),[10] while upon overbidding it, a bidder forgoes this insurance.

What matters for this argument is, of course, only the ratio of the number of bidders across states. Bidding $\bar{b} \in [1/3, 4/10]$ remains an equilibrium (given the other data of the example) whenever $n_\ell = 3n_h$ and $n_\ell \geq 2$.[11] Thus, making the auction large by proportionally increasing the number of bidders does not make the auction more competitive and may not increase the revenue of the seller.

Finally, observe that, if the participation is determined by costly solicitation, the seller's best response to the single atom bidding equilibria of the example is $(n_\ell, n_h) = (1, 1)$ rather than the numbers $(n_\ell, n_h) = (6, 2)$ assumed in the example. Nevertheless, we show

[10] In Atakan and Ekmekci (2014), the winning bidder in a common value auction values information about the state for the sake of a subsequent decision. This may give rise to an atom in the bid distribution because overbidding it would result in the loss of the information inferred from winning at the atom, the probability of which differs across states.

[11] More generally, one can easily show that an equilibrium in which β is constant for *all* x (as in the example) exists whenever $\frac{n_h}{n_\ell} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \leq \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$.

later that an atom can also arise when the solicitation is optimal, though in such a case it may not be that all signals are followed by the same bid.

3.2 With no Participation Curse there are no Atoms

In the case of $n_h \geq n_\ell$ (no participation curse), the bidding equilibrium is essentially free of atoms. If either $n_h > n_\ell$ or $\frac{g_h}{g_\ell}$ is strictly increasing at the bottom of the signal distribution, then the bidding equilibrium in the case of $n_h \geq n_\ell$ cannot have an atom at all. Otherwise, atoms may arise only at the lowest bid. The case of $n_h \geq n_\ell$ includes of course the ordinary common value auction $n_\ell = n_h = n$ as a special case.

Proposition 1 * (**No Atoms if** $n_h \geq n_\ell$) *Suppose that* β *is a bidding equilibrium of* $\Gamma_0(N, \mathbf{n})$, *with* $n_h \geq n_\ell \geq 2$.

- *If* $n_h > n_\ell$, *then* β *is strictly increasing.*
- *If* $n_h = n_\ell$, *let* $\hat{x} = \sup\{x | \frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}\}$.
  - β *is constant on* $(\underline{x}, \hat{x})$,
  - β *is strictly increasing on* $[\hat{x}, \bar{x}]$.

It is well known that there are no atoms in the standard common value auction when $\frac{g_h}{g_\ell}$ is strictly increasing and continuous (see Rodriguez (2000) and McAdams (2007)). This proposition extends these results to the case of $n_h \geq n_\ell$ and $\frac{g_h}{g_\ell}$ weakly increasing or discontinuous.[12]

The key to the absence of atoms in standard common value auctions with strictly increasing $\frac{g_h}{g_\ell}$ is as follows. If a bidding equilibrium β has an atom at some $\bar{b}$ in the interior of the support, then

$$\lim_{\varepsilon \to 0} \mathbb{E}[v|x, \text{win at } \bar{b} + \varepsilon] > \mathbb{E}[v|x, \text{win at } \bar{b}], \tag{6}$$

since, by overbidding the atom, a bidder wins in more favorable circumstances. This is because, conditional on the event that the highest other bid is $\bar{b}$ (which is when the difference between $\bar{b}$ and $\bar{b} + \varepsilon$ matters), the bid $\bar{b} + \varepsilon$ wins for sure in both states, whereas the bid $\bar{b}$ is less likely to win in state h than in state ℓ. This follows from $n_h \geq n_\ell$ and from

$$\Pr(\beta(x) = \bar{b} | \beta(x) \leq \bar{b}, \omega) = \frac{G_\omega\left(x_+(\bar{b})\right) - G_\omega\left(x_-(\bar{b})\right)}{G_\omega\left(x_+(\bar{b})\right)},$$

[12] The second part of the proposition is related to results from Rodriguez (2000), which imply that when $n_l = n_h = 2$ and $\frac{g_h}{g_l}$ is not strictly increasing, then atoms may occur only at the bottom of the bid distribution.

being higher for $\omega = h$ than for $\omega = \ell$ when $\frac{g_h}{g_\ell}$ is strictly increasing.[13] Therefore, overbidding $\bar{b}$ is profitable, since it strictly increases both the expected value conditional on winning (by (6)) and the probability of winning.

Thus, in the case of $n_h > n_\ell$ or $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$, the incentive to overbid an atom is generated both by the improved selection of types at the higher bid, as captured by (6), and by the usual Bertrand logic of a discrete jump in the winning probability. Conversely, as illustrated by the example in the beginning of this section, when $n_h < n_\ell$, an atom may be stable because (6) is reversed, and the worse selection of types following a slight overbid overwhelms the Bertrand effect.

4 The Full Game, Optimal Solicitation and Large Numbers

This section introduces formally the full game and its equilibrium as well as some basic steps towards the subsequent derivation of the characterization results.

4.1 The Full Game and Equilibrium

Let $\Gamma(s)$ be the full game that includes both strategic bidder solicitation by the seller and strategic bidding by the buyers. A bidding strategy β is as before; a solicitation strategy $\mathbf{n} = (n_\ell, n_h)$ prescribes the number of bidders solicited by the seller in each state. The potential number of bidders in $\Gamma(s)$ is N_s with $N_s \geq \frac{v_h}{s}$, which guarantees that it is never profitable for the seller to solicit all potential bidders. Also, let $\mathbb{E}[p|\omega;\beta,n]$ denote the expected *winning* bid in state ω.

A *pure equilibrium* of $\Gamma(s)$ consists of a non-decreasing bidding strategy β and a solicitation strategy $\mathbf{n} = (n_\ell, n_h)$ such that (i) β is a bidding equilibrium of $\Gamma_0(N_s, \mathbf{n})$, and (ii) the solicitation strategy is optimal for the seller,

$$n_\omega \in \underset{n\in\{1,2,\ldots,N_s\}}{\arg\max}\ \mathbb{E}[p|\omega;\beta,n] - ns.$$

Since a pure equilibrium might not exist, we allow for mixed solicitation strategies. Let $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$ denote a mixed solicitation strategy, where $\eta_\omega(n)$ is the probability with which $n = 1, ..., N_s$ bidders are invited in state ω.

The expected payoff $U(b|x;\beta,\boldsymbol{\eta})$ and the probability of winning $\pi_\omega(b|\beta,\boldsymbol{\eta})$ are now functions of the mixed strategy $\boldsymbol{\eta}$. Some explicit expressions of these magnitudes that are needed for the proofs are stated in Subsection 8.2 of the appendix.

[13] In the special case where $n_h = n_\ell$ and $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all signals x for which $\beta(x) \leq \bar{b}$, (6) holds as an equality. In this case, $\frac{G_\omega(x_+)-G_\omega(x_-)}{G_\omega(x_+)}$, is the same in both states.

In a complete analogy to the definitions for pure strategies, $\Gamma_0(N, \boldsymbol{\eta})$ is the bidding game given $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$ and $\Gamma(s)$ is the full game. A bidding equilibrium of $\Gamma_0(N, \boldsymbol{\eta})$ is a strategy β such that, for all x, $b = \beta(x)$ maximizes $U(b|x; \beta, \boldsymbol{\eta})$.

The strategy profile $(\beta, \boldsymbol{\eta})$ is an *equilibrium* of $\Gamma(s)$ if (i) β is a bidding equilibrium of $\Gamma_0(N_s, \boldsymbol{\eta})$ and (ii) the solicitation strategy is optimal,

$$\eta_\omega(n) > 0 \Rightarrow n \in \underset{n \in \{1,2,\ldots,N_s\}}{\arg\max} \; \mathbb{E}[p|\omega; \beta, n] - ns.$$

4.2 Optimal Solicitation: Characterization

The seller's payoff, $\mathbb{E}[p|\omega; \beta, n] - ns$, is strictly concave in n unless β is constant. Consequently, either there is a unique optimal number of sampled bidders or the optimum is attained at two adjacent integers.

Lemma 1 * **Optimal Solicitation** *Given any bidding strategy β, there is an integer n^*_ω such that*

$$\{n^*_\omega, n^*_\omega + 1\} \supseteq \underset{n \in \{1,2,\cdots,N\}}{\arg\max} \; \mathbb{E}[p|\omega; \beta, n] - ns.$$

This result is familiar from other contexts and is an immediate consequence of the concavity of the expectation of the first-order statistic in n, but a self contained proof is provided in the online appendix. Given the lemma, we restrict attention to mixed strategies η whose support contains at most two adjacent integers. Any such mixed strategy η_ω can be described by $n_\omega \in \{1, ..., N\}$ and $\gamma_\omega \in (0, 1]$, where $\gamma_\omega = \eta_\omega(n_\omega) > 0$ and $1 - \gamma_\omega = \eta_\omega(n_\omega + 1) \geq 0$. A solicitation strategy is pure if $\gamma_\omega = 1$. Thus, from here on, when we talk about n_ω in the context of a strategy η_ω, we mean the bottom of the support of η_ω. In fact, since our characterization results pertain to the case of small sampling costs and many bidders, they are not affected by whether the equilibrium strategies are actually pure or mixed. Mixed solicitation strategies matter only for the existence arguments.[14]

4.3 Many Bidders

From here on the discussion focuses on scenarios with many bidders. From a substantive point of view, this is the relevant case for the questions of competitiveness and information aggregation in markets. From an analytical point of view, this case makes it easier to get clean characterization results.

In Section 5 we consider the full game where the number of bidders is determined endogenously. The primitive there is a sequence

$$\left(s^k\right)_{k=1}^{\infty}, \quad s^k > 0 \text{ and } s^k \to 0, \tag{7}$$

[14] The irrelevance of mixing for the case of many bidders is a formal result, which is not included in this paper.

that induces a sequence of games $\Gamma\left(s^k\right)$ and a corresponding sequence of equilibrium bidding and solicitation strategies $(\beta^k, \boldsymbol{\eta}^k)$ with $\boldsymbol{\eta}^k = (\eta_\ell^k, \eta_h^k)$. It turns out that, as $s^k \to 0$, the optimal solicitation would indeed result in ever larger numbers of solicited bidders.

In Section 6 the state dependent numbers of bidders are exogenous. We look at a sequence of bidding games

$$\Gamma_0\left(N^k, \mathbf{n}^k\right) \quad \text{s.t.} \quad n_\omega^k \to \infty, \qquad\qquad \omega = \ell, h,$$

and a corresponding sequence of bidding equilibria β^k.

In either of those cases, we look at the limits of equilibrium magnitudes as $k \to \infty$. For the sake of reducing the complexity, we make the following simplifications. First, when we discuss a fixed sequence $(\beta^k, \boldsymbol{\eta}^k)$, and there is no danger of confusion, magnitudes induced by $(\beta^k, \boldsymbol{\eta}^k)$ is written as $U^k(b|x)$, $\pi_\omega^k(b)$, $\mathbb{E}^k[v|x,\text{win at } b]$ etc. (rather than $U(b|x;\beta^k,\boldsymbol{\eta}^k)$ etc.). Second, the term "limit" (and the operator lim) always refers to a limit over any subsequence such that all the magnitudes of interest are converging, though we will not repeat this qualification each time. Third, since almost all limits we take are with respect to k, we often omit the delimiter $k \to \infty$ from the expression lim.

5 Atoms in Full Equilibrium: Failure of Information Aggregation

This section shows that, under certain conditions, an atom may arise in full equilibrium. This means that strategic solicitation does not prevent the failure of the price to aggregate information.

The example in Section 3.1 established the possibility of atoms in a **bidding** equilibrium, but it was not a full equilibrium. The seller's best response to that bidding behavior is to solicit only one bid, which of course would not in turn induce that bidding behavior. Some reflection would reveal that it is not straightforward to extend a bidding equilibrium into a full equilibrium since it is not easy to see why the pattern of solicitation required to sustain an atom in the bidding equilibrium would indeed be optimal given the bidding it induces. Our approach is constructive. Under certain assumptions—including a sufficiently small solicitation cost—we construct a full equilibrium that exhibits an atom.

The main ideas can be presented in the context of a simple case with two effective signals. We will therefore do so, although the same results are also valid for a more general model.

Good News/Bad News Signal. In the "good news/bad news" case considered here, there is some $\hat{x} \in (\underline{x}, \bar{x})$ such that

$$\frac{g_h(x)}{g_\ell(x)} = \begin{cases} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \text{constant} > 1 & \text{if} \quad x \geq \hat{x}, \\ \frac{g_h(\underline{x})}{g_\ell(\underline{x})} = \text{constant} < 1 & \text{if} \quad x < \hat{x}. \end{cases} \tag{8}$$

Thus, while the model continues to have a continuum of signals, from the information perspective there are only two signals: all signals below $\hat{x}$ have the same information content, and similarly all signals above $\hat{x}$ have the exact same information content as each other. Without further loss of generality, g_ω is assumed to be constant on $[\underline{x}, \hat{x})$ and on $[\hat{x}, \bar{x}]$.

Also assume that the signal is sufficiently informative so that

$$\frac{1}{G_\ell(\hat{x})} < \frac{g_h(\bar{x})}{g_\ell(\bar{x})}. \tag{9}$$

As explained in Section 4.3, we consider a sequence $\left(s^k\right)_{k=1}^{\infty}$, as in (7), and corresponding sequences of games $\Gamma\left(s^k\right)$ with strategy profiles $(\beta^k, \boldsymbol{\eta}^k)$. Recall that $\mathbb{E}[v] \equiv \rho_\ell v_\ell + \rho_h v_h$ is the ex-ante expected value and that $n_\omega^k = \min(\text{support}(\eta_\omega^k))$, $\omega = \ell, h$.

Proposition 2 *Suppose G_ω, $\omega = \ell, h$, satisfy (8) and (9) and $\left(s^k\right)_{k=1}^{\infty}$ is such that $s^k > 0$ and $s^k \to 0$.*

There exists equilibria $(\beta^k, \boldsymbol{\eta}^k)$ such that $\min\{n_\ell^k, n_h^k\} \to \infty$, and, for some $\bar{b}$ and all k large enough,

$$\beta^k(x) = \bar{b} < \mathbb{E}(v) \qquad \forall x \geq \hat{x}.$$

All bidders with signals above $\hat{x}$ bid $\bar{b}$, which is below the ex-ante expected value, $\mathbb{E}[v]$. Since $n_\omega^k \to \infty$ and $\Pr(x \geq \hat{x}|\omega) > 0$, for $\omega = \ell, h$, the winning bid is almost surely $\bar{b}$ in both states in the limit. An increasingly large number of bidders are tied at that bid. The implication is that the winning bid/price does not aggregate any information.

It is useful to observe that the atom is not necessitated by the good news/bad news signal structure. By Proposition 1, in a conventional auction with $n_\ell = n_h$ and this signal structure, bidding equilibrium strategies must be strictly increasing on $[\hat{x}, \bar{x}]$.

