On the Gradual Progression of Irreversible Political Processes

Gilat Levy and Ronny Razin*

August 2010

Abstract

We analyze a dynamic model of progress in society in which players compete in an all-pay-competition in each period to have their ideal action implemented. The winning policy at each competition is implemented at that period, but only if it is better than the previous one, according to some exogenous order. Thus implemented policies can only move in one direction. We show that in any subgame perfect equilibrium of this game, progress is gradual: (i) in each period there is a substantial probability that the game moves forward, i.e., a better action is implemented, (ii) in each period, the probability that the best action is implemented is bounded away from one. Progress is thus inevitable but relatively slow.

1. Introduction

In this paper we are interested in understanding the rate of progress in dynamic political processes which are irreversible. In particular, we analyze an infinite horizon dynamic influence game, in which at each period an action is implemented from a fixed set of actions. The process is constrained however to move in a particular direction: at each period an influence game determines one candidate action which is

*London School of Economics, Emails: g.levy1@lse.ac.uk; r.razin@lse.ac.uk. For valuable comments we thank Alessandro Lizzeri and Ariel Rubinstein. We also thank seminar participants in Penn, Oxford, LSE, Helsinki, Esset 2010, Princeton, Madrid, Warwick and UCL.

implemented only if it is better than the status quo according to an exogenous linear order. Otherwise the status quo action is maintained for this period.

The influence game played at each period takes the following form. A finite number of interested players engage in an all-pay-competition in which they exert resources: the candidate action is selected as a function of these efforts. The main assumption that we make about the all-pay-competition is that a player can always secure a high enough probability of winning by exerting enough resources relative to others'.[1]

Our model of progression through dynamic all-pay competition can capture important features of several environments. The ones detailed below are further analyzed in Section 4:

I. Military conflicts: Suppose that there are two players, a colonial power and an occupied population. There are two possible states/policies that can be implemented: occupation and independence. The colonial power and the occupied population engage in a (military) conflict. As long as the colonial power prevails, occupation is maintained. Once the occupied population manages to gain independence, this becomes an absorbing state.[2] The colonial army retreats and initiating another war is typically very costly.[3]

II. Political processes: Consider a dynamic entry game among candidates, or policies, in a sequence of elections or referenda. The all-pay-competition represents political campaigns or primaries, and society moves forward according to the preferences of the median voter. In particular, policies change, reforms are adopted, or new candidates are elected at the expense of old ones, only if they are perceived to be

[1] Our assumptions about the all-pay competition are satisfied by a general family of influence games which includes many all-pay mechanisms that have been analyzed in the literature (such as all-pay auctions or the Tullock influence functions). The notion of influence games was introduced in Becker (1983). See also Skaperdas (1996) and the survey in Konrad (2009).

[2] Such irreversability is sometimes assumed in the literature on regime changes, which is another application of our model. For example, Acemoglu, Ticchi, and Vindigni (forthcoming) consider military dictatorship to be an absorbing state.

[3] Until its recent entanglement in Iraq, the British forces have not re-occupied any of their old colonies such as Cyprus, Sri Lanka, Palestine or Bangladesh. Similarly, is the case of Italy in Ethiopia, the Netherlands in Indonesia, and the French in Vietnam, Algeria or Senegal among others. In contrast, when there is geographical proximity, independence seem less irreversible, as in the case of Israel and the Palestinian territories.

better than the status quo in the eyes of the median voter. As long as the distribution of preferences in society is relatively stable, such a process exhibits irreversibility.[4]

In this paper we explore how the dynamic competition to promote (or prevent) certain actions unfolds. Our main parameter of interest is the length of each period, ρ, along which players incur utility from the implemented action. If each period is relatively long, competitions are far and few and the implementation of an action lasts for a substantial period of time. Short term incentives are therefore important and give a potential advantage for players with a high intensity to win or those that incur large negative externalities.

When each period is very short, i.e., when $\rho \to 0$, competitions occur in quick succession and the short term incentives are small. Players who are not favoured according to the order of progression have to win a large number of competitions to halt progress, whereas for a player representing the best policy it would suffice to win just one competition to secure a high level of utility and to terminate the game.

Our main result (Theorem 2) shows that even when the length of each period converges to zero, players never "give up". In particular we show that any subgame perfect equilibrium involves *gradualism*: (i) in each period there is a substantial probability that the game moves forward, i.e., that a better action is implemented, (ii) in each period, the probability that the best action is implemented is bounded away from one. Progress is thus inevitable but relatively slow.

To prove this result, we first establish that the willingness to win of players are of comparable magnitude.[5] To do this we must first prove (i) above; showing that the process evolves forward with a strictly positive probability implies that the game will reach the absorbing state in finite time in expectations and thus allows us to bound the magnitude of players' willingness to win. These are of magnitude ρ as players' strategies can affect in expectations only a finite number of periods. Although this is a game with infinite horizon, in equilibrium players fight only for instantaneous gains.

When the magnitudes of players' willingness to win are comparable, we can deduce that the player representing the best action cannot finish the game off in any

[4] The literature on political entry has mainly focused on static entry games with exogenous fixed cost (see Besley and Coate (1997) or Osborne and Slivinski (1996)). In contrast to these papers our central focus is on the endogenous cost of entry and on the dynamics of entry.

[5] Note that in a game with negative externalities, as we analyze here, there is no established way to define willingness to win. Our definition involves the notion of bilateral willingness to win between pairs of players.

one period. Suppose that the best player wins in some period with a probability close to one. This implies that all other players place bids that are marginal compared to their own willingness to win, as (almost) any bid they use in equilibrium only delivers a marginal probability of winning. By our assumption about the all-pay-competition, the best player can secure a win by placing bids that are very high relative to that of the other players; such bids though could still be very small as other players' bids are marginal with respect to their willingness to win. Thus, we can bound all equilibrium bids: They must be marginal compared with the losing players' willingness to win. But then one of these loosing players will want to deviate. By placing a high relative bid he can secure his willingness to win with a small bid.

Theorem 2 provides a lower and upper bound on how fast the process converges to the best policy. To understand better the rate of gradualism and how it is affected by the primitives of the model we analyze two applications. We first examine a bilateral conflict game and analyze how asymmetries in the competition function and intensity of preferences affect the rate of gradualism. We then analyze an application to a political process with one-dimensional policy space and single peaked preferences. We show that there exist environments with *full gradualism* (even when $\rho \to 0$): in such equilibria the player representing the best policy for the median voter competes only once all other policies have been implemented. We show that this particularly slow process hinges on negative externalities and large enough polarization in society.

Our paper contributes to the large literature on influence games.[6] We analyze a dynamic influence game and characterize the results for a general set of influence functions. Also related is the literature on political transitions or regime changes, with pioneering contributions by Acemoglu and Robinson (2001, 2008) and recently Acemoglu, Ticchi, and Vindigni (forthcoming). We complement this literature by focusing on the rate of convergence to steady states for a general family of influence games.[7]

Several papers have analyzed gradualism in different contexts, albeit stemming

[6] Becker (1983) and Grossman and Helpman (1994) have used contest functions or auctions to model how players can directly affect political outcomes. In a political set up, Polborn and Klumpp (2006) analyze a dynamic competition (primaries) between two candidates, via a contest function, to win different districts.

[7] Another related literature is that of endogenous agenda formation, where several papers extend bargaining games to include a stage in which agents compete via an all-pay-auction for the right to propose a policy (see for example Zwiebel and Board (2005), Yildrim (2007) and Evans (1997)).

from different reasons than the one analyzed in our model. Compte and Jehiel (2004) analyze a bargaining game in which the outside option of the players depends on previous offers. Admati and Perry (1991) show that an agent holds back his payments in contribution games to insure that the other agent contributes his share as well. Finally, in a multistage patent race game among two players, Konrad and Kovenock (2009) show that an agent who is losing in the patent race still does not give up, as long as he can win some strictly positive instantaneous prize. We show that this arises even when the instantaneous prize converges to zero.

The rest of the paper is organized as follows. In the next section we present the model. In Section 3 we present our main result of gradualism. Section 4 considers the effects of different parameters on gradualism in two applications. An appendix contains all proofs which are not in the text.

2. The model

We analyze an infinite dynamic game of complete information with a finite set of players, $M = \{1, 2, ..., m\}$.

Feasible actions and preferences: Each player i represents an action $x_i \in X$. The utility of player i from action x_j is v_{ij}, where $-\infty < v_{ij} < 0$ for $i \neq j$ and $v_{ii} = 0$. This specification includes the possibility of negative externalities, that is, when players have different preferences depending on the identity of the winner, or the possibility of pure winning motives (when $v_{ij} = v_{ik}$ for all $j, k \neq i$).

Irreversible progress: At each period $t \in \{1, 2, ...\}$ an action $x^t \in \{x_1, x_2, ..., x_m\}$ is implemented for the length of that period (all periods are of equal length). We now specify how x^t is determined.

We assume that there is an exogenous linear (complete and transitive) order on $\{x_1, x_2, ..., x_m\}$. Without loss of generality, we assume that the index of actions represents the order, i.e., actions with lower index are better. In the beginning of each period t in the game, the players engage in an all-pay competition whose details we specify below. Let the winner of the competition at time t be denoted by i^t. Let s^t be the lowest index of all policies implemented up to (and excluding) time t. We will sometimes refer to s^t as the *status quo* or *state* for period t. The action x^t evolves in the following way: $s^1 = x_m$, $x^t = x_{\min\{s^t, i^t\}}$. Thus, the implemented action in time t changes only if the winner of the period t competition represents a better action according to the order.

Note that the action x_1 is an absorbing state and no player would exert any effort once it has been implemented. Note further that all players can compete at any stage even if their action has already been implemented in the past. These will be players that will either fight to maintain the status quo or perhaps use their bids to change the balance of power between players whose actions are better than the status quo.

Finally, note that in the above model players can only propose the action they represent. Alternatively one can analyze a model in which players strategically choose policies from a set of feasible policies. Our main result, Theorem 2, can be generalized to this alternative model, when the set of feasible policies is finite.

The all-pay competition: We now describe the all-pay-competition that the players engage in at each period t. Each player i places a bid $b_i^t \geq 0$ which he must pay regardless of the outcome. Bids are placed simultaneously at the beginning of period t. To ensure equilibrium existence, suppose that (at each stage) b_i^t is in the compact interval $[0, B]$ for some large finite B. The probability with which player i wins the competition at stage t is determined according to a function $H_i(\mathbf{b}^t)$, where $\mathbf{b}^t$ is the vector of bids. We assume that the function $H_i(.)$ satisfies the following properties:

H1. *For any $K > 0$, there exists a $K' > 0$ such that if $b_i = \max_j b_j$ and $\frac{b_i}{b_j} > K'$ then $\frac{H_i(\mathbf{b})}{H_j(\mathbf{b})} > K$.*

H2. *There exists a $\kappa < \infty$, such that for every ε, for any player i and bids $\mathbf{b}, \mathbf{b}'$, such that $b_j = b_j'$ for all $j \neq i$, and $H_i(\mathbf{b}), H_i(\mathbf{b}') < \varepsilon$, then $|H_j(\mathbf{b}) - H_j(\mathbf{b}')| \leq \kappa\varepsilon$ for all $j \in \{1, 2, .., m\}$.*

H3. $\sum_{i \in M} H_i(\mathbf{b}) = 1$.

