Contracting with Heterogeneous Externalities*

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Abstract

We model situations in which a principal provides incentives to a group of agents to participate in a project (such as a social event or a commercial activity). Agents' benefits from participation depend on the identity of other participating agents. We assume bilateral externalities and characterize the optimal incentive mechanism. We show that the optimal mechanisms provide a ranking of incentives for the agents, which can be described as arising from a tournament among the agents (similar to ones carried out by sports associations). Rather then simply ranking agents according to a measure of popularity, the optimal mechanism makes use of a more refined two-way comparison between the agents. Using the structure of the optimal mechanism we derive results on the principal revenue extraction and the role of the level of externalities asymmetry.

1 Introduction

The success of economic ventures often depends on the participation of a group of agents. In such environments, when an agent decides whether or not to participate she takes into account not only how many other agents are expected to participate but, more importantly, *who* is expected to participate. The focus of this paper is the implications of heterogeneous externalities in a multilateral contracting environment. This emphasis on heterogeneous externalities allows capturing a realistic

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ingredient of multilateral contracts, which are affected by the more complex relationships between the agents.

Consider first a few examples. An owner of a mall needs to convince store owners to "participate" and lease stores in the mall. Standardization agency succeeds in introducing a new standard if it manages to attract a group of firms to adopt the new standard. A firm makes acquisition offers to several owners of target firms. The acquirer's success hinges on gathering market power, which in turn depends on target firms' participation choices (agreeing to sell). Throwing a party or organizing a conference are yet other examples, the success of which depends on the participation of the invited guests.

Multilateral contracting scenarios generate externalities which are rarely symmetric. In a mall, a small store substantially gains from the presence of an anchor store (such as a national brand name), while the opposite externality, induced by the small store, has hardly any effect. The recruitment of a senior star to an academic department can easily attract a young assistant professor to apply to that department, but not the other way around. The adoption of a new standard proposed by a standardization agency induces externalities among the adopting firms but the level of benefits for a given firm crucially depends on the identity of the other adopting firms.

We explore a venture initiated by a certain party (henceforth a principal), which we refer to as *multi-agent initiative.* The success of this initiative depends on the participation of other agents, and thus the principal provides incentive contracts to induce them to participate (incentives could be discounts, gifts, or other benefits). The goal is to design these contracts optimally in view of the prevailing heterogeneous externalities between the agents. Any set of participating agents generates some revenue for the principal, and the principal attempts to maximize his revenue net of the cost of the optimal incentive scheme. Throughout the paper, instead of solving the profit maximization problem, we focus on minimizing the costs of sustaining agents' participation in the multi-agent initiative.

When evaluating the cost of sustaining agents' participation in multi-agent initiatives, we can consider separately two stages: the *selection stage*, in which the principal selects the target audience for the venture, and the *participation stage*, in which the principal introduces a set of contracts in order to induce the participation of his selected group. Clearly, these two stages are related. For example, the set of discounts offered to store owners in a mall depends on the other stores in the mall. To work out the overall solution we solve backwards by first characterizing the optimal mechanism inducing the participation of a given group and this will enable solving the selection part of the problem. We mainly focus on the solution for the second stage.

Our analysis is comprehensive in two respects. First, we allow for all types of externalities, starting with purely positive externalities, continuing with the case of negative externalities, and concluding with the case of mixed externalities, where both positive and negative externalities can coexist in the same problem. For the mixed externalities case, we provide conditions for which the solution for the positive externalities and the solution for the negative externalities cases, are both used in the mixed problem. Second, we derive the optimal mechanism for both *partial implementation*, where the principal sustains agents' participation in some Nash equilibrium, and *full implementation*, where participation is sustained via a unique Nash equilibrium. If the principal cannot coordinate the agents to play her desired equilibrium she will have to pay a premium in terms of higher payments to guarantee participation is a unique equilibrium of the underlying game. We show that this premium varies with the structure of externalities within the group of agents, and depends on the level of asymmetry between the agents.

The heterogeneous externalities among agents are described in our model by a matrix whose entry $w_i(j)$ represents the extent to which agent i is attracted to the initiative when agent j participates. An optimal mechanism is a vector of rewards (offered by the principal to the agents) that sustains full participation at minimal total cost (or maximal total extraction) to the principal. In characterizing the optimal mechanisms we will focus on three main questions: 1. What is the hierarchy of incentives across agents as a function of the externalities; i.e., who should be getting higher-powered incentives for participation? 2. How does the structure of externalities affect the principal's cost of sustaining the group's participation? 3. How does a slight change in the externality that an agent induces on the others affect his reward and the principal's benefits?

Under positive and mixed externalities the optimal contracts are determined by a virtual popularity tournament among the agents. In this tournament agent i *beats* agent j if agent j*'s* benefit from i*'s* participation is greater than i*'s* benefit from j*'s* participation in the initiative. These relations between the agents give rise to a network described by a graph. We use basic graph theory arguments to characterize the optimal mechanism and show that agents who beat a larger number of agents receive higher incentives.

The idea that agents who induce relatively stronger positive externalities receive higher incentives is supported by an empirical paper by Gould et al. (2005). They demonstrate how externalities between stores in malls affect contracts offered by the mall's owners. As in our model, stores are heterogeneous in the externalities they induce on each other. Anchor stores generate large positive externalities by attracting most of the customer traffic to the mall, and therefore increase the sales of non-anchor stores. The most noticeable characteristic of mall contracts is that most anchor stores either do not pay any rent or pay only trivial amounts. On average, anchor stores occupy over 58% of the total leasable space in the mall and yet pay only 10% of the total rent collected by the mall's owner.

A key characteristic of the structure of externalities in a certain group of agents is the level of asymmetry between the pairs of agents, which we show to reduce the principal's cost. Put differently, the principal gains whenever the attraction between any two agents is distributed more asymmetrically (less mutually). Such greater asymmetry allows the principal more leverage in exploiting the externalities to lower costs. This observation has an important implication on the principal's choice of group for the initiative in the selection stage, and also the choice of full or a partial implementation mechanism.

We show that in an asymmetric problem increasing the positive externality that some players enjoy from the participation of some other players will not necessarily increase the principal extraction of revenue in the optimal mechanism. From the perspective of the agents, their reward is *not* a continuous increasing function of the externalities they impose on the others. However, a slight change in these externalities may increase rewards significantly, since a minor change in externalities may change the optimal ranking and thus affect agents' payoffs. This has implications when agents can choose to affect externalities they impose on others.

In multi-agent participation problems one possible and intuitive solution might be to reward agents according to their measure of popularity such that the most popular agents would be rewarded the most. This follows the argument that once a popular agent agrees to participate it is easier to attract the others to join. While "popularity" can be defined in many ways, they all come down to the quality of being widely attracted by others . In our context agent i's popularity will be the sum of externalities it induces on the other agents in the group. However, we show that agents' rewards in the optimal mechanism are determined by something more refined than this standard definition of popularity. Agent i's reward depends on the set of peers that value agent i's participation more than i values theirs. This two-way comparison may result in a different set of incentives than the one imposed by standard definition of popularity.

This work is part of an extensive literature on multi-agent contracting in which externalities arise between the agents. The structure of our game, in which the principal offers a set of contracts and the agents can either accept or reject the offer, is akin to various applications introduced in the literature[1]. Most of the

[1] To give a few examples, these applications include vertical contracting models (Katz and Shapiro 1986a; Kamien, Oren, and Tauman 1992) in which the principal supplies an intermediate good, to N identical downstream firms (agents), which then produce substitute consumer goods; exclusive dealing models (Rasmusen, Ramseyer, and Wiley 1991; Segal and Whinston 2000) in which the principal is an incumbent monopolist who offers exclusive dealing contracts to N identical buyers (agents) in order to deter the entry of a rival; acquisition for monopoly models (Lewis 1983; Kamien and Zang 1990; Krishna 1993) in which the principal makes acquisition

literature assume that externalities depend on the volume of aggregate trade, and not on the identity of the agents. Our emphasis on heterogeneous externalities allows us to capture a more realistic ingredient of the multilateral contracts, which are affected by the more complex relationships between the agents. Heterogeneous externalities were used in Jehiel and Moldovanu (1996) and Jehiel, Moldovanu, and Stachetti (1996), which consider the sale of a single indivisible object by the principal to multiple heterogeneous agents using auctions, when the utilities of the agents depend on which agent ultimately receives the good.

Our general approach is closely related to the seminal papers by Segal (1999, 2003) on contracting with externalities. These papers present a generalized model for the applications mentioned above as well as others. Our paper is also related to the incentive schemes investigated by Winter (2004) in the context of organizations. While we provide a solution for partial implementation, we follow Segal (2003) and Winter (2004) in that we concentrate on situations in which the principal cannot coordinate agents on his preferred equilibrium; that is we are mainly considering contracts that sustain full implementation. Indeed, recent experimental papers (see Brandt and Cooper 2005) indicate that in an environment of positive externalities agents typically are trapped in the bad equilibrium of no-participation.

We point out that since our optimal mechanism is derived by means of a virtual tournament our results are surprisingly connected to the literature on two quite distinct topics: 1. creating a method for ranking sports teams based on tournament results, which has been discussed in the operations research literature and 2. creating a rule for ranking candidates based on the outcome of binary elections. It turns out that Condorcet's (1785) solution to the second problem as well as the method proposed by the OR literature to the first problem are closely related to our solution for the optimal mechanism.