5.1 Equilibrium Construction: Sketch

The complete proof of Proposition 2 is in Appendix 8.3. Some of its main ideas are as follows. We postulate 2-price bidding strategies of the form:

$$\beta^k(x) = \begin{cases} \bar{b} & \text{if} \quad x \geq \hat{x}, \\ \underline{b}^k & \text{if} \quad x < \hat{x}, \end{cases} \tag{10}$$

where $\underline{b}^k < \bar{b}$ and $\lim \underline{b}^k < \bar{b}$.

Given $(\beta^k)_{k=1}^{\infty}$ of this form, any sequence of corresponding optimal solicitation strategies $(\boldsymbol{\eta}^k)_{k=1}^{\infty}$ obviously satisfies $n_\omega^k \to \infty$, $\omega = \ell, h$, since $\bar{b} - \underline{b}^k$ is bounded away from 0 while $s^k \to 0$.

The key step of the proof is that for any $(\boldsymbol{\eta}^k)_{k=1}^{\infty}$ that is optimal given bidding strategies of the form (10),

$$\frac{g_h(\bar{x})}{g_\ell(\bar{x})} \lim \frac{n_h^k}{n_\ell^k} < 1, \tag{11}$$

and $\lim \frac{n_h^k}{n_\ell^k}$ is independent of the choice of $\bar{b}$ and $(\underline{b}^k)$.

These observations are proven (in Step 3 of the proof of Proposition 2) using the marginal conditions for solicitation optimality. Ignoring integer issues—which the formal proof takes into account—the optimality conditions are

$$\begin{aligned}(G_\ell(\hat{x}))^{n_\ell^k}(1-G_\ell(\hat{x}))(\bar{b}-\underline{b}^k) &= s^k,\\ (G_h(\hat{x}))^{n_h^k}(1-G_h(\hat{x}))(\bar{b}-\underline{b}^k) &= s^k.\end{aligned}$$

Substituting out s^k from the two conditions and making a logarithmic transformation, we get

$$n_\ell^k \ln G_\ell(\hat{x}) + \ln(1-G_\ell(\hat{x})) = n_h^k \ln G_h(\hat{x}) + \ln(1-G_h(\hat{x}))\,.$$

Noticing that the first term on each side of the equation dominates the second since $n_\omega^k \to \infty$, we get

$$\lim \frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\hat{x})}{\ln G_h(\hat{x})}. \tag{12}$$

Thus, $\lim \frac{n_h^k}{n_\ell^k} < 1$, and the limit is independent of the choice of choice of $\bar{b}$ and $\left(\underline{b}^k\right)$. Inequality (11) then follows from (12), via $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \frac{1-G_h(\hat{x})}{1-G_\ell(\hat{x})}$ and $\frac{1-z}{\ln z}$ being a decreasing function of z over $(0,1)$.

The significance of inequality (11) is that, for large enough k, the bad news learned from being solicited overwhelms the good news contained in the highest signal. Thus, a bidder with the highest signal is more pessimistic than the prior, $\frac{\rho_h}{\rho_\ell}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h^k}{n_\ell^k} < \frac{\rho_h}{\rho_\ell}$. As we will see, this "overwhelming participation curse" creates an impediment to overbidding.

Another step in proving Proposition 2 is the choice of $\bar{b}$ that would make overbidding it unprofitable. For any $(\beta^k)_{k=1}^{\infty}$ of the form (10) and any corresponding optimal solicitation strategies $\left(\boldsymbol{\eta}^k\right)_{k=1}^{\infty}$,

$$\lim \mathbb{E}^k[v|\bar{x}, \text{win at } \bar{b}] = \mathbb{E}[v] > \lim \mathbb{E}^k[\ v \mid \bar{x}, \text{win at } b' > \bar{b}]. \tag{13}$$

The inequality owes to (11): the likelihood ratio of a bidder with $\bar{x}$ who wins at b', $\frac{\rho_h}{\rho_\ell}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h^k}{n_\ell^k}$, is lower than the prior likelihood ratio, $\frac{\rho_h}{\rho_\ell}$, and this translates to an expected value that is below the ex-ante expected value. The equality in (13) owes to the winning probabilities at $\bar{b}$ being roughly inversely proportional to the number of bidders bidding

$\bar{b}$, i.e., for large k,

$$\frac{\pi_h^k(\bar{b})}{\pi_\ell^k(\bar{b})} \approx \frac{\frac{1}{n_h^k(1-G_h(\hat{x}))}}{\frac{1}{n_\ell^k(1-G_\ell(\hat{x}))}} = \frac{n_\ell^k}{n_h^k}\frac{g_\ell(\bar{x})}{g_h(\bar{x})}.$$

It follows that the likelihood ratio of a bidder with $\bar{x}$ who wins $\bar{b}$ is approximately equal to the prior one,

$$\frac{\rho_h}{\rho\ell}\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{\pi_h^k(\bar{b})}{\pi_\ell^k(\bar{b})} \approx \frac{\rho_h}{\rho\ell}.$$

That is, the information learned from winning at $\bar{b}$ exactly offsets the information learned from being solicited and having a signal $x \geq \hat{x}$, and this translates to an expected value that is approximately equal to the ex-ante expected value, $\mathbb{E}[v]$.[15]

The significance of these two observations is that $\bar{b}$ can be chosen to satisfy

$$\mathbb{E}[v] > \bar{b} > \lim \mathbb{E}^k[\, v \mid \bar{x}, \text{win at } b'],$$

making it profitable for bidders with $x \geq \hat{x}$ to bid $\bar{b}$ and unprofitable to overbid it.

The foregoing explanation presents the more special element in the proof of Proposition 2, capturing the role of the strategic solicitation in generating an atom. To complete the proof, it is verified (in the appendix) that, given any such $\bar{b}$, one can choose a sequence $\left(\underline{b}^k\right)$ with $\lim \underline{b}^k < \bar{b}$ such that the resulting $\left(\beta^k\right)$ together with the corresponding optimal solicitations $\left(\boldsymbol{\eta}^k\right)$ are immune against all other deviations for all values of x.

5.2 Pooling with Many Signals

The assumption that the likelihood ratio takes only two values is not required to establish the existence of a pooling equilibrium. A similar result can be derived in an environment in which the likelihood ratio is a step function with an arbitrary of number steps. One can also allow for an unbounded likelihood ratio, with $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \infty$. The construction of the atom at the top and the argument for why optimal solicitation results in a ratio $\frac{n_h^k}{n_\ell^k}$ that deters even recipients of the best signal from overbidding that atom are essentially as outlined above for the two-signal case. The analysis of the multi-signal case is more complicated in the parts dealing with equilibrium behavior below the atom; see Lauermann and Wolinsky (2013).

6 General Characterization of Bidding Equilibria

We return now to discuss bidding equilibria. As mentioned in the introduction, state dependent participation may arise for a number of reasons. Therefore, the understanding

[15] Alternatively, the equality in (13) follows from the law of iterated expectations. In the limit, as $k \to \infty$, almost surely the winner has a signal $x > \hat{x}$ in both states. Thus, this event contains no information about the state, and the posterior probability conditional on it is equal to the prior.

of bidding equilibria under such circumstances is of interest in its own right, independently of a specific underlying process that determines the participation. We consider a sequence of bidding games $\Gamma_0\left(N^k, \mathbf{n}^k\right)$ such that $n_\omega^k \to \infty$, $\omega = \ell, h$, and $\lim \frac{n_h^k}{n_\ell^k} \in (0, \infty)$ exists.[16] With "many" bidders, only bids associated with signals that are sufficiently close to $\bar{x}$ would have significant probability of winning. Therefore, the object of interest is the equilibrium distribution of the *winning* bid

$$F_\omega\left(p|\beta, n\right) = \left(G_\omega\left(x_+^k(p)\right)\right)^n,$$

and its limits, rather than the distribution of all the bids. We use the shorthand $F_\omega^k(p)$ for $F_\omega\left(p|\beta^k, n_\omega^k\right)$, as we are doing for other functions.

6.1 Winning Bid Distribution: Pooling vs. Separating

Our main results expose a sharp relationship between $\frac{n_h^k}{n_\ell^k}$ and the form of $F_\omega^k(p)$. For large k, this distribution exhibits a large atom at the top if

$$\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1, \tag{14}$$

and is free of atoms if the reverse inequality holds.

The following proposition states this characterization result. Recall that $x_-^k(b) = \inf\{x|\beta^k(x) \geq b\}$ and $x_+^k(b) = \sup\{x|\beta^k(x) \leq b\}$. Thus, if F_ω^k has an atom at b,

$$\Pr(\text{winning bid in } k\text{-th auction} = b|\omega) = G_\omega(x_+^k(b))^{n_\omega^k} - G_\omega(x_-^k(b))^{n_\omega^k} > 0.$$

Proposition 3 ** Consider a sequence of bidding games $\Gamma_0\left(N^k, \mathbf{n}^k\right)$ such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and a corresponding sequence of bidding equilibria β^k.*

1. *If $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$, then there are bids $(b^k)_{k=1}^\infty$ such that, for large enough k, F_ω^k has an atom at b^k and*

   $$\lim \Pr(\textit{winning bid in } k\textit{-th auction} = b^k|\omega) = 1 \qquad \textit{for } \omega = \ell, h.$$

2. *If $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$, then for every sequence of bids $(b^k)_{k=1}^\infty$,*

   $$\lim \Pr(\textit{winning bid in } k\textit{-th auction} = b^k|\omega) = 0 \qquad \textit{for } \omega = \ell, h.$$

The proof is relegated to the online appendix. Part 1 does not mean that most bidders are submitting the same bid, but rather that the probability of the atom's bid is large enough that it wins almost always.

[16] Since we are dealing only with *bidding* equilibria, we will restrict attention to pure solicitation rather than mixed. But all the results will be also valid for mixed solicitation strategies $\boldsymbol{\eta}$ such that the support of η_w has at most two adjacent numbers, as must be the case for $\boldsymbol{\eta}$'s that arise in a full equilibrium.

The proposition's most interesting insight is perhaps the inevitability of a large atom when $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$. In this case, just being included in the auction already involves a "participation curse" that depresses the expected value estimate held by any bidder, since for large k, $\frac{\rho_h}{\rho_\ell}\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \frac{\rho_h}{\rho_\ell}$ and hence $E^k[v|\bar{x}] < E[v]$. Therefore, a bidder who overbids everybody else, would bear the full strength of this "curse" and would avoid overbidding above $E^k[v|\bar{x}]$. But, as noted in the introduction, a winning bid from the "middle" of the winning bid distribution would give rise to a "middle winner's blessing" that partly offsets the "participation curse."

Let us explain this insight in more specific terms. The expected number of bidders with signals above x is $n_\omega^k(1 - G_\omega(x))$. For x close to $\bar{x}$ and large k,

$$\frac{n_h^k}{n_\ell^k}\frac{(1 - G_h(x))}{(1 - G_\ell(x))} \approx \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1. \tag{15}$$

Thus, the expected number of bidders with signals above such an x is larger in state ℓ than in state h. Intuitively, this suggests that, if β^k is strictly increasing, the probability of winning with $\beta^k(x)$ is larger in state h.[17] That is, $\frac{\pi_h^k(\beta^k(x))}{\pi_\ell^k(\beta^k(x))} > \frac{\pi_h^k(\beta^k(\bar{x}))}{\pi_\ell^k(\beta^k(\bar{x}))} = 1$, where the equality owes to the sure win at $\beta^k(\bar{x})$. Therefore, for strictly increasing β^k, x close to $\bar{x}$ and large k, (3) implies

$$\mathbb{E}^k[v|\bar{x}, \text{win at } \beta^k(x)] > \mathbb{E}^k[v|\bar{x}, \text{win at } \beta^k(\bar{x})], \tag{16}$$

in contrast to an ordinary auction where lower bids involve a stronger winner's curse.[18] Now, (16) does not mean that a bidder with signal $\bar{x}$ would benefit from bidding $\beta^k(x)$, since this would involve a lower probability of winning. However, it turns out that the effect (16) dominates. This is because, as $n_\omega^k \to \infty$, the resulting near zero bidders' payoff means that, for large k and all x with a significant probability of winning,

$$\beta^k(x) \approx \mathbb{E}^k[v|x, \text{win at } \beta^k(x)]. \tag{17}$$

Roughly speaking, this implies that, in a first order sense, the effect of a drop in the probability of winning becomes negligible, and hence it is dwarfed by the change in value which remains significant.

The insight of Part 2 is somewhat related to the result of Proposition 1, which showed that there is no atom in the case of $\frac{n_h}{n_\ell} \geq 1$. Observe that, when there are many participants, the ratio of the numbers of bidders with signals near $\bar{x}$ is roughly $\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$. Since

[17] For large n_ω^k and x close to $\bar{x}$, the number of signals $\geq x$ is approximately Poisson distributed with parameter $n_\omega^k(1 - G_\omega(x))$. Therefore $\pi_\omega^k(\beta^k(x)) = \Pr(\text{no signal} \geq x) \approx e^{-n_\omega^k(1-G_\omega(x))}$. This and (15) imply $\pi_h^k(\beta^k(x)) > \pi_\ell^k(\beta^k(x))$.

[18] In an ordinary auction, conditional on winning, the likelihood of h relative to ℓ is lower at lower bids; see Section 3.2.

these bidders are the effective participants, $\lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$ is in some sense the counterpart of the condition from Proposition 1, and a similar intuition applies.

6.2 Expected Revenue and Prices in Large Auctions

Proposition 3 has straightforward implications for the equilibrium prices and revenues. In the "pooling" case" of $\lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$, the auction ends almost certainly with the winning bid b^k in both states. Bidders' individual rationality then implies $\lim b^k \leq \mathbb{E}[v]$, for otherwise bidders' expected ex-ante payoff would be negative. And, as has already been observed (Sections 3.1 and 5.1), this inequality may be strict.

In the "separating" case of $\lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$, the auction ends almost certainly with no tie at the winning bid and, from (17), $\beta^k(x) \approx \mathbb{E}^k[v|x,\text{win at } \beta^k(x)]$, for large k and for x with a significant winning probability.

These observations have immediate implications for the seller's expected revenue $\mathbb{E}^k[p|\omega] \equiv \mathbb{E}\left[p|\omega; \beta^k, n_\omega^k\right]$ for large k. In the "pooling" case, the revenue is the same across the states and may be strictly below the ex-ante expected value, $\mathbb{E}[v]$. In the "separating" case, the seller's ex-ante revenue is approximately equal to the ex-ante expected value, and the interim expected revenue is higher in state h.

Proposition 4 *Consider a sequence of bidding games* $\Gamma_0\left(N^k, \mathbf{n}^k\right)$ *such that* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and a corresponding sequence of bidding equilibria* β^k.

1. *If* $\lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$, *then*
$$\lim \mathbb{E}^k[p|\ell] = \lim \mathbb{E}^k[p|h] \leq \mathbb{E}[v].$$
2. *If* $\lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$, *then*
$$\lim \mathbb{E}^k[p|\ell] < \mathbb{E}[v] < \lim \mathbb{E}^k[p|h],$$
and
$$\rho_\ell \lim \mathbb{E}^k[p|\ell] + \rho_h \lim \mathbb{E}^k[p|h] = \mathbb{E}[v].$$

The proof of the proposition is instructive about the intuition and is therefore presented here without further explanation.

