Non-Exclusive Competition in the Market for Lemons[*]

Andrea Attar[†] Thomas Mariotti[‡] François Salanié[§]

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Abstract

We characterize market equilibria in an exchange economy where buyers compete by offering menus of contracts to a seller who is privately informed about the quality of the traded good. We focus on the situation in which none of the buyers can contract over the aggregate level of trades, that is, contracts are non-exclusive. In this context, we show that equilibrium allocations always exist and are generically unique. In particular, we provide a full strategic foundation for Akerlof's (1970) results: market equilibria involve pooling of different types of the seller, and/or exclusion of the most efficient ones. We contrast our findings with those of standard competitive screening models, which postulate enforceability of exclusive contracts, and we discuss their implications for the empirical tests of adverse selection in financial markets.

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[†]Toulouse School of Economics (IDEI, PWRI) and Università degli Studi di Roma "Tor Vergata."
[‡]Toulouse School of Economics (GREMAQ/CNRS, IDEI).
[§]Toulouse School of Economics (LERNA/INRA).

1 Introduction

Adverse selection is widely recognized as a major obstacle to the efficient functioning of markets. This is especially true on financial markets, where buyers care about the quality of the assets they purchase, and fear that sellers have superior information about it. The same difficulties impede trade on second-hand markets and insurance markets. Theory confirms that adverse selection may indeed have a dramatic impact on economic outcomes. First, all mutually beneficial trades need not take place in equilibrium. For instance, in Akerlof's (1970) model of second-hand markets, only the lowest quality goods are traded at the equilibrium price. Second, there may be difficulties with the very existence of equilibrium. For instance, in Rothschild and Stiglitz's (1976) model of insurance markets, an equilibrium fails to exist whenever the proportion of low-risk agents is too high.

Most contributions to the theory of competition under adverse selection have considered frameworks in which competitors are restricted to make exclusive offers. This assumption is for instance appropriate in the case of car insurance, since law forbids to take out multiple policies on a single vehicle. By contrast, competition on financial markets is typically non-exclusive, as each agent can trade with multiple partners who cannot monitor each others' trades with the agent.[1] This paper supports the view that this difference in the nature of competition may have a significant impact on the way adverse selection affects market outcomes. This has two consequences. First, empirical studies that test for the presence of adverse selection should use different methods depending on whether competition is exclusive or not. Second, the regulation of markets plagued by adverse selection should be adjusted to the type of competition that prevails on them.

To illustrate these points, we consider a stylized model of trade under adverse selection. In our model, a seller endowed with some quantity of a good attempts to trade it with a finite number of buyers. The seller and the buyers have linear preferences over quantities and transfers exchanged. In line with Akerlof (1970), the quality of the good is the seller's private information. Unlike in his model, the good is assumed to be perfectly divisible, so that any fraction of the seller's endowment can potentially be traded. An example that fits these assumptions is that of a firm which floats a security issue by relying on the intermediation services of several investment banks. Buyers compete by simultaneously offering menus of

[1] Examples of this phenomenon abound across industries. In the banking industry, Detragiache, Garella and Guiso (2000), using a sample of small and medium-sized Italian firms, document that multiple banking relationships are very widespread. In the credit card industry, Rysman (2007) shows that US consumers typically hold multiple credit cards from different networks, although they tend to concentrate their spending on a single network. Cawley and Philipson (1999) and Finkelstein and Poterba (2004) report similar findings for the US life insurance market and the UK annuity market.

contracts, or, equivalently, price schedules.[2] After observing the menus offered, the seller decides of her trade(s). Competition is exclusive if the seller can trade with at most one buyer, and non-exclusive if trades with several buyers are allowed.

Under exclusive competition, our conclusions are qualitatively similar to Rothschild and Stiglitz's (1976). In a simple version of the model with two possible levels of quality, pure strategy equilibria exist if and only if the probability that the good is of high quality is low enough. Equilibria are separating: the seller trades her whole endowment when quality is low, while she only trades part of it when quality is high.

The analysis of the non-exclusive competition game yields strikingly different results. Pure strategy equilibria always exist, both for binary and continuous quality distributions. Aggregate equilibrium allocations are generically unique, and have an all-or-nothing feature: depending of whether quality is low or high, the seller either trades her whole endowment or does not trade at all. Buyers earn zero profit on average in any equilibrium. These allocations can be supported by simple menu offers. For instance, one can construct linear price equilibria in which buyers offer to purchase any quantity of the good at a constant unit price equal to the expectation of their valuation of the good conditional on the seller accepting to trade at that price. While other menu offers are consistent with equilibrium, corresponding to non-linear price schedules, an important insight of our analysis is that this is also the unit price at which all trades take place in any equilibrium.

These results are of course in line with Akerlof's (1970) classic analysis of the market for lemons, for which they provide a fully strategic foundation. It is worth stressing the differences between his model and ours. Akerlof (1970) considers a market for a non-divisible good of uncertain quality, in which all agents are price-takers. Thus, by assumption, all trades must take place at the same price, in the spirit of competitive equilibrium models. Equality of supply and demand determines the equilibrium price level, which is equal to the average quality of the goods that are effectively traded. Multiple equilibria may occur in a generic way.[3] By contrast, we allow agents to trade any fraction of the seller's endowment. Moreover, our model is one of imperfect competition, in which a fixed number of buyers choose their offers strategically. In particular, our analysis does not rely on free entry arguments. Finally, buyers can offer arbitrary menus of contracts, including for instance non-linear price schedules. That is, we avoid any a priori restrictions on instruments. The

[2]As established by Peters (2001) and Martimort and Stole (2002), there is no need to consider more general mechanisms in this multiple-principal single-agent setting.

[3]This potential multiplicity of equilibria arises because buyers are assumed to be price-takers. Mas-Colell, Whinston and Green (1995, Proposition 13.B.1) allow buyers to strategically set prices in a market for a non-divisible good where trades are restricted to be zero-one. The equilibrium is then generically unique.

fact that all trades take place at a constant unit price in equilibrium is therefore no longer an assumption, but rather a consequence of our analysis.

A key to our results is that non-exclusive competition expands the set of deviations that are available to the buyers. Indeed, each buyer can strategically use the offers of his competitors to propose additional trades to the seller. Such deviations are blocked by *latent* contracts, that is, contracts that are not traded in equilibrium but which the seller finds it profitable to trade at the deviation stage. These latent contracts are not necessarily complex nor exotic. For instance, in a linear price equilibrium, all the buyers offer to purchase any quantity of the good at a constant unit price, but only a finite number of contracts can end up being traded as long as the seller does not randomize on the equilibrium path. One of the purposes of the other contracts, which are not traded in equilibrium, is to deter cream-skimming deviations that aim at attracting the seller when quality is high. The use of latent contracts has been criticized on several grounds. First, they may allow one to support multiple equilibrium allocations, and even induce an indeterminacy of equilibrium.[4] This is not the case in our model, since aggregate equilibrium allocations are generically unique. Second, a latent contract may appear as a non-credible threat, if the buyer who issues it would make losses in the hypothetical case where the seller were to trade it.[5] Again, this need not be the case in our model. In fact, we construct examples of equilibria in which latent contracts would be strictly profitable if traded.

This paper is closely related to the literature on common agency between competing principals dealing with a privately informed agent. To use the terminology of Bernheim and Whinston (1986), our non-exclusive competition game is a *delegated* common agency game, as the seller can choose a strict subset of buyers with whom she wants to trade. In the specific context of incomplete information, a number of recent contributions use standard mechanism design techniques to characterize equilibrium allocations. The basic idea is that, given a profile of mechanisms proposed by his competitors, the best response of any single principal can be fully determined by focusing on simple menu offers corresponding to direct revelation mechanisms. This allows one to construct equilibria that satisfy certain regularity conditions. This approach has been successfully applied in various delegated agency contexts.[6] Closest to this paper is Biais, Martimort and Rochet (2000), who study competition among principals in a common value environment. In their model, uninformed

[4]For instance, Martimort and Stole (2003) show that, in a complete information setting, latent contracts can be used to support any level of trade between the perfectly competitive outcome and the Cournot outcome.

[5]Latent contracts with negative virtual profits have been for example considered in Hellwig (1983).

[6]See for instance Khalil, Martimort and Parigi (2007) or Martimort and Stole (2009).

market-makers supply liquidity to an informed insider. The insider's preferences are quasi-linear, and quadratic with respect to quantities exchanged. Unlike in our model, the insider has no capacity constraint. Variational techniques are used to construct an equilibrium in which market-makers post convex price schedules. Such techniques do not apply in our model, as all agents have linear preferences, and the seller cannot trade more than her endowment. Instead, we allow for arbitrary menu offers, and we characterize candidate equilibrium allocations in the usual way, that is by checking whether they survive possible deviations. While this approach may be difficult to apply in more complex settings, it delivers interesting new insights, in particular on the role of latent contracts.

The paper is organized as follows. Section 2 introduces the model. Section 3 focuses on a two-type setting. We show that there always exists a market equilibrium where buyers play a pure strategy. In addition, equilibrium allocations are generically unique. We also characterize equilibrium menu offers, with special emphasis on latent contracts. Section 4 analyzes the general framework with a continuum of sellers' types. Section 5 concludes.

2 The Model

There are two kinds of agents: a single seller, and a finite number of buyers indexed by $i = 1, \ldots, n$, where $n \geq 2$. The seller has an endowment consisting of one unit of a perfectly divisible good that she can trade with one or several buyers. Let q^i be the quantity of the good purchased by buyer i, and t^i the transfer he makes in return. The set of feasible trades is the set of vectors $((q^1, t^1), \ldots, (q^n, t^n))$ such that $q^i \geq 0$ and $t^i \geq 0$ for all i, with $\sum_i q^i \leq 1$. Thus the quantity of the good purchased by each buyer must be at least zero, and the sum of these quantities cannot exceed the seller's endowment.

The seller has preferences represented by

$$T - \theta Q,$$

where $Q = \sum_i q^i$ and $T = \sum_i t^i$ denote aggregate quantities and transfers. Here θ is a random variable that stands for the quality of the good as perceived by the seller. Each buyer i has preferences represented by

$$v(\theta) q^i - t^i.$$

Here $v(\theta)$ is a deterministic function of θ that stands for the quality of the good as perceived by the buyers.

We will typically assume that $v(\theta)$ is not a constant function of θ, so that both the seller and the buyers care about θ. Gains from trade arise in this common value environment if $v(\theta) > \theta$ for some realization of θ. However, in line with Akerlof (1970), mutually beneficial trades are potentially impeded because the seller is privately informed of the quality of the good at the trading stage. Following standard terminology, we shall hereafter refer to θ as to the *type* of the seller.

Buyers compete in menus for the good offered by the seller. Trading is non-exclusive in the sense that the seller can pick or reject any of the offers made to her, and can simultaneously trade with several buyers. The following timing of events characterizes our non-exclusive competition game:

1. Each buyer i proposes a menu of contracts, that is, a set C^i of quantity-transfer pairs $(q^i, t^i) \in [0,1] \times \mathbb{R}_+$ that contains at least the no-trade contract $(0,0)$.[7]

2. After privately learning the quality θ, the seller selects one contract (q^i, t^i) from each of the menus C^i's offered by the buyers, subject to the constraint that $\sum_i q^i \leq 1$.

A pure strategy for the seller is a mapping that associates to each type θ and each menu profile $(C^1, \ldots, C^n)$ a vector of trades $((q^1, t^1), \ldots, (q^n, t^n)) \in ([0,1] \times \mathbb{R}_+)^n$ such that $(q^i, t^i) \in C^i$ for all i and $\sum_i q^i \leq 1$. To ensure that the seller's problem

$$\sup \left\{ \sum_i t^i - \theta \sum_i q^i : \sum_i q^i \leq 1 \text{ and } (q^i, t^i) \in C^i \text{ for all } i \right\}$$

has a solution for any type θ and menu profile $(C^1, \ldots, C^n)$, we require the buyers' menus to be compact sets. Throughout the paper, and unless stated otherwise, the equilibrium concept is pure strategy perfect Bayesian equilibrium.

3 The Two-Type Case

In this section, we consider the binary version of our model in which the seller's type can be either low, $\theta = \underline{\theta}$, or high, $\theta = \overline{\theta}$, for some $\overline{\theta} > \underline{\theta} > 0$. Denote by $\nu \in (0,1)$ the probability that $\theta = \overline{\theta}$ and by $\mathbf{E}$ the corresponding expectation operator. We assume that the seller's and the buyers' perceptions of the quality of the good move together, that is $v(\overline{\theta}) > v(\underline{\theta})$, and that it would be efficient to trade no matter the quality of the good, that is $v(\underline{\theta}) > \underline{\theta}$ and $v(\overline{\theta}) > \overline{\theta}$.

[7]As usual, the assumption that each menu must contain the no-trade contract allows one to deal with participation in a simple way.

3.1 The Exclusive Competition Benchmark

As a benchmark, it is helpful to characterize the equilibrium outcomes under exclusive competition, that is, when the seller can trade with at most one buyer, as in standard models of competition under adverse selection. The timing of the exclusive competition game is similar to that of the non-exclusive competition game, except that the second stage is replaced by

2'. After privately learning the quality θ, the seller selects one contract (q^i, t^i) from one of the menus C^i's offered by the buyers.

Given a menu profile $(C^1, \ldots, C^n)$, the seller's problem then becomes

$$\sup\{t^i - \theta q^i : (q^i, t^i) \in C^i \text{ for some } i\}.$$

Let $(\underline{q}^e, \underline{t}^e)$ and $(\overline{q}^e, \overline{t}^e)$ be the contracts traded by each type of the seller in an equilibrium of the exclusive competition game. One has the following result.

Proposition 1 *The following holds*:

(i) Any equilibrium of the exclusive competition game is separating, with

$$(\underline{q}^e, \underline{t}^e) = (1, v(\underline{\theta})) \text{ and } (\overline{q}^e, \overline{t}^e) = \frac{v(\underline{\theta}) - \underline{\theta}}{v(\overline{\theta}) - \underline{\theta}}\,(1, v(\overline{\theta})).$$

(ii) The exclusive competition game has an equilibrium if and only if $\nu \leq \nu^e$, where

$$\nu^e = \frac{\overline{\theta} - \underline{\theta}}{v(\overline{\theta}) - \underline{\theta}}.$$

Hence, when the rules of the competition game are such that the seller can trade with at most one buyer, the structure of market equilibria is formally analogous to that obtaining in the competitive insurance model of Rothschild and Stiglitz (1976). First, any pure strategy equilibrium is separating, with type $\underline{\theta}$ selling her whole endowment, $\underline{q}^e = 1$, and type $\overline{\theta}$ only selling a fraction of her endowment, $0 < \overline{q}^e < 1$. The corresponding contracts are traded at unit prices $v(\underline{\theta})$ and $v(\overline{\theta})$ respectively, yielding each buyer a zero payoff. Second, type $\underline{\theta}$ is indifferent between her equilibrium contract and that of type $\overline{\theta}$, implying

$$\overline{q}^e = \frac{v(\underline{\theta}) - \underline{\theta}}{v(\overline{\theta}) - \underline{\theta}}$$

as stated in Proposition 1(i). The equilibrium is depicted on Figure 1. Point $\underline{A}^e$ corresponds to the equilibrium contract of type $\underline{\theta}$, while point $\overline{A}^e$ corresponds to the equilibrium contract of type $\overline{\theta}$. The two solid lines passing through these points are the equilibrium indifference curves of type $\underline{\theta}$ and type $\overline{\theta}$. The dotted line passing through the origin are indifference curves for the buyers, with slopes $v(\underline{\theta})$ and $v(\overline{\theta})$.