The rest of the paper is organized as follows. In Section 2 we provide a simple two-agent example to illustrate some of the key results in the paper. We introduce the general model in Section 3 and Section 4 provides the solution for participation problem with positive externalities between the agents. In particular, we derive the ranking of incentives in the optimal mechanism by forming a virtual popularity tournament between the agents and explore how the externalities structure affect the principal's costs. In Section 5 we consider the solution of participation problems with negative externalities and show that agents must be fully compensated to sustain a full participation equilibrium. Section 6 provides a solution in the case where positive and negative externalities coexist. In Section 7 we demonstrate how this model can be used to solve selection problems. Section 8 concludes. Proofs are presented in the Appendix.

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offers to N capacity owners (agents); and network externalities models (Katz and Shapiro 1986b).

2 A Simple Two-agent Example

To illustrate some of the key ideas in this paper let's consider a simple two-agent example. Suppose a principal would like to attract agents 1 and 2 to take part in his initiative by offering agent $i \in \{1, 2\}$ a contract that pays v_i if he participates. Let's assume the agents have identical outside options in case they decline the principal's offer of $c > 0$. Furthermore, the decision to participate induces an externality on the other agent. If agent 1 is participating, agent 2's benefit (loss) is $w_2(1)$. Equivalently, if agent 2 is participating agent 1's benefit (loss) is $w_1(2)$. The agents will choose to participate if the payoff from the principal and the benefit/loss from other participating agents, taken together, is greater than the outside option.

Suppose first that the externalities $w_1(2)$ and $w_2(1)$ are strictly positive. A simple mechanism that induces the participation of both agents as a Nash equilibrium is $(v_1, v_2) = (c - w_1(2), c - w_2(1))$, in which agent 1 is offered $c - w_1(2)$ and agent 2 is offered $c - w_2(1)$. However, this mechanism is not satisfactory as it includes an additional equilibrium in which both agents are not participating. We refer to such a mechanism as a *partial implementation* mechanism. In order to sustain the participation of both agents in a unique equilibrium, it is necessary to provide at least one agent, say agent 1, his entire outside option c. Now agent 1 will participate, even if agent 2 declines. Given agent 1's participation, it is sufficient to offer agent 2 only $c - w_2(1)$ to induce his participation, as $w_2(1) > 0$. Hence the mechanism $(c, c - w_2(1))$, while more expensive than the partial implementation, induces participation in a unique equilibrium. We refer to such mechanism as a *full implementation* mechanism, and consider full implementation for the rest of the example.

Let's assume further that externalities are symmetric, hence $w_1(2) = w_2(1) > 0$. In this case, the decision of which agent will receive a higher payoff is arbitrary, as the cost of both mechanisms $(c, c - w_2(1))$ and $(c - w_1(2), c)$ is identical. Suppose now that externalities are asymmetric, say, $w_1'(2) = w_1(2) + \varepsilon$ and $w_2'(1) = w_2(1) - \varepsilon$, when $\varepsilon > 0$, so that $w_1'(2) > w_2'(1)$. Note that the sum of externalities remains unchanged. In this case, clearly, the principal would prefer to offer agent 2 a higher payoff as the payments in mechanism $(c - w_1'(2), c)$ is lower than the alternative full implementation mechanism $(c, c - w_2'(1))$. To get the cheaper full implementation mechanism, the principal exploits the fact that agent 1 favors 2 more than agent 2 favors 1, and thus gives preferential treatment to agent 2 by providing him with a higher incentive. We will later provide a general result, and demonstrate that the set of contracts that minimize the principal's cost in full implementation, is based on these bilateral relationships between the agents.

This simple example also demonstrates that the principal benefits from higher asymmetry between agents' externalities (i.e., lower mutuality). Note that the principal's optimal cost in the full implementation is $2c - w_1'(2) = 2c - w_1(2) - \varepsilon$. This observation is extended later in the paper. Moreover, we show that the cost difference between the more expensive full implementation mechanism and the partial implementation, is decreasing with the level of asymmetry. In this example, the difference between the two types of mechanisms is simply $w_2(1) - \varepsilon$. Therefore, the level of asymmetry is significant both at the agents' selection stage and at the decision of whether to use partial of full implementation mechanism.

To conclude this example, consider the case of negative externalities, i.e., $w_1(2) < 0$ and $w_2(1) < 0$. In this case, the principal has to compensate each agent for the damage caused by the participation of the other. Therefore, the optimal mechanism is simply $v_1 = c + |w_1(2)|$ and $v_2 = c + |w_2(1)|$. In the paper we provide a solution for the mixed externalities case which combines both positive and negative externalities solutions presented above.

3 The Model

A participation problem is given by a triple (N, w, c) where N is a set of n agents. The agents' decision is binary: participate in the initiative or not. The structure of externalities w is an $n \times n$ matrix specifying the bilateral externalities between the agents. An entry $w_i(j)$ represents the added value from participation in the initiative of agent i when agent j is participating. Agents gain no additional benefit from their own participation, so $w_i(i) = 0$. Agents' preferences are additively separable; i.e., agent i's utility from participating jointly with a group of agents M is $\sum_{j \in M} w_i(j)$ for every $M \subseteq N$. In the Appendix we provide an extension of the model in which agents' preferences are non-additive; i.e. externalities are defined over all subsets of agents in group N.

We assume that the externality structure w is fixed and exogenous. Also, c is the vector of the outside options of the agents. For simplicity with a slight abuse of notation, we assume that c is constant over all agents. Our results can easily be generalized to the case of heterogeneous costs.

We assume that contracts offered by the principal are simple and descriptive in the sense that the principal cannot provide payoffs that are contingent on the participation behavior of other agents. Many of the examples discussed above seem to share this feature. Evidently, in Gould et al. (2005), rental contracts in malls include a fixed rental component and an overage rent provision but exclude any contingencies on participation of other stores. Moreover, complex contracts in which payments contingent on the participation of others are also hard to verify in court, especially in environments where participation involves long-term engagement and may be carried out by different players at different points in time.

The set of contracts offered by the principal can be described as an incentive mechanism $v = (v_1, v_2, ..., v_n)$ by which agent i receives a payoff of v_i if he decides to participate and zero otherwise. v_i is not constrained in sign and the principal can either pay or charge the agents but he cannot punish agents for not participating (limited liability). Given a mechanism v agents face a normal form game $G(v)^2$. Each agent has two strategies in the game: participation or default. For a given set M of participating agents, each agent $i \in M$ earns $\sum_{j \in M} w_i(j) + v_i$ and each agent $j \notin M$ earns his outside option. In the first part of the Appendix we propose a more general model in which externalities are multilateral and not just bilateral; i.e., agents' utility from participation is not necessarily additive.

4 Positive Externalities

Positive multilateral externalities are likely to arise in many contracting situations. Network goods, opening stores in a mall and attracting customers, contributing to public goods, are a few such examples. In this section we consider initiatives in which agents benefit in various degrees from the participation of the other agents in the group. Suppose that $w_i(j) > 0$ for all $i, j \in N$, such that $i \neq j$. In this case, agents are more attracted to the initiative the larger the set of participants. We demonstrate how an agent's payment is affected by the externalities that she induces on others as well as by the externalities that others induce on her. We will also refer to how changes in the structure of externalities affect the principal's welfare.

As a first step toward characterizing the optimal full implementation mechanisms, we show in Proposition 1 that optimal mechanisms are part of a general set of mechanisms characterized by the *divide and conquer*[3] property. This set of mechanisms is constructed by ranking agents in an arbitrary fashion, and offering each agent a reward that would induce him to participate in the initiative under the belief that all the agents who precede him in the ranking participate and all the subsequent agents default. Due to positive externalities, "later" agents are induced to participate (implicitly) by the participation of others and thus can be offered smaller (explicit) incentives. More formally, the *divide and conquer* (*DAC*)

[2] We view the participation problem as a reduced form of the global optimization problem faced by the principal which involves both the selection of the optimal group for the initiative and the design of incentives. Specifically, let U be a (finite) universe of potential participants. For each $N \subseteq U$ let $v^*(N)$ be the total payment made in an optimal mechanism that sustains the participation of the set of agents N. The principal will maximize the level of net benefit she can guarantee herself which is given by the following optimization problem: $\max_{N \subseteq U}[u(N) - v^*(N)]$, where $u(N)$ is the principal's gross benefit from the participation of the set N of agents and is assumed to be strictly monotonic with respect to inclusion; i.e., if $T \subsetneq S$, then $u(T) < u(S)$.

[3] See Segal (2003) and Winter (2004) for a similarly structured optimal incentive mechanism.

mechanisms have the following structure:

$$v = (c, c - w_{i_2}(i_1), c - w_{i_3}(i_1) - w_{i_3}(i_2), ..., c - \sum_k w_{i_n}(i_k))$$

where $\varphi = (i_1, i_2, ..., i_n)$ is an arbitrary order of agents. We refer to this order as the ranking of the agents and say that v is a DAC mechanism with respect to the ranking φ. The reward for a certain agent i is increasing along with his position in the ranking; i.e., the higher agent i is located in the ranking, the higher is the payment offered by the principal.

Note that given mechanism v, agent i_1 has a dominant strategy in the game $G(v)$ to participate.[4] Given the strategy of agent i_1, agent i_2 has a dominant strategy to participate as well. Agent i_k has a dominant strategy to participate provided that agents i_1 to i_{k-1} participate as well. Therefore, mechanism v sustains full participation through an iterative elimination of dominated strategies. The following Proposition provides a necessary condition for the optimal mechanism.

