Task Juggling*

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Abstract

We study the work practices of a worker who is assigned a stream of project over time, so fast that he cannot keep up with all of them. This worker deals with overload by choosing how many projects to work on simultaneously. In this framework, we derive an exact functional form for the "production function" of output. This functional form associates an output rate to any combination of: the rate at which the worker opens projects, the difficulty of the projects, and the amount of effort exerted by the worker. We find that when the worker opens new projects at a rate that is too high, the output rate decreases while the number of active projects grows over time and the time it takes to complete each project also grows. We call this phenomenon "task juggling." We ask what forces might cause the worker to juggle tasks. One such force might be lobbying by "clients," each of whom seeks to get the worker to apply effort to his project ahead of the others'. We present a model in which this lobbying leads the worker to work on "too many" projects, i.e., to juggle tasks. Furthermore, we argue informally that task juggling would arise under a fairly general set of assumptions. We also investigate a different perspective, one in which task juggling is in fact optimal. This can happen when output is not the only social goal, but there is value created by having projects achieve certain intermediate stages of completion. We also compare two ways of incentivizing the worker when the worker can multitask across projects of different complexity: rewarding output leads the worker to focus effort on on low-complexity projects, whereas penalizing duration leads the worker to focus on relatively

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more complex projects. We conclude by analyzing certain extensions of the model including one where the worker is forgetful, and so his productivity per task decreases as a function of the time it takes him to complete two consecutive tasks of the same project. The presence of this friction gives rise to the possibility of multiple growth paths.

1 Introduction

Most of us seem to be working on too many projects at once. With too much on one's plate, one's effort gets dispersed and so only incremental work gets done on any one project. We refer to this work practice as *task juggling.* As a result of task juggling projects never get done, deadlines are pushed forward, etc.[1] In our view task juggling is ubiquitous: we (the authors) certainly fall prey to it, and we believe that many other types of workers do, too. In related empirical work (Coviello *et al.* 2010), we show that judges in Italy engage in task juggling, and we measure the effect of this practice on the duration of trials. This paper develops a theory of task juggling. The paper also investigates the forces that could lead a worker (or several workers) to juggle projects.

Let us define task juggling losely as dispersing effort across "too many" projects or, more precisely, as directing too much effort to projects which are at a less advanced stage of completion than other projects. Task juggling leads to excessive duration of projects. The intuition is very simple. Imagine a worker who is assigned a stream of project over time, so fast that he cannot keep up with all of them. The worker can choose to focus his effort on a smaller number of projects. We call these projects *active*, and assume that the worker pushes all active projects along at the same pace. How many projects are kept active is a measure of how much the worker "juggles." A worker may choose to keep more active projects, which means juggling more tasks; alternatively, the worker may keep fewer active projects and, instead, keep a bigger "to do" pile of unstarted projects. Consider now what happens when a new project is opened, that is, taken from the "to do" pile and made active: then, the effort devoted to it reduces the effort available for projects that are closer to completion (assume for the moment that effort is fixed). This externality (a) creates a slowdown in the more advanced projects, (b) without bringing forward the date of completion of the newly opened one. To see that part (b) is true, consider that the newly opened project cannot be completed before *all the tasks relating to the advanced projects as well as to the new*

[1] Of course, it can be efficient to work on a number of projects simultaneously, and our theory will accommodate this possibility. To be precise, therefore, we reserve the term "task juggling" for the case in which more than the efficient number of projects are worked on simultaneously, resulting in a slowdown of project completion.

project are done.[2] Therefore, as far as the date of completing the newly opened project is concerned, it is immaterial whether effort is applied to the old projects first or to the new project. This shows that task juggling (weakly) increases the duration of *all* projects, hence it is Pareto-inferior. This intuition also suggests that large "to do" piles are associated with fast production. This paper develops this intuition within an infinite-horizon dynamic model.

In our model a single worker is assigned projects at a constant rate α, opens them at rate $\nu_t \leq \alpha$, and splits effort η_t equally among all active (i.e., opened but not yet completed) projects. A large ν_t means that many projects are kept active at any given time, and thus the worker is juggling a lot of projects. A small ν_t means that only a few projects are active at any given time, and newly arrived projects are left unopened until the worker is ready to focus on them. Projects are completed at rate ω_t. This paper solves for the output rate ω_t as a function of the other variables, including the worker's effort and also, crucially, the amount of task juggling.

For an example of the kind of empirical settings that this theory is relevant for, consider Figure 1. This figure refers to the "production function" of Italian labor law judges.[3] These judges open (hold the first hearing on) about 130 cases per quarter and close slightly less than that. As a result of this slight discrepancy, the stock of active cases (cases which have received a first hearing but are not yet closed) grows at a steady pace, from about 100 in the year 2000 to more than 200 in 2005. This increase in the number of active cases means that these judges are juggling an increasing number of cases over time. Not surprisingly, therefore, the duration of cases (number of days elapsing between the date a case is assigned to a judge and the date it is completed) increases over time, consistent with the intuition presented above. Finally, and most interesting, the completion time (number of days elapsing between the date in which the first hearing is held and the date the case is completed) is also increasing. This increase in completion time means that it takes judges longer and longer to work through a given case, despite the fact that they are becoming more experienced and they work no less hard over time (see the panel on "standardized effort"). As we will show, our model predicts all of these regularities.

The paper proceeds as follows. In Section 2 we lay out a continuous-time model which describes *how much* the worker works and *how* he works—how many projects are kept active at any point in time. The outcome of interest in this model is the duration of projects: how long it takes between the time a project is assigned and the time it is completed. The duration of projects depends (inversely) on the rate at which projects get done, i.e., the

[2] This is because all active projects are pushed forward at the same pace, and so they necessarily get done in the order they were opened. Even if the "overtaking" of projects were allowed, task juggling would still lead to longer durations.

[3] These judges belong to the Milan labor court. For details on this data see Coviello et al. (2010).

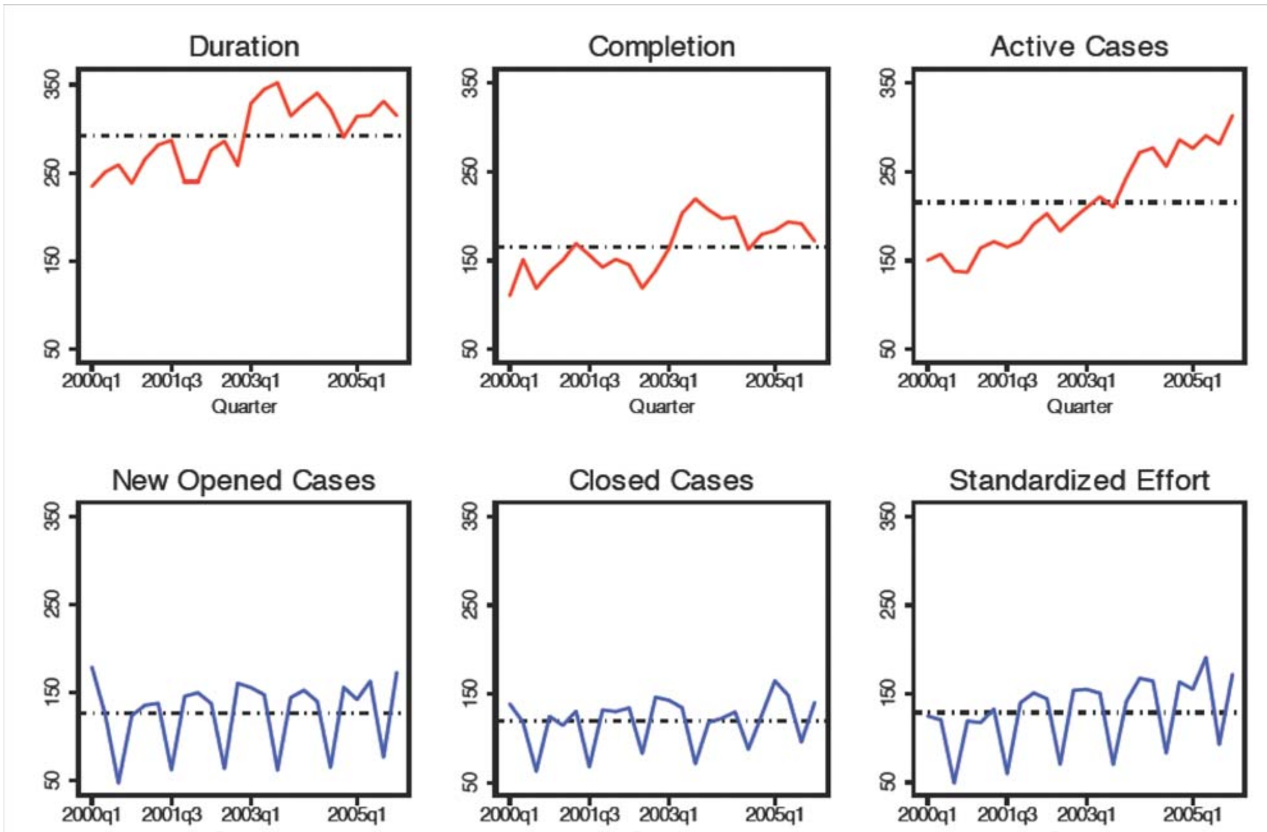

Figure 1: Statistics on the productivity of labor judges. See Coviello *et al.* (2010).

output rate ω_t. Solving for ω_t is the first challenge we are faced with. To appreciate this challenge, observe that the projects that get done in one instant are exactly those projects which, the instant before, were *almost* done. Thus the evolution of ω_t depends on the mass of projects that are almost completed at time t, which we denote by $\varphi_t(0)$, and on the speed at which these projects are moved forward. The latter, in turn, is a function of the worker's effort η_t and of the stock A_t of active projects at time t. Of course, both $\varphi_t(0)$ and A_t are endogenous variables which depend on past choices of ν_t and η_t. Therefore the output rate depends on past history in a complex way. We are able to harness these complexities into a set of four functional equations, equations (1) through (4). These equations constitute a model which will allow us to study questions of economic interest.

In Sections 3 to 5 we provide a closed-form solution for this system of functional equations. The solution fully describes the evolution of: the duration of projects, the worker's inventory of active projects, and the output rate. Finding this solution represents the main mathematical contribution of the paper. In the solution, if the rate at which projects are started and the worker's effort are stationary ($\nu_t = \nu$ and $\eta_t = \eta$), then the mass of active projects A_t grows over time but the output rate remains stationary ($\omega_t = \omega$). Since for policy reasons we are interested mainly in the output rate, expression (5) on page 14 is of particular interest. This expression links the effort rate η, the complexity X of the project, the rate ν at which projects are started, and the output rate ω. This relationship, therefore, represents a production function, the technology that yields output rate ω. According to this production function, if the input rate ν is large enough that the worker is never idle (a reasonable restrictions in many practical cases), then the output rate ω is decreasing in ν. That is, the output rate decreases as the input rate (task juggling) increases.

If task juggling is bad, then why is it so prevalent? Why don't workers keep their input rate down and maximize their output rate? Section 6 offers an explanation that could fit several scenarios. The idea is that effort is allocated "under pressure." We provide a framework in which projects being worked on belong to clients, each of whom wants their project to be done quickly. These clients have the ability to, in each instant, lobby the worker (at a cost) to get their project worked on during that instant.[4] We fully characterize the equilibrium of the lobbying game and show that, no matter how low the cost of lobbying, in equilibrium the input rate is too large and for this reason the output rate is lower than its theoretical maximum. In other words, lobbying causes task juggling. In Section 10.5 we reflect more generally on the fundamental features of an economic environment that make task juggling more likely.

A different conjecture for why task juggling is prevalent, could be that principals foster it in order to elicit non-contractible effort from their workers. If, in fact, juggling were a complement for effort, then forcing task juggling onto the worker would indeed incentivize the worker to exert more effort. However, we show that in our framework ν and η are *strategic substitutes* in the production of output.[5] This means that if a worker is motivated to produce output, anything that makes workers juggle more tasks will also, indirectly, reduce the worker's incentives to exert effort. Our analysis, therefore, suggests that even if a principal faces constraints on the rate of pay for performance (say, in the public sector because money is scarce, or in the private sector because of union rules), still he should not induce his agents to juggle projects. This finding has practical implications for the organization of trials in Italy. Italian judges are required by law to start working on a case as soon as it has been assigned to them. This practice induces task juggling which, in light of our analysis, is detrimental both for the output rate *and for effort.*

A still different view of task juggling is that in fact it could be *efficient.* In Section 7 we

[4] For an academic researcher, the competing clients might be co-authors; for a judge, they might be the plaintiffs in the trials, particularly in countries where trials can last decades.

[5] This means that the cross partial of output with respect to ν and η is negative.

show that if value is created when a project completes early intermediate goals, then it may be optimal to push the input rate ν above the duration-minimizing level. This is because if the primary goal is not (only) to complete the project, but there is also value in *starting* it, for example, then the negative effect of large input rates on output rates is less important. In this case task juggling, with the associated increase in durations, is in fact efficient. As an application, consider certain types of trials in which preliminary injunctions may need to be issued early in the trial, before the case is decided.

In Section 8 we explore variable speed strategies, that is, strategies that calibrate the amount of effort devoted to a case depending on its degree of completion. A special case are the "equal treatment" strategies considered in the rest of the paper, which devote the same proportion of effort to each open case at any point in time. We give conditions under which the optimal variable speed strategy is, in fact, an equal treatment one.

In Section 9 we analyze the different options available to a principal who needs to incentivize the worker. The principal may chose to reward the aggregate output rate, or to penalize the average duration of projects. Although these two performance measures are closely related, if the worker can strategically elect to work on projects of different complexity, then which measure is used matters. Indeed, we show that if the worker is compensated based on aggregate output irrespective of the complexity of the project, then that worker will totally ignore complex projects and focus on easy ones. Under this strategy, complex projects never get done. If instead the worker is penalized based on the average duration of assigned projects, then we show that a worker will work on cases of all complexities, and in fact will devote relatively more effort to complex cases.

Section 10 analyzes several extension to the main model. We deal first with the case in which projects have different degrees of complexity, but they cannot be treated disparately (if they can, then the analysis of Section 9 applies). We characterize the ratio of input to output rates as a function of the complexity of the project and show that more complex projects have a worse input to output ratio. Next we deal with the case in which the worker is forgetful, that is, as the completion time grows and any open project is worked on less and less per unit of time, the worker starts to forget about each individual project and thus before he can make progress he needs additional effort on each case to "remind himself" of where he left off. The additional effort required slows down production relative to a constant growth path. More remarkably, in our model introducing forgetfulness may generate a multiplicity of growth paths. Finally, we discuss how the model would change in the presence of a "time to build" constraint whereby the worker cannot complete a project under a certain time threshold, no matter how large the effort.

1.1 Related Literature

What we call task juggling is an inefficiency that is related to the concept of "bottleneck" in the literatures on project management and project planning (see Model *et al.*, 1983). Our model is also related to the literature on network queuing, originating with Jackson (1963). The focus of these literatures is on how to identify and eliminate bottlenecks in the production or service processes. In our model this is easily done (set the input rate ν equal η/X) and is not the main point of the paper. In contrast, our analysis focuses on the observable features of an inefficient dynamic production function, culminating in equation (5) which, to our knowledge, is a new mathematical result. This result is our starting point for a study of incentives: how private incentives to lobby the worker might result in task juggling, and how to incentivize the worker. Incentives considerations such as the ones we study are largely absent from these literatures.

The result that expanding the number of projects a worker works on will indirectly reduce the worker's incentives to exert effort is reminiscent of a result obtained by Dewatripont *et al.* (1999). In their setup, the worker exerts effort in order to signal his ability. Increasing the number of projects assigned to the worker also, by assumption, increases the amount of noise the signaling activity must overcome. This reduces the worker's returns from signaling. Clearly, this effect is quite different than the one analyzed in this paper.

The model of lobbying by clients presented in Section 6 is a version of common-pool problem, of a kind that has been studied in economics under various guises (see e.g. Ostrom 1990).

Section 9 analyzes the effect of different incentive schemes when the worker can arbitrage across projects of different complexity. Such arbitraging across tasks that cannot be individually incentivized has been called "multitasking" by Holmstrom and Milgrom (1991).

In Section 10.3 we study the case in which coming back to a project too seldom can slow down the productivity on each individual task of the project. This idea is explored in a recent psychological literature on multitasking (this time the word is used in its common acception of doing too many different things at the same time) and task interruptions (see e.g. Mark *et al.* 2008).

2 The Model

In this section we present the basic framework in which the main mathematical result is derived. This model will be extended and generalized later in the paper.

The model lives in continuous time, starting from $t = 0$. There is a continuum of projects.

Each project takes X steps to complete and is characterized, at any point in time, by its degree of completion $x \in [0, X]$. Note that, because x is a continuous variable, we are assuming (with some abuse of language) that there is a continuum of steps for each project. Before a worker starts working on a project, that project's degree of completion is $x = X$. We call a project **completed** when $x = 0$. Here X can be interpreted as a measure of the complexity of the project.