—Insert Figure 1 here—

As in Rothschild and Stiglitz (1976), a pure strategy equilibrium exists under exclusivity only under certain parameter restrictions. Specifically, the equilibrium indifference curve of type $\overline{\theta}$ must lie above the indifference curve for the buyers with slope $\mathbf{E}[v(\theta)]$ passing through the origin, for otherwise there would exist a profitable deviation attracting both types of the seller. As stated in Proposition 1(ii), this is the case if and only if the probability ν that the good is of high quality is low enough.

3.2 Equilibrium Outcomes under Non-Exclusive Competition

We now turn to the analysis of the non-exclusive competition model. We first characterize the restrictions that equilibrium behavior implies for the outcomes of the non-exclusive competition game. Next, we show that this game always has an equilibrium in which buyers post linear prices. Finally, we contrast the equilibrium outcomes with those arising in the exclusive competition model.

3.2.1 Aggregate Equilibrium Allocations

Let $\underline{c}^i = (\underline{q}^i, \underline{t}^i)$ and $\overline{c}^i = (\overline{q}^i, \overline{t}^i)$ be the contracts traded by the two types of the seller with buyer i in equilibrium, and let $(\underline{Q}, \underline{T}) = \sum_i \underline{c}^i$ and $(\overline{Q}, \overline{T}) = \sum_i \overline{c}^i$ be the corresponding aggregate equilibrium allocations. To characterize these allocations, one only needs to require that three types of deviations by a buyer be blocked in equilibrium. In each case, the deviating buyer uses the offers of his competitors as a support for his own deviation. This intuitively amounts to pivoting around the aggregate equilibrium allocation points $(\underline{Q}, \underline{T})$ and $(\overline{Q}, \overline{T})$ in the (Q, T) space. We now consider each deviation in turn.

Attracting type $\underline{\theta}$ by pivoting around $(\underline{Q}, \underline{T})$ The first type of deviations allows one to prove that type $\underline{\theta}$ always trades efficiently in equilibrium.

Lemma 1 $\underline{Q} = 1$ *in any equilibrium.*

One can illustrate the deviation used in Lemma 1 as follows. Observe first that a basic implication of incentive compatibility is that, in any equilibrium, $\overline{Q}$ cannot be higher than $\underline{Q}$.

Suppose then that $\underline{Q} < 1$ in a candidate equilibrium. This situation is depicted on Figure 2. Point $\underline{A}$ corresponds to the aggregate equilibrium allocation $(\underline{Q}, \underline{T})$ traded by type $\underline{\theta}$, while point $\overline{A}$ corresponds to the aggregate equilibrium allocation $(\overline{Q}, \overline{T})$ traded by type $\overline{\theta}$. The two solid lines passing through these points are the equilibrium indifference curves of type $\underline{\theta}$ and type $\overline{\theta}$, with slopes $\underline{\theta}$ and $\overline{\theta}$. The dotted line passing through $\underline{A}$ is an indifference curve for the buyers, with slope $v(\underline{\theta})$.

—Insert Figure 2 here—

Suppose now that some buyer deviates and includes in his menu an additional contract that makes available the further trade $\underline{AA'}$. This leaves type $\underline{\theta}$ indifferent, since she obtains the same payoff as in equilibrium. Type $\overline{\theta}$, by contrast, cannot gain by trading this new contract. Assuming that the deviating buyer can break the indifference of type $\underline{\theta}$ in his favor, he strictly gains from trading the new contract with type $\underline{\theta}$, as the slope $\underline{\theta}$ of the line segment $\underline{AA'}$ is strictly less than $v(\underline{\theta})$. This contradiction shows that one must have $\underline{Q} = 1$ in equilibrium. The assumption on indifference breaking is relaxed in the proof of Lemma 1.

Attracting type $\underline{\theta}$ by pivoting around $(\overline{Q}, \overline{T})$ Having established that $\underline{Q} = 1$, we now investigate the aggregate quantity $\overline{Q}$ traded by type $\overline{\theta}$ in equilibrium. The second type of deviations allows one to partially characterize the circumstances in which the two types of the seller trade different aggregate allocations in equilibrium. We say in this case that the equilibrium is *separating*. An immediate implication of Lemma 1 is that $\overline{Q} < 1$ in any separating equilibrium. Let then $p = \frac{\underline{T} - \overline{T}}{1 - \overline{Q}}$ be the slope of the line connecting the points $(\overline{Q}, \overline{T})$ and $(1, \underline{T})$ in the (Q, T) space. Therefore p is the implicit unit price at which the quantity $1 - \overline{Q}$ can be sold to move from $(\overline{Q}, \overline{T})$ to $(1, \underline{T})$. By incentive compatibility, p must lie between $\underline{\theta}$ and $\overline{\theta}$ in any separating equilibrium. The strategic analysis of the buyers' behavior induces further restrictions on p.

Lemma 2 *In a separating equilibrium, $p < \overline{\theta}$ implies that $p \geq v(\underline{\theta})$.*

In the proof of Lemma 1, we showed that, if $\underline{Q} < 1$, then each buyer has an incentive to deviate. By contrast, in the proof of Lemma 2, we only show that if $p < \min\{v(\underline{\theta}), \overline{\theta}\}$ in a candidate separating equilibrium, then at least one buyer has an incentive to deviate. This makes it more difficult to graphically illustrate why the deviation used in Lemma 2 might be profitable. It is however easy to see why this deviation would be profitable to an entrant or, equivalently, to an inactive buyer that would not trade in equilibrium. This situation is depicted on Figure 3. The dotted line passing through $\overline{A}$ is an indifference curve for the buyers, with slope $v(\underline{\theta})$. Contrary to the conclusion of Lemma 2, the figure is drawn in such a way that this indifference curve is strictly steeper than the line segment $\overline{A}\underline{A}$.

—Insert Figure 3 here—

Suppose now that the entrant offers a contract that makes available the trade $\overline{A}\underline{A}$. This leaves type $\underline{\theta}$ indifferent, since she obtains the same payoff as in equilibrium by trading the aggregate allocation $(\overline{Q}, \overline{T})$ together with the new contract. Type $\overline{\theta}$, by contrast, cannot gain by trading this new contract. Assuming that the entrant can break the indifference of type $\underline{\theta}$ in his favor, he earns a strictly positive payoff from trading the new contract with type $\underline{\theta}$, as the slope p of the line segment $\overline{A}\underline{A}$ is strictly less than $v(\underline{\theta})$. This shows that, unless $p \geq v(\underline{\theta})$, the candidate separating equilibrium is not robust to entry. The assumption on indifference breaking is relaxed in the proof of Lemma 2, which further shows that the proposed deviation is profitable to at least one buyer.

Attracting both types by pivoting around $(\overline{Q}, \overline{T})$ A separating equilibrium must be robust to deviations that attract both types of the seller. This third type of deviations allows one to find a necessary condition for the existence of a separating equilibrium. When this condition fails, both types of the seller must trade the same aggregate allocations in equilibrium. We say in this case that the equilibrium is *pooling*.

Lemma 3 *If* $\mathbf{E}[v(\theta)] > \overline{\theta}$, *any equilibrium is pooling, with*

$$(\underline{Q}, \underline{T}) = (\overline{Q}, \overline{T}) = (1, \mathbf{E}[v(\theta)]).$$

The proof of Lemma 3 consists in showing that if $\mathbf{E}[v(\theta)] > \overline{\theta}$ in a candidate separating equilibrium, then at least one buyer has an incentive to deviate. As for Lemma 2, this makes it difficult to graphically illustrate why this deviation might be profitable. It is however easy to see why this deviation would be profitable to an entrant or, equivalently, to an inactive buyer that would not trade in equilibrium. This situation is depicted on Figure 4. The dotted line passing through $\overline{A}$ is an indifference curve for the buyers, with slope $\mathbf{E}[v(\theta)]$. Contrary to the conclusion of Lemma 3, the figure is drawn in such a way that this indifference curve is strictly steeper than the indifference curves of type $\overline{\theta}$.

—Insert Figure 4 here—

Suppose now that the entrant offers a contract that makes available the trade $\overline{A}\underline{A}'$. This leaves type $\overline{\theta}$ indifferent, since she obtains the same payoff as in equilibrium by trading the aggregate allocation $(\overline{Q}, \overline{T})$ together with the new contract. Type $\underline{\theta}$ strictly gains by trading this new contract. Assuming that the entrant can break the indifference of type $\overline{\theta}$ in his favor, he earns a strictly positive payoff from trading the new contract with both types as the slope $\overline{\theta}$ of the line segment $\overline{A}\underline{A}'$ is strictly less than $\mathbf{E}[v(\theta)]$. This shows that, unless $\mathbf{E}[v(\theta)] \leq \overline{\theta}$, the candidate equilibrium is not robust to entry. Once again, the assumption on indifference breaking is relaxed in the proof of Lemma 3, which further shows that the proposed deviation is profitable to at least one buyer.

The following result provides a partial converse to Lemma 3.

Lemma 4 *If* $\mathbf{E}[v(\theta)] < \overline{\theta}$, *any equilibrium is separating, with*

$$(\underline{Q}, \underline{T}) = (1, v(\underline{\theta})) \text{ and } (\overline{Q}, \overline{T}) = (0, 0).$$

The following is an important corollary of our analysis.

Corollary 1 *Each buyer's payoff is zero in any equilibrium.*

Lemmas 1 to 4 provide a full characterization of the aggregate trades that can be sustained in an equilibrium of the non-exclusive competition game. A key implication of Lemmas 3 and 4 is that the aggregate equilibrium allocation traded by the seller is generically unique.[8] While each buyer always obtains a zero payoff in equilibrium, the structure of equilibrium allocations is directly affected by the severity of the adverse selection problem:

- Whenever $\mathbf{E}[v(\theta)] > \overline{\theta}$, adverse section is mild, which rules out separating equilibria. Indeed, as shown in the proof of Lemma 3, if the aggregate allocation $(\overline{Q}, \overline{T})$ traded by type $\overline{\theta}$ were such that $\overline{Q} < 1$, some buyer would have an incentive to induce both types of the seller to trade this allocation, together with the additional quantity $1 - \overline{Q}$ at a unit price between $\overline{\theta}$ and $\mathbf{E}[v(\theta)]$. Competition among buyers then bids up the price of the seller's endowment to its average value $\mathbf{E}[v(\theta)]$ for the buyers, a price at which both types of the seller are ready to trade. This situation is depicted on Figure 5. The dotted line passing through the origin is the equilibrium indifference curve of the buyers, with slope $\mathbf{E}[v(\theta)]$.

—Insert Figure 5 here—

- Whenever $\mathbf{E}[v(\theta)] < \overline{\theta}$, adverse selection is severe, which rules out pooling equilibria.

[8] The non-generic case where $\mathbf{E}[v(\theta)] = \overline{\theta}$ is discussed after Proposition 2.

This reflects that type $\overline{\theta}$ is no longer ready to trade her endowment at the maximal price $\mathbf{E}[v(\theta)]$ at which buyers would break even in such an equilibrium. More interestingly, our analysis shows that non-exclusive competition induces a specific cost of screening the seller's type in equilibrium. Indeed, any separating equilibrium must be such that no buyer has an incentive to deviate and induce type $\underline{\theta}$ to trade the aggregate allocation $(\overline{Q}, \overline{T})$, together with the additional quantity $1 - \overline{Q}$ at some mutually advantageous price. Lemma 2 shows that to eliminate any incentive for buyers to engage in such trades with type $\underline{\theta}$, the implicit unit price at which this additional quantity $1 - \overline{Q}$ can be sold in equilibrium must be relatively high. As shown in Lemma 4, this implies at most an aggregate payoff $\{\mathbf{E}[v(\theta)] - \overline{\theta}\}\overline{Q}$ for the buyers. Hence type $\overline{\theta}$ can trade actively in a separating equilibrium only in the non-generic case where $\mathbf{E}[v(\theta)] = \overline{\theta}$, while type $\overline{\theta}$ does not trade at all if $\mathbf{E}[v(\theta)] < \overline{\theta}$. This situation is depicted on Figure 6. The dotted line passing through the origin is the equilibrium indifference curve of the buyers, with slope $v(\underline{\theta})$.

—Insert Figure 6 here—

3.2.2 Equilibrium Existence

We now establish that, in contrast with the exclusive competition game of Subsection 3.1, the non-exclusive competition game always has an equilibrium. Specifically, we show that there always exists an equilibrium in which all buyers post linear prices. In such an equilibrium, the unit price at which any quantity can be traded is equal to the expected quality of the goods that are actively traded. Specifically, define

$$p^* = \begin{cases} \mathbf{E}[v(\theta)] & \text{if } \mathbf{E}[v(\theta)] \geq \overline{\theta}, \\ v(\underline{\theta}) & \text{if } \mathbf{E}[v(\theta)] < \overline{\theta}. \end{cases} \tag{1}$$

One then has the following result.

Proposition 2 *The non-exclusive competition game always has an equilibrium in which each buyer offers the menu*

$$\{(q,t) \in [0,1] \times \mathbb{R}_+ : t = p^* q\},$$

and thus stands ready to buy any quantity of the good at a constant unit price p^*.

In the non-generic case where $\mathbf{E}[v(\theta)] = \overline{\theta}$, it is easy to check that there exist two linear price equilibria, a pooling equilibrium with constant unit price $\mathbf{E}[v(\theta)]$ and a separating equilibrium with constant unit price $v(\underline{\theta})$. In addition, there exists in this case a continuum of separating equilibria in which type $\overline{\theta}$ trades actively. Indeed, to sustain an equilibrium trade level $\overline{Q} \in (0,1)$ for type $\overline{\theta}$, it is enough that all buyers offer to buy any quantity of the good at unit price $v(\underline{\theta})$, and that one buyer offers in addition to buy any quantity of the good up to $\overline{Q}$ at unit price $\mathbf{E}[v(\theta)]$. Both types $\underline{\theta}$ and $\overline{\theta}$ then sell a fraction $\overline{Q}$ of their endowment at unit price $\mathbf{E}[v(\theta)]$, while type $\underline{\theta}$ sells the remaining fraction of her endowment at unit price $v(\underline{\theta})$. To avoid this non-generic multiplicity issue and therefore simplify the exposition, we shall assume that $\mathbf{E}[v(\theta)] \neq \overline{\theta}$ in the remainder of this section.