Assumption H1 is a weakening of a standard assumption connecting relative probabilities of winning to relative efforts exerted. In H1 this connection only relates to the highest bidder vis a vis other players. In particular, H1 implies that any player can secure a high enough probability of winning if he is the highest bidder and if his bid is high enough relative to other players' bids.

Assumption H2 is a weakening of an independence assumption often used in the literature. In particular, we apply independence only to the bids of marginal players. H2 implies that a marginal player, one who has a very small probability of winning a competition, can only have a marginal effect on others' chances and in particular cannot dramatically change the balance of power between other players.[8] Assumption H3 is assumed for expositional purposes, and our results hold as long as there is a lower bound on the probability that some player wins.

For simplicity, we assume that H is the same at any period; our results would also hold if H would differ in each period. For example, it may depend on the status quo s. Thus we can let the "balance of power" or "rules of the game" change with different states of progression.

The above set of assumptions are general enough to include many of the functional forms used in the literature, including the generalized Tullock contests.[9] A particular form of this contest which is common in the literature is $H_i(\mathbf{b}) = \frac{b_i^r}{\sum_j b_j^r}$, which for $r = 1$ is referred to as the simple Tullock function, and with $r \to \infty$, approximates the all-pay-auction where the highest bidder wins with probability one. We analyze these two particular examples in Section 4.

Utilities, strategies and equilibria: Let $\rho \in (0,1)$ represent the length of a period along which players incur utility from the implemented action. Without loss of generality, we use $1-\rho$ as the discount rate between periods,[10] and so for a pure strategy profile, $(\mathbf{b}^t)_{t=1}^{\infty}$, the expected utility of player i from the game is:

$$\sum_{t=1}^{\infty}(1-\rho)^{t-1}(\rho\sum_{j=1}^{m} H_j(\mathbf{b}^t)v_{i\min\{j,s^t\}} - b_i^t).$$

Denote the set of probability measures on $[0,B]$ by $P[0,B]$. Given a profile of

[8] Alternatively we could impose a stronger assumption, monotonicity (i.e., that the probability of winning of player i (weakly) increases in his own bid and decreases in others') which along with H3 implies H2 above.

[9] See Skaperdas (1996) for an axiomatic approach that imposes more conditions on the general H and yields the generalized Tullock contest. Skaperdas (1996)'s axiomatization uses an independence axiom. Clark and Riis (1998) also use independence and homogeneity and lose anonymity to get non anonymous Tullock functions.

[10] The model has discrete periods with continuos time flows of payments between periods. An alternative more standard formulation is the exponential discounting of utilities. Let r be the discount rate and $1/k$ the length of each period. The utility function can be written as $\sum_{t=1}^{\infty} e^{\frac{-rt}{k}}(\int_0^{\frac{1}{k}} e^{-\frac{rx}{k}}dx \sum_{j=1}^{n} H_j^{J^t}(\mathbf{b}^t)v_{i\min\{j,J^t\}} - b_i^t)$. In this formulation the equivalent of ρ, or the weight on period t payoffs vis a vis bids, is $\beta(k,r) = \int_0^{\frac{1}{k}} e^{-\frac{rx}{k}}dx$. The equivalent of $1-\rho$, or the discount factor on period t, is $\gamma(k,r) = e^{\frac{-rt}{k}}$. For any $\rho > 0$, there exist (k,r) such that the two models are equivalent. Moreover, our exercise of taking $\rho \to 0$ is analogous to fixing r above and taking $k \to \infty$, i.e., shortening period length. The two formulations are similar as $\lim_{k\to\infty}\beta(k,r) = 0$, $\lim_{k\to\infty}\gamma(k,r) = 1$ and $\lim_{k\to\infty}\frac{1-\beta(k,r)}{\gamma(k,r)} = 1$.

mixed strategies for any period t, $(f_i^t)_{i\in M} \in \prod_{i\in M} P[0,B]$, the above utility function is extended in a straightforward way. For simplicity, we assume that players do not observe the actual bids others place at each stage. Thus the relevant history on which players condition their strategy on is $h = (m, i^1, i^2...i^k)$, which includes the identity of the winners of the competition in each period. We can therefore redefine the state $s(h) = \min\{i|i \in h\}$.

If H is continuous, a Markov Perfect Equilibrium (and hence a SPE) exists by Escobar (2008).[11] Assumption H1 implies however that H is not continuous at $\mathbf{b} = 0$. Nonetheless we can show:

Theorem 1 *Suppose that H satisfies* H1*,* H2*,* H3*, and is continuous except at $\mathbf{b} = 0$. Then there exist an open set of H functions for which a Subgame Perfect Equilibrium exists.*[12]

In the proof we construct a model similar to ours with the exception of strictly positive reservation prices. In this game an MPE exists by Escobar (2008). We then take the reservation prices to zero and show that the limit of the sequence of equilibria is an equilibrium in the limit. To do so, we use the properties of equilibria implied by H1, H2 and H3, and show that along the sequence it cannot be that the strategies of all players place a positive measure on bids converging to zero, which allows us to establish continuity in the limit.

Theorem 2 below holds more generally for H functions that are not continuous at other points than zero (for example, the all-pay-auction) and we therefore proceed without imposing continuity.[13]

3. Progress and gradualism: the main result

At state $s(h)$, competing players are essentially divided into two groups. Players with $i < s(h)$, if win, will enforce progress, whereas players with $i \geq s(h)$, if win, will maintain the status quo. Moreover, any player has to consider both the short and the long term implications of competing. For example, when ρ is large, competitions are far and few, so that each action is implemented for a substantial period. Short

[11]In a Markov Perfect Equilibrium the strategies will only depend on the state s.

[12]The proof shows in fact that an MPE exists. Our main result, Theorem 2, applies to all SPE.

[13]In Section 4 we analyze examples for which an MPE exists when H is the all-pay-auction function.

term incentives are therefore large, giving a potential advantage for players with a high intensity to win (e.g., with low $v'_{ij}s$).

When ρ is small on the other hand, competitions occur in a quick manner and the short term incentive is small. Players with an index higher than $s(h)$ would have to win a large number of competitions to stop progress, whereas for Player 1 for example it would suffice to win just one of these competitions to practically terminate the game. Our main result shows that even when $\rho \to 0$, and players have to repeatedly compete to maintain their action as the current state, they never "give up":

Theorem 2 *There exists an $\varepsilon > 0$ such that in any Subgame Perfect Equilibrium, for any history h and state $s(h) > 1$, for any ρ, (i) some player $i < s(h)$ wins with a probability larger than ε, (ii) any player $i < s(h)$, wins with a probability lower than $1 - \varepsilon$.*

Theorem 2 implies that the process is gradual. First, at any stage, the process moves forward with a strictly positive probability; society will reach state 1, the absorbing state, in finite time in expectations. Second, even if players know that the process will end at state 1, and that the short term gains are vanishingly small, they still fight in a substantial manner relative to one another. Thus, progress is never immediate; actions other than the best one will be implemented with a strictly positive probability.

The full proof is provided in the appendix. In the next two subsections, we first provide the proof for $m = 2$ which illustrates how we make use of our assumptions on H and in particular H1. We then explain what is involved in extending the proof to $m > 2$ and the role played by negative externalities.

3.1 Proof of Theorem 2 for m=2

Consider the case of $m = 2$, with $x_2 \neq x_1$. When $s(h) = 1$, the bidding game terminates as neither player competes anymore. Players 1 and 2 receive 0 and v_{21} respectively from that point onward. We therefore only have to consider histories at which the state is $s(h) = 2$.

Let V_i^h denote the continuation utility of player i following some history h with $s(h) = 2$. Let w_{ij}^h denote the willingness to win (WTW) of player i against player

j for this history: it is the difference in player $i's$ utility from winning vs. losing to player j, following h, and abstracting from any payments made in the current stage.

We therefore have:

$$
\begin{aligned}
w_{12}^h &= \rho(-v_{12}) + (1-\rho)(v_{11} - V_1^{(h,2)}), \\
w_{21}^h &= \rho(-v_{21}) + (1-\rho)(V_2^{(h,2)} - v_{21}).
\end{aligned}
$$

The first term is the short term incentive to win, of magnitude ρ, and the second term is the long term incentive to win. Note that both expressions above are strictly positive as $v_{11} = 0$, $V_i^h \leq 0$, $V_2^h \geq v_{21}$ and $v_{ij} < 0$ for $i \neq j$.

We will often look at sequences of equilibria and then the relevant willingness to win will be $w_{ij|_n}^h$, where we will sometimes suppress the notation for n in these expressions. Often these sequences will relate to a sequence of parameter choices $\{\rho_n\}_{n=1}^{\infty}$ in which $\rho_n \to \rho^*$ where $\rho^* \in [0,1]$. We will say that a sequence $\{a_n\}_{n=1}^{\infty}$ is in the order of ρ or higher (lower) if $\lim_{\rho_n \to \rho^*} \frac{|a_n|}{\rho_n}$ is strictly positive (finite), and is in the order of ρ if $\lim_{\rho_n \to \rho^*} \frac{|a_n|}{\rho_n}$ is both strictly positive and finite. Finally, let $\bar{H}(\tilde{b}_i, b_j)$ be the expectations of H w.r.t. the random variable $\tilde{b}_i$ given some fixed b_j.

We now prove the following:

Proposition 1: *Suppose that $m = 2$. In any SPE, there exists an $\varepsilon > 0$* such that*: (i) the probability that any player wins is larger than ε; (ii) For any sequence of parameter choices, $\{\rho_n\}_{n=1}^{\infty}$, and corresponding sequence of SPE, w_{21}^h and w_{12}^h are of order ρ and all equilibrium bids are of order ρ or lower.*

Note that part (i) of the Proposition corresponds to the statement of Theorem 2 for $m = 2$. Part (ii) of the Proposition characterizes the willingness to win of players as well as the magnitude of their bids, which is a crucial intermediate step in the proof of Theorem 2 when it is extended to more than two players.

Proof of Proposition 1: (i) Suppose to the contrary, that for some history h with $s(h) = 2$ and for any $\varepsilon > 0$ there exists a parameter ρ and an equilibrium in which player $i \in \{1,2\}$ wins with probability smaller than ε. Construct a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ converging to zero, with a corresponding sequence $\{\rho_n\}_{n=1}^{\infty}$ and equilibria, such that player i wins in h with probability smaller than ε_n. As

$$
\bar{H}_i(\tilde{b}) < \varepsilon_n,
$$

then

$$\bar{H}_i(\tilde{b}_j, b_i) < k\varepsilon_n$$

for a measure of at least $1 - \frac{1}{k}$ of bids b_i in the support of player i for all $k > 2$. Choosing a sequence of $k_n \to \infty$ and $k_n\varepsilon_n \to 0$ this implies that for almost any bid b_i in the support of player i,

$$\bar{H}_i(\tilde{b}_j, b_i) < k_n\varepsilon_n.$$

Note though that player i is better off using such b_i rather than zero only if:

$$b_i \leq w_{ij}^h(\bar{H}_j(\tilde{b}_j, 0) - \bar{H}_j(\tilde{b}_j, b_i)),$$

and that when $m = 2$, the worst case scenario for player i when he withdraws his bid is that all his probability of winning shifts to player j. Thus by H3, along the sequence, $\bar{H}_j(\tilde{b}_j, 0) - \bar{H}_j(\tilde{b}_j, b_i)$ is at most $k_n\varepsilon_n$. We can therefore bound almost all equilibrium bids of player i, so that $b_i < k_n\varepsilon_n w_{ij}^h$.

