Behavioral Implications of Shortlisting Procedures

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Abstract

We consider two-stage "shortlisting procedures" in which the menu of alternatives is first pruned by some process or criterion and then a binary relation is optimized. For a given first-stage process, our main "meta-characterization" result supplies a necessary and sufficient condition for choice data to be consistent with a procedure in the associated class. This result applies to any class of procedures with a certain lattice structure, including "consideration filters," "satisficing with salience effects," and "rational shortlist methods." The theory avoids background assumptions made for mathematical convenience; in this and other respects following Richter's classical analysis of preference-maximizing choice in the absence of shortlisting.

1. Introduction

Within the recent literature examining nonstandard models of choice behavior, several contributions study what may be termed "shortlisting procedures."[1] These procedures feature an initial stage in which the menu of available alternatives is pruned by some process or criterion, followed by a second stage in which — as in the standard model — a binary relation is optimized. Notable examples include Lleras et al.'s [16] "consideration filter" and Masatlioglu et al.'s [23] "attention filter" procedures, Tyson's [42] model of "satisficing with salience effects," and Manzini and Mariotti's [18] "rational shortlist methods" (all of which are examined in Section 3 below).

The two stages of a shortlisting procedure can have various interpretations depending on the purpose of the model and the extra assumptions imposed. For example, in Lleras et al. [16] the first stage reflects cognitive constraints that make it infeasible for the decision maker to consider all available options, while the second stage is ordinary preference maximization. In contrast, Tyson [42] introduces a form of imperfect preference maximization at the first stage and uses the second to model differential salience (i.e., success in attracting attention) of the alternatives. In some applications the two stages may even

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[1] The broader literature includes, e.g., Ambrus and Rozen [1], Bossert and Sprumont [4], Caplin and Dean [6], Kalai et al. [15], Mandler et al. [17], Manzini and Mariotti [20], Masatlioglu and Nakajima [22], Masatlioglu and Ok [24], Rubinstein and Salant [29], and Salant and Rubinstein [30].

be controlled by different agents, such as when a headhunting firm assembles a list of job-candidate finalists from which an employer will make the final choice.

As with any decision-theoretic model, several basic questions arise in the analysis of a shortlisting procedure. Firstly, is the model falsifiable in the sense of ruling out some logically-possible combinations of choices? Secondly, given falsifiability, what conditions are necessary and sufficient for observed choice data to be consistent with the procedure? And thirdly, given a consistent set of data, to what extent are the constituents of the model "revealed" (à la Samuelson [31]) by the behavior?

For a shortlisting procedure to be falsifiable, the first stage must have some structure that prevents it from being used to explain any pattern of choices ex post. In Lleras et al. [16] "contraction consistency" of shortlisted alternatives is assumed; in Tyson [42] the implied property is one of "strong expansion consistency"; and other procedures impose their own restrictions on the shortlisting stage. Given some such structural assumption that yields falsifiability, we may then turn to the characterization question: What axioms identify those and only those data sets that could have been generated by a shortlisting procedure of the hypothesized type?

Assume now that the binary relation optimized in the second stage of the procedure is complete and transitive, like a standard preference relation. If the first-stage mechanism were observable, the desired characterization would be supplied by Richter's [27] classical analysis of preference-maximizing choice over an arbitrary collection of menus. Indeed, if we were able to observe the mapping from menus to sets of shortlisted alternatives, then we could treat these shortlisting sets as surrogate menus and apply Richter's result directly.

With an unobservable first stage, however, the situation is more delicate. In this case not only the second-stage relation but also each menu's shortlisting set must be inferred from choices, with a consequent ambiguity: If an alternative was available on but not chosen from a particular menu, is this because it was not shortlisted or due to its being eliminated in the second stage? Characterizing the procedure (or, more precisely, showing sufficiency of proposed axioms) will require us to answer numerous questions of this sort in such a way as to produce both a shortlisting stage with the specified structure and a second-stage relation that is complete and transitive.[2]

In this paper we shall see that — despite the difficulties just described — the classical Richterian analysis can be extended to characterize a range of shortlisting procedures. We proceed abstractly, first defining the space Ξ of "selection functions" that return a subset of each menu in a given domain. A class of shortlisting procedures can then be identified with the set $\Sigma \subset \Xi$ of selection functions permitted as the first stage of the model. The consideration filter procedures of Lleras et al. thus comprise the set Σ^{cf} of functions exhibiting contraction consistency (see Definition 3.1), while Tyson's model of satisficing with salience effects corresponds to the set Σ^{se} of functions exhibiting strong expansion consistency (see Definition 3.8).

A revealed counterpart to the unobservable first-stage mechanism must then have two

[2] A consequence of the ambiguity observed here is that choice data consistent with a class of shortlisting procedures (and with multi-stage models more generally) typically will not have a unique representation. Asking to what extent model constituents are revealed by behavior is equivalent to asking if all valid representations from the specified class of procedures can be guaranteed to agree in some respects.

features. First, it must be in the postulated class Σ of procedures. And second, it must be consistent with the data in the sense that any alternative chosen from a menu must have been shortlisted. At the core of our theory is the following insight: If we can find a selection function that is minimal, in an appropriate sense, among all functions with the two properties just stated, then the Richterian machine will succeed in characterizing the full two-stage model when the image of this "revealed shortlisting map" is used as the collection of surrogate menus.

Our general theory of shortlisting procedures thus replaces the familiar Congruence axiom (Condition 2.5), used by Richter to characterize preference maximization, with a "Σ-Congruence" axiom (Condition 2.9) defined relative to a given class Σ of procedures via the associated revealed shortlisting map. Our main result (Theorem 2.10) identifies when the new condition is necessary and sufficient for choice data to be consistent with a procedure in the class Σ. And since such equivalence holds for a range of classes, this result can be described as a "meta-characterization" of shortlisting procedures.

It remains to determine when a suitable revealed shortlisting map can be found. To this end we note first that when partially ordered by pointwise set inclusion, the space Ξ of selection functions is a complete lattice of which those consistent with the data are a complete sublattice. If, under the same partial order, a particular class Σ of shortlisting procedures is also a complete sublattice, then it follows that the set of selection functions possessing both properties stated above will have a greatest lower bound. And it is this "minimal" function that can play the role of the revealed shortlisting map for the purpose of stating and using Σ-Congruence. Lattice structure therefore emerges as the essential attribute of a class of procedures for our meta-characterization result to be applicable.

To demonstrate the scope of our theory we apply it to a number of specific shortlisting procedures, some present in the literature and others not. It is shown first that the space of consideration filters has the necessary lattice structure, but that the space of attention filters does not. Both the original model of satisficing with salience effects and a variant procedure (leading to weak rather than strong expansion consistency of the shortlisting stage) are seen to permit application of our results, as does the class of rational shortlist methods. And finally, procedures in which the first-stage shortlisting map is "justified" by a binary relation in the sense of Mariotti [21] provide yet another suitable case.[3]

In each application the power of our meta-characterization leaves us with very little work to do. To confirm that the result applies, it suffices to verify the lattice structure of the class of procedures in question. And the only other step needed to obtain a fully operational characterization is to find an explicit expression for the revealed shortlisting map, whose "official" definition (as the greatest lower bound of a set of selection functions) may prove somewhat unwieldy in practice.[4]

Our approach to the behavioral characterization of shortlisting procedures has three distinct advantages. First, its abstract formulation allows us to study multiple classes of procedures simultaneously, and ignores the irrelevant details of specific models. Second,

[3]In regard to these applications our focus will be on the formal problem of behavioral characterization. Discussion of other aspects of each procedure — its intuitive basis, experimental support, usefulness for economic modeling, and so on — can be found in the cited work (where applicable).

[4]Even this second step can be done mechanically in each of the applications we consider, as is shown in Section 4.2.

since most elements of our theory have some analog in the Richterian analysis, we remain on firm ground intuitively — in particular, we are able to avoid convoluted axioms and state our main result in terms of a single condition that generalizes classical Congruence in a natural way. And third, the formal setting in which we operate is completely devoid of background assumptions made strictly for mathematical convenience.

This last advantage merits further elaboration. Among the background assumptions we do *not* impose are:

- *Finiteness of the universal set.* Our universe of alternatives can have any cardinality, and may or may not possess special (e.g., Euclidean) structure.
- *Domain restrictions.* The analysis accepts choice data from an arbitrary collection of menus. There is no requirement that specific (e.g., two-element) menus be either included or excluded.
- *Single-valued choice.* We encode behavior in choice functions defined so as to allow for any mixture of single-valued and multi-valued output. A specialized version of our meta-characterization (Theorem 2.12) covers the purely single-valued case, and here the property is not a background assumption but rather a consequence of the type of shortlisting procedure being characterized.

Needless to say, our theory inherits this high degree of generality from the foundation of Richter [27] on which it builds.

The remainder of the paper is organized as follows. Section 2 describes the modeling environment, discusses the revelation of both shortlisting and "preference" (though the second-stage relation need not bear this interpretation), and states both the ordinary and specialized forms of our meta-characterization result. Section 3 applies the theory to a range of specific shortlisting procedures. Section 4 contains additional results relating to lexicographic preference models and to the algorithmic revelation of shortlisting. All proofs are in the Appendix unless otherwise indicated.

2. Theory

2.1. Preliminaries

Let X be a nonempty set of *alternatives*, and define the set $\mathfrak{X} = \{A : A \subset X\}$ of *menus* drawn from X. Fix a *domain* $\mathfrak{D} \subset \mathfrak{X} \setminus \{\emptyset\}$, and write $\Xi = \{\xi \in \mathfrak{X}^{\mathfrak{D}} : \forall A \in \mathfrak{D} \;\; \xi(A) \subset A\}$ for the space of *selection functions* on $\mathfrak{D}$. Given $\xi_1, \xi_2 \in \Xi$, write $\xi_1 \subset \xi_2$ if $\forall A \in \mathfrak{D}$ we have $\xi_1(A) \subset \xi_2(A)$. For any $\Psi \subset \Xi$ both $\bigwedge \Psi = \bigcap_{\xi \in \Psi} \xi$ and $\bigvee \Psi = \bigcup_{\xi \in \Psi} \xi$ are in Ξ, and hence $\langle \Xi, \subset \rangle$ is a complete lattice. In particular it is bounded, with greatest element $\top$ (the identity mapping) and least element $\bot$ (returning $\emptyset$ everywhere).

The decision maker's behavior is encoded in a nonempty-valued *choice function* $C \in \Xi$. That is to say, for each $A \in \mathfrak{D}$ the associated *choice set* $C(A) \neq \emptyset$ contains those and only those alternatives that can be observed as choices. We write $\Xi_C = \{\xi \in \Xi : C \subset \xi\}$ for the space of selection functions that include C pointwise. Observe that $\langle \Xi_C, \subset \rangle$ is a complete sublattice of $\langle \Xi, \subset \rangle$, with greatest element $\top$ and least element C.

A (*binary*) *relation* on X is any $\mathrm{R} \subset X \times X$, with $\langle x, y\rangle \in \mathrm{R}$ usually written as $x\mathrm{R}y$. Such a relation is a *complete preorder* if it is both complete ($\neg[x\mathrm{R}y]$ only if $y\mathrm{R}x$) and transitive ($x\mathrm{R}y\mathrm{R}z$ only if $x\mathrm{R}z$), and a *complete order* if it is also antisymmetric ($x\mathrm{R}y\mathrm{R}x$ only if $x = y$). A relation is a *strict partial order* if it is both irreflexive ($\forall x \;\; \neg[x\mathrm{R}x]$) and transitive, and a *linear order* if it is also weakly complete ($x \neq y$ only if $x\mathrm{R}y$ or $y\mathrm{R}x$). Any complete relation is reflexive ($\forall x \;\; x\mathrm{R}x$). The *transitive closure* R^* of a relation R is defined by $x\mathrm{R}^*y$ if and only if for some integer $n \geq 2$ there exist $z_1, \ldots, z_n \in X$ such that $x = z_1\mathrm{R}z_2\mathrm{R}\cdots\mathrm{R}z_n = y$. Given $A \in \mathfrak{X}$, write $\mathrm{R}{\Uparrow}(A) = \{x \in A : \forall y \in A \;\; x\mathrm{R}y\}$ for the set of alternatives on menu A (if any) that are greatest with respect to R.

2.2. Classes of shortlisting procedures

The classical theory of choice — describing the behavior of an idealized rational decision maker — can be expressed as the equivalence $C = \mathrm{R}{\Uparrow}$, where R is the agent's preference relation.[5] The following definition generalizes this model to allow preselection of alternatives by a shortlisting map before the preference relation is applied.

Definition 2.1. Given $\Sigma \subset \Xi$, the choice function is a *shortlisting procedure of class* Σ if there exist a $\sigma \in \Sigma$ and a relation R such that $C = \mathrm{R}{\Uparrow} \circ \sigma$. Such a procedure is termed *CP-* or *CO-shortlisting* accordingly as R is a complete preorder or a complete order.

The classical theory may then be recovered as the procedures of class $\Sigma^{\mathrm{id}} = \{\top\}$.

Suppose now that C is a shortlisting procedure of class Σ, but neither the mapping σ nor the relation R is observable. Though we cannot see σ, we know that this function is in Σ. Moreover, any alternative choosable from a menu must be on the relevant shortlist, which is to say that $\sigma \in \Xi_C$. Forming the pointwise intersection of all selection functions that share these two properties thus yields an underestimate of σ with respect to $\subset$.

Definition 2.2. Given $\Sigma \subset \Xi$, let $\hat{\sigma}_\Sigma = \bigwedge[\Sigma \cap \Xi_C]$.