As soon as the worker starts working on a project, that project becomes **active**. The project stops being active when it is completed. At any time t, the worker has A_t active projects, in various degrees of completion. The distribution $\varphi_t(x)$ represents the mass of projects which are exactly x steps away from being done. By definition, the number of active projects at time t is

$$A_t = \int_0^X \varphi_t(x)\, dx \tag{1}$$

We assume that all active projects become more complete at a rate η_t / A_t, where η_t is the rate at which effort is exerted. Informally, this means that in the time interval between t and $t + \Delta$, the worker's work shaves off approximately $(\eta_t / A_t)\,\Delta$ steps from *each* active project.[6] This formulation captures the idea that the worker divides a fixed amount of working hours equally among all projects active at time t. We refer to this procedure as working "in parallel." Parallel work means that all projects proceed at the same speed; this means that after Δ has elapsed, the distribution $\varphi_t(x)$ is translated horizontally to the left (refer to Figure 2), and so for Δ "small enough" we can write intuitively

$$\varphi_{t+\Delta}\left(x - \frac{\eta_t}{A_t}\Delta\right) = \varphi_t(x).$$

To express this condition rigorously, bring $\varphi_t(x)$ to the right-hand side, divide by Δ and let $\Delta \to 0$ to get

$$\frac{\partial \varphi_t(x)}{\partial t} - \frac{\partial \varphi_t(x)}{\partial x}\frac{\eta_t}{A_t} = 0. \tag{2}$$

This partial differential equation embodies the assumption of parallel work.

The projects that fall below 0 (grey mass in Figure 2) are the ones that get completed within the interval Δ. These are the projects whose x at t is smaller than $\frac{\eta_t}{A_t}\Delta$. Therefore, the mass of output between t and $t + \Delta$ is approximately

$$\int_0^{\frac{\eta_t}{A_t}\Delta} \varphi_t(x)\, dx.$$

[6] Note that this formulation requires $A_t > 0$.

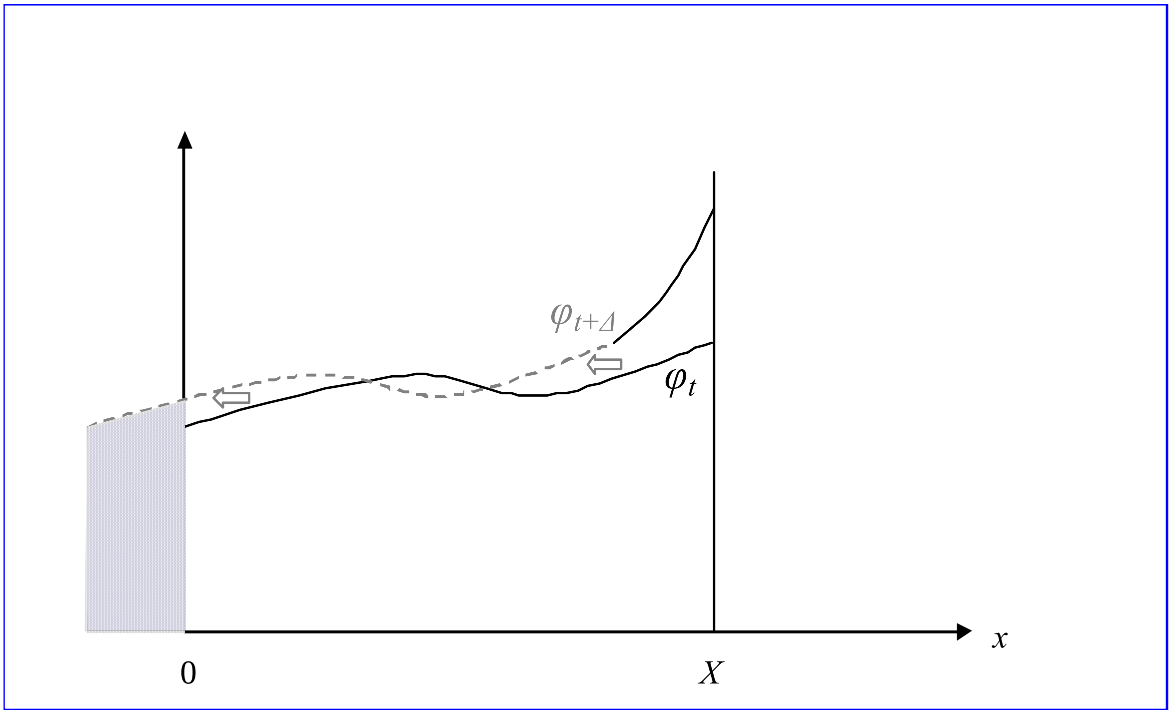

Figure 2: The function φ_t is traslated horizontally to the left as time passes. Newly opened cases are added to the right. The grey mass of cases to the left of zero are completed.

To get the **output rate** ω_t, divide this expression by Δ and let $\Delta \to 0$ to get

$$\omega_t = \lim_{\Delta \to 0} \frac{1}{\Delta} \int_0^{\frac{\eta_t}{A_t}\Delta} \varphi_t(x)\, dx = \frac{\eta_t}{A_t} \varphi_t(0). \tag{3}$$

Cases are assigned to the worker at rate α_t, but the worker is not required to open projects as soon as they are assigned. Rather, we allow the worker to open new projects at a rate ν_t. A larger ν_t will, ceteris paribus, mean more task juggling—more projects being worked on simultaneously. This ν_t is seen either as a choice on the part of the worker, or as determined by lobbying, or else imposed by some regulation. For Δ small, the change in the mass of projects active at t is approximately

$$A_{t+\Delta} - A_t = \nu_t \cdot \Delta - \omega_t \cdot \Delta.$$

Divide both sides by Δ and let $\Delta \to 0$ to get the formally correct expression

$$\frac{\partial A_t}{\partial t} = \nu_t - \omega_t. \tag{4}$$

Graphically, the mass of newly opened projects is squeezed in at the back of the queue in Figure 2, just to the left of X, in whatever space is vacated on the horizontal axis by the progress made in Δ on the pre-existing open projects.

We close this section by defining two measures of productivity.

Definition 1 *For a project assigned at t we define the* ***duration*** *D_t as the time which elapses between t and the completion of the project. For a project opened at t (and thus assigned at a time before t), we define* ***completion time*** *C_t the time which elapses between t and the completion of the project.*

This completes the description of our model. In this model, two variables are interpreted (for now) as exogenously given: η_t and ν_t. The first describes *how much* the worker works, the second *how* he works—how many projects he keeps open at the same time. These two variables will determine, through the process described mathematically by equations (1) through (4), the key variable of interest, the output rate ω_t. This variable, in turn, will determine the duration of a project and its completion time. The two (endogenous) functions A_t and $\varphi_t(x)$ are, perhaps, of merely instrumental interest: they describe the state of the worker's docket at any point in time—how many projects he has open, and the degree of completeness of each. Our first major task is to uncover the law through which η_t and ν_t determine ω_t. We turn to this next.

3 Growth Paths and Fundamental Theorem

Theorem 1 in this section identifies the law through which ν_t and η_t determines ω_t. In terms of language, however, rather than talking about ν_t and η_t *determining* ω_t, it is more convenient to talk about ν_t, η_t and ω_t (and also $\varphi_t(x), A_t$) *belonging to a growth path.*[7] This is the language we use in this section.

Definition 2 *A* ***growth path*** *is a quintuple of positive real functions $[\nu_t, \eta_t, \varphi_t(x), A_t, \omega_t]_{\substack{t\in(0,\infty)\\ x\in[0,X]}}$ that satisfies (1), (2), (3) and (4). A* ***constant growth path*** *is a growth path where input and effort rates are constant, $\nu_t = \nu$ and $\eta_t = \eta$.*

A growth path is a set of functions which are linked together in that they jointly satisfy a system of functional equations. While this definition is somewhat abstract, the way to think

[7] In what follows we will omit the subscripts $t \in (0, \infty)$ and $x \in [0, X]$ when no confusion can arise.

about a growth path is as a time-varying production function. The input and effort rates are given exogenously, and the output rate ω_t is determined by a complicated, time-evolving "production function," which is characterized by the time-varying quantities $\varphi_t(x)$ and A_t.

In general, finding a quadruple that constitutes a growth path is a daunting mathematical problem. However, when ν_t and η_t are constant and equal to ν and η respectively, then a growth path can be fully characterized. We start by guessing a functional form for $\varphi_t(x)$ and A_t. Let

$$\varphi_t^*(x) = \frac{(\nu-\omega)}{\eta}\omega\ t\ e^{\frac{\nu-\omega}{\eta}x},$$

and

$$A_t^* = (\nu-\omega)\ t.$$

The next theorem states that the quintuple $[\nu, \eta, \varphi_t^*(x), A_t^*, \omega]$ satisfies conditions (1) - (4), and thus is a growth path, if and only if ω solves a certain equation involving ν and η.

Theorem 1 *The quintuple $[\nu, \eta, \varphi_t^*(x), A_t^*, \omega_t]$ is a constant growth path if and only if: (a) the output rate is constant, $\omega_t = \omega$; and (b) the triple ν, η, ω solves the equation $X\frac{\nu-\omega}{\eta} = \log(\nu) - \log(\omega)$.*

Proof. One can verify directly that for any K, λ, the pair $\varphi_t(x) = Kte^{\frac{\lambda}{\eta}x}$, $A_t = \lambda t$ solves (2) above. Moreover, for any λ the triple $\varphi_t(x) = Kte^{\frac{\lambda}{\eta}x}$, $A_t = \lambda t$, ω_t satisfies (3) if and only if $K = \frac{\lambda}{\eta}\omega_t$, which implies $\omega_t = \omega$. Finally, the triple ν_t, A_t, ω satisfies (4) if and only if $\lambda = \nu_t - \omega$, which implies $\nu_t = \nu$.

This shows that, for any ν, ω, the quadruple $[\nu, \varphi_t^*(x), A_t^*, \omega]$ satisfies all but one of the equalities which define a constant growth path. However, we do not yet know which values of ν and ω are compatible with each other along a growth path. We now show that the pair $\varphi_t^*(x) = Kte^{\frac{\lambda}{\eta}x}, A_t^* = \lambda t$ solves (1) if and only if $X\frac{\nu-\omega}{\eta} = \log(\nu) - \log(\omega)$. Condition (1) reads

$$A_t^* = \int_0^X \varphi_t^*(x)\ dx.$$

Substituting $\varphi_t^*(x)$ and A_t^* yields

$$\begin{aligned}\lambda t &= \int_0^X Kte^{\frac{\lambda}{\eta}x}\ dx \\ &= Kt\ \left.\frac{\eta}{\lambda}e^{\frac{\lambda}{\eta}x}\right|_{x=0}^{X} \\ &= \frac{\eta}{\lambda}Kt\ \left[e^{\frac{\lambda}{\eta}X} - 1\right].\end{aligned}$$

Now substitute for $K = \frac{\lambda}{\eta}\omega$ and $\lambda = \nu - \omega$ to get

$$\lambda t = \frac{\eta\lambda\omega t}{\eta\lambda}\left[e^{\frac{\lambda}{\eta}X} - 1\right]$$
$$\lambda = \omega\left[e^{\frac{\lambda}{\eta}X} - 1\right]$$
$$\nu - \omega = \omega\left[e^{\frac{(\nu-\omega)}{\eta}X} - 1\right]$$
$$\frac{\nu}{\omega} = e^{\frac{(\nu-\omega)}{\eta}X}.$$

Taking logs yields

$$\log(\nu) - \log(\omega) = X\frac{(\nu - \omega)}{\eta}.$$

Therefore, Theorem 1 is proved. ■

Along a constant growth path the function $\varphi_t^*(x)$ is exponential in x and multiplicative in t, as depicted in Figure 3. As $t \to 0$ the function $\varphi_t^* : [0, X] \to \mathbb{R}$ converges to zero uniformly. As t grows, the function φ_t^* grows multiplicatively in t. Growth in t reflects a progressive increase in the number of active cases, that is, growing task juggling over time.

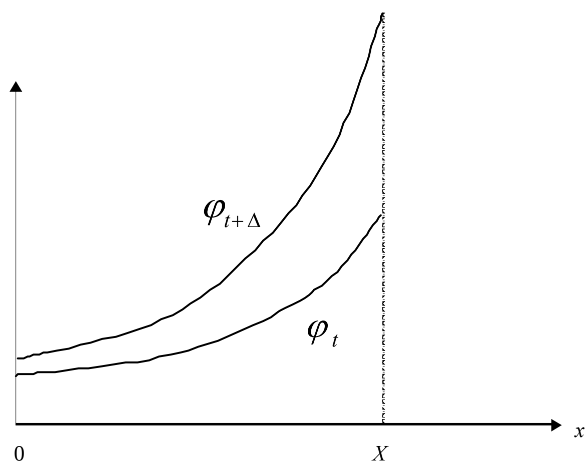

Figure 3: Distribution of active cases, by number of steps away from being done. On the growth path it is exponential.

Growth in task juggling also explains why the function $\varphi_t(x)$ is exponential in x. This is because, when the worker juggles an increasing number of projects over time, projects proceed at a progressively slower pace (that pace is η/A_t, and remember that A_t grows linearly with t). As projects grind along more and more slowly, the constant rate ν of newly inputed cases must squeeze in the progressively smaller "empty segment" available near X. This effect accounts for the exponential shape of $\varphi_t(x)$. Yet, remarkably, despite these complex dynamics the output rate is constant through time. This remarkable property of the output rate results from two opposite effects offsetting each other: on the one hand, cases move through at progressively slower rates, which tends to progressively reduce the output rate. On the other hand, the mass of cases that are almost done increases with time (this is because $\varphi_t(0)$ grows with t), which tends to progressively increase the output rate. These two effects exactly offset each other along a constant growth path, and thus the output rate is time-invariant.

The next result gives expressions for completion time and duration. They both grow linearly with t.

Proposition 1 *In a constant growth path we have* $C_t = \frac{(\nu-\omega)}{\omega}t$ *and* $D_t = \frac{(\alpha-\omega)}{\omega}t$.

Proof. The completion time C_t of a project started at t is the time that it takes all the projects in front of it to clear. These projects are A_t, and given an output rate ω that duration is given by the solution to the following equation

$$\int_t^{t+C_t} \omega \, ds = A_t,$$

which equals

$$\omega C_t = (\nu - \omega) t.$$

Solving for C_t yields the desired expression. Let us now turn to duration. Given an arrival rate α, a project assigned at t finds

$$\alpha t - \omega t$$

projects in front of it. Given an output rate of ω, these projects will take

$$D_t = \frac{(\alpha - \omega)}{\omega} t$$

to complete. This is the duration of a project assigned at t. ■

4 Characterization of the Output Rate

Theorem 1 goes a long way towards characterizing a constant growth path, but there is still some work to do. We need to characterize the relationship that links ν, η and ω along a growth path or, said differently, we need to understand what level of output is possible given certain input and effort rates.

According to Theorem 1, the relationship between ν, η and ω along a growth path is

$$(\nu - \omega)\frac{X}{\eta} = \log(\nu) - \log(\omega). \tag{5}$$

Define

$$h(y) = \frac{X}{\eta}y - \log(y).$$

Then equation (5) reads

$$h(\nu) = h(\omega).$$

The next lemma characterize the function $h(\cdot)$.

Lemma 1 *The function $h(y)$ is strictly convex on $(0, \infty)$, converges to infinity at $y = 0$ and $y = +\infty$, and it has its minimum at $y = \eta/X$.*

Proof. One can easily verify that $h(0) = +\infty = h(\infty), h'(y) = \frac{X}{\eta} - \frac{1}{y}$, and finally $h''(y) = \frac{1}{y^2}$. ■

Figure 4 depicts $h(y)$. For a particular level of ν, the ω that solves equation (5) is represented graphically as the point on the horizontal axis that achieves the same level of the function h. But not all solutions to equation (5) can be part of a growth path. Which solutions are consistent with a growth path is described in the next proposition.

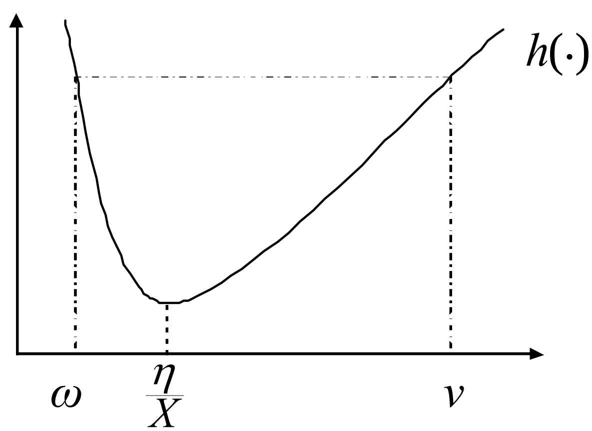

Figure 4: Relationship between input and output rates on a growth path.

Proposition 2 ***(conditions for a growth path)*** *ν is compatible with a constant growth path if and only if $\nu > \frac{\eta}{X}$. In that case, the quintuple $[\nu, \eta, \varphi_t^*(x), A_t^*, \omega]$ is a constant* *growth path if and only if ω is the unique solution that is smaller than $\frac{\eta}{X}$ to the equation $h(\nu) = h(\omega)$.*

Proof. The solution $\omega = \nu$ to equation (5) is not acceptable because then $A_t^* \equiv 0$ and (3) is not well-defined. Nor can we accept solutions where $\omega > \nu$, for then $\varphi_t^*(x)$ and A_t^* would be negative and thus the quadruple identified in Theorem 1 would not meet the definition of a growth path. So we need to find solutions with $\omega < \nu$. This implies $\nu > \frac{\eta}{X}$. The rest of the Proposition follows immediately from Theorem 1. ■

The threshold η/X can be interpreted as the minimum input rate compatible with the worker not being idle; we will discuss this interpretation at the end of this section. Proposition 2 shows how to construct the entire growth path associated with any pair (ν, η). Given a constant input rate $\nu > \frac{\eta}{X}$, one can uniquely identify the corresponding output rate $\omega < \frac{\eta}{X}$ which solves $h(\nu) = h(\omega)$. Then the triple (ν, η, ω) is plugged into the expressions for $\varphi_t^*(x)$ and A_t^* to obtain a full characterization of the growth path.

