Contracting with Non-Exponential Discounting: Moral Hazard and Dynamic Inconsistency*

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Abstract

We study a dynamic principal-agent contracting problem in which both contracting parties have non-exponential discounting and hence have dynamic inconsistencies. We derive the optimal contract as a Markov perfect Nash equilibrium of the game played between the agent's and the principal's future selves. Under the general framework, the optimal contract is characterized by a system of non-linear stochastic differential equations, rather than the classical Hamilton-Jacobi-Bellman equation. We provide a novel existence theorem to the solution of the general framework and demonstrate its applicability by solving two examples explicitly: one in which discounting is quasi-hyperbolic, and another in which discounting is stochastic.

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1 Introduction

The time value of money, or more broadly, the discount of future utility, is fundamental in economic dynamics. In a dynamic principal-agent contracting relationship, how each party discounts the future returns of their actions usually shapes the outcome of the contract.[1] Existing dynamic contracting models mostly assume that contracting parties have exponential discounting. This is analytically convenient but potentially restrictive, because exponential discounting implies that the discount rate between two adjacent periods is constant regardless of how distant into the future those periods are, and calendar dates are mostly irrelevant.

In this paper we relax the exponential discounting assumption and study dynamic contracting when one or both parties have general non-exponential discounting. We model the discounting between two periods through a bi-function which could be either deterministic or stochastic. This generalization enables us to tackle many real-world scenarios: for example the *present bias* that arises either when contracting parties have such inherent preferences (Harris and Laibson (2012), Thaler and Benartzi (2004)) or when one of the parties is a group of individuals like the board of directors, shareholders etc.[2] Alternatively, when the contract involves managing a public firm or a pool of financial assets, the contracting parties' discount rate might be stochastic due to time-varying stock prices or short-term interest rates (as in Vasicek (1977), Cox et al. (1985), Gabaix (2012)). In addition to these well documented cases our general approach allows us to capture inherent values attached to specific dates (such as birthdays, anniversaries, etc) or other behavior such as future bias or any combination thereof. Overall, there is extensive evidence, anecdotal and academical, that "time preferences can be non-exponential" (Laibson (1997)).

Despite the broad evidence of non-exponential discounting, existing studies of economic dynamics often cast their models under exponential discounting because of one specific reason: to avoid *dynamic inconsistencies*. Under dynamic inconsistency, plans for the entirety

[1] In DeMarzo and Sannikov (2006), for example, the difference in discount rate between the principal and the agent determines when payments to the agent are actually made. Lehrer and Pauzner (1999), Ray (2002), and Opp and Zhu (2015) demonstrate various effects of differential discount factors in dynamic games.

[2] Jackson and Yariv (2015) shows that present-bias can emerge even if each member of the group has normal, exponential discounting.

of future that look "optimal" at present may turn out to be suboptimal when the future actually arrives. In particular, if the decision makers are given the chance to re-evaluate their plan, they might do so continuously instead of following the original plan made before. It is important to note that, as Strotz (1955) concludes, in the face of non-exponential discounting "there is nothing patently irrational about the individual who finds that he is in an intertemporal tussle with himself – except that rational behavior requires that he take the prospect of such a tussle into account."

There are usually two remedies for a rational (or sophisticated) decision maker to resolve the dynamic inconsistencies resulting from non-exponential discounting: one is to pre-commit to any action plans the moment they are made, and the other is to take into account the possibility of intertemporal conflict and make plans that will indeed be followed in the future, even though such plans appear suboptimal at present. In the context of contracting, pre-commitment is hardly the optimal approach, because the contract parties could be chronically unhappy once the preferences over time changes and wish to modify or even terminate the contract. In practice, contracts that are supposed to involve long term commitments are usually embedded with clauses that allow some leeway. For example, Mian and Santos (2018) document around 20% of the US syndicated loans are refinanced before maturity, often from the same leading bank. Xu (2017) find similar activities on the corporate bond market, where 60 to 80 billions of dollar of bonds each year are refinanced early through tender offers, repurchases, make-whole calls, etc.[3] In addition, directors often restructure the compensation package of a CEO if the firm is doing well/worse, and a client often restructure the payments to a contractor if their financial situation changes.

In this paper we explore long term contracts that allow contracting parties to revisit the terms if it is agreeable to both. That immediately rules out pre-commitment, and the contract needs to take into account the possibility of intertemporal conflict within each party. In particular, we adopt a game-theoretic framework, where parties forecast their behaviors

[3]Indeed, Xu (2017) writes: "the majority of the literature, both theoretical or empirical, treats maturity of a debt as a one-time decision in which firms choose maturity at issuance and then commit to it until the scheduled due date… In this paper, I identify a number of empirical facts showing that a major portion of early refinancing activities are not conducted through calls, and thus they cannot be explained by interest savings. Instead, firms refinance bonds early to continuously adjust maturity after issuance, leading to a significant discrepancy between maturity at issuance and the effective maturity at retirement."

and treat them as decisions made by different future selves. The intertemporal conflict thus corresponds to a game between those "selves". This approach to resolve dynamic inconsistency is akin to the seminal works of Strotz (1955) and Pollak (1968) and has been widely adopted since. (Ekeland and Lazrak (2010), Ekeland and Pirvu (2008), Björk et al. (2017)). We follow a similar approach and derive the optimal *equilibrium contract* as the Markov perfect Nash equilibrium of the game played between the agent, the principal and their future selves.

The equilibrium contract is characterized by a system of non-linear partial differential equations, instead of the classical Hamilton-Jacobi-Bellman (HJB) equation from standard dynamically consistent contracting benchmarks (e.g., Holmstrom and Milgrom (1987), Sannikov (2008) and He (2011)). This "extended HJB system" consists of a forward part: a value equation, and a backward part: a forward-backward stochastic differential equation (FBSDE). The forward value equation captures the value of the dynamically inconsistent principal by assigning values to each different future self of the principal. The FBSDE has the agents continuation utility as a forward stochastic differential equation, and the principals perceived value at different times as the backward equation. As a whole, the agent's problem is akin to the time-consistent benchmarks with some minor modifications, while the principal's problem requires a significantly different approach.

The extended HJB system approach to dynamic inconsistencies is economically intuitive and has been recently conjectured for general time inconsistent stochastic control problems in Yong (2012), Wang and Wu (2015), Lindensjö (2016), Björk et al. (2017), Wei et al. (2017), etc. However, "[t]he task of proving existence and/or uniqueness of solutions to the extended HJB system seems (...) to be technically extremely difficult" (Björk et al. (2017)). Indeed, without specific explicit forms or very restrictive assumptions the existence of solutions to extended HJB systems is an open question. To characterize the equilibrium contract of our model, we introduce a novel approach and prove the existence of the solution to a broad class of problems that has time inconsistencies arising from non-exponential discounting. Instead of trying to establish well-posedness of the components of the system and their compatibility, we approach the problem by translating the system into an appropriate static game of incomplete information between non-atomic players (à la Schmeidler (1973), Mas-

Colell (1984), Balder (1991)), and prove the existence of the equilibrium of such non-atomic game to establish the existence of a solution to the extended HJB system.

Following the characterization and the existence of the optimal contract, we demonstrate the applicability of our framework by explicitly solving two special cases. In the first case, the principal has *quasi-hyperbolic* discounting, arising naturally when the principal-agent relationship has fixed "terms" but the continuation to another term arrives stochastically. Typical examples include a politician or a board member in an organization that may be promoted or replaced before her current term is over. Principals of this kind are present-biased: they have higher valuations for returns to the agent's actions in the near future when her position is relatively secure, and lower valuations for returns in the far future when the agent's actions may become less relevant to her. We adopt the quasi-hyperbolic discount function of Harris and Laibson (2012) with the same interpretation that the "hyperbolic shock" (β in β-δ preferences of discrete time) to the principals' discounting arrives stochastically (as opposed to arriving in the next period in the traditional β-δ case).

We find that under quasi-hyperbolic discounting the optimal effort is not necessarily monotonic, in contrast to the standard case. The contract takes shape according to three factors; how likely the principal perceives the shock to her discounting is going to happen, how large the drop after the shock is, and how long the remaining contracting horizon is. We demonstrate that there is a "deadline effect" where the likelihood of shock happening before the end of the contracting horizon (the deadline) plays a critical role in shaping the optimal contract.

Our second solved example explores the case of a principal with *stochastic discounting*, arising from contracting problems in which the principal faces time-varying discount rates. For instance, the principal holds stocks and her valuation of future returns is affected by the fluctuation of stock prices and short-term interest rates. We adopt the short-rate model from Gabaix (2007) and show that stochastic discounting results in a "bright future effect": principal provides stronger incentives for higher effort when the realization of past interest rate is low.

The rest of the paper is structured as follows: after a brief literature review, we establish the general framework in Section 2, we first provide the underlying intuition and construction then present the extended HJB system. We then solve the two explicit examples (quasi-hyperbolic and stochastic discounting) in Section 3. Section 4 concludes. All proofs are relegated to the appendix.

1.1 Literature Review

The idea that preferences may be time-inconsistent as the result of non-exponential discounting stems from the seminal work of Strotz (1955) and Pollak (1968), who also suggest the game-theoretic approach that can be applied to solving the optimal dynamic planning problem under time-inconsistency. Obara and Park (2017) define the notion of strongly symmetric sub-game perfect equilibrium for individuals with general discounting in repeated games. They explore games with perfect monitoring and their general form of discounting, includes future bias, present bias, and quasi-hyperbolic discounting. Chade et al. (2008), adopt a Abreu et al. (1990) type recursive characterization of the equilibrium but allow quasi-hyperbolic discounting only.[4] In comparison, our focus is on the principal-agent problem with imperfect monitoring in continuous time.

The basic structure of our model follows the now celebrated literature of continuous-time dynamic principal-agent model with transitory private effort, such as DeMarzo and Sannikov (2006), Biais et al. (2007), Sannikov (2008), He (2011), etc. These, of course, are all models with exponential discounting and infinite contracting horizon. Our model features non-exponential discounting and we demonstrate the effect of different horizons on the contract. A few existing studies combine dynamic contracting with non-standard preferences. Gottlieb and Zhang (2018) study repeated contracting between a risk neutral firm and time inconsistent consumers, and find that of allowing consumers to terminate agreements at will may improve welfare if agents are sufficiently dynamically inconsistent. Gottlieb and Zhang (2018) focuses on adverse selection problems (reporting) rather than moral hazard (hidden effort). See Koszegi (2014) for a survey of contract theory with behavioral preferences.

More broadly, our paper contributes to the literature on the impact of time-inconsistency on various aspects of consumption and production decisions. Harstad (2016) analyzes invest-

[4] See also Kocherlakota (1996), which provides a refinement of sub-game perfect equilibrium with time inconsistent preferences.

ment policy of when the government is time-inconsistent, while Grenadier and Wang (2007) study investment under uncertainty (real option problem with time inconsistent preferences). Basak and Chabakauri (2010) study portfolio choice problem with mean-variance preferences. In macroeconomics, Krusell and Smith (2003) and Cao and Werning (2018) explore the consumption-saving problems the consumption-saving problems with time-inconsistent preferences and Bernheim et al. (2015) analyze the inter-temporal allocation problems with credit constraints under quasi hyperbolic preferences.[5] Typically these analyses are done in discrete time and are limited to a single party to be time inconsistent regardless of the size of the economy. In comparison we tackle the problem where both parties can be time-inconsistent.

Finally, our paper contributes to the mostly mathematical literature of time-inconsistent stochastic control, such as Yong (2012), Wang and Wu (2015), Björk et al. (2017), Lindensjö (2016), and Wei et al. (2017). A common caveat of these studies is that the solution to the resulting extended HJB system is left as an open question.[6] Our novel approach to the existence problem comes from bridging the time-inconsistent control problems to studies of non-atomic games such as Schmeidler (1973), Mas-Colell (1984), Khan and Sun (2002), Balder (1991), Balder (2002). Admittedly our model is set in a more restrictive environment compared to the aforementioned studies, but the restrictions are minimal and are mostly innocuous in economics settings.

2 General Framework

In this section we present a general framework, which introduces generic, non-exponential discounting processes to an otherwise standard dynamic principal-agent model. We first solve the agent's problem and then the principal's problem via a game-theoretic approach. We show how the optimal contract can be defined as a Markov Perfect equilbrium between the agent's and principal's future selves. Finally, we characterize the optimal contract by

[5] See also Amador et al. (2006), Halac and Yared (2014) and Bisin et al. (2015).

[6] Wei et al. (2017) provides a well-posedness result for uncontrolled diffusion processes. Björk et al. (2017) offers a verification of the extended HJB system analogues to us but does not prove existence of the solution. Lindensjö (2016) proves some regularity properties of the system in Björk et al. (2017), but does not address the existence question either.

a system of non-linear partial differential equations, and provide a novel argument for the existence of its solution.

2.1 Basic Environment

Time is continuous. A principal (she) contracts with an agent (he) over a fixed-time horizon $T < \infty$. The principal is risk-neutral, with unlimited liability, and outside option $\underline{V} \geq 0$. The agent is (weakly) risk-averse and has an outside option $\underline{u}$. Monetary cash flow is given by

$$dM_t = \hat{a}_t dt + \sigma dZ_t, \tag{1}$$

where $\hat{a}_t$ is the agent's private action (effort), and Z_t is a standard Brownian motion.

Between time t and t' the *principal's* discount rate is given by a bi-function $R(t, t')$. Such a specification allows for discounting to vary with both the current date and the time in question and not just the time difference. In particular, at time t, the principal uses the mapping defined by $R^t : [0, T] \to [0, 1]$ where $R^t(s) \doteq R(t, s)$ for all s. One can understand $R^t(s)$ as that there are infinitely many discounting functions, one corresponding to each point t in time denoted by $R^t(\cdot)$. We impose the following assumptions on $R^t(\cdot)$:

Assumption 1 *For all $\forall t \geq 0$, the principal's discount function $R^t(\cdot)$ satisfies:*

1. $R^t(s) = 1$ *for all* $s \leq t$ *and* $\lim_{s\to\infty} R^t(s) = 0$.
2. $R^t(s) > 0$ *for* s.
3. $\int_t^\infty R^t(s)ds < +\infty$ *a.s.*
4. $R^t(\cdot)$ *is uniformly Lipschitz continuous.*

The first part of the assumption states that starting from any period any payoff in the current period is not discounted, while any payoff in the infinitely far future has a present value of 0. The second part of the assumption states that any return in a finite future has some positive value, albeit potentially very small. The third part ensures that along any path the discounted values remain finite. The final part is a standard technical assumption.