Since $\langle \Xi_C, \subset\rangle$ is a complete lattice we have $\hat{\sigma}_\Sigma \in \Xi_C$, and plainly $C \in \Sigma \implies \hat{\sigma}_\Sigma = C$. Furthermore, it is immediate that $C \subset \sigma \in \Sigma \implies \hat{\sigma}_\Sigma \subset \sigma$; this is the underestimation property. What is not clear from the definition is whether $\hat{\sigma}_\Sigma \in \Sigma$, a feature that will be needed if we are to use this selection function as a revealed counterpart to the unobserved shortlisting operator σ. The key to our analysis is the following observation, which offers a sufficient condition for the desired property of $\hat{\sigma}_\Sigma$.

Proposition 2.3. *Given $\Sigma \subset \Xi$, if $\langle \Sigma, \subset\rangle$ is a complete lattice then $\hat{\sigma}_\Sigma \in \Sigma$.*

Applying our results (see Section 3) will amount to verifying this lattice structure and finding a more explicit expression for $\hat{\sigma}_\Sigma$, the revealed shortlisting map.

2.3. Revealed preference

We now wish to elicit preference comparisons from choice data, taking into account that some alternatives may not have been shortlisted. To understand how this can be done,

[5] This theory was pioneered by Samuelson [31], Houthakker [13], Arrow [2], Richter [27, 28], Hansson [12], and Suzumura [38], among others. A concise summary appears in Bossert et al. [5].

it is useful first to recall how preferences are revealed in the classical theory without a shortlisting stage.

Definition 2.4. Given $x, y \in X$, we write $x\mathrm{R}^{\mathrm{g}}y$ and say that x is *revealed preferred* to y if $\exists A \in \mathfrak{D}$ such that $y \in A$ and $x \in C(A)$. Moreover, when $x[\mathrm{R}^{\mathrm{g}}]^*y$ we say that x is *indirectly revealed preferred* to y.

Thus a preference is (directly) revealed when one alternative is choosable in the presence of another, while a preference is indirectly revealed when two alternatives are linked by a chain of revealed preferences. Using these definitions, Richter [27] (see also Suzumura [38]) characterizes the classical model with complete preorder preferences as follows.

Condition 2.5 (Congruence)**.** Given $x, y \in A \in \mathfrak{D}$, if both $x \in C(A)$ and $y[\mathrm{R}^{\mathrm{g}}]^*x$ then $y \in C(A)$.

In words, if an alternative is both available and indirectly revealed preferred to a second alternative that is choosable, then the first alternative must itself be choosable.

Theorem 2.6 (Richter [27, p. 639])**.** *There exists a complete preorder* R *such that* $C = \mathrm{R}\Uparrow$ *if and only if Congruence holds.*

The general shortlisting model is treated analogously. We begin by defining notions of revealed preference relative to the output of the revealed shortlisting map $\hat{\sigma}_\Sigma$, which as we know underestimates the true map σ.

Definition 2.7. Given $\Sigma \subset \Xi$ and $x, y \in X$, we write $x\hat{\mathrm{R}}_\Sigma y$ and say that x is Σ-*revealed preferred* to y if $\exists A \in \mathfrak{D}$ such that both $y \in \hat{\sigma}_\Sigma(A)$ and $x \in C(A)$. Moreover, when $x\hat{\mathrm{R}}^*_\Sigma y$ we say that x is *indirectly* Σ-*revealed preferred* to y.

Here the relation $\hat{\mathrm{R}}_\Sigma$ searches for situations in which one alternative is choosable in the presence of another *that has definitely been shortlisted* — the latter qualification needed to ensure that the apparent revealed preference is genuine.

Note that since $\hat{\sigma}_{\Sigma^{\mathrm{id}}} = \top$, we have $\hat{\mathrm{R}}_{\Sigma^{\mathrm{id}}} = \mathrm{R}^{\mathrm{g}}$ in the classical special case. Furthermore, $\Sigma_1 \cap \Xi_C \subset \Sigma_2 \cap \Xi_C$ implies $\hat{\sigma}_{\Sigma_2} \subset \hat{\sigma}_{\Sigma_1}$ and hence $\hat{\mathrm{R}}_{\Sigma_2} \subset \hat{\mathrm{R}}_{\Sigma_1}$. That is to say, the larger is the class of admissible shortlisting maps, the fewer will be the preference comparisons that are unambiguously revealed by a given set of choice data. This is because a more flexible specification of σ can explain more of the observed behavior, leaving less that can be used to make reliable deductions about R. Indeed, since $\hat{\sigma}_\Xi = C$, when the shortlisting map is completely unrestricted all we can infer about the preference relation is that two alternatives are indifferent if they appear together in the same choice set.

The following lemma establishes two facts about Σ-revealed preferences. Firstly, it states that any choosable alternative is always greatest with respect to these preferences among all options returned by $\hat{\sigma}_\Sigma$. And secondly, it confirms that these preferences are always genuine as long as the true shortlisting map is in Σ and the unobserved preference relation is a complete preorder.

Lemma 2.8. *Given* $\Sigma \subset \Xi$*:* **A.** $C \subset \hat{\mathrm{R}}_\Sigma\Uparrow \circ \hat{\sigma}_\Sigma \subset \hat{\mathrm{R}}^*_\Sigma\Uparrow \circ \hat{\sigma}_\Sigma$. **B.** *For any* $\sigma \in \Sigma$ *and complete preorder* R *such that* $C \subset \mathrm{R}\Uparrow \circ \sigma$*, we have* $\hat{\mathrm{R}}_\Sigma \subset \hat{\mathrm{R}}^*_\Sigma \subset \mathrm{R}$.

Observe that $\mathrm{R}\Uparrow \circ \sigma \subset C$ is not a hypothesis of Lemma 2.8B.[6] In addition, note that $\mathrm{R} \subset \hat{\mathrm{R}}^*_\Sigma$ is not a conclusion, meaning that genuine preferences need not be Σ-revealed, even indirectly.

2.4. Meta-characterization results

We characterize CP-shortlisting procedures by modifying Richter's Congruence axiom in a natural way.

Condition 2.9 (Σ-Congruence)**.** Given $\Sigma \subset \Xi$ and $x, y \in A \in \mathfrak{D}$, if $x \in C(A)$, $y \in \hat{\sigma}_\Sigma(A)$, and $y\hat{\mathrm{R}}^*_\Sigma x$ then $y \in C(A)$.

In words, if an alternative is both revealed to have been shortlisted and indirectly revealed preferred to a second alternative that is choosable, then the first alternative too must be choosable.

Observe that this new condition requires both $y \in \hat{\sigma}_\Sigma(A)$ and $y\hat{\mathrm{R}}^*_\Sigma x$ instead of simply $y[\mathrm{R}^{\mathrm{g}}]^*x$, and that these *stronger* hypotheses make the axiom a *weaker* restriction on C. Of course, setting $\Sigma = \Sigma^{\mathrm{id}}$ yields the original Congruence axiom since — as already noted — we have both $\hat{\sigma}_{\Sigma^{\mathrm{id}}} = \top$ and $\hat{\mathrm{R}}_{\Sigma^{\mathrm{id}}} = \mathrm{R}^{\mathrm{g}}$. Moreover, when $\Sigma_1 \cap \Xi_C \subset \Sigma_2 \cap \Xi_C$ it follows that Σ_1-Congruence implies Σ_2-Congruence. And finally, the Ξ-Congruence axiom (which leaves the shortlisting map unrestricted) is easily seen to be vacuous, since it includes $y \in \hat{\sigma}_\Xi(A) = C(A)$ as a hypothesis.

We are now in a position to state our main meta-characterization result.

Theorem 2.10. *Given $\Sigma \subset \Xi$:* **A.** *If the choice function is a CP-shortlisting procedure of class Σ, then Σ-Congruence holds.* **B.** *If Σ-Congruence holds and $\langle \Sigma, \subset \rangle$ is a complete lattice, then the choice function is a CP-shortlisting procedure of class Σ.*

The first part of this result, establishing the necessity of Σ-Congruence, is a more or less direct corollary of Lemma 2.8B. Sufficiency of the axiom is much less transparent, and it is here that we need the lattice structure of $\langle \Sigma, \subset \rangle$ and the resulting fact that $\hat{\sigma}_\Sigma \in \Sigma$ (see Proposition 2.3).

One strength of Theorem 2.10 is that it allows for choice sets with multiple elements. In contrast, many results of this sort adopt the simplifying assumption that choice functions are single-valued.

Condition 2.11 (Univalence)**.** For each $A \in \mathfrak{D}$ we have $x, y \in C(A)$ only if $x = y$.

We can specialize our meta-characterization to this setting by balancing the imposition of single-valued choice with a complete ordering (e.g., no-indifference) requirement on the second-stage relation.

Theorem 2.12. *Given $\Sigma \subset \Xi$:* **A.** *If the choice function is a CO-shortlisting procedure of class Σ, then both Σ-Congruence and Univalence hold.* **B.** *If both Σ-Congruence and Univalence hold and $\langle \Sigma, \subset \rangle$ is a complete lattice, then the choice function is a CO-shortlisting procedure of class Σ.*

[6] This inclusion closes the model, ensuring that if an alternative is both shortlisted and preference-greatest among all shortlisted options, then it is not eliminated in some hypothetical additional stage. Lemma 2.8B remains valid even if the model is not closed in this way.

Incidentally, Σ-Congruence and Univalence can be combined into a single axiom that permits a simpler statement of Theorem 2.12.

Condition 2.13 (Σ-Anticyclicity)**.** Given $\Sigma \subset \Xi$ and $x, y \in X$, we have $x\hat{\mathrm{R}}^*_\Sigma y \hat{\mathrm{R}}^*_\Sigma x$ only if $x = y$.

This condition is clearly necessary when the preference relation R is antisymmetric, since $\hat{\mathrm{R}}^*_\Sigma \subset \mathrm{R}$ by Lemma 2.8B. It implies Σ-Congruence since $x \in C(A)$, $y \in \hat{\sigma}_\Sigma(A)$, and $y\hat{\mathrm{R}}^*_\Sigma x$ yield $x\hat{\mathrm{R}}_\Sigma y \hat{\mathrm{R}}^*_\Sigma x$ and hence $y = x \in C(A)$. And it implies Univalence since $x, y \in C(A)$ only if $x, y \in \hat{\mathrm{R}}^*_\Sigma{\Uparrow} \circ \hat{\sigma}_\Sigma(A)$ by Lemma 2.8A, $x\hat{\mathrm{R}}^*_\Sigma y \hat{\mathrm{R}}^*_\Sigma x$, and thus $x = y$. We conclude the following:

Proposition 2.14. *Given* $\Sigma \subset \Xi$*:* **A.** *If the choice function is a CO-shortlisting procedure of class* Σ*, then* Σ*-Anticyclicity holds.* **B.** *If* Σ*-Anticyclicity holds and* $\langle \Sigma, \subset \rangle$ *is a complete lattice, then the choice function is a CO-shortlisting procedure of class* Σ*.*

When $\langle \Sigma, \subset \rangle$ is a complete lattice, a necessary and sufficient condition for the shortlisting model with complete-order preferences is therefore provided by the requirement that all $\hat{\mathrm{R}}_\Sigma$-cycles be degenerate.[7]

3. Applications

3.1. Consideration/contraction filters

Lleras et al. [16] investigate a procedure defined by the following class of shortlisting maps, which imposes on σ a standard "contraction consistency" condition.[8]

Definition 3.1. We call $\sigma \in \Xi$ a *consideration* (or *contraction*) *filter* and write $\sigma \in \Sigma^{\mathrm{cf}}$ if $\forall A, B \in \mathfrak{D}$ such that $A \subset B$ we have $\sigma(B) \cap A \subset \sigma(A)$.

Here the decision maker is imagined to be cognitively constrained, the relative complexity of different menus is assumed to be aligned with set inclusion, and $\sigma(A)$ is interpreted as the "consideration set" corresponding to menu A.[9] Membership in Σ^{cf} is consistent with a number of heuristic rules, such as considering only the n best alternatives according to a given attribute, or considering only alternatives that are best according to at least one attribute. Essentially the same model is studied by Spears [37] and Tyson [40, pp. 56–65].

It is straightforward to confirm that the theory in Section 2 can be applied to the case of consideration filters.

Proposition 3.2. $\left\langle \Sigma^{\mathrm{cf}}, \subset \right\rangle$ *is a complete lattice.*

[7] In particular, the classical model with complete-order preferences is characterized by the conjunction of Congruence and Univalence, which amounts to the requirement that all R^{g}-cycles be degenerate.

[8] Early uses of this condition appear in Nash [25, p. 159], Chernoff [8, p. 429], and Sen [32, p. 384].

[9] For discussion and references relating to the concept of the consideration set, as well as an application to industrial organization, see Eliaz and Spiegler [10].

Indeed, for $\Psi \subset \Sigma^{\mathrm{cf}}$ and $A, B \in \mathfrak{D}$ such that $A \subset B$, we have

$$[\bigwedge\Psi](B) \cap A = [\bigcap_{\sigma\in\Psi}\sigma(B)] \cap A = \bigcap_{\sigma\in\Psi}[\sigma(B) \cap A] \subset \bigcap_{\sigma\in\Psi}\sigma(A) = [\bigwedge\Psi](A),$$

and hence $\bigwedge\Psi \in \Sigma^{\mathrm{cf}}$ as desired. Theorem 2.10 then yields a specialized characterization.

Corollary 3.3. *The choice function is a CP-shortlisting procedure of class Σ^{cf} if and only if Σ^{cf}-Congruence holds.*

This finding may be compared with related results in Lleras et al. [16, p. 31], Spears [37, p. 6], and Tyson [40, p. 64], all of which are less general due to one or more background assumptions.

We can also give a more explicit expression for the revealed shortlisting map defined by $\hat{\sigma}_{\Sigma^{\mathrm{cf}}} = \bigwedge[\Sigma^{\mathrm{cf}} \cap \Xi_C]$.

Definition 3.4. Define $\hat{\rho}_{\Sigma^{\mathrm{cf}}} \in \Xi$ as follows: For each $x \in A \in \mathfrak{D}$, let $x \in \hat{\rho}_{\Sigma^{\mathrm{cf}}}(A)$ if and only if $\exists B \in \mathfrak{D}$ such that $A \subset B$ and $x \in C(B)$.

