Reallocation Costs and Efficiency*

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Abstract

We study the efficient allocation of a divisible asset when reallocation is costly. Each of two players is initially allocated a share of the asset. At the time of this initial division the players' valuations for the asset are uncertain. After the uncertainty resolves, costly reallocation may take place. Reallocation costs may depend on the amount reallocated and on players' valuations. We first show that contracting on the initial division and the reallocation enables the players to attain the highest possible expected surplus for the specified initial division. We then show that this maximal expected surplus either monotonically increases or monotonically decreases in the larger share of the initial division for a wide range of reallocation cost specifications.

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1 Introduction

The efficient allocation of an asset in environments with uncertainty presents several challenges. When players have private information about their valuations for the asset, the initial allocation determines their outside options and hence whether a subsidy is needed to achieve ex-post efficiency. This is the focus of the literature started by Myerson and Satterthwaite (1983) and Cramton, Gibbons, and Klemperer (1987). When it is possible to invest in improving the asset but investment is not contractible, the allocation of the asset affects players' investment incentives, and hence the surplus from the asset. This is the focus of the literature started by Grossman and Hart (1986) and Hart and Moore (1990). When costly reallocation of the asset may take place after the relevant uncertainty resolves, the initial allocation determines how much will be reallocated ex-post, and hence the surplus from the asset. Studying how to initially allocate the asset in this case is the focus of our paper.

Consider, for example, a firm that wishes to allocate capital among several product lines to determine their capacity. The profitability of each line will be determined after products are put on the market, at which time the firm may wish to reallocate capital to the most profitable line. But capital reallocation is costly because it requires adjusting the product lines. Anticipating this cost, how should the firm initially allocate the capital?

This dilemma also arises in settings with strategic players. Consider, for example, partners in a law firm who begin to handle the cases of a newly-acquired client immediately, so the client will not go elsewhere. After discovering each partner's fit with the client, they may wish to reallocate cases to the partner with the better fit. But doing so is costly because it requires that partner to learn what the other partners did with their cases. How then should the partners initially allocate the cases?

Reallocation in both examples is costly because the asset changes after it is initially allocated. But the cost structure in the two examples is inherently different. In the first example, the costs may be thought of as convex in the amount reallocated. This follows from the standard argument that the firm will choose to reallocate the least costly units first if it has the flexibility to do so, which may be the case if reallocation involves adjusting production processes. But such flexibility may not be available in the context of learning legal cases. If the cases are roughly homogenous and learning amounts to understanding how the other partner handled them, then the cost of learning an additional case decreases in the number of cases that have already been learned, so the reallocation costs are concave.

Concave costs may also arise when reallocation causes a mental loss to the individual whose share decreases, as may be the case when a sibling gives up his share in a family business. In the spirit of Kahneman and Tversky's (1991) model of reference-dependent preferences, with the reference point being the individual's original share, the marginal sensitivity to losses decreases in the size of the loss, so the reallocation costs are concave. Reallocating ownership may also involve legal and other fees that are roughly independent of the amount reallocated.

Our results establish that when reallocation costs are independent of the amount reallocated, players will optimally choose as concentrated an initial division as possible. This continues to hold when the costs are concave in the amount reallocated. In contrast, when reallocation costs are convex in the amount reallocated, players will optimally choose as equal an initial division as possible.

We study a two-stage model in which two symmetric risk-neutral players with quasi-linear utilities wish to allocate between them a divisible asset. At the beginning of stage 0, the players sign a contract that specifies the initial division of the asset, its final allocation, and transfers. The initial division is then carried out, when players' valuations for the asset are still uncertain. Between stage 0 and stage 1 players' valuations are realized, either publicly or privately. In stage 1, players report their valuations and the final allocation is implemented.

When the final allocation differs from the initial division, the players incur reallocation costs that may depend on the amount reallocated and on their valuations for the asset. In the partners example, the dependency on valuations arises because the effort of learning may depend on both partners' fit with the client. Similarly, the emotional loss associated with losing ownership may depend on the individual's valuation for the asset.

Note that the initial division is carried out before players' valuations are realized. This fits settings in which postponing the initial division is too costly, as in the examples above. When this is not the case, the tradeoff between early and delayed allocation can be determined by comparing the value of early allocation, which follows from our analysis, to the first-order statistic of players' valuations net of the delay costs, which is the value of delayed allocation.

In Section 2 we observe that when players' valuations are realized publicly, the final allocation specified in the contract maximizes the sum of players' expected utilities – or the expected surplus – relative to the initial division. Otherwise, there exists a final allocation that generates a higher surplus, which can be shared by the players via ex-ante transfers. When valuations are realized privately, things are more complicated because a player's utility may depend on the other player's valuation through the reallocation costs. We show that maximizing the expected surplus is still possible provided that each player's share of the reallocation costs satisfies an appropriate increasing differences condition. The remainder of the paper thus studies the relationship between the initial division specified in the contract and the maximal expected surplus.

The initial division influences the maximal expected surplus because it determines the maximal amount that can be reallocated. This maximal amount affects the tradeoff between the cost and benefit of reallocation, and hence the actual amount reallocated. In contrast, when reallocation is costless the initial division does not influence the maximal expected surplus, because the efficient final allocation always awards the entire asset to the player with the higher valuation.

Section 3 relates the maximal expected surplus to the initial concentration, which is the larger share in the initial division. For a wide range of reallocation cost functions, we show that the maximal expected surplus is constant when the initial concentration is smaller than some threshold, and is then strictly monotonic in the initial concentration. It is monotonically increasing if the cost function is insensitive to the amount reallocated or concave in the amount reallocated, and it is monotonically decreasing if the cost function is convex in the amount reallocated. This is true regardless of how valuations are distributed and how they affect the reallocation costs. These parameters only affect the threshold above which the monotonicity becomes strict.

For an illustration, consider the case of constant reallocation costs. In this case, for any realization of valuations it is efficient either to reallocate the entire asset to the player with the higher valuation or to maintain the initial division. Suppose that player 1 has the larger initial share, and consider how increasing his initial share affects the efficient final allocation and its expected surplus. First, it changes the set of realizations for which reallocation is optimal. Second, it increases the ex-post surplus generated when the initial division is maintained and player 1 has the higher valuation. Third, it decreases the ex-post surplus generated when the initial division is maintained and player 2 has the higher valuation. The second effect dominates the third because the likelihood that the initial division is maintained is larger when player 1 has the larger valuation than when player 2 does. This is because the benefit of reallocating the asset to a player decreases in the player's initial share. Thus, there is an increase in expected surplus that results from situations in which the initial division is maintained. The first effect further increases the expected surplus.

When reallocation involves an amount insensitive component and a convex variable component, the maximal expected surplus need not be monotonic in the initial concentration. This is because the amount insensitive component pushes toward a more concentrated initial division whereas the convex variable component pushes toward a less concentrated initial division. We show that when one component is "sufficiently dominant," the optimal initial division is still the equal division or a fully concentrated one. We also provide an example in which neither component dominates and the optimal initial division is neither the equal division nor a fully concentrated one.

When there are more than two players, we say that one initial division is more concentrated than another if the first majorizes the second. That is, for any k smaller than the number of players, the sum of the largest k shares in the first division is larger than the sum of the largest k shares in the second division. Similarly to the two-player case, we show that the maximal expected surplus increases in the initial concentration when the reallocation costs are amount insensitive or concave, so the fully concentrated initial divisions are optimal, and decreases in the initial concentration when the reallocation costs are convex, so the equal initial division is optimal.

The main takeaway from our analysis is that reallocation costs affect the initial allocation of a divisible asset in a systematic way. When reallocation involves significant fixed costs, as with ownership transfer, our results predict a highly concentrated initial division. The same prediction applies when reallocation costs are concave, as may be the case with reassigning workload in law firms and other professional partnerships. When reallocation costs are convex, as with reallocating capital or other physical resources, our results predict a relatively equal initial division.

This paper is related to several literatures. In the literature on the implementation of ex-post efficiency, the question is whether there exists an interim incentive-compatible mechanism that awards the asset to the player with the highest valuation while satisfying players' interim participation constraints and without incurring a deficit. In the private-value setting, Myerson and Satterthwaite (1983) showed that under weak conditions the answer is "no" when the asset is initially owned by one of the players, and Cramton, Gibbons, and Klemperer (1987) showed that the answer is "yes" when players are ex-ante symmetric and the initial shares are sufficiently close to being equal. Environments with interdependent valuations have been studied by Fiesler, Kittsteiner, and Moldovanu (2003), Jehiel and Pauzner (2006), and Segal and Whinston (2011).

Our paper differs from this literature in two respects. First, because of the reallocation costs, what is ex-post efficient in our model and the maximal expected surplus depend not only on players' valuations but also on the initial division. Understanding this dependency is the focus of our paper. Second, a deficit can be avoided in our model. This is because players have no property rights or private information at the contracting stage, and are therefore willing to fund a mechanism that is executed after they obtain their private information.