Proposition 2 shows that a constant growth path only exists if the input rate is sufficiently large. What happens otherwise? Then the worker can solve projects faster than he opens them, and in that case our model predicts $A_t \equiv 0$. In this case we do not have a model of task juggling, but rather one of "undercommitment." We conclude this section by analyzing this case. In the analysis we allow for an "initial condition" $A_0 \geq 0$, a possibly positive mass of cases active at time zero. (This hypothesis is not consistent with a constant growth path, where at $t = 0$ the mass of active cases is zero.) The next proposition shows that if $\nu < \frac{\eta}{X}$ then A_t shrinks over time, and if $\nu = \frac{\eta}{X}$ then A_t is constant. Some of these results will be useful in future sections.

Proposition 3 ***(steady-state and shrink paths)*** *If $\nu = \frac{\eta}{X}$ then there are a continuum of steady-state paths, indexed by the mass of projects active at time zero, A_0. In each of these steady states $A_t \equiv A_0$, the output rate is equal to η/X, and the duration of projects is increasing in A_0.*

If $\nu < \frac{\eta}{X}$ then whatever the value of A_0, after a transition period it will be $A_t \equiv 0$ and, from then on, the duration of projects will be zero and the output rate will be equal to ν.

Proof. See the Appendix. ■

The threshold η/X can be interpreted as the minimum input rate compatible with the worker not being idle given that the worker exerts effort at rate η. To understand this interpretation, fix effort η and observe that if $\nu' < \eta/X$ then there exists a smaller effort rate η' such that $\eta'/X = \nu' \geq \omega'$ (the inequality is true because cannot be more cases being completed than there are coming in). This means that if the input rate ν' falls below η/X then the worker could achieve the same level of output ω' by exerting effort at the lower rate η'. This is equivalent to saying that the worker is idle at rate $\eta - \eta'$.

5 Comparative Statics For the Output Rate

In this section we derive a number of comparative static results for the output rate in a constant growth path. To this end, it is convenient to think of the output rate as *generated* by the other parameters of the model. The next definition introduces the notation Ω for the production function of ω. Throughout this section we implicitly assume that $\nu \geq \frac{\eta}{X}$.

Definition 3 *For each pair $(\nu, \eta/X)$ denote by $\Omega(\nu; \eta/X)$ the unique $\omega < \nu$ that solves (5).*

The next proposition presents the comparative statics results for Ω.

Proposition 4 *a) $\Omega(\nu; \eta/X)$ is decreasing in ν.*

b) $\Omega(\nu; \eta/X)$ is increasing in η/X.

c) $\frac{\partial \Omega(\nu;\eta/X)}{\partial \nu \partial \eta} < 0$, which means that ν and η are strategic substitutes in $\Omega(\nu; \eta/X)$.

d) The function $\Omega(\cdot; \cdot)$ is homogeneous of degree 1.

e) $\Omega(\eta/X; \eta/X) = \eta/X$.

Proof. a): Immediate from inspection of Figure 4.

b), c): See the Appendix.

d) Suppose the triple $\left(\nu, \omega, \frac{\eta}{X}\right)$ solves (5). We need to show that for any scalar $r > 0$, the triple $\left(r\nu, r\omega, r\frac{\eta}{X}\right)$ also solves (5). Write

$$\begin{aligned} r\frac{\eta}{X}\left[\log(r\nu) - \log(r\omega)\right] &= r\frac{\eta}{X}\left[\log(\nu) - \log(\omega)\right] \\ &= r(\nu - \omega) = (r\nu - r\omega). \end{aligned}$$

where the second equality follows because the triple $\left(\nu, \omega, \frac{\eta}{X}\right)$ solves (5). The equality between the first and the last element in this chain of equalities shows that the triple $\left(r\nu, r\omega, r\frac{\eta}{X}\right)$ solves (5).

e) Immediate from inspection of Figure 4. ■

Part a) is the effect of task juggling: increasing ν increases the mass of active projects and reduces output. The proposition shows that η/X is the maximum feasible output rate. When $\nu = \omega = \eta/X$ we have a steady state where $A_t \equiv 0$; this is a case treated in Proposition 3. Part b) simply says that if a worker works more then the output rate is larger.

Part c) deals with the complementarity of inputs in the production of the output rate. It says that the returns to effort decrease when ν increases. Intuitively, this is because A_t is larger and so an increase in effort needs to be spread over a greater number of projects.

Part d) admits two economic interpretations. First, the lemma expresses constant returns to scale with respect to "inputs" ν and η. One implication is the following. Imagine that instead of treating all projects in the same pool, the worker splits his projects into two pools: projects only dealt with in the morning, and projects only dealt with in the afternoon. Suppose the worker allocates a fraction $r < 1$ of his projects to the morning, and the remaining $(1 - r)$ to the afternoon. Suppose also that the worker splits his total effort η in the same proportion. Then his total output is the sum of the outputs of morning and afternoon projects, which by the lemma are, respectively, $r\omega$ and $(1 - r)\,\omega$. The total sum is ω, which shows that the worker does not gain or loses from splitting projects into smaller pools. The second interpretation is that r governs the pace at which the system operates. Setting $r > 1$ means that the entire system is working at a faster pace: per unit of time, we have more input, more effort, and more output, all in the same proportion. We build on this interpretation in Section 10.1.

The results reported in Proposition 4 can be used to prove that increasing the input rate, and thus the degree of task juggling, increases the inefficiency.

Corollary 2 *In a constant growth path, completion time C_t and duration D_t are increasing in ν.*

Proof. From Proposition 1 we have

$$C_t = \left(\frac{\nu}{\Omega\,(\nu; \eta/X)} - 1\right) t = \left(\frac{1}{\Omega\,(1; \eta/\nu X)} - 1\right) t,$$

where the second equality follows from Proposition 4 d). From Proposition 4 (b) we have that Ω is increasing in its second argument, whence increasing ν decreases $\Omega\,(1; \eta/\nu X)$ and increases C_t.

As for duration, from Proposition 1 we have

$$D_t = \left(\frac{\alpha}{\Omega\,(\nu; \eta/X)} - 1\right) t.$$

From Proposition 4 (a) we have that Ω is decreasing in its first argument, whence increasing ν decreases $\Omega(\nu; \eta/X)$ and increases D_t. ■

6 Input Rate Determined Through Lobbying Equilibrium

In the previous sections we have assumed that ν_t, the exogenous input rate, is constant through time and, furthermore, that it exceeds the duration-minimizing input rate η/X. We have not discussed how such a ν_t might come about. In this section we "micro-found" ν_t by introducing a game in which the input rate is determined endogenously as an equilibrium phenomenon.

The basic setup is that each project is "owned" by a client who in each instant can lobby the worker to devote a fraction of effort to his project, regardless of its order of assignment. The private benefit of lobbying is that a client avoids its project waiting unopened and gets the worker working on it immediately. But, by forcing the worker to work on their projects, lobbyists force a worker to distribute effort among more projects. This will increase the number of active projects, which slows down all projects. This externality, which is not internalized by the lobbyists, gives rise to an inefficiency. In this game the input rate ν_t is determined endogenously by the clients' lobbying, and the input rate turns out to be constant through time and higher than η/X. Therefore, the results in this section provides a foundation for the model analyzed in the previous sections. At the end of this section we also allow effort to be chosen endogenously by the worker.

The model is as follows. The worker's effort η is constant through time and fixed exogenously (we will relax the second assumption later). Lobbying is modeled as a technology whereby, at any instant t, a client can pay $\kappa \cdot \Delta$ and force activity on his project during the interval $(t, t+\Delta)$. Activity on the project means that the project moves forward by $(\eta/A_t) \cdot \Delta$. The rate κ is interpreted as the per-unit of time cost of lobbying. If κ is not paid then the project sits idle at some x until either lobbying is restarted or the never-lobbied projects of its vintage (those assigned at the same time) *catch up* to x, at which time the project becomes active again and stays active without any need of, or benefit from, further lobbying. In every instant, $\underline{\nu}$ "never lobbied" projects are opened, in the order they were assigned. Once a never-lobbied project is opened, it forever remains active whether or not it is lobbied. The rate $\underline{\nu}$ represents the input rate that would prevail in the absence of any lobbying by the clients.[8] Here A_t denotes the mass of all projects active in instant t and it is composed of

[8] One could be concerned that in equilibrium there might not be enough never-lobbied cases to open, and

the two type of projects: all those that are lobbied in that instant, and some that are not.[9]

We assume that clients minimize B times the duration of their project, from assignment to completion, plus κ times the time spent lobbying. B represents the rate of loss experienced by a client whose project is not completed. We assume no discounting for simplicity.

Since our goal is to explain why lobbying makes the input rate ν inefficiently large, let's tie our hands by stipulating that the input rate of never-lobbied projects $\underline{\nu}$ is "low," that is, it belongs to the interval $\left[0, \frac{\eta}{X}\right]$. This choice of baseline ensures that any slowdown in the output rate cannot be attributed to an excessively large $\underline{\nu}$.

Projects are indexed by the time τ they are assigned and by an index a that runs across the set of the α projects assigned at time τ. We now introduce the notion of lobbying strategy and lobbying equilibrium.

Definition 4 *A **lobbying strategy** for project (a, τ) is a measurable indicator function $S_{a\tau}(t)$ defined on the interval $[\tau, \infty)$ which takes value 1 if project a is lobbied in instant t, and is zero otherwise. A **lobbying equilibrium** is a set of strategies such that, for each project (a, τ), the strategy $S_{a\tau}(t)$ minimizes κ times the time spent lobbying plus B times the project's duration.*

Although strategies are defined for the infinite future, project (a, τ) will be completed at a certain time; the shape of the strategy after that time is payoff-irrelevant. Equilibrium strategies could potentially be quite unwieldy, featuring complex patterns of activity interspersed with periods of no lobbying. The next lemma affords some simplification. It suggests that we should look for equilibria in which clients play just two simple strategies.

Lemma 2 *In any lobbying equilibrium in which the number of active projects grows, two strategies payoff-dominate all others: strategy $\mathbf{1}(\cdot)$ which denotes immediate and perpetual lobbying starting from time of assignment, and strategy $\mathbf{0}(\cdot)$ which denotes never lobbying.*

Proof. See the Appendix. ■

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that therefore it would be more precise to state that in every instant the judge opens the minimum of $\underline{\nu}$ never-lobbied cases and the balance of the never lobbied cases. However, we will see that in equilibrium the balance of never-lobbied cases never falls below $\underline{\nu}$.

[9] Under these rules, for a case that has been lobbied in the past, two scenarios are possible in instant t. First, the case may have been "caught up" by the never-lobbied cases of its own assignement vintage; in other words, the case was lobbied in the past, but then the lobbying lapsed and the case is now at the same stage of advancement (same x) as its never-lobbied assignment vintage. Such a case is worked on without the need for further lobbying and proceeds at speed η/A_t. The second scenario is that the case has not been caught up at time t. In this scenario the case is worked on in the interval Δ and makes $\eta\Delta/A_t$ progress if $\kappa\Delta$ is spent; otherwise, the case does not proceed.

The intuition behind Lemma 2 is the following. Lobbying "buys advancement" at the speed of η/A_t. If it is profitable to lobby at the assignment of the project, then it makes no sense to have interludes of no lobbying. During those interludes the project does not advance, but the mass of active projects A_t keeps growing, making lobbying (once it is restarted) less productive.

Even taking Lemma 2 into account, lobbying equilibria could potentially be very complex because of the possibility of non-constant growth equilibria in which the input rate is not constant through time. Inspired by Lemma 2, we look for a simple class of equilibria in which a time-invariant fraction z of the α newly assigned projects is never lobbied, and the remaining fraction $(1-z)\,\alpha$ is lobbied immediately upon assignment and then continuously during the entire duration of their trial. We will call these equilibria **constant-growth lobbying equilibria**. Note that the definition of constant-growth lobbying equilibrium does not restrict the strategy space.

If players adopt the strategies of a constant-growth lobbying equilibrium, the input rate $\nu(z)$ is determined by z via the identity

$$\nu(z) = \underline{\nu} + (1-z)\,\alpha.$$

The percentage of lobbyists $(1-z^*)$, and hence the input rate $\nu(z^*)$, are determined in equilibrium.

It is not obvious that constant-growth lobbying equilibria should exist. Indeed, for the fraction of lobbied projects $(1-z)$ to be stationary through time, the *relative* payoffs from lobbying and not lobbying must be constant over time. Since both of these payoffs are non-stationary in levels (they decrease over time), that their ratio should be constant is remarkable. Nevertheless, in what follows we show that a constant-growth lobbying equilibrium always exists, and we characterize it.

Proposition 5 *Suppose* $\alpha > \frac{\eta}{X}$. *Then, for any* $\underline{\nu}$ *and any cost of lobbying* κ,

a) a constant-growth lobbying equilibrium exists;

b) in any constant-growth lobbying equilibrium $\nu(z^*) > \frac{\eta}{X}$, *i.e., the input rate exceeds the duration-minimizing one;*

c) the constant-growth lobbying equilibrium is unique;

d) the fraction $(1-z^*)$ *of projects that are lobbied in equilibrium is increasing in* $\frac{\alpha}{\underline{\nu}}$ *and* $\frac{\eta}{X}$, *and decreasing in* $\frac{\kappa}{B}$;

e) the equilibrium input rate $\nu(z^*)$ *is decreasing in* $\frac{\kappa}{B}$ *and increasing in* $\frac{\alpha}{\underline{\nu}}$ *and* $\frac{\eta}{X}$.

Proof. a) We show that there is a time-invariant z such that the value at the time of assignment of two players who follow the two different equilibrium strategies (lobby and not) are the same. The lobbyist's value at the time of assignment for a project assigned at τ, assuming the project is lobbied from assignment through to completion, is $(-\kappa - B)\, C_\tau$ where C_τ is the completion time of a project started at τ. Substituting for C_t from Proposition 1, the value is given by

$$VL_\tau(z) = (-\kappa - B) \quad \left[\frac{\nu(z)}{\Omega(\nu(z); \eta/X)} - 1\right] \tau.$$

The value of the non-lobbyist at the time of assignment for a project assigned at τ, assuming that he never lobbies, is computed as follows. First, the fraction of non-lobbyist projects inputed in each instant is given by $\frac{\underline{\nu}}{\nu(z)}$, and consequently the output rate is made up of a fraction $\frac{\underline{\nu}}{\nu(z)}$ of non-lobbyist projects. Thus, a project assigned at τ finds

$$z\alpha\tau - \frac{\underline{\nu}}{\nu(z)}\Omega(\nu(z); \eta/X)\,\tau$$

non-completed projects in front of it. These projects are completed at rate $\frac{\underline{\nu}}{\nu(z)}\Omega(\nu(z); \eta/X)$, so it takes

$$\left[\frac{z\alpha}{\frac{\underline{\nu}}{\nu(z)}\Omega(\nu(z); \eta/X)} - 1\right]\tau$$

before all non-lobbied projects assigned before τ are completed. Therefore the value of a non-lobbyist at the time of assignment, assuming that he never lobbies in the future, is

$$VN_\tau(z) = -B\left[\frac{z\alpha}{\frac{\underline{\nu}}{\nu(z)}\Omega(\nu(z); \eta/X)} - 1\right]\tau$$

In an equilibrium with lobbyists and non-lobbyists, z^* solves $VL_\tau(z^*) = VN_\tau(z^*)$, or

$$(-\kappa - B) \quad \left[\frac{\nu(z^*)}{\Omega(\nu(z^*); \eta/X)} - 1\right] = -B\left[\frac{\alpha}{\underline{\nu}}z^*\frac{\nu(z^*)}{\Omega(\nu(z^*); \eta/X)} - 1\right] \tag{6}$$

It is important to note that condition is independent of τ. Thus, if a z^* exists that verifies equation (6), this z^* will be time-invariant, consistent with the definition of constant-growth lobbying equilibrium. We conclude the proof by showing that at least one z^* exists that verifies equation (6) and it lies between $\frac{\underline{\nu}}{\alpha}$ and $\frac{\underline{\nu}}{\alpha} + \frac{1}{\alpha}\left(\alpha - \frac{\eta}{X}\right)$.