Proposition 3.5. $\hat{\sigma}_{\Sigma^{\mathrm{cf}}} = \hat{\rho}_{\Sigma^{\mathrm{cf}}}$.

In words, an alternative is revealed to be shortlisted from a particular menu if and only if it is choosable from some weakly larger menu.[10] This formulation substantially simplifies construction of $\hat{\mathrm{R}}_{\Sigma^{\mathrm{cf}}}$ and verification (or falsification) of Σ^{cf}-Congruence.

In a companion paper to [16], Masatlioglu et al. [23] impose a different restriction on consideration sets. A variant of this property appears as Fishburn's [11, p. 976] "Axiom 2," while Johnson and Dean [14, p. 58] refer to it as "Aizerman's Axiom."

Definition 3.6. We call $\sigma \in \Xi$ an *attention* (or *Aizerman*) *filter* and write $\sigma \in \Sigma^{\mathrm{af}}$ if $\forall A, B \in \mathfrak{D}$ such that $\sigma(B) \subset A \subset B$ we have $\sigma(A) = \sigma(B)$.

The interpretation in [23] is that $\sigma(B)$ contains those alternatives on menu B of which the decision maker is aware, and that this set should remain unchanged whenever other options are eliminated.[11]

Attention filters are an example of a class of selection functions that does *not* possess the lattice structure needed to apply the methods of Section 2.

Example 3.7. Let $X = wxyz$ and $\mathfrak{D} = \{wy, wxyz\}$.[12] Define $\sigma_1 \in \Xi$ by $\sigma_1(wy) = w$ and $\sigma_1(wxyz) = xy$; and define $\sigma_2 \in \Xi$ by $\sigma_2(wy) = w$ and $\sigma_2(wxyz) = yz$. We then have $[\sigma_1 \wedge \sigma_2](wy) = w$ and $[\sigma_1 \wedge \sigma_2](wxyz) = y$. It follows that $\sigma_1, \sigma_2 \in \Sigma^{\mathrm{af}}$ and $\sigma_1 \wedge \sigma_2 \notin \Sigma^{\mathrm{af}}$, and so $\langle \Sigma^{\mathrm{af}}, \subset \rangle$ is not a lattice.

Thus Theorems 2.10B and 2.12B cannot be applied in this instance, though the relevant axioms are of course still *necessary* for shortlisting procedures of class Σ^{af}.

[10] Versions of this conclusion appear in [16, p. 14], [37, p. 11], and [40, p. 58].

[11] Under the assumptions that X is finite and $\mathfrak{D} = \mathfrak{X} \setminus \{\emptyset\}$, the condition imposed in Definition 3.6 is expressed in [23] as $\forall B \in \mathfrak{D}\ \ [x \in B \setminus \sigma(B) \Longrightarrow \sigma(B \setminus \{x\}) = \sigma(B)]$.

[12] Note the multiplicative notation for enumerated sets.

3.2. Expansion filters

Tyson [41] models bounded rationality by means of menu-dependent preferences that can become decreasingly fine-grained as the complexity of the choice problem increases. As in the consideration-set environment discussed above, relative complexity is assumed to be aligned with set inclusion. Formally, a *relation system* $\mathcal{R} = \langle \mathrm{R}_A \rangle_{A \in \mathfrak{D}}$ encodes the agent's "perceived preferences," with each R_A a relation on the menu A and choices generated via $C(A) = \mathcal{R}{\Uparrow}(A) = \mathrm{R}_A{\Uparrow}(A)$. The interaction of complexity and cognition is captured by the *nestedness* condition on $\mathcal{R}$ that $\forall A, B \in \mathfrak{D}$ with $A \subset B$, and $\forall x, y \in A$, we have $x\mathrm{R}_A y$ only if $x\mathrm{R}_B y$. Equivalently, this can be expressed as $\neg[x\mathrm{R}_B y]$ only if $\neg[x\mathrm{R}_A y]$; i.e., a strict preference for y over x perceived in the larger choice problem B must also be perceived in the smaller problem A.[13] When the perceived preference system $\mathcal{R}$ is nested and each component R_A is a complete preorder, the resulting behavior is shown to be related to a form of "satisficing" in the sense of Simon [36].

In [42], the nested-relation-system structure is augmented with a second stage that allows the decision maker's "pseudo-indifference" between R_A-greatest alternatives to be broken by their relative salience — a property that could be determined in some contexts by non-informative advertising. Denoting the salience relation by S, choice sets are thus determined as $C(A) = \mathrm{S}{\Uparrow} \circ \mathcal{R}{\Uparrow}(A)$. Viewing the selection function $\mathcal{R}{\Uparrow}$ as a shortlisting map, this two-stage model is covered by our analytical framework, though with a new interpretation under which it is the first rather than the second stage that contains information about the agent's preferences.

When $\mathcal{R}$ is nested and consists of complete preorders, the associated selection function $\sigma = \mathcal{R}{\Uparrow}$ exhibits "strong expansion consistency" (see [41, p. 56]).[14]

Definition 3.8. We call $\sigma \in \Xi$ a *strong-expansion filter* and write $\sigma \in \Sigma^{\mathrm{se}}$ if $\forall A, B \in \mathfrak{D}$ such that $A \subset B$ and $\sigma(B) \cap A \neq \emptyset$ we have $\sigma(A) \subset \sigma(B)$.

Like consideration and unlike attention filters, this class has the lattice structure needed to apply our general theory.

Proposition 3.9. $\langle \Sigma^{\mathrm{se}}, \subset \rangle$ *is a complete lattice.*

Corollary 3.10. *The choice function is a CP-shortlisting procedure of class* Σ^{se} *if and only if* Σ^{se}*-Congruence holds.*

This result reproduces a substantial part of the content of [42, pp. 10–12].

Once again it is useful to have an explicit expression for the revealed shortlisting map $\hat{\sigma}_{\Sigma^{\mathrm{se}}} = \bigwedge[\Sigma^{\mathrm{se}} \cap \Xi_C]$. This is achieved in [42] by defining a relation system $\mathcal{R}^\ell$ that identifies what are termed "revealed pseudo-preferences."[15]

Definition 3.11. For $x, y \in B \in \mathfrak{D}$, we write $x\mathrm{R}^\ell_B y$ if $\exists A \in \mathfrak{D}$ such that $y \in A \subset B$ and $x \in C(A)$.

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[13] In fact, perceived strict preference is the primitive notion in [41], and thus the definition of nestedness directly parallels that of a consideration filter in terms of the perception of preferences or alternatives.

[14] This property is due to Bordes [3, p. 452] and Sen [34, p. 66].

[15] Here the superscript on $\mathcal{R}^\ell$ stands for "local," whereas that on R^{g} (Definition 2.4) stands for "global."

The alternatives revealed to be shortlisted from menu B are then those that are greatest with respect to the transitive closure $[\mathrm{R}^\ell_B]^*$ of the relevant component of $\mathcal{R}^\ell$.

Proposition 3.12. $\hat{\sigma}_{\Sigma^{\mathrm{se}}} = [\mathcal{R}^\ell]^* \Uparrow$.

Suppose now that we relax the assumption that the components of $\mathcal{R}$ are complete preorders, while retaining the nestedness requirement. This enables the model of bounded rationality with salience effects to incorporate other cognitive imperfections vis-à-vis the classical model, such as incompleteness or intransitivity of perceived preferences. When no special ordering assumptions are imposed on $\mathcal{R}$, the shortlisting function $\sigma = \mathcal{R}\Uparrow$ need not be in Σ^{se} but will still exhibit "weak expansion consistency" (see [41, p. 60]).[16]

Definition 3.13. We call $\sigma \in \Xi$ a *weak-expansion filter* and write $\sigma \in \Sigma^{\mathrm{we}}$ if $\forall \mathfrak{B} \subset \mathfrak{D}$ such that $\bigcup_{B\in\mathfrak{B}} B \in \mathfrak{D}$ we have $\bigcap_{B\in\mathfrak{B}} \sigma(B) \subset \sigma(\bigcup_{B\in\mathfrak{B}} B)$.

We can confirm that this class of shortlisting maps has the desired structure.

Proposition 3.14. *$\langle \Sigma^{\mathrm{we}}, \subset \rangle$ is a complete lattice.*

Corollary 3.15. *The choice function is a CP-shortlisting procedure of class Σ^{we} if and only if Σ^{we}-Congruence holds.*

Note that this case is not considered in [42].

It is simple to show that $\Sigma^{\mathrm{we}} \cap \Xi_C \supset \Sigma^{\mathrm{se}} \cap \Xi_C$, and therefore we know a priori that $\hat{\sigma}_{\Sigma^{\mathrm{we}}} \subset \hat{\sigma}_{\Sigma^{\mathrm{se}}} = [\mathcal{R}^\ell]^* \Uparrow$.[17] Indeed, to find the revealed shortlisting map in this instance we need only drop the transitive closure operator.

Proposition 3.16. $\hat{\sigma}_{\Sigma^{\mathrm{we}}} = \mathcal{R}^\ell \Uparrow$.

3.3. Extraction filters

We turn now to shortlisting maps generated by ordinary binary relations, as opposed to the relation systems used in Section 3.2.

Manzini and Mariotti's [18] "rational shortlist methods" involve a primary relation Q used to eliminate alternatives before application of a secondary relation R. The choice set associated with menu A is thus determined as $C(A) = \mathrm{R}\Uparrow \circ \mathrm{Q}\Uparrow(A)$, and the shortlisting map has the simple form $\mathrm{Q}\Uparrow$. The primary and secondary relations are independent of each other and can have various interpretations depending on the context. For example, Manzini and Mariotti imagine "a cautious investor comparing alternative portfolios [who] first eliminates those that are too risky relative to others available, and then ranks the surviving ones on the basis of expected returns."[18]

The properties of a shortlisting map expressible as $\sigma = \mathrm{Q}\Uparrow$ are well known. Under the full-domain assumption $\mathfrak{D} = \mathfrak{X} \setminus \{\emptyset\}$, a map is of this form if and only if it is in the class $\Sigma^{\mathrm{cf}} \cap \Sigma^{\mathrm{we}}$ of selection functions exhibiting both contraction and weak-expansion

[16] This property first appeared in Sen [33, p. 314].

[17] Despite the "strong" and "weak" nomenclature, it is technically not true that $\Sigma^{\mathrm{se}} \subset \Sigma^{\mathrm{we}}$. To ensure that $\sigma \in \Sigma^{\mathrm{se}}$ is also in Σ^{we} we need this function to be nonempty-valued, for which $\sigma \in \Xi_C$ is sufficient.

[18] See also the related models in Cherepanov et al. [7] and Manzini and Mariotti [19].

consistency (see Sen [33, p. 314]).[19] These properties can be merged and strengthened to yield the following requirement, which is necessary and sufficient with an arbitrary domain (and thus equivalent to Richter's [28, p. 33] "V-Axiom").

Definition 3.17. We call $\sigma \in \Xi$ an *extraction filter* and write $\sigma \in \Sigma^{\mathrm{ef}}$ if $\forall A \in \mathfrak{D}$ and $\mathfrak{B} \subset \mathfrak{D}$ such that $A \subset \bigcup_{B \in \mathfrak{B}} B$ we have $[\bigcap_{B \in \mathfrak{B}} \sigma(B)] \cap A \subset \sigma(A)$.

From the statement of this property it is apparent both that $\Sigma^{\mathrm{ef}} \subset \Sigma^{\mathrm{cf}} \cap \Sigma^{\mathrm{we}}$ in general and that this inclusion holds as an equality in the full-domain case (where we know that $\bigcup_{B \in \mathfrak{B}} B \in \mathfrak{D}$).

As usual, our first task is to check the lattice structure of the class of relation-generated shortlists.

Proposition 3.18. *$\langle \Sigma^{\mathrm{ef}}, \subset \rangle$ is a complete lattice.*

Corollary 3.19. *The choice function is a CP-shortlisting procedure of class Σ^{ef} if and only if Σ^{ef}-Congruence holds.*

And it is straightforward to verify that this axiom implies the two conditions identified by Manzini and Mariotti in the full-domain context.[20]

Condition 3.20 (Generalized Weak WARP)**.** Given $A, B, D \in \mathfrak{D}$ and $x, y \in A$ such that $A \subset B \subset D$, if $x \in C(A) \cap C(D)$ and $y \in C(B)$ then $x \in C(B)$.

Condition 3.21 (Weak Expansion)**.** $C \in \Sigma^{\mathrm{we}}$.

Proposition 3.22. *Σ^{ef}-Congruence implies Generalized Weak WARP and Weak Expansion.*

Since $\Sigma^{\mathrm{ef}} \subset \Sigma^{\mathrm{we}}$ we know that $\hat{\sigma}_{\Sigma^{\mathrm{ef}}} \supset \hat{\sigma}_{\Sigma^{\mathrm{we}}} = \mathcal{R}^{\ell}\Uparrow$. And in fact the revealed shortlisting map for extraction filters simply replaces the revealed pseudo-preference system $\mathcal{R}^{\ell}$ with the traditional revealed preference relation R^{g}.

Proposition 3.23. $\hat{\sigma}_{\Sigma^{\mathrm{ef}}} = \mathrm{R}^{\mathrm{g}}\Uparrow$.

3.4. Weak-axiom filters

Our final application is to shortlists generated by binary relations via a stronger form of maximization. In Mariotti's [21, p. 405] terminology, a selection function ξ is *justified* by a relation Q if $\xi = \mathrm{Q}\Uparrow$ and $\forall x, y \in A \in \mathfrak{D}$ with $x \in \xi(A)$ and $y\mathrm{Q}x$ we have $y \in \xi(A)$.[21] Thus justification requires not only that the selected alternatives be those that are greatest with respect to Q, but also that no available but unselected alternative bear the relation Q to any selected one.