Proposition 1 *If v is an optimal full implementation mechanism then it is a divide and conquer mechanism.*

Proof. Let $v = (v_{i_1}, v_{i_2}, ..., v_{i_n})$ be an optimal mechanism of the participation problem (N, w, c). Hence, v generates full participation as a unique Nash equilibrium. Since no-participation is not an equilibrium, at least a single agent, say i_1, receives a reward weakly higher than his outside option c. Otherwise, a no-participation equilibrium exists. Due to the optimality of v his payoff would be exactly c. Agent i_1 chooses to participate under any profile of other agents' decisions. Given that agent i_1 participates and an equilibrium of a single participation is not feasible, at least one other agent, say i_2, must receive a reward weakly greater than $c - w_{i_2}(i_1)$. Since v is the optimal mechanism, i_2's reward cannot exceed $c - w_{i_2}(i_1)$, and under any profile of decisions i_2 will participate. Applying this argument iteratively on the first $k - 1$ agents, at least one other agent, henceforth i_k, must get a payoff weakly higher than $c - \sum_{j=1}^{k-1} w_{i_k}(j)$, but again, since v is optimal, the payoff for agent k must be equal to $c - \sum_{j=1}^{k-1} w_{i_k}(j)$. Hence, the optimal mechanism v must satisfy the *divide and conquer* property and it is a DAC mechanism under a certain ranking φ. ■

4.1 Optimal Ranking

Our construction of the optimal mechanism for the participation problem (N, w, c) relies on Proposition 1, which shows that the optimal mechanism is a *DAC* mech-

[4] Since rewards take continuous values we assume that if an agent is indifferent then he chooses to participate.

anism. We are left to characterize the optimal ranking that yields the DAC mechanism with the lowest payment . We show that under positive externalities the optimal ranking is determined by a virtual popularity tournament among the agents, in which each agent is "challenged" by all other agents. The results of the matches between all pairs of agents are described by a *simple* and *complete*[5] directed graph $G(N, A)$, when N is the set of nodes and A is the set of arcs. N represents the agents, and $A \subset N \times N$ represents the results of the matches, which is a binary relation on N. We refer to such graphs as **tournaments**[6]. More precisely, the set of arcs in tournament $G(N, A)$ is as follows:

(1) $w_i(j) < w_j(i) \iff (i, j) \in A$

(2) $w_i(j) = w_j(i) \iff (i, j) \in A$ *and* $(j, i) \in A$

The interpretation of a directed arc (i, j) in the tournament G is that agent j values mutual participation with agent i more than agent i values mutual participation with agent j. In that case we say that agent i *beats* agent j whenever $w_i(j) < w_j(i)$. In the case of a two-sided arc, i.e., $w_i(j) = w_j(i)$, we say that agent i is *even* with agent j and the match ends in a tie.

The importance of the solution through tournaments is due to the distinction between acyclic and cyclic graphs. We say that a tournament is *cyclic* if there exists at least one node v for which there is a directed path starting and ending at v, and *acyclic* if no such path exists for all nodes.[7] As demonstrated, the solution for cyclic tournaments relies on the acyclic solution, and therefore it is a natural first step.

4.2 Optimal Ranking for Acyclic Tournaments

A ranking φ is said to be **consistent** with tournament $G(N, A)$ if for every pair $i, j \in N$ if i is ranked before j in φ, then i beats j in the tournament G. In other words, if agent i is ranked higher than agent j in a consistent ranking, then agent j values agent i more than agent i values j. We start with the following lemma:

Lemma 1 *If tournament $G(N, A)$ is acyclic, then there exists a unique ranking that is consistent with $G(N, A)$.*

We refer to the unique consistent ranking proposed in Lemma 1 as the *tournament ranking.*[8] From the consistency property, if agent i is ranked above agent j in the tournament ranking, then i beats j. Moreover, each agent's location in

[5] A directed graph $G(N, A)$ is *simple* if $(i, i) \notin A$ for every $i \in N$ and *complete* if for every $i, j \in N$ at least $(i, j) \in A$ or $(j, i) \in A$.

[6] We allow both $(i, j) \in A$ and $(j, i) \in A$.

[7] By definition, if $(i, j) \in A$ and $(j, i) \in A$, then the tournament is cyclic.

[8] The tournament ranking is actually the ordering of the nodes in the unique hamiltonian path of tournament $G(N, A)$.

the tournament ranking is determined by the number of his winnings. Hence, the agent ranked first is the agent who won all matches and the agent ranked last lost all matches. As we demonstrate later, there may be multiple solutions when tournament $G(N, A)$ is cyclic. Proposition 2 provides the solution for participation problems with acyclic tournaments, and shows that the solution is unique.

Proposition 2 *Let (N, w, c) be a participation problem for which the corresponding tournament $G(N, A)$ is acyclic. Let φ be the tournament ranking of $G(N, A)$. The optimal full implementation mechanism of (N, w, c) is given by the DAC mechanism with respect to φ.*

The intuition behind Proposition 2 is based on the notion that if agents $i, j \in N$ satisfy $w_i(j) < w_j(i)$ then the principal should exploit the fact that j favors i more than i favors j by giving preferential treatment to i (putting him higher in the ranking) and using agent i's participation to incentivize agent j. Thus, the principal is able to reduce the cost of incentives by $w_j(i)$, rather than by only $w_i(j)$. Applying this notion upon all pairs of agents minimizes the principal's total payment to the agents, since it maximizes the inherent value of the participants from the participation of the other agents.

The optimal mechanism can be viewed as follows. First the principal pays the outside option c for each one of his agents. Then the agents participate in a virtual tournament that matches each agent against all the other agents. The winner of each match is the agent who imposes a higher externality on his competitor. The loser of each match pays the principal an amount equal to the benefit that he acquires from mutually participating with his competitor. Note that if agent i is ranked higher than agent j in the tournament then it is not necessarily the case that j pays back more than i in total. The total amount paid depends on the size of bilateral externalities and not merely on the number of winning matches. However, the higher agent i is located in the tournament, the lower is the total amount paid to the principal.

An intuitive solution for the participation problem might be to reward agents according to their level of popularity in the group, such that the most popular agents would be rewarded the most. One possible interpretation of popularity in our context would be the sum of externalities imposed on others by participation, i.e., $\sum_{j=1}^{n} w_j(i)$. However, as we have seen, agents' rankings in the optimal mechanism are determined by something more refined than this standard definition of popularity. Agent i's position in our ranking depends on the set of peers that value agent i's participation more than i values theirs. This two-way comparison may result in a different ranking than the one imposed by a standard definition of popularity. This can be illustrated in the following example in which agent 3 is ranked first in the optimal mechanism despite being less "popular" than agent 1.

Example 1 *Consider a group of four agents with an identical outside option* $c = 20$. *The externalities structure of the agents is given by matrix* w, *as shown in Figure 1. The tournament* G *is acyclic and the tournament ranking is* $\varphi = (3, 1, 2, 4)$. *Consequently, the optimal mechanism is* $v = (20, 17, 14, 10)$, *which is the divide and conquer mechanism with respect to the tournament ranking. Note that agent 3 who is ranked first is not the agent who has the maximal* $\sum_{j=1}^{n} w_j(i)$.

$$w = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 4 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 5 & 2 & 0 \end{pmatrix}$$

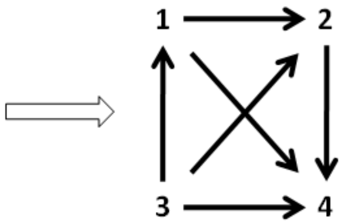

Figure 1

The derivation of the optimal mechanism requires the rather elaborate step of constructing the virtual tournament. However, it turns out that a substantially simpler method can derive the cost of the optimal mechanism for the principal. We next provide a simple formula for the principal's expense in the optimal mechanism. This formula can dispense with the computation of the actual mechanism. Two terms play a role in this formula: the first measures the aggregate level of externalities, i.e., $K_{agg} = \sum_{i\ j} w_i(j)$; the second measures the bilateral asymmetry among the agents, i.e., $K_{asym} = \sum_{i<j} |w_i(j) - w_j(i)|$. Hence, K_{asym} stands for the extent to which agents induce mutual externalities on each other. The smaller the value of K_{asym} the higher the degree of mutuality of the agents. Proposition 3 shows that the cost of the optimal mechanism is additive and declining in these two measures.

Proposition 3 *Let* (N, w, c) *be a participation problem and* V_{full} *be the principal's cost of the optimal full implementation mechanism. If the corresponding tournament* $G(N, A)$ *is acyclic then* $V_{full} = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym})$.

An interesting consequence of Proposition 3 is that for a given level of aggregate externalities, the principal's payment is decreasing with a greater level of asymmetry among the agents, as stated in Corollary 3.1. The intuition behind this result is related to the virtual tournament discussed above. In each match the principal extracts a "fine" from the losing agents. It is clear that these fines are increasing with the level of asymmetry (assuming $w_i(j) + w_j(i)$ is kept constant). Hence, a higher level of asymmetry allows the principal more leverage in exploiting the externalities. This observation may have important implications for the principal's selection stage.