The lowest possible value of z^* is $\frac{\underline{\nu}}{\alpha}$. If z falls below this level, there are not enough non-lobbyists to fill $\underline{\nu}$, and then non-lobbied projects get started immediately. Formally, in this project the expression in brackets on the RHS of (6) is no greater than the brackets on the LHS, whence $VN_\tau(z) > VL_\tau(z)$. So $z \leq \frac{\nu}{\alpha}$ is not consistent with equilibrium. The highest possible value of z^* is that for which $\nu(z^*) = \eta/X$. At this level the LHS of (6) is zero, and so $VN_\tau(z) < VL_\tau(z)$. Intuitively, if z^* were any higher, then $\nu(z^*) < \eta/X$ and then completion times would be zero, and then lobbyists would lobby at zero cost whence Thus such z cannot be part of the equilibrium. To find an expression for this bound, write $\eta/X = \nu^* = \underline{\nu} + (1-z)\alpha$, and solving for z yields $z = \frac{\nu}{\alpha} + \frac{1}{\alpha}\left(\alpha - \frac{\eta}{X}\right)$. We have shown that on the lower bound of the interval $z \in \left(\frac{\nu}{\alpha}, \frac{\nu}{\alpha} + \frac{1}{\alpha}\left(\alpha - \frac{\eta}{X}\right)\right)$ we have $VN_\tau(z) > VL_\tau(z)$, and on the upper bound $VN_\tau(z) < VL_\tau(z)$. Since the two functions $VN_\tau(z)$ and $VL_\tau(z)$ are continuous in z over the interval, they must cross at least once. Any crossing is consistent with an equilibrium.

b) Suppose not, so that $\nu^* \leq \frac{\eta}{X}$. Then $\alpha > \nu^*$, and so a project assigned at τ finds a backlog of $(\alpha - \nu^*)\tau$ unopened projects in front of it. Since projects are opened at rate ν^*, the time it takes the last project in the backlog to be opened is

$$\frac{(\alpha - \nu^*)}{\nu^*}\tau.$$

This expression, which we will call the unopened duration, is positive and grows linearly with τ. This time can be eliminated by lobbying from assignment time all the way through completion, at a total cost that is proportional to completion time. Proposition 3 proves that when $\nu^* \leq \frac{\eta}{X}$ completion time is stationary, i.e., it is the same for projects opened at any τ. Therefore, the strict best response of all projects assigned after a certain $\widehat{\tau}$ is to lobby all the way through completion time, in order to eliminate the unopened duration which exceeds lobbying costs. But then for every $t > \widehat{\tau}$ not lobbying cannot be equally profitable as lobbying. Therefore we have shown that if $\nu^* \leq \frac{\eta}{X}$, a positive mass cannot be not lobbying after $\widehat{\tau}$. Yet the construction requires that in any instant $\alpha - \nu^*$ projects are not lobbied, and this mass is positive because by assumption $\alpha > \frac{\eta}{X} \geq \nu^*$. Contradiction.

c) The equilibrium z^* solves (6), which can be rearranged as

$$\frac{\kappa + B}{B}\left[\frac{\nu(z)}{\Omega(\nu(z);\eta/X)} - 1\right] = \left[\frac{\alpha}{\underline{\nu}}z\frac{\nu(z)}{\Omega(\nu(z);\eta/X)} - 1\right]$$

and rewritten as

$$\left[\frac{\kappa + B}{B} - \frac{\alpha}{\underline{\nu}}z\right]\frac{\nu(z)}{\Omega(\nu(z);\eta/X)} = \left[\frac{\kappa + B}{B} - 1\right]. \tag{7}$$

The LHS in (7) is the product of two positive and decreasing functions of z, and therefore it is decreasing in z. The RHS does not depend on z. Therefore equation (7) admits a unique solution z^*.

d) Rewrite slightly (7) as

$$H\left(z;\frac{\kappa}{B},\frac{\alpha}{\underline{\nu}}\right)=\left[\frac{\kappa}{B}+1-\frac{\alpha}{\underline{\nu}}z\right]\frac{\nu\left(z\right)}{\Omega\left(\nu\left(z\right);\eta/X\right)}=\frac{\kappa}{B}. \tag{8}$$

The function $H\left(z;\frac{\kappa}{B},\frac{\alpha}{\underline{\nu}}\right)$ is decreasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$, so increasing $\frac{\alpha}{\underline{\nu}}$ or $\frac{\eta}{X}$ results in a downward shift of the function. Since the function is decreasing in z, shifting the function downward results in a shift to the left of the intersection point between the function and the constant line $\frac{\kappa}{B}$. Thus z^* is decreasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$.

The function $H\left(z;\frac{\kappa}{B},\frac{\alpha}{\underline{\nu}}\right)$ is increasing in $\frac{\kappa}{B}$, and increasing $\frac{\kappa}{B}$ by δ results in an upward shift of $\delta\frac{\nu(z)}{\Omega(\nu(z);\eta/X)}>1$ in the function. So increasing $\frac{\kappa}{B}$ results in the function shifting upward by more than $\frac{\kappa}{B}$. So, start from a given $\frac{\kappa}{B}$ and focus on the resulting equilibrium z^*, which is the z at which the function H attains height $\frac{\kappa}{B}$. Then increase $\frac{\kappa}{B}$. At z^*, the function H moves up by more than $\frac{\kappa}{B}$. This means that z^* is to the left of the new equilibrium. Thus z^* is increasing in $\frac{\kappa}{B}$.

e) Follows directly from d) and the definition $\nu\left(z\right)=\underline{\nu}+\left(1-z\right)\alpha$. ■

Part b) of the proposition establishes a strong presumption of inefficiently large input rates in our lobbying environment. The intuition is clear: if input rates were efficient, say $\nu<\eta/X$, then completion time would be zero. This means that the cost of lobbying would be zero and, also, that a project which is lobbied would be completed instantaneously. Therefore lobbying is a dominant strategy, which would give rise to an input rate $\nu=\alpha>\eta/X$. Thus an equilibrium input rate ν cannot be smaller than η/X.

Part e) of the proposition also points to κ as a plausible source of heterogeneity in input rates across workers. If a worker is less susceptible to lobbying, which we can model as κ being larger, then the proposition indicates that the worker will have a smaller input rate and a larger output rate. Moreover, the proposition shows that there is more lobbying when the assignment rate is larger, which is intuitive because then the time spent waiting for one's project to be opened becomes larger. Finally, harder working workers and easier projects will give rise to more lobbying. Intuitively, this is because then the completion time gets shorter relative to the duration of a non-lobbied project.

A few words of comment on the causes of inefficiency. The source of a slowdown in output is that, if an additional project is lobbied, that project is able to obtain a small fraction of effort, taking it away from other active projects. In this respect, our model is analogous to models of common resource extraction ("common pool" models) where utilizers cannot be excluded from the pool. We think this is a natural modeling assumption in many cases. In Section 10.5 we discuss at a general level the assumptions that give rise to task juggling.

Finally we turn to the case in which η is chosen by the worker, rather than being exogenously given. Suppose η is determined as the solution to the problem

$$\max_{\eta} \Omega\left(\nu\left(z^{*}\right); \frac{\eta}{X}\right) - c\left(\eta\right). \tag{9}$$

According to this formulation, the worker chooses η by trading off the output rate (increasing in η) against a cost of effort $c(\eta)$. Note that since z^* is taken as given in problem (9), the worker does *not* behave as a Stackelberg leader. This assumption reflects the idea that the worker cannot commit to maintain a given level of effort regardless of lobbying. We now augment the notion of a lobbying equilibrium by allowing the worker's effort η^* to be determined endogenously.

Definition 5 *A* ***lobbying equilibrium with endogenous effort*** *is a lobbying equilibrium in which effort η^* solves (9).*

To ensure that the equilibrium effort level is greater than zero and smaller than αX we assume $c'(0) = 0$, and $c'(\alpha X) = \infty$.

Proposition 6 *Consider a lobbying equilibrium with endogenous effort. If κ increases, then the input rate decreases and the worker's effort increases.*

Proof. See the Appendix. ■

This proposition highlights another dimension of inefficiency associated with lobbying. Not only does lobbying slow down projects, but it also induces the worker to slack off. The intuition behind this result lies in the "strategic substitutes" property stated in Proposition 4 c).

7 Input Rate Determined By Maximization of an Objective Function

In most of the paper we assume that, insofar as the worker has an objective function, it is to maximize the output rate or, equivalently, minimize durations. We also implicitly assume that this is the social goal. Under this assumption the (socially and privately) optimal input rate is, by Proposition 4 a), the lowest possible compatible with no worker idleness, namely η/X. In this section we explore a different hypothesis: that some private or social value may be generated when projects clear *intermediate goals*. So we consider the possibility that value may accrue when a project is merely being opened, or when it gets half-done, etc. One example is a judge who may issue preliminary injunctions early on in the trial, which might increase social welfare.

When weight is placed on clearing intermediate goals, the optimal input rate need no longer equal η/X. To see this, consider an extreme case in which value is generated *only* when a project is opened (e.g., completion does not matter); in this case, the optimal input rate is clearly the largest possible, $\nu = \alpha$. This observation suggests that the framework developed in this section can provide a meaningful theory of the optimal size of the input rate.

In what follows we assume that effort is fixed. We also assume "equal treatment" in the sense that all open projects proceed at the same speed according to equation (2).[10] The equal treatment assumption has bite here, and it will be relaxed in the next section.

The first goal is to specify the (private or social) objective function. To this end, some preliminaries need to be introduced.

Definition 6 ***(rate of clearing intermediate goals)*** *Denote by $\omega_t(x)$ the rate at which, at each instant t, projects clear threshold $x \in [0, X]$.*

For example, $\omega_t(X/2)$ denotes the rate at which projects clear the $X/2$ mark, that is, the rate at which projects become half done. In this notation, the output rate ω_t corresponds to $\omega_t(0)$. We allow for weight to be placed on any number of these "intermediate rates" by postulating the following objective function:

$$\int_0^{\infty} e^{-\rho t} \left[\int_0^X u(\omega_t(x)) \;\; P(dx) \right] \, dt. \tag{P}$$

Here ρ represents a (social or private) discount factor. The function u is increasing and its curvature measures the degree to which low clearing rates are penalized in the objective function. $P(x)$ is a probability measure that specifies the weight placed on intermediate step x, so for example if one third of the value is created when projects clear the $X/2$ mark then $P(X/2) = 1/3$. A slightly different interpretation of $P(X/2) = 1/3$ is that for half the projects two thirds of the value is created by clearing the $X/2$ mark, and for the other half no value is created. In this latter interpretation projects are heterogeneous with respect to when value is created.

[10] Equal treatment can be justified on regulatory grounds—it may not be legal to treat open projects disparately by focusing on some and leaving others behind. It may also be that projects are unobservably heterogeneous, and the worker does not know *ex ante* which projects benefit from clearing which intermediate goal. For example, a judge may not know at which point, if any, a case might settle.

Considerable simplification could be achieved in (**P**) if we knew that $\omega_t(x)$ was stationary, i.e., time-independent. Fortunately the next lemma ensures that is so along a constant growth path, and in addition it offers a convenient expression for $\omega_t(x)$ as a geometric mean of ν and ω with weight x/X.

Lemma 3 *Along a constant growth path,* $\omega_t(x) = \nu^{\frac{x}{X}} \cdot \omega^{1-\frac{x}{X}}$. *Thus, along a constant growth path* $\omega_t(x)$ *is stationary and is denoted by* $\overline{\omega}(x)$.

Proof. The mass of projects that clear x between t and $t+\Delta$ is approximately equal to

$$\int_x^{x+\frac{\eta\Delta}{A_t}} \varphi_t(s)\ ds \approx \varphi_t(x)\frac{\eta\Delta}{A_t},$$

and the rate $\omega_t(x)$ obtains dividing by Δ and letting $\Delta \to 0$. Formally,

$$\omega_t(x) = \frac{\eta}{A_t}\varphi_t(x).$$

We have

$$\omega_t(x) = \frac{\eta\varphi_t(x)}{A_t} = \frac{\eta\varphi_t(0)}{A_t}e^{\frac{(\nu-\omega)}{\eta}x} = \omega e^{\frac{(\nu-\omega)}{\eta}x}, \tag{10}$$

where the second equality holds along a constant growth path. Now,

$$\frac{(\nu-\omega)}{\eta}x = \frac{x}{X}\frac{(\nu-\omega)}{\eta}X = \frac{x}{X}\log\frac{\nu}{\omega},$$

where the last equality follows from equation (5) which must hold along a constant growth path. Taking exponentials on both sides and substituting into (10) yields

$$\omega_t(x) = \omega\left(\frac{\nu}{\omega}\right)^{\frac{x}{X}},$$

which is equivalent to the expression in the lemma. ■

In light of this lemma, the inner integral in problem (**P**) is stationary and so, along a constant growth path, (**P**) simplifies to

$$\frac{1}{\rho}\int_0^X u(\overline{\omega}(x))\ P(dx).$$

If, moreover, we assume $u(\cdot) = \log(\cdot)$, then $(\mathbf{P})$ equals

$$\begin{aligned}
&\frac{1}{\rho}\int_0^X \log\left(\nu^{\frac{x}{X}}\ \omega^{\left(1-\frac{x}{X}\right)}\right)\ P(dx) \\
&= \frac{1}{\rho}\left[\log(\omega) + \frac{\mathbb{E}(P)}{X}\log\left(\frac{\nu}{\omega}\right)\right] \\
&= \frac{1}{\rho}\log\left(\overline{\omega}\left(\mathbb{E}(P)\right)\right). \qquad \text{(P')}
\end{aligned}$$

where $\mathbb{E}(P) = \int_0^X x\ P(dx)$ denotes the expected value of the c.d.f. P. Expression (**P'**) shows that, despite the fact that the measure P may place weight on clearing a great number of intermediate goals, when $u(\cdot) = \log(\cdot)$ the worker's problem can always be reduced to caring about clearing a single "average" goal $\mathbb{E}(P)$. This is a considerable analytical advantage.

The next proposition contains the main result of this section. It shows that, the more we care about clearing early intermediate goals, the higher the optimal input rate ν. Thus we have the basis for a theory optimally chosen input rates that might account for rates exceeding η/X.

Proposition 7 *Suppose the pair (ν, ω) is part of a constant growth path given η/X. If ν is chosen to maximize (**P'**) then ν is strictly increasing in $\mathbb{E}(P)$, and the associated ω is therefore strictly decreasing in $\mathbb{E}(P)$.*

Proof. See the Appendix. ■

This result makes intuitive sense if we think about polar cases. If $\mathbb{E}(P)$ assumes the largest possible value, namely X, then the worker only cares about the rate at which projects are opened. In this case, it makes sense for the worker to chose the largest possible ν because the negative effect on the completion rate is irrelevant. Conversely, if $\mathbb{E}(P)$ assumes the smallest possible value, namely 0, then the worker only cares about the rate at which projects are completed. In this case, it makes sense for the worker to chose the smallest ν compatible with no idleness, that is, η/X.

The analysis in this section assumes equal treatment of all open projects. The equal treatment assumption makes sense in some circumstances. If, however, it is possible to treat projects disparately then it may be optimal to do so. This is the subject of the next section.

8 Variable-Speed Work Strategies

Until now we have assumed that all projects receive the same fraction of the worker's attention and all proceed at the same speed, irrespective of how close they are to being done. In this section we relax this assumption and consider work strategies that allow the worker to focus more effort on just-arrived projects, or almost-done projects, etc. Such strategies can dominate equal treatment strategies when, as in problem (**P**) in Section 7, we care about clearing intermediate goals.

To see why, consider a situation in which all projects are identical, and value is realized only when projects are opened and/or completed. In our notation, $P(0) + P(X) = 1$. If it is possible to move projects along at different rates, then the optimal strategy is the following: all assigned projects are opened immediately, thus setting the input rate equal to α; but, as soon as projects clear X, they are placed in a queue from which projects are retrieved at rate η/X. As soon as a project is retrieved,it is worked on continuously until its completion. This "stop and go" strategy achieves an average value of $\alpha P(X) + \frac{\eta}{X} P(0)$. Since $\alpha \geq \nu$ and $\frac{\eta}{X} \geq \Omega(\nu; \eta/X)$, this average value is larger than $\nu P(X) + \Omega(\nu; \eta/X) P(0)$, which is the average rate that can be achieved with an input rate of ν and equal treatment of projects. This simple example shows that if it is valuable to clear intermediate goals then a variable-speed work strategy can dominate an equal-treatment one.

We now define a strategy which allows effort to be tailored according to a project's level of completion, and partially-completed projects to be kept in a queue.

Definition 7 *Consider a partition of* $[0, X]$ *with generic element* $I_i = (x_{i-1}, x_i)$, *where we posit* $x_0 = 0, x_i < x_{i+1}$, *and* $x_N = X$. *A* ***variable-speed work strategy*** *is a vector of pairs* $(\eta_i, \nu_i)_{i=1}^N$ *such that*

(a) $\sum_i \eta_i = \eta$, *and*

(b) $\nu_i \leq \Omega\left(\nu_{i+1}; \eta_{i+1} \frac{1}{x_{i+1} - x_i}\right)$.

The interpretation is as follows. I_i is an interval of completion levels, η_i the effort devoted to projects whose level of completion at any point in time belongs to I_i, and ν_i is the rate at which projects are allowed to transit into I_i. The worker is allowed to distribute his total effort η in any way he chooses across these intervals, which means that he is allowed to focus on projects with different completion levels. Moreover, the worker is allowed to tailor the input rate ν_i for I_i, that is, the worker chooses how fast to feed into I_i projects coming out of I_{i+1}. Once η_{i+1} and ν_{i+1} are set, the output rate out of interval I_{i+1} is given by $\Omega\left(\nu_{i+1}; \eta_{i+1} \frac{1}{x_{i+1} - x_i}\right)$. Obviously, the worker cannot feed projects into I_i any faster than they come out of I_{i+1}; this accounts for the inequality in part (b) of Definition 7. When strict inequality holds, the worker is slowing down projects coming out of interval I_{i+1} and putting them in a queue of projects waiting to enter interval I_i.