Similarly, between time t and t' the *agent's* discount rate is given by a bi-function $r(t, t')$. In particular, at time t, the agent uses the mapping defined by $r^t : [0, T] \to [0, 1]$ where $r^t(s) \doteq r(t, s)$ for all s. We require the following assumptions on $r^t(\cdot)$:

Assumption 2 *For all $\forall t \geq 0$, the agent's discount function $r^t(\cdot)$ satisfies:*

1. *$r^t(s) = 1$ for all $s \leq t$ and $\lim_{s\to\infty} r^t(s) = 0$.*

2. *$r^t(s) > 0$ for s.*

3. *$\int_t^\infty r^t(s)ds < +\infty$ a.s.*

4. *$r^t(\cdot)$ is uniformly Lipschitz continuous.*

5. *$r^t(s) \leq R^t(s)$ for all s.*

Agent's discounting is analogous to the principal's discounting introduced in assumption 1 with one addition: the principal is weakly more patient then the agent. This is a fairly standard assumption that prevents the principal from making increasing promises until infinity. We present our results initially with deterministic bi-functions to ease the expositional burden, at the end of section 2.3 we provide the assumptions for the discounting to come from a "random field" – the stochastic extension of the bi-functions presented here.

We assume the following forms of utility functions and risk preferences:

Assumption 3 *The agent has a weakly risk averse utility function that is continuous in both arguments and twice differentiable for evaluating instantaneous consumption and action according to $u(c_t, a_t)$, we assume u is convex and decreasing in a, concave and increasing in c. $u(\cdot, \underline{a})$ is invertible with a continuous inverse function. The principal is risk neutral and evaluates instantaneous output net of consumption according to $M_t - c_t$.*

The first requirement on the utility function is fairly standard: effort a is costly with increasing marginal cost and consumption is valuable. The rest of the requirements are largely technical to facilitate the proof of the general existence theorem (Theorem 1), and are sufficient but certainly not necessary conditions. In more specific cases the utility function of the agent can be generalized further.

Denote the probability space as $(\Omega, \mathcal{F}, \mathcal{P})$ and the associated filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Contingent on the filtration, a contract specifies a consumption process $\{c_t\}_{t\geq 0}$ with final payment denoted c_T to the agent and a sequence of recommended action $\{a_t\}_{t\geq 0}$. Set of feasible effort levels comes from a compact set $\mathbb{A} := [\underline{a}, \bar{a}]$ with $\underline{a} > 0$. All quantities are assumed to be integrable and measurable under the usual conditions.

We now define the agent's *continuation utility*, which is the present value of his future utility, and incentive compatible contracts as follows:

Definition 1 *The agent's continuation utility at time t under an incentive compatible contract is:*

$$W_t = E_t \left[\int_t^T r^t(s) u(c_s, \hat{a}_s) ds + r^t(T) u(c_T, \underline{a}) \middle| \mathcal{F}_t \right], \tag{2}$$

A contract is incentive compatible if the agent maximizes his objective function (2) by choosing $\{\hat{a}_t\}_{t\geq 0} = \{a_t\}_{t\geq 0}$.

As usual, the contract can be characterized using the continuation utility as a state variable, along with the discount functions R^t, r^t in case of stochastic discounting.[7]

Without exponential discounting, the valuations of future streams of actions and consumptions might vary during over time. At such instances it is plausible that there are additional gains to be realized by altering the terms (the consumption process and recommended actions) of the of contract. We will call such an alteration a *restructuring* of the contract . Clearly an agent will not be willing to take any restructuring of the contract that delivers lower expected continuation utilities, and the principal would never offer a restructuring of the contract that yields higher expected continuation utility, hence we define a restructuring of a contract as follows:

Definition 2 *A contract* $\{c'_t, a'_t\}_{t\geq 0}$ *is a restructuring of the contract* $\{c_t, a_t\}_{t\geq 0}$ *if*

$$E_t \left[\int_t^T r^t(s) u(c_s, a_s) ds + r^t(T) u(c_T, \underline{a}) \middle| \mathcal{F}_t \right] = E_t \left[\int_t^T r^t(s) u(c'_s, a'_s) ds + r^t(T) u(c'_T, \underline{a}) \middle| \mathcal{F}_t \right]$$

[7] See assumptions, 1', 2' for the stochastic setup.

Allowing restructuring of contracts is what leads to the equilibrium approach as it translates to a partial commitment. We are motivated to explore such contracts as actual long term commitments do allow for restructuring. It is important to underline the relationship between restructuring and contract renegotiation. Renegotiation happens when, along its path, the contract yields suboptimal payoffs to both the agent and principal, and they renegotiate to reestablish efficiency. Suboptimal payoffs can happen due to reasons such as agent's deviations and inefficient punishments. Restructuring, however, holds the continuation utility of the agent constant, and changes the way the continuation payoff is delivered. In the absence of time-inconsistency, restructuring never happens under a principal optimal contract whereas renegotiation might happen if the optimal contract involved inefficient punishments.

Remark 1 *The relationship between restructuring and renegotiation can be highlighted with a rather simple example: consider a contracting problem where an agent chooses between high and low effort and the output can also be high and low. Assume low effort leads to low output certainly and high effort leads to high output with very high probability and there is limited liability on the agent side. It is plausible to imagine a simple contract such that (1) it implements high effort every period, (2) the agent is paid a fixed wage as long as low output does not happen, and (3) low output triggers a punishment phase, during which the agent is paid minimum wage and does not exert effort. However, once the punishment actually becomes necessary, the parties may* renegotiate *to cut the punishment short and re-establish efficiency.*[8] *In the case of* restructuring*, the contract is re-written and the payments are different over time, but the expected payoff that the agent receives (with the potential of punishment included) remains the same. Broadly speaking, restructuring is a very special form of renegotiating the terms of the contract to deliver the* same *payoffs, and it does not concern the inefficiencies of the original contract at all.*

[8]Similar situations arise in other dynamic principal-agent models with time-consistent preferences, such as DeMarzo and Sannikov (2006), Biais et al. (2007) and Sannikov (2008). Renegotiation can happen when the agent's continuation is sufficiently low following a stream of bad shocks, and a "reset" of the agent's continuation utility to a higher level would improve the payoff to the principal as well. See the more recent work of Strulovici (2011), Strulovici (2017) and Maestri (2017) for recent forms of renegotiation proofness in dynamic contracts.

2.2 The Agent's Problem

We now solve the agent's problem. It turns out that even if the agent has non-exponential discounting, his problem is not too different from that under standard exponential discounting.

Lemma 1 *Under assumptions, 1, 2, 3, given any contract and any sequence of the agent's choices, there exists a predictable, finite and L^2 integrable process ψ_t $(0 \leq t \leq T)$ such that W_t evolves according to*

$$dW_t = -\left(\frac{\partial r^t(t)}{\partial t} W_t + u(c_t, a_t)\right) dt + \psi_t \left(dM_t - a_t dt\right) \tag{3}$$

The contract is incentive compatible if and only if

$$\psi_t a_t - u(c_t, a_t) \geq \psi_t \hat{a} - u(c_t, \hat{a}), \quad \forall \hat{a} \in A, \forall t. \tag{4}$$

which implies

$$\psi_t = u_a(c_t, a_t). \tag{5}$$

This continuation utility is akin to the ones in standard time-consistent benchmark models such as Sannikov (2008) with one main difference: it is dependent on the current discount function and the drift changes as the discount function changes. In contrast, continuation utility in standard models with time-consistent preferences is a time-homogeneous Markov process. This close resemblance is not surprising as we are searching for the principal optimal contract and the principal is the one making the contractual offer.

2.3 The Principal's Problem

The principal's problem is significantly more involved. To see that, consider how the contracting problem would look like when only the agent has dynamic-inconsistency due to non-exponential discounting. In that case, as we show in the previous section, the agent's continuation utility will be a difficult to solve explicitly, but not significantly different than a standard setup technically, and the principal's problem would be fairly similar to the existing ones in the literature such as Sannikov (2008), He et al. (2017), Marinovic and Varas (2017). However, due to the non-standard discounting, our principal will have dynamic-inconsistencies and will need to take into account what the future selves will do. Before we delve into the full characterization let us make some observations about the problem that the principal is facing.

First, at the end of the contract term the principal can always retire the agent with a lump sum payment to deliver a promise W_T. Since the principal's valuation of payments vary over time we denote the principal's *retirement benefit* $B(t, W_T)$ evaluated at time t, as

$$B(t, W_T) = -R^t(T)u(c_T, \underline{a}) \tag{6}$$

where c_T is chosen such that $u(c_T, \underline{a}) = W_T$.

The principal's value function depends at time t depends on the promised payment to the agent W_t and the discount function of the current self $R^t(\cdot)$ for each t. Let the vector x_t denote the state variables where the state comprises of W_t and the current value of the stochastic discount functions, if there are any.[9] The principal's controls are denoted by $h_t \equiv [a_t, c_t, \psi_t]$. To derive the principal's value function given her time-inconsistent preference, we adopt a game-theoretic approach following Björk et al. (2017) by constructing a dynamic game between the principal's future selves, and look for Markov perfect equilibria. This approach is used in other time-inconsistent problems, notably Strotz (1955), Ekeland and Pirvu (2008), Ekeland and Lazrak (2010), Yong (2012), Lindensjö (2016), Wei et al. (2017). Importantly, given the principal is sophisticated, any choice the principal makes has to also factor in what she might do in the future under different preferences.

The construction of the dynamic game is as follows:

- Consider the principal is playing a non-cooperative game with her future self. There is one player for each point in time t whom we refer to as "Player t".

- For each fixed t, Player t can only influence the process M_t exactly at time t. She does

[9] There is no need to keep track of the discount bi-function without stochasticity as its progression is deterministic.

that by choosing a strategy of controlling h_t; so the optimal action taken at time t with state x_t is given by $h(t, x_t)$.We thus refer to the function h as the *control law* as it maps the state variables x_t into principal's controls $h_t \equiv [a_t, c_t, \psi_t]$.

- Piece together the control functions for all players, we thus have a feedback control law $h : T \times X \to A \times \mathbb{R} \times \mathbb{R}$.

- Given a feedback control law h, the payoff of Player t is given by the value function

$$J(t, x, h) = \mathrm{E}^h_{t,x} \left[\int_t^T R^t(s) \left(dM_s - c_s ds \right) + B(t, W_T) \right] \tag{7}$$

In discrete time, the standard definition of an equilibrium for this game would be to say that a feedback control law h is a subgame perfect Nash equilibrium if for each t, it has the property that for each future date $s > t$, player s chooses the control $h(s, \cdot)$, then it is optimal for Player t to choose $h(t, \cdot)$.[10] In continuous time, though, this definition is not sufficient since Player t can only choose the exactly at time t, which only influences the control on a time set of Lebesgue measure zero. Put differently, the control chosen by an individual player will have no effect on the dynamics of the process. We therefore adopt a modified definition of the equilibrium concept following Björk et al. (2017) as follows:

Definition 3 *Consider a control law h. For any initial point and state (t, x) and a "small" increment of time Δ, define the agent's "deviation" strategy h_Δ as:*

$$h_\Delta = \begin{cases} \hat{h}(t', x) & \text{for } t \leq t' < t + \Delta \\ h(t', x) & \text{for } t + \Delta \leq t' \leq T \end{cases}$$

where $\hat{h} \neq h$ is another control law, and x is the state vector corresponds to the future date

[10]In discrete time, every subgame perfect Nash equilibrium must be Markov.

t'.[11] *h is called an* ***equilibrium control law*** *if*

$$\liminf_{\Delta \to 0} \frac{J(t,x,h) - J(t,x,h_\Delta)}{\Delta} \geq 0 \tag{8}$$

for all $\hat{h} \neq h$*. For an equilibrium control law* h*, the corresponding value function* $V^h(t,x) = J(t,x,h)$*, as defined in (7), is called an* ***equilibrium value function****,*

With the dynamic game between the time t principal and her future selves constructed, and the equilibrium control law and value function well established, we can now define the optimal contract:

Definition 4 *A contract is optimal if it maximizes the principal's equilibrium value function* $V^h(t,x)$ *for all* t, x *over the set of contracts that 1) are incentive compatible, 2) are equilibrium control laws. Henceforth, we usually suppress* h *and let* $V(t,x)$ *denote the optimal equilibrium value function for an optimal contract.*

Before we proceed with the main theorem we introduce alternative assumptions to show how we can encompass stochasticity in discounting as well. We accomplish this by turning the discount bi-functions to random fields with parameter space $[0,T] \times [0,T]$ so that $R(t,t')$ and $r(t,t')$ are random variables. Now we can introduce analogous assumptions.

Assumption 1' *The principal's discount rate is a random field* $R(t,s) \doteq R^t(s)$ *where for each fixed* t*, the random field with the first parameter* t *corresponds a stochastic process as follows:*

$$R^t(s) = 1 + \int_t^s \mu_{R^t}(y, R^t(y))dy + \int_t^s \sigma_{R^t}(y, R^t(y))dZ^{R^t}$$

This process satisfies the following assumptions:

1. $R^t(s) = 1$ *for all* $s \leq t$ *and* $\lim_{s\to\infty} R^t(s) = 0$*, a.s.* $\forall t \geq 0$

[11] In the language of stochastic optimal control h_Δ is the local spike variation of h. Also this equilibrium approach makes the connection between the discrete time and continuous time clearer. However the full technicality of this connection is beyond the scope of this paper, see Sadzik and Stacchetti (2015) for an excellent treatment of the issue for time consistent case.

2. *There exists a deterministic function* $b(s)$ *with* $lim_{s\to\infty}b(s) = 0$ *such that* $1 \geq R^t(s) \geq b_s$ *for all* s*.*

3. $\int_t^\infty R^t(s)ds < +\infty$ *a.s.*

4. $\mu_{R^t}(y, R^t(y))$ *and* $\sigma_{R^t}(y, R^t(y)))$ *are uniformly Lipschitz continuous in both arguments.*

5. $\sigma_{R^t}(t, \cdot) = 0$ *at* t *and* $\mu_{R^t}(\cdot, \cdot)$ *is bounded.*

Where Z^{R^t} *is a local martingale independent of the production shock* Z_t*.*

Assumption 1' extends the sequence of discounting functions introduced in assumption 1 to a stochastic environment. Parts 1-4 are analogous to the parts in 1, where the non-negativity is strengthened to being bounded below. The final part is a technical requirement that ensures the drift is bounded and the process is deterministic equaling one at $s = t$.