Corollary 3.1 *Let (N, w, c) be a participation problem with an acyclic tournament. Let V_{full} be the principal's cost of the optimal full implementation mechanism. For a given level of aggregate externalities, V_{full} is strictly decreasing with the asymmetry level of the externalities within the group of agents.*

Consider the comparison between the optimal mechanism for full implementation and the one for partial implementation (where the principal suffices with the existence of an equilibrium, not necessarily unique). With partial implementation, the cost for the principal in the optimal mechanism is substantially lower. More specifically, in the least costly mechanism that induces full participation, each agent i receives $v_i = c - \sum_j w_i(j)$. However, this mechanism entails no-participation equilibrium as well; hence coordination is required. The total cost of the partial implementation mechanism is $V_{partial} = n \cdot c - \sum_{i\,j} w_i(j)$. In other words, under partial implementation the principal can extract the full revenue generated by the externalities. Our emphasis on full implementation is motivated by the fact that under most circumstances the principal cannot coordinate the agent to play his most-preferred equilibrium. Brandts and Cooper (2005) report experimental results that speak to this effect. Agents' skepticism about the prospects of the participation of others trap the group in the worst possible equilibrium even when the group is small. Nevertheless, one might be interested in evaluating the cost of moving from partial to full implementation. The following corollary points out that for a given level of aggregate externalities, the premium is decreasing with the level of asymmetry. Hence, the asymmetry level is an important factor in the choice between partial and full implementation mechanisms.

Corollary *3.2 Let V_{full} be the principal's cost of the optimal full implementation mechanism for the problem (N, w, c) with acyclic tournament and $V_{partial}$ the equivalent partial implementation mechanism. Then $V_{full} - V_{partial} = \frac{1}{2}(K_{agg} - K_{asym})$. For a given level of aggregate externalities, $V_{full} - V_{multiple}$ is strictly decreasing with the level of asymmetry.*

For a given level of aggregate externalities, if the asymmetry level is $K_{asym} = 0$ (equivalently, when $w_i(j) = w_j(i)$ for all pairs), then the cost of moving from partial to full implementation is the most expensive. The other extreme case is when the externalities are always one-sided, i.e., for each pair of agents $i, j \in N$ satisfies that either $w_i(j) = 0$ or $w_j(i) = 0$.[9] In this case, the additional cost is zero and full implementation is as expensive as partial implementation.

Finally, it is worthwhile to note that in an asymmetric problem increasing the positive externality that some players enjoy from the participation of some other

[9] Since this section deals with positive externalities, assume that $w_i(j) = \varepsilon$ or $w_j(i) = \varepsilon$ when ε is very small.

players will not necessarily increase the principal extraction of revenue in the optimal mechanism. In particular, in an asymmetric two-person problem raising slightly the externality that the less attractive agent induces on the other one will not change the principal revenue.[10] From the perspective of the agents, their reward is *not* a continuous increasing function of the externalities they impose on the others. However, it is possible that a slight change in these externalities may increase rewards significantly, since a minor change in externalities may change the optimal ranking and thus affect agents' payoffs.

The asymmetric case nicely contrasts with the symmetric case, where the principal's surplus increases with any slight increase of the externalities. With partial implementation, which allows the principal full extraction of surplus, the principal revenue is sensitive to the values of externalities whether the problem is symmetric or asymmetric.

4.3 Optimal Ranking of Cyclic Tournaments

In the previous section we have shown that the optimal full implementation mechanism is derived from a virtual tournament among the agents in which agent i beats agent j if $w_i(j) < w_j(i)$. However, the discussion was based on the tournament being acyclic. If the tournament is cyclic, the choice of the optimal DAC mechanism (i.e., the optimal ranking) is more delicate since Proposition 1 does not hold. Any ranking is prone to inconsistencies in the sense that there must be a pair i, j such that i is ranked above j although j beats i in the tournament. To illustrate this point, consider a three-agent example where agent i beats j, agent j beats k, and agent k beats i. The tournament is cyclic and any ranking of these agents necessarily yields inconsistencies. For example, take the ranking $\{i, j, k\}$, which yields an inconsistency involving the pair (k, i) since k beats i and i is ranked above agent k. This applies to all possible rankings of the three agents.

The inconsistent ranking problem is similar to problems in sports tournaments, which involve bilateral matches that may turn out to yield cyclic outcomes. Various sports organizations (such as the National Collegiate Athletic Association - NCAA) nevertheless provide rankings of teams/players based on the cyclic tournament outcome. Extensive literature in operations research suggests solution procedures for determining the "minimum violation ranking" (e.g., Kendall 1955, Ali et al. 1986, Cook and Kress 1990 and Coleman 2005) that selects the ranking for which the number of inconsistencies is minimized. It can be shown that this ranking is obtained as follows. Take the cyclic (directed) graph obtained by the tournament

[10] It can be shown that in an n-person asymmetric problem one can raise the externalities in half of the matrix's entries (excluding the diagonal) without affecting the principal surplus extraction.

and find the smallest set of arcs such that reversing the direction of these arcs results in an acyclic graph. The desired ranking is taken to be the consistent ranking (per Lemma 1) with respect to the resulting acyclic graph.[11]

One may argue that this procedure can be improved by assigning weights to arcs in the tournament depending on the score by which team i beats team j and then look for the acyclic graph that minimizes the total weighted inconsistency. In fact this approach goes back to Condorcet's (1785) classical voting paper in which he proposed a method for ranking multiple candidates. In the voting game, the set of nodes is the group of candidates, the arcs' directions are the results of pairwise votings, and the weights are the plurality in the votings. The solution to our problem follows the same path. In our framework arcs are not homogeneous and so they will be assigned weights determined by the difference in the bilateral externalities. As in Condorcet's voting paper, we will look for the set of arcs such that their reversal turns the graph into an acyclic one. While Young (1988) characterized Condorcet's method axiomatically, our solution results from a completely different approach, i.e., the design of optimal incentives.

Formally, we define the weight of each arc $(i, j) \in A$ by $t(i, j) = w_j(i) - w_i(j)$. Note that weights are always non-negative as an arc (i, j) refers to a situation in which j favors i more than i favors j. Hence $t(i, j)$ refers to the extent of the one-sidedness of the externalities between the pairs of agents. If an inconsistency in the ranking arises due to an arc (i, j), then this implies that agent j precedes agent i despite the fact that i beats j. Relative to consistent rankings, inconsistencies generate additional costs for the principal. More precisely, the principal has to pay an additional $t(i, j)$ when inconsistency is due to arc $(i, j) \in A$. To illustrate this point, consider a two-agent example in which agent 1 beats agent 2. In the consistent ranking $\phi_1 = \{1, 2\}$ the payment vector is $v_1 = \{c, c - w_2(1)\}$. If an inconsistency arises, i.e., the ranking is $\phi_2 = \{2, 1\}$ then the payment is $v_2 = \{c, c - w_1(2)\}$ and the principal has to pay an additional cost of $w_2(1) - w_1(2)$ since $w_1(2) < w_2(1)$. In other words, the fact that inconsistencies arise in a ranking prevents the principal from fully exploiting the externalities between the agents, as inconsistencies increase the payment relative to the consistent ranking. Therefore the principal's goal would be to select a ranking with the least costly inconsistencies.

For each subset of arcs $S = \{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)\}$ we define $t(S) = \sum_{(i,j)\in S} t(i, j)$, which is the total weight of the arcs in S. For each graph G and subset of arcs S we denote by G_{-S} the graph obtained from G by reversing the arcs in the subset S. Consider a cyclic graph G and let S^* be a subset of arcs that satisfies the following:

(1) G_{-S^*} is acyclic.

[11] Multiple rankings may result from this method.

(2) $t(S^*) \leq t(S)$ for all S such that G_{-S^*} is acyclic.

Then, G_{-S^*} is the acyclic graph obtained from G by reversing the set of arcs with the minimal total weight, and S^* is the set of pairs of agents that satisfy inconsistencies in the tournament ranking of G_{-S^*}. Proposition 4 shows that the optimal ranking of G is the tournament ranking of G_{-S^*} since the additional cost from inconsistencies, $t(S^*)$, is the lowest.

Proposition 4 *Let (N, w, c) be a participation problem with a cyclic tournament G. Let φ be the tournament ranking of G_{-S^*}. Then, the optimal full implementation mechanism is the DAC mechanism with respect to φ.*

In the following example we demonstrate how the optimal mechanism is obtained in the case of cyclic tournaments with positive externalities.

Example 2 *Consider a group of four agents each having identical outside option $c = 20$. The externality structure and the equivalent cyclic tournament are demonstrated in Figure 2. The reversal of the arcs of both subsets $S_1^* = \{(2,4)\}$, $S_2^* = \{(1,2),(3,4)\}$ provide acyclic graphs $G_{-S_1^*}$ and $G_{-S_2^*}$ with minimal weights. The corresponding tournament rankings are $\varphi_1 = (4,3,1,2)$ and $\varphi_2 = (3,2,4,1)$. Hence, the optimal mechanisms are $v_1 = (20,13,13,12)$ and $v_2 = (20,16,10,12)$.Note that the total cost for the principal, 58, is identical for these two mechanisms.*

$$w = \begin{pmatrix} 0 & 1 & 2 & 5 \\ 2 & 0 & 4 & 2 \\ 1 & 2 & 0 & 7 \\ 3 & 4 & 6 & 0 \end{pmatrix} \Longrightarrow$$

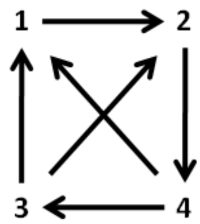

Figure 2

An interesting special case of cyclic participation problems, which is often assumed in the literature, is the one of full symmetry. A participation problem is said to be *symmetric* if $w_i(j) = w_j(i)$ for all pairs $i, j \in N$. In the symmetric case, the principal cannot exploit the externalities among the agents, as $K_{asym} = 0$, and the total payment made by the principal is identical for all rankings. This can be seen to follow from Proposition 4 as well by noting that the tournament has two-way arcs connecting all pairs of agents, and $t(i,j) = 0$ for all i,j and $t(S)$ is uniformly zero. An intriguing feature of the symmetric case is that all optimal mechanisms are discriminative in spite of the fact that all agents are identical.