We will now place an upper bound on the bids of player j. A possible strategy for player j is to bid a sequence of $b_j = \gamma_n b_i$ where $\gamma_n \to \infty$ and $\gamma_n k_n\varepsilon_n \to 0$ so that $b_j \to 0$. By H1, such bid guarantees winning with probability converging to one and a bid converging to zero. The equilibrium bids, denoted b_j, of player j must involve therefore a maximum sequence of bids $b_j^{\max}$ satisfying $\frac{b_j^{\max}}{w_{ij}^h} \to_{\varepsilon_n \to 0} 0$.

We now reach a contradiction: Player i can deviate from his equilibrium strategy and place a sequence of bids b_i' such that $\frac{b_i'}{b_j} \to_{\varepsilon_n \to 0} \infty$ and $\frac{b_i'}{w_{ij}^h} \to_{\varepsilon_n \to 0} 0$. His (relative) gain is in the order of w_{ij}^h while his (relative) cost, b_i', is infinitely smaller than w_{ij}^h, yielding a strictly positive benefit. Therefore, as in the above analysis there is no dependence on ρ, we have shown that there exists an $\bar{\varepsilon} > 0$, such that for all ρ, at any h, $\bar{H}_i(\tilde{b}) > \bar{\varepsilon}$.

We now prove (ii). Let us first consider w_{21}^h and show it is of order ρ. Let ε be given by part (i). Note that an upper bound on the continuation utility following history h' will be given if we set Player 1's probability of winning at its lowest level, ε, and abstract away from bidding costs:

$$\bar{V}_2^{(h,2)} = \varepsilon v_{21} + (1-\rho)(1-\varepsilon)\bar{V}_2^{(h,2)}$$

implying

$$V_2^{(h,2)} \leq \frac{\varepsilon v_{21}}{1-(1-\rho)(1-\varepsilon)}$$

Hence

$$w_{21}^h \leq \frac{\rho(-v_{21})}{1-(1-\rho)(1-\varepsilon)}.$$

Note also that the minimum continuation value following history $(h,2)$ is $\underline{V}^{(h,2)} = v_{21}$ as player 2 can bid zero throughout and guarantee that. Thus

$$\rho(-v_{21}) \leq w_{21}^h \leq \frac{\rho(-v_{21})}{1-(1-\rho)(1-\varepsilon)},$$

which implies that w_{21}^h is of order ρ. This implies that the bids of Player 2 in equilibrium are at most of order ρ.

Let us now consider the WTW of Player 1, w_{12}^h, which is of order ρ or higher. We will now show that it is at most of order ρ. Consider his continuation utility for some bid b after history h:

$$V_1^{(h,2)}(b) \geq \frac{\bar{H}_2(b,\tilde{b}_2)\rho v_{12} - b}{1-(1-\rho)\bar{H}_2(b,\tilde{b}_2)}$$

and in particular consider a bid $b = K\rho$ for a large enough K so that, by H1 and the fact that the bids of Player 2 are of at most order ρ, the probability that Player 2 wins is smaller than a half. As this strategy is not necessarily optimal, we therefore have:

$$V_1^{(h,2)} \geq \frac{\frac{1}{2}\rho v_{12} - K\rho}{1-(1-\rho)\frac{1}{2}}$$

and hence

$$\rho(-v_{12}) \leq w_{12}^h \leq \rho(-v_{12}) + (1-\rho)\left(-\left(\frac{\frac{1}{2}\rho v_{12} - K\rho}{1-(1-\rho)\frac{1}{2}}\right)\right)$$

where the rhs is of order ρ. We therefore find that the WTW of player 1 is of order ρ. The equilibrium bids of player 1 are at most of order ρ as well.■

3.2 Extending the proof to many players

Our general method of proof involves the following. We first prove, by using an induction on the state $s(h)$, that the process must evolve forward with a strictly positive probability (part (i) of the Theorem) along with establishing the magnitudes of the WTW's of all players. These two are intertwined: showing that the process evolves forward with a strictly positive probability implies that the game will reach the absorbing state in finite time in expectations and thus allows us to bound the magnitude of players' WTW. By deriving that the magnitudes of players' WTW are comparable to one another, we can deduce that Player 1, as in Proposition 1 above, cannot win the game too quickly. In addition, the bound on the players' WTW allows us to deduce that equilibrium bids are of order ρ at most.

Extending the arguments from Proposition 1 to more than two players involves considering several issues. First, note that even when $s(h) = 2$, when there are more than two players, many players may compete in equilibrium to defend the policy of Player 2 vis a vis Player 1. We therefore need to impose bounds on the bids of all such players.

Second, with two players, when a player is a marginal player who wins with a small probability, we can use H3 to guarantee that his bids are infinitely smaller than his WTW (as otherwise he would rather withdraw his bid). But when we have many players H3 is not enough, as when a marginal player withdraws his bid, he may substantially change the balance of power between the other many players who compete. Assumption H2 guarantees that this is not the case which again allows us to bound the bids of marginal players.

Finally, negative externalities complicate the proof for many players for another reason; when a losing player withdraws and shifts even a small probability to others, he may shift it to the player he fears most. His bids are therefore infinitely smaller than his WTW against his worst opponent. This implies that we need to consider the worst case scenario for each player, or more generally *all* bilateral comparisons between players. To do this we use the inductive structure of the game, which renders it impossible to move from some state $s(h)$ to any state $s(h') > s(h)$.

Theorem 2 establishes gradualism for a general environment, but does not tell us how slow (or quick) is the rate of gradualism. For example, although we know that at no stage Player 1 wins almost for sure, it is not clear at which stage he actually starts competing. We now consider two simple applications of the model. These illustrate the model's applicability but also allow us to explore in more detail the dynamics of gradualism.

4. Applications

4.1 The rate of gradualism in a bilateral conflict game

Let us consider the particular example outlined in the introduction with two players, a colonizing country (Player 2), associated with the action or state of occupation, and an occupied population (Player 1), associated with the state of independence. In such a game we interpret the resources expanded by each player as military ones, and the H function as representing a conflict or military success function. As long as Player 2 wins the stage competition, occupation continues whereas once Player 1 wins, independence becomes an absorbing state. As a benchmark, assume that $v_{12} = v_{21} = v < 0$ so that the per-period utility of each player from losing is the same.

To assess the rate of gradualism and provide some comparative statics, we consider an all-pay-auction with an advantage for Player 1, which can arise due to some international military assistance or better local knowledge (similar results will arise if Player 2, due to some military capacity, is the advantageous player). Suppose therefore that Player 1 wins the influence game whenever $b_1 > \alpha b_2$, for $0 < \alpha \leq 1$. We analyze a Markov Perfect Equilibrium where players' behavior, on and off equilibrium path, depend only on the status quo/state $s(h) = s$ and not on the history. Recall that we only have to analyze the game at $s = 2$ (the occupation state) as $s = 1$ is an absorbing state. We then find:

Proposition 2: *In the unique Markov Perfect Equilibrium at $s = 2$: (i) Player 2 places an atom on zero of measure $1 - \frac{\alpha}{1+\alpha(1-\rho)}$ and with measure $\frac{\alpha}{1+\alpha(1-\rho)}$ places a bid taken from a uniform distribution on $[0, \rho(-v)]$ whereas player 1 places a bid taken from a uniform distribution over $(0, \alpha\rho(-v)]$. (ii) Player 1 wins every stage with probability $1 - \frac{\alpha}{2(1+\alpha(1-\rho))}$.*

The rate of gradualism which can be captured by the probability that Player 1 wins any stage, depends on ρ and on α. Consider first $\alpha = 1$. At one extreme, when $\rho \to 1$, we have the one-shot all-pay-auction, in which each player wins with probability $\frac{1}{2}$. At the other extreme, when $\rho \to 0$, the probability that Player 1 wins is converging to $\frac{3}{4}$. For any ρ, Player 2 wins every stage with a substantial probability, at least $\frac{1}{4}$, and military efforts expanded are in expectations $\frac{1}{2}\rho|v|$.

Now consider the effect of α on the rate of gradualism: for any ρ, the lower is α so that the advantage of Player 1 is larger, the higher is the probability that he wins the stage competition and thus the game will terminate faster.

Note however that that the effect of the military advantage of Player 1 on gradualism is smaller when the players have to compete frequently.[14] This stems from the dynamic effect: for low values of ρ, a large advantage for Player 1 reflected by a small

[14] As $\frac{\partial^2 (1 - \frac{\alpha}{2(1+\alpha(1-\rho))})}{\partial\rho\partial\alpha} = -\frac{\alpha}{(\alpha - \alpha\rho + 1)^3} < 0$.

α, means that the WTW of Player 1 at any stage is lower, as he knows he is highly likely to win in the future, due to his pronounced advantage. For large values of ρ, this dynamic effect is obviously smaller.

It is easy to extend the analysis and consider asymmetric players' valuations. Suppose now that $|v_{12}| > |v_{21}|$ so that Player 1 values independence more than Player 2 values occupation. In a similar spirit, gradualism will be faster and the probability of winning of Player 1 at any stage in a Markov Perfect equilibrium will be (for $\alpha = 1$), $1 - \frac{|v_{21}|}{2|v_{12}|+(1-\rho)\alpha|v_{21}|)}$; note that again the cross derivative w.r.t. ρ and for example $|v_{21}|$, is negative, so that for small values of ρ, when players engage in repetitive fights the effect of a lower valuation of Player 2 is less pronounced than in a world in which one military conflict determines the result for now and ever.

4.2 Full gradualism: slow convergence to the median voter

We now consider a one-dimensional policy space where players have single-peaked utility functions. Specifically, let $x_1 = 0$, and $|x_i| < |x_{i+1}|$ and assume that players have linear utility functions $v_{ij} = -|x_i - x_j|$. In line with our model, we can think of a society that moves forward according to the preferences of the median voter whose ideal policy is at zero, and his single-peaked preferences are symmetric around zero. This formulation highlights a tension between moderates and extremists: moderates - with ideal policies close to zero- represent better policies for the median voter whereas extremists have a higher intensity to win as, by negative externalities, they fear more than moderates losing to other extremists.

This application of our model can be viewed as a dynamic entry game among candidates, or policies. The all-pay-competition represents in this case political campaigns or primaries. The cost of entry into the political arena is therefore endogenous, and depends on who else competes. Society or the median voter adopt a new policy at each election or referendum only if it is better than the status quo. By Theorem 2, convergence to the median voter will arise in this society in finite time in expectations. This application can therefore provide dynamics foundations to the static median voter results.[15]

[15] It would be easy to extend our results to environments in which the preferences of the median voter are unknown to the players, or even changing from period to period. It would be also be interesting to extend such a model to consider for example the median voter choosing optimally the value of ρ. As Theorem 2 is established for all ρ, the general results would still hold in this case.

Theorem 2 implies that convergence to the median voter's favorite policy is not immediate though. We now use this application to further understand the rate of convergence and how it depends on the distribution of ideal policies and preferences of interested players.

We focus on a particular equilibrium, a *fully gradual equilibrium*, in which the dynamic process evolves so that every player can win with a strictly positive probability only after all other policies which are worse than the one he represents have already been implemented. In such an equilibrium, Player 1 will compete only at $s = 2$.