Another related literature is the literature on the property rights approach to the theory of the firm, pioneered by Grossman and Hart (1986) and Hart and Moore (1990). This literature studies how to allocate asset ownership to induce non-contractible investments. In our model, by way of contrast, there are no explicit investments. The initial division of the asset affects the ex-post surplus because reallocation is costly. Our analysis can therefore be viewed as an investigation of another channel through which the ownership structure affects the ex-post surplus from the asset.

There is also a finance literature that studies costly reallocation in the context of rebalancing a financial portfolio. Typical models in this literature study dynamic settings in which an investor can frequently rebalance his investments in a risk-free asset and a risky asset whose value is determined by a random process. The transaction costs involved in rebalancing may be proportional (linear), constant, or a fraction of the portfolio value.[1] Our work differs from

[1] Cadenillas (2000) surveys infinite horizon, continuous time models of portfolio rebalancing in which a risky asset's value is determined by Brownian motion. There are also two-period models in which the portfolio can be rebalanced between the two periods. Mitchell and Braun (2004) consider this problem in the presence of convex transaction costs, and characterize for a risk-averse investor the efficient frontier that trades off risk and expected

this literature in two respects. First, because we are interested in understanding the effect of reallocation costs in general, rather than in a particular application, our specification of reallocation costs is richer – and of the rest of the environment is simpler – than in this literature. Second, we study how different forms of reallocation costs influence the connection between the initial division of the asset and the surplus generated by optimally reallocating it, rather than taking the initial division as given (if it is at all relevant) and focusing only on optimal rebalancing.

2 Contracting environment

Two risk-neutral players with quasi-linear utilities wish to allocate a divisible asset of size 1 between them. At the beginning of stage 0, the players sign a contract that specifies the initial division of the asset, its final allocation, and transfers. The initial division and any stage 0 transfers are then carried out. At this stage players' valuations for the asset are uncertain and are expected to be drawn from a symmetric joint probability distribution. Between stage 0 and stage 1 players' valuations are realized – either publicly or privately. In stage 1, the final allocation is implemented. There is no discounting between periods.[2]

2.1 Publicly realized valuations

The contract (S, x, t) specifies the following:

1. A stage 0 *initial division* $S = (s, 1 - s)$ of the asset, where $s \in [0, 1]$ is player 1's initial share and $1 - s$ is player 2's initial share.

2. A stage 0 transfer $t \in \mathbb{R}$ from player 1 to player 2 (if t is negative, then $|t|$ is transferred from player 2 to player 1).

3. A stage 1 *final allocation* that specifies, for each public realization (v_1, v_2) of players' valuations, the final share $x(v_1, v_2)$ of player 1 (player 2's share is $1 - x(v_1, v_2)$).

When the final allocation differs from the initial division, each player may incur reallocation costs. These costs may depend on the amount reallocated and on players' valuations, but not on players' identities. The cost of reallocating an amount $\Delta > 0$ from a player with valuation v_L (the Losing player) to a player with valuation v_G (the Gaining player) is $C^L(v_L, v_G, \Delta) \geq 0$

return. Dybvig (2005) considers a similar problem with linear or fixed costs when the investor has mean-variance preferences.

[2]This assumption does not affect any of our results.

to the losing player and $C^G(v_L, v_G, \Delta) \geq 0$ to the gaining player. The total cost $C(v_L, v_G, \Delta) = C^L(v_L, v_G, \Delta) + C^G(v_L, v_G, \Delta)$ is strictly positive.

The per-period utility of a player with valuation v from a share y of the asset is yv. Player 1's expected utility from the initial division and the transfer specified in the contract is $E(sv_1) - t$, where the expectation is taken with respect to the distribution of players' valuations. Given a realization (v_1, v_2) of players' valuations, player 1's stage 1 utility from the contract is

$$u_1(v_1, v_2, x, s) = x(v_1, v_2)v_1 - 1_{x(v_1,v_2)<s}C^L(v_1, v_2, s - x(v_1, v_2)) - 1_{x(v_1,v_2)>s}C^G(v_2, v_1, x(v_1, v_2) - s),$$

where the first term is the player's benefit from the final allocation x, and the other two terms are the player's cost when he loses $s - x(v_1, v_2)$ or gains $x(v_1, v_2) - s$. Player 1's total expected utility from the contract is $E(sv_1 + u_1(v_1, v_2, x, s)) - t$. Player 2's expected utility from the contract is defined similarly.[3]

Because players are ex-ante symmetric, the sum of their expected utilities from the initial division and the transfer, $E[sv_1 + (1 - s)v_2]$, does not depend on s. We thus ignore this expectation, and refer to the sum of players' stage 1 expected utilities as the *expected surplus*:

$$E[x(v_1, v_2)v_1 + (1 - x(v_1, v_2))v_2 - 1_{x(v_1,v_2)<s}C(v_1, v_2, s - x(v_1, v_2)) - 1_{x(v_1,v_2)>s}C(v_2, v_1, x(v_1, v_2) - s)].$$

Given a realization of players' valuations we refer to the sum of players' stage 1 utilities as the *ex-post surplus*. Because any surplus generated by the final allocation can be divided between the players by adjusting the stage 0 transfer, we have the following observation.

Observation 1 *The final allocation specified in the contract maximizes the expected surplus relative to the initial division.*

One final allocation that maximizes the expected surplus is an S-*efficient* final allocation x^S that maximizes the ex-post surplus for any realization (v_1, v_2), that is,

$$x^S(v_1, v_2) \in \arg\max_{x \in [0,1]} xv_1 + (1 - x)v_2 - C(\min\{v_1, v_2\}, \max\{v_1, v_2\}, \max\{x - s, s - x\}).^4$$

Example 1 illustrates an S-efficient final allocation.

Example 1. Suppose that players' valuations are distributed independently and uniformly on $[0, 1]$, and the total reallocation cost is $C < 1/2$. For any initial division S, an S-efficient final allocation has a "bang-bang" form. It allocates the entire asset to player 1 if the average benefit of doing so, $v_1 - v_2$, is strictly larger than the average cost, $C/(1 - s)$, and to player 2 if $v_2 - v_1$ is strictly larger than C/s. Otherwise, it maintains the initial division. Because $C < 1/2$, we have that $\min\{C/s, C/(1 - s)\} < 1$, so with positive probability the final allocation differs from the initial division.$\diamondsuit$

[3] It is $E[(1-s)v_2 + (1 - x(v_1, v_2))v_2 - 1_{x(v_1,v_2)<s}C^G(v_1, v_2, s - x(v_1, v_2)) - 1_{x(v_1,v_2)>s}C^L(v_2, v_1, x(v_1, v_2) - s)] + t$.

[4] We require $C(v_1, v_2, \cdot)$ to be lower semi-continuous in its third argument so that a maximizer exists.

In the setting of Example 1, compare the specified S-efficient final allocation to the final allocation that always assigns the entire asset to the player with the higher valuation. First, the S-efficient final allocation changes with the initial division. Second, it is not symmetric for $s \neq 1/2$. Third, the expected surplus changes with the initial division even when each player's initial share is positive. None of these hold for the final allocation that assigns the entire asset to the player with the higher valuation.[5]

When the contract is incomplete in the sense that each player can walk away with his share of the asset at the beginning of stage 1, players can still maximize the expected surplus relative to the initial division. This is because the ex-post surplus in an S-efficient final allocation is higher than that in the initial division S for every realization of players' valuations, so the contract can specify stage 1 transfers between the players (that depend on the realized valuations and sum to zero) such that the players agree to the S-efficient final allocation. In fact, surplus maximization is achievable even when the contract is incomplete in the sense that it does not specify a final allocation, as long as players can bargain efficiently in stage 1. In this case, the distribution of the ex-post surplus may depend on the initial division of the asset, but anticipating this the players can use the stage 0 transfer t to compensate the player whose stage 1 bargaining position is weaker.

2.2 Privately realized valuations

Implementing an S-efficient final allocation in stage 1, and hence maximizing the expected surplus relative to the initial division S, is also possible when valuations are realized privately. To do this, the contract also specifies:

1. Stage 0 payments $t_1^0, t_2^0 \in \mathbb{R}$, made by players 1 and 2 respectively (instead of the stage 0 payment t).
2. Stage 1 payment schedules that specify, for each pair of reports (v_1, v_2) of players' valuations, the payments $t_1^1(v_1, v_2)$ and $t_2^1(v_1, v_2)$ made by players 1 and 2 respectively.

The payments can be thought of as being made to a third party (or received from that party if they are negative). Below we comment on how to overcome the need for a third party.

Implementation requires that:

(IC) t_1^1 and t_2^1 induce players to reveal their valuations for the asset,

(IR) t_1^1 and t_2^1 provide players with incentives, if needed, to participate in the mechanism after they learn their valuations, and

(BB) the stage 0 payments and the stage 1 expected payments together sum to zero.

[5] In this final allocation, the expected surplus is $2/3 - C$ if $S \notin \{(0,1),(1,0)\}$ and $2/3 - C/2$ otherwise.