We also extend the assumption on the agent's discount rate:

Assumption 2' *The agent's discount rate is a random field* $r(t, s) \doteq r^t(s)$ *where for each fixed* t*, the random field with the first parameter* t *corresponds a stochastic process as follows:*

$$r^t(s) = 1 + \int_t^s \mu_{r^t}(y, r^t(y))dy + \int_t^s \sigma_{r^t}(y, r^t(y))dZ^{r^t}$$

This process satisfies the following assumptions:

1. $r^t(s) = 1$ *for all* $s \leq t$ *and* $\lim_{s\to\infty} r^t(s) = 0,$ *a.s.* $\forall t \geq 0$

2. *There exists a deterministic function* $d(s)$ *with* $lim_{s\to\infty}d(s) = 0$ *such that* $1 \geq r^t(s) \geq d(s)$ *for all* s*.*

3. $\int_t^\infty r^t(s)ds < +\infty$ *a.s.*

4. $\mu_{r^t}(y, r^t(y))$ *and* $\sigma_{r^t}(y, r^t(y)))$ *are uniformly Lipschitz continuous in both arguments.*

5. $\sigma_{r^t}(t, \cdot) = 0$ *at* t *and* $\mu_{r^t}(\cdot, \cdot)$ *is bounded.*

6. $R^t(s) \geq r^t s$ *for all* s

Where Z^{r^t} *is a local martingale independent of the production shock* Z_t*.*

Again the properties introduced in assumption 2' are similar to those in assumption 1' with one addition: the agent is weakly more impatient than the principal. In its most general form the principal being more patient on every path can be achieved by a multitude of ways, a few examples could be by having the fields be correlated or having distinct supports where one dominates the other. First we restate the results corresponding to the agent's side under the new assumptions.

Lemma 1' *Under assumptions 1', 2', 3, given any contract and any sequence of the agent's choices, there exists a predictable, finite and L^2 integrable process ψ_t $(0 \leq t \leq T)$ such that W_t evolves according to*

$$dW_t = -\left(\mu_{r^t}(t, r^t)W_t + u(c_t, a_t)\right)dt + \psi_t\left(dM_t - a_t dt\right) \tag{9}$$

The contract is incentive compatible if and only if

$$\psi_t a_t - u(c_t, a_t) \geq \psi_t \hat{a} - u(c_t, \hat{a}), \quad \forall \hat{a} \in A, \forall t. \tag{10}$$

which implies

$$\psi_t = u_a(c_t, a_t) \tag{11}$$

As a final step, we introduce the following notation of a controlled infinitesimal generator:[12]

Definition 5 *Suppose $x = [x_1\ x_2\ ...\ x_n]$ is a n-dimensional vector of state variables and $Z_t^j,\ j = 1, 2, ..., n$ are independent Brownian motions such that*

$$dx_j = \mu_j(t, x)dt + \sigma_j(t, x)dZ_t^j \tag{12}$$

[12]This notation comes from Björk et al. (2017) which, unlike the most standard notation in stochastic calculus, acts on the time variable as well as on the space variable. (i.e. it includes the term $\partial/\partial t$)

For any process $G(t,x)$ and control h the operator $\mathcal{A}^h$ is defined by

$$\mathcal{A}^h G = \frac{\partial G}{\partial t} + \sum_{j=1}^{n} \mu_j^h(t,x)\frac{\partial G}{\partial x_j} + \frac{1}{2}\sum_{i,j}^{n} \sigma_j^h(t,x)\sigma_i^h(t,x)\frac{\partial^2 G}{\partial x_j x_i} \tag{13}$$

2.4 The Optimal Contract

We are now ready to state our main result regarding the optimal contract:

Theorem 1 *Under assumptions 1, 2, and 3 (or 1', 2', and 3), there exists an optimal contract. The principal's equilibrium value function under this contract $V(t,x)$ with optimal controls $h = (a^*, c^*, \psi)$, is given by:* [13]

$$V(t,x) = \mathrm{E}_{t,x}^h \left[\int_t^T R^t(r)(a_r^* - c_r^*)dr + B(t, W_T) \right]. \tag{14}$$

The equilibrium value function is satisfies the following extended HJB system:

$$\sup_h \{\mathcal{A}^h V(t,x) + a_t - c_t - \mathcal{A}^h f(t,x,t,x) + \mathcal{A}^h f^{tx}(t,x)\} = 0 \tag{15}$$

$$\mathcal{A}^{h^*} f^{sy}(t,x) + R^s(t)(a_t^* - c_t^*) = 0 \tag{16}$$

where for each fixed s, y the function $f^{sy}(t,x) = f(t,x,s,y)$ equation (16) is a Kolmogorov backward equation with the probabilistic interpretation:

$$f^{sy}(t,x) = \mathrm{E}_{t,x}^{h^*} \left[\int_t^T R^s(r)(a_r^* - c_r^*)dr + B(t, W_T) \right] \tag{17}$$

subject to (3 or 9), the IC condition (5 or 11) and boundary conditions:

$$V(T,x) = B(T, W_T) \text{ for all } x \tag{18}$$

where $B(t,W)$ is defined by (6).[14]

Theorem 1 highlights two important differences compared to the standard optimal con-

[13]From here on we will suppress the h in V whenever it is clear.

[14]Note that operator $\mathcal{A}$ only acts on the objects inside the parenthesis. Moreover, on the equilibrium by construction we have $V(t,x) = f(t,x,t,x)$.

tract under exponential discounting. First, there is a system of equations, instead of one single Hamilton-Jacobi-Bellman (HJB) equation consisting of just $\mathcal{A}^h V$ and $(a_t^* - c_t^*)$. The extra equation is a Kolmogorov Backward equation defined using the infinitesimal generator, and captures the "backward induction" intuition that the principal must follow when playing a game against her future selves.[15]

Second, $f(t,x,t,x)$ and $f^{tx}(tx)$ are the "extra" terms in the HJB equation. In exponential discounting benchmark models, the principal's immediate utility $(a_t - c_t)$ and her value function $(V(t,x))$ are sufficient to capture the immediate and future incentives for any action of the agent, and the suprema determines the optimal actions. However, in our model due to dynamic inconsistency we not only need additional terms ($f(t,x,t,x)$ and $f^{tx}(tx)$) but also need to identify these terms through another equation (16).

To understand those extra terms and how they are determined, first observe that the equilibrium value function $V(t,x)$ yields the value of each time t self for each potential state x and relies on what each self will do at all $s > t$. This dependence on "far" terms introduces a non-local PDE, instead of the usual local PDE from an HJB equation. This non-local PDE can be pinned down by another non-local function, the term $f(t,x,t,x)$, which captures the optimal continuation value while allowing for the discount function and the state to change. Similarly, $f^{sy}(t,x)$ captures the principal's valuation of future equilibrium actions by her time-s self in state y. It is another non-local PDE but it only relies on a single discount function (16). In short, those extra terms captures the impact of the principal's different future selves on her previous selves through the backward system.

In general, the backward system in Theorem 1 is very difficult to analyze due to the interaction of non-local PDEs. Indeed, Wei et al. (2017) note the appearance of the backward term poses an essential difficulty and they remark "at the moment we are not able to overcome the difficulty" and pose it as an open problem. Similarly Björk et al. (2017), whose HJB system is identical to ours, note that "the task of proving existence and/or uniqueness of solutions to the extended HJB systems seems (at least to us) to be technically extremely difficult". The challenges mainly come in two ways: first, the backward system is

[15] Iijima and Kasahara (2016) shows another application of backward stochastic differential equations in dynamic games.

actually accompanied by a forward system arising from the state variables, turning it into a forward-backward stochastic differential equation which is known to be very difficult to tackle (Cvitanic and Zhang (2012)). Second, without the solution to the backward system, the value function, which comes from its own non-local PDE, cannot be solved.

We manage to establish the existence of solution to the backward system in Theorem 1 by taking a unique approach: first, we only establish the well-posedness of the backward system under our smoothness assumptions. Then we turn to the value equation, where the solution from the backward system comes in implicitly. We define the game between the different selves as a game of incomplete information between non-atomic players where the utility function of each of these players incorporates these solutions from each of the backward systems. Of course the backward system without an explicit solution only gives a PDE characterization of the utilities of these players. However, using that PDE analogous to the approach used by Mas-Colell (1984) we look for a distribution over utility functions, strategies and information as an equilibria. Once the contracting problem is translated into this non-atomic game of incomplete information, using existing results from game theory (in particular Balder (1991)), we show that an equilibrium exists, which implies there is a solution to extended HJB system.

Theorem 1 is designed to be broadly applicable results under fairly general assumptions regarding the discount processes and utility functions. However, further analysis such as comparative statistics under such generality also is not a trivial task. In the rest of the paper, we explore two important special cases that allows us to derive closed-form solutions.

3 Special Cases

In this section we provide two explicitly solved examples of the optimal contract under simplified conditions. Since the main innovations from introducing dynamic inconsistency come from the principal's problem, we simplify the agent's problem assuming he has CARA utility and has dynamically-consistent, exponential discounting. We then analyze two specific forms of non-exponential discounting processes for the principal: *quasi-hyperbolic* discounting, and *stochastic* discounting. In both cases we obtain closed-form solutions of the optimal contracts and offer comparative statics that shed light on the unique implications of dynamically-inconsistent preferences.

Throughout this section, we maintain the following assumption which strengthens and replaces assumptions 2 (or 2') and 3:

Assumption 4 *The agent has the following discount function and CARA utility.*

$$r^t(s) = e^{-\gamma(s-t)} \text{ for all } t$$

$$u(c,a) = -\frac{1}{\eta}e^{-\eta\left(c-\frac{1}{2}a^2\right)}$$

He also has access to a private savings account whose balance grows at rate γ. The principal is risk-neutral and values instantaneous returns as $M_t - c_t$.

CARA utility plus private saving is a commonly used technique in the contracting literature to simplify the agent's problem.[16] In our model, they leads to the following result:

Lemma 2 *Under assumption 4 the agent's continuation utility will satisfy*

$$dW_t = \gamma(W_t dt - u(c_t,a_t))dt + \psi_t(dM_t - a_t dt) \text{ and} \tag{19}$$

$$u(c_t,a_t) = \gamma W_t, \tag{20}$$

The agent's incentive compatibility condition becomes

$$\psi_t = a_t. \tag{21}$$

The evolution of $\ln(W)$ *is given by*

$$\mathbb{E}\left[\ln(-W_t)\right] = \ln(-W_0) + \frac{1}{2}\int_0^t \eta^2\gamma^2\sigma^2\psi_s^2 ds. \tag{22}$$

Finally, there is no private savings on the equilibrium path.

The proof of lemma 2 is analogous to the proof of He (2011) and hence omitted.

[16]See, e.g. He (2011)), Williams (2015), Marinovic and Varas (2017), etc

3.1 Special Case I: Quasi-Hyperbolic Discounting

Our first example explores the case in which the principal has quasi-hyperbolic Discounting. Such discounting arises when, for example, the principal is a board of directors/group of decision makers that contracts with a single agent. As noted in Jackson and Yariv (2015) collective decision making with different discounting across the decision makers causes present-biased preferences. There is a wide body of literature about present biased preferences including but certainly not limited to O'Donoghue and Rabin (1999), Thaler and Benartzi (2004), Harris and Laibson (2012), Jackson and Yariv (2014), Jackson and Yariv (2015), Bisin et al. (2015). Here we take the stance that the present biased preferences takes the form of quasi-hyperbolic discounting as introduced by Harris and Laibson (2012) where the quasi-hyperbolic discount function is a convex combination of the short run discount-function and a long run discount factor.[17]

Formally, we assume:

Assumption 5 *The principal has the following discount function, with $\beta \in (0,1)$ and $\gamma > \rho + \lambda$.*

$$R^t(s) = (1-\beta)e^{-(\rho+\lambda)(s-t)} + \beta e^{-\rho(s-t)}$$

The representation above is the deterministic characterization of a principal who values "near present" returns higher (discounted by $e^{-\rho(s-t)}$), and "far future" returns lower (discounted by $\beta e^{-\rho(s-t)}$). $\beta < 1$ captures the size of the drop in discount factor (i.e. the principal becomes less patient) in the "far future". The switch between the "near present" and "far future" happens stochastically with arrival intensity λ and the discount function incorporates this expected drop. Notice that for the principal the discounting between period t and $s > t$ only relies on $s-t$ and not t or s. Thus with a slight abuse of notation, we will denote $R^t(s) = (1-\beta)e^{-(\rho+\lambda)(s-t)} + \beta e^{-\rho}(s-t)$ as $R(s-t)$.

With the now simplified agent's continuation utility and the specific discount function, Theorem 1 becomes the following:

Proposition 2 *Under assumptions 4, 5 the value function of the principal satisfies the*

[17] Pan et al. (2015) also offers a continuous time version of quasi-hyperbolic discounting.

following HJB system:

$$\sup_{a_t} V_t + a_t - \left[\frac{1}{2}a_t^2 - \frac{\ln(\eta\gamma)}{\eta} - \frac{1}{\eta}\ln(-W)\right] + \frac{1}{2}\left(\gamma\eta W\psi\sigma\right)^2 V_{WW}$$
$$+ \int_t^T R'(s-t)\left(a_s - \left[\frac{1}{2}a_s^2 - \frac{\ln(\eta\gamma)}{\eta} - \frac{1}{\eta}\ln(-W_s)\right]\right) ds$$
$$+ \frac{\ln(-W_T)}{\eta}R'(T-t) = 0$$
$$\mathcal{A}^{h^*} f(t, W_t)^{s,W_s} + R(s-t)(a_t^* - c_t^*) = 0$$

subject to (19) and incentive compatibility condition (21) and boundary condition

$$V(T, x) = B(T, W_T) \text{ for all } x \tag{23}$$

where $B(t, W)$ is defined by (6).

The relatively simpler form of the extended HJB enables us to calculate the optimal contract in closed form, although such calculation still requires substantial effort

Corollary 3 *Under assumptions 4, 5 in the optimal contract a_t and c_t satisfy the following:*

$$a_t = \left[1 + \frac{\eta\gamma^2\sigma^2}{\rho(\lambda+\rho)}\left(-\beta\lambda(1-\beta)\left(e^{-(\lambda+\rho)(T-t)} - e^{-\rho(T-t)}\right) + (1-\eta+\rho\eta+\lambda\eta)\rho + (1-\eta)\beta\lambda\right)\right]^{-1}$$
$$c_t = \frac{1}{2}a_t^2 - \frac{\ln(\gamma\eta)}{\eta} - \frac{1}{\eta}\ln(-W_t)$$

Given the closed form of the optimal contract we can explore the implications of quasi-hyperbolic discounting in details. We summarize our key findings as follows:

Proposition 4 *The optimal contract derived in corollary 3 has the following properties:*

1. *If $\beta = 1$ ($\beta = 0$) optimal contract converges to dynamically-consistent principal with discount rate ρ ($\rho + \lambda$). Optimal effort a_t is (weakly) monotonic in time t.*

2. *As $T \to \infty$ optimal contract converges to an optimal contract with time consistent-principal with discount rate $\frac{\rho(\lambda+\rho)}{\rho+\beta\lambda}$.*

3. *For every $\rho \in (0, 1)$, if λ is high enough a_t becomes non-monotone function over time.*

The first property is straightforward: if $\beta = 1$ ($\beta = 0$) the principal has the same discount rate for "near present" returns and "far future" returns. That is, she has exponential discounting. Not surprisingly, the optimal solution is analogous to the celebrated solution of Holmstrom and Milgrom (1987) which induces monotone effort regardless of the time horizon.[18]

The second property demonstrates that the effect of quasi-hyperbolic exponential discounting becomes indistinguishable if the time horizon becomes arbitrarily large. Björk et al. (2017) shows for any time-inconsistent optimal control problem that has a solution, there is a time-consistent problem that has the same solution with a *different instantaneous utility function*. We show that if the contracting horizon is infinitely long, the solution to the time-inconsistent problem agrees with a time-consistent problem that has *exactly the same instantaneous utility function* but with a *different exponential discount factor*, which we pin down explicitly. Intuitively, because the principal is sophisticated, as the horizon increases the problem that each of her future selves faces becomes similar. When the horizon in infinite, the principal's problem becomes stationary. Under quasi-hyperbolic discounting, stationarity is enough to ensure that the solution agrees with that of a principal with the exactly the same instantaneous utility function.