3.2.3 Comparison with the Exclusive Competition Model

Our analysis provides a fully strategic foundation for Akerlof's (1970) original intuition: if adverse selection is severe enough, only goods of low quality are traded in any market equilibrium. This contrasts sharply with the predictions of standard models of competition under adverse selection, in which, as in the exclusive competition game of Subsection 3.1, exclusivity clauses are assumed to be enforceable at no cost. Specifically, the equilibrium outcomes of the non-exclusive competition game differ in three crucial ways from that of the exclusive competition game:

- First, the exclusive competition game has an equilibrium only if the probability that the good is of high quality is low enough. By contrast, the non-exclusive competition game always has an equilibrium.

- Second, when it exists, the equilibrium of the exclusive competition game is always separating, while for certain parameter values all the equilibria of the non-exclusive competition game are pooling.

- Third, even when all equilibria of the non-exclusive competition game are separating, their structure is very different from that of the exclusive competition game. In the latter case, type $\underline{\theta}$ is indifferent between her equilibrium contract and that of type $\overline{\theta}$, who trades a strictly positive fraction of her endowment. By contrast, in the former case, type $\underline{\theta}$ strictly prefers her aggregate equilibrium allocation to that of type $\overline{\theta}$, who does not trade in equilibrium.

With regard to the last point, simple computations show that the threshold $\nu^e = \frac{\overline{\theta}-\underline{\theta}}{v(\overline{\theta})-\underline{\theta}}$ for ν below which the exclusive competition game has an equilibrium is strictly greater than the threshold $\nu^{ne} = \max\left\{0, \frac{\overline{\theta}-v(\underline{\theta})}{v(\overline{\theta})-v(\underline{\theta})}\right\}$ for ν below which all equilibria of the non-exclusive competition game are separating. Thus if one assumes that $\nu \leq \nu^e$, so that equilibria exist under both exclusivity and non-exclusivity, two situations can arise. When $0 < \nu < \nu^{ne}$, the equilibrium is separating under both exclusivity and non-exclusivity, and more trade takes place in the former case. By contrast, when $\nu^{ne} < \nu \leq \nu^{e}$, the equilibrium is separating under exclusivity and pooling under non-exclusivity, and more trade takes place in the latter case. Therefore, from an ex-ante viewpoint, exclusive competition leads to a more efficient outcome under severe adverse selection, while non-exclusive competition leads to a more efficient outcome under mild adverse selection.

3.3 Equilibrium Menus and Latent Contracts

We now explore in more depth the structure of the menus offered by the buyers in equilibrium. We first provide equilibrium restrictions for the price of issued and traded contracts. Next, we show that a large number of latent contracts needs to be issued in equilibrium. Then, we relate our analysis to the literature on communication in common agency games. Finally, we show that the aggregate equilibrium allocations can also be sustained through non-linear price schedules.

3.3.1 Price Restrictions

Our first result provides equilibrium restrictions on the price of all issued contracts.

Proposition 3 *The unit price of any contract issued in an equilibrium of the non-exclusive competition game is at most p^*.*

The intuition for this result is as follows. First, if $\mathbf{E}[v(\theta)] > \overline{\theta}$ and some buyer offered to purchase some quantity at a unit price above $\mathbf{E}[v(\theta)]$, any other buyer would have an incentive to induce both types of the seller to trade this contract and to sell him the remaining fraction of their endowment at a unit price slightly below $\mathbf{E}[v(\theta)]$. Second, if $\mathbf{E}[v(\theta)] < \overline{\theta}$ and some buyer offered to purchase some quantity at a unit price above $v(\underline{\theta})$, then any other buyer would have an incentive to induce type $\underline{\theta}$ to trade this contract and to sell him the remaining fraction of her endowment at a unit price slightly below $v(\underline{\theta})$. As a corollary, one obtains a straightforward characterization of the price of traded contracts.

Corollary 2 *The unit price of any contract traded in an equilibrium of the non-exclusive competition game is p^*.*

3.3.2 Latent Contracts

With these preliminaries at hand, we can investigate which contracts need to be issued to sustain the aggregate equilibrium allocations. From a strategic viewpoint, what matters for each buyer is the outside option of the seller, that is, what aggregate allocations she can achieve by trading with the other buyers only. For each buyer i, and for each menu profile $(C^1, \ldots, C^n)$, this is described by the set of aggregate allocations that remain available if buyer i withdraws his menu offer C^i. One first has the following result.

Proposition 4 *In any equilibrium of the non-exclusive competition game, the aggregate allocation* $(1, p^*)$ *remains available if any buyer withdraws his menu offer.*

The aggregate equilibrium allocation must therefore remain available even if a buyer deviates from his equilibrium menu offer. The reason is that this buyer would otherwise have an incentive to offer both types to sell their whole endowment at a price slightly below $\mathbf{E}[v(\theta)]$ (if $\mathbf{E}[v(\theta)] > \overline{\theta}$), or to offer type $\underline{\theta}$ to sell her whole endowment at price $v(\underline{\theta})$ while offering type $\overline{\theta}$ to sell a smaller fraction of her endowment on more advantageous terms (if $\mathbf{E}[v(\theta)] < \overline{\theta}$). The flip side of this observation is that no buyer is essential in providing the seller with her aggregate equilibrium allocation. This rules out standard Cournot outcomes in which the buyers would simply share the market and in which all issued contracts would actively be traded by some type of the seller, as in Biais, Martimort and Rochet (2000). As an illustration, when there are two buyers, there is no equilibrium in which each buyer would only offer to purchase half of the seller's endowment.

Because of the non-exclusivity of competition, equilibrium in fact involves much more restrictions on menus offers than those prescribed by Propositions 3 and 4. For instance, if $\mathbf{E}[v(\theta)] > \overline{\theta}$, there is no equilibrium in which each buyer only offers the allocation $(1, \mathbf{E}[v(\theta)])$ besides the no-trade contract. Indeed, any buyer could otherwise deviate by offering to purchase a quantity $\overline{q} < 1$ at some price $\overline{t} \in (\mathbf{E}[v(\theta)] - \overline{\theta}(1 - \overline{q}), \mathbf{E}[v(\theta)] - \underline{\theta}(1 - \overline{q}))$. By construction, this is a cream-skimming deviation that attracts only type $\overline{\theta}$, and that yields the deviating buyer a payoff

$$\nu[v(\overline{\theta})\overline{q} - \overline{t}] > \nu\{v(\overline{\theta})\overline{q} - \mathbf{E}[v(\theta)] + \underline{\theta}(1 - \overline{q})\},$$

which is strictly positive for $\overline{q}$ close enough to 1. To block such deviations, latent contracts must be issued that are not actively traded in equilibrium but which the seller has an incentive to trade if some buyer attempts to break the equilibrium. In order to play this deterrence role, the corresponding latent allocations must remain available if any buyer withdraws his menu offer. For instance, in the case $\mathbf{E}[v(\theta)] > \overline{\theta}$, the cream-skimming deviation described above is blocked if the quantity $1 - \overline{q}$ can always be sold at unit price

$\mathbf{E}[v(\theta)]$ at the deviation stage, since both types of the seller then have the same incentives to trade the contract proposed by the deviating buyer. This corresponds to the linear price equilibrium described in Proposition 2. In this equilibrium, the number of latent contracts is large; indeed, the menus offered by the buyers are infinite collections of contracts. The following result shows that this is a robust feature of any equilibrium.

Proposition 5 *In any equilibrium of the non-exclusive competition game, there are infinitely many aggregate allocations that remain available if any buyer withdraws his menu offer.*

The intuition for this result is as follows. As suggested by the above discussion, one of the roles of latent contracts is to prevent cream-skimming deviations that only attract type $\overline{\theta}$. Each buyer issues these contracts anticipating that type $\underline{\theta}$ will have an incentive to trade them following a cream-skimming deviation by any of the other buyers. Now, there are infinitely many such deviations. Consistent with this, the proof of Proposition 5 proceeds by showing that if only finitely many latent contracts were offered at equilibrium by buyers $j \neq i$, it would be possible to construct a cream-skimming deviation for buyer i that would yield him a strictly positive payoff.

3.3.3 Menus, Communication, and the Failure of the Revelation Principle

Our results on the necessary role played by latent contracts to support equilibrium allocations have a natural interpretation in the language of the common agency literature, whose aim is to analyze situations where several principals compete through mechanisms for the services of a single agent.[9] In our context, given a set M^i of messages from the seller to buyer i, a mechanism for buyer i is a mapping $\gamma^i : M^i \to [0,1] \times \mathbb{R}_+$ that associates to each message sent by the seller to buyer i a quantity-transfer pair or contract. Denote by Γ^i the set of mechanisms available to buyer i and let $\Gamma = \prod_{i=1}^n \Gamma^i$. In the common agency game relative to Γ, the seller takes her participation and communication decisions after having observed the profile of mechanisms $(\gamma^1, \ldots, \gamma^n) \in \Gamma$ offered by the different buyers. Peters (2001) and Martimort and Stole (2002) have proven the following result, often referred to as the *Delegation Principle*: the equilibrium outcomes relative to a given set of mechanisms Γ correspond to the outcomes that can be supported at equilibrium in a game where the sellers offer menus over the allocations induced by Γ.

In our setting, buyers compete over menus of contracts for the trade of a divisible good. From Proposition 5, we know that equilibrium menus should contain an infinite number of

[9]See for instance Martimort (2007) for a recent overview of that literature.

contracts. In view of the Delegation Principle, this suggests that to support our Akerlof-like equilibrium outcomes when competition over mechanisms is considered, a rich structure of communication has to be postulated. That is, an infinite number of messages should be available to the seller, allowing her to effectively act as a coordinating device among buyers, so as to guarantee the existence of an equilibrium. In particular, these allocations cannot be supported if buyers are restricted to compete through simple direct mechanisms of the form $\hat{\gamma}^i : \{\underline{\theta}, \overline{\theta}\} \to [0,1] \times \mathbb{R}_+$ through which the seller can only communicate her type to the buyers. Indeed, if the buyers are restricted to direct mechanisms, only a finite set of offers will be available to the seller, which, as we have seen, makes it impossible to support our equilibrium allocations. Critically, direct mechanisms do not provide enough flexibility to buyers to make a strategic use of the seller in deterring cream-skimming deviations.[10] The possibility to support some equilibrium allocations relative to an arbitrary set of indirect mechanisms, but not in the corresponding direct mechanism game, has been acknowledged as a failure of the Revelation Principle in common agency games, and documented in purely abstract game-theoretic examples.[11] One of the contribution of our analysis is to exhibit a natural and relevant economic setting that exhibits this feature. Note furthermore that, in contrast with the exclusive competition context, where market equilibria can without any loss of generality be characterized through simple direct mechanisms, the restriction to such mechanisms turns out to be devastating under non-exclusivity: indeed, in this context, an immediate implication of our analysis is that no allocation can be supported at equilibrium in the direct mechanism game.

3.3.4 Non-Linear Equilibria

An important insight of our analysis is that one can also construct non-linear equilibria in which latent contracts are issued at a unit price different from that of the aggregate allocation that is traded in equilibrium.

Proposition 6 *The following holds*:

(i) If $\mathbf{E}[v(\theta)] > \overline{\theta}$, *then, for each* $\phi \in [\overline{\theta}, \mathbf{E}[v(\theta)])$, *the non-exclusive competition game has an equilibrium in which each buyer offers the menu*

[10] This difficulty remains intact even if stochastic direct mechanisms are allowed. Indeed, in any pure strategy equilibrium of a direct mechanism game where buyers use stochastic mechanisms, the seller will send messages before observing the realization of uncertainty. At equilibrium, only a finite number of lotteries over allocations will be offered. Bilateral risk-neutrality then makes this situation equivalent to one in which only deterministic allocations are proposed. One should moreover observe that it is problematic to interpret stochastic mechanisms in our model, where the seller operates under a capacity constraint.

[11] See for instance Peck (1997), Peters (2001) and Martimort and Stole (2002).

$$\left\{(q,t) \in \left[0, \frac{v(\overline{\theta}) - \mathbf{E}[v(\theta)]}{v(\overline{\theta}) - \phi}\right] \times \mathbb{R}_+ : t = \phi q\right\} \cup \{(1, \mathbf{E}[v(\theta)])\}.$$

(ii) If $\mathbf{E}[v(\theta)] < \overline{\theta}$, then, for each $\psi \in \left(v(\underline{\theta}), v(\underline{\theta}) + \frac{\overline{\theta} - \mathbf{E}[v(\theta)]}{1-\nu}\right]$, the non-exclusive competition game has an equilibrium in which each buyer offers the menu

$$\{(0,0)\} \cup \left\{(q,t) \in \left[\frac{\psi - v(\underline{\theta})}{\psi}, 1\right] \times \mathbb{R}_+ : t = \psi q - \psi + v(\underline{\theta})\right\}.$$

This results shows that the unique aggregate equilibrium allocation can also be supported through non-linear prices. In such equilibria, the price each buyer is willing to pay for an additional unit of the good is not the same for all quantities purchased. For instance, in the equilibrium for the severe adverse selection case described in Proposition 6(i), buyers are not ready to pay anything for all quantities up to the level $\frac{\psi - v(\underline{\theta})}{\psi}$, while they are ready to pay ψ for each additional unit of the good above this level. The price schedule posted by each buyer is such that, for any $q < 1$, the unit price $\max\left\{0, \psi - \frac{\psi - v(\underline{\theta})}{q}\right\}$ at which he offers to purchase the quantity q is strictly below $\underline{\theta}$, while the marginal price ψ at which he offers to purchase an additional unit given that he has already purchased a quantity $q \geq \frac{\psi - v(\underline{\theta})}{\psi}$ is strictly above $\underline{\theta}$. Therefore the equilibrium budget set of the seller

$$\left\{(Q,T) \in [0,1] \times \mathbb{R}_+ : Q = \sum_i q^i \text{ and } T \leq \sum_i t^i \text{ where } (q^i, t^i) \in C^i \text{ for all } i\right\}$$

is not convex in this equilibrium. As a result of this, the seller has a strict incentive to deal with a single buyer: market equilibria can be supported with a single active buyer, provided that the other buyers coordinate by offering appropriate latent contracts. It follows in particular that non-exclusive competition does not necessarily entail that the seller enters into multiple contracting relationships. This contrasts with recent work on competition in non-exclusive mechanisms under incomplete information, where attention is typically restricted to equilibria in which the informed agent has a convex budget set in equilibrium, or, what amounts to the same thing, where the set of allocations available to her is the frontier of a convex budget set.[12] In our model, this would for instance arise if all buyers posted concave price schedules. It is therefore interesting to notice that, as a matter of fact, our non-exclusive competition game admits no equilibrium in which each buyer i posts a strictly concave price schedule T^i. The reason is that the aggregate price schedule $\mathfrak{T}$ defined by $\mathfrak{T}(Q) = \sup\{\sum_i T^i(q^i) : \sum_i q^i = Q\}$ would otherwise be strictly concave in the aggregate

[12] See for instance Biais, Martimort and Rochet (2000), Khalil, Martimort and Parigi (2007) or Martimort and Stole (2009). Piaser (2007) offers a general discussion of the role of latent contracts in incomplete information settings.

quantity traded Q. This would in turn imply that contracts are issued at a unit price strictly above $\mathfrak{T}(1)$, which, as shown by Proposition 3, is impossible in equilibrium.