Corollary 4.1 *When the externality structure w is symmetric then all DAC mechanisms are optimal.*

We can now provide the analogue version of Proposition 3 for the cyclic case. In this case, the optimal ranking has an additional term $K_{cyclic} = t(S^*)$ representing the cost of making the tournament acyclic, i.e., the cost the principal needs to bear due to the inconsistencies.

Proposition 5 *Let (N, w, c) be a participation problem. Let V_{full} be the principal's optimal cost of a full implementation mechanism inducing participation. Then $V_{full} = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym}) + K_{cyclic}$.*

Corollary 3.1 still holds for pairs of agents that are not in S^*. More specifically, if we increase the level of asymmetry between pairs of agents that are outside of S^*, we reduce the total expenses that the principal incurs in the optimal mechanism.

5 Negative Externalities

So far we have limited our discussion to environments in which an agent's participation positively affects the willingness of other agents to participate. We now turn to the case in which externalities are all negative. In Section 6 we discuss a more general case that incorporates mixed externalities.

Environments of negative externalities are those of congestions. Traffic, market entry, and competition among applicants all share the property that the more agents participate, the lower the utility of each participant is. The heterogenous property in our framework seems quite descriptive in some of these examples. In the context of competition it is clear that a more competitive candidate/firm induces a larger externality (in absolute value) than a less competitive one. It is also reasonable to assume, at least for some of these environments, that the principal desires a large number of participants in spite of the negative externalities that they induce on each other.

We show that in order to sustain full participation as a unique Nash equilibrium under negative externalities the principal has to fully compensate all agents for the participation of the others. As we have seen, positive externalities allow the principal to exploit the participation of some agents in order to incentivize others. With negative externalities this is not the case since agents' incentives to participate decline with the participation of others. Hence it remains for the principal simply to reimburse the agents for the disutility arising from the participation of the others.

Proposition 6 *Let* (N, w, c) *be a participation problem with negative externalities. Then an optimal full implementation mechanism* v *sustains full participation is given by* $v_i = c + \sum_{i \neq j} |w_i(j)|$ *, and* v *is unique.*

We can solve the negative externalities problem using the results of the previous section. By providing each agent with an initial compensation equals to the sum of negative externalities to which he is exposed, we receive a new participation problem in which all externalities are zero (symmetric externalities structure). By Corollary 4.1, all rankings with respect to the DAC mechanism of the new problem are optimal, and thus adding these incentives to the initial compensation yields the optimal incentive scheme.

6 Mixed Externalities

We next consider participation problems with mixed externalities; i.e., $w_i(j)$ may get both positive and negative values in the same participation problem. Naturally, real-world multi-agent initiatives may capture both types of externalities. In social events, individuals may highly benefit from some of the invited guests, while preferring to avoid others. In a mall, the entry of a new store will benefit some stores as it attracts more customers, but impose negative externalities on its competitors.

Our analysis of the mixed externalities case is based on the following binary relation. We say that agent i is *non-averse* to agent j if $w_i(j) \geq 0$, and we write it as $i \succeq j$. We will assume that $\succeq$ is symmetric and transitive, i.e., $i \succeq j \implies j \succeq i$ and if $i \succeq j$ and $j \succeq k$ then $i \succeq k$. Note that this assumption does not imply any constraint on the magnitude of the externalities, but just on their sign. In particular, it imposes that j gains a weakly positive externality from i if i gains a weakly positive externality from j. Also, j gains a weakly positive externality from k if there are weakly positive externalities between i and k. While the symmetry and transitivity of the *non-averse* relation seem rather intuitive assumptions, not all strategic environments satisfy them. These assumptions are particularly relevant to environments where the selected population is partitioned into social, ethnic, or political groups with animosity potentially occurring only between groups but not within groups. We analyze a specific example of this sort of environment in the next section.

It turns out that the optimal solution of participation problems with symmetry and transitivity of the *non-averse* relation is derived by a decomposition of the participation problem into two separate participation problems: one that involves only positive externalities, and the other that involves only negative externalities. This is done by simply decomposing the externalities matrix into a negative and a positive matrix. In the following Proposition we show that the *decomposition mechanism,* a mechanism which is the sum of the two optimal mechanisms of the two decomposed participation problems, is the optimal mechanism for the mixed externalities participation problem.

Proposition 7 *Consider a participation problem* (N, w, c) *. Let* (N, w^+, c) *be a participation problem such that* $w_i^+(j) = w_i(j)$ *if* $w_i(j) \geq 0$ *and* $w_i^+(j) = 0$ *if* $w_i(j) < 0$*, and let* u^+ *be the optimal full implementation mechanism of* (N, w^+, c)*. Let* $(N, w^-, 0)$ *be a participation problem such that* $w_i^-(j) = w_i(j)$ *if* $w_i(j) < 0$ *and* $w_i^-(j) = 0$ *if* $w_i(j) \geq 0$*, and let* u^- *be the optimal full implementation mechanism of* $(N, w^-, 0)$*. Then, the decomposition mechanism* $v = u^+ + u^-$ *is an incentive-inducing mechanism. Moreover, if agents satisfy symmetry and transitivity with respect to the non-averse relation,* v *is the optimal mechanism.*

Proposition 7 shows that the virtual popularity tournament discussed in earlier sections plays a central role also in the mixed externalities case as it determines payoffs for the positive component of the problem. When symmetry and transitivity hold, the principal can exploit the positive externalities to reduce payments. In this tournament i beats j whenever (1) $w_j(i) \geq 0$ and $w_j(i) \geq 0$, and (2) $w_j(i) > w_i(j)$. Note that under the *non-averse* assumptions, the principal provides complete compensation for the agents who suffer from negative externalities, as with the negative externalities case. We illustrate the solution in the following example.

Example 3 *Consider a group of four agents each having identical outside option* $c = 20$*. The externality structure of the agents is demonstrated by matrix* w*, as shown in Figure 3. The directed graph of the decomposed positive participation problem* (N, w^+, c) *is also noted in Figure 3. This yields multiple optimal ranking. More specifically, the optimal ranking for the positive participation problem is any ranking in which agent 2 precedes agent 1, and agent 4 precedes agent 3. Pick ranking* $\varphi = (2, 1, 4, 3)$*. The corresponding optimal mechanism of the positive participation problem is* $u^+ = (20, 18, 20, 17)$*. Note that* $S^* = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$*. The optimal mechanism for the mixed externalities problem is the decomposition mechanism* $v = u^+ + u^- = (25, 23, 27, 21)$*.*

$$w = \begin{pmatrix} 0 & 2 & -2 & -3 \\ 1 & 0 & -1 & -4 \\ -3 & -2 & 0 & 4 \\ -6 & -1 & 3 & 0 \end{pmatrix} \Longrightarrow$$

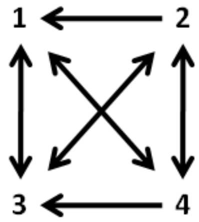

Figure 3

We conclude this section by deriving the analogous result to Propositions 3 and 5 in the case of mixed externalities. Since the principal provides complete compensation for agents who suffer from negative externalities, define $K_{neg} = \sum_{i,j} \left|w_i^-(j)\right|$. We show that the principal's cost of incentivizing his agents is decomposed in pretty much the same way as in the positive externalities case, except that now the principal has to add the compensation for the negative externalities. Specifically:

Proposition 8 *Let (N, w, c) be a mixed participation problem and V_{full} be the payment of the optimal full implementation mechanism v. Let K_{agg}, K_{asym}, and K_{cyclic} be the characteristics of the amended participation problem (N, w^+, c), and K_{neg} be the characteristic of the participation problem $(N, w^-, 0)$. Then, if agents satisfy symmetry and transitivity with respect to the non-averse relation, $V_{full} = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym}) + K_{cyclic} + K_{neg}$.*

Proposition 8 follows trivially from Propositions 5, 6 and 7.

7 Group Identity and Selection

In this section we consider special externalities structures to demonstrate how the selection stage can be incorporated, once we have solved the participation problem. Assume that the externalities take values of 0 or 1. We interpret it as an environment in which an agent either benefits from the participation of his peer or gains no benefit. We provide three examples of group identities in which the society is partitioned into two groups and agents have hedonic preferences over members in these groups. We demonstrate how the optimal mechanism proposed in previous sections may affect the selection of the agents in the planning of the initiative.

(1) **Segregation** - agents benefit from participating with their own group's members and enjoy no benefit from participating with members from the

other group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 1$. Otherwise, $w_i(j) = 0$.

(2) **Desegregation**[12] - agents benefit from participating with the other group's members and enjoy no benefit from participating with members of their own group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 0$. Otherwise, $w_i(j) = 1$.

(3) **Status** - the society is partitioned into two status groups, high and low. Each member of the society benefits from participating with each member of the high-status group and enjoys no benefit from participating with members of the low-status group. Formally, let B_1 be the high status group and set $w_i(j) = 1$ if and only if $j \in B_1$ (otherwise $w_i(j) = 0$).