The third property highlights the possibility of non-monotonic actions under finite horizon. From the previous two properties we know effort is (weakly) monotonic in time without quasi-hyperbolic discounting or without finite horizon. Under quasi-hyperbolic discounting and finite horizon though, a "near-future" principal with the short-run discount factor and a "far-future" principal with the long-run discount factor may prefer different levels of actions. Suppose the former prefers high effort and the later prefers low effort. Because the switch between the two types of principals happens stochastically, the sophisticated "near-future" principal anticipates her less patient "far-future" self will arrive at some point. Thus she designs a path of effort that gradually decreases towards the level preferred by her "near-future" self. However, at some point as she approaches the end of the contracting horizon (the deadline), the probability of the switch happening before the deadline becomes smaller

[18] Note that the difference arises due to the boundary conditions. More recently with He (2011), Marinovic and Varas (2017), He et al. (2017) reach similar conclusions.

over time, as the contract is "running out of time." Consequently, the principal acts more likely her "near future" patient self and revert the recommended effort until it converges to the time-consistent benchmark level at the deadline, causing a U-shaped path. Similarly effort could first increase then decrease towards the end, causing a hump-shaped path. We refer to this reverting of actions towards the end of the contracting horizon as the "deadline effect". The turning point is determined by λ, the arrival intensity of the drop in discount. The higher the λ, the sooner the action path changes course. If λ is sufficiently low, the drop in discount is so remote that the "deadline effect" does not kick in and the path of optimal actions converges to the (weakly) monotonic case.

We illustrate the properties above and the comparison between quasi-hyperbolic discounting and the time-consistent benchmark in Figure 1

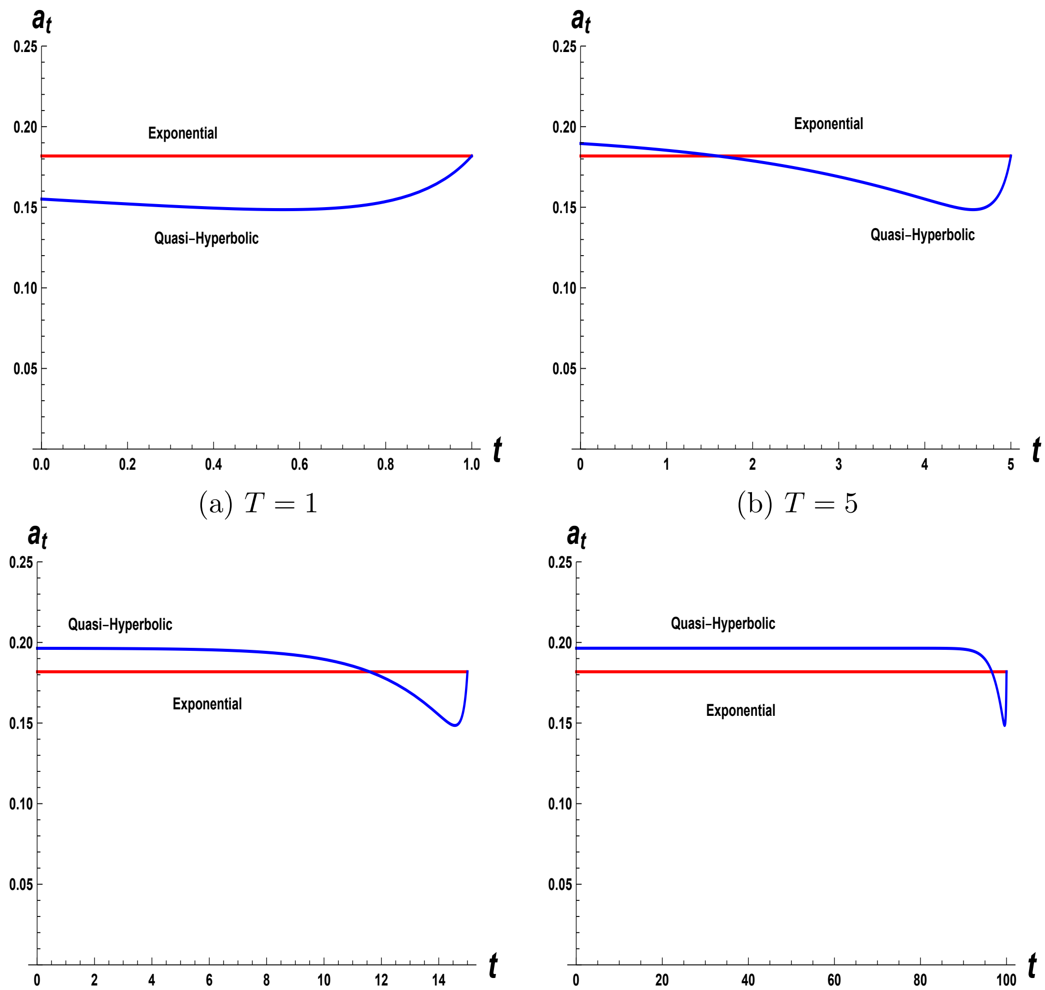
3.2 Special Case II: Stochastic Discounting

Our second special case considers a principal with stochastic discounting. This arises when, for example, the principal uses the prevailing short-term market interest rates as her discount rate. Stochastic short rate is a standard assumption in finance and there are various short rate models designed for different purposes.[19] The class of short rate models we explore are initiated by Gabaix (2007, 2012); Filipović et al. (2017) and are mostly applicable in characterizing the prices of zero-coupon bonds. In particular, we adopt a version of short rate from Filipovic et al. (2018) with simplified parameters for tractability. A fully parameterized solution is offered in appendix for readers who are interested.

Assumption 6 *We assume instantaneous discount rate is given by a diffusion process r_t and principal discounts future with the function $R^t(s)$,*

$$R^t(s) = e^{-\int_t^s r_u du}.$$

[19]See, e.g., Vasicek (1977), Cox et al. (1985), Hull and White (1990), Ho and Lee (1986), Black et al. (1990).



(a) $T = 1$ (b) $T = 5$ (c) $T = 15$ (d) $T = 100$

Figure 1: These plots illustrate the action paths over time recommended by the optimal contract. The blue line indicates the action path under quasi-hyperbolic discounting with parameter values $\eta = 2, \sigma = 1, \beta = 0.5, \lambda = 5, \rho = 0.5, \gamma = 1.5$. The red line indicates the action path under exponential discounting with $\rho = 1$ and the rest of the parameters the same. Each plot corresponds to a different time horizon (different T).

We are going to assume dynamics of r_t is governed as follows

$$r_t = \alpha + \kappa - \frac{\kappa\theta}{P_t}$$

$$dP_t = \kappa(\theta - P_t)dt + \kappa\left(\nu - P_t\right)dZ_t^P$$

where Z_t^P is a Brownian motion independent of the cash flow shock Z_t. For positive interest rates we assume, $\kappa > 0$, $\theta > \nu \geq 0$, and we also assume $\alpha + \kappa - \kappa\theta/\nu < \gamma$ so that the principal is more patient for all possible values of P_t.

The following lemma, due to Filipovic et al. (2018), highlights the key properties of this discount rate.

Lemma 3 (Filipovic 2018) *At time t, the expected discounted value of a single util in period $t' > t$ is valued at*

$$E(R^t(t')) = E\left(e^{-\int_t^{t'} r_s ds}\right) = e^{-\alpha(T-t)} \frac{\theta + e^{-(T-t)\kappa}(P_t - \theta)}{P_t} \tag{24}$$

Moreover, letting ξ_t denote $R^0(t)$ we also have

$$\xi_t = e^{-\alpha t} P_t$$
$$R^t(t') = \frac{\xi_{t'}}{\xi_t}$$

Let us briefly discuss the underlying structure of this short rate model. The term P_t is a mean reverting process with mean θ and a reflecting boundary at ν. It is akin to the Ornstein-Uhlenbeck (OU) process, the most commonly known mean reverting process, except with a reflecting boundary. The process P_t multiplies a standard discount factor $e^{\alpha t}$ to generate the discounting process ξ_t. The reflecting boundary and the parameter restrictions serves the purpose of ensuring the discount rate remains positive and the principal remains more patient than the agent.

Figure 2 illustrates a sample path of the short rate r_t (blue line) and the resulting discounting factor ξ_t (red line). The relationship between ξ_t and $\xi_{t'}$ enables us to keep track of ξ as a state variable which turns out to simplify some of the algebra.

With a specific form on the discount rate, the extended HJB system of theorem 1 simplifies to the following:

Proposition 5 *Under assumptions 4 and 6, the principal's value function satisfies the following HJB system*

$$\sup_{a_t} \left[a_t - \frac{1}{2}a_t^2 + \frac{\ln(-\eta)}{\eta} + \frac{\ln(\gamma)}{\eta} + \frac{\ln(W_t)}{\eta} + \mathcal{A}^h f(t, \xi_t, W_t)^{t,\xi_t,W_t} \right] = 0 \tag{25}$$

$$\mathcal{A}^{h^*} f(t, \xi_t, W_t)^{s,\xi_s,W_s} + \frac{\xi_t}{\xi_s}(a_t^* - c_t^*) = 0 \tag{26}$$

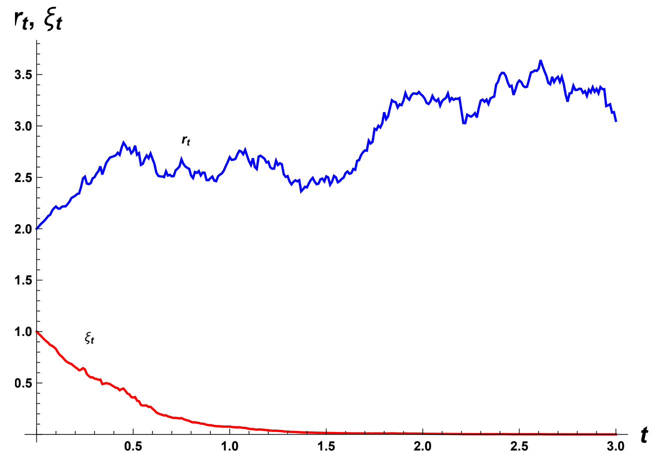


Figure 2: This figure plots a simulated path of the short rate r_t (blue line) the resulting discounting factor ξ_t (red line) according to Lemma 3. Parameter values are $\alpha = 3, \kappa = 1, \theta = 3, \nu = 1, T = 3$

subject to (19) and incentive compatibility condition (21) and boundary condition

$$f(T, \xi_T, W_T)^{s,\xi_s,W_s} = \frac{\xi_T}{\xi_s} \frac{\ln(-W_T)}{\eta} \tag{27}$$

We can also obtain closed-form solution to the optimal effort and consumption path:

Corollary 6 *Under assumptions 4, and 6, a_t and c_t satisfy the following:*

$$\begin{aligned} a_t =& \frac{\xi_t + \sigma_\xi \gamma\eta\sigma A(t)}{\xi_t \left(1 + (\gamma\eta\sigma)^2 \left(A(t) + B(t)\right)\right)} \\ c_t =& \frac{1}{2} a_t^2 - \frac{\ln(\gamma\eta)}{\eta} - \frac{1}{\eta}\ln(-W_t). \end{aligned}$$

Given the closed-form solution we can explore the implications of stochastic discounting, summarized as follows:

Proposition 7 *The optimal contract derived in corollary 6 has the following properties:*

1. *As ξ_t increases, a_t increases.*
2. *As $\kappa \to 0$, suggested action a_t converges to time consistent principal with discount rate α*
3. *As $T \to \infty$ $\lim_{t\to\infty} a_t$ oscillates according to Z_t.*

The first property highlights the connection between the past interest rate and the optimal future action. According to Lemma 3, higher values of ξ_t implies in the past, the principal valued the future more. Therefore she provided strong incentives for high future effort, and such effort materialize when that future becomes present. We refer to this as the"bright futures effect" of stochastic discounting.

The second property is essentially a stability result. Observe that as the volatility of the mean reverting shocks to the discount factor disappears, the discount factor of the principal also converges to exponential discounting. The second property states that as the discount factors converge, the actions under the optimal contract converges to their time-consistent benchmark as well.

The third property demonstrates the long-run effect of the short rate process, which behaves like an OU process. Initially the short rate process may contain time trends and that leads to paths of actions with trends. However, as the OU process converges to a distribution around the mean as time goes to infinity, any time trend dissipates after sufficiently long history. The long-run optimal effort behaves only according to the randomness of the interest rate and thus oscillates as the interest rate oscillate.

In sum, by assuming a time-consistent agent with CARA utility, we explore the implications of special non-exponential discounting processes on the principal's side in details. The resulting optimal contract can feature non-monotonic effort and consumption, in contrast with the (weakly) monotonic effort and consumption under time-consistent benchmarks. The non-monotonicity is brought by a "deadline effect" under quasi-hyperbolic discounting and by a "bright future effect" under stochastic discounting.

4 Conclusion

It is well-known in the literature that individuals with time-inconsistent preferences make dynamically inconsistent plans, because what appears optimal today may turn out to be suboptimal tomorrow. As a result, is it possible for long-term contracts between parties with such preferences to exist, particularly if the parties are sophisticated in anticipating the changes to their preferences in the future? If so, how do the contracts differ from known time-consistent benchmarks? We answer these questions by establishing a general framework combining moral hazard with time-inconsistent preferences resulting from non-exponential discounting. This contracting problem can be equivalently expressed as a dynamic game played between the agent, the principal and their future selves, except the characterization of its solution has been an open problem in past literature. We provide a novel argument for the solution by translating this game into a non-atomic game with incomplete information and prove the existence of equilibrium this non-atomic game. Given it's generality, such approach is potentially applicable in other settings besides dynamic contracting. Moreover, we offer two examples: quasi-hyperbolic discounting and stochastic discounting, for which we manage to obtain closed-formed solutions. The resulting optimal contract produces testable implications, such as time-varying, non-monotonic effort and consumption paths.