Definition 8 *An **equal treatment** work strategy is a special variable-speed strategy for which* $\nu_i = \Omega\left(\nu_{i+1}; \eta_{i+1}\frac{1}{x_{i+1}-x_i}\right)$ *for all* i.

The strategies considered in the previous sections are in fact equal treatment strategies. To see this, observe that in a strategy where all projects move to the right at the same rate $\eta\frac{1}{A_t}$ we can fix any x and think of projects that cross x (moving from right to the left) as projects that have just outputed "completion level below x" and are just being inputed into "completion rate higher than x." Obviously, these two artificial output and input rates are the same.

Since equal treatment strategies are a special case of variable-speed strategies, obviously the latter are going to be at least as good along whatever dimension we choose to measure. When is the converse true? That is, when is the "best" variable-speed strategy actually an equal treatment one? The next proposition addresses this question.

Proposition 8 *(a) For any variable-speed work strategy, there is an equal treatment strategy that yields the same output rate and requires (weakly) less effort.*

(b) Consider the profile of intermediate output rates $\overline{\omega}(x)$ *generated by an equal treatment strategy with effort* η*. The sequence of variable-speed work strategies which in the limit yields the same profile requires the same amount of effort* η *in the limit.*

(c) Fix η *and* ν*, and let* ω *be the output rate along the associated constant growth path. Suppose we want to maximize (**P**) with* $u(\cdot) = \log(\cdot)$ *and intermediate outputs being valued according to* $\overline{P}(dx) = \overline{\omega}(x)\ dx$*. Then equal treatment strategies do just as well as variable-speed work strategies.*

Proof. See the Appendix. ■

Part (a) of this proposition shows that any output rate that can be achieved by a variable-speed strategy can also be achieved, more cheaply, by an equal treatment strategy. In this sense, there is no better strategy than an equal treatment strategy.[11] This result justifies the focus on equal treatment strategies *if we only care about output rates.* Because this result holds only if we care exclusively about output rates, it does not apply to the setup of Section

[11] However, equal treatment strategies are not necessarily optimal if the objective is not only to increase the output rate, but also to clear intermediate goals. This point will be addressed at the end of Section 7.

7. That is to say, part (a) of this proposition applies in a world in which equal treatment functions with $\nu > \omega$ are necessarily suboptimal.

Parts (b) and (c) rehabilitate equal treatment strategies when $\nu > \omega$. In part (b), for any pair $(\nu, \eta/X)$ these strategies are shown to be the most efficient way to get the profile of intermediate outputs they generate. So, to the extent that we deem these strategies "inefficient" when $\nu > \eta/X$, it is only because we are evaluating their profile of intermediate outputs according to a criterion that they do not meet (perhaps because we only value final output). But if we wish to generate the intermediate output profile they generate, there is no other strategy that attains it for cheaper. In part (c) we show that equal treatment strategies with $\nu > \omega$ can be optimal within the class of variable-speed strategies, by reverse-engineering the parameters that ensure that the variable-speed strategy that maximizes (**P**) is, in fact, an equal treatment strategy.

9 Incentives and Multitasking

In this section we set aside the question of the inefficiencies associated with too large an input rate, in order to focus on a different question: how to properly incentivize the worker in a setting where the worker can strategically direct his effort to projects of different complexity, but the incentive scheme cannot condition on the complexity of the projects. Instead, the incentive scheme can only condition on "aggregate productivity."

This is a "multitasking" setup (Holmstrom and Milgrom 1991) which we believe arises frequently, for example when the worker is an "expert" vis a vis a less-expert principal, or when the complexity of the projects is not verifiable to a third party (a judge, say) and so incentive contracts based on such information cannot be enforced.

We compare the incentive effects of compensating the worker based on two different measures of aggregate productivity: the aggregate output rate, and the average duration of assigned projects. We find that compensating the worker based on the aggregate output rate leads to severe multitasking problems (in the sense of Holmstrom and Milgrom 1991) where the worker totally focuses his effort on those projects requiring the fewest steps and totally neglects to work on the other projects. In contrast, penalizing large average durations leads to a more "Rawlsian" behavior whereby the worker focuses relatively more effort on projects requiring more steps.

The model is as follows. In each instant, the worker is assigned α_1 projects that will take X_1 tasks to complete, α_2 projects that will take X_2 tasks to complete, ... , up to α_N projects that will take X_N tasks to complete. Without loss of generality we set $X_i < X_{i+1}$. In this setup the vector $\{\alpha_i, X_i\}_i$ fully describes how many projects are assigned of which length.

The worker chooses the rate ν_i at which to open projects of type i, and the effort η_i to devote to each type of project.

A worker who is rewarded based on aggregate output maximizes

$$\int_0^\infty e^{-\rho t}\left[w\sum_{i=1}^N \omega_i - c\left(\sum_{i=1}^N \eta_i\right)\right] dt, \tag{11}$$

where ρ represents the worker's discount factor, w is a parameter that captures the magnitude of the incentives, ω_i is the output rate of projects of type $i, c(\cdot)$ is a convex cost of effort, and $\sum_{j=1}^N \eta_j$ represents the total effort exerted by the worker. A worker who is penalized linearly based on the average duration of his projects maximizes

$$\int_0^\infty e^{-\rho t}\left[-w\sum_{i=1}^N \alpha_i D_{it} - c\left(\sum_{i=1}^N \eta_i\right)\right] dt, \tag{12}$$

where D_{it} represents the duration of a project of type i assigned at t.

Proposition 9 *(a) Suppose the worker is rewarded based on aggregate output rates. There there exists a threshold $\overline{X}$ such that the worker will immediately open all assigned projects requiring fewer than $\overline{X}$ steps to complete and immediately complete them. All projects requiring more than $\overline{X}$ steps are never worked on.*

(b) A worker who is penalized linearly based on the average duration of his projects will devote positive effort to all projects, even those that take many steps to complete. Moreover, $(\alpha_j)^2 X_j > (\alpha_i)^2 X_i$ implies that either $\eta_j^* > \eta_i^*$ or else $\eta_j^* = \alpha_j X_j$. In other words, ceteris paribus the worker will work more on projects requiring more steps to complete, and on those assigned at a higher rate.

Proof. See the Appendix. ■

10 Extensions and Discussion

In this section we consider several important extension of the main model. We also discuss why, when effort is allocated to projects based on some competitive or strategic mechanism, then there is reason to expect some amount of task juggling.

10.1 Non-Stationary Input and Effort Rates

In this section we relax the assumption of time-invariant input and effort rates. We analyze the admittedly special case in which input and effort rates, while not stationary, evolve at the same pace. In this case there is an easy argument that makes all our previous theory applicable almost directly. The idea is that, if input and effort rates evolve at the same pace, then we can construct an artificial time scale under which these two are time-invariant. We may call this time scale "worker time," as opposed to the conventional "calendar time." So, for example, when there is a (calendar) time interval in which input and output rate are higher than average, we allow worker time to "speed up" in this interval such that the input and effort rate *per unit of worker time* are no higher than average. Since under this artificial time scale the input and effort rates are constant, then the previous theory ensures that the output rate is constant *in worker time*. Finally, we recover the (non-stationary) output rate *in calendar time* by inverting the transformation operated by worker time. Using this technique we obtain the following proposition.

Proposition 10 *Suppose ν_t, η_t are not stationary but they vary at the same rate, that is, η_t/ν_t is stationary and equal to η_0/ν_0. Then the (non-stationary) output rate ω_t equals $\omega_0 \cdot (\nu_t/\nu_0)$, where $\omega_0 = \Omega(\nu_0, \eta_0/X)$.*

Proof. See the Appendix. ■

10.2 Heterogeneous, Equally Treated Cases

Until now we have usually assumed that all projects take the same number of tasks X to complete. Let us now consider the situation in which projects are heterogeneous in the number of tasks they take to complete. We still assume "equal treatment," however, in the sense that once opened, all projects proceed at the same speed according to equation (2). Such equal treatment will arise if, for example, the worker cannot distinguish which projects take fewer tasks to complete, as may be the case for legal cases that settle *unexpectedly* during trial. Another reason why equal treatment may prevail is that disparate treatment of projects may not be legal.

In this section we provide exact formulas that characterize how the input/output ratio varies across projects with different X's. To build some intuition for the results that follow, consider two projects which are opened at the same time: project 1 taking X_1 tasks to complete, and project 2 taking X_2. If $X_1 < X_2$, then we should expect the output rate of projects of type 1 to be larger, *relative to their input rate*, compared to projects of type 2. To get some intuition, consider the polar case in which type-1 projects take so few tasks to complete that $X_1 \approx 0$. Then $\omega_1 \approx \nu_1$, and thus the ratio of input to output rates approaches its theoretical maximum.

The model is as follows. Fix the worker's effort level at η. In each instant, the worker opens ν_1 projects that will take X_1 tasks to complete, ν_2 projects that will take X_2 tasks to complete, ... , up to ν_N projects that will take X_N tasks to complete. We allow for the possibility that $N = \infty$, in which case the set of different types of projects is countable.[12] In this setup the vector $\{\nu_i, X_i\}_i$ fully describes how many projects are opened of which length. For every open project, in the time interval between t and $t + \Delta$, the worker's work shaves off approximately

$$\frac{\eta}{A_t}\Delta,$$

where A_t represents the mass of all projects open at time t.

The next proposition provides an exact characterization of the output rates as a function of the input rates and of the characteristics of the project.

Proposition 11 *Fix η and a constellation of $\{\nu_i, X_i\}_i$. If $\eta < \sum_{i=1}^N \nu_i X_i$ then completion time is positive for all projects, and there exists a constant $K < 1$ such that $\omega_i = \nu_i \cdot K^{X_i}$ for all i.*

If $\eta > \sum_{i=1}^N \nu_i X_i$ completion time is zero for all projects and $\frac{\nu_i}{\omega_i} = 1$ for all i.

This proposition informs us about the *relative* magnitudes of the input/output ratios $\frac{\nu_i}{\omega_i}$ and $\frac{\nu_j}{\omega_j}$. According to the proposition, $X_i < X_j$ implies $\frac{\nu_i}{\omega_i} < \frac{\nu_j}{\omega_j}$, that is, projects that take more tasks to complete have a worse input/output rate ratio. Note that the constant K in the proposition is an unspecified function of effort and of the entire vector $\{\nu_i, X_i\}$. Therefore Proposition 11 should not be construed as informing us about the *level* of any particular ω_i.

10.3 Forgetful Worker

In this section we deal with the case in which, as completion time grows and any open project is worked on less and less frequently per unit of time, the worker progressively forgets about the details of each individual project. Thus, every time the worker picks up a project again, he needs to spend some additional effort to "remind himself" of where he left off before he can make progress.

[12] The analysis in this section generalizes immediately to the case in which the set of possible types of cases has the power of the continuum.

We model this phenomenon by assuming that in the time interval between t and $t+\Delta$, the worker's effort shaves off approximately $\frac{\eta}{A_t+F_t}\Delta$ steps from each active project. The factor $F_t>0$ captures a "forgetfulness penalty." We assume that F_t becomes larger over time; its exact form of will be specified later. The presence of forgetfulness requires amending equations (2) and (3) from Section 2. The two amended equations read

$$\frac{\partial\varphi_t(x)}{\partial t}-\frac{\partial\varphi_t(x)}{\partial x}\frac{\eta}{A_t+F_t}=0. \tag{13}$$

and

$$\omega_t=\frac{\eta}{A_t+F_t}\varphi_t(0). \tag{14}$$

Equations (1) and (4) remain unchanged.

Definition 9 *The quintuple $[\nu,\eta,\varphi_t(x),A_t,\omega]$ is a* ***constant growth path with forgetful worker*** *if conditions (1), (4), (13) and (14) are verified.*

Now let us specify F_t. We want to capture the notion of "time elapsed between the accomplishment of two consecutive steps," even though in our model steps are continuous and so strictly speaking there are no two consecutive steps. In our model, instead, we can think about the time that elapses between the accomplishment of given percentiles of completion, say between 20% and 30% of completion. A large completion time C_t corresponds to bigger stretches of time elapsing between the achievement of any two percentiles of completion, so we assume that the "forgetfulness penalty" F_t is proportional to the completion time C_t according to a factor of proportionality f. Formally, we assume

$$\begin{aligned}F_t&=f\cdot C_t\\&=f\,\frac{\nu-\omega}{\omega}\,t\\&=F\cdot t.\end{aligned}$$

where the second equality follows from Proposition 1, and the third is simply a definition of the real number F. Note that, by making F_t proportional to C_t, we have made F_t endogenous.

Now we spell out what a constant growth path with forgetful worker looks like. Define

$$\varphi_t^{**}(x)=\frac{(\nu-\omega)+F}{\eta}\omega\ t\ e^{\frac{(\nu-\omega)+F}{\eta}x},$$

and recall the previous definition

$$A_t^*=(\nu-\omega)\ t.$$

Theorem 3 *The quintuple* $[\nu, \eta, \varphi_t^{**}(x), A_t^*, \omega]$ *is a constant growth path with forgetful worker if and only if the triple* ν, η, ω *solves the equation* $\frac{X}{\eta}(\nu - \omega + F) = \log(\nu) - \log(\omega)$.

Proof. See the Appendix. ■

Substituting for F we get that if the triple ν, η, ω are part of a constant growth path with forgetful worker then they must satisfythe equation

$$\log(\nu) - \log(\omega) = \frac{X}{\eta}\left(\nu - \omega + f\,\frac{\nu - \omega}{\omega}\right). \tag{15}$$

This expression can be analyzed with similar methods as were used for equation (5), and the following results can be obtained.

Proposition 12 ***(a)*** *If* $\nu \leq \frac{\eta}{X}$ *then the completion time in a constant growth path with forgetful worker is zero, and hence this paths coincides with the constant growth path described in Sections 3 and 4, regardless of* f.

(b) (forgetful worker case) *If* $\nu > \frac{\eta}{X}$ *then completion time in a constant growth path with forgetful worker is positive, and the output rate is smaller than in the constant growth path described in Sections 3 and 4.*

Proof. See the Appendix. ■

Remarkably, for a given ν there may be several values of ω that satisfy equation (15). To each of these is associated a different constant growth path with forgetful worker. We now present a numerical example displaying multiple ω that solve equation (15). Rewrite equation (15) as

$$\frac{\log\left(\frac{\nu}{\omega}\right)}{\nu - \omega} = \frac{X}{\eta}\left(1 + \frac{f}{\omega}\right)$$

Now, set $\frac{X}{\eta} = 1,\ \nu = 1.1,\ f = 0.1$ and let's plot the LHS and RHS of the equation

$$\frac{\log\left(\frac{1.1}{\omega}\right)}{(1.1 - \omega)} = 1 + \frac{0.1}{\omega}.$$

The horizontal axis in Figure 5 is ω, the LHS is the solid black line and the RHS is the red dashed one. We see two intersection points in the interval $(0, \eta/X)$, corresponding to two solutions to the equation.

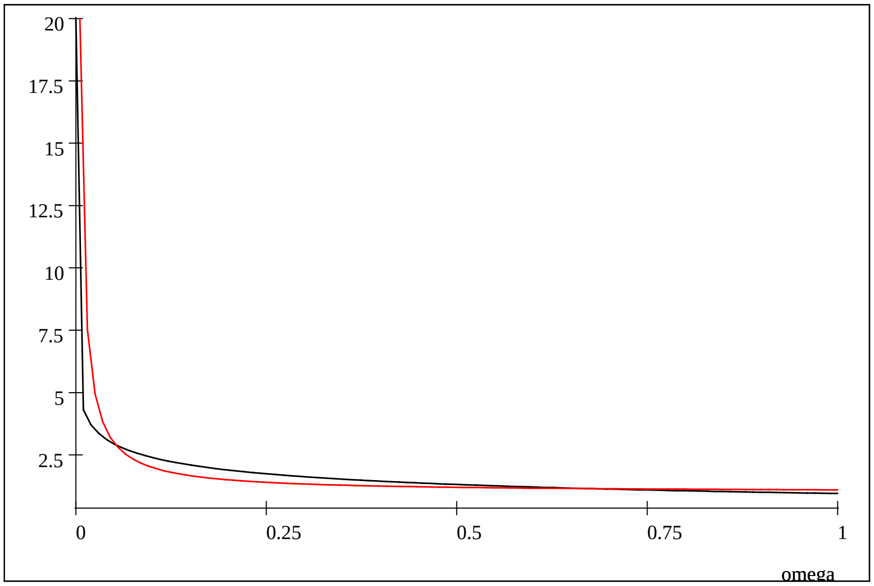

Figure 5: Multiple growth paths with forgetful worker.