Proposition 9 *Let (N, w, c) be a participation problem. Let n_1 and n_2 be the number of agents selected from groups B_1 and B_2 respectively such that $n_1+n_2 = n$. Denote by $v(n_1, n_2)$ the principal cost of incentivizing agents under the optimal mechanism given that the group composition is n_1 and n_2. The following holds:*
1) under Segregation $v(n_1, n_2)$ is decreasing with $\mid n_1 - n_2 \mid$.
2) under Desegregation $v(n_1, n_2)$ is increasing with $\mid n_1 - n_2 \mid$.
3) under Status $v(n_1, n_2)$ is decreasing with n_1.

In the case of Segregation, the principal's cost of incentives is increasing with the mixture of groups; hence in the selection stage the principal would prefer to give precedence to one group over the other. In the Desegregation case the principal's cost is declining with mixture; hence in the selection stage the principal would like to balance between members of the groups. In the Status case the cost is declining with the number of agents recruited from B_1, which will be strongly preferred to members from B_2.

8 Conclusion

In this paper we analyzed a multi-agent contracting framework in which externalities are heterogeneous. Introducing a complicated structure of heterogeneous externalities allowed us to explore a few aspects of the multi-agent contracting environments that are not apparent in the homogeneous case. These include the impact of externalities asymmetry on payments, the implications of externalities

[12] An example could be a singles party.

structure on the hierarchy of incentives, and the effect of variations in structures of externalities on the principal's payments and agents' rewards.

More specifically, greater asymmetry between the agents' benefits reduced the principal's payment in the full implementation problem. This is an important implication for the selection stage of the initiative. In addition, externalities asymmetry turns out to play a role also in the selection between partial and full implementation, as it affects the premium required to sustain full participation as a unique equilibrium. Greater asymmetry decreases this premium, and thus makes full implementation more likely.

The hierarchy of incentives in the positive and mixed externalities case is determined by a ranking that results from a virtual popularity tournament. In the simplest case, an agent i is ranked above agent j if agent i benefits less from the joint participation with agent j than agent j's benefit from agent i. We demonstrated that this ranking of incentives is different from the standard ranking that is based on agents' popularity.

We provided a few comparative statics of changes in externalities structures. In an asymmetric participation problem increasing the positive externalities that some players enjoy from the participation of some other players will not necessarily increase the principal extraction of revenue in the optimal mechanism. This is an important consideration also in the stage of forming the group of agents. In addition, we show that from the agents' perspective a slight change in externalities can lead to substantial impact on the rewards due to changes in ranking. Hence, there is a discontinuity in the principal's payment to induce participation.

This discontinuity in rewards may suggest a preliminary game in which agents invest effort to increase the positive externalities that they induce on others. For example, agents can invest in their social skills to make themselves more attractive guests to social events. A firm may invest to increase its market share in order to improve its ranking position in an acquisition game. Under certain circumstances such an investment may turn out to be quite attractive as we have seen that a slight change in externalities may result in a substantial gain, due to a change in the ranking. The preliminary game on externalities can be thought of as a network formation game similar to the ones discussed in the network formation literature (see Jackson 2003 for a comprehensive survey).

Specifically, consider a selection[13] of an optimal mechanism function that maps each matrix of externalities to a payoff vector $\Gamma : w \rightarrow \pi$ (payoffs for agents include both the transfer from the principal as well as the intrinsic benefits from participation). One can think of the matrix of externalities as a generalized network in the sense that it specifies the intensity[14] of arcs, in contrast to standard networks

[13] We refer to selection because the optimal mechanism may not be unique.

[14] For such models, see Calvo, Lasaga, and van den Nouweland (1999), Calvo-Armengol and

which only specify whether a link exists. If we assume that agents can increase bilateral externalities according to a given cost function then the externalities become endogenous in the model. The new game will now have two stages. The first stage is a network formation game (which determines the externalities) and the second stage is the participation game. The analysis of such a game is beyond the scope of this paper but seems to be a natural next step.

Jackson (2001, 2001b), Goyal and Moraga (2001), and Page, Wooders, and Kamat (2001).

References

[1] Brandts, J. and D. Cooper (2006) "A Change Would Do You Good... An Experimental Study on How to Overcome Coordination Failure in Organizations," *American Economic Review* 96, 669-693.

[2] Calvo, E., J. Lasaga, and A. van den Nouweland (1999) "Values of Games with Probabilistic Graphs," *Mathematical Social Sciences* 37, 79-95.

[3] Calvò, A. and M.O. Jackson (2004) "The Effect of Social Networks on Employment and Inequality," *American Economic Review* 94, 426-454.

[4] Condorcet, Marquis de (1785) *Essai sur l'application de l'analyse a la probabilite des decisions rendues a la probabilite des voix,* Paris: De l'imprimerie royale.

[5] Gould, E., P. Pashigian, and C. Prendergast (2005) "Contracts, Externalities, and Incentives in Shopping Malls," *Review of Economics and Statistics* 87, 411-422

[6] Goyal, S. and J. L. Moraga (2001) "R & D Networks," *RAND Journal of Economics* 32, 686-707.

[7] Jackson, M.O. (2004) "A Survey of Models of Network Formation: Stability and Efficiency," in *Group Formation in Economics; Networks, Clubs and Coalitions*, edited by G. Demange and M. Wooders, Cambridge University Press.

[8] Jehiel, P., B. Moldovanu, and E. Stachetti (1996) "How (Not) to Sell Nuclear Weapons," *American Economic Review* 86, 814-829.

[9] Jehiel, P. and B. Moldovanu (1996) "Strategic Nonparticipation," *RAND Journal of Economics* 27, 84-98.

[10] Kamien, M., S. Oren, and Y. Tauman (1992) "Optimal Licensing of Cost-Reducing Innovation," *Journal of Mathematical Economics* 31, 483-508.

[11] Kamien, M. and I. Zang (1990) "The Limits of Monopolization through Acquisition," *Quarterly Journal of Economics* 105, 465-499.

[12] Katz, M. and C. Shapiro (1986a) "How to License Intangible Property," *Quarterly Journal of Economics* 101, 567-589.

[13] Katz, M. and C. Shapiro (1986b) "Technology Adoption in the Presence of Network Externalities," *Journal of Political Economy* 94, 822-841.

[14] Kendall, M. (1962), *Rank Correlation Methods,* 3rd ed., Hafner: New York.

[15] Krishna, K. (1993) "Auctions with Endogenous Valuations: The Persistence of Monopoly Revisited," *American Economic Review* 83, 147-160.

[16] Lewis, T. (1983) "Preemption, Divestiture, and Forward Contracting in a Market Dominated by a Single Firm," *American Economic Review* 73, 1092-1101.

[17] Mutuswami, S. and E. Winter (2002) "Subscription Mechanisms for Network Formation," *Journal of Economic Theory* 106, 242-264.

[18] Page, F., M. Wooders, and S. Kamat (2001) "Networks and Farsighted Stability," Discussion Paper, University of Warwick.

[19] Rasmusen, E., M. Ramseyer, and J. Wiley (1991) "Naked Exclusion," *American Economic Review* 81, 1137-1145.

[20] Segal I. (1999) "Contracting with Externalities," *Quarterly Journal of Economics* 114, 337-388.

[21] Segal I. and Whinston M. (2000) "Naked Exclusion: Comment," *American Economic Review* 90, 296-309.

[22] Segal, I. (2003) "Coordination and Discrimination in Contracting with Externalities: Divide and Conquer?" *Journal of Economic Theory* 113, 147-327.

[23] Winter, E. (2004) "Incentives and Discrimination," *American Economic Review* 94, 764-773.

[24] Winter, E. (2006) "Optimal Incentives for Sequential Production," *RAND Journal of Economics* 37, 376-390.

9 Appendix

9.1 Non additive Preferences

We propose here an extension of the model in which agents' preferences are non-additive. A participation problem is described by a group of agents N, and an outside options vector c as noted previously. We assume a general externality structure, which is given by non-additive preferences of the agents over all subsets of agents in the group N. More specifically, for each i, $u_i : 2^{N\backslash\{i\}} \longrightarrow R$. The function $u_i(S)$ stands for the benefit of agent i from the participation with the subset $S \subseteq N$. We normalize $u(\emptyset) = 0$. The condition of positive externalities reads now: for each i and subsets S, T such that $T \subset S$ we have $u_i(S) \geq u_i(T)$.

Arguments similar to those used in Proposition 1, show that the optimal mechanism that sustains full participation as a unique equilibrium is also a divide and conquer mechanism. Hence, to construct the optimal mechanism we need to construct the optimal ranking of the agents.

Consider a three agents example, with the following order $\phi = \{i_1, i_2, i_3\}$. The payoff vector in a DAC mechanism with ranking ϕ is $\{c, c - u_{i_2}(i_1), c - u_{i_3}(i_1, i_2)\}$. Hence, the optimal order would maximize the intrinsic value of participation of other agents, $u_{i_2}(i_1) + u_{i_3}(i_1, i_2)$. More generally, the principal has to choose ϕ to solve the following optimization problem:

$$\max_{\phi} \sum_{j=2}^{n} u_{i_j}(i_1, ..., i_{j-1})$$

We say that agent i beats j if for all $S \subset N$ such that $i, j \notin S$ we have $u_i(S \bigcup j) - u_i(S) < u_j(S \bigcup i) - u_j(S)$[15]. Intuitively, i beats j if i's marginal contribution to the utility of j is greater than j's marginal contribution to the utility of i, regardless of subset S at which marginal contributions are being calculated. Assuming this binary relation to be complete (and not necessarily transitive) enables us to construct a complete directed graph $G(N, A)$ when N is the set of nodes (which represent the agents), and A is the set of arcs that are defined in the following way: If agent i beats j then $(i, j) \in A$. The following result is based on similar arguments to those used in Proposition 2:

Proposition 10 *Let (N, c) be a participation problem with non-additive preferences, for which the corresponding directed graph $G(N, A)$ is complete and acyclic. Let φ be the tournament ranking of $G(N, A)$. The optimal full implementation mechanism of (N, c) is given by the DAC mechanism with respect to φ.*

[15] With $S = \emptyset$ we get the condition we had with the additively separarable preferences.