Incentive Compatibility (IC) is required to implement the final allocation; Individual Rationality (IR) is relevant when players can walk away with their initial share of the asset at the beginning of stage 1; Budget Balance (BB) guarantees that the players capture the entire surplus from the asset.

The difficulty in guaranteeing (IC) arises because each player's utility may depend on the other player's valuation through the reallocation costs.[6] The following observation, which uses the notion of increasing differences,[7] identifies sufficient conditions on the reallocation cost functions for implementing an S-efficient allocation as an ex-post equilibrium. In an ex-post equilibrium, given that the other player reports truthfully, a player prefers to report his valuation truthfully for any valuation of the other player.

Observation 2 *For any initial division S, an S-efficient final allocation is implementable as an ex-post equilibrium if the reallocation cost functions C^G and C^L satisfy increasing differences in (v_L, Δ) and decreasing differences in (v_G, Δ).*

The conditions in Observation 2 say that the marginal reallocation costs increase in the losing player's valuation and decrease in the gaining player's valuation, both for the gaining player and for the losing player. The proofs of this, and all other results, are in the appendix.

Given t_1^1 and t_2^1 that guarantee (IC), these payment schedules can be increased by a constant to guarantee (IR), without affecting players' incentives for truthful revelation. The payments t_1^0 and t_2^0 can then be chosen to simultaneously guarantee (BB) and divide the surplus between the players. This allows players to overcome any expected deficit that may arise in stage 1 when (IR) is needed.[8]

Relaxing (IC) to hold as a Bayesian equilibrium enables achieving ex-post budget balance (rather than ex-ante budget balance). That is, the payment schedules t_1^1 and t_2^1 can be chosen so that for any realization (v_1, v_2), $t_1^1(v_1, v_2) + t_1^2(v_1, v_2) = -(t_1^0 + t_2^0)$.[9] In this case, there is no need for a third party to balance the budget ex-post: positive payments t_i^0 are put in a "safe"

[6] When this is not the case, a Groves (1973) mechanism can be used to implement an S-efficient final allocation in dominant strategies without imposing any conditions on the cost functions.

[7] Let V and X be two sets of real numbers. A function $f : V \times X \to \mathbb{R}$ satisfies (strict) increasing differences in (v, x) if $g(v) = f(v, x') - f(v, x)$ (strictly) increases in v for any $x' > x$. If $g(v)$ (strictly) decreases in v, then f satisfies (strict) decreasing differences in (v, x).

[8] Such a deficit arises in standard settings with costless reallocation when the initial division is sufficiently concentrated, as shown by Myerson and Satterthwaite (1983) and Cramton, Gibbons, and Klemperer (1987).

[9] If players' valuations are stochastically independent, this can be achieved with payment schedules similar to that of Arrow (1979) and d'Aspremont and Gérard-Varet (1979). If players' valuations are not stochastically independent but meet certain (generic) distributional assumptions, then the payment schemes of Kosenock and Severinov (2008) can be used.

in stage 0, and payments t_i^1 are put in or taken from the safe in stage 1; negative payments t_i^0 are taken from the safe at the end of stage 1, after payments t_i^1 have been made.

To summarize, whether players' valuations are realized publicly or privately, the players can attain the maximal expected surplus relative to the initial division specified in the contract. We therefore proceed to study how the maximal expected surplus changes with the initial division.

Comment. Studying how the maximal expected surplus changes with the initial division is also relevant in single-person settings, such as the capital allocation example described in the Introduction. In such settings, the asset can be initially allocated between two uses, and xv_i is the surplus generated by allocating x shares of the asset to use i. The reallocation costs capture the loss of surplus that arises from reallocating the asset. Maximizing the expected surplus corresponds to maximizing the single-person's expected utility from the asset.

3 Optimal initial divisions

We first study how the maximal expected surplus changes with the initial division when the reallocation costs are insensitive to the amount reallocated. We then consider costs with either a concave or a convex variable component.

3.1 Amount-insensitive reallocation costs

Some reallocation costs do not depend on the amount reallocated. One example is the legal and other fees associated with transferring ownership of a property, which may essentially be the same regardless of how much of the property changes hands. The magnitude of this overhead may be affected by players' valuations when they correspond to the use to which the property is put (residential, commercial, etc.). We denote by $AIC(v_L, v_G)$ the total amount-insensitive cost of transferring any amount $\Delta > 0$ from a player with valuation v_L to a player with valuation v_G. Efficiency implies that $v_G > v_L$.

Let $S = (s, 1-s)$ be some initial division. We first specify an S-efficient final allocation, which achieves the maximal expected surplus relative to S. For this, we denote by $s_{change}(v_L, v_G)$ the minimal amount for which the benefit of reallocation exceeds the cost:

$$s_{change}(v_L, v_G) = \inf\{x \leq 1 : x(v_G - v_L) \geq AIC(v_L, v_G)\},$$

with $s_{change}(v_L, v_G) = 1$ if no such x exists. The S-efficient final allocation x^S allocates the entire asset to the player with the strictly higher valuation if the other player's initial share is strictly larger than s_{change}, and otherwise maintains the initial division S:

$$x^S(v_1, v_2) = \begin{cases} 1 & \text{if } v_1 > v_2 \text{ and } 1 - s > s_{change}(v_1, v_2) \\ 0 & \text{if } v_2 > v_1 \text{ and } s > s_{change}(v_2, v_1) \\ s & \text{otherwise.} \end{cases} \quad (1)$$

To understand how the maximal expected surplus changes with the initial division, let S' be an initial division that is more concentrated than S, where the concentration of an initial division is the larger share in the division. We now compare the expected surplus in x^S when the initial division is S with the expected surplus in a "modified" final allocation $\hat{x}^{S'}$ when the initial division is S'. The modified final allocation $\hat{x}^{S'}$ maintains the more concentrated initial division S' for those realizations for which x^S maintains S, and is otherwise identical to x^S.

To facilitate the comparison, fix two valuations $v_G > v_L$ and consider the change in the *sum* of the ex-post surpluses of the two realizations (v_G, v_L) and (v_L, v_G). There are three cases to consider. If the concentration of S is smaller than $s_{change}(v_L, v_G)$, then x^S maintains the initial division S for both realizations, so $\hat{x}^{S'}$ maintains the initial division S' for both realizations. The sum of the ex-post surpluses is then $v_L + v_G$ in both final allocations. If the concentration of S is strictly smaller than $1 - s_{change}(v_L, v_G)$, then x^S reallocates the asset to the player with valuation v_G independently of his initial share, and so does $\hat{x}^{S'}$. The sum of the ex-post surpluses is then $2(v_G - AIC(v_L, v_G))$ in both final allocations.[10]

Otherwise, the concentration of S is larger than $s_{const}(v_L, v_G) = \max\{s_{change}(v_L, v_G), 1 - s_{change}(v_L, v_G)\}$ and x^S reallocates the asset to the player with valuation v_G if he has the smaller initial share, and maintains the initial division if he has the larger initial share. When x^S reallocates the asset, moving to $\hat{x}^{S'}$ and S' has no effect on the ex-post surplus because the reallocation costs are amount insensitive. But when x^S maintains the initial division, moving to $\hat{x}^{S'}$ and S' strictly increases the ex-post surplus because the initial share of the player with the higher valuation is larger in S' than in S. Therefore, the sum of the ex-post surpluses of (v_L, v_G) and (v_G, v_L) strictly increases.

This implies that if the concentration of S is larger than s_{const} for a positive measure of realizations, then moving to $\hat{x}^{S'}$ and S' strictly increases the maximal expected surplus. Therefore, letting $s_{AIC} = \inf\{k : s_{const}(v_L, v_G) \leq k \text{ for a positive measure of realizations}(v_L, v_G)\}$ we obtain the following result.

Proposition 1 *The maximal expected surplus increases in the concentration of the initial division. The increase is strict if and only if the initial concentration is larger than the threshold* s_{AIC}.

There are two reasons why the maximal expected surplus may be constant around the equal initial division. The first is that the reallocation costs may be large relative to the benefit of

[10]If $S' \in \{(0,1), (1,0)\}$, then the sum increases to $2v_G - AIC(v_L, v_G)$.

reallocating half of the asset. When this happens, the initial division is always maintained when it is close to being equal. As the initial concentration increases, the benefit of reallocating to the player with the smaller initial share increases. Once the initial concentration exceeds s_{change}, the asset is reallocated to the player with the smaller initial share (but not to the player with the larger initial share), and the maximal expected surplus becomes strictly monotonic in the initial concentration. The second reason is that the reallocation costs may be small relative to the benefit of reallocating half of the asset. When this happens, the asset is always reallocated when the initial division is close to being equal. As the initial concentration increases, the benefit of reallocating to the player with the larger share decreases. Once the initial concentration exceeds $1 - s_{change}$, the asset is no longer reallocated to the player with the larger initial share (but continues to be reallocated to the player with the smaller initial share), and the maximal expected surplus becomes strictly monotonic in the initial concentration. Example 2 illustrates these two possibilities.