Proof: Part 1 is an immediate corollary of Part 1 of Proposition 3. It follows from the individual rationality of optimal bids.

For the cases covered by Part 2, we have already noted in (17) that, for high k and for x with a significant probability of winning, $\beta^k(x) \approx \mathbb{E}^k[v|x,\text{win at } \beta^k(x)]$. Let $y_{[\mathbf{n}^k]}$ be the random variable describing the highest signal given participation $\mathbf{n}^k = (n_\ell^k, n_h^k)$.[19] Then

$$\beta^k(x) \approx \mathbb{E}[v|y_{[\mathbf{n}^k]} = x; \beta^k, \mathbf{n}^k] \equiv \mathbb{E}^k[v|y_{[\mathbf{n}^k]} = x]. \tag{18}$$

Observe that, for large k,[20]

$$\mathbb{E}^k[p|\omega] = \mathbb{E}\left[\beta^k(y_{[\mathbf{n}^k]})|\omega\right] \approx \mathbb{E}\left[\mathbb{E}^k[v|y_{[\mathbf{n}^k]}]|\omega\right]. \tag{19}$$

This implies the equality of the ex-ante expected revenue to the ex-ante expected value: $\mathbb{E}\left(\mathbb{E}^k[p|\omega]\right) = \mathbb{E}\left(\mathbb{E}\left[\mathbb{E}^k[v|y_{[\mathbf{n}^k]}]|\omega\right]\right) = \mathbb{E}[v]$, which is just an instant of the law of iterated expectations

To establish $\lim \mathbb{E}^k[p|h] > \lim \mathbb{E}^k[p|\ell]$, it is useful to argue in terms of the quantiles of the distribution, since the set of relevant x's (with significant winning probability) converges to $\bar{x}$. So, let $x^k(\alpha)$ denote the α-quantile of the distribution of $y_{[\mathbf{n}^k]}|\ell$. That is, for any $\alpha \in (0,1)$, $x^k(\alpha)$ is defined by $\left(G_\ell\left(x^k(\alpha)\right)\right)^{n_\ell^k} = \alpha$.

Rewriting the expected value on the right-hand side of (19) as an integral and changing the integration variable via the transformation $\alpha = \left(G_\ell\left(x^k(\alpha)\right)\right)^{n_\ell^k}$,

$$\begin{aligned}
\mathbb{E}^k[p|\ell] &\approx \int_{\underline{x}}^{\bar{x}} \mathbb{E}^k[v|y_{[\mathbf{n}^k]} = x] d(G_\ell(x))^{n_\ell} = \int_0^1 \mathbb{E}^k[v|y_{[\mathbf{n}^k]} = x^k(\alpha)] d\alpha, \\
\mathbb{E}^k[p|h] &\approx \int_{\underline{x}}^{\bar{x}} \mathbb{E}^k[v|y_{[\mathbf{n}^k]} = x] d(G_h(x))^{n_h} = \int_0^1 \mathbb{E}^k[v|y_{[\mathbf{n}^k]} = x^k(\alpha)] d\left(G_h\left(x^k(\alpha)\right)\right)^{n_h^k}.
\end{aligned}$$

The final step in the proof derives the limits of $\left(G_h\left(x^k(\alpha)\right)\right)^{n_h^k}$ and $\mathbb{E}^k[v|y_{[\mathbf{n}^k]} = x^k(\alpha)]$. Obviously, $x^k(\alpha) \to \bar{x}$. Therefore,

$$\lim \frac{\left(1 - G_h\left(x^k(\alpha)\right)\right) n_h^k}{\left(1 - G_\ell\left(x^k(\alpha)\right)\right) n_\ell^k} = \lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \equiv \lambda > 1. \tag{20}$$

Since for large k,[21]

$$\left(G_\omega\left(x^k(\alpha)\right)\right)^{n_\omega^k} \approx e^{-\lim\left(1 - G_\omega\left(x^k(\alpha)\right)\right) n_\omega^k},$$

(20) implies

$$\left(G_h\left(x^k(\alpha)\right)\right)^{n_h^k} \approx \left[\left(G_\ell\left(x^k(\alpha)\right)\right)^{n_\ell^k}\right]^\lambda = \alpha^\lambda.$$

[19] $y_{[\mathbf{n}^k]}$ is not an ordinary first order statistic: In state ω, it is the first order statistic $x_{(n_\omega^k)}$.

[20] Note that here $\mathbb{E}^k[v|y_{[\mathbf{n}^k]}]$ is a random variable while $\mathbb{E}^k[v|y_{[\mathbf{n}^k]} = x]$ is a number.

[21] Recall that $\lim_{n\to\infty}\left(1 - \frac{(1-x_n)n}{n}\right)^n = e^{-\lim n(1-x_n)}$ given any sequence (x_n) with $\lim_{n\to\infty} x_n = 1$.

From (3), we have

$$\mathbb{E}^k[v|y_{[\mathbf{n}^k]}=x]=\frac{v_\ell+\frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{n_h^k}{n_\ell^k}\frac{(G_h(x))^{n_h^k-1}}{(G_\ell(x))^{n_\ell^k-1}}v_h}{1+\frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{n_h^k}{n_\ell^k}\frac{(G_h(x))^{n_h^k-1}}{(G_\ell(x))^{n_\ell^k-1}}}.$$

Since $x^k(\alpha)\to\bar{x}$ and hence $\left(G_\ell\left(x^k(\alpha)\right)\right)^{n_\ell^k}\approx\left(G_\ell\left(x^k(\alpha)\right)\right)^{n_\ell^k-1}$ for large k,

$$\lim\mathbb{E}^k[v|y_{[\mathbf{n}^k]}=x^k(\alpha)]=\frac{v_\ell+\frac{\rho_h}{\rho_\ell}\lambda\frac{\alpha^\lambda}{\alpha}v_h}{1+\frac{\rho_h}{\rho_\ell}\lambda\frac{\alpha^\lambda}{\alpha}}.$$

Therefore,

$$\lim\mathbb{E}^k[p|\ell] = \int_0^1\lim\mathbb{E}^k[v|y_{[\mathbf{n}^k]}=x^k(\alpha)]d\alpha, \tag{21}$$

$$\lim\mathbb{E}^k[p|h] = \int_0^1\lim\mathbb{E}^k[v|y_{[\mathbf{n}^k]}=x^k(\alpha)]d(\alpha^\lambda). \tag{22}$$

Now, $\lim\mathbb{E}^k[p|h]>\lim\mathbb{E}^k[p|\ell]$ follows because $\lim\mathbb{E}^k[v|y_{[\mathbf{n}^k]}=x^k(\alpha)]$ is a strictly increasing function of α and the measure $d\left(\alpha^\lambda\right)$ stochastically dominates the measure $d\alpha$. ∎

6.3 The Extent of Information Aggregation

As a fairly immediate corollary of the above arguments, we get the following observation on the extent of information aggregation by the price.

Proposition 5 *Consider a sequence of bidding games $\Gamma_0\left(N^k,\mathbf{n}^k\right)$ such that $\min\{n_\ell^k,n_h^k\}\to\infty$ and $\lim\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}>1$, and a corresponding sequence of bidding equilibria β^k. For any $\varepsilon>0$, there are Δ and δ such that $\Delta>\delta>1$ and*

$$\text{if}\quad\lim\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} > \Delta,\quad\text{then}\quad|\lim\mathbb{E}^k[p|\omega]-v_\omega|<\varepsilon,\quad\omega=\ell,h,$$

$$\text{if}\quad\lim\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \delta,\quad\text{then}\quad|\lim\mathbb{E}^k[p|\omega]-\mathbb{E}[v]|<\varepsilon,\quad\omega=\ell,h.$$

Thus, in the separating case of $\lim\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}>1$, the price aggregates the information well when $\lim\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ is large and poorly when it is near 1. Of course, we already know from Proposition 3 that, in the case of $\lim\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}<1$ that is not covered by Proposition 5, the price fails to aggregate the information.

Proof: The first part of the proposition follows from $\lim\mathbb{E}^k[v|y_{[\mathbf{n}^k]}=x^k(\alpha)]\cong\mathbb{E}[v]$ for all α when λ is close to one and from equations (21) and (22). The second part follows from $\lim \mathbb{E}^k[v|y_{[\mathbf{n}^k]} = x^k(\alpha)] \cong v_\ell$ for all $\alpha < 1$ for large λ. This and (21) implies that $\lim \mathbb{E}^k[p|\ell] \cong v_\ell$ for large λ. Then, the equality of the ex-ante expected revenue and $\mathbb{E}[v]$ from the second part of Proposition 4 requires $\lim \mathbb{E}^k[p|h] \cong v_h$. ∎

6.4 A Comment on the Borderline Case

In the borderline case of $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = 1$, the distribution of the winning bid becomes degenerate on $\mathbb{E}[v]$, so that the expected price is independent of the state, with $\lim \mathbb{E}_h^k(p) = \lim \mathbb{E}_\ell^k(p) = \mathbb{E}[v]$. We do not know whether β^k is strictly increasing for large k (but becoming increasingly flat at the top) or whether β^k itself contains atoms (is flat for given k).

Proposition 6 ** Consider a sequence of bidding games $\Gamma_0(N^k, \boldsymbol{\eta}^k)$ such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = 1$. If β^k is a bidding equilibrium of $\Gamma_0(N^k, \mathbf{n}^k)$ for all k, then the limit of the winning bid distribution is a mass point on $\mathbb{E}[v]$, i.e., $\lim F_\omega^k(b) = 1$ for all $b > \mathbb{E}[v]$ and $= 0$ for all $b < \mathbb{E}[v]$.*

7 Discussion

7.1 Robustness of Atoms

One of our more distinct insights concerns the emergence of a significant atom in the winning bid distribution. This is a robust result with intuitive underpinning. Recall the example from Section 3.1. First, the stark atom of our simple model need not be taken too literally. In a more noisy version of the model, the same forces would generate a cluster of close but non-identical bids. To see this, consider a noisy bidding variation of the model: When a bidder selects bid $\hat{b}$, the actual bid is $b = \hat{b} + \varepsilon$, where $\varepsilon \sim U[-\delta, \delta]$ for some small $\delta > 0$. Essentially the same arguments that were used in Section 3.1 establish that the equilibria of that example remain equilibria in this case as well.

More generally, as we explained before, when $\frac{n_h}{n_\ell}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$, just being included in the auction already involves a "participation curse," which is partly offset by a "middle winner's blessing" associated with winning with a "middle" bid. This induces bidders to aim at a "middle" bid that reduces the relative probability of winning at state ℓ. Consequently, the outcome ends up resembling pooling, even in a nosier environment that would preclude the sharp pooling of our model.

Second, if n_ω, $\omega = \ell, h$, in the example of Section 3.1 are changed from 6 and 2 to $6m$ and $2m$ respectively, then Proposition 3 implies that, for sufficiently large m, all equilibria must necessarily involve an atom. Thus, any robustness criterion that rules out atoms would rule out all equilibria.

Finally, the existence of an atom is robust in other ways as well. Assume in the example a second-price auction. Essentially the same arguments continue to imply the existence of equilibrium with an atom. In particular, a bidder who overbids the atom at $\bar{b}$ wins in both states, and pays $\bar{b}$, which exceeds the expected value conditional on winning.

7.2 Affiliation of First-Order Statistic and State

The emergence of atoms can be also explained by the failure of the affiliation between the value and the highest signal. Recall that $y_{[\mathbf{n}]}$ denotes the highest signal realization given participation $\mathbf{n} = (n_\ell, n_h)$. The likelihood ratio conditional on $y_{[\mathbf{n}]} = x$ is

$$\frac{n_h}{n_\ell} \frac{g_h(x)}{g_\ell(x)} \frac{(G_h(x))^{n_h-1}}{(G_\ell(x))^{n_\ell-1}}. \tag{23}$$

In standard auctions with $n_h = n_\ell = n$, this likelihood ratio is increasing in x. Thus, the statistic $y_{[\mathbf{n}]}$, which in this case coincides with the first order statistic $x_{(n)}$, is affiliated with the value. In contrast, with state dependent participation, the likelihood ratio (23) need not be increasing—it is in fact *decreasing* for x sufficiently close to $\bar{x}$ if $\frac{n_h}{n_\ell}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$. Therefore, $y_{[\mathbf{n}]}$ might not be affiliated with the value.

7.3 Signaling: Observable Number of Bidders

In a variation on our model in which the seller's solicitation of n bidders is observed prior to the bidding,[22] there are two types of pure strategy signaling equilibria—separating and pooling. In the pooling equilibrium, $n_\ell = n_h$. The pooling equilibria are the same as those of the standard common value auction, since n is independent of the state. Multiple pooling equilibria can be supported by off-path beliefs that place probability 1 on state ℓ following a seller's deviation. In the separating equilibrium, $n_\ell = 2$ and $n_h > 2$, bidders bid v_ℓ and v_h respectively and the seller's payoff is $v_\ell - 2s$ in both states. Therefore, if s is small, the seller's revenue is lower than it is when n is not observable, as in the model of this paper. Incentive compatibility requires $v_h - n_h s = v_\ell - 2s$, which, owing to integer constraints, may hold only for special configurations of s, v_ℓ and v_h.[23]

7.4 Sticky Prices

The phenomenon of "sticky prices"—prices that do not respond to changes in the fundamentals of the environment has been commonly explained in the relevant literatures by the presence of menu costs. Our analysis suggests another source—adverse selection—for explaining this phenomenon. This is seen through the fact that in the pooling equilibrium

[22] The focus of this paper is trading on environments in which the seller cannot verifiably communicate the number of solicited bidders. The variation of this subsection is merely an exercise aimed at providing sharper understanding of the model.

[23] But there always exists a nearby partially separating mixed strategy equilibrium in which the seller mixes slightly between n_ℓ and n_h, the bids are the expected value conditional on the observed number and the seller's payoff is close to $v_\ell - 2s$.

identified in this paper trade may take place (almost certainly) at some $\bar{b} < \mathbb{E}[v]$, which need not be sensitive to (small) changes in the fundamentals—the values v_ω and the prior ρ_ω.

7.5 Information Aggregation and Efficiency

In the simple common values environment of this paper, it is efficient to transfer the unit to any buyer and this transfer indeed occurs in equilibrium regardless of how well the information is aggregated. But this does not mean that the question of information aggregation has no importance in a common values environment. Straightforward enrichments of the simple model of this paper will introduce efficiency consequences for information aggregation. For example, if the seller's cost is $c \in (v_\ell, v_h)$, efficiency requires that trade takes place only in state h. In this case, a failure of information aggregation implies allocative inefficiencies. Alternatively, if the seller has an opportunity to invest in quality improvements prior to trade, a failure of information aggregation could imply inefficiently weak investment incentives.

8 Appendix

8.1 Winning Probability at Atoms

The following lemma derives an expression for the winning probability in the case of a tie. Recall from (5) that $x_+(b) = \sup\{x|\beta(x) \le b\}$ and $x_-(b) = \inf\{x|\beta(x) \ge b\}$, so that an atom at b means $x_-(b) < x_+(b)$. We omit the argument and write x_- and x_+ when it is clear from the context.