10.4 Time To Build

In certain contexts, there may be technological limitations on how fast a project can be completed. For example, a judge may need to allow the lawyers time, between two successive hearings, to produce certain evidence and to evaluate and respond to the evidence produced by the adversary. For an academic researcher, before being able to produce a final draft, time may be required to absorb one's intermediate findings and "put the puzzle together". And so on. The common thread in all these these examples is that one cannot complete a project under a certain time threshold, no matter how large the effort. Let us call this threshold $\underline{T}$. We introduce the "time to build" constraint in our model via the constraint

$$\underline{C}_t = \max\{C_t, \underline{T}\},$$

where $\underline{C}_t$ denotes the completion time for a case started at t in the model with a time-to-build constraint, and $C_t = t(\nu - \omega)/\omega$ is the completion time in a model without that constraint. One way to think about this constraint is to imagine that the system evolves exactly like in the case without constraint, except that cases that were completed in less than $\underline{T}$ are "held back" and not "released" until $\underline{T}$ has elapsed from the time they were opened. Note that, if $\nu > \omega$ then C_t grows linearly with time and so after a certain time we have $\underline{C}_t = C_t$. Thus in our formulation the time-to-build constraint stops binding after a certain time because, in a constant growth path, eventually projects take long enough to complete. In particular, the completion times for all projects started after a $\widehat{t}$ such that $C_{\widehat{t}} = \underline{T}$ are unaffected by the time-to-build constraint. One can easily verify that

$$\widehat{t} = \frac{\omega}{(\nu - \omega)} \underline{T}.$$

The output rate will, after a certain time $\widehat{\widehat{t}}$, coincide with the constant-growth-path one which is ω. The value of $\widehat{\widehat{t}}$ is given by

$$\widehat{\widehat{t}} = \widehat{t} + C_{\widehat{t}} = \frac{\nu}{(\nu - \omega)} \underline{T}.$$

For $t < \underline{T}$, the output rate is zero which is the time-to-build effect. For $t \in \left[\widehat{t}, \widehat{\widehat{t}}\right]$, the output rate will be higher than ω and decreasing.

10.5 Discussion: Factors Conducive to Task Juggling

In this section we reflect, in an admittedly loose way, on the factors that might generate task juggling. We argue that, when effort is allocated among different "owners" or clients of the projects, and the owners are rationed in the sense that at least some of them have to wait for service, then the no-juggling allocation is "incentive *incompatible*" in a rather fundamental way.

The argument for the incentive-incompatibility of the no-task juggling allocation is the following. In the no-task juggling allocation, all the client's time is spent waiting for his project to be opened, but as soon as a project is opened, it is immediately completed. This means that, for any given t, there are clients which are an instant away from being done and, simultaneously, clients who need to wait D_t before they are done. Since D_t grows linearly with time whenever $\alpha > \eta/X$ (which in our model this means that clients are rationed), it follows that the incentives to deviate from the no-task juggling allocation are increasingly strong for clients who are at the back of the line, compared with clients at the head of the line. This being so, clients exhibit differences in willingness to pay for a bit of effort which, as time goes by, grow to infinity. In this (admittedly informal) sense, the no-task juggling allocation is "incentive incompatible."

In order for this "incentive incompatibility" to be acted upon, it must be possible for clients at the back of the line to "compete for effort". Therefore, the second ingredient that is conducive to task juggling is that binding intertemporal commitments to a future profile of effort for a project are not available. This ensures that at any time t all effort η_t cannot have been pre-committed in the past, and so the newly arrived clients are able to compete for effort. If, in contrast, the allocation rules are such that it is possible for clients to obtain from the worker a binding intertemporal commitment to an effort schedule over time, then it is more likely (though by no means obvious) that the no-task juggling outcome may arise as the equilibrium of a game.[13]

Summing up, when projects are owned by clients who are rationed, and effort is allocated without a commitment to an intertemporal profile of effort, then we should expect task juggling to prevail. In our opinion these conditions are rather unrestrictive. For example, they would apply to a model where workers compete for clients by offering bits of efforts.[14] We interpret this argument as saying that the assumptions required to sustain the no-task juggling allocation are restrictive. When they are not met, there is reason to expect some task juggling.

11 Conclusion

We have developed a theory of a worker who deals with overload by choosing how many projects to work on simultaneously. In this framework, we have derived an exact functional form for the "production function" of output. This functional form associates an output rate to any combination of: the rate at which the worker opens projects, the difficulty of the projects, and the amount of effort exerted by the worker. This functional form allows one, for example, to forecast how much output can be achieved with a given amount of

[13] The intertemporal effort profile associated with the no-task juggling allocation is for every assigned case to receive zero effort until the time when it receives a concentrated burst of effort and it gets done. With task juggling, instead, some cases are started "too early," which means that they receive bits of positive effort at a time when according to the no-task juggling allocation they shouldn't receive any effort at all. Task-juggling equals front-loading of the time-profile of effort.

[14] In this case, too, we should expect just-arrived clients to get at least some bit of effort, in violation of the no-juggling prescription. To flesh out this idea a little more, consider an environment in which several workers compete for an exogenously given stream of customers/projects. Competition takes place over time in a sequence of spot markets in which workers promise customers "lumps of effort" towards their project in that date. The customer chooses the worker that offers him the most effort in that date. Customers cannot commmit to exclusively deal with one worker and so are perpetually the beneficiaries of competition. Assume that there are completion-dependent "switching costs": a worker trying to steal a customer needs to pay a fixed cost to serve that customer which is proportional to the degree of project completion. In other words, a newly-won project which is already far along is more difficult/expensive for the worker than a project has just been started. This kind of progressive switching cost is natural in many environments in which the workers build project-specific knowledge which cannot (or would not) be transferred to a competitor. (Accountants and lawyers might be good examples of workers who build customer-specific knowledge.) These progressive switching costs lead to progressive customer lock-in, which implies that competition will be softer for projects which are near completion. This, in turn, will lead these projects to receive less effort relative to projects that are just started. This intertemporal front-loading of effort where new projects receive more effort than they "should" is precisely the distinctive signature of task juggling.

effort, and therefore it is of practical relevance for empirical and policy purposes. In this production function, we found that when the worker opens new projects at a rate that is too high, the number of active projects grows over time and the time it takes to complete each project also grows, while the output rate decreases. We have called this phenomenon "task juggling," because it arises from the worker dispersing a given amount of effort across too many projects.

We have asked what forces might cause the worker to juggle tasks. One such force might be lobbying by "clients" who seek to get the worker to apply effort to their project ahead of the others. We presented a model in which this lobbying leads the worker to work on "too many" projects, i.e., to juggle tasks. Furthermore, we have argued informally that task juggling would arise under a fairly general set of assumptions. Our take-away from this analysis is that we should expect some degree of task juggling in many situations.

We have also investigated a different perspective, one in which task juggling is in fact optimal. This can happen when output is not the only social goal, but there is value created by having projects achieve certain intermediate stages of completion. In this framework we have shown that the optimal input rate exceeds that which is optimal when the only goal is to maximize completion rates. This may be another reason why we would observe task juggling.

We have investigated how different incentive schemes induce the worker to focus his efforts on projects with different complexity. A worker who is rewarded based on aggregate output rates will focus all his effort on non-complex projects and will never work on complex projects. In contrast, a worker who is penalized for the average duration of the projects assigned to him will work on all projects, and focus proportionally more effort on more complex projects. Which of these two schemes produces the most desirable allocation of effort will depend on the specific application.

We view this paper and its companion (Coviello *et al.* 2010) as initial steps into the theoretical and empirical analysis of work scheduling. Although there is a strong intuition that a "juggled" work schedule will result in slower processing and lower output, measuring the quantitative effects of task juggling is far from straightforward, even at a purely theoretical level. This is because juggled work schedules come in an almost infinite range of variations, as many as the ways in which X steps of each of N projects can be ordered, a very large cardinality indeed. This paper cuts through this complexity by providing an approach in which task juggling is parameterized by one number, the rate at which new projects are opened. Because of its simplicity, this approach is amenable to empirical work. In our companion paper (Coviello et al 2010) we use this framework and a sample of italian Judges to guide an empirical analysis that estimates the causal effect of an exogenously induced increase in parallel working. We find that judges do juggle tasks, and that the slowdown in output resulting from more task juggling is large.

References

[1] Coviello, Decio, Andrea Ichino and Nicola Persico (2010) "Don't spread yourself too thin: The impact of task juggling on workers' speed of job completion." Manuscript, NYU.

[2] Dewatripont, Mathias, Ian Jewitt, and Jean Tirole (1999) "The Economics of Career Concerns, Part II: Application to Missions and Accountability of Government Agencies." *The Review of Economic Studies* 66(1) pp. 199-217.

[3] Holmstrom, Bengt and Paul Milgrom (1991) "Multitask Principal-Agent Analyses: Incentive Contracts, Asset Ownership, and Job Design." *Journal of Law, Economics, & Organization*, Vol. 7.

[4] Jackson, James R. (1963) "Jobshop-like Queueing Systems." *Management Science*, Vol. 10, No. 1 (Oct., 1963), pp. 131-142.

[5] Mark, G., D. Gudith, and U. Klocke. "The Cost of Interrupted Work: More Speed and Stress." In *CHI '08: Proceedings of the SIGCHI conference on Human factors in computing systems*, pages 107–110, Florence,Italy, April 2008. ACM Press.

[6] Moder,J, C Phillips, E Davis (1983) *Project management with CPM, PERT and project diagramming.* Van Nostrand Reinhold Publishers.

[7] Ostrom, Elinor (1990) *Governing the Commons.* Cambridge University Press.

A Proofs and Technical Results

Proof of Proposition 3

Proof. Let's start with the case $\nu < \frac{\eta}{X}$. In this case the setup of the model described in Section 2 is no longer applicable, since that setup implicitly required that $A_t > 0$, which now cannot be guaranteed. In fact, if we start at time 0 with $A_0 > 0$ and open projects at rate $\nu < \frac{\eta}{X}$, we expect $\omega_t > \nu$, and so we are on a temporary "shrink path" where over time A_t will shrink down to zero. After A_t hits zero, the worker completes projects instantaneously as soon as they are opened, and the system settles into a long-run path with $\omega_t = \nu < \frac{\eta}{X}$, and $A_t = C_t = D_t = 0$. In this long-run steady state, increasing ν increases ω contrary to Proposition 4.

In the case $\nu = \frac{\eta}{X}$, let us conjecture $\nu = \omega$ and so by (4) we have $A_t = A$. Fix any $A_0 > 0$. Note that this requires assuming an initial load of projects at time zero. Then (3) reads

$$\omega = \frac{\eta}{A_0}\varphi_t(0),$$

whence for all $t > 0$

$$\varphi_t(0) = \frac{A_0}{\eta}\omega. \tag{16}$$

Now, by definition we have that for all $x > 0$ we have $\varphi_t(0) = \varphi_\tau(x)$ for some $\tau < t$. This observation, together with (16), implies

$$\varphi_\tau(x) = \frac{A_0}{\eta}\omega \text{ for all } x, \tau.$$

Then (1) reads

$$A_0 = \int_0^X \varphi_t(x)\,dx = \frac{X}{\eta}A_0\omega.$$

Note that this equality reduces to the identity $\omega = \eta/X$, which yields no new information. This means that any A_0 is compatible with the steady state path when $\nu = \eta/X$. Whatever is the initial condition of open projects A_0, choosing $\nu = \eta/X$ will exactly perpetuate that mass of open projects.

The completion time of a newly opened project is the interval of time it takes the worker to process the A_0 projects that have precedence over it. We are looking for the time interval

C_t it takes for a worker to complete A_0 projects. At a completion rate ω, C_t solves

$$\begin{aligned} A_0 &= \int_t^{t+C_t} \omega \, ds \\ &= \omega C_t = \frac{\eta}{X} C_t, \end{aligned}$$

whence the completion time of a newly activated project is $C_t = \frac{A_0}{\eta} X$, which is increasing in A_0. Given an arrival rate α, a project assigned at t finds

$$A_0 + \alpha t - \omega t$$

projects in front of it. The duration of a project assigned at t is the time it takes to complete these projects given an output rate ω. Thus the duration of a project assigned at t is also increasing in A_0. ■

Proof of Proposition 4 a), b).

Proof. a) There are three types of solutions to the equation $h(\nu) = h(\omega)$. The first one is $\nu = \omega$. This solution is not compatible with the analysis we have carried out because then $A_t = 0$. Then there are two kinds of solutions, one where $\nu < \frac{\eta}{X} < \omega$, which is not acceptable for then $A_t < 0$. The remaining kind of solution is $\nu > \frac{\eta}{X} > \omega$. Under this restriction, the shape of $h(\cdot)$ guarantees the required property.

b) Fix ν, and consider two values $\eta > \eta'$ with associated ω and ω'. The output rates ω and ω' solve

$$\begin{aligned} h(\omega; \eta/X) &= h(\nu; \eta/X) \\ h(\omega'; \eta'/X) &= h(\nu; \eta'/X). \end{aligned}$$

Combining these equalities yields

$$h(\omega'; \eta'/X) - h(\omega; \eta/X) = h(\nu; \eta'/X) - h(\nu; \eta/X). \tag{17}$$

Now, an easy to verify property of $h(y; \eta/X)$ that, for any $y_1 < y_2$,

$$h(y_1; \eta'/X) - h(y_1; \eta/X) < h(y_2; \eta'/X) - h(y_2; \eta/X).$$

Setting $y_1 = \omega, y_2 = \nu$, and combining with (17) gives

$$\begin{aligned} h(\omega; \eta'/X) - h(\omega; \eta/X) &< h(\omega'; \eta'/X) - h(\omega; \eta/X) \\ h(\omega; \eta'/X) &< h(\omega'; \eta'/X) \end{aligned} \tag{18}$$

Now, remember that $\omega' < \eta'/X$. Then either $\omega > \eta'/X$, in which project $\omega > \omega'$ and there is nothing to prove, or else $\omega < \eta'/X$. In this project both ω and ω' lie on the decreasing portion of the function $h(\cdot;\eta'/X)$. Then equation (18) yields $\omega > \omega'$. ■

Next we prove a technical lemma that is necessary to prove Proposition 4 c).

Lemma 4 *Take any triple* $\left(\nu, \omega, \frac{\eta}{X}\right)$ *where* $\omega = \Omega(\nu; \eta/X)$. *Then* $\left|\nu - \frac{\eta}{X}\right| > \left|\omega - \frac{\eta}{X}\right|$. *That is, along a growth path the* ***actual output*** *rate is closer to the* ***efficient output*** *rate than is the input rate.*

Proof. For any $\nu > \frac{\eta}{X} > \omega$ we can write

$$h(v) = h\left(\frac{\eta}{X}\right) + \int_0^{\nu - \frac{\eta}{X}} h'\left(\frac{\eta}{X} + s\right) ds \tag{19}$$

$$h\left(\frac{\eta}{X}\right) = h(\omega) + \int_0^{\frac{\eta}{X} - \omega} h'(\omega + r)\, dr.$$

Make the change of variable $r = -\omega + \frac{\eta}{X} - s$ in the second equation, and one gets

$$\begin{aligned} h\left(\frac{\eta}{X}\right) &= h(\omega) - \int_{-\omega + \frac{\eta}{X}}^{0} h'\left(\frac{\eta}{X} - s\right) ds \\ &= h(\omega) + \int_0^{-\omega + \frac{\eta}{X}} h'\left(\frac{\eta}{X} - s\right) ds. \end{aligned}$$

Substitute into equation (19) to get

$$h(v) = h(\omega) + \int_0^{-\omega + \frac{\eta}{X}} h'\left(\frac{\eta}{X} - s\right) ds + \int_0^{\nu - \frac{\eta}{X}} h'\left(\frac{\eta}{X} + s\right) ds.$$

Since the triple $\left(\nu, \omega, \frac{\eta}{X}\right)$ solves (5), it follows that $h(v) = h(\omega)$ and so we may rewrite equation (19) once more as

$$\int_0^{\frac{\eta}{X} - \omega} -h'\left(\frac{\eta}{X} - s\right) ds = \int_0^{\nu - \frac{\eta}{X}} h'\left(\frac{\eta}{X} + s\right) ds \tag{20}$$

Now, from the proof of Lemma 1 we have $h'(y) = \frac{X}{\eta} - \frac{1}{y}$ and so

$$h'\left(\frac{\eta}{X} + s\right) = \frac{X}{\eta} - \frac{1}{\frac{\eta}{X} + s} = \frac{X}{\eta}\left(1 - \frac{1}{1 + s\frac{X}{\eta}}\right) = \frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1 + s\frac{X}{\eta}}\right)$$
$$h'\left(\frac{\eta}{X} - s\right) = \frac{X}{\eta} - \frac{1}{\frac{\eta}{X} - s} = \frac{X}{\eta}\left(1 - \frac{1}{1 - s\frac{X}{\eta}}\right) = -\frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1 - s\frac{X}{\eta}}\right)$$

for any s such that $h'\left(\frac{\eta}{X} - s\right)$ is well defined, that is, $s < \frac{\eta}{X}$. If in addition $s > 0$ then

$$h'\left(\frac{\eta}{X} + s\right) = \frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1 + s\frac{X}{\eta}}\right) < \frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1 - s\frac{X}{\eta}}\right) = -h'\left(\frac{\eta}{X} - s\right). \tag{21}$$

Now let us turn to equation (20) and let us suppose, by contradiction, that $\nu - \frac{\eta}{X} < \frac{\eta}{X} - \omega$. We may then rewrite that equation as

$$\int_0^{\frac{\eta}{X}-\omega} -h'\left(\frac{\eta}{X} - s\right) ds - \int_0^{\nu-\frac{\eta}{X}} h'\left(\frac{\eta}{X} + s\right) ds = 0$$
$$\int_0^{\nu-\frac{\eta}{X}} \left[-h'\left(\frac{\eta}{X} - s\right) - h'\left(\frac{\eta}{X} + s\right)\right] ds + \int_{\nu-\frac{\eta}{X}}^{\frac{\eta}{X}-\omega} -h'\left(\frac{\eta}{X} - s\right) ds = 0$$