A further implication of Proposition 6 is that latent contracts supporting the equilibrium allocations can be issued at a profitable price for the issuer. For instance, in the equilibrium described in Proposition 6(ii), any contract in the set $\left\{\left[\frac{\psi-v(\underline{\theta})}{\psi},1\right)\times\mathbb{R}_+ : t=\psi q-\psi+v(\underline{\theta})\right\}$ would yield its issuer a strictly positive payoff, even if it were traded by type $\underline{\theta}$ only. In equilibrium, no mistakes occur, and buyers correctly anticipate that none of these contracts will be traded. Nonetheless, removing these contracts would break the equilibrium. One should notice in that respect that the role of latent contracts in non-exclusive markets has usually been emphasized in complete information environments in which the agent does not trade efficiently in equilibrium.[13] In these contexts, latent contracts can never be profitable. Indeed, if they were, there would always be room for proposing an additional latent contract at a less profitable price and induce the agent to accept it. In our model, by contrast, type $\underline{\theta}$ sells her whole endowment in equilibrium. It follows from Proposition 3 that there cannot be any latent contract that would make losses. In addition, there is no incentive for any single buyer to raise the price of these contracts and make the seller willing to trade them.

4 The Continuous-Type Case

[To be completed]

5 Conclusion

In this paper, we have studied a simple imperfect competition model of trade under adverse selection. When competition is exclusive, the existence of equilibria is problematic, while equilibria always exist when competition is non-exclusive. In this latter case, aggregate quantities and transfers are generically unique, and correspond to the allocations that obtain in Akerlof's (1970) model. Linear price equilibria can be constructed in which buyers stand ready to purchase any quantity at a constant unit price.

The fact that possible market outcomes tightly depend on the nature of competition suggests that the testable implications of competitive models of adverse selection should be evaluated with care. Indeed, these implications are typically derived from the study of exclusive competition models, such as Rothschild and Stiglitz's (1976) two-type model of

[13] See for instance Hellwig (1983), Martimort and Stole (2003), Bisin and Guaitoli (2004) or Attar and Chassagnon (2009).

insurance markets. By contrast, our analysis shows that more competitive outcomes can be sustained in equilibrium under non-exclusive competition, and that these outcomes can involve a substantial amount of pooling.

These results offer new insights into the empirical literature on adverse selection. For instance, several studies have taken to the data the predictions of theoretical models of insurance provision, without reaching clear conclusions.[14] Cawley and Philipson (1999) argue that there is little empirical support for the adverse selection hypothesis in life insurance. In particular, they find no evidence that marginal prices raise with coverage. Similarly, Finkelstein and Poterba (2004) find that marginal prices do not significantly differ across annuities with different initial annual payments. The theoretical predictions tested by these authors are however derived from models of exclusive competition,[15] while our results clearly indicate that they do not hold when competition is non-exclusive, as in the case of life insurance or annuities. Indeed, non-exclusive competition might be one explanation for the limited evidence of screening and the prevalence of nearly linear pricing schemes on these markets. As a result, more sophisticated procedures need to be designed in order to test for the presence of adverse selection in markets where competition is non-exclusive.

[14]See Chiappori and Salanié (2003) for a survey of this literature.

[15]Chiappori, Jullien, Salanié and Salanié (2006) have derived general tests based on a model of exclusive competition, that they apply to the case of car insurance.

Appendix

Proof of Proposition 1. The proof follows more or less standard lines (see for instance Mas-Colell, Whinston and Green (1995, Chapter 13, Section D)) and goes through a series of steps.

Step 1 Denote by $(\underline{q}, \underline{t})$ and $(\overline{q}, \overline{t})$ the contracts traded by the two types of the seller in equilibrium. These contracts must satisfy the following incentive constraints:

$$\underline{t} - \underline{\theta}\underline{q} \geq \overline{t} - \underline{\theta}\overline{q},$$

$$\overline{t} - \overline{\theta}\overline{q} \geq \underline{t} - \overline{\theta}\underline{q}.$$

Since the buyers always have the option not to trade, each of them must obtain at least a zero payoff in equilibrium. Suppose that some buyer's equilibrium payoff is strictly positive. Then the buyers' aggregate equilibrium payoff is strictly positive,

$$\nu[v(\overline{\theta})\overline{q} - \overline{t}] + (1 - \nu)[v(\underline{\theta})\underline{q} - \underline{t}] > 0.$$

Any buyer i obtaining less than half of this amount in equilibrium can deviate by offering a menu consisting of the no-trade contract and of two new contracts. The first one is

$$\underline{c}^i(\underline{\varepsilon}) = (\underline{q}, \underline{t} + \underline{\varepsilon}),$$

for some strictly positive number $\underline{\varepsilon}$, and is designed to attract type $\underline{\theta}$. The second one is

$$\overline{c}^i(\overline{\varepsilon}) = (\overline{q}, \overline{t} + \overline{\varepsilon}),$$

for some strictly positive number $\overline{\varepsilon}$, and is designed to attract type $\underline{\theta}$. To ensure that type $\underline{\theta}$ trades $\underline{c}^i(\underline{\varepsilon})$ and type $\overline{\theta}$ trades $\overline{c}^i(\overline{\varepsilon})$ with him, buyer i can choose $\underline{\varepsilon}$ to be equal to $\overline{\varepsilon}$ when both types' equilibrium incentive constraints are simultaneously binding or slack, and choose $\underline{\varepsilon}$ and $\overline{\varepsilon}$ to be different but close enough to each other when one of these constraints is binding and the other is slack. The change in buyer i's payoff induced by this deviation is at least

$$\frac{1}{2}\left\{\nu[v(\overline{\theta})\overline{q} - \overline{t}] + (1 - \nu)[v(\underline{\theta})\underline{q} - \underline{t}]\right\} - \nu\overline{\varepsilon} - (1 - \nu)\underline{\varepsilon},$$

which is strictly positive for $\underline{\varepsilon}$ and $\overline{\varepsilon}$ close enough to zero. Thus each buyer's payoff is zero in any equilibrium.

Step 2 Suppose that there exists a pooling equilibrium with both types of the seller trading the same contract (q^p, t^p). It follows from Step 1 that $t^p = \mathbf{E}[v(\theta)]q^p$ and that both types of the seller must trade with the same buyer j. Any buyer $i \neq j$ can deviate by offering a menu consisting of the no-trade contract and of the contract

$$\overline{c}^i(\varepsilon) = (q^p - \varepsilon, t^p - \underline{\theta}\varepsilon(1+\varepsilon)),$$

for some strictly positive number ε. Trading $\overline{c}^i(\varepsilon)$ decreases type $\underline{\theta}$'s payoff by $\underline{\theta}\varepsilon^2$ compared to what she obtains by trading (q^p, t^p) with buyer j. Hence type $\underline{\theta}$ does not trade $\overline{c}^i(\varepsilon)$ following buyer i's deviation. By contrast, if $\varepsilon < \frac{\overline{\theta}}{\underline{\theta}} - 1$, trading $\overline{c}^i(\varepsilon)$ allows type $\overline{\theta}$ to increase her payoff by $[\overline{\theta} - (1+\varepsilon)\underline{\theta}]\varepsilon$ compared to what she obtains by trading (q^p, t^p) with buyer j. Hence type $\overline{\theta}$ trades $\overline{c}^i(\varepsilon)$ following buyer i's deviation. The payoff for buyer i induced by this deviation is

$$\nu\{v(\overline{\theta})q^p - t^p - [v(\overline{\theta}) - \underline{\theta}(1+\varepsilon)]\varepsilon\},$$

which is strictly positive for ε close enough to zero since $t^p = \mathbf{E}[v(\theta)]q^p$ and $v(\overline{\theta}) > \mathbf{E}[v(\theta)]$. This, however, is impossible by Step 1. Thus any equilibrium must be separating, with the two types of the seller trading different contracts.

Step 3 Suppose that $v(\underline{\theta})\underline{q} > \underline{t}$, so that the contract $(\underline{q}, \underline{t})$ yields the buyer who trades it with type $\underline{\theta}$ a strictly positive payoff. Any buyer i can deviate by offering a menu consisting of the no-trade contract and of the contract

$$\underline{c}^i(\varepsilon) = (\underline{q}, \underline{t} + \varepsilon),$$

for some strictly positive number ε. Type $\underline{\theta}$ trades $\underline{c}^i(\varepsilon)$ following buyer i's deviation, and also possibly type $\overline{\theta}$. The payoff for buyer i induced by this deviation is thus at least

$$(1-\nu)[v(\underline{\theta})\underline{q} - \underline{t} - \varepsilon],$$

which is strictly positive for ε close enough to zero if $v(\underline{\theta})\underline{q} > \underline{t}$. Since this is impossible by Step 1, it must be that $\underline{t} \geq v(\underline{\theta})\underline{q}$. Suppose next that $v(\overline{\theta})\overline{q} > \overline{t}$, so that the contract $(\overline{q}, \overline{t})$ yields the buyer j who trades it with type $\overline{\theta}$ a strictly positive payoff. Any buyer $i \neq j$ can deviate by offering a menu consisting of the no-trade contract and of the contract

$$\overline{c}^i(\varepsilon) = (\overline{q} - \varepsilon, \overline{t} - \underline{\theta}\varepsilon(1+\varepsilon)),$$

for some strictly positive number ε. As in Step 2, it is easy to check that type $\underline{\theta}$ does not trade $\overline{c}^i(\varepsilon)$ following buyer i's deviation, while type $\overline{\theta}$ does so provided $\varepsilon < \frac{\overline{\theta}}{\underline{\theta}} - 1$. The payoff for buyer i induced by this deviation is

$$\nu\{v(\overline{\theta})\overline{q} - \overline{t} - [v(\overline{\theta}) - \underline{\theta}(1+\varepsilon)]\varepsilon\},$$

which is strictly positive for ε close enough to zero if $v(\overline{\theta})\overline{q} > \overline{t}$. Since this is impossible by Step 1, it must be that $\overline{t} \geq v(\overline{\theta})\overline{q}$. This, along with the facts that $\underline{t} \geq v(\underline{\theta})\underline{q}$ and that the buyers' aggregate equilibrium payoff is zero, implies that $\underline{t} = v(\underline{\theta})\underline{q}$ and $\overline{t} = v(\overline{\theta})\overline{q}$. Thus the contracts $(\underline{q}, \underline{t})$ and $(\overline{q}, \overline{t})$ are traded at unit prices $v(\underline{\theta})$ and $v(\overline{\theta})$, and no cross-subsidization across types can take place in equilibrium.

Step 4 Suppose that type $\underline{\theta}$ sells a quantity $\underline{q} < 1$ in equilibrium. Any buyer i can deviate by offering a menu consisting of the no-trade contract and of the contract

$$\underline{c}^i(\varepsilon) = (1, \underline{t} + [v(\underline{\theta}) - \varepsilon](1 - \underline{q})),$$

for some strictly positive number ε. As long as $\varepsilon < v(\underline{\theta}) - \underline{\theta}$, trading $\underline{c}^i(\varepsilon)$ allows type $\underline{\theta}$ to increase her payoff by $[v(\underline{\theta}) - \underline{\theta} - \varepsilon](1 - \underline{q})$ compared to what she obtains by trading $(\underline{q}, \underline{t})$. Hence type $\underline{\theta}$ trades $\underline{c}^i(\varepsilon)$ following buyer i's deviation, and also possibly type $\overline{\theta}$. The payoff for buyer i induced by this deviation is thus at least

$$(1 - \nu)\{v(\underline{\theta}) - \underline{t} - [v(\underline{\theta}) - \varepsilon](1 - \underline{q})\} = (1 - \nu)(1 - \underline{q})\varepsilon,$$

where use was made of the fact that $\underline{t} = v(\underline{\theta})\underline{q}$ by Step 3. Since $\varepsilon > 0$, this payoff is strictly positive, which is impossible by Step 1. Thus type $\underline{\theta}$ sells her whole endowment in any equilibrium, and $(\underline{q}, \underline{t}) = (\underline{q}^e, \underline{t}^e)$ as defined in Proposition 1.

Step 5 The contract $(\overline{q}^e, \overline{t}^e)$ is characterized by two properties: it has a unit price $v(\overline{\theta})$ and type $\underline{\theta}$ is indifferent between $(\underline{q}^e, \underline{t}^e)$ and $(\overline{q}^e, \overline{t}^e)$. One cannot have $\overline{q} > \overline{q}^e$, for $(\overline{q}, \overline{t})$ is traded at unit price $v(\overline{\theta})$ by Step 3, and any contract in which a quantity strictly higher than $\overline{q}^e$ is traded at unit price $v(\overline{\theta})$ is strictly preferred by type $\underline{\theta}$ to $(\underline{q}^e, \underline{t}^e)$. Now suppose that type $\underline{\theta}$ trades $(\underline{q}^e, \underline{t}^e)$ with buyer j in equilibrium and that $\overline{q} < \overline{q}^e$. Then type $\underline{\theta}$ strictly prefers $(\underline{q}^e, \underline{t}^e)$ to $(\overline{q}, \overline{t})$, that is $\underline{t}^e - \underline{\theta}\underline{q}^e > \overline{t} - \underline{\theta}\overline{q}$. Any buyer $i \neq j$ can deviate by offering a menu consisting of the no-trade contract and of the contract

$$\overline{c}^i(\varepsilon) = (\overline{q} + \varepsilon, \overline{t} + \overline{\theta}\varepsilon(1 + \varepsilon)),$$

for some strictly positive number ε. Trading $\overline{c}^i(\varepsilon)$ decreases type $\underline{\theta}$'s payoff by

$$\underline{t}^e - \underline{\theta}\underline{q}^e - \overline{t} + \underline{\theta}\overline{q} - [\overline{\theta}(1 + \varepsilon) - \underline{\theta}]\varepsilon$$

compared to what she obtains by trading $(\underline{q}^e, \underline{t}^e)$ with buyer j. Since $\underline{t}^e - \underline{\theta}\underline{q}^e > \overline{t} - \underline{\theta}\overline{q}$, type $\underline{\theta}$ does not trade $\overline{c}^i(\varepsilon)$ following buyer i's deviation if ε is close enough to zero. By contrast, trading $\overline{c}^i(\varepsilon)$ allows type $\overline{\theta}$ to increase her payoff by $\overline{\theta}\varepsilon^2$ compared to what she obtains in equilibrium. Hence type $\overline{\theta}$ trades $\overline{c}^i(\varepsilon)$ following buyer i's deviation. The payoff for buyer i induced by this deviation is

$$\nu[v(\overline{\theta})(\overline{q}+\varepsilon)-\overline{t}-\overline{\theta}\varepsilon(1+\varepsilon)]=\nu[v(\overline{\theta})-\overline{\theta}(1+\varepsilon)],$$

where use was made of the fact that $\overline{t}=v(\overline{\theta})\overline{q}$ by Step 3. When $\varepsilon<\frac{v(\overline{\theta})}{\overline{\theta}}-1$, this payoff is strictly positive, which is impossible by Step 1. Thus type $\underline{\theta}$ sells a fraction $\overline{q}^e$ of her endowment in any equilibrium, and $(\overline{q},\overline{t})=(\overline{q}^e,\overline{t}^e)$ as defined in Proposition 1.