Lemma 2 *Suppose β is non-decreasing and, for some $\bar{b}$, $x_-\left(\bar{b}\right) < x_+\left(\bar{b}\right)$. Then,*

$$\pi_\omega\left(\bar{b}|\beta, n\right) = \frac{G_\omega(x_+)^n - G_\omega(x_-)^n}{n\left(G_\omega(x_+) - G_\omega(x_-)\right)} = \int_{x_-}^{x_+} \frac{\left(G_\omega(x)\right)^{n-1} g_\omega(x)\, dx}{G_\omega(x_+) - G_\omega(x_-)}. \tag{24}$$

Observe that the last expression is the expected probability of a randomly drawn signal from $[x_+, x_-]$ to be the highest. Thus, $\pi_\omega\left(\bar{b}|\beta, n\right)$ "averages" what would be the winning probabilities of the types in $[x_+, x_-]$, if β were strictly increasing.

Proof of Lemma 2: Since β is non-decreasing, $G_\omega(\{x|\beta(x) < b\}) = G_\omega(x_-)$ and $G_\omega(\{x|\beta(x) > b\}) = 1 - G_\omega(x_+)$. The winning probability at b is then:

$$\begin{aligned}
&\pi_\omega(b|\beta, n) \\
= \; & \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{1}{i+1} G_\omega(x_-)^{n-i-1} [G_\omega(x_+) - G_\omega(x_-)]^i \\
= \; & \frac{1}{n} \sum_{i=0}^{n-1} \binom{n}{i+1} G_\omega(x_-)^{n-i-1} [G_\omega(x_+) - G_\omega(x_-)]^i \\
= \; & \frac{\sum_{k=1}^{n} \binom{n}{k} G_\omega(x_-)^{n-k} [G_\omega(x_+) - G_\omega(x_-(b))]^k}{n [G_\omega(x_+) - G_\omega(x_-(b))]} \\
= \; & \frac{\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} G_\omega(x_-)^{n-k} [G_\omega(x_+) - G_\omega(x_-)]^k - G_\omega(x_-)^n}{n [G_\omega(x_+) - G_\omega(x_-)]} \\
= \; & \frac{(G_\omega(x_-) + G_\omega(x_+) - G_\omega(x_-))^n - G_\omega(x_-)^n}{n [G_\omega(x_+) - G_\omega(x_-)]} \\
= \; & \frac{G_\omega(x_+)^n - G_\omega(x_-)^n}{n [G_\omega(x_+) - G_\omega(x_-)]}.
\end{aligned}$$

The critical step is to apply the binomial theorem, $\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^{n-k} b^k = (a+b)^n$. The second equality from the lemma is immediate. ∎

8.2 Notation for Mixed Strategies

Given a mixed solicitation strategy $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$, let

$$\overline{n}_\omega(\eta_\omega) \triangleq \sum_{n=1}^{N} n\eta_\omega(n), \text{ and } \overline{\pi}_\omega[b|\beta, \eta_\omega] \triangleq \sum_{n=1}^{N} \eta_\omega(n)\, n\pi_\omega(b|\beta, n) / \overline{n}_\omega. \tag{25}$$

These are the expected number of bidders and the weighted average probability of winning in state ω. To make the expressions less dense, we omit here and later the argument of $\overline{n}_\omega(\eta_\omega)$ and write just $\overline{n}_\omega$ instead. The counterpart of (1)—the expected payoff to a bidder who bids b given a mixed solicitation strategy $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$—is

$$U(b|x; \beta, \boldsymbol{\eta}) = \frac{\rho_\ell g_\ell(x)\, \overline{n}_\ell \overline{\pi}_\ell(b|\beta, \eta_\ell)(v_\ell - b) + \rho_h g_h(x)\, \overline{n}_h \overline{\pi}_h(b|\beta, \eta_h)(v_h - b)}{\rho_\ell g_\ell(x)\, \overline{n}_\ell + \rho_h g_h(x)\, \overline{n}_h}. \tag{26}$$

Expressions (2)—(4) can also be adapted to mixed strategies, with $\overline{n}_\omega$ and $\overline{\pi}_\omega$ taking the place of n_ω and π_ω.

Just as for pure strategies, the expected utility can be written as

$$U(b|x; \beta, \boldsymbol{\eta}) = \Pr[\text{ win at } b \mid x; \beta, \boldsymbol{\eta}] \left(\mathbb{E}[v|x, \text{win at } b; \beta, \boldsymbol{\eta}] - b\right), \tag{27}$$

where

$$\mathbb{E}[\; v \mid x, \text{ win at } b; \beta, \boldsymbol{\eta}] \;=\; \frac{\rho_\ell g_\ell(x)\,\overline{n}_\ell \overline{\pi}_\ell(b|\beta,\eta_\ell)\, v_\ell + \rho_h g_h(x)\,\overline{n}_h \overline{\pi}_h(b|\beta,\eta_h)\, v_h}{\rho_\ell g_\ell(x)\,\overline{n}_\ell \overline{\pi}_\ell(b|\beta,\eta_\ell) + \rho_h g_h(x)\,\overline{n}_h \overline{\pi}_h(b|\beta,\eta_h)} \tag{28}$$

$$= \frac{v_\ell + \frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{\overline{n}_h}{\overline{n}_\ell}\frac{\overline{\pi}_h(b|\beta,\eta_h)}{\overline{\pi}_\ell(b|\beta,\eta_\ell)} v_h}{1 + \frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{\overline{n}_h}{\overline{n}_\ell}\frac{\overline{\pi}_h(b|\beta,\eta_h)}{\overline{\pi}_\ell(b|\beta,\eta_\ell)}},$$

and

$$\Pr[\text{ win at } b \mid x; \beta, \boldsymbol{\eta}] \;=\; \frac{\rho_\ell g_\ell(x)\,\overline{n}_\ell \overline{\pi}_\ell(b|\beta,\eta_\ell) + \rho_h g_h(x)\,\overline{n}_h \overline{\pi}_h(b|\beta,\eta_h)}{\rho_\ell g_\ell(x)\,\overline{n}_\ell + \rho_h g_h(x)\,\overline{n}_h} \tag{29}$$

$$= \frac{\overline{\pi}_\ell(b|\beta,\eta_\ell) + \frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{\overline{n}_h}{\overline{n}_\ell}\overline{\pi}_h(b|\beta,\eta_h)}{1 + \frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{\overline{n}_h}{\overline{n}_\ell}}.$$

8.3 Pooling Equilibrium in the Full Game

Proof of Proposition 2: First, the proof derives implications for optimal solicitation $(\boldsymbol{\eta}^k)_{k=1}^\infty$ in the face of a 2-price bidding strategy $\beta^k(x)$ and $s^k \to 0$. Then, it identifies values of $\underline{b}^k$ and $\bar{b}$ with which $\beta^k(x)$ is a bidding equilibrium, given a solicitation strategy of the form derived above. Finally, a fixed-point argument is used to confirm existence.

Recall that $\lim s^k = 0$ by the hypothesis of the proposition. In what follows we consider $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^\infty$ such that

$$\begin{aligned} &\beta^k(x) = \begin{cases} \bar{b} & \text{if } \; x \geq \hat{x} \\ \underline{b}^k & \text{if } \; x < \hat{x}, \end{cases} \\ &\underline{b}^k < \bar{b} \text{ and } \lim \underline{b}^k < \bar{b}, \\ &\boldsymbol{\eta}^k \text{ is optimal given } \beta^k \text{ and } s^k. \end{aligned} \tag{30}$$

Recall that the support of η_ω is $\{n_\omega, n_\omega + 1\}$ with $\gamma_\omega = \eta_\omega(n_\omega) > 0$.

Step 1: For every k with $n_\omega^k \geq 2$,

$$\frac{(1 - G_\ell(\hat{x}))}{(1 - G_h(\hat{x}))}\frac{1}{G_h(\hat{x})} \geq \frac{G_h(\hat{x})^{n_h^k - 1}}{G_\ell(\hat{x})^{n_\ell^k - 1}} \geq G_\ell(\hat{x})\frac{(1 - G_\ell(\hat{x}))}{(1 - G_h(\hat{x}))}, \tag{31}$$

and,

$$\text{if } \; \gamma_\ell^k < 1, \; \text{ then } \; \frac{G_h(\hat{x})^{n_h^k}}{G_\ell(\hat{x})^{n_\ell^k}} \leq \frac{(1 - G_\ell(\hat{x}))}{(1 - G_h(\hat{x}))}. \tag{32}$$

Proof of Step 1: Since n_ω^k maximizes the seller's expected payoff[24], $\mathbb{E}_\omega\left[p|\beta^k, n_\omega^k\right] - n_\omega^k s$, it satisfies

$$\mathbb{E}_\omega\left[p|\beta^k, n_\omega^k\right] - \mathbb{E}_\omega\left[p|\beta^k, n_\omega^k - 1\right] \geq s^k \geq \mathbb{E}_\omega\left[p|\beta^k, n_\omega^k + 1\right] - \mathbb{E}_\omega\left[p|\beta^k, n_\omega^k\right]. \tag{33}$$

[24] In this proof, we will not omit the arguments β^k, n_ω^k of the functions like π_ω and E_ω, since sometimes, as in the next equation, n_ω^k is not fixed for all expressions.

Since, by (30),

$$\mathbb{E}_\omega\left[p|\beta^k, n_\omega^k\right] = \left(1 - G_\omega(\hat{x})^{n_\omega^k}\right)\bar{b} + G_\omega(\hat{x})^{n_\omega^k}\underline{b}^k,$$

(33) can be rewritten as

$$G_\omega(\hat{x})^{n_\omega^k - 1}(1 - G_\omega(\hat{x}))\left(\bar{b} - \underline{b}^k\right) \geq s^k \geq G_\omega(\hat{x})^{n_\omega^k}(1 - G_\omega(\hat{x}))\left(\bar{b} - \underline{b}^k\right) \qquad \omega = \ell, h. \tag{34}$$

Therefore,

$$\begin{aligned} G_h(\hat{x})^{n_h^k - 1}(1 - G_h(\hat{x})) &\geq G_\ell(\hat{x})^{n_\ell^k}(1 - G_\ell(\hat{x})), \\ G_\ell(\hat{x})^{n_\ell^k - 1}(1 - G_\ell(\hat{x})) &\geq G_h(\hat{x})^{n_h^k}(1 - G_h(\hat{x})), \end{aligned}$$

which implies (31).

Now, if $\gamma_\ell^k < 1$, then $s^k = G_\ell(\hat{x})^{n_\ell^k}(1 - G_\ell(\hat{x}))\left(\bar{b} - \underline{b}^k\right)$. Therefore, the last inequality can be replaced by $G_\ell(\hat{x})^{n_\ell^k}(1 - G_\ell(\hat{x})) \geq G_h(\hat{x})^{n_h^k}(1 - G_h(\hat{x}))$, yielding (32). □

Step 2: $n_\omega^k \to \infty$ for $\omega \in \{\ell, h\}$.

Proof of Step 2: Since $\lim s^k = 0$ and $\lim\left(\bar{b} - \underline{b}^k\right) > 0$, it follows from the second inequality in (34)—which does not require $n_\omega^k \geq 2$—that $\lim G_\omega(\hat{x})^{n_\omega^k} = 0$, so $\lim n_\omega^k = \infty$. □

Step 3:

$$\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\lim\frac{n_h^k}{n_\ell^k} < 1.$$

Proof of Step 3: From (31),

$$\lim\frac{G_h(\hat{x})^{n_h^k}}{G_\ell(\hat{x})^{n_\ell^k}} = \lim\left(\frac{G_h(\hat{x})^{\frac{n_h^k}{n_\ell^k}}}{G_\ell(\hat{x})}\right)^{n_\ell^k} \in (0, \infty).$$

Since $n_\ell^k \to \infty$, this requires that

$$\lim\frac{G_h(\hat{x})^{\frac{n_h^k}{n_\ell^k}}}{G_\ell(\hat{x})} = \frac{G_h(\hat{x})^{\lim\frac{n_h^k}{n_\ell^k}}}{G_\ell(\hat{x})} = 1.$$

Applying a logarithmic transformation to both sides, $\ln G_h(\hat{x})\lim\frac{n_h^k}{n_\ell^k} = \ln G_\ell(\hat{x})$. Hence,

$$\lim\frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\hat{x})}{\ln G_h(\hat{x})}.$$

Therefore,

$$\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\lim\frac{n_h^k}{n_\ell^k} = \frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{\ln G_\ell(\hat{x})}{\ln G_h(\hat{x})} = \frac{1 - G_h(\hat{x})}{1 - G_\ell(\hat{x})}\frac{\ln G_\ell(\hat{x})}{\ln G_h(\hat{x})} < 1.$$

where the second equality follows from $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{1-\hat{x}}{1-\hat{x}} = \frac{1-G_h(\hat{x})}{1-G_\ell(\hat{x})}$ and the inequality follows from the fact that $\left|\frac{1-z}{\ln z}\right|$ is strictly increasing in $z \in (0,1)$ and that $G_\ell(\hat{x}) > G_h(\hat{x})$. $\square$

Step 4: Recall $\overline{n}_\omega(\eta_\omega)$ and $\overline{\pi}_\omega(b|\beta, \eta_\omega)$, $\omega = \ell, h$, from (25).

$$\lim \frac{\overline{n}_h^k}{\overline{n}_\ell^k} = \lim \frac{n_h^k}{n_\ell^k} = \frac{\ln G_\ell(\hat{x})}{\ln G_h(\hat{x})}. \tag{35}$$

For all $b \geq \underline{b}^k$,

$$\lim \frac{\overline{\pi}_h(b|\beta^k, \eta_h^k)}{\overline{\pi}_\ell(b|\beta^k, \eta_\ell^k)} = \lim \frac{\gamma_h^k \pi_h(b|\beta^k, n_h^k) + (1-\gamma_h^k)\pi_h(b|\beta^k, n_h^k+1)}{\gamma_\ell^k \pi_\ell(b|\beta^k, n_\ell^k) + (1-\gamma_\ell^k)\pi_\ell(b|\beta^k, n_\ell^k+1)}, \tag{36}$$

and

$$\lim \frac{\overline{\pi}_h(\bar{b}|\beta^k, \eta_h^k)}{\overline{\pi}_\ell(\bar{b}|\beta^k, \eta_\ell^k)} = \lim \frac{\pi_h(\bar{b}|\beta^k, n_h^k)}{\pi_\ell(\bar{b}|\beta^k, n_\ell^k)} = \lim \frac{n_\ell^k}{n_h^k}\frac{g_\ell(\bar{x})}{g_h(\bar{x})}. \tag{37}$$

Proof of Step 4: Since $\eta_\omega^k\left(n_\omega^k\right) = \gamma_\omega^k = 1 - \eta_\omega^k\left(n_\omega^k + 1\right)$, we have $\overline{n}_\omega^k = \gamma_\omega^k n_\omega^k + (1-\gamma_\omega^k)\left(n_\omega^k + 1\right)$. Steps 2 and 3 now imply (35).