Example 2. Suppose that player 1's valuation is distributed uniformly on $[0, 1/3] \cup [2/3, 1]$, that player 2's valuation is $v_2 = 1 - v_1$, and that the reallocation cost is C. For $C > 1/2$ the expected surplus is constant around the equal initial division because the benefit of reallocating half of the asset is at most 1/2, so the asset is never reallocated there. For $C < 1/6$ the expected surplus is constant around the equal initial division because the asset is always reallocated to the player with the higher valuation. This is because the difference between players' valuations is at least 1/3, so reallocating half of the asset generates a benefit of at least 1/6. Figure 1 illustrates how the threshold s_{AIC} changes as a function of C. When $C < 1/6$, the threshold s_{AIC} decreases in C because the range of initial divisions for which the asset is always reallocated shrinks. When $1/6 \leq C \leq 1/2$, the threshold s_{AIC} is constant at 1/2 because whenever the initial division is not equal there are realizations for which the asset is reallocated to the player with the higher valuation when he has the larger initial share but not when he has the smaller initial share. When $C > 1/2$, the threshold s_{AIC} increases in C because the range of initial divisions for which the asset is never reallocated increases.[11] $\diamondsuit$

Proposition 1 shows that in the presence of amount-insensitive reallocation costs increasing the concentration of the initial division increases the maximal expected surplus. This is true for any symmetric distribution of players' valuations, and regardless of the magnitude of the reallocation costs.

[11] Formally, if $C < 1/6$, then $s_{AIC} = 1 - 3C$, because $s_{change}(v_L, v_G) \leq 3C \leq 1/2$ for any v_L and v_G, so $s_{const}(v_L, v_G) = 1 - s_{change}(v_L, v_G) \geq 1 - 3C$ (and $s_{AIC} \geq 1 - 3C$), and $s_{const}(1/3, 2/3) = 1 - s_{change}(1/3, 2/3) = 1 - 3C$ (so $s_{AIC} \leq 1/3C$). If $1/6 \leq C \leq 1/2$ then $s_{AIC} = 1/2$, because by definition $s_{const}(v_L, v_G) \geq 1/2$ for any v_L and v_G, and $s_{const}(v_L, v_G) = s_{change}(v_L, v_G) = 1/2$ when $v_G - v_L = 2C$. If $1/2 < C \leq 1$, then $s_{AIC} = C$ because $s_{const}(v_L, v_G) \geq s_{change}(v_L, v_G) \geq C$ for any v_L and v_G (so $s_{AIC} \geq C$) and $s_{const}(0, 1) = s_{change}(0, 1) = C$ (so $s_{AIC} \leq C$). If $C \geq 1$ then $s_{AIC} = 1$ because no reallocation takes place.

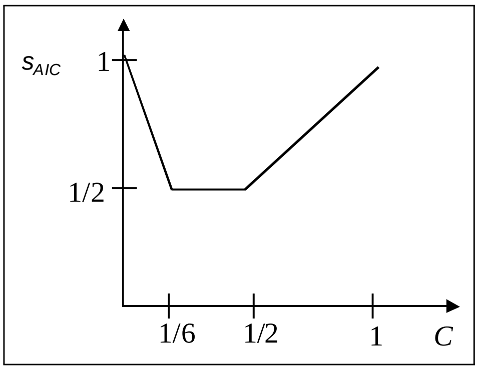

Figure 1: The threshold s_{AIC} as a function of the reallocation cost C in Example 2

An immediate implication is that the contract should specify a fully concentrated initial division, in which one player has the entire asset. This initial division is strictly better than any other initial division if and only if there is a positive measure of realizations for which the difference between the players' valuations is strictly larger than the reallocation costs associated with the valuations.[12] In Example 2, this happens if and only if $C < 1$, because the maximal difference between players' valuations is 1.

Another implication of Proposition 1 relates to situations in which players are constrained in choosing the initial division but not the final allocation. This happens, for example, when learning one's valuation for the asset requires a positive initial share, or when players have time or other constraints that are likely to be resolved before valuations are realized. In such cases, the players should specify as concentrated an initial division as possible.

Finally, when the initial division is not constrained, the contract should specify a fully concentrated initial division and a final allocation that either maintains the initial division or reallocates the entire asset to the player with the higher valuation. This implies that ex-post one of the players has the entire asset, which is similar to the final allocation in the setting of Cramton, Gibbons, and Klemperer (1987), in which reallocation is costless. But in contrast to that setting, the player who ends up with the asset may sometimes be the one with the lower valuation.

3.2 Concave reallocation costs

Reallocation costs may have a variable component that is concave in the amount reallocated. As mentioned in the Introduction, this may be the case if reallocation requires learning by the gainer or results in an emotional loss for the loser. The total reallocation cost function is then

[12]If the condition holds, then there is an $\varepsilon > 0$ such that there is a positive measure of realizations for which $s_{const} < 1 - \varepsilon$, so $s_{AIC} < 1 - \varepsilon$. If the condition does not hold, then for a measure 1 of realizations and any initial division it is optimal to maintain the initial division.

concave regardless of whether there are also amount-insensitive costs.

Concavity implies that, fixing an initial division S, there is an S-efficient final allocation x^S that for any realization of valuations either reallocates the entire asset to the player with the higher valuation or maintains the initial division. This is because the marginal reallocation cost decreases in the amount reallocated, whereas the marginal benefit is constant.

In terms of the expected surplus in x^S, the main difference from the setting with amount-insensitive costs is that increasing the concentration of the initial division increases the cost of reallocating the asset to the player with the higher valuation when he has the smaller initial share. But this increase is offset by the benefit generated when the player with the higher valuation has the larger initial share. To see this, fix two realizations (v_L, v_G) and (v_G, v_L), and suppose that in x^S when the initial division is S the asset is reallocated to the player with the higher valuation when he has the smaller initial share. If the asset is also reallocated to the player with the higher valuation when he has the larger initial share, then concavity implies that the cost decrease that results from increasing the concentration (and using the same final allocation) is larger than the cost increase mentioned above. The other possibility is that the initial division is maintained when the player with the higher valuation has the larger initial share. By concavity, the marginal increase in the reallocation costs mentioned above is smaller than the average reallocation cost; this average reallocation cost is in turn smaller than the average benefit of reallocation, $v_G - v_L$ (because x^S is S-efficient); and this average benefit equals the marginal benefit from increasing the concentration when the player with the higher valuation has the larger initial share and the more concentrated initial division is maintained.

Therefore, letting $s_{concave}$ (formally defined in the appendix) denote the smallest share for which the expected benefit of increasing the concentration is strictly larger than the expected cost, we obtain the following result.

Proposition 2 *The maximal expected surplus increases in the concentration of the initial division. The increase is strict if and only if the initial concentration is larger than* $s_{concave}$.

Propositions 1 and 2 extend to environments with more than two periods in which players' valuations in each period are drawn independently from their valuations in previous periods and the asset can be reallocated after each period's valuations are realized. Maximizing the (possibly discounted) sum of the expected surpluses across periods can be done as follows. Allocate the entire asset initially to a single player, and in each period, based on players' realized valuations, either do nothing or reallocate the entire asset to the other player. This maximizes the per-period surplus, because the entire asset is allocated to a single player at the beginning of every period, before the valuations for that period are realized. And the sum of expected surpluses cannot exceed what is obtained by maximizing the per-period expected surplus.

Proposition 2 may hint at how to assign ownership when players are loss averse. More concretely, suppose that a player views his initial share as a reference point, and experiences a mental loss when his final share is smaller than this reference point.[13] According to Kahneman and Tversky (1991), the marginal sensitivity to losses decreases in the amount lost, so reallocation costs are concave. Proposition 2 then shows that the players should initially allocate the asset to a single player, who will transfer ownership to the other player ex-post if the other player's valuation is sufficiently higher than his.[14] This continues to hold if the initial division is instead determined by a planner who seeks to maximize players' sum of expected utilities. For example, when parents wish to divide family assets among their children, they may be concerned that any future reallocation of the assets may result in mental losses to the children whose shares in a particular asset decrease. In this case, the parents should concentrate the ownership of each asset in the hands of one child, and can preserve equality by allocating additional monetary assets.

3.3 Convex reallocation costs

Unlike with mental losses, variable reallocation costs are often convex in the amount reallocated. For example, when two firms that jointly own a production facility reallocate its output, this involves reneging on commitments to supply the output; a firm may then have the flexibility to renege on the least costly commitments first, which leads to convex variable costs.

Convexity implies that it is generally not ex-post efficient either to reallocate the entire asset to one player or to maintain the initial division. Moreover, unlike with concave variable costs, the qualitative relationship between the initial division and the maximal expected surplus depends on the magnitude of the amount-insensitive costs. We first consider convex variable costs without an amount-insensitive component, and then add this component.