We now define this formally. In what follows, we say that a player is *active* if the measure of non-zero bids in his support is strictly positive. Then:

Definition 1: In a *fully gradual equilibrium,* at any state $s > 1$, only players s and $s - 1$ are: (i) active, and (ii) win with a strictly positive probability.[16]

We establish whether a fully gradual MPE exists both for the all-pay-auction and for the simple Tullock function (where $H_i(b) = \frac{b_i}{\sum_j b_j}$ if $\exists b_j > 0$ and some $H_i(\mathbf{0})$ otherwise). Moreover, for simplicity, we focus on $m = 3$, with $x_1 = 0$, $x_2 > 0$ and $|x_3| > x_2$. We first consider existence of a fully gradual equilibrium when $x_3 < 0$. We then illustrate the effect of the distribution of ideal policies by fixing $|x_3|$ but switching to $x_3 > 0$. Finally, for any x_3, we focus on pure winning motives and assume that $v_{ij} = v < 0$ for all $i \neq j$.

Note that at $s = 2$, for all the cases we analyze, it is an equilibrium for only players 1 and 2 to compete. To construct a fully gradual equilibrium, at $s = 3$, only players 2 and 3 can be active. As we show in the appendix, the binding condition that needs to be satisfied is that given a competition between players 2 and 3, Player 1 prefers not to be active.

4.2.1 Existence of a fully gradual equilibrium when $\mathbf{x_3 < 0}$: When ρ is large (for example close to one), it is easy to sustain a fully gradual equilibrium. In such a case, there are few competitions, and extreme players with strong intensity to win have a relative advantage compared with moderate ones (whose advantage is in a dynamic game with many competitions). Intuitively, players 2 and 3's WTW's vis a vis each other are roughly $|x_3| + |x_2|$ whereas that of Player 1 is at most $|x_3|$ and

[16] An extension of a fully gradual equilibrium is that at some state s, player $s-1$ competes against some player $i > s$ where player i "defends" the policy of s. Our results are robust to a more general definition of full gradualism.

he is therefore priced out of the competition. We show however that full gradualism arises also for small values of ρ:

Proposition 3: *Suppose that* $x_3 < 0$ *and consider the all-pay-auction and the simple Tullock function. For sufficiently small values of* ρ *: (i) There exists a* $\gamma > 0$ *such that a fully gradual MPE exists when* $\frac{|x_3|}{x_2} \geq \gamma$*. (ii) The cutoff* γ *is larger under the simple Tullock function than under the all-pay-auction.*

According to Proposition 3, it is the relative values of x_2 and $|x_3|$ that are important in characterizing the equilibria above.[17] Somewhat counter-intuitively, Player 1 stays out if Player 3 - which represents his worst case scenario, is located far enough from him. The reason is that the distance between these players affects mainly the willingness to win of Player 3, who knows that in expected finite time Player 1 will win (by Theorem 2). On the other hand, when considering whether to deviate, Player 1 knows that Player 2 wins the competition with Player 3 more often (due to his advantage of being more moderate), and that he will win it in finite time in expectations by Theorem 2. Thus his willingness to win in $s = 3$ is guided mainly by his distance from Player 2. In the case of the all-pay-auction, ensuring that the willingness to win of Player 1 is lower than that of Player 3 (or the highest bid in the game, which is the best deviation for Player 1) yields the condition on $\frac{|x_3|}{x_2}$.

In the simple Tullock function, Player 3 has to be even further away for Player 1 to stay out for two reasons. First, fixing equilibrium behavior as in the all-pay-auction, this competition is less aggressive and Player 1 can beneficially deviate even with a small bid and thus is more likely to do so. Second, as equilibrium behavior is less aggressive than in the all-pay-auction, Player 3 wins more often at $s = 3$ which encourages Player 1 to become active to avoid his worst-case scenario.

4.2.2 The effect of the distribution of ideal policies on full gradualism: We now compare between the case above in which $x_3 < 0$ -a "two-sided" polity- and $x_3 > 0$ -a "one-sided" one. To make this comparison meaningful we fix $|x_3|$, so that the distances of players 2 and 3 from Player 1 are fixed.

Proposition 4: *Fix* $|x_3|$*. For both the all-pay-auction and the simple Tullock function, for sufficiently small values of* ρ *: (i) when* $x_3 > 0$, *there exists* $\gamma' > 0$ *such*

[17] This prediction is different from political entry models in which the payments in equilibria are exogenously fixed and so absolute values of ideal policies may matter (see Besley and Coate (1997) and Osborne et al (2004)).

that a fully gradual MPE exists when $\frac{|x_3|}{x_2} \geq \gamma'$*; (ii) The cutoff* γ' *under* $x_3 > 0$ *is larger than the cutoff* γ *under* $x_3 < 0$.

Although $|x_3|$ is fixed, Player 1 is more motivated to deviate when the distribution is one-sided. The reason is that in the expected equilibrium between player 2 and 3, Player 3 is not as disadvantaged when he is on the same side of Player 2. Thus, he wins more often, which increases the incentive of Player 1 to deviate (as Player 3 provides him with his worst outcome). In the "one-sided" influence game, convergence to the median voter is therefore faster.

4.2.3. The importance of negative externalities in generating full gradualism: In many environments, politicians or interest groups may care only about winning and not about the identity of the winner if they lose.[18] Consider then the extreme case of such pure winning motives, where $v_{ij} = v < 0$ for all i, j.[19] We still consider three players, whereas x_3 can be either positive or negative (note that although valuations are equal, players still differ in their attractiveness to the median voter). We can then show:

Proposition 5: *For both the all-pay auction and the simple Tullock function, for all* ρ*, there is no fully gradual MPE: In all MPE, Player 1 is active in every stage and wins every stage with a strictly positive probability.*[20]

Intuitively, the absence of negative externalities sharpens the advantage of player 1 and so convergence to the median voter can arise in this case in the first stage of the game, or in any stage of the game, with a strictly positive probability.

5. Conclusion

We have presented a model of a dynamic process in which society can only move forward in one direction and players compete to influence whether and how fast the process moves. The model can be applied to varied environments such as international conflicts, decision making in committees, or possibly R&D races. Our

[18] Our results above are robust to the inclusion of small office or winning motivations.

[19] The case of equal valuations for all players is the natural case to consider as an extension of our political economy model. If valuations differ, we can construct valuation vectors in which extreme players have relatively large valuations, to somewhat mimic the negative externalities that exist in our model.

[20] In the appendix we establish that indeed an MPE exists in the all-pay-auction.

main assumption about the competition function at each period is that a player can guarantee a high enough probability of winning whenever his bid is high enough relative to that of other players. We show that society moves forward with a strictly positive probability and that players fight only for instantaneous benefits. Thus, even if these are marginal, players still compete, implying that the process is gradual. We have presented two specific applications that allow us to shed some light on the rate of gradualism. In particular, our application to a political setup in which society converges to the median voter's outcome, has highlighted the role of negative externalities in sustaining full gradualism.

6. Appendix

6.1 Notation and preliminaries

Let V_i^h be the continuation utility of player i following some history $h = (m, i^1, ..i^k)$. Let $s(h) = \min\{i | i \in h\}$ be the state at history h. Let u_{ij}^h be the utility of player i when player j wins at history h, abstracting from the possible payments made by player i right after this history. That is, $u_{ij}^h = \rho(-v_{ij}) + (1-\rho)V_i^{(h,j)}$. Let $w_{ij}^h = u_{ii}^h - u_{ij}^h$ denote the willingness to win (WTW) of player i against player j in history h.

We will often look at sequences of equilibria and then the relevant willingness to win will be $w_{ij|_n}^h$. We will often suppress the notation for n in these expressions, writing w_{ij}^h for $w_{ij|_n}^h$. Often these sequences will relate to a sequence of parameter choices $\{\rho_n\}_{n=1}^{\infty}$ in which $\rho_n \to \rho^*$ where $\rho^* \in [0, 1]$.

In what follows we use the following terms to refer to the magnitudes of variables computed for a sequence of parameters $\{\rho_n\}_{n=1}^{\infty}$ and equilibrium strategies. We say that w_{ij}^h is of *order* ρ if $0 < \lim_{\rho_n \to \rho^*} \left| \frac{w_{ij|n}^h}{\rho_n} \right| < \infty$ for any sequence of equilibria computed for a sequence $\{\rho_n\}_{n=1}^{\infty}$. Similarly we say that w_{ij}^h is of *order* ρ *or lower* if $0 \leq \lim_{\rho_n \to \rho^*} \left| \frac{w_{ij|n}^h}{\rho_n} \right| < \infty$ and of an order ρ *or higher* if $0 < \lim_{\rho_n \to \rho^*} \left| \frac{w_{ij|n}^h}{\rho_n} \right| \leq \infty$. We will also sometimes write $x(\rho) \approx y(\rho)$ for two functions $x(\rho)$ and $y(\rho)$ to imply that $\lim_{\rho_n \to \rho^*} \frac{x(\rho_n)}{y(\rho_n)} = 1$.

Finally, note that the WTW of player 1 vis a vis any other player is strictly positive and of order ρ or higher. To see this note that,

$$w_{1i}^h = \rho(-v_{1i}) + (1-\rho)(v_{11} - V_1^{(h,i)}) \geq \rho(-v_{21}), \; i = 2, ..., m,$$

where $v_{11} = 0$, $-v_{1i} > 0$ and $V_1^h \leq 0$ for all h. The same is true for the WTW of Player 2 vis a vis Player 1:

$$w_{21}^h = \rho(-v_{21}) + (1-\rho)(V_2^{(h,2)} - v_{21}) \geq \rho(-v_{21}),$$

as Player 2 could choose not to bid after history h and for the rest of the game, so that $V_2^h \geq v_{21}$ for any h.

6.2 Proof of Theorem 1

We construct a sequence of games that converge to the game analyzed. Let $\{r_n\}_{n=1}^{\infty}$ be a decreasing sequence of real numbers in $(0, B]$ such that $r_n \to 0$. For any n, the action set for each player is $[r_n, B] \cup \{0\}$. We assume that for any $r > 0$, H is continuous on $([r, B] \cup \{0\})^m$. By Escobar (2008) there exists a behavior strategy Markov Perfect equilibrium for any n. We follow a sequence of equilibrium strategies, s^n, that converges to some strategy profile s. We now show that s is an equilibrium in the limit game in which the set of action profiles is $([0, B])^m$.

Suppose to the contrary that s is not an equilibrium in the limit game. Therefore there exist a player i and a strategy s_i' such that $U_i(s_i', s_{-i}) > U_i(s)$.

Step 1:

$$U_i(s^n) \to U_i(s)$$

We show that H is continuous on the equilibrium strategies.[21] Note that H is continuous on all bidding strategies but for the case in which all players bid zero. We will now prove that strategy profiles which put a strictly positive measure on all players bidding zero do not arise in equilibrium in the limit.

Suppose that under s there is a strictly positive measure on all players bidding zero. Let α_{-i} be the measure of the event in which all players besides player i place zero bids under s. For any s_i^n take $\hat{b}_i^n = \inf b_i^n$ such that b_i^n is in the support of s_i^n. We take a sequence of bids $b_i^{m,n}$ in the support of s_i^n such that $b_i^{m,n} \underset{m\to\infty}{\to} \hat{b}_i^n$. Compute $\alpha_i \equiv \lim_n \lim_m \bar{H}_i(b_i^{m,n}, s_{-i}^n)$.