There are several directions through which our study can be potentially extended. First and foremost the extended HJB systems and their variations arise in many time-inconsistent optimal control problems, whether they are arising from non-exponential discounting or other sources of time inconsistency. The existence of a solution was an open problem and we hope our indirect approach that bridges time inconsistent control to non-atomic games can be generalized to tackle the existence of optimal control under more general scenarios as well. Additionally even though the general existence theorem is rather broad, our solved examples highlight the direct applicability of the techniques as well; in particular it might be interesting to explore time inconsistent contracting problem under other specifications such as ambiguity, habit formation and mean-variance risk preferences. We leave these questions for future research.

Appendix A

Proof of Lemma 1

To see the agent's continuation payoff follows lemma 1 observe the following. Agent's payoff at time 0 if he is truthful can be written as follows:

$$W_0 = E\left[\int_0^T r^0(s)u(c_s, a_s)ds + r(0,T)u(c_T, \underline{a})\right] \tag{28}$$

Now fix a contract process. Suppose we are at time t agent's total expected payoff from t onward in this truthful contract is given by

$$U_t = \int_0^t r^t(s)u(c_s, a_s)ds + r^t(t)W_t \tag{29}$$

Notice that this continuation value has to be a martingale. Thus by martingale representation theorem there is exists a progressively measurable process ψ_t such that:

$$U_t = \int_0^t \psi_s dZ_s \tag{30}$$

Where Z_s is a standard Brownian motion on $\mathcal{F}_t$. Now differentiating equation 29 yields

$$dU_t = r^t(t)u(c_t, a_t)dt + r^{t'}(t)W_t + r^t(t)dW_t \tag{31}$$

differentiation of equation 30 yields

$$dU_t = \psi_t dZ_t \tag{32}$$

Combining both equations and replacing dZ_t using the identity $dM_t = a_t dt + \sigma dZ_t$. yields the identity:

$$dW_t = -(\frac{\partial r^t(t)}{\partial t}W_t + u(c_t, a_t)) + \psi_t(dM_t - a_t dt)$$

Now the incentive condition will be satisfied for all t if we have,

$$\forall a \in A, \quad \psi_t a_t - u(c_t, a_t) \geq \psi_t \hat{a} - u(c_t, \hat{a}), \quad \forall t. \tag{33}$$

To see that IC is satisfied in this setup, observe the following. Consider the process a_t that satisfies the equation 33 and consider an alternative strategy $\tilde{a}_t$. Now, consider the agent's total payoff when he plays $\tilde{A}_t$ until t and follows A afterwards calculated at period t.

$$U_t^{\tilde{a},a} = \int_0^t r^t(s)u(c_s, (\tilde{a}_s))ds + r^t(t)W_t \tag{34}$$

Differentiation and using equations 29 and 30 yields

$$dU_t^{\tilde{a},a} = r^t(t)u(c_t,\tilde{a}_t))dt - r^t(t)u(c_t,a_t)dt + \psi_t dZ_t \tag{35}$$

Notice that this Brownian part Z_t defined under a in equation 30. Now we can replace it with a Brownian motion $\tilde{Z}_t$ under $\tilde{a}$, which is defined as $\tilde{Z}_t = Z_t - \frac{\int_0^t(\tilde{a}_s - a_s)ds}{\sigma}$ to get,

$$dU_t^{\tilde{a},a} = r^t(t)u(c_t,\tilde{a}_t)dt - r^t(t)u(c_t,a_t)dt + \psi_t d\tilde{Z}_t - \frac{\psi_t}{\sigma}(\tilde{a}_t - a_t)dt \tag{36}$$

Using the fact that $r^t(t) = 1$ and reorganizing leads to

$$dU_t^{\tilde{a},a} = \left[(\psi_t\tilde{a}_t - u(c_t,\tilde{a}_t)) - (\psi_t a_t - u(c_t,a_t))\right]dt + \psi_t d\tilde{Z}_t \tag{37}$$

With the condition 33 holding, the term in brackets cannot be positive so for any arbitrary strategy $\tilde{a}$ so the agent doesn't have an incentive to deviate. The proof of lemma 1' is analogous where $\frac{\partial r^t(t)}{\partial t}$ term is replaced by $dr^t(t)$. The quadratic variation and the immediate effect of the diffusion term disappears due to part 5 of assumption 2' yielding the identity in lemma 1'.

Proof of Theorem 1

The proof of the theorem involves heuristic derivation, two steps and a verification argument. We provide the proof under assumptions 1', 2', hence the discount factors comprise part of the state. The proof just simplifies when the discount factors are no longer part of the state.

Derivation of the Extended HJB System

In this section of the appendix we derive the extended HJB equation. This derivation is analogous to the derivation provided in Björk et al. (2017) and Yong (2012) so we provide only a short argument. By construction, of the h_Δ we have

$$\begin{aligned} J(t+\Delta, X_{t+\Delta}, h_\Delta) = J(t,x,h) - \mathrm{E}_{t,x}^{h_\Delta}\left[\int_t^{t+\Delta} R(t,s)\,(dM_s - c_s ds)\right] \\ + \mathrm{E}_{t,x}^{h} f(t+\Delta, x_{t+\Delta}^h, t+\Delta, x_{t+\Delta}^h) - \mathrm{E}_{t,x}^{h} f(t+\Delta, x_{t+\Delta}^h, t, x). \end{aligned}$$

Since $J(t+\Delta, X_{t+\Delta}, h_\Delta) = V(t+\Delta, X_{t+\Delta})$, we can rewrite the equation as follows:

$$\begin{aligned} \mathrm{E}_{t,x}^{h} V(t+\Delta, X_{t+\Delta}) = V(t,X_t) - \mathrm{E}_{t,x}^{h_\Delta}\left[\int_t^{t+\Delta} R(t,s)\,(dM_s - c_s ds)\right] \\ + \mathrm{E}_{t,x}^{h} f(t+\Delta, x_{t+\Delta}^h, t+\Delta, x_{t+\Delta}^h) - \mathrm{E}_{t,x}^{h} f(t+\Delta, x_{t+\Delta}^h, t, x). \end{aligned}$$

Add and subtract $f(t,x,t,x)$ to the right hand side rearrange by dividing both sides to Δ and take the limit as $\Delta \to 0$. So, we reach

$$\sup_h \{\mathcal{A}^h V(t,x) + a_t - c_t - \mathcal{A}^h f(t,x,t,x) + \mathcal{A}^h f^{tx}(t,x)\} = 0.$$

Moreover for every fixed X, t, $f^{t,x}(t,x)$ corresponds to a martingale process therefore it must satisfy the following PDE

$$\mathcal{A}^{h^*} f^{sy}(t,x) + R^s(t)(a_t^* - c_t^*) = 0.$$

in addition it must satisfy the boundary conditions

$$\begin{aligned} V(T,x) &= B(T, W_T) \text{ for all } x \\ f(T,x,T,x) &= B(T, W_T) \text{ for all } x \end{aligned}$$

so we reach the extended HJB system of theorem 1.

The Existence of a Solution to the Backward System

In order to prove existence we first start with an arbitrary incentive compatible control law $\hat{h}$ and consider the second part of the HJB system, the backward equation:

$$\mathcal{A}^{\hat{h}} f^{sy}(t,x) + R^s(t)(a_t^{\hat{h}} - c_t^{\hat{h}}) = 0.$$

For each t, x the backward equation

$$\mathcal{A}^{\hat{h}} f^{tx}(t,x) + R^t(t)(a_t^{\hat{h}} - c_t^{\hat{h}}) = 0$$

is a semilinear parabolic Partial Differential Equation (PDE). Observe that if the PDE had a solution $f_{\hat{h}}^{tx}(t,x)$ then for every s, y we could consider a version of $x^{s,y}$ that starts at time s in value y and reaches x by time t. This would imply we would have $f_{\hat{h}}^{sy}(t,x)$, where we underline the dependence on the control $\hat{h}$. An equivalent representation of the backward system is

$$f_{\hat{h}}^{tx}(t,x) = f_{\hat{h}}^{tx}(T, x_T) - \int_t^T R^t(r)(a_r^{\hat{h}} - c_r^{\hat{h}})dr - \int_t^T Y_r^{\hat{h}} dZ_r$$

For an adapted process $Y^{\hat{h}}$. A solution to the backward system alone would correspond to a pair of processes $f_{\hat{h}}^{tx}(t,x)$ and $Y_t^{\hat{h}}$. However, notice that for a given arbitrary control $\hat{h}$, the c_t is pinned down by the incentive condition. In particular, observe that $a_t \in [\underline{a}, \bar{a}]$ and due to assumption 3 for any given a_t, ψ_t, x_t there is a unique c_t, denoted by $c_t^{IC}(a_t, \psi_t)$. Moreover given that any control is a mapping $\hat{h}(t, x_t)$ the backward system is accompanied by a forward system

$$
\begin{aligned}
dR^t &= \mu_{R^t}(y, R^t(y))dy + \sigma_{R^t}(t, R^t(t)t)dZ_t^R \\
dr^t &= \mu_{r^t}(y, r^t(y))dy + \sigma_{r^t}(t, r^t(t))dZ_t^r \\
dW_t &= -\left(\mu_{r^t}(t, r^t(t))W_t + u(c_t, a_t)\right)dt + \psi\sigma dZ_t
\end{aligned}
$$

However, also notice that the solution to the backward system $f_{\hat{h}}^{tx}(t,x)$ and $Y_t^{\hat{h}}$ does not appear in the forward system, hence this is a *decoupled* forward-backward stochastic differential equation (FBSDE). For a textbook treatment of FBSDE's we refer the readers to Ma et al. (1999). Our goal is to ensure that the for any incentive compatible control $\hat{h}$ the FBSDE system is well-posed and therefore has a unique solution, and that solution is also continuous.

Given that the system is decoupled, first let us write down the generator of the backward system. Since we are only looking at Markovian, incentive compatible controls, the generator only depends on the forward part.

$$
R^t(s)(a_s^{\hat{h}(x_t)} - c_s^{\hat{h}(x_t)})
$$

From Cvitanic and Zhang (2012), section 9.5, proposition 9.5.2 we know that to have the FBSDE well posed we need to establish that the forward component has a unique solution that satisfies the Markov Property and the backward part has a unique solution. For the forward part we observe that under assumptions 1', and 2' from standard results in SDE theory we know that the diffusions R^t and r^t have unique, Markovian solutions. For W_t analogous to Sannikov (2008) under Markovian controls lemma 1 will also have a unique, Markovian solution. Now given that the forward system is Markovian, we need to establish that the backward system is well posed. Now we observe that since the backward system does not have the backward terms in the generator but only the forward ones, the generator $R^t(s)(a_s^{\hat{h}(x_t)} - c_s^{\hat{h}(x_t)})$ is trivially uniformly Lipschitz continuous in $V_t^{\hat{h}}$ and Y_t.[20] Thus by theorem 9.3.5 of Cvitanic and Zhang (2012) the backward system has a unique solution. Now since the backward system has a unique solution and the forward system has a unique Markovian solution by proposition 9.5.2 of Cvitanic and Zhang (2012) the FBSDE is well posed and there is a triple $x^{\hat{h}}$, $f_{\hat{h}}^{tx}(t,x)$ and $Y_t^{\hat{h}}$ for any incentive compatible Markovian control.

Finding a Fixed Point to The Extended HJB System

From the previous part we know that for incentive compatible control $\hat{h}$ there exists a unique process $V^{\hat{h}} = f_{\hat{h}}^{tx}(t,x)$ that satisfies the backward system. For any incentive compatible control $\tilde{h}$, consider the infinitesimal generator with control $\tilde{h}$ on the process $f_{\hat{h}}^{tx}(t,x)$, denoted by $\mathcal{A}^{\tilde{h}} f_{\hat{h}}^{tx}(t,x)$. Fix a time t and a state x pair and consider the static optimization problem

[20]The backward terms not appearing in generator is not a suprising when the backward system is a sort of "continuation utiltiy", see Duffie and Epstein (1992), El Karoui et al. (1997)

solved for every t and x

$$\sup_{\tilde{h}}\{a_t^{\tilde{h}} - c_t^{\tilde{h}} - \mathcal{A}^{\tilde{h}} f_{\hat{h}}^{tx}(t,x)\} = 0.$$

Essentially, for any given control $\hat{h}$ the sup above generates another control h using the value function generated from the backward system. So, if one were to consider a mapping from controls to controls we are trying to find a fixed point of such a mapping, with additional difficulties arising from the filtered information structure and rather large control space. The difficulty here is that we do not have convexity of the best response correspondance and we do not have quasi-concavity of the utility function on ones strategy.

Remember that we had defined the time inconsistent control problems as the limit of a game between time t selves. In case of finitely many such selves the existence of an equilibria can be found by backward induction. However as noted in Wei et al. (2017) taking the formal limit for infinitely many such selves in general is very difficult. Well-posedness is established in Wei et al. (2017) for cases where the diffusion term is not controlled, hence their methods are not applicable. Existence and uniqueness remain very much an open problem in general time inconsistent stochastic optimal control problems as noted by Björk et al. (2017).

In order to tackle this difficulty we define the game between the selves properly and search for existence of equilibria in the corresponding game. Towards that goal we utilize techniques from static games with non-atomic players such as the ones explored in Schmeidler (1973), Mas-Colell (1984), Khan and Sun (2002) and also closely related to mean-field games Guéant et al. (2011) and aggregate games Acemoglu and Jensen (2015). In particular, we will translate our game to the incomplete information game identified in Balder (1991). [21]

Consider the exogenously random parts of the model $\Omega = [R, r, Z]_{[0,T]}$ where Z is the standard Brownian motion that governs the randomness in the M and W. The probability triple is denoted by $(\Omega, \mathcal{P}, \mathcal{F})$. Notice that when defined this way, a realization ω identifies a whole path of the exogenous random factors. W can be identified by ω and given controls. Let $S = A \times C$ denotes the space of strategies. The player space is identified as $[0, T]$. $\mathcal{P}$ is common knowledge among the players. We characterize players having incomplete observation of a realization as differential information. In particular, the information component of a player t is characterized by a sub $\sigma-$ algebra of $\mathcal{F}$, denoted by $\mathcal{F}_t$, which corresponds to the natural filtration since our players' observation is differing according to time. Let $\mathfrak{M}$ denote the space of all measures on S. Furthermore, let $\mathfrak{S}$ denote the set of all measurable decision rules δ, $\delta : \Omega \to \mathfrak{M}$. Hence, players are allowed to randomize.