With no amount-insensitive costs, the total cost function $C(v_L, v_G, \Delta)$ is convex in the amount reallocated. To understand how the maximal expected surplus changes with the initial division, fix two realizations, (v_L, v_G) and (v_G, v_L), and compare the sums of the ex-post surpluses of these realizations in two efficient final allocations that correspond to two initial divisions, with the second initial division being less concentrated than the first. These sums differ in the following two cases, which depend on how players' initial shares compare to the optimal unconstrained amount that would be reallocated to the player with the higher valuation if the other player initially had the entire asset.[15]

[13]This reference point differs from that of Koszegi and Rabin (2006), which is the individual's expectation about his final asset ownership. As Kahneman and Tversky (1991, page 1046) note, "Although the reference state usually corresponds to the decision maker's current position, it can also be influenced by aspirations, expectations, norms, and social comparisons [...]."

[14]Proposition 2 also applies if players experience a mental gain when their final share exceeds their reference point, provided that for any reallocation the sum of the mental gain and loss components is concave.

[15]This optimal amount is $\inf \{\arg\max_{s \leq 1} \{s(v_G - v_L) - C(v_L, v_G, s)\}\}$.

In the simple case in which the larger share in the first initial division is smaller than the optimal unconstrained amount, the entire asset is allocated to the player with the higher valuation in both final allocations, independently of his initial share. Convexity then implies that the sum of the ex-post surpluses is higher in the final allocation that corresponds to the less concentrated initial division. Now suppose that the optimal unconstrained amount is between the smaller and larger shares in the first initial division. In this case, this optimal amount is reallocated to the player with the higher valuation if he has the smaller initial share, and the entire asset is allocated to the player with the higher valuation if he has the larger initial share. The same is true when the initial division is slightly less concentrated. Relative to the more concentrated initial division, the reallocation costs increase when the player with the higher valuation has the larger initial share, and the ex-post surplus increases when the player with the higher valuation has the smaller initial share. The former increase is smaller than the latter because the smaller share in the first initial division is smaller than the optimal unconstrained amount.

Therefore, letting s_{convex} (formally defined in the appendix) denote the smallest share for which the expected benefit of decreasing the concentration is strictly larger than the expected cost, we obtain the following result.

Proposition 3 *The maximal expected surplus decreases in the concentration of the initial division. The decrease is strict if and only if the initial concentration is larger than s_{convex}.*

Propositions 2 and 3 have different predictions about the optimal initial division of assets in the presence of reallocation costs. Whereas concave costs push toward concentrated initial divisions, convex costs push toward an equal initial division. This qualitative difference is robust to the magnitude of the reallocation costs, how the costs depend on valuations, and the distribution of valuations. The boundary case is linear reallocation costs (with no amount-insensitive costs), in which the maximal expected surplus does not depend on the initial division.

Propositions 1 and 3 also have different predictions about the optimal initial division: amount-insensitive costs push toward a concentrated initial division, whereas convex variable costs push toward an equal initial division. When both costs are present and each player has only two possible valuations $v_G > v_L$, one of the costs dominates, so an optimal initial division is either a fully concentrated one or the equal one. To understand which of them is optimal, denote by $v^s_{opt}(v_L, v_G)$ the maximal surplus that can be generated by transferring up to s shares from a player with valuation v_L to a player with valuation v_G when ignoring amount-insensitive costs, that is,

$$v^s_{opt}(v_L, v_G) = \max_{y \leq s} \{y(v_G - v_L) - VC(v_L, v_G, y)\},$$

where VC denotes the variable convex costs. We then have the following result.

Observation 3 *If the amount-insensitive costs are higher than $2v^{1/2}_{opt}(v_L, v_G) - v^1_{opt}(v_L, v_G)$, then the expected surplus is maximized at the fully concentrated initial divisions. If the amount-*

insensitive costs are lower than $2v_{opt}^{1/2}(v_L, v_G) - v_{opt}^1(v_L, v_G)$*, then the expected surplus is maximized at the equal initial division.*

Convexity implies that $2v_{opt}^{1/2}(v_L, v_G) \geq v_{opt}^1(v_L, v_G)$, so the equal initial division is optimal for low amount-insensitive costs, and the fully concentrated ones are optimal for high amount-insensitive costs. Variable reallocation costs that are linear in the amount reallocated have $2v_{opt}^{1/2} = v_{opt}^1$, so with amount-insensitive costs the only optimal initial divisions are the fully concentrated ones.[16]

Observation 3 clearly extends to the case in which players have more than two valuations, provided that its conditions hold for a measure 1 of realizations. This implies that a sufficient condition for the fully concentrated initial divisions to be optimal is that the amount-insensitive costs are always larger than $v_{opt}^{1/2}$, because v_{opt}^1 is always larger than $v_{opt}^{1/2}$.

In general, however, when each player has more than two possible valuations it may be that neither the equal initial division nor the fully concentrated ones are optimal. Moreover, in contrast to the case of two possible valuations, the optimal initial concentration need not increase in the magnitude of the amount insensitive costs. This is illustrated by the following example.

Example 3. Suppose that each player has three possible valuations, 0, 5/4, and 2, that the distribution of players' valuations is uniform over the four realizations $(0, 5/4), (5/4, 0), (0, 2)$, and $(2, 0)$, and that the variable cost is Δ^2. If the amount-insensitive cost is 3/8, then a straightforward calculation shows that the maximal expected surplus in an initial division with concentration 9/16 is larger than in a fully concentrated one or the equal one, so the optimal initial concentration is higher than 1/2. If the amount insensitive cost increases to 25/64, then the optimal initial concentration decreases to 1/2.$\Diamond$

4 Concluding remarks

This paper studied how to initially divide an asset when there is uncertainty about its best use and when reallocation is costly. We showed that the nature of the reallocation costs determines the optimal initial division. When reallocation costs are amount insensitive or concave in the amount reallocated, the more concentrated the initial division the larger the expected surplus. When reallocation costs are convex in the amount reallocated, the more concentrated the initial division the smaller the expected surplus. We conclude with possible modifications of our symmetry and two-player assumptions.

Asymmetric Players. The symmetry assumption allowed us to isolate the effect of the reallocation costs on the optimal initial division. When players are asymmetric, the asymmetry

[16]This case is also covered by Proposition 2, because the total cost is then concave.

interacts with the reallocation costs to determine the optimal initial division.

One possible asymmetry is in the distribution of players' valuations. Suppose, for example, that for any pair of possible valuations player 1's valuation is likely to be higher than player 2's valuations, that is , $f(v_G, v_L) > f(v_L, v_G)$ for every $v_G > v_L$, where f is the density of the distribution of valuations. In this case, comparing two initial divisions with the same concentration, the maximal expected surplus is larger when player 1 has the larger initial share than when player 2 does, for any reallocation costs. This implies that in any optimal initial division player 1 has the larger initial share. When reallocation costs are amount-insensitive or concave, the forces identified in the proof of Proposition 2 continue to hold, so the optimal initial division assigns the entire asset to player 1. When reallocation costs are convex, the asymmetry implies that the equal division is no longer optimal, and that in the optimal initial division player 1's share is strictly larger than player 2's share. This is true independently of the stage 0 expected surplus, which also pushes toward a higher initial share for player 1.

Another possible asymmetry is in the reallocation costs. Suppose, for example, that the marginal cost of reallocating shares from player 2 to player 1 is higher than in the other direction. That is, if $C_i(v_i, v_j, \Delta)$ is the total cost of transferring Δ shares from player i with valuation v_i to player $j \neq i$ with valuation v_j, then for any $v_G > v_L$ we have $\partial C_2 (v_L, v_G, \Delta) / \partial \Delta > \partial C_1 (v_L, v_G, \Delta) / \partial \Delta$. Then the same conclusions hold as in the case of asymmetric valuations.

More than two players. Extending our analysis to $N > 2$ players requires a more general notion of concentration. Given two initial divisions $S = (s_1, \ldots, s_N)$ and $T = (t_1, \ldots, t_N)$, we say than S is more concentrated than T if S majorizes T, that is, if when the shares in S and in T are arranged in decreasing order, we have that $\sum_{i=1}^{k} s_i \geq \sum_{i=1}^{k} t_i$ for every $k = 1, \ldots, N$. In settings with two players, majorization induces a complete ordering on the set of initial divisions. With three or more players, the induced ordering is a partial one. Assuming that the reallocation costs are additive across reallocations, and that the costs of each reallocation depend only on the amount reallocated and on the valuations of the gaining player and the losing player, we obtain the following generalization of our results.

Proposition 4 *Suppose that S majorizes T. Then,*

(1) If the reallocation costs are amount-insensitive or concave in the amount reallocated, then the maximal expected surplus associated with S is higher than that associated with T.

(2) If the reallocation costs are convex in the amount reallocated, then the maximal expected surplus associated with T is higher than that associated with S.

In particular, because the initial divisions that assign the asset to a single player majorize every other initial division, they are optimal when the reallocation costs are amount insensitive or concave. And because the equal initial division is majorized by every other initial division, this division is optimal when the reallocation costs are convex.