If $\alpha_1 < \alpha_{-1}$, then for any ε, there exists an n_ε large enough such that $\lim_m \bar{H}_1(b_1^{m,n}, s_{-1}^n) < \alpha_1 + \varepsilon$ and the measure of all bids smaller than ε of all others is larger than $\alpha_{-1} - \varepsilon$. If we take ε to converge to zero, by H1, we can find a deviation of player 1 to a bid which converges to zero that will capture the gap $\alpha_{-1} - \alpha_1$ and as shown above his bilateral WTW (and hence is total willingness to win) is strictly positive and hence he would rather capture this gap. Moreover by H1 and H2, we know that this deviation will alter only a measure zero of any other probabilities of winning in the game.

Suppose $\alpha_1 \geq \alpha_{-1}$. This implies $\alpha_2 = 0$. We can repeat the same deviation for 2 whose WTW against 1 is strictly positive. We have therefore reached a contradiction to the proposed strategies.

[21] Given the compactness of strategies and utilities, the expected utility from the dynamic game will be continuous whenever H is continuous.

Step 2: For any t_i there exists some sequence $t_i^n \to t_i$ such that

$$\lim U(t_i^n, s_{-i}^n) \geq U(t_i, s_{-i})$$

If $t_i > 0$, $U(t_i^n, S_{-i}^n) \to U(t_i, S_{-i})$ as H is continuos in this case. If $t_i = 0$, and there is another player at least whose strategy has a measure zero on bids converging to zero, then H is continuous on s_{-i}^n and t_i^n. Suppose that this is not the case. Then we take some sequence $t_i^n \to 0$ and compute the probabilities of winning of each player in each sequence. We then assume this as the tie breaking rule for H when all players bid zero. Note that there exist then an open set around this particular H that would satisfy that $\lim U(t_i^n, S_{-i}^n) \geq U(0, .., 0)$.

We can now conclude the proof. By Step 1,

$$U_i(s^n) \to U_i(s)$$

and by Step 2, for s_i', there exists some sequence $s_i'^n \to s_i'$ such that

$$\lim_{t_i \to t_i} U(s_i'^n, s_{-i}^n) \geq U(s_i', s_{-i}).$$

Therefore, there must exist some n for which $U(s_i'^n, s_{-i}^n) > U_i(s^n)$, a contradiction to the sequence s^n being a sequence of equilibria.□

6.3 Proof of Theorem 2

We first prove the following Lemma.

Lemma A1: *(i) There exists an $\bar{\varepsilon} > 0$, such that for all ρ, the probability that some player $i < s(h)$ wins is larger than $\bar{\varepsilon}$. (ii) For any sequence of parameter choices $\{\rho_n\}_{n=1}^{\infty}$ with $\rho_n \to 0$ and corresponding equilibria: a. for any $j \neq 1$, w_{ji}^h is of order ρ or lower, $V_j^h \approx v_{j1}$ and w_{21}^h is of order ρ. b. w_{1j}^h is of order ρ and all bids are of order ρ or lower.*

Proof of Lemma A1:

We will prove the Lemma by induction on the state of a history.

Step 1: Proving the Lemma for histories h such that $s(h) = 2$.

(i) We will first show that the probability that Player 1 wins in such histories is bounded away from zero.

Suppose to the contrary, that for any $\varepsilon > 0$ there exists a ρ and an equilibrium in which Player 1 wins in history h with probability smaller than ε. Construct a sequence of such equilibria with sequences $\{\varepsilon_n\}_{n=1}^{\infty}$ and $\{\rho_n\}_{n=1}^{\infty}$ in which ε_n converges to zero. Without loss of generality we consider mixed strategies and suppress the notation for the element of the sequence and the history. We denote these mixed strategies by f_i^n.

As $E(\Pr(1 \text{ wins})) \equiv \bar{H}_1(\tilde{b}) = \displaystyle\int_{b_1} f_1(b_1)\bar{H}_1(b_1, \tilde{b}_{-1})db_1 < \varepsilon$, then

$$\bar{H}_1(b_1, \tilde{b}_{-1}) < k\varepsilon$$

for a measure of at least $1 - \frac{1}{k}$ of bids in the support of player 1. Choosing a sequence of $k_{\varepsilon_n} \to_{\varepsilon_n \to 0} \infty$ and $k_{\varepsilon_n}\varepsilon_n \to 0$ this implies that for a measure $1 - \frac{1}{k_{\varepsilon_n}} \to 1$ of bids, b_1^*, in the support of player 1,

$$\bar{H}_1(b_1^*, \tilde{b}_{-1}) < k_{\varepsilon_n}\varepsilon_n.$$

We now compare the utility from each bid in the support of player 1 with that from a bid of zero. Given the strategies of all other players at this period, player 1 is better off using b_1^* rather than zero only if:

$$b_1^* \leq w_{12}^h \sum_{i \neq 1} (\bar{H}_i(0, \tilde{b}_{-1}) - \bar{H}_i(b_1^*, \tilde{b}_{-1}))$$

For almost all b_1^*, by H1 and H2, $\bar{H}_i(0, \tilde{b}_{-1}) - \bar{H}_i(b_1^*, \tilde{b}_{-1}) \leq \kappa k_{\varepsilon_n}\varepsilon_n$, implying that $b_1^* < (n-1)\kappa k_{\varepsilon_n}\varepsilon_n w_{12}^h$.

Note that some players with $i > 2$ may still fight in order to defend the state 2 as the current state. Consider then all active players besides player 1. Consider such a player, j_n, along the sequence. A possible strategy for j_n is to bid $\gamma_n b_1^*$ where $\gamma_n \to \infty$ and $\gamma_n k_{\varepsilon_n}\varepsilon_n \to 0$ so that his bid converges to zero. By H1, such a bid guarantees that some agent with $i \geqslant 2$ wins (and thus maintaining state 2) with probability converging to 1 and a bid converging to zero. Thus the equilibrium strategy of all other active players with $i > 2$ must involve a maximal bid $b_{-1}^{\max *}$ with $\frac{b_{-1}^{\max *}}{w_{12}^h} \to_{\varepsilon_n \to 0} 0$.

We now reach a contradiction, as player 1 can deviate from his equilibrium strategy. To see this consider a sequence of bids b_1' such that $\frac{b_1'}{b_{-1}^{\max *}} \to_{\varepsilon_n \to 0} \infty$ and $\frac{b_1'}{w_{12}^h} \to_{\varepsilon_n \to 0} 0$. By H1, his (relative) gain is almost w_{12}^h while his (relative) cost is infinitely smaller than w_{12}^h, yielding a strictly positive benefit far enough into the sequence. Therefore, we have shown that there exists an $\bar{\varepsilon} > 0$, such that for all ρ, $\bar{H}_1(\tilde{b}) > \bar{\varepsilon}$.

(ii) *a.* Note first that for all $j \geq 2$, $w^h_{j2} = 0$. We therefore have to consider the magnitude of w^h_{j1}.

Consider an individual j, for which $v_{j1} < v_{j2}$. Note that given the particular strategies, summarized by the random variable $\tilde{\mathbf{b}}$, played at this history,

$$V^h_j = \bar{H}_1(\tilde{\mathbf{b}})v_{j1} + \rho(1 - \bar{H}_1(\tilde{\mathbf{b}}))v_{j2}) + (1-\rho)(1 - \bar{H}_1(\tilde{\mathbf{b}}))V^{(h,2)}_j - \tilde{b}_j.$$

Thus, at any SPE, we can construct an upper bound on the continuation value for any such history. To do so, we plug for the maximum continuation value also in the future, and choose the most favorable equilibrium probabilities, γ, (which are bounded in $[\bar{\varepsilon}, 1-\bar{\varepsilon}]$) but abstract from the player's actual bids:

$$\bar{V}^h_j = \frac{\gamma v_{j1} + (1-\gamma)\rho v_{j2}}{1 - (1-\rho)(1-\gamma)}$$

and hence an upper bound on the willingness to win is:

$$\begin{aligned} \bar{w}^h_{j1} &= \rho(v_{j2} - v_{j1}) + (1-\rho)(\bar{V}^h_j - v_{j1}) \\ &= \frac{\rho(v_{j2} - v_{j1})}{1 - (1-\rho)(1-\gamma)} \end{aligned}$$

and hence the willingness to win w^h_{j1} is of order ρ or lower. Note also that by bidding zero always, each such player j can guarantee v^2_{j1}. We therefore have $w^h_{j1} \geq \rho(v_{j2} - v_{j1})$ and thus w^h_{j1} is of an order ρ any such $j \neq 1$. Note that this holds for Player 2 as well and thus proves *b*.

Consider now $j's$ such that $v_{j1} > v_{j2}$, note that

$$V^h_j - v_{j1} < 0$$

and so $w^h_{j1} < 0$. This implies that for such players, at these histories, every equilibrium bid is zero. Thus following history h,

$$V^h_j = \frac{(1 - \bar{H}_1(\tilde{\mathbf{b}}))\rho(v_{j2} - v_{j1})}{1 - (1-\rho)(1 - \bar{H}_1(\tilde{\mathbf{b}}))}$$

and therefore

$$w^h_{j1} = \rho(v_{j2} - v_{j1}) + (1-\rho)\frac{(1 - \bar{H}_1(\tilde{\mathbf{b}}))\rho(v_{j2} - v_{j1})}{1 - (1-\rho)(1 - \bar{H}_1(\tilde{\mathbf{b}}))}$$

which is of order ρ.

Finally, note that as w^h_{j1} and w^h_{j2} are of order ρ or lower for all $j \neq 1$ this implies that at any history, for a sequence of parameter choices $\{\rho_n\}_{n=1}^{\infty}$ with $\rho_n \to 0$,

$$V_j^h \approx v_{j1}.$$

b. We now consider w^h_{12} and the magnitude of the bids. Note that all players other than 1 will never place bids larger than their willingness to win and hence all bids of these players are of at most order ρ. Consider now Player 1. A possible strategy for player 1, by H1, is to place a bid $K\rho$ and to win with probability $1-\varepsilon(K)$. Choosing a large enough K, his continuation utility satisfies,

$$V_1^h \geq \frac{\varepsilon(K)\rho v_{12} - K\rho}{1-(1-\rho)\varepsilon(K)}$$

and thus his willingness to win is of order ρ as (by $V_1^h \leq 0$),

$$\rho(-v_{12}) \leq w^h_{12} \leq \rho(-v_{12}) - (1-\rho)\frac{\varepsilon(K)\rho v_{12} - K\rho}{1-(1-\rho)\varepsilon(K)}.$$

Note that as all the bilateral willingess to win of all players are of at most order ρ, all bids are of at most order ρ.

This completes the proof of Step 1.

Step 2: The induction hypothesis: *Assume that for all histories h with $s(h) \leq l-1$: (i) There exists an $\bar{\varepsilon} > 0$, such that for all ρ, the probability that some player $i < s(h)$ wins is larger than $\bar{\varepsilon}$. (ii) For any sequence of parameter choices $\{\rho_n\}_{n=1}^{\infty}$ with $\rho_n \to 0$ and corresponding equilibria: a. for any $j \neq 1$, w^h_{ji} is of order ρ or lower, $V_j^h \approx v_{j1}$ and w^h_{21} is of order ρ, b. w^h_{1j} is of order ρ and all bids are of order ρ or lower.*

Step 3: Proving the Lemma for histories h with $s(h) = l$.

(i). We first show that the probability that some player $i < l$ wins is bounded away from zero.