[19]Stronger consistency requirements would be implied if we were to impose ordering properties on Q such as completeness or transitivity. (On this point, see Section 4.1.)

[20]More precisely, Manzini and Mariotti specify "Weak WARP," a version of Condition 3.20 for single-valued choice functions, together with weak expansion consistency for pairs of sets rather than arbitrary collections as in Condition 3.21. In each case our version of the condition is slightly more general.

[21]Clark [9] refers to this relationship as "strict rationalization."

When the shortlisting map σ is justified by a relation it is clearly in the class Σ^{ef}, and hence also in $\Sigma^{\text{cf}} \cap \Sigma^{\text{we}}$. Clark [9, p. 488] and Mariotti [21, p. 405] determine the implied restriction on σ more precisely, showing that a selection function on an arbitrary domain is justified if and only if it satisfies the familiar "weak axiom [of revealed preference]."[22]

Definition 3.24. We call $\sigma \in \Xi$ a *weak-axiom filter* and write $\sigma \in \Sigma^{\text{wa}}$ if $\forall A, B \in \mathfrak{D}$ such that $\sigma(B) \cap A \neq \emptyset$ we have $\sigma(A) \cap B \subset \sigma(B)$.

Here adding the hypothesis $A \subset B$ would yield the definition of a strong-expansion filter, so we have $\Sigma^{\text{wa}} \subset \Sigma^{\text{se}}$. Moreover, the required lattice structure is present with or without this hypothesis.

Proposition 3.25. $\langle \Sigma^{\text{wa}}, \subset \rangle$ *is a complete lattice.*

Corollary 3.26. *The choice function is a CP-shortlisting procedure of class* Σ^{wa} *if and only if* Σ^{wa}*-Congruence holds.*

The revealed shortlisting map $\hat{\sigma}_{\Sigma^{\text{wa}}} = \bigwedge[\Sigma^{\text{wa}} \cap \Xi_C]$ is most easily identified by means of an algorithmic construction.

Definition 3.27. A. Define $\hat{\tau}^0_{\Sigma^{\text{wa}}} \in \Xi$ by $\hat{\tau}^0_{\Sigma^{\text{wa}}} = C$. **B.** For each $k \geq 0$, define $\hat{\tau}^{k+1}_{\Sigma^{\text{wa}}} \in \Xi$ inductively as follows: For each $x \in B \in \mathfrak{D}$, let $x \in \hat{\tau}^{k+1}_{\Sigma^{\text{wa}}}(B)$ if and only if $\exists y \in A \in \mathfrak{D}$ such that $x \in \hat{\tau}^k_{\Sigma^{\text{wa}}}(A)$ and $y \in \hat{\tau}^k_{\Sigma^{\text{wa}}}(B)$. **C.** Define $\hat{\tau}_{\Sigma^{\text{wa}}} \in \Xi$ by $\hat{\tau}_{\Sigma^{\text{wa}}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\text{wa}}}$.

Observe that here $\hat{\tau}^{k+1}_{\Sigma^{\text{wa}}}(B) \supset \hat{\tau}^k_{\Sigma^{\text{wa}}}(B)$, since we can always take $y = x$ and $A = B$. The construction builds up $\hat{\sigma}_{\Sigma^{\text{wa}}}$ iteratively from the observed choices in C, by adding at each stage the alternatives whose shortlisting can be deduced via the weak axiom property of the map σ.

Proposition 3.28. $\hat{\sigma}_{\Sigma^{\text{wa}}} = \hat{\tau}_{\Sigma^{\text{wa}}}$.

3.5. Summary of applications

A summary of our applications of Theorem 2.10 appears in Figure 1. Here the left panel shows logical relationships among the axioms characterizing shortlisting procedures of five classes: Σ^{cf} (with σ satisfying contraction consistency), Σ^{se} (with $\sigma = \mathcal{R}\Uparrow$ and $\mathcal{R}$ a nested system of complete preorders), Σ^{we} (with $\sigma = \mathcal{R}\Uparrow$ and $\mathcal{R}$ a nested relation system), Σ^{ef} (with $\sigma = \mathrm{Q}\Uparrow$), and Σ^{wa} (with σ justified by Q). The trivial class $\Sigma^{\text{id}} = \{\top\}$, which prohibits meaningful shortlisting and thus yields the standard model, is included for the sake of comparison. (The related class Σ^{sa} is discussed in Section 4.1.) Σ^{wa}-Congruence implying Σ^{ef}-Congruence, for example, reflects the fact that σ can be justified by Q only if $\sigma = \mathrm{Q}\Uparrow$.

The right panel in Figure 1 shows pointwise inclusions among the revealed shortlisting maps associated with our various classes of procedures. For example, we have $\hat{\sigma}_{\Sigma^{\text{wa}}} \supset \hat{\sigma}_{\Sigma^{\text{ef}}}$ (a consequence of $\Sigma^{\text{wa}} \subset \Sigma^{\text{ef}}$), leading to $\hat{\mathrm{R}}_{\Sigma^{\text{wa}}} \supset \hat{\mathrm{R}}_{\Sigma^{\text{ef}}}$ and the aforementioned implication between congruence conditions. The figure also records the explicit construction of each revealed shortlisting map; for example, $\hat{\sigma}_{\Sigma^{\text{ef}}} = \bigwedge[\Sigma^{\text{ef}} \cap \Xi_C]$ can be expressed as $\mathrm{R}^{\text{g}}\Uparrow$. The number of the relevant Proposition is shown above each nontrivial equality.

[22] This is Arrow's [2, p. 123] condition "C5," a generalization of Samuelson's [31, p. 65] "Postulate III."

Axioms	Revealed shortlisting maps
Σ^{id}-Congruence $\Longleftrightarrow$ Σ^{sa}-Congruence	$\top = \hat{\sigma}_{\Sigma^{\mathrm{id}}} \supset \hat{\sigma}_{\Sigma^{\mathrm{sa}}} \overset{4.6}{=} [\mathrm{R}^{\mathrm{g}}]^{*}\Uparrow \overset{4.9\mathrm{B}}{=} \hat{\tau}_{\Sigma^{\mathrm{sa}}}$
$\Downarrow$ $\quad\Downarrow$	$\cup \quad \cup$
Σ^{wa}-Congruence $\Longrightarrow$ Σ^{se}-Congruence	$\hat{\tau}_{\Sigma^{\mathrm{wa}}} \overset{3.28}{=} \hat{\sigma}_{\Sigma^{\mathrm{wa}}} \supset \hat{\sigma}_{\Sigma^{\mathrm{se}}} \overset{3.12}{=} [\mathcal{R}^{\ell}]^{*}\Uparrow \overset{4.9\mathrm{A}}{=} \hat{\tau}_{\Sigma^{\mathrm{se}}}$
$\Downarrow$ $\quad\Downarrow$	$\cup \quad \cup$
Σ^{ef}-Congruence $\Longrightarrow$ Σ^{we}-Congruence	$\hat{\tau}^{1}_{\Sigma^{\mathrm{ef}}} \overset{4.13\mathrm{C}}{=} \mathrm{R}^{\mathrm{g}}\Uparrow \overset{3.23}{=} \hat{\sigma}_{\Sigma^{\mathrm{ef}}} \supset \hat{\sigma}_{\Sigma^{\mathrm{we}}} \overset{3.16}{=} \mathcal{R}^{\ell}\Uparrow \overset{4.13\mathrm{B}}{=} \hat{\tau}^{1}_{\Sigma^{\mathrm{we}}}$
$\Downarrow$	$\cup$
Σ^{cf}-Congruence	$\hat{\tau}^{1}_{\Sigma^{\mathrm{cf}}} \overset{4.13\mathrm{A}}{=} \hat{\rho}_{\Sigma^{\mathrm{cf}}} \overset{3.5}{=} \hat{\sigma}_{\Sigma^{\mathrm{cf}}}$

Figure 1: Summary of applications of Theorem 2.10. Depicted are logical relationships among the axioms that characterize several classes of shortlisting procedures (left panel), together with inclusions among the associated revealed shortlisting maps (right panel).

Recall that for each application an analogous characterization for single-valued choice functions follows from Theorem 2.12.

4. Additional results

4.1. Strong-axiom filters

As mentioned above in Section 3.3, an extraction filter is a shortlisting map generated by a binary relation that need not possess any particular ordering properties. Suppose now that we require this "primary" relation to be a complete preorder. The associated class of CP-shortlisting procedures will then contain choice functions of the form $C = \mathrm{R}\Uparrow \circ \mathrm{Q}\Uparrow$ with Q and R both complete and transitive. As a consequence of Theorem 2.6, the map $\sigma = \mathrm{Q}\Uparrow$ will in this case satisfy the Richterian congruence axiom stated in terms of C as Condition 2.5. For general selection functions, this requirement can be expressed as follows.

Definition 4.1. Given $\xi \in \Xi$ and $x, y \in X$, we write $x[\![\xi]\!]y$ if $\exists A \in \mathfrak{D}$ such that $y \in A$ and $x \in \xi(A)$.

Definition 4.2. We call $\sigma \in \Xi$ a *strong-axiom filter* and write $\sigma \in \Sigma^{\mathrm{sa}}$ if $\forall x, y \in A \in \mathfrak{D}$ such that $x \in \sigma(A)$ and $y[\![\sigma]\!]^{*}x$ we have $y \in \sigma(A)$.

And we then have both that $[\![C]\!] = \mathrm{R}^{\mathrm{g}}$ and that $C \in \Sigma^{\mathrm{sa}}$ restates Congruence.

There is no difficulty in showing that Theorem 2.10 applies to the class of strong-axiom filters.

Proposition 4.3. *$\langle \Sigma^{\mathrm{sa}}, \subset \rangle$ is a complete lattice.*

Corollary 4.4. *The choice function is a CP-shortlisting procedure of class Σ^{sa} if and only if Σ^{sa}-Congruence holds.*

However, the choice functions characterized in this way are not a new subset of the space of selection functions; they are precisely those that are consistent with the classical model.

Proposition 4.5. Σ^{sa}-*Congruence is logically equivalent to Congruence.*

In contrast to our other results of the same sort, Corollary 4.4 is therefore not a true refinement of Theorem 2.6.

Proposition 4.5 establishes that in terms of behavioral implications, imposing $\sigma \in \Sigma^{\mathrm{sa}}$ collapses the general shortlisting model to its classical (no-shortlisting) special case. The reason for this is easily appreciated: When Q and R are both complete preorders, we have $\mathrm{R}\Uparrow \circ \mathrm{Q}\Uparrow = \mathrm{L}\Uparrow$ for L defined as the lexicographic composition of Q and R (i.e., by $x\mathrm{L}y$ if and only if $x\mathrm{Q}y$ and either $x\mathrm{R}y$ or $\neg[y\mathrm{Q}x]$). And since in this case L itself will be a complete preorder, it follows that $C = \mathrm{L}\Uparrow$ will satisfy Congruence.

While the Σ^{id} and Σ^{sa} classes of procedures are behaviorally equivalent, they do not share the same revealed shortlisting map. On the one hand, it is immediate that $\hat{\sigma}_{\Sigma^{\mathrm{id}}} = \top$. On the other, we can show that in regard to revealed shortlisting the class Σ^{sa} bears a relationship to Σ^{ef} resembling that of Σ^{se} to Σ^{we}: In each case the difference is the presence or absence of a transitive closure operator.

Proposition 4.6. $\hat{\sigma}_{\Sigma^{\mathrm{sa}}} = [\mathrm{R}^{\mathrm{g}}]^{*}\Uparrow$.

In summary, the CP-shortlisting procedures of classes Σ^{id} and Σ^{sa} are identical and are demarcated by Congruence. These choice functions are explained by the two hypotheses in different ways, however. Disallowing shortlisting leads us to interpret the behavior as $C = [\mathrm{R}^{\mathrm{g}}]^{*}\Uparrow \circ \top$, with alternatives eliminated only at the second stage. In contrast, if we permit shortlisting of the form $\sigma = \mathrm{Q}\Uparrow$ but require Q to be a complete preorder, then our construction will yield $C = \top \circ [\mathrm{R}^{\mathrm{g}}]^{*}\Uparrow$ and feature elimination only at the first stage. This illustrates the fact that our analysis uses the shortlisting map to explain as much of the behavior as possible, employing the second stage only when it is genuinely needed.[23]

Observe that the class Σ^{sa} and the associated map $\hat{\sigma}_{\Sigma^{\mathrm{sa}}}$ are included in the summary of applications in Figure 1.

4.2. Algorithmic revelation of shortlisting

As Figure 1 indicates, Propositions 3.5, 3.12, 3.16, 3.23, 3.28, and 4.6 provide explicit expressions for the maps $\hat{\sigma}_{\Sigma} = \bigwedge[\Sigma \cap \Xi_C]$ corresponding to the classes $\Sigma = \Sigma^{\mathrm{cf}}$, Σ^{se}, Σ^{we}, Σ^{ef}, Σ^{wa}, and Σ^{sa}. Among these results, Proposition 3.28 is unique in constructing $\hat{\sigma}_{\Sigma^{\mathrm{wa}}}$ algorithmically via a sequence $\langle \hat{\tau}^{0}_{\Sigma^{\mathrm{wa}}}, \hat{\tau}^{1}_{\Sigma^{\mathrm{wa}}}, \hat{\tau}^{2}_{\Sigma^{\mathrm{wa}}}, \ldots \rangle$ of selection functions. We now show that the other five results can be seen in the same light, unifying this aspect of the theory.

The cases of Σ^{se} and Σ^{sa} directly parallel that of Σ^{wa}.