5 Appendix

Proof of Observation 2. We demonstrate that there exists a payment schedule $t_1^1(\cdot,\cdot)$ that supports ex-post incentive compatibility for player 1; the treatment of player 2 is analogous. By part (a) of Lemma 2 in Segal and Whinston (2011), a sufficient condition is that

(1) $x^S(v_1, v_2)$ increases in v_1 for every v_2, and

(2) $u(v_1, v_2, x, s)$ satisfies increasing differences in (v_1, x) for every v_2 and s.

For (1), fix v_2 and consider valuations $v_1 \geq v_2$, which implies that $x^S(v_1, v_2) \geq s$. Because $x^S(v_1, v_2)$ maximizes the sum of the players' utilities (surplus), we can apply Topkis's (1998) theorem 2.8.7 to show that $x^S(v_1, v_2)$ increases in v_1. For this it suffices to show that

$$f(v_1, x) \equiv u_1(v_1, v_2, x, s)+u_2(v_1, v_2, 1-x, 1-s) = xv_1 - C^G(v_2, v_1, x-s)+(1-x)v_2 - C^L(v_2, v_1, x-s)$$

$$= v_1 x + (1-x)\, v_2 - C(v_2, v_1, x-s)$$

satisfies strict increasing differences in (v_1, x) when $v_1 \geq v_2$ and x is in $[s, 1]$.[17] Since $v_1 = v_G$ and C^G and C^L satisfy decreasing differences in $(v_G, x-s)$, we have that $-C = -C^G - C^L$ satisfies increasing differences in (v_1, x). In addition, $v_1 x$ satisfies strict increasing differences in (v_1, x) and $(1-x)\, v_2$ satisfies increasing differences in (v_1, x). Therefore, f satisfies strict increasing differences in (v_1, x), so $x(v_1, v_2)$ increases in v_1. A similar argument shows that $x^S(v_1, v_2)$ increases in v_1 for $v_1 \leq v_2$. Finally, because for v_1 and v_1' such that $v_1' \geq v_2 \geq v_1$ we have $x^S(v_1', v_2) \geq s \geq x^S(v_1, v_2)$, we conclude that $x^S(v_1, v_2)$ increases in v_1.

For (2), suppose that x is in $[s, 1]$, and hence $u(v_1, v_2, x, s) = xv_1 - C^G(v_2, v_1, x-s)$. Since xv_1 satisfies increasing differences in (v_1, x) and C^G satisfies decreasing differences in $(v_1, x-s)$, we are done. A similar argument applies to x in $[0, s]$. Finally, consider x' and x such that $x' \geq s \geq x$. We have that

$$u(v_1, v_2, x', s) - u(v_1, v_2, x, s) = x'v_1 - C^G(v_2, v_1, x'-s) - \big(xv_1 - C^L(v_1, v_2, s-x)\big)$$

$$= (x'-x)\, v_1 - C^G(v_2, v_1, x'-s) + C^L(v_1, v_2, s-x)\,.$$

To show that the expression increases in v_1, it suffices to show that $C^G(v_2, v_1, x'-s)$ decreases in v_1 and $C^L(v_2, v_1, s-x)$ increases in v_1. Because C^G satisfies decreasing differences in (v_G, Δ), for $v_1' > v_1$ we have that $C^G(v_2, v_1', x'-s) - C^G(v_2, v_1', 0) \leq C^G(v_2, v_1, x'-s) - C^G(v_2, v_1, 0)\,,$ and because $C^G(\cdot,\cdot,0) = 0$, we have that $C^G(v_2, v_1', x'-s) \leq C^G(v_2, v_1, x'-s)\,.$ Similarly, $C^L(v_2, v_1', s-x) \geq C^L(v_2, v_1, s-x)$.

[17] This would satisfy the conditions of Topkis's theorem because f is trivially quasisupermodular in x, and strict increasing differences in (v_1, x) implies the strict single crossing property in (v_1, x).

Proof of Propositions 1 and 2. We first formally define the threshold $s_{concave}$. We then observe that Proposition 1 is a corollary of Proposition 2. We then prove Proposition 2.

Let $C(v_L, v_G, \Delta) = AIC(v_L, v_G) + VC(v_L, v_G, \Delta)$ be the total reallocation cost function, where VC is concave in Δ. For every two valuations $v_G > v_L$, let $s_{change}(v_L, v_G)$ denote the minimal share that is worth transferring, i.e., $s_{change}(v_L, v_G) = \inf\{s : s(v_G - v_L) \geq C(v_L, v_G, s)\}$ with $s_{change}(v_L, v_G) = 1$ if no such s exists. Let $s_{lin}(v_L, v_G)$ denote the largest number s such that the total reallocation cost function is linear in the amount reallocated in the interval $[1-s, s]$, i.e., $s_{lin}(v_L, v_G) = \sup\{s : C(v_L, v_G, \Delta) \text{ is linear in } \Delta \text{ on } [1-s, s]\}$. Let

$$s_{const}(v_L, v_G) = \begin{cases} \min\{s_{lin}(v_L, v_G), 1 - s_{change}(v_L, v_G)\} & \text{if } s_{change}(v_L, v_G) < 1/2 \\ s_{change}(v_L, v_G) & \text{Otherwise.} \end{cases}$$

Then,

$$s_{concave} = \inf\{k : s_{const}(v_L, v_G) \leq k \text{ for a positive measure of realizations } (v_L, v_G)\}.$$

Proposition 1 is a corollary of proposition 2, because when costs are amount insensitive, $s_{lin}(v_L, v_G) = 1$, and the definition of s_{const} for concave costs coincides with the definition of s_{const} for amount-insensitive costs.

We now prove Proposition 2. Suppose that the larger share in the initial division S, denoted by $s_{\max}$, is smaller than 1. Consider the S-efficient final allocation x^S given by (1) (where the definition of s_{change} is with respect to C and not AIC) and a modified final allocation that maintains the more concentrated initial division for those realizations for which x^S maintains S, and otherwise coincides with x^S. Choose two valuations $v_G > v_L$, and compare the sum of the ex-post surpluses of the two realizations (v_G, v_L) and (v_L, v_G) in x^S with the initial division S and in the modified final allocation with a higher initial concentration. There are three cases:

1. If $s_{\max} \leq s_{change}(v_G, v_L)$, then the initial division is maintained for both realizations, so the sum is $v_L + v_G$ in both final allocations.

2. If $s_{\max} < 1 - s_{change}(v_G, v_L)$, then the player with v_G gets the entire asset in both realizations, so the sum in x^S is $v_G - C(v_L, v_G, s_{\max}) + v_G - C(v_L, v_G, 1 - s_{\max})$, and in the modified final allocation with a higher concentration $s_{\max} + \varepsilon$ it is

$$v_G - C(v_L, v_G, s_{\max} + \varepsilon) + v_G - C(v_L, v_G, 1 - s_{\max} - \varepsilon).$$

   The second sum is larger than the first by concavity of $C(v_L, v_G, \cdot)$. The sums are equal for small $\varepsilon > 0$ if and only if $s_{\max} < s_{lin}(v_L, v_G)$.

3. If $s_{\max} > s_{change}(v_G, v_L) \geq 1 - s_{\max}$, then the player with v_G gets the entire asset if his initial share is $1 - s_{\max}$ and otherwise the initial division is maintained, so the sum in x^S

is $v_G - C(v_L, v_G, s_{\max}) + s_{\max} v_G + (1 - s_{\max}) v_L$, and in the modified final allocation with a higher initial concentration $s_{\max} + \varepsilon$ it is

$$v_G - C(v_L, v_G, s_{\max} + \varepsilon) + (s_{\max} + \varepsilon) v_G + (1 - s_{\max} - \varepsilon) v_L.$$

The second sum is larger than the first if and only if

$$v_G - v_L \geq \frac{C(v_L, v_G, s_{\max} + \varepsilon) - C(v_L, v_G, s_{\max})}{\varepsilon}.$$

This inequality holds because

$$v_G - v_L \geq \frac{C(v_L, v_G, s_{\max})}{s_{\max}} \geq \frac{C(v_L, v_G, s_{\max} + \varepsilon) - C(v_L, v_G, s_{\max})}{\varepsilon},$$

where the first inequality follows from $s_{\max} \geq s_{change}(v_L, v_G)$ and the second inequality follows from the concavity of $C(v_L, v_G, \cdot)$. One of the inequalities holds strictly. Otherwise, if the second inequality is an equality then $C(v_L, v_G, \cdot)$ is linear on $[0, s_{\max}]$. If, in addition, the first inequality holds as an equality, then by definition $s_{change}(v_G, v_L) = 0$. But this is impossible, because $s_{\max} < 1$ and $0 < 1 - s_{\max} \leq s_{change}$.