Let ε be the probability that some player with $j < l$ wins. We will show that there cannot be a sequence of parameter choices, $\{\rho_n\}_{n=1}^{\infty}$, and equilibria such that ε converges to zero. Suppose there exists such a sequence. By arguments similar to above, almost all bids must be infinitely smaller than $\max_j \max_i w_{ji}^h$ for $j < l$ and $i \leq l$.

Now consider Player 1. His utility is at most $(1-\varepsilon)u_{1l}^h + \varepsilon \bar{u}_{1i}^h$ where $\bar{u}_{1i}^h$ is the expectations over the utility from players $i < l$ winning. On the other hand, there exists some sequence of bids b_1' with $\frac{b_1'}{\max_j \max_i w_{ji}^h} \to 0$ that guarantees winning with probability 1 in the limit, and $b_1' \to 0$. Thus, from such a deviation his utility approaches $u_{11}^h - b_1'$ so that his gain is $(1-\varepsilon)w_{1l}^h + \varepsilon \bar{w}_{1i}^h - b_1'$.

Note that w_{1l}^h is of order ρ or higher. If $\max_j \max_i w_{ji}^h = w_{j'i'}^h$ for some $i', j' < l$, then by the induction $\frac{\max_j \max_i w_{ji}^h}{w_{1l}^h}$ is bounded. Assume therefore that $\max_j \max_i w_{ji}^h = w_{j'i'}^h$ for some $j' < l$ and $i' = l$, $j \neq 1$. Note that it must be then by the induction strictly positive and at least of order ρ. Then player j' can deviate to the sequence b_1' and his gain is $(1-\varepsilon)w_{j'i'}^h + \varepsilon \bar{w}_{ji}^h - b_1'$ which is strictly positive (as $\bar{w}_{ji}^h$ is lower than $w_{j'i'}^h$ and is of at most order ρ by the induction), a contradiction. Therefore, we have proved that there exists a $\bar{\varepsilon} > 0$ such that $\varepsilon > \bar{\varepsilon} > 0$.

(ii) *a.* Note that $w_{21}^h = \rho(-v_{21}) + (1-\rho)(V_2^{(h,2)} - v_{21})$ and that $s(h,2) = 2$ and thus by Step 1, $V_2^{(h,2)} \to v_{21}$ and so w_{21}^h is of order ρ.

We now show that w_{ji}^h is of order ρ or lower for all $j \neq 1$ (with the exception of w_{21}^h) and that $V_j^h \approx v_{j1}$ when $\rho \to 0$.

Note that $w_{ji}^h = 0$ whenever $j, i \geq s(h)$ and that by the induction hypothesis w_{ji}^h for $j, i < s(h)$ is of order ρ or lower. We now show that for all other cases, w_{ji}^h is of order ρ or lower.

For brevity let the (expected) probability that the state remains l be $1 - z$. In addition, let p_k^h be the expected probability that player $k < l$ wins after history h given the equilibrium strategies, with $\sum_{k<l} p_k^h = z$. We have two cases to consider.

Case 1: Consider first w_{ji}^h for $j \geq l > i$, where $w_{ji}^h = \rho(v_{jl} - v_{ji}) + (1-\rho)(V_j^{(h,j)} - V_j^{(h,i)})$ where $s(h,j) = l$ and $s(h,i) = i$.

By similar arguments to the analysis for $s(h) = 2$, we have:

$$\begin{aligned} V_j^{(h,j)} - V_j^{(h,i)} &\lessapprox \frac{\rho((1-z)v_{jl} + \sum_{k<l} p_k^l v_{jk}) + (1-\rho)\sum_{k<l} p_k^l v_{j1}}{1-(1-\rho)(1-z)} - v_{j1} \\ &= \frac{\rho(1-z)v_{jl} + \rho \sum_{k<l} p_k^l v_{jk} - v_{j1}\rho}{1-(1-\rho)(1-z)} \end{aligned}$$

and hence the willingness to win, w_{ji}^h, is of at most order ρ if it is positive. On the other hand, we want to make sure it is not a negative number of order higher than ρ. To find a lower bound for the continuation utility, we plug for the minimum continuation utility in the future and keep player $j's$ equilibrium bids:

$$V_j^{(h,j)} - V_j^{(h,i)} \gtrapprox \frac{\rho(1-z)v_{jl} + \rho \sum_{k<l} p_k^l v_{jk} - v_{j1}\rho - \tilde{b}_j}{1-(1-\rho)(1-z)}$$

But note that $\tilde{b}_j < \max_{i'} w_{ji'}^h$, assume this is $w_{ji^*}^h$ (note that $i^* < l$). We now establish the magnitude of $w_{ji^*}^h$; this will allow us to bound and thus w_{ji}^h.

To establish the magnitude of $w_{ji^*}^h$, we plug it as the maximum payment instead of $\tilde{b}_j$, and then obtain:

$$w_{ji^*}^h \geq \rho(v_{jl} - v_{ji^*})\frac{1-(1-\rho)(1-z)}{2-(1-\rho)(1-z)} + (1-\rho)\frac{\rho(1-z)v_{jl} + \rho \sum_{k<l} p_k^l v_{jk} - v_{j1}\rho}{2-(1-\rho)(1-z)}$$

implying that $w_{ji^*}^h$ is of order ρ. This implies that $\tilde{b}_j$ is of at most order ρ and thus more generally w_{ji}^h for all $i < l$ is of order ρ or lower.

Note also that:

$$w_{ji}^h = \rho(v_{jl} - v_{ji}) + (1-\rho)(V_j^{(h,j)} - V_j^{(h,i)})$$

where by the induction $V_j^{(h,i)} \approx v_{j1}$ and as w_{ji}^h is of order ρ or lower we have that $V_j^h \approx v_{j1}$.

Case 2: Consider now w_{jk}^h for $j < l \leq k$: we have $w_{jk}^h = \rho(-v_{jl}) + (1-\rho)(V_j^{(h,j)} - V_j^{(h,k)})$ where $s(h,j) = j$ and $s(h,k) = l$. We now consider some strategy b for player j. For this strategy $V_j^{(h,k)}(\tilde{\mathbf{b}}_{-j}, b) \geq V_j^{(h,k)}$, i.e., continuation utility must be at least as high in equilibrium as under the bid b. We choose b in the following way. If in equilibrium, following a zero bid, the probability that the state remains l is less than 1, we choose $b = 0$. If it is not, then there must exist some K large enough, so that by H1, if a bid $b = K\rho$ is placed in equilibrium, the probability that the state remains l is less than 1, as players with $i \geq l$ place bids of at most order ρ by the above. We then choose such sequence of $K's$ so that $b = K\rho$ and the probability that state l is maintained is bounded from above by $1 - z(b)$. With the chosen b, using $V_j^{(h,j)} \approx v_{j1}$ for any $i < l$, we have that

$$V_j^{(h,k)} \gtrapprox \frac{\rho(1-z(b))v_{jl} + \rho \sum_{k<l} p_k^l(b) v_{jk} + (1-\rho)\sum_{k<l} p_k^l(b) v_{j1} - b}{1-(1-\rho)(1-z(b))}$$

We then have

$$V_j^{(h,j)} - V_j^{(h,k)} \lessapprox \frac{\rho(1-z(b))(v_{j1}-v_{jl}) + \rho\sum_{k<l} p_k^l(b)(v_{j1}-v_{jk}) + b}{1-(1-\rho)(1-z(b))}$$

and thus w_{jl}^h is of at most order ρ.

To compute an upper bound for the continuation utilities, we maintain the equilibrium probabilities (as if player j is using his equilibrium strategy) but set player $j's$ bid to zero, so that:

$$V_j^{(h,k)} \lessapprox \frac{\rho(1-z)(v_{jl}) + \rho\sum_{k<l} p_i^l v_{jk} + (1-\rho)\sum_{k<l} p_k^l v_{j1}}{1-(1-\rho)(1-z)}$$

and thus w_{jl}^h is of at least of order ρ. This also implies that the equilibrium bids of all players (other than 1) are of at most order ρ. Note also that as by the induction $V_j^{(h,j)} \approx v_{j1}$ and as we have shown that w_{jk}^h is of order ρ or lower, we also have that $V_j^h \approx v_{j1}$.

(ii) b. We now consider the magnitude of the bids and show that w_{1j}^h is of order ρ. In particular we have to show that his WTW is at most of order ρ as we know trivially that it is at least of order ρ.

At state l, for all players other than player 1, w_{ij}^h at any history is of at most order ρ and thus their bids will be of at most order ρ. We can, as above, find some bid $b = K\rho$ which wins against all other bids with probability $1-\varepsilon(K)$. We then have:

$$V_1^h \geq \rho\varepsilon(K)\min_{0<j\leq l} v_{1j} + (1-\rho)\varepsilon(K)\min_{0<j\leq l} V_1^{(h,j)} - K\rho$$

Thus the maximum willingness to win of player 1 is of order ρ so that his bids are of order at most ρ as well.

This completes the proof of Step 3 and of Lemma A1.■

We can now prove the Theorem. Lemma A1 establishes (i). We now show (ii). Suppose by way of contradiction that at some history h with $s(h) = l$, there is a sequence of parameter choices and equilibrium strategies such that Player 1 wins with probability $1-\varepsilon$ converging to one. Similar arguments as in Lemma A1 imply that for all $j \neq 1$, for almost all bids b_j^* in the support of j, $\frac{b_j^*}{\max_{i\leq l} w_{ji}^h} \to 0$ and thus Player 1's bids must satisfy $\frac{b_1^*}{\max_j \max_{i\leq l} w_{ji}^h} \to 0$. Now consider Player 2 for whom $w_{21}^h > 0$ and is of order ρ by Lemma A1. Player 2 can deviate to some bid b_2' with $\frac{b_2'}{\max_j \max_{i\leq l} w_{ji}^h} \to 0$

and $\frac{b_2'}{b_1^*} \to \infty$ which will guarantee winning with probability converging to 1, and therefore, relative to his equilibrium strategy, a gain of at least $(1-\varepsilon)w_{21}^h + \varepsilon\bar{w}_{2j}^h - b_2'$ which is strictly positive as $\bar{w}_{2j}^h$ is of order ρ or lower, a contradiction.

Now suppose that at some history h with $s(h) = l$, there is a sequence of parameter choices and equilibrium strategies such that player $k < l, k \neq 1$ wins with probability $1-\varepsilon$ converging to one. We can repeat the same analysis as above with Player 1 now deviating, for whom $w_{1k}^h > 0$ and is of order ρ and thus a deviation to overcome player k will provide him a strictly positive gain, a contradiction.

This completes the proof of Theorem 2.■

6.4 Proofs for applications

We start with useful results about MPE in which only two players are active. With some abuse of notation, we denote the WTW (continuation value) of player i vis a vis player j as w_{ij}^s (V_i^s), i.e., replace the history superscript with that of the state, which is the only information relevant for an MPE.

Lemma A2: *Consider all-pay-auctions. (i) Suppose m=2. In the unique MPE, at any state s,* $\min\{w_{ij}^s, w_{ji}^s\} > 0$ *and the players' cumulative distribution over bids* F_i *and* F_j *are determined by (for* $\min\{w_{ij}^s, w_{ji}^s\} = w_{ij}^s$*):*

$$F_j(b) = \frac{b}{w_{ij}^s};\ F_i(b) = \frac{w_{ji}^s - w_{ij}^s + b}{w_{ji}^s} \text{ for all } b \in [0, w_{ij}^s]$$

(ii) For any m, suppose that in equilibrium only players i and j are active and behave as in (i). Then for any other player k, if $w_{ki}^s + w_{kj}^s > 0$*, then* $\bar{H}_i(\tilde{b}_i, \tilde{b}_j, \mathbf{b}_{-\{i,j\}})w_{ki}^s + (1 - \bar{H}_i(\tilde{b}_i, \tilde{b}_j, \mathbf{b}_{-\{i,j\}}))w_{kj}^s \leq w_{ij}^s$ *where* $\mathbf{b}_{-\{i,j\}} = \mathbf{0}$.