Definition 4.7. **A.** Define $\hat{\tau}^{0}_{\Sigma^{\mathrm{se}}} \in \Xi$ by $\hat{\tau}^{0}_{\Sigma^{\mathrm{se}}} = C$. **B.** For each $k \geq 0$, define $\hat{\tau}^{k+1}_{\Sigma^{\mathrm{se}}} \in \Xi$ inductively as follows: For each $x \in B \in \mathfrak{D}$, let $x \in \hat{\tau}^{k+1}_{\Sigma^{\mathrm{se}}}(B)$ if and only if $\exists y \in A \in \mathfrak{D}$ such that $A \subset B$, $x \in \hat{\tau}^{k}_{\Sigma^{\mathrm{se}}}(A)$, and $y \in \hat{\tau}^{k}_{\Sigma^{\mathrm{se}}}(B)$. **C.** Define $\hat{\tau}_{\Sigma^{\mathrm{se}}} \in \Xi$ by $\hat{\tau}_{\Sigma^{\mathrm{se}}} = \bigcup_{k=0}^{\infty} \hat{\tau}^{k}_{\Sigma^{\mathrm{se}}}$.

[23] Note that expressing C as either $[\mathrm{R}^{\mathrm{g}}]^{*}\Uparrow \circ \top$ or $\top \circ [\mathrm{R}^{\mathrm{g}}]^{*}\Uparrow$ does not in itself establish the sufficiency of Congruence for CP-shortlisting procedures of class Σ^{id} or Σ^{sa}, respectively. The reason is that $[\mathrm{R}^{\mathrm{g}}]^{*}$, while transitive, need not be complete: Indeed, the heart of Richter's proof of Theorem 2.6 is his construction of a complete preorder Q with the property that $\mathrm{Q}\Uparrow = [\mathrm{R}^{\mathrm{g}}]^{*}\Uparrow$ on $\mathfrak{D}$.

Definition 4.8. **A.** Define $\hat{\tau}^0_{\Sigma^{\mathrm{sa}}} \in \Xi$ by $\hat{\tau}^0_{\Sigma^{\mathrm{sa}}} = C$. **B.** For each $k \geq 0$, define $\hat{\tau}^{k+1}_{\Sigma^{\mathrm{sa}}} \in \Xi$ inductively as follows: For each $x \in A \in \mathfrak{D}$, let $x \in \hat{\tau}^{k+1}_{\Sigma^{\mathrm{sa}}}(A)$ if and only if $\exists y \in A$ such that $y \in \hat{\tau}^k_{\Sigma^{\mathrm{sa}}}(A)$ and $x[\![\hat{\tau}^k_{\Sigma^{\mathrm{sa}}}]\!]^* y$. **C.** Define $\hat{\tau}_{\Sigma^{\mathrm{sa}}} \in \Xi$ by $\hat{\tau}_{\Sigma^{\mathrm{sa}}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\mathrm{sa}}}$.

Proposition 4.9. **A.** $\hat{\sigma}_{\Sigma^{\mathrm{se}}} = \hat{\tau}_{\Sigma^{\mathrm{se}}}$. **B.** $\hat{\sigma}_{\Sigma^{\mathrm{sa}}} = \hat{\tau}_{\Sigma^{\mathrm{sa}}}$.

Here $\hat{\tau}^{k+1}_{\Sigma^{\mathrm{se}}} \supset \hat{\tau}^k_{\Sigma^{\mathrm{se}}}$ and $\hat{\tau}^{k+1}_{\Sigma^{\mathrm{sa}}} \supset \hat{\tau}^k_{\Sigma^{\mathrm{sa}}}$, and again the functions $\hat{\sigma}_{\Sigma^{\mathrm{se}}}$ and $\hat{\sigma}_{\Sigma^{\mathrm{sa}}}$ are built up by iterating the relevant consistency condition (respectively, strong expansion and Richterian congruence). Indeed, Propositions 3.12 and 4.6 show that by forming $[\mathcal{R}^{\ell}]^*\Uparrow$ and $[\mathrm{R}^{\mathrm{g}}]^*\Uparrow$ we are in effect carrying out these iterations.

We can similarly iterate the contraction, weak expansion, and extraction consistency conditions to construct the revealed shortlisting maps for the remaining classes.

Definition 4.10. **A.** Define $\hat{\tau}^0_{\Sigma^{\mathrm{cf}}} \in \Xi$ by $\hat{\tau}^0_{\Sigma^{\mathrm{cf}}} = C$. **B.** For each $k \geq 0$, define $\hat{\tau}^{k+1}_{\Sigma^{\mathrm{cf}}} \in \Xi$ inductively as follows: For each $x \in A \in \mathfrak{D}$, let $x \in \hat{\tau}^{k+1}_{\Sigma^{\mathrm{cf}}}(A)$ if and only if $\exists B \in \mathfrak{D}$ such that $A \subset B$ and $x \in \hat{\tau}^k_{\Sigma^{\mathrm{cf}}}(B)$. **C.** Define $\hat{\tau}_{\Sigma^{\mathrm{cf}}} \in \Xi$ by $\hat{\tau}_{\Sigma^{\mathrm{cf}}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\mathrm{cf}}}$.

Definition 4.11. **A.** Define $\hat{\tau}^0_{\Sigma^{\mathrm{we}}} \in \Xi$ by $\hat{\tau}^0_{\Sigma^{\mathrm{we}}} = C$. **B.** For each $k \geq 0$, define $\hat{\tau}^{k+1}_{\Sigma^{\mathrm{we}}} \in \Xi$ inductively as follows: For each $x \in A \in \mathfrak{D}$, let $x \in \hat{\tau}^{k+1}_{\Sigma^{\mathrm{we}}}(A)$ if and only if $\exists \mathfrak{B} \subset \mathfrak{D}$ such that $\bigcup_{B \in \mathfrak{B}} B = A$ and $x \in \bigcap_{B \in \mathfrak{B}} \hat{\tau}^k_{\Sigma^{\mathrm{we}}}(B)$. **C.** Define $\hat{\tau}_{\Sigma^{\mathrm{we}}} \in \Xi$ by $\hat{\tau}_{\Sigma^{\mathrm{we}}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\mathrm{we}}}$.

Definition 4.12. **A.** Define $\hat{\tau}^0_{\Sigma^{\mathrm{ef}}} \in \Xi$ by $\hat{\tau}^0_{\Sigma^{\mathrm{ef}}} = C$. **B.** For each $k \geq 0$, define $\hat{\tau}^{k+1}_{\Sigma^{\mathrm{ef}}} \in \Xi$ inductively as follows: For each $x \in A \in \mathfrak{D}$, let $x \in \hat{\tau}^{k+1}_{\Sigma^{\mathrm{ef}}}(A)$ if and only if $\exists \mathfrak{B} \subset \mathfrak{D}$ such that $\bigcup_{B \in \mathfrak{B}} B \supset A$ and $x \in \bigcap_{B \in \mathfrak{B}} \hat{\tau}^k_{\Sigma^{\mathrm{ef}}}(B)$. **C.** Define $\hat{\tau}_{\Sigma^{\mathrm{ef}}} \in \Xi$ by $\hat{\tau}_{\Sigma^{\mathrm{ef}}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\mathrm{ef}}}$.

These three cases are much simpler, however, in that they each complete the construction in a single step.

Proposition 4.13. **A.** $\hat{\sigma}_{\Sigma^{\mathrm{cf}}} = \hat{\tau}^1_{\Sigma^{\mathrm{cf}}} = \hat{\tau}_{\Sigma^{\mathrm{cf}}}$. **B.** $\hat{\sigma}_{\Sigma^{\mathrm{we}}} = \hat{\tau}^1_{\Sigma^{\mathrm{we}}} = \hat{\tau}_{\Sigma^{\mathrm{we}}}$. **C.** $\hat{\sigma}_{\Sigma^{\mathrm{ef}}} = \hat{\tau}^1_{\Sigma^{\mathrm{ef}}} = \hat{\tau}_{\Sigma^{\mathrm{ef}}}$.

Note that these conclusions and those of Proposition 4.9 are incorporated in Figure 1.

This unified algorithmic perspective on the construction of revealed shortlisting maps sheds some light on the scope of our main "meta-characterization" results. In Section 3.1 we have seen that the class of attention filters lacks the lattice structure needed to apply Theorems 2.10B and 2.12B. Correspondingly, it would be misguided to iterate Aizerman's axiom in the hope of obtaining $\hat{\sigma}_{\Sigma^{\mathrm{af}}}$. The axiom states (in part) that if $\sigma(B) \subset A \subset B$ and $x \in \sigma(A)$ then $x \in \sigma(B)$. But we cannot express this implication algorithmically as $\hat{\tau}^k_{\Sigma^{\mathrm{af}}}(B) \subset A \subset B$ and $x \in \hat{\tau}^k_{\Sigma^{\mathrm{af}}}(A)$ only if $x \in \hat{\tau}^{k+1}_{\Sigma^{\mathrm{af}}}(B)$, since the hypothesis $\sigma(B) \subset A$ required for a valid inference does not follow from $\hat{\tau}^k_{\Sigma^{\mathrm{af}}}(B) \subset A$, even if $\hat{\tau}^k_{\Sigma^{\mathrm{af}}}(B) \subset \sigma(B)$. This suggests a relationship between the consistency conditions that generate non-lattice classes of shortlisting maps and those for which the associated iterations are defective.[24]

The primary contribution of the algorithmic approach, however, is to remove in many instances the need to guess or infer the revealed shortlisting map. In our applications of the theory, we have first proved assertions such as $\hat{\sigma}_{\Sigma^{\mathrm{cf}}} = \hat{\rho}_{\Sigma^{\mathrm{cf}}}$ and $\hat{\sigma}_{\Sigma^{\mathrm{se}}} = [\mathcal{R}^{\ell}]^*\Uparrow$ directly, and only later shown how these maps can be obtained algorithmically. But if we did not

[24] Other consistency conditions of this sort include idempotence (see [35, p. 1372]) and, more generally, path independence (see [26, p. 1080]).

know in advance (e.g., from [16] and [42]) what the explicit forms of $\hat{\sigma}_{\Sigma^{\mathrm{cf}}}$ and $\hat{\sigma}_{\Sigma^{\mathrm{se}}}$ would turn out to be, we could find them easily by iterating the relevant consistency condition.

In practice, applying our meta-characterizations is thus often simpler than previously advertised. While it remains necessary to verify the lattice structure of each target class, finding an explicit expression for the revealed shortlisting map — a step that is essential for Σ-Congruence to be testable — can in many cases be left up to an algorithm whose structure is immediately apparent from the definition of the class.

A. Appendix

Proof of Proposition 2.3. The assertion follows immediately from the definition of a complete lattice. □

Proof of Lemma 2.8. **A.** Given $A \in \mathfrak{D}$, let $x \in C(A) \subset \hat{\sigma}_\Sigma(A)$. For each $y \in \hat{\sigma}_\Sigma(A)$ we have $x\hat{\mathrm{R}}_\Sigma y$, and so $x \in \hat{\mathrm{R}}_\Sigma{\Uparrow} \circ \hat{\sigma}_\Sigma(A)$. The second inclusion is immediate.

B. The first inclusion is immediate. Given $x, y \in X$, if $x\hat{\mathrm{R}}_\Sigma y$ then $\exists A \in \mathfrak{D}$ such that $y \in \hat{\sigma}_\Sigma(A)$ and $x \in C(A) \subset \mathrm{R}{\Uparrow} \circ \sigma(A)$. Since $C \subset \sigma \in \Sigma$, we have $\hat{\sigma}_\Sigma \subset \sigma$ and thus $y \in \sigma(A)$. But then $x\mathrm{R}y$, so $\hat{\mathrm{R}}_\Sigma \subset \mathrm{R}$. Hence $\hat{\mathrm{R}}^*_\Sigma \subset \mathrm{R}^* \subset \mathrm{R}$ since R is transitive. □

Lemma A.1 (extracted from Richter [27, pp. 639–640])**.** *For any reflexive relation* Q *on* X *there exists a complete preorder* $\mathrm{S} \supset \mathrm{Q}^*$ *such that* $\forall x, y \in X$ *we have* $x\mathrm{S}y\mathrm{Q}^*x$ *only if* $x\mathrm{Q}^*y$.

Proof of Lemma A.1. Since Q is reflexive, the asymmetric part T of Q^* is a strict partial order and the symmetric part E of Q^* is a congruence with respect to T. Write $\phi(x)$ for the E-equivalence class containing a given $x \in X$, and define a strict partial order $\gg$ on $\Phi = \{\phi(x) : x \in X\}$ by $\phi(x) \gg \phi(y)$ if and only if $x\mathrm{T}y$. By Szpilrajn's Theorem [39] we can then embed $\gg$ in a linear order $\ggg$ on Φ, proceeding to define the complete preorder S by $x\mathrm{S}y$ if and only if $\neg[\phi(y) \ggg \phi(x)]$. It follows that $x\mathrm{Q}^*y$ only if either $\phi(x) \gg \phi(y)$ or $\phi(x) = \phi(y)$. But then $\phi(x) \ggg \phi(y)$ or $\phi(x) = \phi(y)$, and in either case $\neg[\phi(y) \ggg \phi(x)]$ and $x\mathrm{S}y$. Hence $\mathrm{Q}^* \subset \mathrm{S}$. Moreover, given $x, y \in X$ with $x\mathrm{S}y\mathrm{Q}^*x$, we have $\neg[\phi(y) \gg \phi(x)]$ and so $\neg[y\mathrm{T}x]$. But since $y\mathrm{Q}^*x$, this implies that $x\mathrm{Q}^*y$. □

Proof of Theorem 2.10. **A.** Let $C = \mathrm{R}{\Uparrow} \circ \sigma$ for some $\sigma \in \Sigma$ and complete preorder R. Given $x, y \in A \in \mathfrak{D}$ such that $x \in C(A) = \mathrm{R}{\Uparrow} \circ \sigma(A)$, $y \in \hat{\sigma}_\Sigma(A) \subset \sigma(A)$, and $y\hat{\mathrm{R}}^*_\Sigma x$, we have $y\mathrm{R}x$ by Lemma 2.8B. It follows that $y \in \mathrm{R}{\Uparrow} \circ \sigma(A) = C(A)$ since R is a complete preorder, and so Σ-Congruence holds.