Step 6 It follows from Steps 4 and 5 that if an equilibrium exists, the contracts that are traded in this equilibrium are $(\underline{q}^e,\underline{t}^e)$ and $(\overline{q}^e,\overline{t}^e)$. To conclude the proof, one only needs to determine under which circumstances it is possible to support this allocation in equilibrium. Suppose first that $\nu>\nu^e$. Any buyer i can deviate by offering a menu consisting of the no-trade contract and of the contract

$$\tilde{c}^i(\varepsilon)=(1,v(\overline{\theta})\overline{q}^e+\overline{\theta}(1-\overline{q}^e)+\varepsilon),$$

for some strictly positive number ε. Using the fact that type $\underline{\theta}$ is indifferent between $(\underline{q}^e,\underline{t}^e)$ and $(\overline{q}^e,\overline{t}^e)$, one can check that trading $\tilde{c}^i(\varepsilon)$ allows type $\underline{\theta}$ to increase her payoff by

$$v(\overline{\theta})\overline{q}^e+\overline{\theta}(1-\overline{q}^e)+\varepsilon-v(\underline{\theta})=(\overline{\theta}-\underline{\theta})(1-\overline{q}^e)+\varepsilon$$

compared to what she obtains by trading $(\underline{q}^e,\underline{t}^e)$. Hence type $\underline{\theta}$ trades $\tilde{c}^i(\varepsilon)$ following buyer i's deviation. Similarly, trading $\tilde{c}^i(\varepsilon)$ allows type $\overline{\theta}$ to increase her payoff by ε compared to what she obtains by trading $(\overline{q}^e,\overline{t}^e)$. Hence type $\overline{\theta}$ trades $\tilde{c}^i(\varepsilon)$ following buyer i's deviation. Simple computations show that the payoff for buyer i induced by this deviation is

$$\mathbf{E}[v(\theta)]-v(\overline{\theta})\overline{q}^e-\overline{\theta}(1-\overline{q}^e)-\varepsilon=[v(\overline{\theta})-v(\underline{\theta})](\nu-\nu^e)-\varepsilon,$$

which is strictly positive for ε close enough to zero. Since this is impossible by Step 1, it follows that no equilibrium exists when $\nu>\nu^e$. Suppose then that $\nu\leq\nu^e$. Consider a candidate equilibrium in which each buyer proposes the menu consisting of the no-trade contract and of the contracts $(\underline{q}^e,\underline{t}^e)$ and $(\overline{q}^e,\overline{t}^e)$. Then, on the equilibrium path, it is a best response for type $\underline{\theta}$ to trade $(\underline{q}^e,\underline{t}^e)$ and for type $\underline{\theta}$ to trade $(\overline{q}^e,\overline{t}^e)$. By Step 3, this yields each buyer a zero payoff. To verify that this constitutes an equilibrium, one first needs to check that no buyer can strictly increase his payoff by proposing a single contract besides the no-trade contract. By Steps 3, 4 and 5, there is no profitable deviation that would attract only one type of the seller. Moreover, a profitable pooling deviation exists if and only if, given the menus offered in equilibrium, both types of the seller would have a strict incentive to sell their whole endowment at price $\mathbf{E}[v(\theta)]$. This is the case if and only if $\mathbf{E}[v(\theta)] > v(\overline{\theta})\overline{q}^e + \overline{\theta}(1-\overline{q}^e)$, or equivalently $\nu > \nu^e$. Thus when $\nu \leq \nu^e$, no menu consisting of a single contract besides the no-trade contract can constitute a profitable deviation. To conclude the proof, one only needs to check that no buyer can strictly increase his payoff by offering two contracts besides the no-trade contract, that attract both types of the seller. The maximum payoff that any buyer can achieve in this way is given by

$$\max_{(\underline{q},\underline{t},\overline{q},\overline{t})} \{\nu[v(\overline{\theta})\overline{q} - \overline{t}] + (1-\nu)[v(\underline{\theta})\underline{q} - \underline{t}]\}$$

subject to the following incentive and participation constraints:

$$\underline{t} - \underline{\theta}\underline{q} \geq \overline{t} - \underline{\theta}\overline{q},$$

$$\overline{t} - \overline{\theta}\overline{q} \geq \underline{t} - \overline{\theta}\underline{q},$$

$$\underline{t} - \underline{\theta}\underline{q} \geq \underline{t}^e - \underline{\theta}\underline{q}^e,$$

$$\overline{t} - \overline{\theta}\overline{q} \geq \overline{t}^e - \overline{\theta}\overline{q}^e.$$

Note from the incentive constraints that $\overline{q} \leq \underline{q}$. It is clear that at least one of the participation constraints must be binding. Suppose first that type $\underline{\theta}$'s participation constraint is binding. If $\underline{q} \leq \overline{q}^e$, then the relevant constraint for type $\overline{\theta}$ is her incentive constraint. It is then optimal to let type $\overline{\theta}$ be indifferent between $(\underline{q},\underline{t})$ and $(\overline{q},\overline{t})$. Since $v(\underline{\theta}) > \underline{\theta}$, $v(\overline{\theta}) > \overline{\theta}$ and $\underline{q} \leq \overline{q}^e$, the maximum payoff that the deviating buyer can achieve in this way is obtained by offering $(\overline{q},\overline{t}) = (\underline{q},\underline{t}) = (\overline{q}^e,\overline{t}^e)$, and is therefore strictly negative. If $\underline{q} > \overline{q}^e$, then the relevant constraint for type $\overline{\theta}$ is her participation constraint. It is then optimal to let type $\overline{\theta}$ be indifferent between $(\overline{q},\overline{t})$ and $(\overline{q}^e,\overline{t}^e)$. One cannot have $\overline{q} > \overline{q}^e$, for otherwise type $\underline{\theta}$ would strictly prefer $(\overline{q},\overline{t})$ to $(\underline{q},\underline{t})$. Since $v(\underline{\theta}) > \underline{\theta}$, $v(\overline{\theta}) > \overline{\theta}$ and $\overline{q} \leq \overline{q}^e$, the maximum payoff that the deviating buyer can achieve in this way is obtained by offering the equilibrium contracts $(\underline{q}^e,\underline{t}^e)$ and $(\overline{q}^e,\overline{t}^e)$. Suppose finally that type $\overline{\theta}$'s participation constraint is binding. If $\overline{q} \leq \overline{q}^e$, then the relevant constraint for type $\underline{\theta}$ is her participation constraint. It is then optimal to let type $\underline{\theta}$ be indifferent between $(\underline{q},\underline{t})$ and $(\underline{q}^e,\underline{t}^e)$. Again, since $v(\underline{\theta}) > \underline{\theta}$, $v(\overline{\theta}) > \overline{\theta}$ and $\overline{q} \leq \overline{q}^e$, the maximum payoff that the deviating buyer can achieve in this way is obtained by offering the equilibrium contracts $(\underline{q}^e,\underline{t}^e)$ and $(\overline{q}^e,\overline{t}^e)$. If $\overline{q} > \overline{q}^e$, then the relevant constraint for type $\underline{\theta}$ is her incentive constraint. It is then optimal to let type $\underline{\theta}$ be indifferent between $(\underline{q},\underline{t})$ and $(\overline{q},\overline{t})$. Simple computations show that the payoff for the deviating buyer is

$$\{\nu[v(\overline{\theta}) - \underline{\theta}] - \overline{\theta} + \underline{\theta}\}\overline{q} + (1-\nu)[v(\underline{\theta}) - \underline{\theta}]\underline{q} - \overline{t}^e + \overline{\theta}\overline{q}^e.$$

Since $\nu \leq \nu^e$, $v(\underline{\theta}) > \underline{\theta}$ and $\overline{q} > \overline{q}^e$, this is at most equal to the payoff that the deviating buyer would obtain by offering the equilibrium contracts $(\underline{q}^e, \underline{t}^e)$ and $(\overline{q}^e, \overline{t}^e)$. The result follows. ■

Proof of Lemma 1. Suppose instead that $\underline{Q} < 1$. Any buyer i can deviate by offering a menu consisting of the no-trade contract and of two new contracts. The first one is

$$\underline{c}^i(\varepsilon) = (\underline{q}^i + 1 - \underline{Q}, \underline{t}^i + (\underline{\theta} + \varepsilon)(1 - \underline{Q})),$$

for some strictly positive number ε, and is designed to attract type $\underline{\theta}$. The second one is

$$\overline{c}^i(\varepsilon) = (\overline{q}^i, \overline{t}^i + \varepsilon^2),$$

and is designed to attract type $\overline{\theta}$. The key feature of this deviation is that type $\underline{\theta}$ can sell her whole endowment by trading $\underline{c}^i(\varepsilon)$ together with the contracts $\underline{c}^j$, $j \neq i$. Since the unit price at which buyer i offers to purchase the quantity increment $1 - \underline{Q}$ in $\underline{c}^i(\varepsilon)$ is $\underline{\theta} + \varepsilon$, this guarantees her a payoff increase $(1 - \underline{Q})\varepsilon$ compared to what she obtains in equilibrium. When ε is close enough to zero, she cannot obtain as much by trading $\overline{c}^i(\varepsilon)$ instead. Indeed, even if this were to increase her payoff compared to what she obtains in equilibrium, the corresponding increase would be at most $\varepsilon^2 < (1 - \underline{Q})\varepsilon$. Hence type $\underline{\theta}$ trades $\underline{c}^i(\varepsilon)$ following buyer i's deviation. Consider now type $\overline{\theta}$. By trading $\overline{c}^i(\varepsilon)$ together with the contracts $\overline{c}^j$, $j \neq i$, she can increase her payoff by ε^2 compared to what she obtains in equilibrium. By trading $\underline{c}^i(\varepsilon)$ instead, the most she can obtain is her equilibrium payoff, plus the payoff from selling the quantity increment $1 - \underline{Q}$ at unit price $\underline{\theta} + \varepsilon$. For ε close enough to zero, $\underline{\theta} + \varepsilon < \overline{\theta}$ so that this unit price is too low from the point of view of type $\overline{\theta}$. Hence type $\overline{\theta}$ trades $\overline{c}^i(\varepsilon)$ following buyer i's deviation. The change in buyer i's payoff induced by this deviation is

$$-\nu\varepsilon^2 + (1 - \nu)[v(\underline{\theta}) - \underline{\theta} - \varepsilon](1 - \underline{Q})$$

which is strictly positive for ε close enough to zero if $\underline{Q} < 1$. Thus $\underline{Q} = 1$, as claimed. ■

Proof of Lemma 2. Suppose that $p < \overline{\theta}$ in a separating equilibrium. Any buyer i can deviate by offering a menu consisting of the no-trade contract and of two new contracts. The first one is

$$\underline{c}^i(\varepsilon) = (\overline{q}^i + 1 - \overline{Q}, \overline{t}^i + (p + \varepsilon)(1 - \overline{Q})),$$

for some strictly positive number ε, and is designed to attract type $\underline{\theta}$. The second one is

$$\overline{c}^i(\varepsilon) = (\overline{q}^i, \overline{t}^i + \varepsilon^2),$$

and is designed to attract type $\overline{\theta}$. The key feature of this deviation is that type $\underline{\theta}$ can sell her whole endowment by trading $\underline{c}^i(\varepsilon)$ together with the contracts $\overline{c}^j$, $j \neq i$. Since the unit price at which buyer i offers to purchase the quantity increment $1-\overline{Q}$ in $\underline{c}^i(\varepsilon)$ is $p+\varepsilon$, this guarantees her a payoff increase $(1-\overline{Q})\varepsilon$ compared to what she obtains in equilibrium. As in the proof of Lemma 1, it is easy to check that when ε is close enough to zero, she cannot obtain as much by trading $\overline{c}^i(\varepsilon)$ instead. Hence type $\underline{\theta}$ trades $\underline{c}^i(\varepsilon)$ following buyer i's deviation. Consider now type $\overline{\theta}$. By trading $\overline{c}^i(\varepsilon)$ together with the contracts $\overline{c}^j$, $j \neq i$, she can increase her payoff by ε^2 compared to what she obtains in equilibrium. As in the proof of Lemma 1, it is easy to check that when $p+\varepsilon < \overline{\theta}$, she cannot obtain as much by trading $\underline{c}^i(\varepsilon)$ instead. Hence type $\overline{\theta}$ trades $\overline{c}^i(\varepsilon)$ following buyer i's deviation. The change in buyer i's payoff induced by this deviation is

$$-\nu\varepsilon^2 + (1-\nu)\{v(\underline{\theta})(\overline{q}^i - \underline{q}^i) - \overline{t}^i + \underline{t}^i + [v(\underline{\theta}) - p - \varepsilon](1-\overline{Q})\},$$

which must be at most zero for any ε close enough to zero. Since $\underline{Q} = 1$ by Lemma 1, summing over the i's and letting ε go to zero then yields

$$v(\underline{\theta})(\overline{Q}-1) - \overline{T} + \underline{T} + n[v(\underline{\theta}) - p](1-\overline{Q}) \leq 0,$$

which, from the definition of p and the fact that $\overline{Q} < 1$, implies that

$$(n-1)[v(\underline{\theta}) - p] \leq 0.$$

Since $n \geq 2$, it follows that $p \geq v(\underline{\theta})$, as claimed. ■

Proof of Lemma 3. Suppose that a separating equilibrium exists. Any buyer i can deviate by offering a menu consisting of the no-trade contract and of the contract

$$\tilde{c}^i(\varepsilon) = (\overline{q}^i + 1 - \overline{Q}, \overline{t}^i + (\overline{\theta}+\varepsilon)(1-\overline{Q})),$$

for some strictly positive number ε, that is designed to attract both types of the seller. The key feature of this deviation is that both types can sell their whole endowment by trading $\tilde{c}^i(\varepsilon)$ together with the contracts $\overline{c}^j$, $j \neq i$. Since the unit price at which buyer i offers to purchase the quantity increment $1-\overline{Q}$ in $\tilde{c}^i(\varepsilon)$ is $\overline{\theta}+\varepsilon$, and since $\overline{\theta} \geq p$, this guarantees both types of the seller a payoff increase $(1-\overline{Q})\varepsilon$ compared to what they obtain in equilibrium. Hence both types trade $\tilde{c}^i(\varepsilon)$ following buyer i's deviation. The change in buyer i's payoff induced by this deviation is

$$\{\mathbf{E}[v(\theta)] - \overline{\theta} - \varepsilon\}(1-\overline{Q}) + (1-\nu)[v(\underline{\theta})(\overline{q}^i - \underline{q}^i) - \overline{t}^i + \underline{t}^i],$$

which must be at most zero for any ε. Since $\underline{Q} = 1$ by Lemma 1, summing over the i's and letting ε go to zero then yields

$$n\{\mathbf{E}[v(\theta)] - \overline{\theta}\}(1 - \overline{Q}) + (1 - \nu)[v(\underline{\theta})(\overline{Q} - 1) - \overline{T} + \underline{T}] \leq 0,$$

which, from the definition of p and the fact that $\overline{Q} < 1$, implies that

$$n\{\mathbf{E}[v(\theta)] - \overline{\theta}\} + (1 - \nu)[p - v(\underline{\theta})] \leq 0.$$