We equip $\mathfrak{S}$ with the weak topology, which is the weakest topology where functions identified below are continuous on $\mathfrak{S}$:

$$\delta \to \int_\Omega \phi(\omega) \left[\int_S c(s)\delta(\omega) ds\right] P(d\omega), \phi \in L^1(\Omega, \mathcal{P}, \mathcal{F}), c \in C_B(S).$$

Where $L^1(\Omega, \mathcal{P}, \mathcal{F})$ denote the space of $\mathcal{P}$ integrable functions and $C_B(A)$ denote the space of bounded and continuous functions, we also assume with the usual L^1 norm on $(\Omega, \mathcal{P}, \mathcal{F})$. With this notation a pure strategy profile is a distribution over $\mathfrak{S}$, we denote the

[21] See also, more recent work Balder (2002), Jovanovic and Rosenthal (1988).

set of all probability distributions over $\mathfrak{S}$ as $M(\mathfrak{S})$.

Finally for each t let $\mathfrak{S}_t \subset \mathfrak{S}$ denote the set of all $\mathcal{F}_t$ measurable decision rules δ, $\delta : \Omega \to \mathfrak{M}$. Now from here we define the set

$$D = \{(\mathcal{F}_t, \delta) : \mathcal{F}_t \in \{\mathcal{F}_t\}_{t\in[0,T]} \text{ and } \delta \in \mathfrak{S}_t\}.$$

Since $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration $\mathcal{F}$ is the appropriate σ-algebra in which the conditional expectations are measurable.[22] Due to lemma 2 of Balder (1991) D is $\mathcal{F} \times \mathcal{B}(\mathfrak{S})$ measurable and $\mathfrak{S}_t$ is a compact subset of $\mathfrak{S}$ for every $\mathcal{F}_t \in \{\mathcal{F}_t\}_{t\in[0,T]}$. [23] The restriction of the σ-algebra $\mathcal{F}$ to D is the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$.

First we observe that any control h defined in the first part satisfies $h \in M(\mathfrak{S})$ with the additional property that each $h_t \in \mathfrak{S}_t$. Observe that for any given t, ω and $\mathcal{F}_t$ using the backward system for any control we can define:

$$f_h^{tx}(t,x) = \mathrm{E}_{t,x}^h \left[\int_t^T R(t,s)(a_s^h - c_s^h)dr + B(t, W_T) \right].$$

Above object only depends on h and ω and is calculated according to the information at $\mathcal{F}_t$. Now for any player t, and any $\delta \in \mathfrak{S}$ and $h \in M(\mathfrak{S})$ we can define the utility function as:

$$u_t(\delta, \nu) = \mathbb{E}\left[a(\delta) - c(\delta) - \mathcal{A}^{(a(\delta),c(\delta))} f_h^{tx}(t,x) | \mathcal{F}_t\right].$$

First we make the observation that u_t is continuous in both its arguments and is measurable by definition. Notice that with the randomness baked in to the $f^{tx}(t,x)$ part, a player is identified by the characteristics $(t, \mathcal{F}_t)$, however we have not restricted δ to be $\mathcal{F}_t$ measurable.

Analogous to Balder (1991) we can define the *utility function for the game* as a function $U : [0,T] \times D \times M(\mathfrak{S}) \to \mathbb{R}$. Therefore, a game is identified by $([0,T] \times \mathcal{F}, \mu, \{\mathfrak{S}_t\}, U\}$, where $[0,T] \times \mathcal{F}$ denotes the set of characteristics, μ is a given distribution over characteristics, $\mathfrak{S}_t$ denotes the set of strategies available to a player with characteristics $(t, \mathcal{F}_t)$ and U is the utility function for the game.

Now, we can identify a *characteristic - strategy*(CS) distribution, which is a distribution over $[0,T] \times D$ which specifies the way possible characteristic-strategy combinations are distributed in the game. Letting λ be CS distribution the respective marginals of the distribution $\lambda_{|[0,T]\times\mathcal{F}}$ coincides with a distribution over characteristics and $\lambda_{|\mathfrak{S}}$ coincides with a distribution over strategies.

Definition 6 *A CS distribution λ is an equilibrium if*

- *The marginal of $\lambda_{|[0,T]\times\mathcal{F}} = \mathcal{P} \times \lambda_U^T$*
- $\lambda(\{((t, \mathcal{F}_t), \delta) \in [0,T] \times D : \delta \in \arg\max_{\mathfrak{S}_t} U(t, \mathcal{F}_t, \cdot, \lambda_{|\mathfrak{S}}))= 1$

where λ_U^T denote the uniform distribution over $[0,T]$.

[22] Balder (1991) explores more general sub σ-algebras hence has to also introduce the pointwise convergence topology which is due to Cotter (1986) for measurability of conditional expectations.

[23] Where $\mathcal{B}(\mathfrak{S})$ is the Borel sigma algebra defined on $\mathfrak{S}$.

First, let us observe that in the game between type t selves the distribution over $[0,T]\times\mathcal{F}$ is identified by the filtered probability space generated by $[R,r,Z]_{[0,T]}$. Now, we observe that $\mathfrak{S}$ is defined as above complete and separable, $\mathfrak{S}_t$ is compact as noted above, $U(t,\mathcal{F}_t,\cdot,\cdot)$ is continuous in both arguments and $U(\cdot,\cdot,\cdot,\nu)$ for $\nu\in\mathfrak{S}$ is measurable, hence by Theorem 1 of Balder (1991), there exists an equilibrium distribution. Hence there exists a control law h^* such that for every t,x

$$(a^{h^*},c^{h^*})\in\arg\sup_{\tilde{h}}\{a_t^{\tilde{h}}-c_t^{\tilde{h}}-\mathcal{A}^{\tilde{h}}f_{h^*}^{tx}(t,x)\}=0.$$

This implies there exist a solution to the extended HJB system.

Verification Theorem

In this section we are going to prove the verification theorem. Assume that for all y the functions $V(t,x)$, $f^{y,t}(t,x)$ and $\hat{h}(t,x)$ have the following properties

Theorem 8 *Assume the following*

- $V(t,x)$, $f^{y,t}(t,x)$ *solves the HJB equation*
- $V(t,x)$ *and* $f^{y,t}(t,x)$ *are smooth.* [24]
- $\hat{h}$ *is an admissible control and argmax of the* V *equation*

Proof of Verification

Step 1 Probabilistic interpretation for f
Apply Ito's formula $f^{sy}(t,x)$ by extended HJB conditions we know that it has a drift of $R(s,r)(a_s-c_s)$ and use the boundary condition at time T, to write as

$$f^{sy}(t,x)=\mathrm{E}_{t,x}^{h}\left[\int_t^T R^s(r)(a_r-c_r)dr+B(t,W_T)\right]$$

Step 2 We are going to show that $V(t,x)=J(t,x,h^*)$ for all (t,x).
Define function H as $H(t,x,t,x,h):=R(t,t)(a_t-c_t)$. From the extended HJB system

$$\mathcal{A}^hV(t,x)+\mathcal{A}^hf^{t,x}(t,x)-\mathcal{A}^hf(t,x,t,x)+H(t,x,t,x,h)=0$$

Again from the extended HJB system

$$H(t,x,t,x,h^*)+\mathcal{A}^{h^*}f^{t,x}(t,x)=0$$

so we end up with

$$\mathcal{A}^hV(t,x)=\mathcal{A}^hf(t,x,t,x)$$

[24] It is enough to be in C^2 respect to x and C^1 respect to t.

for all t and x. Since V is smooth we can apply Ito's lemma, therefore

$$\mathbb{E}V(T, X_T^U) = V(t,x) + \mathbb{E}\left[\int_t^T \mathcal{A}^u V(t,s)ds\right]$$

so we can rewrite as

$$\mathbb{E}V(T, X_T^U) = V(t,x) + \mathbb{E}\left[\int_t^T \mathcal{A}s^u f(t,s,t,s)ds\right]$$

Applying the same reasoning $f^{t,x}(t,x)$

$$\mathbb{E}f^{t,x}(T,t,X_T,X_T)$$

by the Boundary conditions for V and f implies

$$V(t,x) = J(t,x,\hat{u})$$

Step 3 Optimality of h^*
Next step is to show that h^* is an equilibrium control law. Suppose agents use an arbitrary control law $\hat{h}$ over period lenght $\Delta > 0$.

$$J(t,x,\hat{h}_\Delta)$$

Combining the expressions above

$$\liminf_{\Delta \to 0} \frac{J(t,x,h^*) - J(t,x,h_\Delta)}{\Delta} \geq 0$$

$$\begin{aligned} \mathrm{E}_{t,x}^h V(t+\Delta, X_{t+\Delta}) = V(t,X_t) - \mathrm{E}_{t,x}^{h_\Delta}\left[\int_t^{t+\Delta} R^t(s)\left(dM_s - c_s ds\right)\right] \\ + \mathrm{E}_{t,x}^h f(t+\Delta, x_{t+\Delta}^h, t+\Delta, x_{t+\Delta}^h) - \mathrm{E}_{t,x}^h f(t+\Delta, x_{t+\Delta}^h, t, x). \end{aligned}$$

Proof of Lemma 2

To proceed with a solution, first we are going to assume that the agent has access to private savings and the contract has to be terminated at a deterministic deadline T. [25]The no-savings condition of the agent implies, by Lemma 3 of He (2011) in the optimal contract (proof to appendix)

$$u(c_t, a_t) = \gamma W_t,$$

therefore,

$$c_t = \frac{1}{2}a_t^2 - \frac{\ln(\gamma\eta)}{\eta} - \frac{1}{\eta}\ln(-W_t).$$

[25]Savings assumption simplifies the contract but it is not necessary.

In the optimal contract by Martingale representation theorem (for instance see Sannikov (2008)) we can write agent's continuation utility as follows:

$$dW_t = -\gamma\eta W_t \sigma \psi_t dZ_t \tag{38}$$

moreover in this case agent's incentive compatibility condition becomes

$$\psi_t = a_t. \tag{39}$$

Then by Ito's Rule we can calculate the evolution of $\ln(W)$ as follows

$$\mathbb{E}\left[\ln(-W_t)\right] = \ln(-W_0) + \frac{1}{2}\int_0^t \eta^2\gamma^2\sigma^2\psi_s^2 ds.$$

Proofs for Quasi-Hyperbolic Case

Under assumptions 4, 5 the value function of the principal satisfies the following HJB system:

$$\begin{aligned}\sup_{a_t} V_t + a_t - \left[\frac{1}{2}a_t^2 - \frac{\ln(\eta\gamma)}{\eta} - \frac{1}{\eta}\ln(-W)\right] + \frac{1}{2}\left(\gamma\eta W\psi\sigma\right)^2 V_{WW} \\ + \int_t^T R'(s-t)\left(a_s - \left[\frac{1}{2}a_s^2 - \frac{\ln(\eta\gamma)}{\eta} - \frac{1}{\eta}\ln(-W_s)\right]\right) ds \\ + \frac{\ln(-W_T)}{\eta}R'(T-t) = 0 \\ \mathcal{A}^{h^*} f(t,W_t)^{s,W_s} + R^s(t)(a_t^* - c_t^*) = 0\end{aligned}$$

subject to (19), the IC condition (21), and the boundary conditions.

We are going to guess the principal's value function has the following functional form

$$F(t,W) = A(t)\ln(-W) + B(t)$$

with the boundary condition $A(T) = \frac{1}{\eta}$. Given our guess,

$$\dot{F}_t = \dot{A}_t \ln(-W) + \dot{B}(t), F_W = \frac{A(t)}{W}, F_{WW} = -\frac{A(t)}{W^2}.$$

By plugging our functional form guess to extended HJB equation we reach

$$1 - a_t - a_t\eta^2\gamma^2 A(t) = 0 \Rightarrow a_t = \frac{1}{1+\eta^2\gamma^2\sigma^2 A(t)}.$$

To get the $\dot{A}(t)$, we can collect the log terms

$$\dot{A}_t = \frac{1}{\eta}\left(1 + \int_t^T R'(s-t)\, ds + R'(T-t)\right)$$

or equivalently

$$\dot{A}_t = \frac{1}{\eta} \left(R(T-t) + R'(T-t) \right).$$

plug in the quasi hyperbolic discounting function

$$\dot{A}_t = \frac{1}{\eta} \left(\beta(1-\rho)e^{-\rho(T-t)} + (1-\beta)(1-(\rho+\lambda))e^{-(\rho+\lambda)(T-t)} \right)$$

also note that this case boundary condition becomes $A(T) = \frac{1}{\eta}$. Depending on the λ action can be non-monotone function over time. Especially if λ is very high.
If we are in the time consistent world

$$\dot{A}_t = \frac{1}{\eta} \left(1-r \right) e^{-r(T-t)}$$

and

$$A(t) = \frac{2r-1+(1-r)e^{-r(T-t)}}{\eta r}$$

observe that depending on the r, $\dot{A}_t$ is either always positive or negative, however, it can not be non-monotone.

$$A_t = \frac{1}{\eta} \left(\frac{\beta(1-\rho)}{\rho} e^{-\rho(T-t)} + (1-\beta) \frac{1-(\rho+\lambda)}{\rho+\lambda} e^{-(\rho+\lambda)(T-t)} + \frac{2\rho(\rho+\lambda)-(\beta\lambda+\rho)}{\rho(\rho+\lambda)} \right)$$

so we can solve for a_t.

Proof for Proposition 4

Part 1) It is easy to see that by inspection if $\beta = 1$, $\dot{A}(t)$ equals to $\frac{1}{\eta}\left(\rho+\lambda-1\right)$. Similarly if$\beta = 0$ $\dot{A}(t)$ equals to $\frac{1}{\eta}\left(\rho-1\right)$ which is equivalent to time-consistent solutions.

Part 2) Fix any $t < T$, then look at the limit

$$\lim_{T\to\infty} a(t) = \frac{1}{1+\eta^2\gamma^2\sigma^2\frac{1}{\eta}\left(\frac{2\rho(\rho+\lambda)-(\beta\lambda+\rho)}{\rho(\rho+\lambda)}\right)}.$$

Which corresponds to the time consistent solution in Holmstrom and Milgrom (1987).
Parts 3) Recall the closed form solution for $a(t)$:

$$a_t = \frac{1}{1+\eta^2\gamma^2\sigma^2\frac{1}{\eta}\left(\frac{\beta(1-\rho)}{\rho}e^{-\rho(T-t)} + (1-\beta)\frac{1-(\rho+\lambda)}{\rho+\lambda}e^{-(\rho+\lambda)(T-t)} + \frac{2\rho(\rho+\lambda)-(\beta\lambda+\rho)}{\rho(\rho+\lambda)}\right)}.$$

Then it is easy to see that dynamics of $a(t)$ pinned down by the term $\beta\frac{(1-\rho)}{\rho}e^{-\rho(T-t)}$ +

$(1-\beta)\,\frac{1-(\rho+\lambda)}{\rho+\lambda}e^{-(\rho+\lambda)(T-t)}$. It's time derivative equals to

$$\beta(1-\rho)e^{-\rho(T-t)}+(1-\beta)(1-(\rho+\lambda))e^{-(\rho+\lambda)(T-t)}$$

It is easy to see that there is a unique point in which sign of the derivative can change (if it ever happens). In particular for t small enough the first term dominates, but for t large enough with $)1-(\rho-\lambda))(1-\beta)$ being negative, the second term will dominate to change the sign of the derivative.