The proof is provided in the next part of the Appendix. The framework presented here is more general than the separable additive preferences in that the marginal contribution of agent i to the utility of agent j is not constant as assumed in the additive separable case, but depends on the set of other agents who participate in the initiative. Nevertheless, the general structure of the solution remains unchanged.

9.2 Proofs

Proof of Lemma 1 We will demonstrate that there is a single node with $n-1$ outgoing arcs. Since the tournament is a complete, directed, and acyclic graph there cannot be two such nodes. If such a node does not exist, then all nodes in G have both incoming and outgoing arcs. Since the number of nodes is finite, we get a contradiction for G being acyclic. We denote this node as i_1 and place its corresponding agent first in the ranking (hence this agent beats all other agents). Now let us consider a subgraph $G(N^1, A^1)$ which results from the removal of node i_1 and its corresponding arcs. Graph $G(N^1, A^1)$ is directed, acyclic, and complete and, therefore, following the previous argument, has a single node that has exactly $n-2$ outgoing arcs. We denote this node as i_2, and place its corresponding agent at the second place in the ranking. Note that agent i_1 beats agent i_2 and therefore the ranking is consistent so far. After the removal of node i_2 and its arcs we get subgraph $G(N^2, A^2)$ and consequently node i_3 is the single node that has $n-3$ outgoing arcs in subgraph $G(N^2, A^2)$. Following this construction, we can easily observe that the ranking $\varphi = (i_1, i_2, ..., i_n)$ is consistent among all pairs of agents and due to its construction also unique. ■

Proof of Proposition 2 According to Proposition 1 the optimal mechanism is a DAC mechanism. Hence the optimal mechanism is derived from constructing the optimal ranking and is equivalent to minimizing the sum of incentives, V_{full} :

$$\begin{aligned} V_{full} &= \min_{(j_1, j_2, ..., j_n)} \left[n \cdot c - \left\{ \sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k) \right\} \right] \\ &= \max_{(j_1, j_2, ..., j_n)} \left[\sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k) \right] \end{aligned}$$

Since no externalities are imposed on nonparticipants, the outside options of the agents have no role in the determination of the optimal mechanism. We will show that the ranking that solves the maximization problem of the principal is the tournament ranking. Let us assume, without loss of generality, that the tournament ranking φ is the identity permutation: hence $\varphi(i) = i$, and

$W_\varphi = \sum_{k=1}^{2} w_2(k) + ... + \sum_{k=1}^{n} w_n(k)$, when W_φ is the principal's revenue extraction. By contradiction, assume that there exists $\varphi \neq \sigma$ such that $W_\varphi \leq W_\sigma$. First, assume that σ is obtained from having two *adjacent* agents i and j in φ trade places such that i precedes j in φ and j precedes i in σ. By Lemma 1, agent i beats agent j; thus $W_\sigma = W_\varphi - w_j(i) + w_i(j)$ and $W_\sigma < W_\varphi$.

Note that since φ is the tournament ranking, agent 1 beats all agents, agent 2 beats all agents but agent 1, and so on. Now consider unconstrained $\sigma = \{i_1, ..., i_n\}$ such that $\varphi \neq \sigma$. If agent 1 is not located first, by a sequence of adjacent swaps $(1, i_j)$, we move agent 1 to the top of the ranking. In each of the substitutions agent 1 beats i_j. Next, if agent 2 is not located at the second place, by a sequence of adjacent substitutions $(2, i_j)$, we move agent 2 to the second place. Again, agent 2 beats all agents i_j. The process ends in at most n stages and produces the desired order φ. As demonstrated, any adjacent substitution results in a higher extraction, and therefore $W_\sigma < W_\varphi$. Therefore, the DAC mechanism with respect to the tournament ranking is unique and optimal. ■

Proof of Proposition 3 Without loss of generality, assume that the tournament ranking φ is the identity permutation. Hence, under the optimal mechanism, the principal's payment is $V_{full} = n \cdot c - \left[\sum_{j=1}^{1} w_1(j) + ... + \sum_{j=1}^{n} w_n(j)\right]$. Denote $s_i(j) = [w_i(j) + w_j(i)]$ and $a_i(j) = [w_i(j) - w_j(i)]$. We can represent K_{agg} and K_{asym} in the following manner: $K_{agg} = \sum\limits_{i\ j} w_i(j) = \sum\limits_{i<\ j} (w_i(j) + w_j(i)) = \sum\limits_{i<j} s_i(j)$ and $K_{asym} = \sum\limits_{i<j} |a_i(j)|$. Since $w_i(j) = \frac{1}{2}(s_i(j) + a_i(j))$ we can rewrite the principal's payment as:

$$\begin{aligned} V_{full} &= n \cdot c - \frac{1}{2}\left[\sum_{j=1}^{1} \{s_1(j) + a_1(j)\} + ... + \sum_{j=1}^{n} \{s_n(j) + a_n(j)\}\right] \\ &= n \cdot c - \frac{1}{2}\left(\sum_{i>j} s_i(j) + \sum_{i>j} a_i(j)\right) \end{aligned}$$

Note that $s_i(j) = s_j(i)$ and $a_i(j) = -a_j(i)$. In addition $a_i(j) > 0$ when $i > j$ as the tournament is acyclic and ranking is consistent. Therefore, $V_{full} = n \cdot c - \frac{1}{2}\left(\sum_{i<j} s_i(j) - \sum_{i<j} |a_i(j)|\right) = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym})$. ■

Proof of Corollary 3.2 The result follows immediately from Proposition 3, where we show that $V_{full} = n \cdot c - \frac{1}{2}\sum_{i\ j} w_i(j) - \frac{1}{2}\sum_{i<j} |w_i(j) - w_j(i)|$, and from $V_{partial} = n \cdot c - \sum_{i\ j} w_i(j)$. Taken together, the two yield $V_{full} - V_{partial} = \frac{1}{2}\sum_{i\ j} w_i(j) - \frac{1}{2}\sum_{i<j} |w_i(j) - w_j(i)| = \frac{1}{2}(K_{agg} - K_{asym})$. ■

Proof of Proposition 4 Let $G(N, A)$ be a cyclic graph. Consider a subset of arcs S such that G_{-S} is acyclic, and the tournament ranking of G_{-S} is $\varphi =$

$(j_1, j_2, ..., j_n)$. The payment of the principal V_{full} under the DAC mechanism with respect to φ is $V_{full} = n \cdot c - \left\{\sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k)\right\}$. Note that each $(i,j) \in S$ satisfies an inconsistency in tournament ranking φ. More specifically, if $(i,j) \in S$, then i beats j, and agent j is positioned above agent i. In addition, $w_i(j) = w_j(i) - t(i,j)$, where $w_i(j) < w_j(i)$ and $t(i,j) > 0$. Consider the following substitution: If $(i,j) \in S$ then $w_i(j) = \hat{w}_j(i) - t(i,j)$; otherwise $w_i(j) = \hat{w}_i(j)$. This allows us to rewrite the principal's payment as $V_{full} = n \cdot c - \left\{\sum_{k=1}^{1} \widehat{w}_{j_1}(j_k) + ... + \sum_{k=1}^{n} \widehat{w}_{j_n}(j_k)\right\} + t(S)$. Note that $\hat{w}_i(j) = \max(w_i(j), w_j(i))$. Therefore, different rankings affect only the level of $t(S)$, as the first two terms in V_{full} remain indifferent to variations in the ranking. This implies that the subset S with the lowest $t(S)$ brings V_{full} to a minimum. Hence, the optimal mechanism is the DAC mechanism with respect to the tournament ranking of G_{-S^*}. ■

Proof of Proposition 5 As demonstrated in Proposition 4, the optimal payment of the principal is the DAC mechanism with respect to the tournament ranking of G_{-S^*}. According to Proposition 4, this can be written as $V_{full} = n \cdot c - \left\{\sum_{k=1}^{1} \widehat{w}_{j_1}(j_k) + ... + \sum_{k=1}^{n} \widehat{w}_{j_n}(j_k)\right\} + t(S)$ when $\hat{w}_i(j) = \max(w_i(j), w_j(i))$. Following the argument of Proposition 3, denote $s_i(j) = [\hat{w}_i(j) + \hat{w}_j(i)]$ and $a_i(j) = [\hat{w}_i(j) - \hat{w}_j(i)]$ and the principal's payment is $V_{full} = n \cdot c - \frac{1}{2}\left(\sum_{i<j} s_i(j) + \sum_{i<j} |a_i(j)|\right) + t(S) = n \cdot c - \frac{1}{2}\left(K_{agg} + K_{asym}\right) + K_{cyclic}$. ■

Proof of Proposition 6 Given mechanism v, participation is a dominant strategy for all agents, under the worst-case scenario in which all other agents participate since $u_i = \sum_{i=1}^{n} w_i(j) + v_i = c$ for every $i \in N$. To show that v is optimal, consider a mechanism m for which $m_i < v_i$ for some agents and $m_i = v_i$ for the rest. By contradiction, assume full participation equilibrium holds under mechanism m. Consider an agent i for which $m_i < v_i$. If all other agents are participating, then agent i's best response is to default since $u_i = \sum_{i=1}^{n} w_i(j) + m_i < c$. Hence, v is a unique and optimal mechanism. ■

Proof of Proposition 7 To prove the Proposition we use the following lemma.