Starting from this inequality, two cases must be distinguished. If $p < \overline{\theta}$, then Lemma 2 applies, and therefore $p \geq v(\underline{\theta})$. It then follows that $\mathbf{E}[v(\theta)] \leq \overline{\theta}$. If $p = \overline{\theta}$, the inequality can be rearranged so as to yield

$$(n - 1)\{\mathbf{E}[v(\theta)] - \overline{\theta}\} + \nu[v(\overline{\theta}) - \overline{\theta}] \leq 0.$$

Since $n \geq 2$ and $v(\overline{\theta}) > \overline{\theta}$, it follows again that $\mathbf{E}[v(\theta)] \leq \overline{\theta}$, which shows the first part of the result. Consider next some pooling equilibrium, and denote by $(1, T)$ the corresponding aggregate equilibrium allocation. To show that $T = \mathbf{E}[v(\theta)]$, one needs to establish that the buyers' aggregate payoff is zero in equilibrium. Let B^i be buyer i's equilibrium payoff, which must be at least zero since each buyer always has the option not to trade. Buyer i can deviate by offering a menu consisting of the no-trade contract and of the contract

$$\hat{c}^i(\varepsilon) = (1, T + \varepsilon),$$

for some strictly positive number ε. It is immediate that both types trade $\hat{c}^i(\varepsilon)$ following buyer i's deviation. The change in payoff for buyer i induced by this deviation is

$$\mathbf{E}[v(\theta)] - T - \varepsilon - B^i,$$

which must be at most zero for any ε. Letting ε go to zero yields

$$B^i \geq \mathbf{E}[v(\theta)] - T = \sum_j B^j$$

where the equality follows from the fact that each type of the seller sells her whole endowment in a pooling equilibrium. Since this inequality holds for each i and all the B^i's are at least zero, they must all in fact be equal to zero. Hence $T = \mathbf{E}[v(\theta)]$, as claimed. ■

Proof of Lemma 4. Suppose first that a pooling equilibrium exists, and denote by $(1, T)$ the aggregate allocation traded by both types in this equilibrium. Then the buyers' aggregate payoff is $\mathbf{E}[v(\theta)] - T$. One must have $T - \overline{\theta} \geq 0$ otherwise type $\overline{\theta}$ would not trade. Since the buyers' aggregate payoff must be at least zero in equilibrium, it follows that $\mathbf{E}[v(\theta)] \geq \overline{\theta}$, which shows the first part of the result. Next, observe that in any separating equilibrium, the buyers' aggregate payoff is equal to

$$(1-\nu)[v(\underline{\theta}) - \underline{T}] + \nu[v(\overline{\theta})\overline{Q} - \overline{T}] = (1-\nu)[v(\underline{\theta}) - p(1-\overline{Q})] + \nu v(\overline{\theta})\overline{Q} - \overline{T}$$

by definition of p. One shows that $p \geq v(\underline{\theta})$ in any such equilibrium. If $p < \overline{\theta}$, this follows from Lemma 2. If $p = \overline{\theta}$, this follows from Lemma 3, which implies that $\overline{\theta} \geq \mathbf{E}[v(\theta)] > v(\underline{\theta})$ whenever a separating equilibrium exists. Using this claim along with the fact that $\overline{T} \geq \overline{\theta Q}$, one obtains that the buyers' aggregate payoff is at most $\{\mathbf{E}[v(\theta)] - \overline{\theta}\}\overline{Q}$. Since this must be at least zero, one necessarily has $(\overline{Q}, \overline{T}) = (0, 0)$ whenever $\mathbf{E}[v(\theta)] < \overline{\theta}$. In particular, the buyers' aggregate payoff $(1-\nu)[v(\underline{\theta}) - p]$ is then equal to zero. It follows that $p = v(\underline{\theta})$ and thus $\underline{T} = v(\underline{\theta})$, which shows the second part of the result. ■

Proof of Corollary 1. In the case of a pooling equilibrium, the result has been established in the proof of Lemma 3. In the case of a separating equilibrium, it has been shown in the proof of Lemma 4 that the buyers' aggregate payoff is at most $\{\mathbf{E}[v(\theta)] - \overline{\theta}\}\overline{Q}$. As a separating equilibrium exists only if $\mathbf{E}[v(\theta)] \leq \overline{\theta}$, it follows that the buyers' aggregate payoff is at most zero in any such equilibrium. Since each buyer always has the option not to trade, the result follows. ■

Proof of Proposition 2. Assume first that $\mathbf{E}[v(\theta)] \geq \overline{\theta}$, so that $p^* = \mathbf{E}[v(\theta)]$. The proof goes through a series of steps.

Step 1 Given the menus offered, any best response of the seller leads to an aggregate trade $(1, \mathbf{E}[v(\theta)])$ irrespective of her type. Assuming that each buyer trades the same quantity with both types of the seller, all buyers obtain a zero payoff.

Step 2 No buyer can profitably deviate in such a way that both types of the seller trade the same contract (q, t) with him. Indeed, such a deviation is profitable only if $\mathbf{E}[v(\theta)]q > t$. However, given the menus offered by the other buyers, the seller always has the option to trade quantity q at unit price $\mathbf{E}[v(\theta)]$. She would therefore be strictly worse off trading the contract (q, t) no matter her type. Such a deviation is thus infeasible.

Step 3 No buyer can deviate in such a way that he obtains a strictly positive payoff from trading with type $\underline{\theta}$. Indeed, an additional contract $(\underline{q}, \underline{t})$ attracts type $\underline{\theta}$ only if $\underline{t} \geq \mathbf{E}[v(\theta)]\underline{q}$, since she has the option to trade any quantity at unit price $\mathbf{E}[v(\theta)]$. The corresponding payoff for the deviating buyer is then at most $\{v(\underline{\theta}) - \mathbf{E}[v(\theta)]\}\underline{q}$ which is at most zero.

Step 4 From Step 3, a profitable deviation must attract type $\overline{\theta}$. An additional contract $(\overline{q}, \overline{t})$ attracts type $\overline{\theta}$ only if $\overline{t} \geq \mathbf{E}[v(\theta)]\overline{q}$, since she has the option to trade any quantity at unit price $\mathbf{E}[v(\theta)]$. However, type $\underline{\theta}$ can then also weakly increase her payoff by mimicking type $\overline{\theta}$'s behavior. One can therefore construct the seller's strategy in such a way that it is impossible for any buyer to deviate by trading with type $\overline{\theta}$ only.

Step 5 From Steps 3 and 4, a profitable deviation must involve trading with both types. Whatever the contract traded by the seller with the deviating buyer, and no matter her type, she can sell to the other buyers the remaining fraction of her endowment at unit price $\mathbf{E}[v(\theta)]$. Hence each type of the seller faces the same problem, namely to use optimally the deviating buyer's and the other buyers' offers to sell her whole endowment at the maximum price. One can therefore construct the seller's strategy in such a way that each type selects the same contract from the deviating buyer's menu. By Step 2, this makes such a deviation non profitable. Hence the result.

Assume next that $\mathbf{E}[v(\theta)] < \overline{\theta}$, so that $p^* = v(\underline{\theta})$. Again, the proof goes through a series of steps.

Step 1 Given the menus offered, any best response of the seller leads to aggregate trades $(1, v(\underline{\theta}))$ for type $\underline{\theta}$ and $(0, 0)$ for type $\overline{\theta}$, and all buyers obtain a zero payoff.

Step 2 No buyer can profitably deviate in such a way that both types of the seller trade the same contract (q, t) with him. Indeed, such a deviation is profitable only if $\mathbf{E}[v(\theta)]q > t$. Since $\overline{\theta} > \mathbf{E}[v(\theta)]$, this however implies that $t - \overline{\theta}q < 0$, so that type $\overline{\theta}$ would be strictly worse off trading the contract (q, t). Such a deviation is thus infeasible.

Step 3 No buyer can deviate in such a way that he obtains a strictly positive payoff from trading with type $\underline{\theta}$. Indeed, an additional contract $(\underline{q}, \underline{t})$ attracts type $\underline{\theta}$ only if $\underline{t} \geq v(\underline{\theta})\underline{q}$, since she always has the option to trade quantity $\underline{q}$ at unit price $v(\underline{\theta})$. The corresponding payoff for the deviating buyer is then at most zero.

Step 4 From Step 3, a profitable deviation must attract type $\overline{\theta}$. An additional contract $(\overline{q}, \overline{t})$ attracts type $\overline{\theta}$ only if $\overline{t} \geq \overline{\theta}\overline{q}$. However, since $\overline{\theta} > \mathbf{E}[v(\theta)] > v(\underline{\theta})$, type $\underline{\theta}$ can then strictly increase her payoff by trading the contract $(\overline{q}, \overline{t})$ and selling to the other buyers the remaining fraction of her endowment at unit price $v(\underline{\theta})$. It is thus impossible for any buyer to deviate by trading with type $\overline{\theta}$ only.

Step 5 From Steps 3 and 4, a profitable deviation must involve trading with both types.

Given the menus offered, the most profitable deviations involve trading some quantity $\overline{q}$ at unit price $\overline{\theta}$ with type $\overline{\theta}$, and trading a quantity 1 at unit price $\overline{\theta}\overline{q}+v(\underline{\theta})(1-\overline{q})$ with type $\underline{\theta}$. By construction, type $\underline{\theta}$ is indifferent between trading the contract $(1, \overline{\theta}\overline{q}+v(\underline{\theta})(1-\overline{q}))$ and trading the contract $(\overline{q}, \overline{\theta}\overline{q})$ while selling to the other buyers the remaining fraction of her endowment at unit price $v(\underline{\theta})$. As for type $\overline{\theta}$, she is indifferent between trading the contract $(\overline{q}, \overline{\theta}\overline{q})$ and not trading at all. The corresponding payoff for the deviating buyer is then

$$\nu[v(\overline{\theta})-\overline{\theta}]\overline{q}+(1-\nu)[v(\underline{\theta})-\overline{\theta}\overline{q}-v(\underline{\theta})(1-\overline{q})]=\{\mathbf{E}[v(\theta)]-\overline{\theta}\}\overline{q},$$

which is at most zero when $\mathbf{E}[v(\theta)]<\overline{\theta}$. Hence the result. ■

Proof of Proposition 3. Assume first that $\mathbf{E}[v(\theta)]>\overline{\theta}$, so that $p^*=\mathbf{E}[v(\theta)]$. Suppose an equilibrium exists in which some buyer i offers a contract $c^i=(q^i,t^i)$ at unit price $\frac{t^i}{q^i}>\mathbf{E}[v(\theta)]$. Notice that one must have $\mathbf{E}[v(\theta)]-t^i\geq\overline{\theta}(1-q^i)$ otherwise c^i would give type $\overline{\theta}$ more than her equilibrium payoff. Similarly, one must have $q^i<1$ otherwise c^i would give both types more than their equilibrium payoff. Any other buyer j could offer a menu consisting of the no-trade contract and of the contract

$$c^j(\varepsilon)=(1-q^i,\mathbf{E}[v(\theta)]-t^i+\varepsilon),$$

with $0<\varepsilon<t^i-q^i\mathbf{E}[v(\theta)]$. If both c^i and $c^j(\varepsilon)$ were available, both types of the seller would sell their whole endowment at price $\mathbf{E}[v(\theta)]+\varepsilon$ by trading c^i with buyer i and $c^j(\varepsilon)$ with buyer j, thereby increasing their payoff by ε compared to what they obtain in equilibrium. Buyer j's equilibrium payoff is thus at least

$$\mathbf{E}[v(\theta)](1-q^i)-\{\mathbf{E}[v(\theta)]-t^i+\varepsilon\}=t^i-q^i\mathbf{E}[v(\theta)]-\varepsilon>0,$$

which is impossible since each buyer's payoff is zero in any equilibrium by Corollary 1. Hence no contract can be issued at a price strictly above $\mathbf{E}[v(\theta)]$. The result follows.

Assume next that $\mathbf{E}[v(\theta)]<\overline{\theta}$, so that $p^*=v(\underline{\theta})$. Suppose an equilibrium exists in which some buyer i offers a contract $c^i=(q^i,t^i)$ at unit price $\frac{t^i}{q^i}>v(\underline{\theta})$. Notice that one must have $t^i\leq\overline{\theta}q^i$ otherwise c^i would give type $\overline{\theta}$ more than her equilibrium payoff. Similarly, one must have $v(\underline{\theta})-t^i\geq\underline{\theta}(1-q^i)$ and $q^i<1$ otherwise c^i would give type $\underline{\theta}$ more than her equilibrium payoff. Any other buyer j could offer a menu consisting of the no-trade contract and of the contract

$$c^j(\varepsilon)=(1-q^i,v(\underline{\theta})-t^i+\varepsilon),$$

where $0<\varepsilon<\min\{t^i-q^iv(\underline{\theta}),\overline{\theta}-v(\underline{\theta})\}$. If both c^i and $c^j(\varepsilon)$ were available, type $\underline{\theta}$ would sell her whole endowment at price $v(\underline{\theta})+\varepsilon$ by trading c^i with buyer i and $c^j(\varepsilon)$ with buyer j, thereby increasing her payoff by ε compared to what she obtains in equilibrium. Moreover, since $v(\underline{\theta}) + \varepsilon < \overline{\theta}$, type $\overline{\theta}$ would strictly lose from trading $c^j(\varepsilon)$ with buyer j. Buyer j's equilibrium payoff is thus at least

$$(1-\nu)\{v(\underline{\theta})(1-q^i) - [v(\underline{\theta}) - t^i + \varepsilon]\} = (1-\nu)[t^i - q^i v(\underline{\theta}) - \varepsilon] > 0,$$

which is impossible since each buyer's payoff is zero in any equilibrium by Corollary 1. Hence no contract can be issued at a price strictly above $v(\underline{\theta})$. The result follows. ■

Proof of Corollary 2. Assume first that $\mathbf{E}[v(\theta)] > \overline{\theta}$, so that $p^* = \mathbf{E}[v(\theta)]$. From Proposition 3, no contract is issued, and a fortiori traded, at a unit price strictly above $\mathbf{E}[v(\theta)]$. Suppose now that a contract with unit price strictly below $\mathbf{E}[v(\theta)]$ is traded in equilibrium. Then, since the aggregate allocation traded by both types is $(1, \mathbf{E}[v(\theta)])$, at least one buyer must be trading a contract at a unit price strictly above $\mathbf{E}[v(\theta)]$, a contradiction. Hence the result.