B. Suppose Σ-Congruence holds and $\langle \Sigma, \subset \rangle$ is a complete lattice. Define Q by $x\mathrm{Q}y$ if and only if $x\hat{\mathrm{R}}_\Sigma y$ or $x = y$; so that $\hat{\mathrm{R}}_\Sigma \subset \mathrm{Q}$. Now define S by $x\mathrm{S}y$ if and only if $x\hat{\mathrm{R}}^*_\Sigma y$, $\neg[y\hat{\mathrm{R}}^*_\Sigma x]$, or $x = y$. Observe that $C \subset \hat{\mathrm{R}}^*_\Sigma{\Uparrow} \circ \hat{\sigma}_\Sigma \subset \mathrm{Q}^*{\Uparrow} \circ \hat{\sigma}_\Sigma$, using Lemma 2.8A. Given $x \in A \in \mathfrak{D}$, if $x \in \hat{\sigma}_\Sigma(A) \setminus C(A)$ then $\exists y \in C(A) \subset \hat{\mathrm{R}}^*_\Sigma{\Uparrow} \circ \hat{\sigma}_\Sigma(A)$, so both $y\hat{\mathrm{R}}^*_\Sigma x$ and $y \neq x$. We have also $\neg[x\hat{\mathrm{R}}^*_\Sigma y]$ by Σ-Congruence, so $\neg[x\mathrm{S}y]$ and $x \notin \mathrm{S}{\Uparrow} \circ \hat{\sigma}_\Sigma(A)$. It follows that $\mathrm{S}{\Uparrow} \circ \hat{\sigma}_\Sigma \subset C$ by contraposition. Since Q is reflexive, by Lemma A.1 there exists a complete preorder $\mathrm{R} \supset \mathrm{Q}^*$ with $\mathrm{R} \subset \mathrm{S}$. But then $C \subset \mathrm{Q}^*{\Uparrow} \circ \hat{\sigma}_\Sigma \subset \mathrm{R}{\Uparrow} \circ \hat{\sigma}_\Sigma \subset \mathrm{S}{\Uparrow} \circ \hat{\sigma}_\Sigma \subset C$ and so $C = \mathrm{R}{\Uparrow} \circ \hat{\sigma}_\Sigma$, with $\hat{\sigma}_\Sigma \in \Sigma$ by Proposition 2.3 and R a complete preorder. □

Proof of Theorem 2.12. **A.** Let $C = \mathrm{R}\Uparrow \circ \sigma$ for some $\sigma \in \Sigma$ and complete order R. Since any complete order is a complete preorder, Σ-Congruence then holds by Theorem 2.10. Moreover, if for some $A \in \mathfrak{D}$ we have $x, y \in C(A) = \mathrm{R}\Uparrow \circ \sigma(A)$, then $x\mathrm{R}y\mathrm{R}x$ and so $x = y$ since R is a complete order. Hence Univalence holds.

B. Suppose that both Σ-Congruence and Univalence hold and $\langle \Sigma, \subset \rangle$ is a complete lattice. By Theorem 2.10 there exist a $\sigma \in \Sigma$ and a complete preorder Q such that $C = \mathrm{Q}\Uparrow \circ \sigma$. Define S by $x\mathrm{S}y$ if and only if $x\mathrm{Q}y$ and $\neg[y\mathrm{Q}x]$. Then S is a strict partial order, and it follows by Szpilrajn's [39] Embedding Theorem that there exists a linear order $\mathrm{T} \supset \mathrm{S}$. Now define R by $x\mathrm{R}y$ if and only if $x\mathrm{T}y$ or $x = y$, so that $\mathrm{T} \subset \mathrm{R}$, and observe that R is a complete order. Given $x \in A \in \mathfrak{D}$, if $x \in C(A) = \mathrm{Q}\Uparrow \circ \sigma(A)$ then for all $y \in \sigma(A)$ such that $y \neq x$ we have $y \notin C(A)$ by Univalence. It follows that $x\mathrm{S}y$, $x\mathrm{T}y$, and $x\mathrm{R}y$, and thus $x \in \mathrm{R}\Uparrow \circ \sigma(A)$ since R is reflexive. Hence $C \subset \mathrm{R}\Uparrow \circ \sigma$. To confirm the reverse inclusion, let $x \in A \in \mathfrak{D}$ be such that $x \in \mathrm{R}\Uparrow \circ \sigma(A)$ and take any $y \in \sigma(A)$ such that $y \neq x$. We then have $x\mathrm{R}y$, $\neg[y\mathrm{R}x]$ since R is a complete order, $\neg[y\mathrm{T}x]$, and $\neg[y\mathrm{S}x]$. But this implies that $x\mathrm{Q}y$ since Q is a complete preorder, and so $x \in \mathrm{Q}\Uparrow \circ \sigma(A) = C(A)$. Hence $\mathrm{R}\Uparrow \circ \sigma \subset C$ and $C = \mathrm{R}\Uparrow \circ \sigma$, with $\sigma \in \Sigma$ and R a complete order. □

Proof of Proposition 2.14. In text. □

Proof of Proposition 3.2. In text. □

Proof of Proposition 3.5. Clearly $\hat{\rho}_{\Sigma^{\mathrm{cf}}} \in \Sigma^{\mathrm{cf}} \cap \Xi_C$, so that $\hat{\sigma}_{\Sigma^{\mathrm{cf}}} \subset \hat{\rho}_{\Sigma^{\mathrm{cf}}}$. Moreover, for any $\sigma \in \Sigma^{\mathrm{cf}} \cap \Xi_C$ and $x \in A \in \mathfrak{D}$ we have $x \in \hat{\rho}_{\Sigma^{\mathrm{cf}}}(A)$ only if $\exists B \in \mathfrak{D}$ such that $A \subset B$ and $x \in C(B)$. But then $x \in \sigma(B)$ since $\sigma \in \Xi_C$, whereupon $x \in \sigma(A)$ since $\sigma \in \Sigma^{\mathrm{cf}}$. Thus $\hat{\rho}_{\Sigma^{\mathrm{cf}}} \subset \sigma$, and it follows that $\hat{\rho}_{\Sigma^{\mathrm{cf}}} \subset \hat{\sigma}_{\Sigma^{\mathrm{cf}}}$. Hence $\hat{\sigma}_{\Sigma^{\mathrm{cf}}} = \hat{\rho}_{\Sigma^{\mathrm{cf}}}$. □

Proof of Proposition 3.9. Given $\Psi \subset \Sigma^{\mathrm{se}}$ and $A, B \in \mathfrak{D}$ such that $A \subset B$ and

$$\emptyset \neq [\textstyle\bigwedge \Psi](B) \cap A = [\textstyle\bigcap_{\sigma \in \Psi} \sigma(B)] \cap A = \displaystyle\bigcap_{\sigma \in \Psi} [\sigma(B) \cap A],$$

for each $\sigma \in \Psi$ we have $\sigma(B) \cap A \neq \emptyset$. But then

$$[\textstyle\bigwedge \Psi](A) = \displaystyle\bigcap_{\sigma \in \Psi} \sigma(A) \subset \bigcap_{\sigma \in \Psi} \sigma(B) = [\textstyle\bigwedge \Psi](B).$$

Hence $\bigwedge \Psi \in \Sigma^{\mathrm{se}}$. □

Proof of Proposition 3.12. Clearly $C \subset \mathcal{R}^\ell\Uparrow \subset [\mathcal{R}^\ell]^*\Uparrow$, and so $[\mathcal{R}^\ell]^*\Uparrow \in \Xi_C$. Moreover, given $A, B \in \mathfrak{D}$ such that $A \subset B$ and $[\mathcal{R}^\ell]^*\Uparrow(B) \cap A \neq \emptyset$, we have that $\exists y \in [\mathcal{R}^\ell]^*\Uparrow(B) \cap A$. It follows that $x \in [\mathcal{R}^\ell]^*\Uparrow(A)$ only if $x[\mathrm{R}^\ell_A]^*y$, and so $x[\mathrm{R}^\ell_B]^*y$ since $[\mathcal{R}^\ell]^*$ is nested. But then $x \in [\mathcal{R}^\ell]^*\Uparrow(B)$, so $[\mathcal{R}^\ell]^*\Uparrow(A) \subset [\mathcal{R}^\ell]^*\Uparrow(B)$. Hence we can conclude that $[\mathcal{R}^\ell]^*\Uparrow \in \Sigma^{\mathrm{se}}$, and therefore $\hat{\sigma}_{\Sigma^{\mathrm{se}}} \subset [\mathcal{R}^\ell]^*\Uparrow$.

Given $\sigma \in \Sigma^{\mathrm{se}} \cap \Xi_C$ and $x \in B \in \mathfrak{D}$, let $x \in [\mathcal{R}^\ell]^*\Uparrow(B)$. For any $y \in C(B)$, we have $y \in \sigma(B)$ since $\sigma \in \Xi_C$. Moreover, there exist an integer $n \geq 2$ and $z_1, \ldots, z_n \in B$ such that $x = z_1 \mathrm{R}^\ell_B z_2 \mathrm{R}^\ell_B \cdots \mathrm{R}^\ell_B z_n = y$, and we have $z_n = y \in \sigma(B)$. Now for $k \in \{1, \ldots, n-1\}$, suppose $z_{k+1} \in \sigma(B)$. Since $z_k \mathrm{R}^\ell_B z_{k+1}$, we have that $\exists A_k \in \mathfrak{D}$ such that $z_{k+1} \in A_k \subset B$ and $z_k \in C(A_k) \subset \sigma(A_k)$. But then $z_k \in \sigma(B)$ since $\sigma \in \Sigma^{\mathrm{se}}$. By induction it follows that $x = z_1 \in \sigma(B)$, and hence $[\mathcal{R}^\ell]^*\Uparrow \subset \sigma$. Therefore $[\mathcal{R}^\ell]^*\Uparrow \subset \hat{\sigma}_{\Sigma^{\mathrm{se}}}$, and so $\hat{\sigma}_{\Sigma^{\mathrm{se}}} = [\mathcal{R}^\ell]^*\Uparrow$. □

Proof of Proposition 3.14. Given $\Psi \subset \Sigma^{\mathrm{we}}$ and $\mathfrak{B} \subset \mathfrak{D}$ such that $\bigcup_{B\in\mathfrak{B}} B \in \mathfrak{D}$, we have

$$\bigcap_{B\in\mathfrak{B}} [\bigwedge\Psi](B) = \bigcap_{B\in\mathfrak{B}} \bigcap_{\sigma\in\Psi} \sigma(B) = \bigcap_{\sigma\in\Psi} \bigcap_{B\in\mathfrak{B}} \sigma(B) \subset \bigcap_{\sigma\in\Psi} \sigma(\textstyle\bigcup_{B\in\mathfrak{B}} B) = [\bigwedge\Psi](\bigcup_{B\in\mathfrak{B}} B).$$

Hence $\bigwedge\Psi \in \Sigma^{\mathrm{we}}$. □

Proof of Proposition 3.16. Clearly $C \subset \mathcal{R}^\ell\Uparrow$, and so $\mathcal{R}^\ell\Uparrow \in \Xi_C$. Moreover, given $\mathfrak{B} \subset \mathfrak{D}$ with $\bigcup_{B\in\mathfrak{B}} B \in \mathfrak{D}$, if $x \in \bigcap_{B\in\mathfrak{B}} \mathcal{R}^\ell\Uparrow(B)$ then $\forall y \in B \in \mathfrak{B}$ we have $x\mathrm{R}^\ell_B y$. It follows that $\forall y \in \bigcup_{B\in\mathfrak{B}} B$ we have $x\mathrm{R}^\ell_{[\bigcup_{B\in\mathfrak{B}} B]} y$ since $\mathcal{R}^\ell$ is nested. But this is equivalent to $x \in \mathcal{R}^\ell\Uparrow(\bigcup_{B\in\mathfrak{B}} B)$, so $\bigcap_{B\in\mathfrak{B}} \mathcal{R}^\ell\Uparrow(B) \subset \mathcal{R}^\ell\Uparrow(\bigcup_{B\in\mathfrak{B}} B)$. Hence $\mathcal{R}^\ell\Uparrow \in \Sigma^{\mathrm{we}}$, and therefore $\hat{\sigma}_{\Sigma^{\mathrm{we}}} \subset \mathcal{R}^\ell\Uparrow$.