Proofs for Stochastic Discounting Case

Proof of Proposition 5

We are going to introduce the following notation, principal discounts the future using function ξ

$$\xi_t = e^{-\int_0^t r_s ds}$$

where short rate interest rate denoted as r_t. The dynamics of ξ_t governed by the function P_t, which satisfies the following stochastic differential equation

$$dP_t = \kappa(\theta - P_t)dt + \kappa\left(\nu - ZP_t\right)dZ_t^P$$

where $dZ_t^P$is a local martingale (for simplicity we assume it is a standard Brownian Motion). We are going to use the following equality

$$\xi_t = e^{-\alpha t}P_t,$$

after taking time derivative we can write the evolution of ξ as follows

$$d\xi_t = -\alpha e^{-\alpha t}\left(P_t\right)dt + e^{-\alpha t}dP_t.$$

By plugging dP_t, we can write the evolution of ξ_t as follows

$$d\xi_t = -\alpha e^{-\alpha t}\left(P_t\right)dt + e^{-\alpha t}\kappa\left[(\theta - P_t)dt + \left(\nu - P_t\right)dZ_t^P\right].$$

Note that using above equation,

$$P(t,T) := \mathbb{E}[\frac{\xi_T}{\xi_t}] = e^{-\alpha(T-t)}\frac{\theta + e^{-(T-t)\kappa}(P_t - \theta)}{P_t}$$

moreover

$$\mathbb{E}_t[P_T] = \theta + e^{-\kappa(T-t)}(P_t - \theta), \quad t \le T.$$

Therefore, short-rate can be equivalently written as

$$r_t = \alpha - \frac{\kappa(\theta - P_t)}{P_t}.$$

Define α^* as follows

$$\alpha^* := \sup_{P\in E} \frac{\kappa(\theta - P)}{P}$$

which guarantees that short rate (r) stays positive. That is we need $\alpha \geq \alpha^*$ The last constraint we have is $P > 0$ for all $P \in E$. This means we have to bound the domain of P. Easiest way to do this is choose $\kappa > 0$ and $\theta > \nu \geq 0$ therefore $E = [\nu, \infty)$.
We can rewrite the r_t as,

$$r_t = \alpha + \kappa - \frac{\kappa\theta}{P_t}.$$

By Ito's lemma we can write the evolution of the short rate as

$$dr_t = -\sigma_{P_t} \frac{\kappa\theta}{(P_t)^3} dt + \frac{\kappa\theta}{(P_t)^2} dP_t.$$

We can solve for the P_t and reach

$$P_t = \xi_t e^{\alpha t}.$$

Plugging in to the evolution of ξ_t

$$d\xi_t = -\alpha\xi_t dt + e^{-\alpha t}\kappa \left(\left(\theta - \xi_t e^{\alpha t}\right) dt + \left(\nu - \xi_t e^{\alpha t}\right) dZ_t^P \right)$$

after simplification

$$d\xi_t = \left(-\left(\alpha + \kappa\right)\xi_t + e^{-\alpha t}\kappa\theta \right) dt + e^{-\alpha t}\kappa\nu dM_t - \kappa\xi_t dZ_t^P.$$

In particular ξ_t is a linear stochastic differential equation, so

$$f(t, \xi_t, W_t, s, \xi_s, W_s) = \frac{\xi_t}{\xi_s} E_{t,W_t,r_t} \left[\int_t^T \frac{\xi_k}{\xi_t}(a_k - C_k)dk + \frac{\xi_T}{\xi_t} \frac{\ln(-W_T)}{\eta} dk \right]$$

using the linearity of the SDE probabilistic interpretation can be rewritten as

$$\begin{aligned} f(t, \xi_t, W_t, s, \xi_s, W_s) = \frac{1}{\xi_s} E_{t,W_t,\xi_t} \Bigg[\int_t^T (\Phi(t,k)\xi_t + \int_t^k D(t,s)ds + \int_t^k d(t,s)dZ_s)(a_k - C_k)dk \\ + \left((\Phi(t,k)\xi_t + \int_t^k D(t,k)ds + \int_t^k d(t,s)dZ_s) \right) \frac{\ln(-W_T)}{\eta} dk \Bigg] \end{aligned} \tag{40}$$

where Φ, D, d are defined in the Appendix C for the fully general model of stochastic discounting. Extended HJB system can be written as

$$\sup_{a_t}\left[a_t-\frac{1}{2}a_t^2+\frac{\ln(-\eta)}{\eta}+\frac{\ln(\gamma)}{\eta}+\frac{\ln(W_t)}{\eta}+\mathcal{A}^h f(t,\xi_t,W_t)^{t,\xi_t,W_t}\right]=0 \tag{41}$$

$$\mathcal{A}^{h^*} f(t,\xi_t,W_t)^{s,\xi_s,W_s}+\frac{\xi_t}{\xi_s}(a_t^*-c_t^*)=0 \tag{42}$$

$$f(T,\xi_T,W_T)^{s,\xi_s,W_s}=\frac{\xi_T}{\xi_s}\frac{\ln(-W_T)}{\eta} \tag{43}$$

with probabilistic interpretation $f(t,\xi_t,W_t)^{t,\xi_t,W_t}=V(t,\xi_t,W_t)$ and h^* denotes the equilibrium control

$$\begin{aligned}\mathcal{A}^h f(t,\xi_t,W_t)^{s,\xi_s,W_s}=&f_t(t,\xi_t,W_t)^{s,\xi_s,W_s}+\mu_\xi f_\xi(t,\xi_t,W_t)^{s,\xi_s,W_s}\\&+\frac{1}{2}\sigma_\xi^2 f_{\xi\xi}(t,\xi_t,W_t)^{s,\xi_s,W_s}+\frac{1}{2}\left(\gamma\eta W\psi\sigma\right)^2 f_{WW}(t,\xi_t,W_t)^{s,\xi_s,W_s}\\&+\sigma_\xi\left(\gamma\eta W\psi\sigma\right) f_{W\xi}(t,\xi_t,W_t)^{s,\xi_s,W_s}\end{aligned}$$

Let's take a closer look at each of the objects, starting with the derivative respect to ξ_t

$$f_\xi(t,\xi_t,W_t,s,\xi_s,W_s)=\frac{1}{\xi_s}E_{t,W_t,\xi_t}\left[\int_t^T(\Phi(t,k)(a_k-C_k)dk+\Phi(t,k)\frac{\ln(-W_T)}{\eta}dk\right]$$

$$f_{\xi\xi}(t,\xi_t,W_t)^{s,\xi_s,W_s}=0.$$

Let's take derivative respect to W, we use the fact that $a_t-c_t=a_t-\frac{1}{2}a_t^2+\frac{\ln(-\eta)}{\eta}+\frac{\ln(\gamma)}{\eta}+\frac{\ln(W_t)}{\eta}$

$$f_{\xi W}(t,\xi_t,W_t)^{s,\xi_s,W_s}=\frac{1}{\xi_s}E_{t,W_t,\xi_t}\left[\int_t^T\Phi(t,k)\frac{1}{W_t\eta})dk+\Phi(t,T)\frac{1}{W_T\eta}\right]$$

We are going to have the following guess:

$$f(t,\xi_t,W_t)^{s,\xi_s,W_s}=\frac{1}{\xi_s}\ln(W)\left[\xi_tA(t)+B(t)\right]+C(t).$$

This guess implies that, principal's value function will have the following functional form

$$V(t,\xi,W)=\frac{\ln(W)}{\xi_t}D(t)+E(t)\ln(W)+M(t)$$

First order condition implies that

$$1-a_t-a_t\left(\gamma\eta W\sigma\right)^2(A(t)\frac{\xi_t}{\xi_s}+B(t))\frac{1}{W^2}+\sigma_\xi\left(\gamma\eta W\sigma\right)A(t)\frac{1}{\xi_s}\frac{1}{W}=0.$$

Solving for a_t

$$a_t = \frac{\xi_t + \sigma_\xi \gamma \eta \sigma A(t)}{\xi_t \left(1 + (\gamma\eta\sigma)^2 \left(A(t) + B(t)\right)\right)}.$$

We need to plug $\sigma_\xi = e^{-\alpha t}\kappa\nu + \kappa\xi_t$.
Collecting the terms of $\frac{\xi_t}{\xi_s}\ln(W_t))$ terms to pin down $A(t)$

$$\dot{A}(t)\frac{\xi_t}{\xi_s}\ln W_t - (\alpha+\kappa)\,\xi_t \ln W_t A(t)\frac{1}{\xi_s} + \frac{\xi_t}{\xi_s}\frac{lnW_t}{\eta} = 0$$

Then $A(t)$ evolves as

$$\dot{A}(t) = (\alpha+\kappa)\,A(t) - \frac{1}{\eta}$$

collecting the terms of $\frac{1}{\xi_s}\ln(W_t)$ to pin down $B(t)$

$$\dot{B}(t) + e^{-\alpha t}\kappa\theta A(t) = 0$$

with the boundary conditions $A(T) = \frac{1}{\eta}, B(T) = 0$. We can solve for $A(t)$ and $B(t)$

$$A(t) = \frac{(\alpha+\kappa-1)e^{-(T-t)(\alpha+\kappa)} + 1}{\eta(\alpha+\kappa)}$$

$$B(t) = \frac{\theta\left((\alpha+\kappa)(\alpha-1)\,e^{-\alpha T} + \kappa e^{-\alpha t} + \alpha\,(1-\alpha-\kappa)\,e^{-(\alpha+\kappa)T+\kappa t}\right)}{\alpha\eta(\alpha+\kappa)}$$

Proof of Proposition 7

Part 1 Remember that

$$a_t = \frac{\xi_t + \sigma_\xi \gamma \eta \sigma A(t)}{\xi_t \left(1 + (\gamma\eta\sigma)^2 \left(A(t) + B(t)\right)\right)}.$$

Note that both $A(t)$ is positive all the time and $B(t)$ is weakly positive all the time. This implies, as ξ_t increases a_t increases.
Part 2 Then it is easy to see that

$$\lim_{T\to\infty} B(t) = \frac{\kappa\theta e^{-\alpha t}}{\alpha\eta\,(\alpha+\kappa)} = Ce^{-\alpha t}$$

For $C = \kappa\theta/\alpha\eta(\alpha+\kappa)$ and

$$\lim_{T\to\infty} A(t) = \frac{1}{\eta(\alpha+\kappa)} = A^*$$

Plugging σ_ξ to a_t

$$a_t = \frac{\xi_t + (e^{-\alpha t}\kappa\nu + \kappa\xi_t)\gamma\eta\sigma A(t)}{\xi_t \left(1 + (\gamma\eta\sigma)^2 \left(A(t) + B(t)\right)\right)}.$$

Rearranging,

$$a_t = \frac{1 + \kappa\gamma\eta\sigma A(t)}{\left(1 + (\gamma\eta\sigma)^2 \left(A(t) + B(t)\right)\right)} + \frac{e^{-\alpha t}\kappa\nu + \gamma\eta\sigma A(t)}{\xi_t \left(1 + (\gamma\eta\sigma)^2 \left(A(t) + B(t)\right)\right)}.$$

After simplifying and rearranging again we have

$$a_t = \frac{1 + \kappa\gamma\eta\sigma A(t) + \left(\kappa\nu\gamma\eta\sigma A(t)\right)\left(\xi_t e^{\alpha t}\right)^{-1}}{\left(1 + (\gamma\eta\sigma)^2 \left(A(t) + B(t)\right)\right)}.$$

Now plugging in Z_t instead of ξ_t, and taking $\lim T \to \infty$ then we have $A(t) \to A^*$ and $B(t) \to Ce^{-\alpha t}$ and finally have

$$\lim_{T\to\infty} a_t = \frac{1 + \kappa\gamma\eta\sigma A^* + \left(\kappa\nu\gamma\eta\sigma A^*\right)\left(\frac{1}{Z_t}\right)}{\left(1 + (\gamma\eta\sigma)^2 \left(A^* + Ce^{-\alpha t}\right)\right)}$$

For the second part of the proposition we can now take the limit as $t \lim \to \infty$ as well to

$$\lim_{t\to\infty}\lim_{T\to\infty} a_t = \frac{1 + \kappa\gamma\eta\sigma A^* + \left(\kappa\nu\gamma\eta\sigma A^*\right)\left(\frac{1}{Z_t}\right)}{\left(1 + (\gamma\eta\sigma)^2 \left(A^*\right)\right)}$$

Which oscillates according to Z_t around θ.