Assume next that $\mathbf{E}[v(\theta)] < \overline{\theta}$, so that $p^* = v(\underline{\theta})$. From Proposition 3, no contract is issued, and a fortiori traded, at a unit price strictly above $v(\underline{\theta})$. Suppose now that a contract with unit price strictly below $v(\underline{\theta})$ is traded in equilibrium. Then, since the aggregate allocation traded by type $\underline{\theta}$ is $(1, v(\underline{\theta}))$, at least one buyer must be trading a contract at a unit price strictly above $v(\underline{\theta})$, a contradiction. Hence the result. ■

Proof of Proposition 4. Fix some equilibrium with menu offers $(C^1, \ldots, C^n)$, and let

$$\mathfrak{A}^{-i} = \left\{ \sum_{j \neq i} (q^j, t^j) : \sum_{j \neq i} q^j \leq 1 \text{ and } (q^j, t^j) \in C^j \text{ for all } j \neq i \right\}$$

be the set of aggregate allocations that remain available if buyer i withdraws his menu offer C^i. By construction, $\mathfrak{A}^{-i}$ is a compact set. One must show that $(1, p^*) \in \mathfrak{A}^{-i}$.

Assume first that $\mathbf{E}[v(\theta)] > \overline{\theta}$, so that $p^* = \mathbf{E}[v(\theta)]$. Suppose the aggregate allocation $(1, \mathbf{E}[v(\theta)])$ traded by both types does not belong to $\mathfrak{A}^{-i}$. Since $\mathfrak{A}^{-i}$ is compact, there exists some open set of $[0,1] \times \mathbb{R}_+$ that contains $(1, \mathbf{E}[v(\theta)])$ and that does not intersect $\mathfrak{A}^{-i}$. Moreover, any allocation $(Q^{-i}, T^{-i}) \in \mathfrak{A}^{-i}$ is such that $T^{-i} \leq \mathbf{E}[v(\theta)]Q^{-i}$ by Proposition 3. Since $\overline{\theta} < \mathbf{E}[v(\theta)]$ under mild adverse selection, this implies that $\mathfrak{A}^{-i}$ does not intersect the set of allocations that are weakly preferred by both types to $(1, \mathbf{E}[v(\theta)])$. By continuity of the seller's preferences, it follows that there exists some strictly positive number ε such that the contract $(1, \mathbf{E}[v(\theta)] - \varepsilon)$ is strictly preferred by each type to any allocation in $\mathfrak{A}^{-i}$.

Thus, if this contract were available, both types would trade it. This implies that buyer i's equilibrium payoff is at least ε, which is impossible since each buyer's payoff is zero in any equilibrium by Corollary 1. Hence $(1, \mathbf{E}[v(\theta)]) \in \mathfrak{A}^{-i}$. The result follows.

Assume next that $\mathbf{E}[v(\theta)] < \overline{\theta}$, so that $p^* = v(\underline{\theta})$. Suppose the aggregate allocation $(1, v(\underline{\theta}))$ traded by type $\underline{\theta}$ does not belong to $\mathfrak{A}^{-i}$. Since $\mathfrak{A}^{-i}$ is compact, there exists an open set of $[0,1] \times \mathbb{R}_+$ that contains $(1, v(\underline{\theta}))$ and that does not intersect $\mathfrak{A}^{-i}$. Moreover, any allocation $(Q^{-i}, T^{-i}) \in \mathfrak{A}^{-i}$ is such that $T^{-i} \leq v(\underline{\theta})Q^{-i}$ by Proposition 3. Since $\underline{\theta} < v(\underline{\theta})$, this implies that $\mathfrak{A}^{-i}$ does not intersect the set of allocations that are weakly preferred by type $\underline{\theta}$ to $(1, v(\underline{\theta}))$. Since the latter set is closed and $\mathfrak{A}^{-i}$ is compact, it follows that there exists a contract $(\overline{q}^i, \overline{t}^i)$ with unit price $\frac{\overline{t}^i}{\overline{q}^i} \in (\overline{\theta}, v(\overline{\theta}))$ such that the allocation $(1, v(\underline{\theta}))$ is strictly preferred by type $\underline{\theta}$ to any allocation obtained by trading the contract $(\overline{q}^i, \overline{t}^i)$ together with some allocation in $\mathfrak{A}^{-i}$.[16] Moreover, since $\frac{\overline{t}^i}{\overline{q}^i} > \overline{\theta}$, the contract $(\overline{q}^i, \overline{t}^i)$ guarantees a strictly positive payoff to type $\overline{\theta}$. Thus, if both $(1, v(\underline{\theta}))$ and $(\overline{q}^i, \overline{t}^i)$ were available, type $\underline{\theta}$ would trade $(1, \underline{\theta})$ and type $\overline{\theta}$ would trade $(\overline{q}^i, \overline{t}^i)$. This implies that buyer i's equilibrium payoff is at least $\nu[v(\overline{\theta})\overline{q}^i - \overline{t}^i] > 0$, which is impossible since each buyer's payoff is zero in any equilibrium by Corollary 1. Hence $(1, v(\underline{\theta})) \in \mathfrak{A}^{-i}$. The result follows. ■

Proof of Proposition 5. Fix some equilibrium and some buyer i, and define the set $\mathfrak{A}^{-i}$ as in the proof of Proposition 4. One must show that $\mathfrak{A}^{-i}$ is infinite. Define

$$z^{-i}(\theta, Q) = \max\{T^{-i} - \theta Q^{-i} : (Q^{-i}, T^{-i}) \in \mathfrak{A}^{-i} \text{ and } Q^{-i} \leq Q\}$$

to be the highest payoff that a seller of type θ can get from trading with buyers $j \neq i$, when her remaining stock is Q. Notice that $z^{-i}(\theta, Q)$ is positive and increasing in Q. Observe that

$$T^{-i} - \overline{\theta}Q^{-i} = T^{-i} - \underline{\theta}Q^{-i} + (\underline{\theta} - \overline{\theta})Q^{-i} \geq T^{-i} - \underline{\theta}Q^{-i} + (\underline{\theta} - \overline{\theta})Q$$

as long as $Q^{-i} \leq Q$. Taking maximums on both sides of this inequality yields

$$z^{-i}(\overline{\theta}, Q) \geq z^{-i}(\underline{\theta}, Q) + (\underline{\theta} - \overline{\theta})Q \tag{2}$$

for all $Q \in [0,1]$. Now, let $U(\theta)$ be the equilibrium payoff of type θ. It follows from Proposition 4 that this payoff remains available to type θ if buyer i withdraws his menu offer. Suppose that buyer i deviates by offering a menu consisting of the no-trade contract and of a contract $(\overline{q}, \overline{t})$ that is designed to attract only type $\overline{\theta}$. To ensure that this is so, one

[16]This follows directly from the fact that if K is compact and F is closed in some normed vector space X, and if $K \cap F = \emptyset$, then for any vector u in X, $(K + \lambda u) \cap F = \emptyset$ for any sufficiently small scalar λ.

imposes the following incentive compatibility constraints:

$$U(\underline{\theta}) > \overline{t} - \underline{\theta}\overline{q} + z^{-i}(\underline{\theta}, 1 - \overline{q}),$$

$$\overline{t} - \overline{\theta}\overline{q} + z^{-i}(\overline{\theta}, 1 - \overline{q}) > U(\overline{\theta}).$$

Clearly these constraints together require that

$$\underline{\theta}\overline{q} - z^{-i}(\underline{\theta}, 1 - \overline{q}) + U(\underline{\theta}) > \overline{\theta}\overline{q} - z^{-i}(\overline{\theta}, 1 - \overline{q}) + U(\overline{\theta}). \tag{3}$$

The resulting profit is then $v(\overline{\theta})\overline{q} - \overline{t}$, which must be at most zero by Corollary 1. Since $\overline{t}$ can be as close as one wishes to $\overline{\theta}\overline{q} - z^{-i}(\overline{\theta}, 1 - \overline{q}) + U(\overline{\theta})$, one thus obtains the following implication: if $\overline{q}$ satisfies (3), then

$$[v(\overline{\theta}) - \overline{\theta}]\overline{q} \leq U(\overline{\theta}) - z^{-i}(\overline{\theta}, 1 - \overline{q}). \tag{4}$$

Two cases must now be distinguished.

Assume first that $\mathbf{E}[v(\theta)] > \overline{\theta}$, so that $U(\underline{\theta}) = \mathbf{E}[v(\theta)] - \underline{\theta}$ and $U(\overline{\theta}) = \mathbf{E}[v(\theta)] - \overline{\theta}$ by Lemma 3. Then (4) is false if and only if

$$z^{-i}(\overline{\theta}, 1 - \overline{q}) > \mathbf{E}[v(\theta)] - \overline{\theta} - [v(\overline{\theta}) - \overline{\theta}]\overline{q}. \tag{5}$$

Define $\overline{q}^* = \frac{\mathbf{E}[v(\theta)] - \overline{\theta}}{v(\overline{\theta}) - \overline{\theta}}$, and observe that $0 < \overline{q}^* < 1$. For $\overline{q} > \overline{q}^*$, the right-hand side of (5) is negative, and thus (5) holds. Hence (4) is false, and therefore (3) is false as well:

$$z^{-i}(\overline{\theta}, 1 - \overline{q}) \leq z^{-i}(\underline{\theta}, 1 - \overline{q}) + (\underline{\theta} - \overline{\theta})(1 - \overline{q}).$$

Letting $Q = 1 - \overline{q}$ and combining this inequality with (2), we obtain that

$$z^{-i}(\overline{\theta}, Q) = z^{-i}(\underline{\theta}, Q) + (\underline{\theta} - \overline{\theta})Q \tag{6}$$

for all $Q < 1 - \overline{q}^*$. One now shows that (6) implies that for any such Q, and for any solution $(Q^{-i}(\underline{\theta}, Q), T^{-i}(\underline{\theta}, Q))$ to the maximization problem that defines $z^{-i}(\underline{\theta}, Q)$, one has $Q^{-i}(\underline{\theta}, Q) = Q$. To see this, observe that the trade $(Q^{-i}(\underline{\theta}, Q), T^{-i}(\underline{\theta}, Q))$ is also feasible for type $\overline{\theta}$ in the maximization problem that defines $z^{-i}(\overline{\theta}, Q)$. Thus one must have

$$z^{-i}(\overline{\theta}, Q) \geq T^{-i}(\underline{\theta}, Q) - \overline{\theta}Q^{-i}(\underline{\theta}, Q) = z^{-i}(\overline{\theta}, Q) + (\underline{\theta} - \overline{\theta})Q^{-i}(\underline{\theta}, Q). \tag{7}$$

The inequality in (7) cannot be strict, for otherwise $z^{-i}(\overline{\theta}, Q) > z^{-i}(\overline{\theta}, Q) + (\underline{\theta} - \overline{\theta})Q$ as $Q^{-i}(\underline{\theta}, Q) \leq Q$, which would contradict (6). It follows that (7) holds as an equality, which implies that $Q^{-i}(\underline{\theta}, Q) = Q$ by (6). Since this equality is true for all $Q \in [0, 1 - \overline{q}^*)$, it follows from the definition of $z^{-i}(\underline{\theta}, \cdot)$ that there exists a continuum of distinct points in $\mathfrak{A}^{-i}$. Hence the result.

Assume next that $\mathbf{E}[v(\theta)] < \overline{\theta}$, so that $U(\underline{\theta}) = v(\underline{\theta}) - \underline{\theta}$, $U(\overline{\theta}) = 0$ and $z^{-i}(\overline{\theta}, \cdot) = 0$ by Lemma 4. Then the right-hand side of (4) is zero, while the left-hand side is strictly positive as long as $\overline{q}$ is strictly positive. Therefore (3) cannot hold for any such $\overline{q}$, which implies that

$$v(\underline{\theta}) - \underline{\theta} - (\overline{\theta} - \underline{\theta})\overline{q} \leq z^{-i}(\underline{\theta}, 1 - \overline{q})$$

for all $\overline{q} \in (0, 1]$. Moreover, by Proposition 3, no contract can be issued at a price strictly above $p^* = v(\underline{\theta})$. Thus

$$z^{-i}(\underline{\theta}, 1 - \overline{q}) \leq [v(\underline{\theta}) - \underline{\theta}](1 - \overline{q})$$

for all $\overline{q} \in (0, 1]$. Letting $Q = 1 - \overline{q}$ and combining these two inequalities, one obtains the following lower and upper bounds for $z^{-i}(\underline{\theta}, Q)$:

$$v(\underline{\theta}) - \overline{\theta} + (\overline{\theta} - \underline{\theta})Q \leq z^{-i}(\underline{\theta}, Q) \leq [v(\underline{\theta}) - \underline{\theta}]Q$$

for all $Q \in [0, 1)$. Since these bounds are strictly increasing in Q and coincide at $Q = 1$, it follows from the definition of $z^{-i}(\underline{\theta}, \cdot)$ that there exists a sequence in $\mathfrak{A}^{-i}$ composed of distinct points that converges to $(1, v(\underline{\theta}))$. Hence the result. ■

Proof of Proposition 6. (i) The proof goes through a series of steps.

Step 1 Given the menus offered, any best response of the seller leads to an aggregate trade $(1, \mathbf{E}[v(\theta)])$ irrespective of her type. Since $\phi < \mathbf{E}[v(\theta)]$, it is optimal for each type of the seller to trade her whole endowment with a single buyer. Assuming that each type of the seller trades with the same buyer, all buyers obtain a zero payoff. Note also that if any buyer withdraws his menu offer, the most the seller can achieve by trading with the other buyers consists in trading with a single buyer.

Step 2 No buyer can profitably deviate in such a way that both types of the seller trade the same contract (q, t) with him. Indeed, such a deviation is profitable only if $\mathbf{E}[v(\theta)]q > t$. Since $\phi < \mathbf{E}[v(\theta)]$, the highest payoff the seller can achieve by purchasing the contract (q, t) together with some contract in the menu offered by the other buyers is less than the payoff from trading the contract $(1, \mathbf{E}[v(\theta)])$, which remains available at the deviation stage. She would therefore be strictly worse off trading the contract (q, t) no matter her type. Such a deviation is thus infeasible.

Step 3 No buyer can deviate in such a way that he obtains a strictly positive payoff from trading with type $\underline{\theta}$. Indeed, trading an additional contract $(\underline{q}, \underline{t})$ with type $\underline{\theta}$ is profitable only if $v(\underline{\theta})\underline{q} > \underline{t}$. The same argument as in Step 2 then shows that type $\underline{\theta}$ would be strictly worse off trading the contract $(\underline{q}, \underline{t})$ rather than the contract $(1, \mathbf{E}[v(\theta)])$, which remains available at the deviation stage. Such a deviation is thus infeasible.