The range of s in the above equation is at most $\left(0, \frac{\eta}{X} - \omega\right) \subset \left(0, \frac{\eta}{X}\right)$, and therefore (21) applies. This guarantees that the first integral is strictly positive. The second integral is strictly positive as well. Hence the equation cannot be verified. We therefore contradict our assumption that $\nu - \frac{\eta}{X} < \frac{\eta}{X} - \omega$. ■

Proof of Proposition 4 c)

Proof. Equation (5) reads

$$(\nu - \Omega(\nu; \eta/X)) = \frac{\eta}{X}\left[\log(\nu) - \log(\Omega(\nu; \eta/X))\right]. \tag{22}$$

Fix ν and differentiate both sides of (22) with respect to η to get

$$-\frac{\partial\Omega(\nu; \eta/X)}{\partial\eta} = \frac{1}{X}\left[\log(\nu) - \log(\Omega(\nu; \eta/X))\right] - \frac{\eta}{X}\frac{1}{\Omega(\nu; \eta/X)}\frac{\partial(\Omega(\nu; \eta/X))}{\partial\eta}.$$

Rearranging we get

$$\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\eta}\left[\frac{\eta}{X}\frac{1}{\Omega\left(\nu;\eta/X\right)}-1\right]=\frac{1}{X}\left[\log\left(\nu\right)-\log\left(\Omega\left(\nu;\eta/X\right)\right)\right] \tag{23}$$

$$=\frac{1}{\eta}\left(\nu-\Omega\left(\nu;\eta/X\right)\right), \tag{24}$$

where the second equation susbtitutes from (22). Now, fix η and differentiate (23) with respect to ν. This yields

$$\frac{\partial^2\Omega\left(\nu;\eta/X\right)}{\partial\eta\partial\nu}\left[\frac{\eta}{X}\frac{1}{\Omega\left(\nu;\eta/X\right)}-1\right]-\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\eta}\frac{\eta}{X}\frac{1}{\left(\Omega\left(\nu;\eta/X\right)\right)^2}\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\nu}$$
$$=\frac{1}{X}\left[\frac{1}{\nu}-\frac{1}{\Omega\left(\nu;\eta/X\right)}\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\nu}\right],$$

which can be rewritten as

$$\frac{\partial^2\Omega\left(\nu;\eta/X\right)}{\partial\eta\partial\nu}\left[\frac{\eta}{X}\frac{1}{\Omega\left(\nu;\eta/X\right)}-1\right]=\frac{1}{X}\left[\frac{1}{\nu}+\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\nu}\frac{1}{\Omega\left(\nu;\eta/X\right)}\left(\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\eta}\eta\frac{1}{\Omega\left(\nu;\eta/X\right)}-1\right)\right]. \tag{25}$$

The term in brackets on the left-hand side is positive, so $\frac{\partial^2\Omega(\nu;\eta/X)}{\partial\eta\partial\nu}$ has the same sign as the term in brackets on the right hand side of (25). We need to sign this term. To this end, substitute for $\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}$ from (24) so that the term in brackets on the right hand side of (25) reads

$$\frac{1}{\nu}+\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\nu}\frac{1}{\Omega\left(\nu;\eta/X\right)}\left[\frac{\left(\nu-\Omega\left(\nu;\eta/X\right)\right)}{\left(\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}-1\right)}\frac{1}{\Omega\left(\nu;\eta/X\right)}-1\right]$$
$$=\frac{1}{\nu}+\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\nu}\frac{1}{\Omega\left(\nu;\eta/X\right)}\left[\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega\left(\nu;\eta/X\right)}\right]. \tag{26}$$

Now, to get an expression for $\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}$, fix η and differentiate both sides of (22) with respect to ν to get

$$1-\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\nu}=\frac{\eta}{X}\left[\frac{1}{\nu}-\frac{1}{\Omega\left(\nu;\eta/X\right)}\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\nu}\right]$$
$$\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\nu}\left[\frac{\eta}{X}\frac{1}{\Omega\left(\nu;\eta/X\right)}-1\right]=\frac{\eta}{X\nu}-1$$
$$\frac{\partial\Omega\left(\nu;\eta/X\right)}{\partial\nu}=\frac{\Omega\left(\nu;\eta/X\right)}{\nu}\frac{\frac{\eta}{X}-\nu}{\frac{\eta}{X}-\Omega\left(\nu;\eta/X\right)}.$$

Substituting into (26) yields

$$\begin{aligned}
&\frac{1}{\nu}-\frac{1}{\nu}\left(\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega\left(\nu;\eta/X\right)}\right)^{2} \\
&=\frac{1}{\nu}\left[1-\left(\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega\left(\nu;\eta/X\right)}\right)^{2}\right]. \qquad (27)
\end{aligned}$$

By Lemma 4,

$$\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega\left(\nu;\eta/X\right)}>1$$

and so equation (27) is negative. Thus the right hand side of (25) is negative, which implies $\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu\partial\eta}<0$. ■

Proof of Lemma 2.

Proof. We prove that any strategy $S_\tau(\cdot)$ (typically displaying "intermittent" lobbying) is dominated either by strategy $\mathbf{0}(\cdot)$ or by strategy $\mathbf{1}(\cdot)$. Let us show this next. First, if $S_\tau(\cdot)$ is caught up, then it is dominated by the strategy $\mathbf{0}(\cdot)$ which achieves the same completion date at a lower lobbying cost. This is because after a strategy is caught up, it cannot go any faster than its assignment vintage. Suppose then that $S_\tau(\cdot)$ is not caught up.

Denote

$$\chi(t)=\int_\tau^t S_\tau(s)\ ds$$

where by construction $\chi(\cdot)$ is non-decreasing, $\chi(\tau)=0$ and $\chi(t)\le t-\tau$. The function $\chi(t)$ can be interpreted as a measure representing how much activity has occurred on the project between τ and t or, equivalently, the state of advancement of the project. When strategy S_τ is employed, the project's advancement at time t is given by

$$\begin{aligned}
x_S(t)&=X-\int_\tau^t \dot{x}_S(r)\ dr \\
&=X-\int_\tau^t \frac{\eta}{A_r}\ d\chi(r).
\end{aligned}$$

Denote by T the time at which the project is done, that is, T is the smallest value that solves $x_S(T)=0$. Create a new strategy $\widetilde{S}(t)$ which equals 1 for $t\in[\tau,\tau+\chi(T)]$ and 0 for

$t > \tau + \chi(T)$. Then we have

$$\begin{aligned} 0 &= x_S(T) \\ &= X - \int_\tau^T \frac{\eta}{A_r}\, d\chi(r) \\ &= X - \int_\tau^{\tau+\chi(T)} \frac{\eta}{A_{\chi^{-1}(y-\tau)}}\, dy \\ &\geq X - \int_\tau^{\tau+\chi(T)} \frac{\eta}{A_y}\, dy \\ &= X - \int_\tau^{\tau+\chi(T)} \frac{\eta}{A_y} \widetilde{S}_\tau(y)\; dy = x_{\widetilde{S}}(\tau + \chi(T)) \end{aligned}$$

where the third equality reflects a change of variable $y = \tau + \chi(r)$, and the inequality follows because $\chi(y) \leq y - \tau$, hence $\chi^{-1}(y-\tau) \geq y$ and $A_{\chi^{-1}(y-\tau)} \geq A_y$. The inequality shows that strategy S is just done at time T, whereas strategy $\widetilde{S}$ is more than done already by time $\tau + \chi(T) \leq T$. This means that the duration under strategy $\widetilde{S}$ is smaller than that under strategy S. . Denote by $\widetilde{T} \leq \tau + \chi(T)$ the time strategy $\widetilde{S}$ is done. Let us now turn to lobbying expenditures. Strategy S's lobbying expenditure is given by $\kappa\chi(T)$. Strategy $\widetilde{S}$'s lobbying expenditure is given by $\kappa\left(\widetilde{T} - \tau\right)$. Since $\widetilde{T} \leq \tau + \chi(T)$, strategy $\widetilde{S}$'s lobbying expenditure is smaller than strategy S's.

Summing up, we have shown that duration and lobbying expenditure are smaller under strategy $\widetilde{S}$ than under strategy S. Thus strategy $\widetilde{S}$ dominates S. Notice that, since under $\widetilde{S}$ a project ends at $\widetilde{T} \leq \tau + \chi(T)$, strategy $\widetilde{S}$ is payoff-equivalent to strategy $\mathbf{1}(\cdot)$. Thus strategy S is dominated by strategy $\mathbf{1}(\cdot)$. ■

Proof of Proposition 6

Proof. Suppose by contradiction that, as κ increases to $\widehat{\kappa}$, we have $\widehat{z} < z^*$. Then by definition we have $\nu(\widehat{z}) > \nu(z^*)$. Since by Proposition 4 c) $\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu\partial\eta} < 0$, it follows from problem (9) that $\widehat{\eta} < \eta^*$. Then $\Omega(\nu(\widehat{z}); \widehat{\eta}/X) < \Omega(\nu(z^*); \eta^*/X)$, and then the entire LHS of equation (8) becomes larger. Since the RHS stays unchanged, equation (8) can no longer be satisfied, and so we do not have an equilibrium. Therefore it must be that as κ increases to $\widehat{\kappa}$, the input rate decreases. It then follows from problem (9) that the worker's effort increases. ■

Proof of Proposition 7

Proof. Along a constant growth path (**P'**) is proportional to

$$U\left(\nu;\widetilde{P}\right)=\log\left(\Omega\left(\nu;\eta/X\right)\right)+\frac{\mathbb{E}\left(P\right)}{X}\log\left(\frac{\nu}{\Omega\left(\nu;\eta/X\right)}\right)$$

We know from Proposition 4 (a) that $\log\left(\frac{\nu}{\Omega(\nu;\eta/X)}\right)$ is strictly increasing in ν. It follows that, if $\mathbb{E}\left(\widetilde{P}\right)>\mathbb{E}\left(P\right)$, then the expression

$$U\left(\nu;\widetilde{P}\right)-U\left(\nu;P\right)$$

is an increasing function of ν. Then for any pair $\nu<\nu'$ we have

$$U\left(\nu;\widetilde{P}\right)-U\left(\nu;P\right)<U\left(\nu';\widetilde{P}\right)-U\left(\nu';P\right).$$

Rearranging yields

$$U\left(\nu';\widetilde{P}\right)-U\left(\nu;\widetilde{P}\right)>U\left(\nu';P\right)-U\left(\nu;P\right).$$

Now set $\nu'=\nu^*=\sup\left\{\arg\max U\left(\nu;P\right)\right\}$. Then the right-hand side is no smaller than zero, which implies that $U\left(\nu^*;\widetilde{P}\right)>U\left(\nu;\widetilde{P}\right)$ for any $\nu<\nu^*$. This shows that the maximizer(s) of $U\left(\cdot;\widetilde{P}\right)$ must be at least as large as ν^*. To finish the proof we need to show that the maximizer(s) of $U\left(\cdot;\widetilde{P}\right)$ are in fact strictly larger than ν^*. The fact that the function $\Omega\left(\nu\right)$ is differentiable in ν guarantees that $\partial U\left(\nu;P\right)/\partial\nu$ is zero at ν^*. But then $\partial U\left(\nu;\widetilde{P}\right)/\partial\nu$ cannot be zero at ν^* (recall that $U\left(\nu;\widetilde{P}\right)-U\left(\nu;P\right)$ is a strictly increasing function of ν). Therefore the maximizer(s) of $U\left(\cdot;\widetilde{P}\right)$ cannot include ν^* and thus they must be strictly larger than ν^*. ■

Proof of Proposition 8

Proof. (a) Suppose there is an $j>0$ such that

$$\nu_j<\Omega\left(\nu_{j+1};\eta_{j+1}\frac{1}{x_{j+1}-x_j}\right). \tag{28}$$

Then we can decrease η_{j+1} slightly without changing any other element of $\left(\eta_i,\nu_i\right)_{i=1}^N$ and obtain a new completion-dependent strategy which requires less effort than the original one, and which has the same output since ν_j and all the other variables with index j or lower are unaffected. Moreover, the new strategy satisfies part (b) of Definition 7 for all j if the decrease in η_{j+1} is small enough. Continue this process until (28) binds for all j, and an equal treatment strategy is obtained with the desired property.

(b) Fix η and ν in an equal treatment strategy, and let ω be the associated completion rate along the constant growth path. These ν and ω generate an entire profile $\overline{\omega}(x)$ according to Lemma 3. We turn now to the variable-speed strategy which is going to approximate the profile $\overline{\omega}(x)$. For each N consider the partition $\left(0, \frac{X}{N}, \frac{2X}{N}..., X\right)$. The cheapest way to generate a given output rate ω out of an interval of size $\frac{X}{N}$ is to have input rate ω and effort $\frac{X}{N}\omega$. So the cheapest way to generate intermediate output rates $\left(\omega, \overline{\omega}\left(\frac{X}{N}\right), \overline{\omega}\left(\frac{2X}{N}\right)..., \nu\right)$ is with a variable-speed strategy $(\eta_i, \nu_i)_{i=0}^{N} = \left(\frac{X}{N}\overline{\omega}\left(\frac{i}{N}X\right), \overline{\omega}\left(\frac{i}{N}X\right)\right)_{i=0}^{N}$. The effort required by this strategy is

$$\eta = \sum_{i=0}^{N} \eta_i = \sum_{i=0}^{N} \frac{X}{N}\overline{\omega}\left(\frac{i}{N}X\right).$$

Taking the limit as $N \to \infty$, this effort converges to

$$\begin{aligned}\int_0^X \overline{\omega}(x)\ dx &= \int_0^X \nu^{\frac{x}{X}} \cdot \omega^{1-\frac{x}{X}}\ dx \\ &= \omega \int_0^X \left(\frac{\nu}{\omega}\right)^{\frac{x}{X}}\ dx\end{aligned}$$

where the first equality follows from Lemma 3. Performing the change of variable $y = x/X$ yields

$$\begin{aligned}\eta &= \omega \int_0^1 \left(\frac{\nu}{\omega}\right)^{y} X\ dy \\ &= \frac{\omega X}{\log\left(\frac{\nu}{\omega}\right)} \left(\frac{\nu}{\omega}\right)^{y}\Bigg|_{y=0}^{1} \\ &= \frac{\omega X}{\log\left(\frac{\nu}{\omega}\right)} \left(\frac{\nu}{\omega} - 1\right).\end{aligned}$$

Rearranging yields $(\nu - \omega)\frac{X}{\eta} = \log(\nu) - \log(\omega)$, which is exactly equation (5). So the effort required by the limit of the sequence of cheapest variable-speed strategies is the same effort that generates ω in the equal-treatment strategy.

c) For each N consider the partition $\left(0, \frac{X}{N}, \frac{2X}{N}..., X\right)$ and the associated variable-speed strategy $(\eta_i, \nu_i)_{i=0}^N$. Given this strategy space, the maximization of problem (**P**) reads

$$\max_{(\eta_i,\nu_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left([\nu_i]^{\frac{x}{X}} \cdot \left[\Omega\left(\nu_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}} \right) \overline{P}(dx)$$
$$\text{s.t. } \nu_i \leq \Omega\left(\nu_{i+1}, N\eta_{i+1}\right) \text{ for all } i,$$
$$\sum_i \eta_i = \eta.$$

Since $\frac{N}{X}\eta_i \leq \nu_i$, the objective function is smaller than

$$\max_{(\eta_i,\nu_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left([\nu_i]^{\frac{x}{X}} \cdot \left[\Omega\left(\frac{N}{X}\eta_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}} \right) \overline{P}(dx).$$

and also the constraint is more restrictive than

$$\nu_i \leq \Omega\left(\frac{N}{X}\eta_{i+1}, \frac{N}{X}\eta_{i+1}\right) \text{ for all } i.$$

We therefore define the relaxed problem as

$$\max_{(\eta_i,\nu_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left([\nu_i]^{\frac{x}{X}} \cdot \left[\Omega\left(\frac{N}{X}\eta_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}} \right) \overline{P}(dx).$$
$$\text{s.t. } \nu_i \leq \Omega\left(\frac{N}{X}\eta_{i+1}, \frac{N}{X}\eta_{i+1}\right) \text{ for all } i.$$
$$\sum_i \eta_i = \eta.$$

In the solution to the relaxed problem the constraints on each ν_i bind and so the relaxed problem reads

$$\max_{(\eta_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left(\left[\Omega\left(\frac{N}{X}\eta_{i+1}, \frac{N}{X}\eta_{i+1}\right)\right]^{\frac{x}{X}} \cdot \left[\Omega\left(\frac{N}{X}\eta_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}} \right) \overline{P}(dx)$$
$$\text{s.t. } \sum_i \eta_i = \eta.$$