Proof of Lemma A2: (i) Consider the first order conditions for player i and j :

$$\begin{aligned} f_j(b)w_{ij}^s &= 1 \\ f_i(b)w_{ji}^s &= 1. \end{aligned}$$

These imply the form of the distribution function above, with an atom on zero for F_i.[22] (ii) For some player k, any utility maximizing bid must satisfy the first

[22] Standard arguments from auction theory imply the continuity, non atomness and same support of the distribution functions used in equilibrium.

order condition $f_i F_j w^s_{ki} + f_j F_i w^s_{kj} - 1 = 0$. The second order condition, using (i), is $f_i f_j (w^s_{ki} + w^s_{kj}) \geq 0$. Hence utility maximizing bids are either 0 or the maximum bid which is w^s_{ij}. So for player k not to enter, we must have that his utility from a bid of zero is higher than the utility from the maximum bid, which implies the condition in the Lemma.■

Lemma A3: *Consider the simple Tullock function. Suppose that at some state s, only two players, i and j, are active in a pure strategy MPE. Then (i) $H_i(b_i, b_j, \mathbf{b}_{-\{i,j\}}) w^s_{ji} = H_j(b_i, b_j, \mathbf{b}_{-\{i,j\}}) w^s_{ij} = b_i + b_j$ where $\mathbf{b}_{-\{i,j\}} = \mathbf{0}$, and (ii) for any other player k, $H_j(b_i, b_j, \mathbf{b}_{-\{i,j\}}) w^s_{kj} + (1 - H_j(b_i, b_j, \mathbf{b}_{-\{i,j\}})) w^s_{ki} \leq H_j(b_i, b_j, \mathbf{b}_{-\{i,j\}}) w^s_{ij}$.*

Proof of Lemma A3: The conditions in (i) are the first order conditions for i and j that must hold in equilibrium (and are also sufficient for the simple Tullock function). For player k, expected utility from the equilibrium, given some bid b is

$$H_i(b_k, b_i, b_j, 0, .., 0)(-w^s_{ki}) + H_j(b_k, b_i, b_j, 0, .., 0)(-w^s_{kj}) - b_k$$

and the first order condition is

$$\frac{H_i(b_k, b_i, b_j, 0, .., 0)}{(b_k + b_i + b_j)} w^s_{ki} + \frac{H_j(b_k, b_i, b_j, 0, .., 0)}{(b_k + b_i + b_j)} w^s_{kj} - 1$$

Note that if the first order condition is positive at some point, then it also must be positive for $b_k = 0$. Thus, to check a possible deviation, it is sufficient to check that the condition is positive at $b_k = 0$. This together with the conditions in (i) of players i and j imply the condition in (ii).■

Proof of Proposition 2: We first solve for the biased all-pay-auction in the one-shot game with different valuations, v_1 and v_2, satisfying $v_2 < v_1$. It is then easy to see that, in the spirit of Hillman and Riley (1989), Player 2 places a bid of zero with measure $1 - \alpha \frac{v_2}{v_1}$, and with measure $\alpha \frac{v_2}{v_1}$ mixes uniformly on $(0, v_2)$. For player 1 the largest bid is αv_2 and hence he mixes uniformly on $(0, \alpha v_2)$. The wining probability of player 1 is $1 - \alpha \frac{v_2}{v_1} + \int_0^{\alpha v_2} \frac{b}{v_1} \frac{1}{\alpha v_2} db = 1 - \frac{\alpha v_2}{2 v_1}$.

Now consider the dynamic game with equal per-period valuation v. We have:

$$\begin{aligned} w^2_{12} &= \rho(-v) + (1-\rho)(-V^2_1) \\ w^2_{21} &= \rho(-v) + (1-\rho)(V^2_2 - v) \end{aligned}$$

Conjecture that player 2 places an atom on zero and hence $V^2_2 = v$ so $w^2_{21} = \rho(-v)$ and hence as $V^2_1 = \frac{\alpha \rho^2 v^2}{v_1 + (1-\rho)\alpha \rho v}$, and thus $w^2_{12} = \rho(-v)(1 + \alpha(1-\rho))$. Plugging w^2_{12} for

v_1 and w_{21}^2 for v_2 in the one-shot game solution above yields the result, for which the probability of Player 1 winning is $1-\frac{\alpha}{2(1+\alpha(1-\rho))}$. Finally, conjecturing that player 1 places an atom on zero yields a contradiction.■

Proof of Proposition 3: *All-pay-auctions:* We compute first the equilibrium for $s=2$. Only players 1 and 2 can be active, and it is easy to see that player 3 does not wish to deviate (as this can just increase the probability of player 2 winning). The analysis follows Lemma A2: We conjecture that in equilibrium player 2 has a lower willingness to win[23], and is therefore the one who places an atom on zero of size $1-\frac{w_{21}^2}{w_{12}^2}$, where

$$\begin{aligned} w_{21}^2 &= \rho x_2 + (1-\rho)(V_2^2 - V_2^1), \\ w_{12}^2 &= \rho x_2 + (1-\rho)(V_1^1 - V_1^2). \end{aligned}$$

By Lemma A2,

$$V_2^1 = -x_2;\ V_1^2 = \frac{w_{21}^2}{w_{12}^1}(-\rho x_2 + (1-\rho)V_1^2)$$

and plugging for these values, we can solve for the ratio $\frac{w_{21}^2}{w_{12}^2} = \frac{1}{2-\rho}$, implying that the atom is of size $\frac{1-\rho}{2-\rho}$. Thus, $V_1^2 = -\rho x_2$, $V_2^2 = -x_2$, and $V_3^2 = \frac{-3|x_3| - x_2\rho + |x_3|\rho}{3-\rho}$.

Now consider the equilibrium in which players 2 and 3 only are active at $s=3$ and both ignore Player 1. The willingness to win of each player is:

$$\begin{aligned} w_{23}^3 &= \rho(|x_3| + x_2) + (1-\rho)(V_2^2 - V_2^3); \\ w_{32}^3 &= \rho(|x_3| + x_2) + (1-\rho)(V_3^3 - V_3^2); \end{aligned}$$

Conjecture that the atom on zero is on player 3 (the opposite cannot arise). Let the size of the atom be δ. Then:

$$\begin{aligned} V_2^3 &= \delta(1-\rho)(-x_2) + (1-\delta)(-\rho(|x_3| + x_2) + (1-\rho)V_2^3 \\ V_2^3 &= \frac{\delta(1-\rho)(-x_2) - (1-\delta)\rho(|x_3| + x_2)}{1-(1-\delta)(1-\rho)} \\ V_2^2 - V_2^3 &= \frac{\rho(|x_3|(1-\delta) - \delta x_2)}{\delta(1-\rho)+\rho} \\ V_3^3 - V_3^2 &= -\rho(|x_3| + x_2 + V_3^2) \end{aligned}$$

We can solve for $\delta = 1 - \frac{w_{32}^3}{w_{23}^3}$ to find:

$$\delta(\rho) = \frac{3|x_3| + 3x_2\rho - 4|x_3|\rho - 5x_2\rho^2 + 2x_2\rho^3 + |x_3|\rho^2}{6|x_3| + 7x_2\rho - 5|x_3|\rho - 7x_2\rho^2 + 2x_2\rho^3 + |x_3|\rho^2} \underset{\rho\to 0}{\longrightarrow} \frac{1}{2}.$$

[23] Indeed, conjecturing the opposite leads to a contradiction.

Now assume that Player 1 is not active, but is part of the game. When Player 2 considers placing a bid of zero, given the atom of Player 3 on zero, Player 1 may now win with some probability. Note however that it is not optimal for Player 2 to place a bid on zero, as a marginal bid wins against both other players, a profitable deviation for Player 2, as his WTW against each is positive. This implies that the analysis above is valid for the case in which Player 1 exists but is not active.

Note that $\delta(\rho) \geq 0$ for all ρ. This allows to compute

$$\begin{aligned} & \frac{1+\delta(\rho)}{2} w_{12}^3 + \frac{1-\delta(\rho)}{2} w_{13}^3 \\ = \; & \rho\left(\frac{1+\delta(\rho)}{2} x_2 + \left(\frac{1-\delta(\rho)}{2}\right)|x_3|\right) + \\ & (1-\rho)\left(-\frac{1+\delta(\rho)}{2} V_1^2 - \left(\frac{1-\delta(\rho)}{2}\right) V_1^3\right) \end{aligned}$$

where

$$V_1^3 = -\frac{\frac{1+\delta}{2}(\rho x_2 + (1-\rho)\rho x_2) + \frac{1-\delta}{2}\rho|x_3|}{1-(1-\rho)\frac{1-\delta}{2}}.$$

Recall that the willingness to win of player 1 vis a vis any other player is positive. We can therefore use the condition in Lemma A2. To check that $\frac{1+\delta(\rho)}{2} w_{12}^3 + \frac{1-\delta(\rho)}{2} w_{13}^3 < w_{23}^3$ we note that the *lhs* is maximal for $\rho \to 0$. We therefore compute $\lim_{\rho \to 0}[\frac{1+\delta(\rho)}{2} w_{12}^3(\rho) + \frac{1-\delta(\rho)}{2} w_{13}^3(\rho) - w_{32}^3(\rho)] = 3x_2 - |x_3|$ to get the required condition on the relative size of x_3 and x_2.