Step 4 From Step 3, a profitable deviation must attract type $\overline{\theta}$. An additional contract $(\overline{q}, \overline{t})$ that is profitable when traded with type $\overline{\theta}$ attracts her only if $\overline{t} + \phi(1 - \overline{q}) \geq \mathbf{E}[v(\theta)]$, that is, only if she can weakly increase her payoff by trading the contract $(\overline{q}, \overline{t})$ and selling to the other buyers the remaining fraction of her endowment at unit price ϕ. That this is feasible follows from the fact that, when $\overline{t} + \phi(1 - \overline{q}) \geq \mathbf{E}[v(\theta)]$ and $v(\overline{\theta})\overline{q} > \overline{t}$, the quantity $1 - \overline{q}$ is less than the maximal quantity $\frac{v(\overline{\theta}) - \mathbf{E}[v(\theta)]}{v(\overline{\theta}) - \phi}$ that can be traded at unit price ϕ with the other buyers. Moreover, the fact that $\phi \geq \overline{\theta}$ guarantees that it is indeed optimal for type $\overline{\theta}$ to behave in this way at the deviation stage. However, type $\underline{\theta}$ can then also weakly increase her payoff by mimicking type $\overline{\theta}$'s behavior. One can therefore construct the seller's strategy in such a way that it is impossible for any buyer to deviate by trading with type $\overline{\theta}$ only.

Step 5 From Steps 3 and 4, a profitable deviation must involve trading with both types. Whatever the contract traded by the seller with the deviating buyer, and no matter her type, she will sell to the other buyers the remaining fraction of her endowment at unit price ϕ. Hence, each type of the seller faces the same problem, namely to use optimally the deviating buyer's and the other buyers' offers to sell her whole endowment at the maximum price. One can therefore construct the seller's strategy in such a way that each type selects the same contract from the deviating buyer's menu. By Step 2, this makes such a deviation non profitable. The result follows.

(ii) The proof goes through a series of steps.

Step 1 Given the menus offered, any best response of the seller leads to an aggregate trade $(1, v(\underline{\theta}))$ for type $\underline{\theta}$ and $(0, 0)$ for type $\overline{\theta}$. Since each buyer is not ready to pay anything for quantities up to $\frac{\psi - \theta}{\psi}$ and offers to purchase each additional unit at a constant marginal price ψ above this level, it is optimal for type $\underline{\theta}$ to trade her whole endowment with a single buyer, and all buyers obtain a zero payoff. Note also that if any buyer withdraws his menu offer, the most the seller can achieve by trading with the other buyers consists in trading with a single buyer.

Step 2 No buyer can profitably deviate in such a way that both types of the seller trade the same contract (q, t) with him. This can be shown as in Step 2 of the first part of the proof of Proposition 2.

Step 3 No buyer can deviate in such a way that he obtains a strictly positive payoff from trading with type $\underline{\theta}$. Indeed, trading an additional contract $(\underline{q}, \underline{t})$ with type $\underline{\theta}$ is profitable only if $v(\underline{\theta})\underline{q} > \underline{t}$. Since $\psi > v(\underline{\theta})$, the highest payoff type $\underline{\theta}$ can achieve by purchasing the contract (q, t) together with some contract in the menu offered by the other buyers is less than the payoff from trading the contract $(1, v(\underline{\theta}))$, which remains available at the deviation stage. She would therefore be strictly worse off trading the contract (q, t). Such a deviation is thus infeasible.

Step 4 From Step 3, a profitable deviation must attract type $\overline{\theta}$. An additional contract $(\overline{q}, \overline{t})$ attracts type $\overline{\theta}$ only if $\overline{t} \geq \overline{\theta}\overline{q}$. Two cases must be distinguished. If $\overline{q} \leq \frac{v(\underline{\theta})}{\psi}$, then type $\underline{\theta}$ can trade the contract $(\overline{q}, \overline{t})$ and sell to some other buyer the remaining fraction of her endowment at price $\psi(1 - \overline{q}) - \psi + v(\underline{\theta})$. The price at which she can sell her whole endowment is therefore at least $(\overline{\theta} - \psi)\overline{q} + v(\underline{\theta})$, which is strictly higher than the price $\underline{\theta}$ that she obtains in equilibrium since $\overline{\theta} > v(\underline{\theta}) + \frac{\overline{\theta} - \mathbf{E}[v(\theta)]}{1-\nu} \geq \psi$. If $\overline{q} > \frac{v(\underline{\theta})}{\psi}$, then by trading the contract $(\overline{q}, \overline{t})$, type $\underline{\theta}$ obtains at least a payoff $\frac{(\overline{\theta} - \underline{\theta})v(\underline{\theta})}{\psi}$, which, since $\overline{\theta} > \psi > v(\underline{\theta})$, is more than her equilibrium payoff $v(\underline{\theta}) - \underline{\theta}$. Thus type $\underline{\theta}$ can always strictly increase her payoff by trading the contract $(\overline{q}, \overline{t})$. It is therefore impossible for any buyer to deviate by trading with type $\overline{\theta}$ only.

Step 5. From Steps 3 and 4, a profitable deviation must involve trading with both types. Given the menus offered, the most profitable deviations lead to trading some quantity $\overline{q} \leq \frac{v(\underline{\theta})}{\psi}$ at unit price $\overline{\theta}$ with type $\overline{\theta}$, and trading a quantity 1 at unit price $\overline{\theta}\overline{q} + v(\underline{\theta}) - \psi\overline{q}$ with type $\underline{\theta}$. By construction, type $\underline{\theta}$ is indifferent between trading the contract $(1, \overline{\theta}\overline{q} + v(\underline{\theta}) - \psi\overline{q})$ and trading the contract $(\overline{q}, \overline{\theta}\overline{q})$ while selling to the other buyers the remaining fraction of her endowment at price $\psi(1 - \overline{q}) - \psi + v(\underline{\theta})$. As for type $\overline{\theta}$, she is indifferent between trading the contract $(\overline{q}, \overline{\theta}\overline{q})$ and not trading at all. The corresponding payoff for the deviating buyer is then

$$\nu[v(\overline{\theta}) - \overline{\theta}]\overline{q} + (1 - \nu)\{v(\underline{\theta}) - [\overline{\theta}\overline{q} + v(\underline{\theta}) - \psi\overline{q}]\} = [\nu v(\overline{\theta}) + (1 - \nu)\psi - \overline{\theta}]\overline{q},$$

which is at most zero since $\psi \leq v(\underline{\theta}) + \frac{\overline{\theta} - \mathbf{E}[v(\theta)]}{1-\nu}$. The result follows. ■

References

[1] Akerlof, G.A. (1970): "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 84(3), 488–500.

[2] Attar, A., and A. Chassagnon (2009): "On Moral Hazard and Nonexclusive Contracts," *Journal of Mathematical Economics*, forthcoming.

[3] Bernheim, B.D., and M.D. Whinston (1986): "Common Agency," *Econometrica*, 54(4), 923–942.

[4] Biais, B., D. Martimort, and J.-C. Rochet (2000): "Competing Mechanisms in a Common Value Environment," *Econometrica*, 68(4), 799–837.

[5] Bisin, A., and D. Guaitoli (2004): "Moral Hazard and Nonexclusive Contracts," *RAND Journal of Economics*, 35(2), 306–328.

[6] Cawley, J., and T.J. Philipson (1999): "An Empirical Examination of Information Barriers to Trade in Insurance," *American Economic Review*, 89(4), 827–846.

[7] Chiappori P.-A., B. Jullien, B. Salanié, and F. Salanié (2006): "Asymmetric Information in Insurance: General Testable Implications," *RAND Journal of Economics*, 37(4), 783–798.

[8] Chiappori, P.-A., and B. Salanié (2003): "Testing Contract Theory: A Survey of Some Recent Work," in M. Dewatripont, L.P. Hansen and S.J. Turnovsky (Eds.), *Advances in Economics and Econometrics*: *Theory and Applications*, *Eighth World Congress*, Volume 1, pp. 115–149, Cambridge, Cambridge University Press.

[9] Detragiache, E., P. Garella, and L. Guiso (2000): "Multiple versus Single Banking Relationships: Theory and Evidence," *Journal of Finance*, 55(3), 1133–1161.

[10] Finkelstein, A., and J.M. Poterba (2004): "Adverse Selection in Insurance Markets: Policyholder Evidence from the U.K. Annuity Market," *Journal of Political Economy*, 112(1), 183–208.

[11] Hellwig, M. (1983): "On Moral Hazard and Non-Price Equilibria in Competitive Insurance Markets," Discussion Paper No. 109, Sonderforschungsbereich 21, University of Bonn.

[12] Khalil, F.A., D. Martimort, and B. Parigi (2007): "Monitoring a Common Agent: Implications for Financial Contracting," *Journal of Economic Theory*, 135(1), 35–67.

[13] Martimort, D. (2007): "Multi-Contracting Mechanism Design," in R. Blundell, W.K. Newey and T. Persson (Eds.), *Advances in Economics and Econometrics*: *Theory and Applications*, *Ninth World Congress*, Volume 1, pp. 57–101, Cambridge, Cambridge University Press.

[14] Martimort, D., and L. Stole (2002): "The Revelation and Delegation Principles in Common Agency Games," *Econometrica*, 70(4), 1659–1673.

[15] Martimort, D., and L. Stole (2003): "Contractual Externalities and Common Agency Equilibria," *Advances in Theoretical Economics*, 3(1), Article 4.

[16] Martimort, D., and L. Stole (2009): "Market Participation under Delegated and Intrinsic Common-Agency Games," *RAND Journal of Economics*, 40(1), 78–102.

[17] Mas-Colell, A., M.D. Whinston, and J.R. Green (1995): *Microeconomic Theory*, New York, Oxford, Oxford University Press.

[18] Peck, J. (1997): "A Note on Competing Mechanisms and the Revelation Principle," Mimeo, Ohio State University.

[19] Peters, M. (2001): "Common Agency and the Revelation Principle," *Econometrica*, 69(5), 1349–1372.

[20] Piaser, G. (2007): "Direct Mechanisms, Menus and Latent Contracts," CREFI-LSF Working Paper No. 07-09, Université du Luxembourg.

[21] Rothschild, M., and J.E. Stiglitz (1976): "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information," *Quarterly Journal of Economics*, 90(4), 629–649.

[22] Rysman, M. (2007): "An Empirical Analysis of Payment Card Usage," *Journal of Industrial Economics*, 60(1), 1–36.

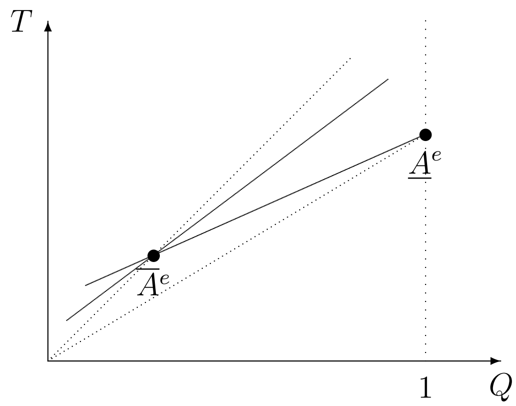

Figure 1 Equilibrium allocations under exclusive competition

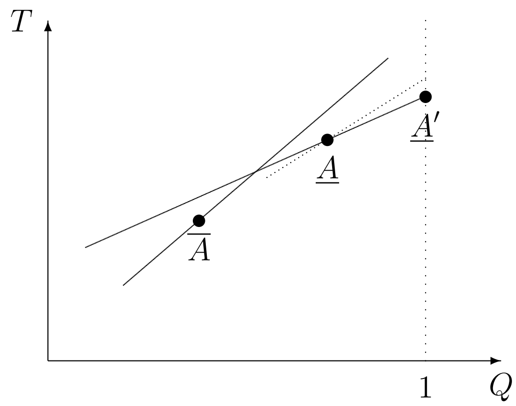

Figure 2 Attracting type $\underline{\theta}$ by pivoting around $(\underline{Q}, \underline{T})$

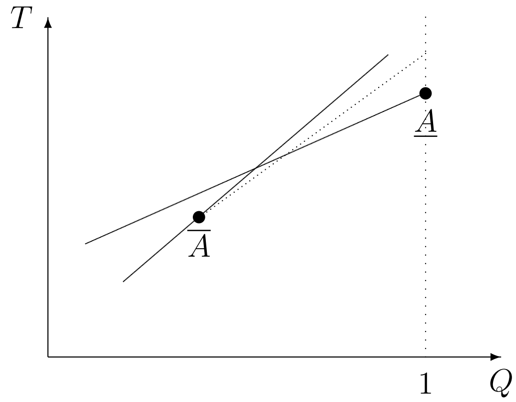

Figure 3 Attracting type $\underline{\theta}$ by pivoting around $(\overline{Q}, \overline{T})$

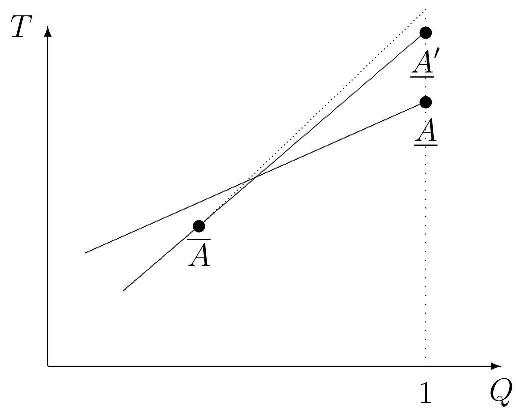

Figure 4 Attracting both types by pivoting around $(\overline{Q}, \overline{T})$

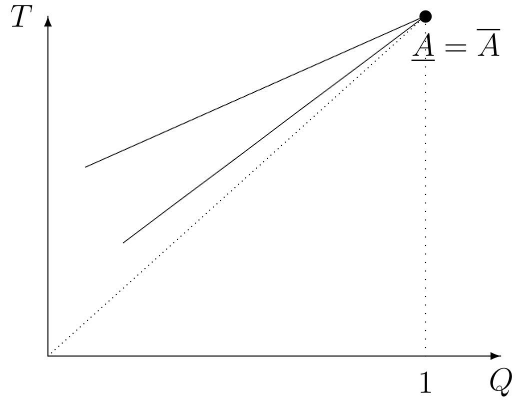

Figure 5 Aggregate equilibrium allocations when $\mathbf{E}[v(\theta)] > \overline{\theta}$

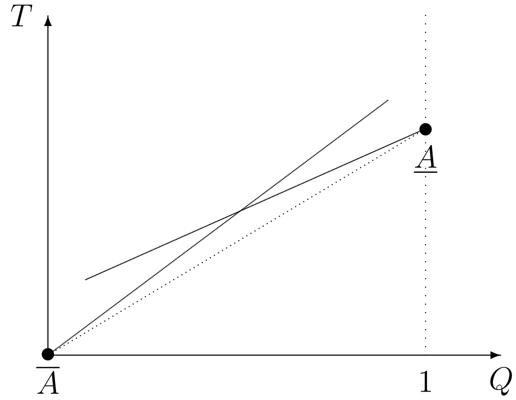

Figure 6 Aggregate equilibrium allocations when $\mathbf{E}[v(\theta)] < \overline{\theta}$