Appendix B

Solution to Linear SDE's

$$dX_t = (A(t)X(t) + a(t))\,dt + (B(t)X(t) + b(t))\,dW_t$$

define the fundamental equation as follows

$$\Phi(t) = e^{\int_0^t \left(A(s) - \frac{B(s)^2}{2}\right) ds + \int_0^t B(s) dW(s)},$$

then we reach the following solution

$$X(t) = \Phi(t)\left(X(0) + \int_0^t \Phi^{-1}(s)\,(a(s) - B(s)b(s))\,ds + \int_0^t \Phi^{-1}(s)b(s)dW(s)\right).$$

Appendix C

$$\xi_t = e^{-\int_0^t r_s ds}. \tag{44}$$

We are going to assume dynamics of ξ_t is governed as follows

$$\xi_t = e^{-\alpha t}\left(\phi + \psi Z_t\right), \qquad dZ_t = \kappa(\theta - Z_t)dt + \kappa\left(\nu - Z_t\right)dP_t \tag{45}$$

where P_t is another Brownian motion independent of the cash flow shock Z_t. At time t, the expected discounted value of consumption C in the period t' is valued at

$$E(R(t,t')) = E\left(e^{-\int_t^{t'} r_s ds}\right) = e^{-\alpha(T-t)}\frac{\phi + \psi\theta + \psi e^{-(T-t)\kappa}(Z_t - \theta)}{\phi + \psi Z_t} \tag{46}$$

We are going to introduce the following notation, principal discounts the future using function ξ

$$\xi_t = e^{-\int_0^t r_s ds}$$

where short rate interest rate denoted as r_t. The dynamics of ξ_t governed by the function Z_t, which satisfies the following stochastic differential equation

$$dZ_t = \kappa(\theta - Z_t)dt + \kappa\left(\nu - Z_t\right)dM_t$$

where $dM_t$is a local martingale (for simplicity we assume it is a standard Brownian Motion). We are going to use the following equality

$$\xi_t = e^{-\alpha t}\left(\phi + \zeta Z_t\right),$$

after taking time derivative we can write the evolution of ξ as follows

$$d\xi_t = -\alpha e^{-\alpha t}\left(\phi + \zeta Z_t\right)dt + e^{-\alpha t}\zeta dZ_t.$$

By plugging dZ_t, we can write the evolution of ξ_t as follows

$$d\xi_t = -\alpha e^{-\alpha t}\left(\phi + \zeta Z_t\right)dt + e^{-\alpha t}\kappa\zeta\left[(\theta - Z_t)dt + (\nu - Z_t)\,dM_t\right].$$

Note that using above equation,

$$P(t,T) := \mathbb{E}[\frac{\xi_T}{\xi_t}] = e^{-\alpha(T-t)}\frac{\phi + \zeta\theta + \zeta e^{-(T-t)\kappa}(Z_t - \theta)}{\phi + \zeta Z_t}$$

moreover

$$\mathbb{E}_t[Z_T] = \theta + e^{-\kappa(T-t)}(Z_t - \theta), \quad t \leq T.$$

Therefore, short-rate can be equivalently written as

$$r_t = \alpha - \frac{\zeta\kappa(\theta - Z_t)}{\phi + \zeta Z_t}.$$

Define α as follows

$$\alpha := \sup_{Z \in E}\frac{\zeta\kappa(\theta - Z)}{\phi + \zeta Z}$$

which guarantees that short rate (r) stays positive. The last constraint we have is $\phi + \zeta Z > 0$ for all $Z \in E$. This means we have to bound the domain of Z. Easiest way to do this is choose $\kappa < 0$ and $\nu > \theta \geq 0$ therefore $E = [\nu, \infty)$.
We can rewrite the r_t as,

$$r_t = \alpha + \kappa - \frac{\kappa\left(\zeta\theta + \phi\right)}{\phi + \zeta Z_t}.$$

By Ito's lemma we can write the evolution of the short rate as

$$dr_t = -\sigma_{Z_t}\frac{\zeta^2\kappa\left(\zeta\theta + \phi\right)}{\left(\phi + \zeta Z_t\right)^3}dt + \frac{\zeta\kappa\left(\zeta\theta + \phi\right)}{\left(\phi + \zeta Z_t\right)^2}dZ_t.$$

We can solve for the Z_t and reach

$$Z_t = \frac{\xi_t e^{\alpha t} - \phi}{\zeta}.$$

Plugging in to the evolution of ξ_t

$$d\xi_t = -\alpha\xi_t dt + e^{-\alpha t}\kappa\zeta\left(\left(\theta - \frac{\xi_t e^{\alpha t} - \phi}{\zeta}\right)dt + \left(\nu - \frac{\xi_t e^{\alpha t} - \phi}{\zeta}\right)dM_t\right)$$

after simplification

$$d\xi_t = \left(-\left(\alpha + \kappa\right)\xi_t + e^{-\alpha t}\kappa\zeta\left(\theta + \frac{\phi}{\zeta}\right)\right)dt + e^{-\alpha t}\kappa\zeta\left(\nu + \frac{\phi}{\zeta}\right)dM_t - \kappa\xi_t dM_t.$$

Moreover observe that ξ_t is Linear Stochastic Differential equation has closed form solution exits and the soluton is linear in ξ (see Appendix B). Recall our probabilistic interpretation,

$$f(t,\xi_t,W_t,s,\xi_s,W_s) = \frac{\xi_t}{\xi_s}E_{t,W_t,r_t}\left[\int_t^T \frac{\xi_k}{\xi_t}(a_k - C_k)dk + \frac{\xi_T}{\xi_t}\frac{\ln(-W_T)}{\eta}dk\right]$$

using the linearity of the SDE probabilistic interpretation can be rewritten as

$$\begin{aligned} f(t,\xi_t,W_t,s,\xi_s,W_s) = \frac{1}{\xi_s}E_{t,W_t,\xi_t}\Bigg[\int_t^T (\Phi(t,k)\xi_t + \int_t^k D(t,s)ds + \int_t^k d(t,s)dZ_s)(a_k - C_k)dk \\ + \left((\Phi(t,k)\xi_t + \int_t^k D(t,k)ds + \int_t^k d(t,s)dZ_s)\right)\frac{\ln(-W_T)}{\eta}dk\Bigg] \end{aligned} \tag{47}$$

where d, D and Φ are derived using Appendix B. [26] Our extended HJB system can be written as:

$$\sup_{a_t}\left[a_t - \frac{1}{2}a_t^2 + \frac{\ln(-\eta)}{\eta} + \frac{\ln(\gamma)}{\eta} + \frac{\ln(W_t)}{\eta} + \mathcal{A}^u f(t,\xi_t,W_t)^{t,\xi_t,W_t}\right] = 0 \tag{48}$$

$$\mathcal{A}^{u^*} f(t,\xi_t,W_t)^{s,\xi_s,W_s} + \frac{\xi_t}{\xi_s}(a_t^* - c_t^*) = 0 \tag{49}$$

$$f(T,\xi_T,W_T)^{s,\xi_s,W_s} = \frac{\xi_T}{\xi_s}\frac{\ln(-W_T)}{\eta} \tag{50}$$

with probabilistic interpretation $f(t,\xi_t,W_t)^{t,\xi_t,W_t} = V(t,\xi_t,W_t)$ and u^* denotes the equilibrium control.

$$\begin{aligned} \mathcal{A}^u f(t,\xi_t,W_t)^{s,\xi_s,W_s} =& f_t(t,\xi_t,W_t)^{s,\xi_s,W_s} + \mu_\xi f_\xi(t,\xi_t,W_t)^{s,\xi_s,W_s} \\ &+ \frac{1}{2}\sigma_\xi^2 f_{\xi\xi}(t,\xi_t,W_t)^{s,\xi_s,W_s} + \frac{1}{2}\left(\gamma\eta W\psi\sigma\right)^2 f_{WW}(t,\xi_t,W_t)^{s,\xi_s,W_s} \\ &+ \sigma_\xi\left(\gamma\eta W\psi\sigma\right) f_{W\xi}(t,\xi_t,W_t)^{s,\xi_s,W_s} \end{aligned}$$

Let's take a closer look at each of the objects, starting with the derivative respect to ξ_t

$$f_\xi(t,\xi_t,W_t,s,\xi_s,W_s) = \frac{1}{\xi_s}E_{t,W_t,\xi_t}\left[\int_t^T (\Phi(t,k)(a_k - C_k)dk + \Phi(t,k)\frac{\ln(-W_T)}{\eta}dk\right]$$

$$f_{\xi\xi}(t,\xi_t,W_t)^{s,\xi_s,W_s} = 0$$

let's take derivative respect to W, we use the fact that $a_t - c_t = a_t - \frac{1}{2}a_t^2 + \frac{\ln(-\eta)}{\eta} + \frac{\ln(\gamma)}{\eta} + \frac{\ln(W_t)}{\eta}$

[26] To be more precise Φ, D, d defined as follows: $\Phi(t,k) = e^{\int_t^k -\left(\alpha+\kappa+\frac{1}{2}\kappa^2\right)ds + \int_t^k \kappa dZ_s}$ and $D(t,k) = \Phi(t,k)\int_t^k \Phi^{-}1(t,s)\left(e^{-\alpha(s-t)}\kappa\zeta\left(\theta + \frac{\phi}{\zeta}\right) - \kappa e^{-\alpha(s-t)}\kappa\zeta\left(\nu + \frac{\phi}{\zeta}\right)\right)ds$ and $d(t,k) = \Phi(t,k)\int_t^k e^{-\alpha(s-t)}\kappa\zeta(\nu + \frac{\phi}{\zeta}ds$.

$$f_{\xi W}(t,\xi_t,W_t)^{s,\xi_s,W_s} = \frac{1}{\xi_s} E_{t,W_t,\xi_t}\left[\int_t^T \Phi(t,k)\frac{1}{W_t\eta})dk + \Phi(t,T)\frac{1}{W_T\eta}\right]$$

We are going to have the following guess:

$$f(t,\xi_t,W_t)^{s,\xi_s,W_s} = \frac{1}{\xi_s}\ln(W)\left[\xi_t A(t) + B(t)\right] + C(t).$$

This guess implies that, principal's value function will have the following functional form

$$V(t,\xi,W) = \frac{\ln(W)}{\xi_t} D(t) + E(t)\ln(W) + M(t)$$

First order condition implies that

$$1 - a_t - a_t\left(\gamma\eta W\sigma\right)^2\left(A(t)\frac{\xi_t}{\xi_s} + B(t)\right)\frac{1}{W^2} + \sigma_\xi\left(\gamma\eta W\sigma\right)A(t)\frac{1}{\xi_s}\frac{1}{W} = 0.$$

Solving for a_t

$$a_t = \frac{\xi_t + \sigma_\xi\gamma\eta\sigma A(t)}{\xi_t\left(1 + (\gamma\eta\sigma)^2\left(A(t) + B(t)\right)\right)}.$$

We need to plug new $\sigma_\xi = e^{-\alpha t}\kappa\zeta\left(\nu + \frac{\phi}{\zeta}\right) - \kappa\xi_t$, note that if $\left(\nu + \frac{\phi}{\zeta}\right) = 0$ suggested action becomes independent of the discount factor ξ_t and solely function of time. However, this is impossible given parameter restrictions of Linear-Rational models of short rates.
Collecting the terms of $\frac{\xi_t}{\xi_s}\ln(W_t))$ terms to pin down $A(t)$

$$\dot{A}(t)\frac{\xi_t}{\xi_s}\ln W_t - (\alpha+\kappa)\,\xi_t\ln W_t A(t)\frac{1}{\xi_s} + \frac{\xi_t}{\xi_s}\frac{\ln W_t}{\eta} = 0$$

Then $A(t)$ evolves as

$$\dot{A}(t) = (\alpha+\kappa)\,A(t) - \frac{1}{\eta}$$

collecting the terms of $\frac{1}{\xi_s}\ln(W_t)$ to pin down $B(t)$

$$\dot{B}(t) + e^{-\alpha t}\kappa\zeta A(t)\left(\theta + \frac{\phi}{\zeta}\right) = 0$$

with the boundary conditions $A(T) = \frac{1}{\eta}, B(T) = 0$. We can solve for $A(t)$ and $B(t)$

$$A(t) = \frac{(\alpha+\kappa-1)e^{-(T-t)(\alpha+\kappa)} + 1}{\eta(\alpha+\kappa)}$$

$$B(t) = \frac{(\theta\zeta+\phi)\left((\alpha+\kappa)(\alpha-1)\,e^{-\alpha T} + \kappa e^{-\alpha t} + \alpha\,(\alpha-\kappa-1)\,e^{-(\alpha+\kappa)T+\kappa t}\right)}{\alpha\eta(\alpha+\kappa)}$$

Note that $A(t)$ is positive all the time. $B(t)$ is either positive or negative all the time depending on the sign of the term $\kappa\zeta\left(\theta+\frac{\phi}{\zeta}\right)$. If that term is negative $B(t)$ is negative all the time otherwise it is positive all the time. This implies, as ξ_t increases a_t decreases if $B(t)$ is positive otherwise, it depends on the sign of $A(t)+B(t)$.

Plugging σ_ξ to a_t

$$a_t=\frac{\xi_t+(e^{-\alpha t}\kappa\zeta\left(\nu+\frac{\phi}{\zeta}\right)+\kappa\xi_t)\gamma\eta\sigma A(t)}{\xi_t\left(1+(\gamma\eta\sigma)^2\left(A(t)+B(t)\right)\right)}.$$

gives the optimal contract in the fully parametrized of Filipovic et al. (2018). Unfortunately, with such generality number of parameters hinders the ability to the comparative statics analytically. However, this parametrization suitable for empirical analysis.

Appendix D

Quasi-hyperbolic HJB equation In this section, we are going to derive the HJB equation for the special case of Quasi-hyperbolic.

$$\begin{aligned}
V(t,x) &\geq J(t,x,h_\Delta)- \\
&= J(t,x,h_\Delta)-\mathrm{E}^h_{t,x}\left[J(t+\Delta,X_{t+\Delta})\right]+\mathrm{E}^{t,x}V(t+\Delta,x+\Delta) \\
&= \mathrm{E}^{t,x}\left[\int_0^T R(s-t)\left(a^{h_\Delta}(s)-ch_\Delta(s)\right)ds-\int_{t+\Delta}^T R(s-t-\Delta)\left(a^h(s)-c^h(s)\right)ds\right]+\mathrm{E}^{t,x}V(t+ \\
&\approx \Delta\mathrm{E}^{t,x}\left[a^{h_\Delta}(t+\Delta)-ch_\Delta(t+\Delta)-\int_{t+\Delta}^T R'(s-t-\Delta)\left(a^{h_\Delta}(s)-ch_\Delta(s)\right)ds\right]+\mathrm{E}^{t,x}V(t+\Delta
\end{aligned}$$

Then by first order Taylor expansion,

$$\begin{aligned}
\int_t^T R(s-t)\left(a_s^{h_\Delta}-c_s^{h_\Delta}\right)ds \approx& \int_{t+\Delta}^T R(s-t-\Delta)\left(a_s^{h_\Delta}-c_s^{h_\Delta}\right)ds \\
&+\Delta\left(a^{h_\Delta}_{(t+\Delta)}-c^{h_\Delta}_{(t+\Delta)}+\int_{t+\Delta}^T R'(s-t-\Delta)\left(a_s^{h_\Delta}-c_s^{h_\Delta}\right)ds\right)
\end{aligned}$$

Taking $\Delta\to 0$

$$\sup_h A^h V(t,x)+\mathrm{E}^h_{t,x}\left[\int_t^T R'(s-t)\right]=0$$

Appendix E

In this section we show that there exists an equivalent time consistent problem for the principal with a different objective function.

Proposition 9 *For every time inconsistent principal in this model, there is a time consistent principal wit a different objective function*

Suppose for now assume that there exist an optimal contract with equilibrium control law h, then using the h we can construct the following function K as follows:

$$K(t, x, h) = R(t, t)(a_t^* - c_t^*) - \mathcal{A}^h f(t, x, t, x) + \mathcal{A}^h f^{tx}(t, x)$$

Given the above K, maximizing the following problem becomes standard time consistent optimal control problem.

$$\mathrm{E}_{t,x} \int_t^T K(s, x, h_s) ds$$

Major caveat of this approach is that once needed to use the equilibrium h to construct the time-consistent optimal control problem.

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