Moreover, cases 1 and 3 show that if $s_{change}(v_L, v_G) \geq 1/2$ the increases is strict if and only if $s_{\max} \geq s_{change}(v_L, v_G)$. If $s_{change}(v_L, v_G) < 1/2$, then cases 2 and 3 and the definition of $s_{const}(v_L, v_G)$ show that the increase is strict for small $\varepsilon > 0$ if and only if $s_{\max} \geq s_{const}(v_L, v_G)$. Now, if $s_{\max} \leq s_{const}(v_L, v_G)$, then if we use the modified final allocation with an initial concentration lower than $s_{\max}$, the sum does not change.[18] So the sum in an $s_{\max}$-efficient allocation as a function of $s_{\max}$ is constant on $[1/2, s_{const}(v_L, v_G)]$ and strictly increases on $[s_{const}(v_L, v_G), 1]$.

Proof of Proposition 3. Denote by $s_{opt}(v_L, v_G) = \inf\{\arg\max_{s \leq 1}\{s(v_G - v_L) - C(v_L, v_G, s)\}\}$ the optimal unconstrained amount, and let

$$s_{const}(v_L, v_G) = \begin{cases} 1 - s_{opt}(v_L, v_G) & \text{if } s_{opt}(v_L, v_G) < 1/2 \\ \min\{s_{lin}(v_L, v_G), s_{opt}(v_L, v_G)\} & \text{otherwise,} \end{cases}$$

where s_{lin} is defined as in the previous proof. Then,

$$s_{convex} = \inf\{k : s_{const}(v_L, v_G) \leq k \text{ for a positive measure of realizations } (v_L, v_G)\}.$$

Consider an initial division S with $s_{\max} > 1/2$, where $s_{\max}$ is the larger share in S. Let $S^{-\varepsilon}$ be an initial division with concentration $s_{\max} - \varepsilon$ for a small $\varepsilon > 0$. For any $v_L < v_G$, it is easy to see that it is optimal to reallocate $\min\{s, s_{opt}(v_L, v_G)\}$ of the asset to the player with v_G,

[18] This includes the case $s_{\max} = 1$, because then Case 2 does not arise.

where s is the initial share of the player with v_L. Therefore, the following final allocation x^S is S-efficient:

$$x^S(v_1,v_2)=\begin{cases}1 & v_1>v_2 \text{ and } 1-s\le s_{opt}(v_2,v_1)\\ s+s_{opt}(v_2,v_1) & v_1>v_2 \text{ and } 1-s> s_{opt}(v_2,v_1)\\ 0 & v_2>v_1 \text{ and } s\le s_{opt}(v_1,v_2)\\ s-s_{opt}(v_2,v_1) & v_2>v_1 \text{ and } s> s_{opt}(v_1,v_2)\end{cases}.$$

Choose two valuations $v_G > v_L$, and compare the sum of the ex-post surpluses of the two realizations (v_G, v_L) and (v_L, v_G) in x^S with the initial division S and in $x^{S^{-\varepsilon}}$ with the initial division $S^{-\varepsilon}$. There are three cases:

1. If $s_{\max} \le s_{opt}(v_L, v_G)$, then the player with v_G gets the entire asset in both realizations, so the sum in x^S is $v_G - C(v_L, v_G, s_{\max}) + v_G - C(v_L, v_G, 1-s_{\max})$, and in $x^{S^{-\varepsilon}}$ it is $v_G - C(v_L, v_G, s_{\max}+\varepsilon) + v_G - C(v_L, v_G, 1-s_{\max}-\varepsilon)$. The second sum is larger than the first by convexity of $C(v_L, v_G, \cdot)$. The sums are equal if and only if $s_{\max} \le s_{lin}(v_L, v_G)$.

2. If $s_{\max} \le 1 - s_{opt}(v_G, v_L)$, then the player with valuation v_G is reallocated $s_{opt}(v_L, v_G)$ in both realizations, so the sums are equal.

3. If $s_{\max} > s_{opt}(v_G, v_L) > 1 - s_{\max}$, then the player with v_G gets the entire asset if his initial share is $s_{\max}$, and otherwise he is reallocated $s_{opt}(v_L, v_G)$, so the sum in x^S is

$$v_G - C(v_L, v_G, 1-s_{\max})$$
$$+(1-s_{\max}+s_{opt}(v_L,v_G))\,v_G + (s_{\max}-s_{opt}(v_L,v_G))\,v_L - C(v_L, v_G, s_{opt}(v_L,v_G)),$$

and similarly in $x^{S^{-\varepsilon}}$. The sum of surpluses is larger in $x^{S^{-\varepsilon}}$ if and only if

$$\varepsilon(v_G - v_L) \ge C(v_L, v_G, 1-s_{\max}+\varepsilon) - C(v_L, v_G, 1-s_{\max}).$$

This inequality holds strictly, because $s_{opt} > 1-s_{\max}$ implies that after reallocating $1-s_{\max}$ shares to the player with valuation v_G, the benefit of reallocating an additional ε is strictly larger than the cost of doing so.

This shows that slightly decreasing the initial concentration increases the sum of the maximal ex-post surpluses for the two realizations (v_L, v_G) and (v_G, v_L). Because this sum is continuous in the initial concentration,[19] the sum decreases in the initial concentration on the interval $[1/2, 1]$. To identify where the decrease is strict, suppose first that $s_{opt}(v_L, v_G) \ge 1/2$. Then cases 1 and 3 show that the sum strictly decreases if and only if $s_{\max} > \min\{s_{opt}(v_L, v_G), s_{lin}(v_L, v_G)\}$. If $s_{opt}(v_L, v_G) < 1/2$, then cases 2 and 3 show that the sum strictly decreases if and only if $s_{\max} > 1 - s_{opt}(v_L, v_G)$. Therefore, the sum as a function of $s_{\max}$ is constant on $[1/2, s_{const}(v_L, v_G)]$ and strictly decreases on $[s_{const}(v_L, v_G), 1]$.

[19]Convexity implies that $C(v_L, v_G, \cdot)$ is continuous on $[0,1]$, so it is equicontinuous there, as is $v_G - v_L$. This implies that the maximal ex-post surplus is continuous in the initial concentration.

Proof of Observation 3. If both players have the same valuation, the asset is not reallocated. We therefore focus on the two realizations (v_L, v_G) and (v_G, v_L). We first show that the sum of the ex-post surpluses for these realizations is maximized in either a fully concentrated initial division or the equal one. Denote by $s_0(v_L, v_G)$ the minimal unconstrained share for which reallocation is beneficial, that is,

$$s_0(v_L, v_G) = \inf \{s : s(v_G - v_L) - VC(v_L, v_G, s) \geq AIC(v_L, v_G)\},$$

and $s_0(v_L, v_G) = 1$ if this set is empty. Fix some initial division S. Suppose first that the smaller initial share in S is larger than s_0. In this case, reallocating some of the asset to the player with the higher valuation is beneficial independently of his initial share. Because the variable reallocation cost is convex, the arguments developed for Proposition 3 show that the sum of the ex-post surpluses is larger in the equal initial division than in S. Now suppose that the smaller initial share is smaller than s_0. In this case, the initial division S is maintained when the player with the lower valuation has the smaller initial share. The sum of ex-post surpluses is $v_L + v_G$ plus any additional benefit generated by reallocating shares to the player with the higher valuation when he has the smaller initial share. This additional benefit increases in the concentration because more of the asset can be transferred to this player.

For the first statement in the proposition, suppose that

$$AIC(v_L, v_G) \geq 2v_{opt}^{1/2}(v_L, v_G) - v_{opt}^{1}(v_L, v_G).$$

If the initial division is maintained, the sum of the ex-post surpluses for the two realizations is $v_L + v_G$. Consider the following cases:

1. If $s_0(v_L, v_G) > 1/2$, then an efficient final allocation for an equal initial division is to maintain the initial division.
   In a fully concentrated initial division, maintaining the initial division gives the same sum of ex-post surpluses, so the sum in an efficient final allocation is larger.

2. If $s_0(v_L, v_G) \leq 1/2$, then the sum of the ex-post surpluses in an efficient final allocation for the equal initial division is $v_L + v_G + 2v_{opt}^{1/2}(v_L, v_G) - 2AIC(v_L, v_G)$, and for a fully concentrated one is $v_L + v_G + v_{opt}^{1}(v_L, v_G) - AIC(v_L, v_G)$, which is larger by assumption.

For the second statement in the proposition, suppose that

$$AIC(v_L, v_G) \leq 2v_{opt}^{1/2}(v_L, v_G) - v_{opt}^{1}(v_L, v_G). \tag{2}$$

This implies that $s_0(v_L, v_G) \leq 1/2$. Otherwise, $v_{opt}^{1/2}(v_L, v_G) < AIC(v_L, v_G)$ and $v_{opt}^{1/2}(v_L, v_G) \leq v_{opt}^{1}(v_L, v_G)$ contradict (2). The result now follows as in case 2 above.

Proof of Proposition 4. The proof requires the following definition. For any two players, we say that the concentration of the two players increases (decreases), if some of the initial share of the player with the lower (higher) initial share is added to the initial share of the player with the higher (lower) initial share. We refer to this process as increasing (decreasing) the concentration of the two players.