Given $\sigma \in \Sigma^{\mathrm{we}} \cap \Xi_C$ and $x \in B \in \mathfrak{D}$, let $x \in \mathcal{R}^\ell\Uparrow(B)$. For any $y \in B$ we have $x\mathrm{R}^\ell_B y$, and so $\exists A_y \in \mathfrak{D}$ such that $y \in A_y \subset B$ and $x \in C(A_y) \subset \sigma(A_y)$ since $\sigma \in \Xi_C$. But then $\bigcup_{y\in B} A_y = B \in \mathfrak{D}$, and since $\sigma \in \Sigma^{\mathrm{we}}$ it follows that $x \in \sigma(B)$. Hence $\mathcal{R}^\ell\Uparrow \subset \sigma$, so we have $\mathcal{R}^\ell\Uparrow \subset \hat{\sigma}_{\Sigma^{\mathrm{we}}}$ and $\hat{\sigma}_{\Sigma^{\mathrm{we}}} = \mathcal{R}^\ell\Uparrow$. □

Proof of Proposition 3.18. Given $\Psi \subset \Sigma^{\mathrm{ef}}$, $A \in \mathfrak{D}$, and $\mathfrak{B} \subset \mathfrak{D}$ such that $A \subset \bigcup_{B\in\mathfrak{B}} B$, we have

$$[\textstyle\bigcap_{B\in\mathfrak{B}} [\bigwedge\Psi](B)] \cap A = [\bigcap_{B\in\mathfrak{B}} \bigcap_{\sigma\in\Psi} \sigma(B)] \cap A =$$
$$\bigcap_{\sigma\in\Psi} [\textstyle\bigcap_{B\in\mathfrak{B}} \sigma(B)] \cap A \subset \displaystyle\bigcap_{\sigma\in\Psi} \sigma(A) = [\textstyle\bigwedge\Psi](A).$$

Hence $\bigwedge\Psi \in \Sigma^{\mathrm{ef}}$. □

Proof of Proposition 3.22. Let Σ^{ef}-Congruence hold. Given $A, B, D \in \mathfrak{D}$ and $x, y \in A$ such that $A \subset B \subset D$, let $x \in C(A) \cap C(D)$ and $y \in C(B)$. Then both $x \in \hat{\sigma}_{\Sigma^{\mathrm{ef}}}(D) \cap B$ and $y \in \hat{\sigma}_{\Sigma^{\mathrm{ef}}}(B) \cap A$, and so since $\hat{\sigma}_{\Sigma^{\mathrm{ef}}} \in \Sigma^{\mathrm{ef}}$ we have $x \in \hat{\sigma}_{\Sigma^{\mathrm{ef}}}(B)$ and $y \in \hat{\sigma}_{\Sigma^{\mathrm{ef}}}(A)$, respectively. But $x \in C(A)$ then implies that $x\hat{\mathrm{R}}_{\Sigma^{\mathrm{ef}}} y$, and since $y \in C(B)$ it follows that $x \in C(B)$ by Σ^{ef}-Congruence. Hence Generalized Weak WARP holds.

Now, given $\mathfrak{B} \subset \mathfrak{D}$ such that $\bigcup_{B\in\mathfrak{B}} B \in \mathfrak{D}$, let $x \in \bigcap_{B\in\mathfrak{B}} C(B)$. Then $\exists y \in C(\bigcup_{B\in\mathfrak{B}} B)$ and $A \in \mathfrak{B}$ such that $y \in A$ and $x \in C(A)$. We have also $y \in \hat{\sigma}_{\Sigma^{\mathrm{ef}}}(\bigcup_{B\in\mathfrak{B}} B)$, so $y \in \hat{\sigma}_{\Sigma^{\mathrm{ef}}}(A)$ since $\hat{\sigma}_{\Sigma^{\mathrm{ef}}} \in \Sigma^{\mathrm{ef}}$. It follows that $x\hat{\mathrm{R}}_{\Sigma^{\mathrm{ef}}} y$. Moreover, we have $x \in \bigcap_{B\in\mathfrak{B}} \hat{\sigma}_{\Sigma^{\mathrm{ef}}}(B)$ and hence $x \in \hat{\sigma}_{\Sigma^{\mathrm{ef}}}(\bigcup_{B\in\mathfrak{B}} B)$, again since $\hat{\sigma}_{\Sigma^{\mathrm{ef}}} \in \Sigma^{\mathrm{ef}}$. But then $x \in C(\bigcup_{B\in\mathfrak{B}} B)$ by Σ^{ef}-Congruence, so Weak Expansion holds. □

Proof of Proposition 3.23. Clearly $C \subset \mathrm{R}^{\mathrm{g}}\Uparrow$, and so $\mathrm{R}^{\mathrm{g}}\Uparrow \in \Xi_C$. Moreover, given $x \in A \in \mathfrak{D}$ and $\mathfrak{B} \subset \mathfrak{D}$ such that $A \subset \bigcup_{B\in\mathfrak{B}} B$, if $x \in \bigcap_{B\in\mathfrak{B}} \mathrm{R}^{\mathrm{g}}\Uparrow(B)$ then $\forall y \in A$ we have $x\mathrm{R}^{\mathrm{g}} y$. Hence $x \in \mathrm{R}^{\mathrm{g}}\Uparrow(A)$, so $[\bigcap_{B\in\mathfrak{B}} \mathrm{R}^{\mathrm{g}}\Uparrow(B)] \cap A \subset \mathrm{R}^{\mathrm{g}}\Uparrow(A)$. It follows that $\mathrm{R}^{\mathrm{g}}\Uparrow \in \Sigma^{\mathrm{ef}}$, and therefore $\hat{\sigma}_{\Sigma^{\mathrm{ef}}} \subset \mathrm{R}^{\mathrm{g}}\Uparrow$.

Given $\sigma \in \Sigma^{\mathrm{ef}} \cap \Xi_C$ and $x \in A \in \mathfrak{D}$, let $x \in \mathrm{R}^{\mathrm{g}}\Uparrow(A)$. For any $y \in A$ we have $x\mathrm{R}^{\mathrm{g}} y$, and so $\exists B_y \in \mathfrak{D}$ such that $y \in B_y$ and $x \in C(B_y) \subset \sigma(B_y)$ since $\sigma \in \Xi_C$. We then have $A \subset \bigcup_{y\in A} B_y$, and since $\sigma \in \Sigma^{\mathrm{ef}}$ it follows that $x \in \sigma(A)$. Hence $\mathrm{R}^{\mathrm{g}}\Uparrow \subset \sigma$, so we have $\mathrm{R}^{\mathrm{g}}\Uparrow \subset \hat{\sigma}_{\Sigma^{\mathrm{ef}}}$ and $\hat{\sigma}_{\Sigma^{\mathrm{ef}}} = \mathrm{R}^{\mathrm{g}}\Uparrow$. □

Proof of Proposition 3.25. Given $\Psi \subset \Sigma^{\mathrm{wa}}$ and $A, B \in \mathfrak{D}$ such that

$$\emptyset \neq [\bigwedge\Psi](B) \cap A = [\bigcap_{\sigma\in\Psi}\sigma(B)] \cap A = \bigcap_{\sigma\in\Psi}[\sigma(B) \cap A],$$

for each $\sigma \in \Psi$ we have $\sigma(B) \cap A \neq \emptyset$. But then

$$[\bigwedge\Psi](A) \cap B = [\bigcap_{\sigma\in\Psi}\sigma(A)] \cap B = \bigcap_{\sigma\in\Psi}[\sigma(A) \cap B] \subset \bigcap_{\sigma\in\Psi}\sigma(B) = [\bigwedge\Psi](B).$$

Hence $\bigwedge\Psi \in \Sigma^{\mathrm{wa}}$. □

Proof of Proposition 3.28. Since $C = \hat{\tau}^0_{\Sigma^{\mathrm{wa}}} \subset \hat{\tau}_{\Sigma^{\mathrm{wa}}}$, we have $\hat{\tau}_{\Sigma^{\mathrm{wa}}} \in \Xi_C$. Moreover, given $x, y \in A, B \in \mathfrak{D}$ such that $x \in \hat{\tau}_{\Sigma^{\mathrm{wa}}}(A)$ and $y \in \hat{\tau}_{\Sigma^{\mathrm{wa}}}(B)$, there exist $i, j \geq 0$ such that $x \in \hat{\tau}^i_{\Sigma^{\mathrm{wa}}}(A)$ and $y \in \hat{\tau}^j_{\Sigma^{\mathrm{wa}}}(B)$. Let $m = \max\{i, j\}$. Then $x \in \hat{\tau}^m_{\Sigma^{\mathrm{wa}}}(A)$ and $y \in \hat{\tau}^m_{\Sigma^{\mathrm{wa}}}(B)$, and it follows that $x \in \hat{\tau}^{m+1}_{\Sigma^{\mathrm{wa}}}(B) \subset \hat{\tau}_{\Sigma^{\mathrm{wa}}}(B)$. Thus $\hat{\tau}_{\Sigma^{\mathrm{wa}}} \in \Sigma^{\mathrm{wa}}$, and so $\hat{\sigma}_{\Sigma^{\mathrm{wa}}} \subset \hat{\tau}_{\Sigma^{\mathrm{wa}}}$.

Given $\sigma \in \Sigma^{\mathrm{wa}} \cap \Xi_C$, we have $\hat{\tau}^0_{\Sigma^{\mathrm{wa}}} = C \subset \sigma$. Now for $k \geq 0$ suppose that $\hat{\tau}^k_{\Sigma^{\mathrm{wa}}} \subset \sigma$, and let $x \in B \in \mathfrak{D}$. If $x \in \hat{\tau}^{k+1}_{\Sigma^{\mathrm{wa}}}(B)$ then $\exists y \in A \in \mathfrak{D}$ such that $x \in \hat{\tau}^k_{\Sigma^{\mathrm{wa}}}(A) \subset \sigma(A)$ and $y \in \hat{\tau}^k_{\Sigma^{\mathrm{wa}}}(B) \subset \sigma(B)$. Since $\sigma \in \Sigma^{\mathrm{wa}}$ we then have $x \in \sigma(B)$, and therefore $\hat{\tau}^{k+1}_{\Sigma^{\mathrm{wa}}} \subset \sigma$. By induction it follows that $\hat{\tau}_{\Sigma^{\mathrm{wa}}} = \bigcup_{k=0}^{\infty}\hat{\tau}^k_{\Sigma^{\mathrm{wa}}} \subset \sigma$. Hence $\hat{\tau}_{\Sigma^{\mathrm{wa}}} \subset \hat{\sigma}_{\Sigma^{\mathrm{wa}}}$ and $\hat{\sigma}_{\Sigma^{\mathrm{wa}}} = \hat{\tau}_{\Sigma^{\mathrm{wa}}}$. □

Proof of Proposition 4.3. Given $\Psi \subset \Sigma^{\mathrm{sa}}$ and $x, y \in A \in \mathfrak{D}$ such that $x \in [\bigwedge\Psi](A) = \bigcap_{\sigma\in\Psi}\sigma(A)$ and $y[\![\wedge\Psi]\!]^*x$, there exist an integer $n \geq 2$ and $z_1, \ldots, z_n \in X$ such that $y = z_1[\![\wedge\Psi]\!]z_2[\![\wedge\Psi]\!]\cdots[\![\wedge\Psi]\!]z_n = x$. Then for $k \in \{1, \ldots, n-1\}$ there exists a $B_k \in \mathfrak{D}$ such that $z_{k+1} \in B_k$ and $z_k \in [\bigwedge\Psi](B_k) = \bigcap_{\sigma\in\Psi}\sigma(B_k)$. It follows that $\forall\sigma \in \Psi$ we have $z_k[\![\sigma]\!]z_{k+1}$, and thus $y[\![\sigma]\!]^*x$ and $y \in \sigma(A)$. But then $y \in \bigcap_{\sigma\in\Psi}\sigma(A) = [\bigwedge\Psi](A)$. Hence $\bigwedge\Psi \in \Sigma^{\mathrm{sa}}$. □

Proof of Proposition 4.5. It is immediate that Congruence implies Σ^{sa}-Congruence. To show the converse, suppose that Σ^{sa}-Congruence holds and let $x, y \in A \in \mathfrak{D}$ be such that $x \in C(A)$ and $y[\mathrm{R}^{\mathrm{g}}]^*x$. Since $x \in [\mathrm{R}^{\mathrm{g}}]^*\Uparrow(A)$, we have $y \in [\mathrm{R}^{\mathrm{g}}]^*\Uparrow(A) = \hat{\sigma}_{\Sigma^{\mathrm{sa}}}(A)$. Moreover, there exist an integer $n \geq 2$ and $z_1, \ldots, z_n \in X$ with $y = z_1\mathrm{R}^{\mathrm{g}}z_2\mathrm{R}^{\mathrm{g}}\cdots\mathrm{R}^{\mathrm{g}}z_n = x$. For $k \in \{1, \ldots, n-1\}$ there exists a $B_k \in \mathfrak{D}$ such that $z_{k+1} \in B_k$ and $z_k \in C(B_k)$. Since $z_{k+1}[\mathrm{R}^{\mathrm{g}}]^*x\mathrm{R}^{\mathrm{g}}y[\mathrm{R}^{\mathrm{g}}]^*z_k \in [\mathrm{R}^{\mathrm{g}}]^*\Uparrow(B_k)$, we then have $z_{k+1} \in [\mathrm{R}^{\mathrm{g}}]^*\Uparrow(B_k) = \hat{\sigma}_{\Sigma^{\mathrm{sa}}}(B_k)$ and so $z_k\hat{\mathrm{R}}_{\Sigma^{\mathrm{sa}}}z_{k+1}$. It follows that $y\hat{\mathrm{R}}^*_{\Sigma^{\mathrm{sa}}}x$, and therefore $y \in C(A)$ by Σ^{sa}-Congruence. Hence Congruence holds. □

Proof of Proposition 4.6. Clearly $C \subset \mathrm{R}^{\mathrm{g}}\Uparrow \subset [\mathrm{R}^{\mathrm{g}}]^*\Uparrow$, and so $[\mathrm{R}^{\mathrm{g}}]^*\Uparrow \in \Xi_C$. Moreover, given $x, y \in A \in \mathfrak{D}$ such that $x \in [\mathrm{R}^{\mathrm{g}}]^*\Uparrow(A)$ and $y[\![[\mathrm{R}^{\mathrm{g}}]^*\Uparrow]\!]^*x$, there exist an integer $n \geq 2$ and $z_1, \ldots, z_n \in X$ such that $y = z_1[\![[\mathrm{R}^{\mathrm{g}}]^*\Uparrow]\!]z_2[\![[\mathrm{R}^{\mathrm{g}}]^*\Uparrow]\!]\cdots[\![[\mathrm{R}^{\mathrm{g}}]^*\Uparrow]\!]z_n = x$. It follows that for $k \in \{1, \ldots, n-1\}$ there exists a $B_k \in \mathfrak{D}$ such that $z_{k+1} \in B_k$ and $z_k \in [\mathrm{R}^{\mathrm{g}}]^*\Uparrow(B_k)$. But then $z_k[\mathrm{R}^{\mathrm{g}}]^*z_{k+1}$, and thus $y[\mathrm{R}^{\mathrm{g}}]^*x \in [\mathrm{R}^{\mathrm{g}}]^*\Uparrow(A)$ and $y \in [\mathrm{R}^{\mathrm{g}}]^*\Uparrow(A)$. Hence we can conclude that $[\mathrm{R}^{\mathrm{g}}]^*\Uparrow \in \Sigma^{\mathrm{sa}}$, and therefore $\hat{\sigma}_{\Sigma^{\mathrm{sa}}} \subset [\mathrm{R}^{\mathrm{g}}]^*\Uparrow$.