To establish (36), rewrite the definition of $\overline{\pi}_\omega$ to get

$$\begin{aligned} &\frac{\overline{\pi}_\omega(b|\beta^k, \eta_\omega^k)}{\gamma_\omega^k \pi_\omega(b|\beta^k, n_\omega^k) + (1-\gamma_\omega^k)\pi_\omega(b|\beta^k, n_\omega^k+1)} \\ = &\frac{\frac{n_\omega^k}{\overline{n}_\omega^k}\left(\gamma_\omega^k \pi_\omega(b|\beta^k, n_\omega^k) + (1-\gamma_\omega^k)\pi_\omega(b|\beta^k, n_\omega^k+1)\right) + \frac{(1-\gamma_\omega^k)}{\overline{n}_\omega^k}\pi_\omega(b|\beta^k, n_\omega^k+1)}{\gamma_\omega^k \pi_\omega(b|\beta^k, n_\omega^k) + (1-\gamma_\omega^k)\pi_\omega(b|\beta^k, n_\omega^k+1)}. \end{aligned}$$

Since, by Step 2, $n_\omega^k \to \infty$, the RHS converges to 1, implying (36).

To establish the second equality in (37), note that (24) implies $\pi_\omega(\bar{b}|\beta^k, n_\omega^k) = \frac{1-G_\omega(\hat{x})^{n_\omega^k}}{n_\omega^k[1-G_\omega(\hat{x})]}$. Since $G_\omega(\hat{x}) < 1$ and, by Step 2, $n_\omega^k \to \infty$, it follows that $G_\omega(\hat{x})^{n_\omega^k} \to 0$. Also, since signals satisfy (8), $\frac{1-G_h(\hat{x})}{1-G_\ell(\hat{x})} = \frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{1-\hat{x}}{1-\hat{x}}$. Therefore, we have

$$\lim \frac{\pi_h(\bar{b}|\beta^k, n_h^k)}{\pi_\ell(\bar{b}|\beta^k, n_\ell^k)} = \lim \frac{n_\ell^k}{n_h^k}\frac{g_\ell(\bar{x})}{g_h(\bar{x})}. \tag{38}$$

For the first equality of (37), note that

$$\lim \frac{\pi_\omega(\bar{b}|\beta^k, n_\omega^k+1)}{\pi_\omega(\bar{b}|\beta^k, n_\omega^k)} = \lim \frac{\frac{1-G_\omega(\hat{x})^{n_\omega^k+1}}{\left(n_\omega^k+1\right)[1-G_\omega(\hat{x})]}}{\frac{1-G_\omega(\hat{x})^{n_\omega^k}}{n_\omega^k[1-G_\omega(\hat{x})]}} = 1. \tag{39}$$

Now, (39) and (36) yield the desired equality. $\square$

Steps 5-7 derive bounds, v^*, v^{**} and v^{***}, on the (limits of the) expected values conditional on winning at $\bar{b}$, $b' > \bar{b}$, and $\underline{b}^k$, respectively. Importantly, these bounds hold uniformly for all sequences $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^{\infty}$ that satisfy (30).

Step 5: There exists a number $v^{***} < \mathbb{E}[v]$ such that

$$\lim \mathbb{E}[v|\ x, \text{ win at } \bar{b}; \beta^k, \boldsymbol{\eta}^k] = \begin{cases} \mathbb{E}[v] & \text{if } \quad x \geq \hat{x}, \\ v^{***} & \text{if } \quad x < \hat{x}. \end{cases}$$

Proof of Step 5: Using (37) and (35) to evaluate the limit of (28),

$$\lim \mathbb{E}[\ v\ |x, \text{ win at } \bar{b}; \beta^k, \boldsymbol{\eta}^k] = \frac{v_\ell + \frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{g_\ell(\bar{x})}{g_h(\bar{x})}v_h}{1 + \frac{\rho_h}{\rho_\ell}\frac{g_h(x)}{g_\ell(x)}\frac{g_\ell(\bar{x})}{g_h(\bar{x})}}. \tag{40}$$

If $x \geq \hat{x}$, then $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$, and hence the RHS $= \rho_\ell v_\ell + \rho_h v_h \equiv \mathbb{E}[v]$, as claimed.

For $x < \hat{x}$, define v^{***} to be the RHS of (40) at $x = \underline{x}$. Therefore, v^{***} is independent of the choice of $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^{\infty}$, as required. Since $\frac{g_h(x)}{g_\ell(x)}\frac{g_\ell(\bar{x})}{g_h(\bar{x})} = \left(\frac{g_h(\underline{x})}{g_\ell(\underline{x})}\right) / \left(\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\right) < 1$, we have $v^{***} < \rho_\ell v_\ell + \rho_h v_h \equiv \mathbb{E}[v]$. □

Step 6: There exists a number $v^{**} < \mathbb{E}[v]$ such that, for any $x \geq \hat{x}$,

$$\lim \mathbb{E}[\ v\ |\ x, \text{ win at } b > \bar{b}; \beta^k, \boldsymbol{\eta}^k] = v^{**}.$$

Proof of Step 6: For $x \geq \hat{x}$, $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$. For $b > \bar{b}$, $\pi_\ell(b|\beta^k, \eta_\ell^k) = \pi_h(b|\beta^k, \eta_h^k) = 1$. Thus, $\frac{\bar{\pi}_h(b^k|\beta^k, \eta_h^k)}{\bar{\pi}_\ell(b^k|\beta^k, \eta_\ell^k)} = 1$. Use both of these observations, Step 4 and (28) to get, for any $x \geq \hat{x}$,

$$\lim \mathbb{E}[\ v\ |\ x, \text{ win at } b > \bar{b}; \beta^k, \boldsymbol{\eta}^k] = \frac{v_\ell + \frac{\rho_h}{\rho_\ell}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{\ln G_\ell(\hat{x})}{\ln G_h(\hat{x})}v_h}{1 + \frac{\rho_h}{\rho_\ell}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{\ln G_\ell(\hat{x})}{\ln G_h(\hat{x})}} \triangleq v^{**},$$

By Step 3, $v^{**} < \rho_\ell v_\ell + \rho_h v_h \equiv \mathbb{E}[v]$, and, by its definition, v^{**} is independent of $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^{\infty}$, as required. □

Step 7: There exists a number v^* such that, for any $x < \hat{x}$,

$$\mathbb{E}\left[v|x, \text{ win at } \underline{b}^k; \beta^k, \boldsymbol{\eta}^k\right] \leq v^* < \mathbb{E}[v].$$

Proof of Step 7: By (24), $\pi_\omega(\underline{b}^k|\beta^k, n) = \frac{G_\omega(\hat{x})^n - 0}{n(G_\omega(\hat{x}) - 0)} = \frac{G_\omega(\hat{x})^{n-1}}{n}$. Substituting this into the definition of $\bar{\pi}_\omega$ gives

$$\frac{\bar{\pi}_h(\underline{b}^k|\beta^k, \eta_h^k)}{\bar{\pi}_\ell(\underline{b}^k|\beta^k, \eta_\ell^k)} = \frac{\overline{n}_\ell^k}{\overline{n}_h^k}\frac{\gamma_h^k G_h(\hat{x})^{n_h^k - 1} + (1 - \gamma_h^k)G_h(\hat{x})^{n_h^k}}{\gamma_\ell^k G_\ell(\hat{x})^{n_\ell^k - 1} + (1 - \gamma_\ell^k)G_\ell(\hat{x})^{n_\ell^k}}.$$

Then substituting this into (28) and noting that from (8), for $x < \hat{x}$, $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})} = \frac{G_h(\hat{x})}{G_\ell(\hat{x})}$, we get

$$\mathbb{E}\left[v|\underline{x},\ \text{win at } \underline{b}^k;\beta^k,\boldsymbol{\eta}^k\right] = \frac{\rho_\ell v_\ell + \rho_h \frac{\gamma_h^k G_h(\hat{x})^{n_h^k} + (1-\gamma_h^k) G_h(\hat{x})^{n_h^k+1}}{\gamma_\ell^k G_\ell(\hat{x})^{n_\ell^k} + (1-\gamma_\ell^k) G_\ell(\hat{x})^{n_\ell^k+1}} v_h}{\rho_\ell + \rho_h \frac{\gamma_h^k G_h(\hat{x})^{n_h^k} + (1-\gamma_h^k) G_h(\hat{x})^{n_h^k+1}}{\gamma_\ell^k G_\ell(\hat{x})^{n_\ell^k} + (1-\gamma_\ell^k) G_\ell(\hat{x})^{n_\ell^k+1}}}. \tag{41}$$

It follows from Step 1 that

$$\frac{\gamma_h^k G_h(\hat{x})^{n_h^k} + (1-\gamma_h^k) G_h(\hat{x})^{n_h^k+1}}{\gamma_\ell^k G_\ell(\hat{x})^{n_\ell^k} + (1-\gamma_\ell^k) G_\ell(\hat{x})^{n_\ell^k+1}} \leq \frac{1}{G_\ell(\hat{x})} \frac{1-G_\ell(\hat{x})}{1-G_h(\hat{x})}.$$

Specifically, if $\gamma_\ell^k = 1$, the LHS is bounded from above by $\frac{G_h(\hat{x})^{n_h^k}}{G_\ell(\hat{x})^{n_\ell^k}}$ and the inequality follows from the first inequality in (31); if $\gamma_\ell^k < 1$, the LHS is bounded from above by $\frac{G_h(\hat{x})^{n_h^k}}{G_\ell(\hat{x})^{n_\ell^k+1}}$ and the inequality follows from (32). Hence,

$$\mathbb{E}\left[v|\underline{x},\ \text{win at } \underline{b}^k;\beta^k,\boldsymbol{\eta}^k\right] \leq \frac{\rho_\ell}{\rho_h \frac{1}{G_\ell(\hat{x})}\frac{1-G_\ell(\hat{x})}{1-G_h(\hat{x})} + \rho_\ell} v_\ell + \frac{\rho_h \frac{1}{G_\ell(\hat{x})}\frac{1-G_\ell(\hat{x})}{1-G_h(\hat{x})}}{\rho_h \frac{1}{G_\ell(\hat{x})}\frac{1-G_\ell(\hat{x})}{1-G_h(\hat{x})} + \rho_\ell} v_h \triangleq v^*.$$

From Assumption (9), $\frac{1}{G_\ell(\hat{x})}\frac{1-G_\ell(\hat{x})}{1-G_h(\hat{x})} < 1$. Therefore, $v^* < \rho_\ell v_\ell + \rho_h v_h \equiv \mathbb{E}[v]$. By its definition, v^* is independent of the choice of $\left(\beta^k,\boldsymbol{\eta}^k\right)_{k=1}^{\infty}$. □

The remaining steps use the bounds from Steps 5-7 to construct the equilibrium β^k's.

Step 8: If

$$\underline{b}^k = \mathbb{E}[v|\ \underline{x}, \text{win at } b = \underline{b}^k;\beta^k,\boldsymbol{\eta}^k] \leq \max\{v^*, v^{**}, v^{***}\} < \bar{b} < \mathbb{E}[v], \tag{42}$$

then bidding $\bar{b}$ is a best response for $x \geq \hat{x}$ and large k.

Proof of Step 8: Fix some $x \geq \hat{x}$. From (26), Step 5, Step 6, (42), and the positive winning probability at both $\bar{b}$ and $b' > \bar{b}$, it follows that, for large k, $U(\bar{b}|x;\beta^k,\boldsymbol{\eta}^k) \approx \Pr[\text{ win at } \bar{b} \mid x;\beta^k,\boldsymbol{\eta}^k](\mathbb{E}[v] - \bar{b}) > 0$ and $U(b'|x;\beta^k,\boldsymbol{\eta}^k) \approx (v^{**} - b') < 0$. Thus, it is profitable to bid $\bar{b}$, but unprofitable to overbid it.

Consider $b'' < \bar{b}$. From (26), $U(b''|x;\beta^k,\boldsymbol{\eta}^k) < \Pr[\text{ win at } b'' \mid x;\beta^k,\boldsymbol{\eta}^k]v_h$. By (29), for any b, $\Pr[\text{ win at } b \mid x;\beta^k,\boldsymbol{\eta}^k]$ is a weighted sum of the $\bar{\pi}_\omega(b|\beta^k,\eta_\omega^k)$, $\omega = \ell, h$, with weights that are independent of b. By (24), for large k, $\bar{\pi}_\omega(\bar{b}|\beta^k,\eta_\omega^k)$ is on the order of $\frac{1}{n_\omega^k(1-G_\omega(\hat{x}))}$, while $\bar{\pi}_\omega(b''|\beta^k,\eta_\omega^k)$ is at most on the order of $G_\omega(\hat{x})^{n_\omega^k-1}$. Hence, $\frac{\bar{\pi}_\omega(b''|\beta^k,\eta_\omega^k)}{\bar{\pi}_\omega(\bar{b}|\beta^k,\eta_\omega^k)} \to 0$, $\omega = \ell, h$, implying $\frac{\Pr[\text{ win at } b'' \mid x;\beta^k,\boldsymbol{\eta}^k]}{\Pr[\text{ win at } \bar{b} \mid x;\beta^k,\boldsymbol{\eta}^k]} \to 0$. This and $\mathbb{E}[v] - \bar{b} > 0$ from (42) imply $\frac{U(b''|x,\beta^k,\boldsymbol{\eta}^k)}{U(\bar{b}|x,\beta^k,\boldsymbol{\eta}^k)} \to 0$. Thus, there is no incentive to undercut $\bar{b}$ for large k. □

Step 9: Bidding $\underline{b}^k$ is a best response for $x < \hat{x}$ and large k.

Proof of Step 9: Fix some $x' < \hat{x}$. By the choice of $\underline{b}^k$ in (42), $U(\underline{b}^k|x',\beta^k,\boldsymbol{\eta}^k) = 0$ for all k. Consider now possible deviations. Obviously, for $b < \underline{b}^k$, $U(b|x',\beta^k,\boldsymbol{\eta}^k) = 0$. From

Step 5 and (42), $U(\bar{b}|x', \beta^k, \boldsymbol{\eta}^k) < 0$ for large k. For $b > \bar{b}$, it follows from $\frac{g_h(x')}{g_\ell(x')} < \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$, Step 6, and (42) that $U(b|x', \beta^k, \boldsymbol{\eta}^k) < U(b|\bar{x}, \beta^k, \boldsymbol{\eta}^k) < 0$ for large k.