If S majorizes T, then by Lemma 2 on page 47 of Hardy, Littlewood, and Pólya (1952) there exists a finite sequence of initial divisions $\bar{s}^1, \ldots, \bar{s}^m$, (where $\bar{s}^i = (\bar{s}^i_1, \ldots, \bar{s}^i_n)$) such that (1) $\bar{s}^1 = T$, (2) $\bar{s}^m = S$, and (3) for every $i < m$, there are players j_i and k_i such that $\bar{s}^{i+1}$ is derived from $\bar{s}^i$ by increasing the concentration of players j_i and k_i. Proposition 4 follows from applying the following two Lemmas to the sequence of initial divisions.

Lemma 1 *If the reallocation costs are amount insensitive or concave in the amount reallocated, then increasing the concentration of any two players increases the maximal expected surplus.*

Proof. Because the marginal cost of reallocating decreases in the amount reallocated, we consider an efficient final allocation in which if for a given realization of valuations some amount is transferred from player i to player j, then player i's entire initial share is transferred to player j. Without loss of generality consider players 1 and 2, some realization of all players' valuations, and the realization in which v_1 and v_2 are reversed. Suppose that $s_1 \geq s_2 > 0$, where s_i is player i's initial share. Let $v_L = \min\{v_1, v_2\}$ and $v_G = \max\{v_1, v_2\}$, with $v_L < v_G$.[20] Consider the sum of the ex-post surpluses associated with the two realizations of players' valuations in the efficient final allocation. In this sum, consider the sum of surpluses associated with shares s_1 and s_2 that are initially assigned to the player with valuation v_L. We now show that this sum increases if the concentration of players 1 and 2 increases and a modified final allocation is implemented, which transfers the entire share of player i to player j when the original efficient final allocation does so, and otherwise maintains the new initial division (the same argument applies to the sum of the surpluses associate with shares s_1 and s_2 that are initially assigned to the player with valuation v_G, which concludes the proof). There are three cases:

1. The initial share of the player with valuation v_L is not reallocated whether it is s_1 or s_2. Then, the sum equals $(s_1 + s_2)\, v_L$ in both final allocations.

2. The initial share of the player with valuation v_L is reallocated to a player with valuation $\bar{v} > v_L$ when the initial share is s_1, but not when it is s_2. Then, the sum is

$$s_1\bar{v} - C(v_L, \bar{v}, s_1) + s_2 v_L.$$

Because the initial share s_1 is reallocated, we have $s_1\bar{v} - C(v_L, \bar{v}, s_1) \geq s_1 v_L$, or equivalently $s_1(\bar{v} - v_L) \geq C(v_L, \bar{v}, s_1)$, which together with the concavity of $C(v_L, \bar{v}, \cdot)$ implies that for

[20] An almost identical proof applies when $v_L = v_G$.

positive $\varepsilon \leq s_2$ we have $\varepsilon(\bar{v} - v_L) \geq C(v_L, \bar{v}, s_1 + \varepsilon) - C(v_L, \bar{v}, s_1)$. This last inequality implies that the sum in the modified final allocation satisfies

$$(s_1 + \varepsilon)\bar{v} - C(v_L, \bar{v}, s_1 + \varepsilon) + (s_2 - \varepsilon) v_L \geq s_1\bar{v} - C(v_L, \bar{v}, s_1) + s_2 v_L.$$

3. The initial share of the player with valuation v_L is reallocated to a player with valuation $\bar{v} > v_L$ when the initial share is s_1, and to a player with valuation $\tilde{v} > v_L$ when the initial share is s_2 ($\tilde{v}$ may or may not equal $\bar{v}$). Then, the sum is

$$\underbrace{s_1\bar{v} - C(v_L, \bar{v}, s_1)}_{f(s_1)} + \underbrace{s_2\tilde{v} - C(v_L, \tilde{v}, s_2)}_{g(s_2)}.$$

   To show that this sum increases if the concentration of players 1 and 2 increases, it suffices to show that the marginal value of f at s_1 is higher than that of g at s_2. Suppose this is not the case. First note that $f(s_2) \leq g(s_2)$, otherwise it would have been better to reallocate the initial share s_2 to the player with valuation $\bar{v}$ instead of to the player with valuation $\tilde{v}$. Now, because the marginal values of f and g increase from s_2 to s_1 (reallocation costs are concave), and the marginal value of f at s_1 is strictly lower than than of g at s_2 (our assumption), we have that the marginal value of f at x is strictly lower than that of g at x for every x in $[s_2, s_1]$. Together with $f(s_2) \leq g(s_2)$ this implies that $f(s_1) < g(s_1)$. But then it would have been better to reallocate the initial share s_1 to the player with valuation $\tilde{v}$ instead of to the player with valuation $\bar{v}$.

Note that there is no "Case 4," because if s_2 is reallocated when it is the initial share of the player with valuation v_L, then by concavity of the costs s_1 is reallocated when it is the initial share of the player with valuation v_L. ■

Lemma 2 *If the reallocation costs are convex in the amount reallocated, then decreasing the concentration of any two players increases the maximal expected surplus.*

Proof. Without loss of generality consider players 1 and 2, some realization of all players' valuations, and the realization in which v_1 and v_2 are reversed. Suppose that $s_1 > s_2$, where s_i is player i's initial share. Let $v_L = \min\{v_1, v_2\}$ and $v_G = \max\{v_1, v_2\}$, with $v_L < v_G$. Consider the set of sequences that maximize the ex-post surplus generated by optimally reallocating s shares that were initially allocated to the player with valuation v_L,

$$\arg \max_{(x_1, x_2, \ldots, x_n)} \left\{ x_1 v_L + x_2 v_G - C(v_L, v_G, x_2) + \sum_{i=3}^{n} (x_i v_i - C(v_L, v_i, x_i)) : \sum_{i=1}^{n} x_i = s \right\}, \quad (3)$$

and let $x_{opt}(s)$ be the maximal sequence $(x_1, \ldots, x_n)$ in this set according to the lexicographic order (that is, for any other sequence $(y_1, \ldots, y_n)$ in the set, if there is some $i \leq n$ such that $y_i > x_i$, then there is some $j < i$ such that $x_j > y_j$). Consider the sum of the ex-post surpluses associated with the two realizations of players' valuations in some efficient final allocation. In this sum, consider the sum of surpluses associated with shares s_1 and s_2 that are initially assigned to the player with valuation v_L. This sum is given by the expression in the curly brackets in (3) with $(x_1, \ldots, x_n)$ given by $x_{opt}(s_1)$ plus the same expression with $(x_1, \ldots, x_n)$ given by $x_{opt}(s_2)$. We now show that this sum increases if the concentration of players 1 and 2 decreases, so that $x_{opt}(s_1)$ is replaced with $x_{opt}(s_1 - \varepsilon)$ and $x_{opt}(s_2)$ is replaced with $x_{opt}(s_2 + \varepsilon)$ for some small $\varepsilon > 0$ (an almost identical argument applies to the sum of the surpluses associated with shares s_1 and s_2 that are initially assigned to the player with valuation v_G, which concludes the proof).

Fix some $\varepsilon < s_1 - s_2$. By the definition of x_{opt} and the convexity of the reallocation costs, for every i, the i^{th} coordinate of $x_{opt}(s_2 + \varepsilon)$ is larger than the i^{th} coordinate of $x_{opt}(s_2)$ by some $\varepsilon_i \geq 0$, with $\sum_{i=1}^{n} \varepsilon_i = \varepsilon$. The increase in the surplus associated with share s_2 that is initially assigned to the player with valuation v_L resulting from changing $x_{opt}(s_2)$ to $x_{opt}(s_2 + \varepsilon)$ is

$$\varepsilon_1 v_L + \varepsilon_2 v_G - (C(v_L, v_G, x_2 + \varepsilon_2) - C(v_L, v_G, x_2)) + \sum_{i=3}^{n} (\varepsilon_i v_i - (C(v_L, v_i, x_i + \varepsilon_i) - C(v_L, v_i, x_i))), \tag{4}$$

where x_i is the i^{th} coordinate of $x_{opt}(s_2)$. The expression (4) is larger than

$$\varepsilon_1 v_L + \varepsilon_2 v_G - (C(v_L, v_G, y_2) - C(v_L, v_G, y_2 - \varepsilon_2)) + \sum_{i=3}^{n} (\varepsilon_i v_i + (C(v_L, v_i, y_i) - C(v_L, v_i, y_i - \varepsilon_i))), \tag{5}$$

where y_i is the i^{th} coordinate of $x_{opt}(s_1)$, because the reallocation costs are convex and (by definition of x_{opt}) $y_i \geq x_i + \varepsilon_i$. The expression (5) is an upper bound on the decrease in the surplus associated with share s_1 that is initially assigned to the player with valuation v_L resulting from changing $x_{opt}(s_1)$ to $x_{opt}(s_1 - \varepsilon)$. ■

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