Given $\sigma \in \Sigma^{\mathrm{sa}} \cap \Xi_C$ and $x \in A \in \mathfrak{D}$, let $x \in [\mathrm{R}^{\mathrm{g}}]^*\Uparrow(A)$. For any $y \in C(A)$, we have $y \in \sigma(A)$ since $\sigma \in \Xi_C$. Moreover, there exist an integer $n \geq 2$ and $z_1, \ldots, z_n \in X$ such that $x = z_1\mathrm{R}^{\mathrm{g}}z_2\mathrm{R}^{\mathrm{g}}\cdots\mathrm{R}^{\mathrm{g}}z_n = y$. Now for $k \in \{1, \ldots, n-1\}$ there exists a $B_k \in \mathfrak{D}$ such that $z_{k+1} \in B_k$ and $z_k \in C(B_k) \subset \sigma(B_k)$. But then $z_k[\![\sigma]\!]z_{k+1}$, and thus $x[\![\sigma]\!]^*y$. It follows that $x \in \sigma(A)$ since $\sigma \in \Sigma^{\mathrm{sa}}$, and hence $[\mathrm{R}^{\mathrm{g}}]^*\Uparrow \subset \sigma$. Therefore $[\mathrm{R}^{\mathrm{g}}]^*\Uparrow \subset \hat{\sigma}_{\Sigma^{\mathrm{sa}}}$, and so $\hat{\sigma}_{\Sigma^{\mathrm{sa}}} = [\mathrm{R}^{\mathrm{g}}]^*\Uparrow$. □

Proof of Proposition 4.9. **A.** Since $C = \hat{\tau}^0_{\Sigma^{\rm se}} \subset \hat{\tau}_{\Sigma^{\rm se}}$, we have $\hat{\tau}_{\Sigma^{\rm se}} \in \Xi_C$. Moreover, given $x, y \in A, B \in \mathfrak{D}$ such that $A \subset B$, $x \in \hat{\tau}_{\Sigma^{\rm se}}(A)$ and $y \in \hat{\tau}_{\Sigma^{\rm se}}(B)$, there exist $i, j \geq 0$ such that $x \in \hat{\tau}^i_{\Sigma^{\rm se}}(A)$ and $y \in \hat{\tau}^j_{\Sigma^{\rm se}}(B)$. Let $m = \max\{i, j\}$. Then $x \in \hat{\tau}^m_{\Sigma^{\rm se}}(A)$ and $y \in \hat{\tau}^m_{\Sigma^{\rm se}}(B)$, and it follows that $x \in \hat{\tau}^{m+1}_{\Sigma^{\rm se}}(B) \subset \hat{\tau}_{\Sigma^{\rm se}}(B)$. Thus $\hat{\tau}_{\Sigma^{\rm se}} \in \Sigma^{\rm se}$, and so $\hat{\sigma}_{\Sigma^{\rm se}} \subset \hat{\tau}_{\Sigma^{\rm se}}$.

Given $\sigma \in \Sigma^{\rm se} \cap \Xi_C$, we have $\hat{\tau}^0_{\Sigma^{\rm se}} = C \subset \sigma$. Now for $k \geq 0$ suppose that $\hat{\tau}^k_{\Sigma^{\rm se}} \subset \sigma$, and let $x \in B \in \mathfrak{D}$. If $x \in \hat{\tau}^{k+1}_{\Sigma^{\rm se}}(B)$ then $\exists y \in A \in \mathfrak{D}$ such that $A \subset B$, $x \in \hat{\tau}^k_{\Sigma^{\rm se}}(A) \subset \sigma(A)$ and $y \in \hat{\tau}^k_{\Sigma^{\rm se}}(B) \subset \sigma(B)$. Since $\sigma \in \Sigma^{\rm se}$ we then have $x \in \sigma(B)$, and therefore $\hat{\tau}^{k+1}_{\Sigma^{\rm se}} \subset \sigma$. By induction it follows that $\hat{\tau}_{\Sigma^{\rm se}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\rm se}} \subset \sigma$. Hence $\hat{\tau}_{\Sigma^{\rm se}} \subset \hat{\sigma}_{\Sigma^{\rm se}}$ and $\hat{\sigma}_{\Sigma^{\rm se}} = \hat{\tau}_{\Sigma^{\rm se}}$.

B. Since $C = \hat{\tau}^0_{\Sigma^{\rm sa}} \subset \hat{\tau}_{\Sigma^{\rm sa}}$, we have $\hat{\tau}_{\Sigma^{\rm sa}} \in \Xi_C$. Moreover, given $x, y \in A \in \mathfrak{D}$ such that $y \in \hat{\tau}_{\Sigma^{\rm sa}}(A)$ and $x[\![\hat{\tau}_{\Sigma^{\rm sa}}]\!]^* y$, there exist $i, j \geq 0$ such that $y \in \hat{\tau}^i_{\Sigma^{\rm sa}}(A)$ and $x[\![\hat{\tau}^j_{\Sigma^{\rm sa}}]\!]^* y$. Let $m = \max\{i, j\}$. Then $y \in \hat{\tau}^m_{\Sigma^{\rm sa}}(A)$ and $x[\![\hat{\tau}^m_{\Sigma^{\rm sa}}]\!]^* y$, and therefore $x \in \hat{\tau}^{m+1}_{\Sigma^{\rm sa}}(A) \subset \hat{\tau}_{\Sigma^{\rm sa}}(A)$. Thus $\hat{\tau}_{\Sigma^{\rm sa}} \in \Sigma^{\rm sa}$, and so $\hat{\sigma}_{\Sigma^{\rm sa}} \subset \hat{\tau}_{\Sigma^{\rm sa}}$.

Given $\sigma \in \Sigma^{\rm sa} \cap \Xi_C$, we have $\hat{\tau}^0_{\Sigma^{\rm sa}} = C \subset \sigma$. Now for $k \geq 0$ suppose that $\hat{\tau}^k_{\Sigma^{\rm sa}} \subset \sigma$, and let $x \in A \in \mathfrak{D}$. If $x \in \hat{\tau}^{k+1}_{\Sigma^{\rm sa}}(A)$ then $\exists y \in A$ such that $y \in \hat{\tau}^k_{\Sigma^{\rm sa}}(A)$ and $x[\![\hat{\tau}^k_{\Sigma^{\rm sa}}]\!]^* y$, and it follows that $y \in \sigma(A)$ and $x[\![\sigma]\!]^* y$. Since $\sigma \in \Sigma^{\rm sa}$ we then have $x \in \sigma(A)$, and therefore $\hat{\tau}^{k+1}_{\Sigma^{\rm sa}} \subset \sigma$. By induction it follows that $\hat{\tau}_{\Sigma^{\rm sa}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\rm sa}} \subset \sigma$. Hence $\hat{\tau}_{\Sigma^{\rm sa}} \subset \hat{\sigma}_{\Sigma^{\rm sa}}$ and $\hat{\sigma}_{\Sigma^{\rm sa}} = \hat{\tau}_{\Sigma^{\rm sa}}$. □

Proof of Proposition 4.13. **A.** Given $x \in A \in \mathfrak{D}$, we have $x \in \hat{\tau}^1_{\Sigma^{\rm cf}}(A)$ if and only if $\exists B \in \mathfrak{D}$ such that $A \subset B$ and $x \in \hat{\tau}^0_{\Sigma^{\rm cf}}(B) = C(B)$. But this is equivalent to $x \in \hat{\rho}_{\Sigma^{\rm cf}}(A)$, and hence $\hat{\tau}_{\Sigma^{\rm cf}} \supset \hat{\tau}^1_{\Sigma^{\rm cf}} = \hat{\rho}_{\Sigma^{\rm cf}} = \hat{\sigma}_{\Sigma^{\rm cf}}$ by Proposition 3.5.

Given $\sigma \in \Sigma^{\rm cf} \cap \Xi_C$, we have $\hat{\tau}^0_{\Sigma^{\rm cf}} = C \subset \sigma$. Now for $k \geq 0$ suppose that $\hat{\tau}^k_{\Sigma^{\rm cf}} \subset \sigma$, and let $x \in A \in \mathfrak{D}$. If $x \in \hat{\tau}^{k+1}_{\Sigma^{\rm cf}}(A)$ then $\exists B \in \mathfrak{D}$ such that $A \subset B$ and $x \in \hat{\tau}^k_{\Sigma^{\rm cf}}(B) \subset \sigma(B)$. Since $\sigma \in \Sigma^{\rm cf}$ we then have $x \in \sigma(A)$, and therefore $\hat{\tau}^{k+1}_{\Sigma^{\rm cf}} \subset \sigma$. By induction it follows that $\hat{\tau}_{\Sigma^{\rm cf}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\rm cf}} \subset \sigma$. Hence $\hat{\tau}_{\Sigma^{\rm cf}} \subset \hat{\sigma}_{\Sigma^{\rm cf}}$.

B. Given $x \in A \in \mathfrak{D}$, we have $x \in \hat{\tau}^1_{\Sigma^{\rm we}}(A)$ if and only if $\exists \mathfrak{B} \subset \mathfrak{D}$ such that $\bigcup_{B \in \mathfrak{B}} B = A$ and $x \in \bigcap_{B \in \mathfrak{B}} \hat{\tau}^0_{\Sigma^{\rm we}}(B) = \bigcap_{B \in \mathfrak{B}} C(B)$. This is equivalent to the assertion that $\forall y \in A$ there exists a $B_y \in \mathfrak{D}$ such that $y \in B_y \subset A$ and $x \in C(B_y)$, which is to say that $x \in \mathcal{R}^\ell {\Uparrow}(A)$. Hence $\hat{\tau}_{\Sigma^{\rm we}} \supset \hat{\tau}^1_{\Sigma^{\rm we}} = \mathcal{R}^\ell {\Uparrow} = \hat{\sigma}_{\Sigma^{\rm we}}$ by Proposition 3.16.

Given $\sigma \in \Sigma^{\rm we} \cap \Xi_C$, we have $\hat{\tau}^0_{\Sigma^{\rm we}} = C \subset \sigma$. Now for $k \geq 0$ suppose that $\hat{\tau}^k_{\Sigma^{\rm we}} \subset \sigma$, and let $x \in A \in \mathfrak{D}$. If $x \in \hat{\tau}^{k+1}_{\Sigma^{\rm we}}(A)$ then $\exists \mathfrak{B} \subset \mathfrak{D}$ such that $\bigcup_{B \in \mathfrak{B}} B = A$ and $x \in \bigcap_{B \in \mathfrak{B}} \hat{\tau}^k_{\Sigma^{\rm we}}(B) \subset \bigcap_{B \in \mathfrak{B}} \sigma(B)$. Since $\sigma \in \Sigma^{\rm we}$ we then have $x \in \sigma(A)$, and thus $\hat{\tau}^{k+1}_{\Sigma^{\rm we}} \subset \sigma$. By induction it follows that $\hat{\tau}_{\Sigma^{\rm we}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\rm we}} \subset \sigma$. Hence $\hat{\tau}_{\Sigma^{\rm we}} \subset \hat{\sigma}_{\Sigma^{\rm we}}$.

C. Given $x \in A \in \mathfrak{D}$, we have $x \in \hat{\tau}^1_{\Sigma^{\rm ef}}(A)$ if and only if $\exists \mathfrak{B} \subset \mathfrak{D}$ such that $\bigcup_{B \in \mathfrak{B}} B \supset A$ and $x \in \bigcap_{B \in \mathfrak{B}} \hat{\tau}^0_{\Sigma^{\rm ef}}(B) = \bigcap_{B \in \mathfrak{B}} C(B)$. This is equivalent to the assertion that $\forall y \in A$ there exists a $B_y \in \mathfrak{D}$ such that $y \in B_y$ and $x \in C(B_y)$, which is to say that $x \in \mathrm{R}^{\mathrm{g}} {\Uparrow}(A)$. Hence $\hat{\tau}_{\Sigma^{\rm ef}} \supset \hat{\tau}^1_{\Sigma^{\rm ef}} = \mathrm{R}^{\mathrm{g}} {\Uparrow} = \hat{\sigma}_{\Sigma^{\rm ef}}$ by Proposition 3.23.

Given $\sigma \in \Sigma^{\rm ef} \cap \Xi_C$, we have $\hat{\tau}^0_{\Sigma^{\rm ef}} = C \subset \sigma$. Now for $k \geq 0$ suppose that $\hat{\tau}^k_{\Sigma^{\rm ef}} \subset \sigma$, and let $x \in A \in \mathfrak{D}$. If $x \in \hat{\tau}^{k+1}_{\Sigma^{\rm ef}}(A)$ then $\exists \mathfrak{B} \subset \mathfrak{D}$ such that $\bigcup_{B \in \mathfrak{B}} B \supset A$ and $x \in \bigcap_{B \in \mathfrak{B}} \hat{\tau}^k_{\Sigma^{\rm ef}}(B) \subset \bigcap_{B \in \mathfrak{B}} \sigma(B)$. Since $\sigma \in \Sigma^{\rm ef}$ we then have $x \in \sigma(A)$, and thus $\hat{\tau}^{k+1}_{\Sigma^{\rm ef}} \subset \sigma$. By induction it follows that $\hat{\tau}_{\Sigma^{\rm ef}} = \bigcup_{k=0}^{\infty} \hat{\tau}^k_{\Sigma^{\rm ef}} \subset \sigma$. Hence $\hat{\tau}_{\Sigma^{\rm ef}} \subset \hat{\sigma}_{\Sigma^{\rm ef}}$. □

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