For $b \in \left(\underline{b}^k, \bar{b}\right)$, $\pi_\omega(b|\beta^k, n) = G_\omega\left(\hat{x}\right)^{n-1}$. This and (36) imply

$$\lim \frac{\overline{\pi}_h(b|\beta^k, \eta_h^k)}{\overline{\pi}_\ell(b|\beta^k, \eta_\ell^k)} = \lim \frac{\gamma_h^k G_h\left(\hat{x}\right)^{n_h^k - 1} + (1-\gamma_h^k) G_h\left(\hat{x}\right)^{n_h^k}}{\gamma_\ell^k G_\ell\left(\hat{x}\right)^{n_\ell^k - 1} + (1-\gamma_\ell^k) G_\ell\left(\hat{x}\right)^{n_\ell^k}}.$$

Then, substitute this and $g_\omega\left(x'\right) = g_\omega\left(\underline{x}\right) = G_\omega\left(\hat{x}\right)/\hat{x}$ into (28) to get

$$\lim \mathbb{E}\left[v|x',\ \text{win at } b \in \left(\underline{b}^k, \bar{b}\right); \beta^k, \boldsymbol{\eta}^k\right] = \frac{\rho_\ell v_\ell + \rho_h \lim \frac{n_h^k}{n_\ell^k} \frac{\gamma_h^k G_h(\hat{x})^{n_h^k} + (1-\gamma_h^k) G_h(\hat{x})^{n_h^k+1}}{\gamma_\ell^k G_\ell(\hat{x})^{n_\ell^k} + (1-\gamma_\ell^k) G_\ell(\hat{x})^{n_\ell^k+1}} v_h}{\rho_\ell + \rho_h \lim \frac{n_h^k}{n_\ell^k} \frac{\gamma_h^k G_h(\hat{x})^{n_h^k} + (1-\gamma_h^k) G_h(\hat{x})^{n_h^k+1}}{\gamma_\ell^k G_\ell(\hat{x})^{n_\ell^k} + (1-\gamma_\ell^k) G_\ell(\hat{x})^{n_\ell^k+1}}}.$$

This, together with (41), (42), and $\lim \frac{n_h^k}{n_\ell^k} < 1$ (from Step 3) imply that for large k,

$$\mathbb{E}\left[v|x',\ \text{win at } b \in \left(\underline{b}^k, \bar{b}\right); \beta^k, \boldsymbol{\eta}^k\right] < \mathbb{E}\left[v|x', \text{win at } \underline{b}^k; \beta^k, \boldsymbol{\eta}^k\right] = \underline{b}^k.$$

Hence, for $b \in \left(\underline{b}^k, \bar{b}\right)$, $U(b|x', \beta^k, \boldsymbol{\eta}^k) < 0$. Thus, for any $x' < \hat{x}$ and large k, there is no profitable deviation from $\underline{b}^k$. □

Step 10: There exists a sequence $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^\infty$ that satisfies (30) and (42).

Proof of Step 10: Let $\hat{v} :\equiv \max\left[v^*, v^{**}, v^{***}\right]$. Steps 5-7 imply that, if $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^\infty$ satisfies (30), then

$$\mathbb{E}[v|\underline{x}, \text{win at } \underline{b}^k; \beta^k, \boldsymbol{\eta}^k] \le \hat{v} < \mathbb{E}\left[v\right], \tag{43}$$

Choose any $\bar{b}$ to satisfy (42).

Given any $b \in [0, \hat{v}]$, let β_b be as in (30) with $\underline{b}^k = b$ and $\bar{b}$ as fixed before. Define the correspondence $\Psi_\omega^k : [0, \hat{v}] \rightrightarrows \Delta\left\{1, ..., N_{s^k}\right\}$ by

$$\Psi_\omega^k\left(b\right) \triangleq \underset{\eta \in \Delta\left\{1, ..., N_{s^k}\right\}}{\arg\max} \sum \eta\left(n\right)\left(\mathbb{E}_\omega\left[p|\beta_b, n\right] - ns^k\right),$$

and $\Psi^k \triangleq \Psi_\ell^k \times \Psi_h^k$. Define the function $\hat{\underline{b}} : \left(\Delta\left\{1, ..., N_{s^k}\right\}\right)^2 \times [0, \hat{v}] \to [0, \hat{v}]$ by

$$\hat{\underline{b}}\left(\boldsymbol{\eta}, b\right) = \mathbb{E}\left[v|\ \underline{x},\ \text{win at } b; \beta_b, \boldsymbol{\eta}\right].$$

By the theorem of the maximum and the continuity of the expected winning bid in $\boldsymbol{\eta}$ and b, Ψ^k is a non-empty and upper hemicontinuous correspondence. By Lemma 1, it is convex valued. A bidder's posterior conditional on being solicited is continuous in $\boldsymbol{\eta}$ (note that $\eta_\omega\left(0\right) = 0$ by definition) and hence $\hat{\underline{b}}\left(\boldsymbol{\eta}, b\right)$ is continuous in $\boldsymbol{\eta}$ as well. The function $\hat{\underline{b}}\left(\boldsymbol{\eta}, b\right)$ is constant in b, since the expectation defining it is the same for all b, and $\hat{\underline{b}}$ maps into $[0, \hat{v}]$ by (43).

Thus, $(\Psi^k, \hat{\underline{b}})$ is a non-empty, convex valued, and upper hemicontinuous correspondence from $\left(\Delta\left\{1,...,\bar{N}\right\}\right)^2 \times [0,\hat{v}]$ into itself. Therefore, by Kakutani's theorem it has a fixed point. Denote one such fixed point as $(\boldsymbol{\eta}^k, \underline{b}^k)$. The fixed-point satisfies the requirements in the statement of the Lemma: $\boldsymbol{\eta}^k$ is a best-response to β^k for all k, and $\underline{b}^k, \bar{b}$ are constructed to satisfy (42), which also implies the remaining conditions of (30), namely, $\underline{b}^k < \bar{b}$ and $\lim \underline{b}^k < \bar{b}$. □

By Step 10, there exists a sequence $\left(\beta^k, \boldsymbol{\eta}^k\right)_{k=1}^{\infty}$ that satisfies (30) and (42). For all k sufficiently large, $\left(\beta^k, \boldsymbol{\eta}^k\right)$ forms an equilibrium: By (30), $\boldsymbol{\eta}^k$ is a best-response to β^k for all k. By Steps 8 and 9, β^k is a best response to $\boldsymbol{\eta}^k$ for all k large enough. This completes the proof. ∎

References

Atakan, A. and M. Ekmekci (2014), "Auctions, Actions, and the Failure of Information Aggregation." *American Economic Review*, 2014-2048.

Atakan, A. and M. Ekmekci (2015), "Market Selection and Information Content of Prices, Mimeo.

Broecker, T. (1990), "Credit-Worthiness Tests and Interbank Competition," *Econometrica*, 429-452.

Burdett, K. and K. Judd (1983), "Equilibrium Price Dispersion," *Econometrica*, 955-969.

Comanor, W. and M. Schankerman (1976), "Identical Bids and Cartel Behavior," *The Bell Journal of Economics*, 281–286.

Kremer, I. (2002), "Information Aggregation in Common Value Auctions," *Econometrica,* 1675–1682.

Lauermann, S. and A. Wolinsky (2013), "A Common Value Auction with Bidder Solicitation—The Complete Analysis," Working Paper.

Lauermann, S. and A. Wolinsky (2016), "Search with Adverse Selection," *Econometrica*, 243–315.

McAdams, D. (2007), "Monotonicity in Asymmetric First-Price Auctions with Affiliation," *International Journal of Game Theory,* 427-453.

Milgrom, P. (1979), "A Convergence Theorem for Competitive Bidding with Differential Information," *Econometrica*, 679-688.

Milgrom, P. (1981): "Rational Expectations, Information Acquisition, and Competitive Bidding," *Econometrica,* 921-943.

Mund, V. (1960), "Identical Bid Prices," *Journal of Political Economy,* 150–69.

Murto, P. and J. Välimäki (2015), "Common Value Auctions with Costly Entry," Mimeo.

Riordan, M. (1993), "Competition and Bank Performance: A Theoretical Perspective," in: Capital Markets and Financial Intermediation, C. Mayer and X. Vives, eds., Cambridge University Press, 328-343.

Rodriguez, G. (2000), "First Price Auctions: Monotonicity and Uniqueness," *International Journal of Game Theory*, 413-432.

Subramanian, G. (2010), "Negotiauctions: New dealmaking strategies for a competitive marketplace," WW Norton & Company.

Wilson, R. (1977), "A Bidding Model of Perfect Competition," *Review of Economic Studies,* 511-518.

9 Appendix—For Online Publication Only

9.1 Proof of Proposition 1

Proof of Proposition 1: Suppose that $n_h \geq n_\ell \geq 2$ and β is a non-decreasing bidding equilibrium given (n_ℓ, n_h).

Recall from (5) that $x_+(b) = \sup\{x|\beta(x) = b\}$ and $x_-(b) = \inf\{x|\beta(x) = b\}$, so that an atom at b means $x_-(b) < x_+(b)$. We omit the arguments and write x_- and x_+ when it is clear from the context.

Step 1: $\frac{G_h(x)^{n_h}}{G_\ell(x)^{n_\ell}}$ is weakly increasing on any interval (y, z) if $n_\ell \leq n_h$. It is constant if and only if $n_\ell = n_h$ and $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < z$.

Proof of Step 1: For $z > y$,

$$\begin{aligned}\frac{G_h(z)^{n_h}}{G_\ell(z)^{n_\ell}} - \frac{G_h(y)^{n_h}}{G_\ell(y)^{n_\ell}} &= \left(\frac{G_h(z)}{G_\ell(z)}\right)^{n_\ell} (G_h(z))^{n_h - n_\ell} - \left(\frac{G_h(y)}{G_\ell(y)}\right)^{n_\ell} (G_h(y))^{n_h - n_\ell} \\ &\geq \left(\frac{G_h(z)}{G_\ell(z)}\right)^{n_\ell} - \left(\frac{G_h(y)}{G_\ell(y)}\right)^{n_\ell} \geq 0,\end{aligned}$$

where the first inequality is from $n_h \geq n_\ell$ and $G_h(z) > G_h(y)$, and the second inequality is from the MLRP. Hence, $\frac{G_h(x)^{n_h}}{G_\ell(x)^{n_\ell}}$ is weakly increasing on (y, z), and it is strictly increasing unless both $n_h = n_\ell$ and $\frac{G_h(x)}{G_\ell(x)}$ is constant on (y, z), which requires $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < z$.

To see the latter assertion, suppose that $\frac{G_h(x)}{G_\ell(x)} = C$ (a constant) on $[y, z]$. Thus, $\left(\frac{G_h(x)}{G_\ell(x)}\right)'_x = 0$, which implies $\frac{g_h(x)}{g_\ell(x)} = C$ on (y, z). Now, if there were some $\underline{x} < x' < y$ such that $\frac{g_h(x')}{g_\ell(x')} < C$, then by the monotonicity of $\frac{g_h}{g_\ell}$ and $\frac{g_h(x)}{g_\ell(x)} = C$ on (y, z),

$$\begin{aligned}C = \frac{G_h(z)}{G_\ell(z)} &= \frac{\int_{\underline{x}}^{y} \frac{g_h(x)}{g_\ell(x)} g_\ell(x)\, dx + \int_{x_-}^{z} \frac{g_h(x)}{g_\ell(x)} g_\ell(x)\, dx}{G_\ell(z)} \\ &< \frac{C\left[\int_{\underline{x}}^{y} g_\ell(x)\, dx + \int_{y}^{z} g_\ell(x)\, dx\right]}{G_\ell(z)} = C,\end{aligned}$$

a contradiction. Therefore, $\frac{g_h(x)}{g_\ell(x)} = C$ for all $\underline{x} < x' < z$. Finally, $\frac{g_h(\underline{x})}{g_\ell(\underline{x})} = C$ by right-continuity at $\underline{x}$. □

Step 2: If $x_-(b) < x_+(b)$ and $n_h \geq n_\ell$, then

$$\frac{G_h(x_-)^{n_h - 1}}{G_\ell(x_-)^{n_\ell - 1}} \leq \frac{\pi_h(b|\beta, n_h)}{\pi_\ell(b|\beta, n_\ell)} \leq \frac{G_h(x_+)^{n_h - 1}}{G_\ell(x_+)^{n_\ell - 1}},$$

with equalities holding if and only if $n_\ell = n_h$ and $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < x_+$. Otherwise, if either $n_\ell > n_h$ or $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for some $x < x_+$ or both, both inequalities are strict.

Proof of Step 2: Expanding the expression for $\pi_\omega[b|\beta, n_\omega]$ from (24) yields

$$\frac{\pi_h(b|\beta, n_h)}{\pi_\ell(b|\beta, n_\ell)} = \frac{n_\ell}{n_h} \frac{G_h(x_-)^{n_h-1} + G_h(x_-)^{n_h-2} G_h(x_+) + \cdots + G_h(x_+)^{n_h-1}}{G_\ell(x_-)^{n_\ell-1} + G_\ell(x_-)^{n_\ell-2} G_\ell(x_+) + \cdots + G_\ell(x_+)^{n_\ell-1}}.$$

Divide through by $\frac{G_h(x_+)^{n_h-1}}{G_\ell(x_+)^{n_\ell-1}}$ to obtain

$$\begin{aligned}
\left(\frac{\pi_h(b|\beta, n_h)}{\pi_\ell(b|\beta, n_\ell)}\right) / \left(\frac{G_h(x_+)^{n_h-1}}{G_\ell(x_+)^{n_\ell-1}}\right) &= \frac{\left[1 + \frac{G_h(x_-)}{G_h(x_+)} + \left(\frac{G_h(x_-)}{G_h(x_+)}\right)^2 + \cdots + \left(\frac{G_h(x_-)}{G_h(x_+)}\right)^{n_h-1}\right] / n_h}{\left[1 + \frac{G_\ell(x_-)}{G_\ell(x_+)} + \left(\frac{G_\ell(x_-)}{G_\ell(x_+)}\right)^2 + \cdots + \left(\frac{G_\ell(x_-)}{G_\ell(x_+)}\right)^{n_\ell-1}\right] / n_\ell} \\
&\leq \frac{\left[1 + \frac{G_h(x_-)}{G_h(x_+)} + \left(\frac{G_h(x_-)}{G_h(x_+)}\right)^2 + \cdots + \left(\frac{G_h(x_-)}{G_h(x_+)}\right)^{n_\ell-1}\right] / n_\ell}{\left[1 + \frac{G_\ell(x_-)}{G_\ell(x_+)} + \left(\frac{G_\ell(x_-)}{G_\ell(x_+)}\right)^2 + \cdots + \left(\frac{G_\ell(x_-)}{G_\ell(x_+)}\right)^{n_\ell-1}\right] / n_\ell} \leq 1.
\end{aligned}$$

The first inequality follows from the fact that, since $\frac{G_h(x_-)}{G_h(x_+)} < 1$, the numerator after the inequality is an average of the largest n_ℓ terms out of the n_h terms that are averaged on the numerator before the inequality sign. The second inequality follows from $\frac{G_h(x_-)}{G_h(x_+)} \leq \frac{G_\ell(x_-)}{G_\ell(x_+)}$, which in turn follows from $\frac{G_h(x_-)}{G_\ell(x_-)} \leq \frac{G_h(x_+)}{G_\ell(x_+)}$ which holds by MLRP.

Analogously, dividing through by $\frac{G_h(x_-)^{n_h-1}}{G_\ell(x_-)^{n_\ell-1}} > 1$ and exactly reversing the previous arguments,

$$\left(\frac{\pi_h(b|\beta, n_h)}{\pi_\ell(b|\beta, n_\ell)}\right) / \left(\frac{G_h(x_-)^{n_h-1}}{G_\ell(x_-)^{n_\ell-1}}\right) \geq 1,$$

where the inequalities are explained by noting that $\frac{G_h(x_+)}{G_h(x_-)} > 1$ and reversing the previous arguments.