Substituting for $\Omega(y,y) = y$, the relaxed problem reads

$$\max_{(\eta_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \left[\frac{x}{X}\log\left(\frac{N}{X}\eta_{i+1}\right) + \left(1-\frac{x}{X}\right)\log\left(\frac{N}{X}\eta_i\right)\right]\overline{P}(dx)$$
$$= \max_{(\eta_i)_{i=0}^N} \sum_i \left[\log\left(\frac{N}{X}\eta_{i+1}\right)\int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \frac{x}{X}\overline{P}(dx) + \log\left(\frac{N}{X}\eta_i\right)\int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \left(1-\frac{x}{X}\right)\overline{P}(dx)\right],$$

subject to the effort constraint. Define

$$\overline{P}_i = \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \frac{x}{X}\overline{P}(dx) + \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \left(1-\frac{x}{X}\right)\overline{P}(dx)$$

We can rewrite the relaxed problem as

$$\max_{(\eta_i)_{i=0}^N} \sum_i \overline{P}_i \log\left(\frac{N}{X}\eta_i\right)$$
$$\text{s.t. } \sum_i \eta_i = \eta.$$

This is a concave problem so the solution is identified from the first order conditions of the associated Lagrangean. At the optimum these conditions imply that $\overline{P}_i \frac{1}{\eta_i^*}$ is the same for all i. Since η_i^* converges to zero as $N \to \infty$, it is convenient to write the first order conditions as

$$\overline{P}_i \frac{1}{\frac{N}{X}\eta_i^*} = \text{const for all } i.$$

As $N \to \infty, \overline{P}_i$ converges to $\overline{P}(x_i)$. Moreover, since $\frac{N}{X}\eta_i^* = \Omega\left(\frac{N}{X}\eta_i^*, \frac{N}{X}\eta_i^*\right)$, $\frac{N}{X}\eta_i^*$ is also equal to an intermediate output rate, which as $N \to \infty$ converges to the intermediate output rate $\omega^*(x_i)$ in the relaxed problem. Therefore, in the solution to the relaxed problem the intermediate output rate the first order conditions imply

$$\omega^*(x_i) = \text{const} \cdot \overline{P}(x_i) = \text{const} \cdot \overline{\omega}(x_i),$$

where the last equality follows by definition of $\overline{P}(\cdot)$. This means that in the solution to the relaxed problem, the profile of intermediate output rates is proportional to that associated by an equal treatment strategy with effort η. The budget constraint then ensures that the constant is equal to 1, and thus that the solution to the relaxed problem is in fact an equal treatment strategy with effort η. Since this strategy is obviously feasible in the original problem, the proof is done. ■

Proof of Proposition 9

Proof. (a) The square bracket in the integral of (11) is time-invariant, so it can be factored out of the integral. Then maximizing (11) is equivalent to solving

$$\max_{\{\nu_i,\eta_i\}} \; w\sum_{i=1}^{N}\left[\Omega\left(\nu_i;\eta_i/X_i\right)\right] \; - c\left(\sum_{j=1}^{N}\eta_j\right),$$

Fix any constellation of $(\eta_1,...,\eta_N)$ (not necessarily optimal). Given that constellation, the worker will optimally chose the input rate $\nu_i = \frac{\eta_i}{X_i}$, because this choice achives the maximal feasible output rate given η_i which is $\Omega\left(\eta_i/X_i;\eta_i/X_i\right) = \frac{\eta_i}{X_i}$ We can therefore rewrite the worker's problem as

$$\max_{\eta_i} \; w\sum_{i=1}^{N}\frac{\eta_i}{X_i} - c\left(\sum_{j=1}^{N}\eta_j\right).$$

The maximand is concave and so the first order conditions identify a maximum. They read

$$\frac{w}{X_i} - c'\left(\sum_{j=1}^{N}\eta_j^*\right) \leq 0 \text{ for all } i \text{ such that } \eta_i^* = 0,$$

$$\frac{w}{X_k} - c'\left(\sum_{j=1}^{N}\eta_j^*\right) = 0 \text{ for the unique } k \text{ such that } \eta_k^* \in \left(0,\alpha_k X_k\right), \text{ and}$$

$$\frac{w}{X_i} - c'\left(\sum_{j=1}^{N}\eta_j^*\right) \geq 0 \text{ for all } i \text{ such that } \eta_i^* = \alpha_i X_i.$$

These conditions imply that the worker will set η_i at its maximum on projects with the lowest X_i. All other projects are ignored.

(b) Substitute from Proposition 1 into (12) to get

$$\int_0^\infty e^{-\rho t}\left[-w\sum_{i=1}^{N}\alpha_i\left(\frac{\alpha_i}{\Omega\left(\nu_i;\eta_i/X_i\right)}-1\right)t - c\left(\sum_{i=1}^{N}\eta_i\right)\right]dt$$

$$= -\frac{w}{\rho^2}\sum_{i=1}^{N}\alpha_i\left(\frac{\alpha_i}{\Omega\left(\nu_i;\eta_i/X_i\right)}-1\right) - \frac{1}{\rho}c\left(\sum_{i=1}^{N}\eta_i\right),$$

where we have used the identities $\int_0^\infty e^{-\rho t}t\,dt = -\frac{1}{\rho}\left(t+\frac{1}{\rho}\right)e^{-\rho t}\Big|_0^\infty = 1/\rho^2$ and $\int_0^\infty e^{-\rho t}\,dt = 1/\rho$. Like in part (a), the worker will optimally chose the input rate $\nu_i = \frac{\eta_i}{X_i}$, so the maximization problem simplifies to

$$\max_{\{\eta_i\}} -\frac{w}{\rho^2}\sum_{i=1}^{N}\alpha_i\left(\frac{\alpha_i X_i}{\eta_i}-1\right)-\frac{1}{\rho}c\left(\sum_{i=1}^{N}\eta_i\right).$$

The maximand is concave in each η_i and so the first order conditions identify an interior maximum. They read

$$\frac{w}{\rho}(\alpha_i)^2 X_i\left[\frac{1}{\eta_i^*}\right]^2-c'\left(\sum_{j=1}^{N}\eta_j^*\right)\leq 0 \text{ for all } i \text{ such that } \eta_i^*=0,$$
$$\frac{w}{\rho}(\alpha_k)^2 X_k\left[\frac{1}{\eta_k^*}\right]^2-c'\left(\sum_{j=1}^{N}\eta_j^*\right)=0 \text{ for all } k \text{ such that } \eta_k^*\in(0,\alpha_k X_k)\,, \text{ and}$$
$$\frac{w}{\rho}(\alpha_i)^2 X_i\left[\frac{1}{\eta_i^*}\right]^2-c'\left(\sum_{j=1}^{N}\eta_j^*\right)\geq 0 \text{ for all } i \text{ such that } \eta_i^*=\alpha_i X_i.$$

From these conditions we see that the project $\eta_i^*=0$ is not possible for any i. Then, either $\eta_i^*=\alpha_i X_i$ for all i, or else there is an i such that η_i^* is interior, that is, $\eta_i^*<\alpha_i X_i$. If η_i^* is interior then for all j such that $(\alpha_j)^2 X_j>(\alpha_i)^2 X_i$ we have

$$\begin{aligned} c'\left(\sum_{j=1}^{N}\eta_j^*\right) &= \frac{w}{\rho}(\alpha_i)^2 X_i\left[\frac{1}{\eta_i^*}\right]^2 \\ &< \frac{w}{\rho}(\alpha_j)^2 X_j\left[\frac{1}{\eta_i^*}\right]^2 \\ &\leq \frac{w}{\rho}(\alpha_j)^2 X_j\left[\frac{1}{\eta_j^*}\right]^2, \end{aligned}$$

where the last inequality holds only for $\eta_j^*\leq\eta_i^*$. Thus if $\eta_j^*\leq\eta_i^*$ then derivative of the objective function with respect to η_j is positive at η_j^*. By the first order conditions this means $\eta_j^*=\alpha_j X_j$. Summing up, we have shown that if $(\alpha_j)^2 X_j>(\alpha_i)^2 X_i$ then either $\eta_j^*>\eta_i^*$ or else $\eta_j^*=\alpha_j X_j$. ■

Proof of Proposition 10

Proof. Define

$$\dot{\mathfrak{T}}(t)=\nu_t/\nu_0,$$

so that

$$\nu_t=\nu_0\cdot\dot{\mathfrak{T}}(t)$$

and

$$\eta_t = \frac{\eta_t}{\nu_t} \cdot \nu_t = \frac{\eta_0}{\nu_0} \cdot \dot{\mathfrak{T}}(t) \cdot \nu_0 = \eta_0 \cdot \dot{\mathfrak{T}}(t).$$

Now use the function $\dot{\mathfrak{T}}(t)$ to define a synthetic "worker time scale" where

$$\mathfrak{T}(t) = \int_0^t \dot{\mathfrak{T}}(y)\ dy$$

represents how much worker time has elapsed between calendar times zero and t. Note that since $\mathfrak{T}(0) = 0$ both clocks, the worker clock and the calendar clock, start at the same time.We want to show that ν_t, η_t are stationary in worker time (even though they are not stationary in calendar time). To show this we compute that the mass of input which accrues between worker times $\mathfrak{t}$ and $\mathfrak{t} + \mathfrak{D}$ (here $\mathfrak{D}$ represents a unit of worker time). This mass is given by the mass of input which accrues between calendar times $\mathfrak{T}^{-1}(\mathfrak{t})$ and $\mathfrak{T}^{-1}(\mathfrak{t} + \mathfrak{D})$, which is

$$\begin{aligned}\int_{\mathfrak{T}^{-1}(\mathfrak{t})}^{\mathfrak{T}^{-1}(\mathfrak{t}+\mathfrak{D})} \nu_t\ dt &= \int_{\mathfrak{T}^{-1}(\mathfrak{t})}^{\mathfrak{T}^{-1}(\mathfrak{t}+\mathfrak{D})} \nu_0 \cdot \dot{\mathfrak{T}}(t)\ dt \\ &= \nu_0 \cdot \left[\mathfrak{T}(t)\Big|_{\mathfrak{T}^{-1}(\mathfrak{t})}^{\mathfrak{T}^{-1}(\mathfrak{t}+\mathfrak{D})} \right] = \nu_0 \cdot \mathfrak{D}.\end{aligned}$$

Dividing by $\mathfrak{D}$ gives the input rate in worker time, which is constant and equal to ν_0. The same argument shows that the effort rate is constant in worker time. Now, if $\nu_\mathfrak{t}$ and $\eta_\mathfrak{t}$ are constant in worker time $\mathfrak{t}$, then the theory developed in the previous sections applies with respect to worker time and guarantees that there is an output rate $\omega_\mathfrak{t}$ which is also constant in worker time $\mathfrak{t}$. These constant input, effort, and output rate are given by ν_0, η_0 and $\omega_0 = \Omega(\nu_0, \eta_0/X)$. Since the output rate is constant in worker time, it evolves through calendar time as $\omega_0 \cdot \dot{\mathfrak{T}}(t)$. This concludes the proof. ■

Proof of Proposition 11

Proof. Think of the worker as grouping projects by type, and working on each group of projects separately. Accordingly, we denote by η_{it} the (still to be computed) amount of effort allocated to A_{it}, the mass of projects of type i at time t. By definition, $\sum_i \eta_{it} = \eta$ and $\sum_i A_{it} = A_t$. In order for this representation to be valid, the η_{it}'s must be such that all groups of projects move at the same speed, so for all i, j we must have

$$\frac{\eta_{it}}{A_{it}} \Delta = \frac{\eta_{jt}}{A_{jt}} \Delta. \tag{29}$$

We conjecture, and later verify, that there exists a unique set of time-invariant $\{\eta_{it}\}_i = \{\eta_i\}_i$ that solves this equation. In this case each group of projects follows a constant growth path, and so from Proposition 1 we have $A_{it} = (\nu_i - \omega_i)\,t$. Substituting into equation (29) yields

$$\frac{\eta_i}{(\nu_i - \omega_i)} = \frac{\eta_j}{(\nu_j - \omega_j)}. \tag{30}$$

For all i, ν_i and ω_i are linked by expression (5) and so we may replace ω_i with $\Omega\left(\nu_i; \eta_i / X_i\right)$. Since $\Omega\left(\nu; \eta / X\right)$ is increasing in η, the left- and right-hand sides of equation (30) are increasing in η_i and η_j respectively. This fact implies that there exists a unique set $\{\eta_i\}_{i=1}^N$ which solves equation (30) and simultaneously meets the constraint $\sum_i \eta_{it} = \eta$. This verifies that our conjecture was correct.

Now, recall that expression (5) reads

$$(\nu_i - \omega_i)\,\frac{X_i}{\eta_i} = \log\left(\frac{\nu_i}{\omega_i}\right).$$

Substituting into (30) yields

$$\frac{\log\left(\frac{\nu_i}{\omega_i}\right)}{X_i} = \frac{\log\left(\frac{\nu_j}{\omega_j}\right)}{X_j},$$

or equivalently

$$\left(\frac{\nu_i}{\omega_i}\right)^{X_j} = \left(\frac{\nu_j}{\omega_j}\right)^{X_i}. \tag{31}$$

Thus, if $X_i < X_j$ then $\frac{\nu_i}{\omega_i} < \frac{\nu_j}{\omega_j}$. Moreover, equation (31) is verified we replace ω_i with $\nu_i \cdot K^{X_i}$, and do the same for ω_j. This means that, given η and a constellation of $\{\nu_i, X_i\}_i$, there exists a constant K such that for all i

$$\omega_i = \nu_i \cdot K^{X_i}.$$

The constant K cannot exceed 1 for otherwise $\omega_i > \nu_i$ for all i. The constant equals 1 only if completion time is zero, which requires $\eta \geq \sum_{i=1}^N \nu_i X_i$. Otherwise, $K < 1$. ■

Proof of Theorem 3.

Proof. Condition (13) is verified because

$$\begin{aligned}
&\frac{\partial \varphi_t^{**}(x)}{\partial t} - \frac{\partial \varphi_t^{**}(x)}{\partial x}\frac{\eta}{A_t + F_t} \\
&= \frac{(\nu-\omega)+F}{\eta}\omega\ e^{\frac{(\nu-\omega)+F}{\eta}x} - \frac{(\nu-\omega)+F}{\eta}\omega\ t\ e^{\frac{(\nu-\omega)+F}{\eta}x}\frac{(\nu-\omega)+F}{\eta}\frac{\eta}{(\nu-\omega+F)\,t} \\
&= \frac{(\nu-\omega)+F}{\eta}\omega\ e^{\frac{(\nu-\omega)+F}{\eta}x} - \frac{(\nu-\omega)+F}{\eta}\omega\quad e^{\frac{(\nu-\omega)+F}{\eta}x} \\
&= 0.
\end{aligned}$$

Condition (14) is verified because

$$\begin{aligned}
\omega &= \frac{\eta}{A_t + F_t}\varphi_t^{**}(0) \\
&= \frac{\eta}{(\nu - \omega + F)\,t}\frac{(\nu-\omega)+F}{\eta}\omega\ t \\
&= \omega.
\end{aligned}$$

Condition (4) can be verified immediately.

Condition (1) reads

$$A_t^* = \int_0^X \varphi_t^*(x)\ dx.$$

Substituting for $\varphi_t^{**}(x)$ and A_t^* yields

$$\begin{aligned}
(\nu-\omega)\,t &= \int_0^X \frac{(\nu-\omega)+F}{\eta}\omega\ t\ e^{\frac{(\nu-\omega)+F}{\eta}x}\ dx \\
&= \frac{(\nu-\omega)+F}{\eta}\omega\ t \int_0^X e^{\frac{(\nu-\omega)+F}{\eta}x}\ dx \\
&= \frac{(\nu-\omega)+F}{\eta}\omega\ t\ \frac{\eta}{(\nu-\omega)+F}\ e^{\frac{(\nu-\omega)+F}{\eta}x}\Big|_{x=0}^{X} \\
&= \omega\ t\quad e^{\frac{(\nu-\omega)+F}{\eta}x}\Big|_{x=0}^{X} \\
&= \omega\ t\quad \left[e^{\frac{(\nu-\omega)+F}{\eta}X} - 1\right].
\end{aligned}$$

We can rewrite this equality as

$$\begin{aligned}
\frac{\nu}{\omega} - 1 &= \left[e^{\frac{(\nu-\omega)+F}{\eta}X} - 1\right] \\
\frac{\nu}{\omega} &= e^{\frac{(\nu-\omega)+F}{\eta}X} \\
\log(\nu) - \log(\omega) &= (\nu - \omega + F)\frac{X}{\eta}
\end{aligned}$$

■

Proof of Proposition 12

Proof. (a) In this case $C_t \equiv 0$ and then we are back to the standard case of non-forgetful worker.

(b). Fix any ν and let ω^{**} be the output rate in a constant growth path with forgetful worker. Then ω^{**} solves equation (15), which can be written as

$$h(\omega) = h(\nu) + f \cdot \frac{\nu - \omega}{\omega} \frac{X}{\eta}. \tag{32}$$

Suppose $f > 0$, and by contradiction, that the output rate in a constant growth path with non-forgetful worker, call it ω^*, is smaller than ω^{**}. Since obviously, $\omega^{**} < \nu$, we have $\omega^* \leq \omega^{**} < \nu$. By definition of ω^* we have $h(\omega^*) = h(\nu)$, and since the function h is convex, it follows that $h(\omega^{**}) \leq h(\nu)$. But then since $f > 0$, the right-hand side in (32) must exceed the left-hand side, and so the equation cannot be satisfied. We have reached a contradiction. ■