Simple Tullock functions: We first compute the equilibrium when $s = 2$ and players 1 and 2 compete. The expected utility of player 1 from bid b_1 is

$$\frac{b_1}{b_1+b_2} u_{11}^2 + \frac{b_2}{b_1+b_2} u_{12}^2 - b_1$$

where the first order condition is

$$\frac{b_2}{(b_1+b_2)^2} u_{11}^2 - \frac{b_2}{(b_1+b_2)^2} u_{12}^2 - 1 = 0 \tag{1}$$

which together with the *foc* for player 2 implies that

$$\frac{w_{12}^2}{w_{21}^2} = \frac{b_1}{b_2} \tag{2}$$

where,

$$\begin{aligned}
w_{12}^2 &= \rho x_2 + (1-\rho)(-V_1^2) \\
w_{21}^2 &= \rho x_2 + (1-\rho)(V_2^2 + x_2) \\
V_1^2 &= \frac{\frac{b_2}{b_1+b_2}(-\rho x_2) - b_1}{1 - \frac{b_2}{b_1+b_2}(1-\rho)} \\
V_2^2 + x_2 &= \frac{-b_2 + x_2 \frac{b_2}{b_1+b_2}\rho}{1 - \frac{b_2}{b_1+b_2}(1-\rho)}.
\end{aligned}$$

Solving the system of the first order equations (1) and (2), we find that:

$$\begin{aligned}
b_1 &= \rho b_2 \frac{x_2}{-2b_2 + 2\rho b_2 + \rho x_2} \underset{\rho\to 0}{\approx} \frac{\rho x_2}{2} \\
b_2 &= x_2 \frac{\sqrt{-\rho + \rho^2 + 1} - 1}{2\rho - 2} \underset{\rho\to 0}{\approx} \frac{\rho x_2}{4}
\end{aligned}$$

In the limit, as $\rho \to 0$:

$$\begin{aligned}
V_1^2 &\approx -\frac{5\rho x_2}{4}, \\
V_2^2 + x_2 &\approx -\frac{3\rho x_2}{8} \\
V_3^2 &\approx (-x_3) + \frac{1}{2}\rho(-x_3 - x_2)
\end{aligned}$$

Now consider the game between players 2 and 3 at $s = 3$. Let d_i be the equilibrium bid of player i. First order conditions for 2 and 3 are:

$$\begin{aligned}
\frac{w_{23}^3}{w_{32}^3} &= \frac{d_2}{d_3} \\
\frac{d_3}{(d_2+d_3)^2} w_{23}^3 &= 1
\end{aligned}$$

where,

$$\begin{aligned}
w_{23}^3 &= \rho(x_2 + x_3) + (1-\rho)(V_2^2 - V_2^3) \\
w_{32}^3 &= \rho(x_2 + x_3) + (1-\rho)(V_3^3 - V_3^2) \\
V_2^2 - V_2^3 &= \frac{\frac{-b_2 - \frac{b_1}{b_1+b_2}x_2}{1-\frac{b_2}{b_1+b_2}(1-\rho)}\rho + d_2 + \frac{d_3}{d_2+d_3}\rho(x_2+x_3)}{1 - \frac{d_3}{d_2+d_3}(1-\rho)} \\
V_3^3 - V_3^2 &= \frac{-d_3 - \frac{d_2}{d_2+d_3}\rho(x_2+x_3) - \rho\frac{\frac{b_1}{b_1+b_2}(-x_3) + \frac{b_2}{b_1+b_2}\rho(-x_3-x_2)}{1-(1-\rho)\frac{b_2}{b_1+b_2}}}{1 - \frac{d_3}{d_2+d_3}(1-\rho)}
\end{aligned}$$

With these expressions we solve the set of first order conditions above in the limit as $\rho \to 0$ we find that

$$\begin{aligned} d_2 &\approx \frac{1}{2} x_3 \rho \\ d_3 &\approx \frac{1}{4} x_3 \rho \end{aligned}$$

Now consider a deviation from player 1. The expected utility of player 1 from a bid b given the equilibrium is:

$$\begin{aligned} u_1(b) &= \frac{d_2}{b + d_2 + d_3}(\rho(-x_2) + (1-\rho)V_1^2) \\ &\quad + \frac{d_3}{b + d_2 + d_3}(\rho(-x_3) + (1-\rho)V_1^3) - b \end{aligned}$$

where:

$$\begin{aligned} V_1^3 &= \frac{\frac{d_2}{d_2+d_3}(\rho(-x_2) + (1-\rho)V_1^2) + \frac{d_3}{d_2+d_3}\rho(-x_3)}{1 - (1-\rho)\frac{d_3}{d_2+d_3}} \\ &\underset{\rho \to 0}{\approx} \frac{\frac{2}{3}(\rho(-x_2) - (1-\rho)\frac{5\rho x_2}{4}) + \frac{1}{3}\rho(-x_3)}{1 - (1-\rho)\frac{1}{3}} \end{aligned}$$

Plugging this and V_1^2 in his expected utility, assuming that $b = \gamma \rho x_3$, dividing utility by ρ, and taking the limit, we get:

$$\lim_{\rho \to 0} \frac{U_1(b)}{\rho} = -\gamma x_3 - 0.25 \frac{x_3}{0.75 x_3 + \gamma x_3}(2.25 x_2 + 1.5 x_3) - 1.125 x_2 \frac{x_3}{0.75 x_3 + \gamma x_3}$$

The derivative of this expression evaluated at $\gamma = 0$ is,

$$\frac{\partial(\lim_{\rho \to 0} \frac{U_1(\gamma)}{\rho})|_{\gamma=0}}{\partial \gamma} < 0 \Leftrightarrow x_3 > 9x_2$$

Note that if $\frac{\partial(\lim_{\rho \to 0} \frac{U_1(\gamma)}{\rho})}{\partial \gamma}$ is positive, it is positive for $\gamma = 0$. Therefore we conclude that for small ρ an equilibrium will exist iff $x_3 > 9x_2$.■

Proof of Proposition 4: We first prove:

Claim A1: *Suppose $x_3 > 0$ and that at $s = 2$, players 1 and 2 compete and that at $s = 3$, players 2 and 3 compete. In both the all-pay-auction and the simple Tullock function, Player 3 wins the stage game at $s = 3$ with a higher probability than he does when $x_3 < 0$.*

Proof of Claim A1: Let x denote the distance between players 2 and 3. We want to show that the probability that 3 wins the game with 2 is decreasing with x

when $|x_3| > x_2$ and x_2 is fixed. Consider the equilibrium for some x and now decrease x. If the players use the same strategies, for both the instantaneous gain increases by $\rho\Delta x$. Let z be the (expected) probability that 2 wins. The future payoffs:

$$V_2^2 - V_2^3 = V_2^2 - \frac{z(1-\rho)V_2^2 - (1-z)\rho x - \tilde{b}_2}{1-(1-z)(1-\rho)} = \frac{\rho V_2^2 + (1-z)\rho x + \tilde{b}_2}{1-(1-z)(1-\rho)}$$

as V_2^2 remains the same, this decreases with x when strategies remain the same. Thus it decreases by $\frac{(1-z)\rho}{1-(1-z)(1-\rho)}\Delta x$. For player 3:

$$\begin{aligned} V_3^3 - V_3^2 &= \frac{z(1-\rho)V_3^2 - z\rho x - \tilde{b}_3}{1-(1-z)(1-\rho)} - V_3^2 = \frac{-\rho V_3^2 - z\rho x + \tilde{b}_3}{1-(1-z)(1-\rho)} \\ &= \frac{-\rho\frac{z_1|x_3| - z_2\rho x}{1-(1-\rho)z_2} - z\rho x + \tilde{b}_3}{1-(1-z)(1-\rho)} \end{aligned}$$

which changes by $\frac{\frac{z_2\rho^2}{1-(1-\rho)z_2} - z\rho}{1-(1-z)(1-\rho)}\Delta x$ when x decreases. If $\frac{z_2\rho^2}{1-(1-\rho)z_2} - z\rho < 0$, then we know that $\frac{w_{32}}{w_{23}}$ increases, implying by both Lemma A2 and Lemma A3 that player 3 has to bid more aggressively so that player 3 will win more often. Suppose that it is positive, we then want to show that $\frac{z_2\rho^2}{1-(1-\rho)z_2} - z\rho < (1-z)\rho$ which holds as $z_2 < 1$ and so we have the same result.■

Claim A1 implies that $\bar{H}_3 w_{13}^3 + (1-\bar{H}_3)w_{12}^3$ increases when we switch from $x_3 < 0$ to $x_3 > 0$. To see that, recall that distances of Player 1 from each player remain the same so the only change from his point of view is the increase in $\bar{H}_3$ which also increases w_{13}^3. Morever, $w_{13}^3 > w_{12}^3$ as $V_1^3 < V_1^2$.

The WTW of Player 3 converges, when $\rho \to 0$, to ρx_3 disregarding its distance form x_2 and so we know that the condition for full gradualism becomes harder to sustain in the all-pay-auction. Moreover, we can follow the same strategy as in the proof of Proposition 4, and find that the fully gradual equilibrium holds for all ρ iff $4x_2 < x_3$ and that player 3 wins the stage game when $s = 3$ with a higher probability than when $x_3 < 0$.

In the simple contest function, note that as H_2 decreases and w_{32}^3 is roughly the same, again it is easier that the condition for deviation which is $H_3 w_{13}^3 + (1-H_3)w_{12}^3 > H_2 w_{32}^3$ will be satisfied and so harder to sustain a fully gradual MPE.■

Proof of Proposition 5: Let α be the probability that player 2 wins in equilibrium in $s = 3$ when only players 2 and 3 are active and only they win with a strictly positive probability. We first prove:

Lemma A4: *Suppose that Player 1 is not active and wins with a zero probability at* $s = 3$*. Then for any H,* $\alpha w_{12}^3 + (1-\alpha) w_{13}^3 > w_{23}^3$.

Proof of Lemma A4:

(i) $w_{12}^2 \geq w_{21}^2$:

$$w_{12}^2 = \rho(-v) + (1-\rho)(-V_1^2) \geq \rho(-v) + (1-\rho)(V_2^2 - v) = w_2^2$$

as $V_1^2 + V_2^2 \leq v$.

(ii) $w_{21}^2 > w_{23}^3$ as $V_2^3 \geq v$:

$$w_{21}^2 = \rho(-v) + (1-\rho)(V_2^2 - v) \geq \rho(-v) + (1-\rho)(V_2^2 - V_2^3) = w_{23}^3.$$

(iii) $w_{13}^3 > w_{12}^2$:

$$w_{13}^3 = \rho(-v) + (1-\rho)(-V_1^3) > \rho(-v) + (1-\rho)(-V_1^2) = w_{12}^2$$

as $V_1^2 > V_1^3$: in state $s = 3$, as long as the game remains in this state, player 1's instantaneous utility is v, and $V_1^2 \geq v$.

From (i), (ii), and (iii), $\alpha w_{12}^3 + (1-\alpha) w_{13}^3 > w_{23}^3$.$\square$

By Lemmatta A2, A3 and A4, there exists a deviation in both the all-pay auction and the simple Tullock function whenever player 1 wins with probability converging to zero.

We will now show that an MPE exist in the all-pay-auction for N players; for convenience let us normalize the payoff so that the payoff from winning is $v > 0$ and the payoff from losing is 0.

Note that players' continuation values are at least 0 at any stage game. Second, consider $s = 2$ and note that the only players that may potentially submit strictly positive bids are 1 and 2. The atom must be on 2 and the solution involves $V_1^2 = v(1-\rho)$, $w_{12}^2 = \rho v(2-\rho)$, $V_2^2 = 0$ and $w_{21}^2 = \rho v$. Suppose, by way of induction, that for every state $l < s$, the equilibrium is as in Baye et al (1996) in which player 1 has the highest willingness to pay. In particular this implies that there is an atom on bid zero for all players beside Player 1 and so $v > V_1^l > 0$ and $V_i^l = 0$ for all $1 < i \leq l$, implying that $w_{1l}^l > \rho v$ and that $w_{ij}^l = \rho v$ for all $1 < i \leq l$, $j \leq l$.

Suppose we are now at state s. Note that players $i > s$ do not participate. For the remaining players, consider the family of equilibria as in Baye et al (1996)[24] in

[24] There is in fact a continuum of equilibria.

which (i) $\Pi_i F_i(0) = \frac{1-\rho}{2-\rho}$ for all $i \neq 1$ that participate, and so $F_i(0) > 0$ for any such i, (ii) player 1 participates in every stage and he wins with a higher probability than any other. We then have that $w_{ij}^s = \rho v + (1-\rho)(V_i^i - V_i^j) = \rho v$ for all $i \neq 1 < s, j \leq s$, and $w_{1i}^s = \rho v + (1-\rho)(v - V_1^i) > \rho v$ as $V_1^i < v$ by the induction and by the above for all $i \leq s$. The conjectured family of equilibria is therefore sustained.■

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