In both cases, the two inequalities hold as equalities if and only if $n_h = n_\ell$ and $\frac{G_h(x_-)}{G_h(x_+)} = \frac{G_\ell(x_-)}{G_\ell(x_+)}$. The last equality is equivalent to $\frac{G_h(x_-)}{G_\ell(x_-)} = \frac{G_h(x_+)}{G_\ell(x_+)}$ which holds if and only if $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < x_+$, as argued before. □

Step 3: If for some p, $x_+(p) > x_-(p)$, then $v_\ell \leq p < v_h$.

Proof of Step 3: $p \geq v_h$ would imply strictly negative payoff. Slightly overbidding an atom at $p < v_\ell$ is necessarily profitable. □

Step 4: If for some p, $x_+(p) > x_-(p)$, then $n_h = n_\ell$, $U(p|x;\beta,\mathbf{n}) = 0$ for $x \in (x_-, x_+)$ and $\frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for all $x < x_+$.

Proof of Step 4: Use (2)—(4) to rewrite

$$\begin{aligned}
&U(p|x;\beta,\mathbf{n}) \qquad (44)\\
= \; &\frac{\rho_\ell g_\ell(x) n_\ell \pi_\ell(p|\beta, n_\ell)}{\rho_\ell g_\ell(x) n_\ell + \rho_h g_h(x) n_h} \left[(v_\ell - p) + \frac{\rho_h}{\rho_\ell} \frac{g_h(x)}{g_\ell(x)} \frac{n_h}{n_\ell} \frac{\pi_h(p|\beta, n_h)}{\pi_\ell(p|\beta, n_\ell)} (v_h - p)\right].
\end{aligned}$$

The expected payoff of a bidder with signal $x \in (x_-, x_+)$ who bids "just above" p is

$$\begin{aligned} &\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} U(p+\varepsilon|x;\beta,\mathbf{n}) \qquad (45) \\ = \quad &\frac{\rho_\ell g_\ell(x) n_\ell G_\ell(x_+)^{n_\ell - 1}}{\rho_\ell g_\ell(x) n_\ell + \rho_h g_h(x) n_h} \left[(v_\ell - p) + \frac{\rho_h}{\rho_\ell} \frac{g_h(x)}{g_\ell(x)} \frac{n_h}{n_\ell} \frac{G_h(x_+)^{n_h - 1}}{G_\ell(x_+)^{n_\ell - 1}} (v_h - p) \right]. \end{aligned}$$

By optimality, $U(p|x;\beta,\mathbf{n}) \geq 0$ for all $x \in (x_-,\, x_+)$ and, by Lemma 2, $G_\ell(x_+)^{n_\ell - 1} > \pi_\ell(p|\beta, n_\ell)$. Therefore, it follows from (44), (45), the MLRP, Step 2 and Step 3 that $\lim_{\varepsilon>0,\varepsilon\to 0} U(p+\varepsilon|x;\beta,\mathbf{n}) \geq U(p|x;\beta,\mathbf{n})$ for all $x \in (x_-,\, x_+)$. The inequality is strict if $U(p|x;\beta,\mathbf{n}) > 0$ or $n_h > n_\ell$ or $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ for some $x < x_+$ (or any combination). Therefore, optimality of β implies the claim. □

Step 5: If $n_h = n_\ell \geq 2$ then $\beta(x'') = \beta(x')$ for all $x'', x' \in (\underline{x}, \hat{x})$, where $\hat{x} = \sup\left\{x | \frac{g_h(x)}{g_\ell(x)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}\right\}$.

Proof of Step 5: Suppose to the contrary that $\beta(x'') < \beta(x')$ for some $x'', x' \in (\underline{x}, \hat{x})$. Note that $U(\beta(\underline{x})|\underline{x};\beta,\mathbf{n}) = 0$ in every bidding equilibrium. (If there is no atom at $\underline{x}$, this is immediate from the monotonicity of β; if there is an atom at $\underline{x}$, this follows from the Step 4.) By the choice of $\hat{x}$, $\frac{g_h(x')}{g_\ell(x')} = \frac{g_h(x'')}{g_\ell(x'')} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$. This and $n_h = n_\ell$ imply via Steps 1 and 4 that $\frac{\pi_h(\beta(x'')|\beta,n_h)}{\pi_\ell(\beta(x'')|\beta,n_\ell)} = \frac{\pi_h(\beta(x')|\beta,n_h)}{\pi_\ell(\beta(x')|\beta,n_\ell)} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$. To see this, consider the second equality. If there is no atom at $\beta(x')$, it follows from Step 1; if there is an atom, Step 4 implies $x_+(\beta(x')) \leq \hat{x}$ and it follows from Step 2. The argument for the equality between the first and third terms is completely analogous. In addition, $\frac{g_h(x'')}{g_\ell(x'')} = \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ and optimality implies $U(\beta(x'')|x'';\beta,\mathbf{n}) = 0$. Therefore, since $\pi_\omega(\beta(x)|\beta,n_\omega) > 0$ for any $x > \underline{x}$ and $\omega \in \{\ell, h\}$ (by monotonicity of β), evaluating (44) yields

$$\begin{aligned} 0 \quad &= \quad (v_\ell - \beta(x'')) + \frac{\rho_h}{\rho_\ell} \frac{g_h(x'')}{g_\ell(x'')} \frac{n_h}{n_\ell} \frac{\pi_h(\beta(x'')|\beta,n_h)}{\pi_\ell(\beta(x'')|\beta,n_\ell)} (v_h - \beta(x'')) \\ &= \quad (v_\ell - \beta(x'')) + \frac{\rho_h}{\rho_\ell} \frac{g_h(x')}{g_\ell(x')} \frac{n_h}{n_\ell} \frac{\pi_h(\beta(x')|\beta,n_h)}{\pi_\ell(\beta(x')|\beta,n_\ell)} (v_h - \beta(x'')) \\ &> \quad (v_\ell - \beta(x')) + \frac{\rho_h}{\rho_\ell} \frac{g_h(x')}{g_\ell(x')} \frac{n_h}{n_\ell} \frac{\pi_h(\beta(x')|\beta,n_h)}{\pi_\ell(\beta(x')|\beta,n_\ell)} (v_h - \beta(x')) \\ &= \quad U(\beta(x')|x';\beta,\mathbf{n}), \end{aligned}$$

where the second equality follows from the argument in the previous paragraph, and the inequality follows from the hypothesis $\beta(x'') < \beta(x')$. But $U(\beta(x')|x';\beta,\mathbf{n}) < 0$ contradicts the optimality of $\beta(x')$. Therefore, $\beta(x'') = \beta(x')$. □

Steps 4 establishes the first part and the second item of the second part of Proposition **1**; Step 5 establishes the first item of the second part. ∎

9.2 Proof of Lemma 1

Proof of Lemma 1: $\Pr(\text{winning bid} \le p) = \left(G_\omega\left(\beta^{-1}\left([0,p]\right)\right)\right)^n$, where $\beta^{-1}\left([0,p]\right) = \{x : \beta(x) \in [0,p]\}$. Since the expectation of a positive random variable is equal to the integral of its decumulative distribution function,

$$\mathbb{E}_\omega\left[p|\beta,n\right] = \int_0^{v_h}\left(1-\left(G_\omega\left(\beta^{-1}\left([0,t]\right)\right)\right)^n\right)dt.$$

Treating n as a continuous variable, $\frac{d\mathbb{E}_\omega[p|\beta,n]}{dn^2} < 0$. Hence, $\mathbb{E}_\omega\left[p|\beta,n\right]$ is strictly concave, which implies the lemma. ∎

9.3 Lemmas for the Proof of Proposition 3

We state and prove an auxiliary lemma first and then three lemmas that address the main parts of the proof. The auxiliary lemma provides a sharp relationship between the first-order statistics in the two states (in the limit).

Lemma 3 **(Poisson Approximation)** *Consider some sequence* $\left(x^k, \mathbf{n}^k\right)$ *with* $\min\left\{n_\ell^k, n_h^k\right\} \to \infty$ *and* $\lim \frac{n_h^k}{n_\ell^k} < \infty$*. If* $\lim\left(G_\ell\left(x^k\right)\right)^{n_\ell^k} = q \in [0,1]$ *then letting* $\lambda \equiv \lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$,

$$\lim\Big(G_h(x^k)\Big)^{n_h^k} = q^\lambda.$$

Proof of Lemma 3. Let $Q_\omega = \lim\left(1-G_\omega\left(x^k\right)\right)n_\omega^k$. Observe that

$$\lim\Big(G_\omega\left(x^k\right)\Big)^{n_\omega^k} = \lim\left(1-\frac{1-G_\omega\left(x^k\right)}{n_\omega^k}n_\omega^k\right)^{n_\omega^k} = e^{-Q_\omega}.$$

Since, by assumption, the limit Q_ℓ indeed exists and since $\lim \frac{1-G_h\left(x^k\right)}{1-G_\ell\left(x^k\right)}\frac{n_h^k}{n_\ell^k} = \lambda$, the limit Q_h exists as well and $Q_h = \lambda Q_\ell$. Therefore, since $q = e^{-Q_\ell}$,

$$\lim\Big(G_h(x^k)\Big)^{n_h^k} = e^{-Q_h} = e^{-\lambda Q_\ell} = q^\lambda.$$

∎

Lemma 4 **("Zero Profit")** *For any* $\varepsilon > 0$*, there is an* $M(\varepsilon)$ *such that, if* $n_\omega > M(\varepsilon)$*,* $\omega = \ell, h$*, then* $U(\beta(x)|x) < \varepsilon$ *for all* x *in every bidding equilibrium* β*.*

Proof of Lemma 4. Let $\varepsilon > 0$. Since, by (44), $(U(b|\cdot;\beta,\mathbf{n}))_{b,\beta,\mathbf{n}}$ is a family of equicontinuous functions, there is $z_\varepsilon > 0$ such that

$$\left|U(b|x';\beta,\mathbf{n}) - U(b|x;\beta,\mathbf{n})\right| \le \frac{\varepsilon}{2},$$

for all b, all $(\beta,\mathbf{n})$ and all x, x' s.t. $|x-x'| \le z_\varepsilon$.

Suppose $U(\beta(x)|x;\beta,\mathbf{n})=\varepsilon$, for some x. From β being a bidding equilibrium, for all x,x' s.t. $|x-x'|\leq z_\varepsilon$,

$$\left|U(\beta(x)|x;\beta,\mathbf{n})-U(\beta(x')|x';\beta,\mathbf{n})\right|\leq\frac{\varepsilon}{2}. \tag{46}$$

Therefore, for any x,

$$\begin{aligned}&U(\beta(x)|x;\beta,\mathbf{n})-\frac{\varepsilon}{2}\leq\inf_{x'\in[x-z_\varepsilon,x+z_\varepsilon]}U(\beta(x')|x';\beta,\mathbf{n})\\&\leq\sum_{\omega=\ell,h}\rho_\omega\frac{\int_{x-z_\varepsilon}^{x+z_\varepsilon}[v_\omega-\beta(x')]\pi_\omega(\beta(x')|\beta,n_\omega)dG_\omega(x')}{G_\omega(x+z_\varepsilon)-G_\omega(x-z_\varepsilon)}\leq\sum_{\omega=\ell,h}\rho_\omega\frac{v_\omega\int_{\underline{x}}^{\bar{x}}\pi_\omega(\beta(x')|\beta,n_\omega)dG_\omega(x')}{G_\omega(x+z_\varepsilon)-G_\omega(x-z_\varepsilon)}\\&\leq\sum_{\omega=\ell,h}\frac{\rho_\omega v_\omega}{n_\omega\left(G_\omega(x+z_\varepsilon)-G_\omega(x-z_\varepsilon)\right)}\leq\frac{\mathbb{E}[v]}{\min_{\omega\in\{\ell,h\}}\left(n_\omega\left(G_\omega(x+z_\varepsilon)-G_\omega(x-z_\varepsilon)\right)\right)},\end{aligned}$$

where the first inequality follows from (46), the second is immediate from the definition of U, the third owes to increasing the term in the numerator, and the fourth from the fact that the expected probability of winning over all signals is $1/n_\omega$. Now, let $M(\varepsilon)$ be large enough so that, for $n_\omega\geq M(\varepsilon)$, the RHS is smaller than $\frac{\varepsilon}{2}$. Therefore, for any $\mathbf{n}$ such that $n_\omega\geq M(\varepsilon)$, $U(\beta(x)|x;\beta,\mathbf{n})<\varepsilon$. ∎

Corollary 2 *Let $(\mathbf{n}^k)_{k=1}^\infty$ be such that $\min\{n_\ell^k,n_h^k\}\to\infty$ and $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of bidding equilibrium strategies.*
(i)

$$\lim\sup_{x\in[\underline{x},\bar{x}]}U^k(\beta^k(x)|x)=0. \tag{47}$$

(ii) If, for some sequence $(b^k)_{k=1}^\infty$ of bids and some ω, $\lim\pi_\omega^k\left[b^k\right]>0$, then for any sequence $(x^k)_{k=1}^\infty$,

$$\lim\mathbb{E}^k[v|x^k,\textit{win at }b^k]\leq\lim b^k. \tag{48}$$

(iii) If $\lim\pi_\omega^k\left[\beta^k\left(x^k\right)\right]>0$ for some ω sequence $(x^k)_{k=1}^\infty$, then

$$\lim\beta^k\left(x^k\right)=\lim\mathbb{E}^k[v|x^k,\textit{win at }\beta^k\left(x^k\right)]. \tag{49}$$

Proof of Corollary 2: Part (i) and (ii) follow immediately from Lemma 4 that would be contradicted if (47) or (48) did not hold. Part (iii) is immediate from (48) and the individual rationality condition,

$$\beta^k\left(x^k\right)\leq\mathbb{E}^k[v|x^k,\text{win at }\beta^k\left(x^k\right)].$$

∎

Lemma 5 *Let $\mathbf{n}^k$ be such that $\min\{n_\ell^k,n_h^k\}\to\infty$ and $\lim\frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}<1$, and let $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of non-decreasing bidding equilibrium strategies. If $(b^k)_{k=1}^\infty$ is* *a sequence of bids such that* β^k *has no atom at* b^k *and* $\lim \pi_\ell^k\left(b^k\right) \in (0,1)$, *then,*

$$\lim \mathbb{E}^k[v|\bar{x}, \textit{win at } b^k] > \lim \mathbb{E}^k[v|\bar{x}, \textit{win at } \beta^k(\bar{x})].$$

Proof of Lemma 5. By (28), we have to show that

$$\lim \frac{\pi_h^k\left[b^k\right]}{\pi_\ell^k\left[b^k\right]} > \lim \frac{\pi_h^k\left[\beta^k(\bar{x})\right]}{\pi_\ell^k\left[\beta^k(\bar{x})\right]}.$$

Let $x^k = \sup\{x|\beta^k(x) \le b^k\}$. Thus, if b^k is in the range of β^k, then $x^k = \left(\beta^k\right)^{-1}(b^k)$ and otherwise β^k has an upward jump at x^k and b^k falls in that gap. Since, by hypothesis of the Lemma, $\pi_\omega^k\left(b^k\right) = \left(G_\omega\left(x^k\right)\right)^{n_\omega^k-1} \in (0,1)$. we have to show that

$$\lim \frac{G_h\left(x^k\right)^{n_h^k-1}}{G_\ell(x^k)^{n_\ell^k-1}} > \lim \frac{\pi_h^k\left[\beta^k(\bar{x})\right]}{\pi_\ell^k\left[\beta^k(\bar{x})\right]}. \tag{50}$$

Let

$$\hat{q} \triangleq \lim \left(G_\ell\left(x^k\right)\right)^{n_\ell^k-1}.$$

Recall the shorthand $\lambda \equiv \lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$. Now, $\hat{q} \in (0,1)$, Lemma 3, and $\lambda < 1$ imply

$$\lim \frac{G_h\left(x^k\right)^{n_h^k-1}}{G_\ell(x^k)^{n_\ell^k-1}} = \hat{q}^{\lambda-1} > 1. \tag{51}$$

Let

$$\bar{x}_-^k \triangleq x_-^k\left(\beta^k(\bar{x})\right) \text{ and } q \triangleq \lim\left(G_\ell\left(\bar{x}_-^k\right)\right)^{n_\ell^k}.$$

Since, by the hypothesis, $x^k \le \bar{x}_-^k$, we have $q \ge \hat{q}$.

Case 1. Suppose that $q = 1$. Then, either $\bar{x}_-^k = \bar{x}$ and hence $\pi_\omega^k\left(\beta^k(\bar{x})\right) = 1$, or $\bar{x}_-^k < \bar{x}$ in which case it follows from Lemma 2 that

$$\pi_\omega^k\left(\beta^k(\bar{x})\right) = \frac{1-\left(G_\omega\left(\bar{x}_-^k\right)\right)^{n_\omega^k}}{n_\omega^k\left(1-G_\omega\left(\bar{x}_-^k\right)\right)} = \frac{1+G_\omega\left(\bar{x}_-^k\right)+\ldots+\left(G_\omega\left(\bar{x}_-^k\right)\right)^{n_\omega^k-1}}{n_\omega^k} \ge \frac{n_\omega^k\left(G_\omega\left(\bar{x}_-^k\right)\right)^{n_\omega^k-1}}{n_\omega^k}. \tag{52}$$

By Lemma 3, $\lim\left(G_h\left(\bar{x}_-^k\right)\right)^{n_h^k} = q^\lambda = 1$ as well, it follows that $\lim \pi_\omega^k\left(\beta^k(\bar{x})\right) = 1$ for $\omega = \ell, h$. Hence, in either case

$$\lim \frac{\pi_h^k\left[\beta^k(\bar{x})\right]}{\pi_\ell^k\left[\beta^k(\bar{x})\right]} = 1.$$

This, and (51) implies (50).

Case 2. Suppose that $q < 1$. So, there is an atom at $\beta^k(\bar{x})$. First, consider $\lambda \in (0,1)$. From the first equality in (52) and from Lemma 3 and its proof,

$$\lim \pi_\ell^k \left[\beta^k\left(\bar{x}\right)\right] = \frac{1-q}{-\ln q} \text{ and } \lim \pi_h^k \left[\beta^k\left(\bar{x}\right)\right] = \frac{1-q^\lambda}{-\ln q^\lambda}.$$

Hence,

$$\lim \frac{\pi_h^k\left[\beta^k\left(\bar{x}\right)\right]}{\pi_\ell^k\left[\beta^k\left(\bar{x}\right)\right]} = \frac{\frac{1-q^\lambda}{-\lambda \ln q}}{\frac{1-q}{-\ln q}} = \frac{1-q^\lambda}{\lambda\left(1-q\right)} < q^{\lambda-1}, \tag{53}$$

where the last inequality follows from $\lambda \in (0,1)$, $q \in (0,1)$, and straightforward manipulation.[25] By the hypothesis that $\lim\left(G_\ell\left(x^k\right)\right)^{n_\ell^k} > 0$ and $x^k \le \bar{x}_-^k$, we have $q \ge \hat{q} > 0$. This and $\lambda < 1$ imply $\hat{q}^{\lambda-1} \ge q^{\lambda-1}$. Now, this together with (51) and (53) imply (50).

If $\lambda = 0$, by Lemma 3, $\lim G_h\left(x^k\right)^{n_h^k} = 1$, and hence $\lim \pi_h^k\left[\beta^k\left(\bar{x}\right)\right] = 1$. Thus, (50) follows from $\left(G_\ell\left(x^k\right)\right)^{n_\ell^k} < \pi_\ell^k\left[\beta^k\left(\bar{x}\right)\right]$ ∎

Lemma 6 *Let* $\mathbf{n}^k$ *be such that* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and* $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$. *Let* $(\beta^k)_{k=1}^\infty$ *be a corresponding sequence of bidding equilibria. If* $(\beta^k)_{k=1}^\infty$ *exhibits a sequence of non-vanishing atoms* $(b^k)_{k=1}^\infty$, *i.e.,* $\lim\left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k} > \lim\left(G_\ell\left(x_-^k\left(b^k\right)\right)\right)^{n_\ell^k}$, *then*

$$\lim_{k\to\infty} b^k < \lim_{k\to\infty}\lim_{\varepsilon\to 0} \mathbb{E}^k[v|x_+^k, \textit{win at } b^k+\varepsilon].$$

Proof of Lemma 6. By bidders' individual rationality, $\mathbb{E}^k[v|x_-^k,\text{win at } b^k] \ge b^k$. Therefore, the claim will follow from $\mathbb{E}^k[v|x_+^k,\text{win at } b^k+\varepsilon] > \mathbb{E}^k[v|x_-^k,\text{win at } b^k]$, which in turn follows from

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)}\frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)}\frac{\left(G_h\left(x_+^k\left(b^k\right)\right)\right)^{n_h^k}}{\left(G_\ell\left(x_+^k\left(b^k\right)\right)\right)^{n_\ell^k}}. \tag{54}$$

Let $q_- = \lim G_\ell\left(x_-^k\right)^{n_\ell^k}$ and $q_+ = \lim G_\ell\left(x_+^k\right)^{n_\ell^k}$. Note that $G_\omega\left(x_+^k\right)^{n_\omega^k} \cong G_\omega\left(x_+^k\right)^{n_\omega^k-1}$ for large k. By the hypothesis of the lemma, $q_+ > 0$. By Lemma 3, $\lim G_h\left(x_-^k\right)^{n_h^k} = (q_-)^\lambda$ and $\lim G_h\left(x_+^k\right)^{n_h^k} = (q_+)^\lambda$. Recall from Lemma 2 that

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \lim \frac{n_\ell^k}{n_h^k}\frac{G_\ell\left(x_+^k\right)-G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right)-G_h\left(x_-^k\right)}\frac{G_h\left(x_+^k\right)^{n_h^k}-G_h\left(x_-^k\right)^{n_h^k}}{G_\ell\left(x_+^k\right)^{n_\ell^k}-G_\ell\left(x_-^k\right)^{n_\ell^k}}.$$

Using this and the above observations,

$$\lim \frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)}\frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \lim\left(\frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)}\frac{G_\ell\left(x_+^k\right)-G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right)-G_h\left(x_-^k\right)}\right)\frac{g_h\left(\bar{x}\right)}{g_\ell\left(\bar{x}\right)}\frac{(q_+)^\lambda-(q_-)^\lambda}{\lambda\left(q_+-q_-\right)}.$$

[25] With $f(q,x) = q^{x-1}(x+q-xq)$, (53) is equivalent to $f(q,x) > 1$ for $q, x \in (0,1)$, which in turn follows from $f(1,x) = 1$ and $\frac{\partial}{\partial q}f(\cdot,x) = (-1)\,q^{x-2}(1-x)\,x\,(1-q) < 0$ for $q,x \in (0,1)$.

Now, $\lim \frac{g_h\left(x_+^k\right)}{g_\ell\left(x_+^k\right)} = \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ and by MLRP

$$\frac{g_h\left(x_-^k\right)}{g_\ell\left(x_-^k\right)} \frac{G_\ell\left(x_+^k\right) - G_\ell\left(x_-^k\right)}{G_h\left(x_+^k\right) - G_h\left(x_-^k\right)} \leq 1.$$

Therefore, we may establish (54) by showing that,

$$\frac{(q_+)^\lambda - (q_-)^\lambda}{\lambda\left(q_+ - q_-\right)} < (q_+)^{\lambda-1}. \tag{55}$$

Letting $Q = \frac{q_-}{q_+} < 1$, (55) is equivalent to $Q^\lambda - \lambda Q + \lambda > 1$. Since $\lambda > 1$, the LHS is decreasing in Q over $[0,1)$ and is equal to 1 at $Q = 1$. Therefore, (55) holds and so does (54). ∎

9.4 Proof of Proposition 3

Proof of Proposition 3. To prove the first part, suppose $\lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$. We show that there is a sequence $\left(x_-^k, x_+^k\right)$, with $x_-^k < x_+^k$ such that β^k is constant on (x_-^k, x_+^k) and

$$\lim \left(G_\omega\left(x_+^k\right)\right)^{n_\omega^k} - \left(G_\omega\left(x_-^k\right)\right)^{n_\omega^k} = 1 \qquad \text{for } \omega \in \{\ell, h\}.$$

Consider any sequence (x^k) for which $\lim \left(G_h\left(x^k\right)\right)^{n_h^k} \in (0,1)$. Let

$$y_-^k = x_-^k\left(\beta^k\left(x^k\right)\right) \text{ and } y_+^k = x_+^k\left(\beta^k\left(x^k\right)\right).$$

We show that

$$\lim \left(G_h\left(y_-^k\right)\right)^{n_h^k} = 0 \text{ and } \lim \left(G_h\left(y_+^k\right)\right)^{n_h^k} = 1.$$

In fact, it is sufficient to prove that $\lim \left(G_h\left(y_+^k\right)\right)^{n_h^k} = 1$, because the sequence (x^k) can be chosen to make $\lim \left(G_h\left(x^k\right)\right)^{n_h^k}$ arbitrarily small.

Suppose to the contrary that

$$\lim \left(G_h\left(y_+^k\right)\right)^{n_h^k} < 1. \tag{56}$$

Then, there is some sequence of bids $(b^k)_{k=1}^\infty$ such β^k has no atom at b^k (i.e., either b^k is in the range of β^k and $x_+^k\left(b^k\right) = x_-^k\left(b^k\right)$, or β^k has an upward jump and b^k falls in the gap) and $\lim \pi_\ell^k\left(b^k\right) \in (0,1)$. To construct such a sequence, pick any sequence $(z^k)_{k=1}^\infty$ with $z^k > y_+^k$ for all k and $\lim \left(G_h\left(z^k\right)\right)^{n_h^k} = \lim \left(G_h\left(y_+^k\right)\right)^{n_h^k}$. By the definition of y_+^k, $\beta^k\left(z^k\right) > \beta^k\left(x^k\right)$. Since there are at most countably many atoms, for every k, there is some bid b^k with $\beta^k\left(x^k\right) < b^k < \beta^k\left(z^k\right)$ such that β^k has no atom at b^k. Therefore,

$$\left(G_h\left(y_+^k\right)\right)^{n_h^k} \leq \pi_h^k(b^k) \leq \left(G_h\left(z^k\right)\right)^{n_h^k},$$

implying

$$\lim \pi_h^k(b^k) = \lim \left(G_h\left(y_+^k\right)\right)^{n_h^k} \in (0,1)\,. \tag{57}$$

Thus, the zero-profit condition (48) from Corollary 2 requires that

$$\lim b^k \geq \lim \mathbb{E}^k[v|\bar{x}, \text{win at } b^k]. \tag{58}$$

Now, since β^k has no atom at b^k, $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$, and, by (57), $\lim \pi_h^k(b^k) \in (0,1)$, Lemma 5 implies that

$$\lim \mathbb{E}^k[v|\bar{x}, \text{win at } b^k] > \lim \mathbb{E}^k[v|\bar{x}, \text{win at } \beta^k(\bar{x})]. \tag{59}$$

In addition, (57) and the monotonicity of β^k imply that $\lim_{k\to\infty} \pi_h^k[\beta^k(\bar{x})] > 0$. Therefore, (49) from Corollary 2 requires that

$$\lim \mathbb{E}^k[v|\bar{x}, \text{win at } \beta^k(\bar{x})] = \lim \beta^k(\bar{x})\,. \tag{60}$$

Hence, (58)-(60) together imply

$$\lim b^k > \lim \beta^k(\bar{x})\,,$$

in contradiction to $b^k < \beta^k\left(z^k\right)$ and the monotonicity of β^k. Thus, (56) cannot hold, which proves the claim.

To prove the second part, suppose $\lim \frac{n_h^k}{n_\ell^k}\frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$ and suppose to the contrary that $\beta^k(x) = b^k$ for all $x \in (x_-^k, x_+^k)$ and $\lim \left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k} > \lim \left(G_\ell\left(x_-^k\right)\right)^{n_\ell^k} \geq 0$. Thus,

$$\lim_{k\to\infty} \lim_{\varepsilon^k\to 0} \pi_\ell^k\left[b^k + \varepsilon^k\right] = \lim_{k\to\infty}\left(G_\ell\left(x_+^k\right)\right)^{n_\ell^k} > 0. \tag{61}$$

This and Lemma 6 implies that

$$\lim_{k\to\infty} \lim_{\varepsilon^k\to 0} U^k(b^k + \varepsilon^k | x_+^k) > 0.$$

contradicting the zero-profit condition (47). Thus, there can be no atom. ∎

9.5 Proof of Proposition 6

Proof of Proposition 6: From bidders' individual rationality,

$$\rho_\ell \lim \mathbb{E}^k[p|\ell] + \rho_h \lim \mathbb{E}^k[p|h] \leq \mathbb{E}[v]\,. \tag{62}$$

We show that, for any $p < \mathbb{E}[v]$, $\lim F_\omega^k(p) = 0$. This together with (62) implies the proposition, since if $\lim F_\omega^k(p) < 1$, for some $p > \mathbb{E}[v]$, (62) would be violated.

Suppose to the contrary that, for some p, $\lim F_\omega^k(p) > 0$. Therefore, there is $p' < \mathbb{E}[v]$, such that $q \triangleq \lim \pi_\ell^k(p') > 0$. Then, there is a sequence $\left(b^k\right)_{k=1}^\infty$ such that β^k has no atom at b^k for any k, $b^k \geq p'$, and $\lim b^k = p'$. Letting $\hat{q} \triangleq \lim \pi_\ell^k(b^k)$, Lemma 3 and

$\lambda \equiv \lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} = 1$ imply

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{\hat{q}^\lambda}{\hat{q}} = 1.$$

Thus, from (3), $\lim \mathbb{E}^k[v|\bar{x},\text{win at } b^k] = \mathbb{E}[v] > \lim b^k$. Since also $\lim \pi_\omega^k(b^k) > 0$ from $b^k > p'$ and $\lim \pi_\ell^k(p') > 0$, we have

$$\lim U^k(b^k|\bar{x}) > 0,$$

contradicting the zero-profit condition (47). Thus, such $\left(b^k\right)_{k=1}^\infty$ cannot exist. Therefore, $\lim \pi_\omega^k(p) = 0$ for all $p < \mathbb{E}[v]$, as needed. ∎