Lemma 7.1: Consider a participation problem (N, w, c) with mixed externalities. Then the decomposition mechanism sustains full participation as a unique equilibrium.

Proof: Consider a decomposition mechanism v of the participation problem (N, w, c), when w includes mixed externalities. By definition $v_i = u_i^+ + u_i^-$, when u^+ is the optimal full implementation mechanism for the positive participation problem, and u^- is the optimal mechanism for the negative participation problem. Let $\phi = \{i_1, ..., i_N\}$ the optimal ranking in the positive participation problem (N, w^+, c). As demonstrated in Proposition 6, the optimal mechanism for the negative participation problem is a full compensation of agents for the negative externalities, i.e., $u_i^- = \sum_{j \in D_i} |w_i(j)|$ when $D_i = \{j \mid w_i(j) < 0 \ s.t. \ j \in N\}$. We will demonstrate that v sustains full participation as a unique equilibrium. To deter no participation equilibrium, at least a single agent has to participate, regardless of the participation choices of the other agents. Agent i_1 gets $u_{i_1}^+ = c$, and $u_{i_1}^- = \sum_{j \in D_{i_1}} |w_{i_1}(j)|$; therefore regardless of others' choices, agent i_1 will participate in the initiative. Next, it is necessary to prevent an equilibrium in which only a single agent participates, and note that given agent i_1's participation, agent i_2 will choose to participate, regardless of the choices of agents $i_3, ..., i_N$. This is true since $u_{i_2}^+ = c - w_{i_2}^+(i_1)$ and $u_{i_1}^- = \sum_{j \in D_{i_2}} |w_{i_2}(j)|$, and, given the participation of agent i_1 and the compensation for negative externalities, agent i_2 will participate. Applying the same logic for all subsequent agents, we get that the only possible equilibrium is full participation, since all agents are willing to participate given the participation of the agents who preceded them in the optimal ranking of the positive participation problem, and since they are being compensated for negative externalities. **End of Lemma.**

Now we turn to show that if the matrix of externalities satisfies symmetry and transitivity with respect to the *non-averse* relation, then the decomposition mechanism is the optimal mechanism for full participation. Note that since we have defined $w_{ii} = 0$ for each agent i, the *non-averse* relation $\succeq$ is reflexive (i.e., $i \succeq i$) and together with the symmetry and transitivity assumptions it is an equivalence relation. Therefore the set of agents can be partitioned into equivalence classes $S_1, ...S_m$ such that $\cup_j S_j = N$ and $S_j \cap S_i = \oslash$ for all i, j, and for each $k, l \in S_i$ we have that k and l induce non-negative externalities on each other. Let us now define m new participation problems by restricting the matrix of externalities to the set S_j where $1 \leq j \leq m$. All these problems are non-negative and their optimal mechanisms are all DAC mechanisms. Let u_j^+ be an optimal mechanism for the problem defined on the set S_j, which is a vector of size $\#S_j$. Write $u^+ = (u_1^+, ..., u_m^+)$. For each agent $i \in N$ denote by d_i the total of negative externalities imposed on agent i, i.e., $d_i = \Sigma_{\{j; w_{ij} < 0\}} w_{ij}$, and set $u^- = (d_1, ...d_n)$. We will show that $u = u^+ + u^-$ is an optimal mechanism for the mixed problem.

Let w be the original matrix of externalities and let $w = w^+ + w^-$ the decomposition of w , where w^+ is a non-negative matrix and w^- is a non-positive matrix. Clearly, u^+ is an optimal mechanism for w^+. This is true since under w^+ the externalities between agents of the same class are non-negative and between agents of different classes they are zero. By a similar argument, u^- is the optimal mechanism for w^-. Hence $u = u^+ + u^-$ is a decomposition mechanism and as such by Lemma 1 it sustains full participation as a unique equilibrium.

To show that $u = u^+ + u^-$ is optimal for the participation problem (N, w, c) we have to show that the principal cannot extract more from the agents if he wants all of them to participate in a unique equilibrium. Let ij denote agent at the i's location in the optimal order of the DAC mechanism within class S_j. Consider first an alternative mechanism u', which is identical to u in terms of incentives for all agents except for a single agent, who is placed first in the optimal order of some class j. This agent is getting less in u', i.e., $u'_{1j} < u_{1j}$. Under the mechanism u' there exists a Nash equilibrium in which the set of participating agents is $N \backslash S_j$. This is true since $1j$ does not have a dominant strategy to participate and therefore all agents in S_j are failing to participate in a Nash equilibrium on the game restricted to S_j, and hence also on the entire game. Furthermore, any agent in $N \backslash S_j$ chooses to participate when S_j participates, and even more so if the S_j class stays out (given the negative externalities between S_j and the rest). Hence u' does not qualify as an optimal mechanism. A similar argument holds whenever we reduce the payoff of more than one agent while limiting ourselves to one class only, say S_j. As we proved in Propositions 2 and 4, this induces an equilibrium on the game restricted to S_j with partial participation and this equilibrium also applies to the entire game; hence u' does not induce a unique full participation equilibrium.

We now move on to the case where the alternative mechanism u' reduces the payoff to more than a single agent and from more than a single class. Assume first that for some k classes $\{l_1, ..., l_k\}$, with $k \leq m$, agents $1j$, $j \in \{l_1, ..., l_k\}$, are paid less in u' than in u and that all other agents are paid the same. It must be the case that under u' full participation is a Nash equilibrium; otherwise it is definitely not the optimal unique full participation Nash equilibrium. We assume therefore that under u' full participation is a Nash equilibrium and we shall show that it is nevertheless not a unique equilibrium. Specifically, we will show that under u' we can construct a Nash equilibrium with the set of participants being $N \backslash S_{j^*}$, when S_{j*} is an arbitrary class such that $j^* \in \{l_1, ..., l_k\}$.

We first note that there exists an equilibrium in which none of the members in S_{j*} participate. Indeed, given that all agents in $N \backslash S_{j*}$ participate, participation is no longer a dominant strategy for agent $1j^*$ and hence there exists an equilibrium in which all agents in S_{j*} are staying out of the game restricted to S_{j*}, and therefore this equilibrium holds also for the entire game. Since for all agents in $N \backslash S_{j*}$ it was a best response to participate even when all members in S_{j*} (with whom they have negative externalities) participate it is certainly still a best response for these agents when S_{j*} are are not participating. Hence partial participation is a Nash equilibrium and we obtained the desired contradiction. Next, we assume a mechanism u' in which an arbitrary group of agents is paid less than the payoff in u. Let j^* be a class such that $j^* \in \{l_1, ..., l_k\}$ in which some agent ij is paid less. Using the argument in the first part of the proof and in Propositions 2 and 4, there is a Nash equilibrium in the game restricted to S_{j*} where there is only partial participation. Furthermore, the profile in which the set of participants is exactly

$N \backslash S_{j*}$ in addition to the subset of participants in S_{j*} is a Nash equilibrium in the entire game. This again follows from the fact that under u' the best response of each agent in $N \backslash S_{j*}$ to full participation by the rest must be to participate, and $N \backslash S_{j*}$ will definitely participate if a subset of S_{j*} is not participating.

We have shown so far that for a given mechanism u' in which some agents get less than u and the rest get the same as u, there is an equilibrium in which some agents do not participate. It is therefore left to show that for a given mechanism u' in which we reduce the payoff for some agents and increase the payoff for others either we still have an equilibrium in which some agents do not participate or alternatively the total payment of the principal increases. Consider first that u is unique; hence the optimal solution u_{+}^{j} for each positive participation problem in each class is unique. Assume that u' is achieved by lowering the payoff for the agent ranked first in class j^*, i.e., agent $1j^*$. To rule out the equilibrium in which class S_{j^*} is not participating it is necessary to increase the payoff of another agent, say kj^*, in such a way that she will have a dominant strategy to participate, and induce the participation of the agent whose payoff was lowered. This means moving the kj^* agent to become first in the class. But this must cost more to the principal as v^{j*} is unique, a contradiction. Alternatively, if u is not a unique, then there are multiple solutions for each positive participation problem. The argument is very similar. Say that u was chosen from a group of decomposition mechanisms. Again, to avoid partial participation equilibrium it is necessary to provide higher incentives to agent kj^*, who becomes first. If the total payment is identical to the payment in v^{j*} then we have reached a different optimal solution to the participation problem of class j^*. However, if we did not reach a different decomposition mechanism, this implies that the payment in this case is higher. The same logic can be applied to situations in which we lower the payoffs for other agents within the different classes. Therefore we conclude that the decomposition mechanism is the optimal mechanism that sustains full participation as a unique equilibrium. ■

Proof of Proposition 8 In Proposition 7 we show that when agents satisfy symmetry and transitivity of the *non-averse* relation then the problem can be separated into positive and negative participation problems (N, w^+, c) and $(N, w^-, 0)$, respectively, such that v is the optimal mechanism, $v = u^+ + u^-$. Thus, the costs can be decomposed in the following manner: $V_{full} = V^+ + V^-$. From Proposition 5 we know that $V^+ = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym}) + K_{cyclic}$. From Proposition 6, due to the negative externalities, the principal provides complete compensation; hence $V^- = \sum_{i\ j} \left| w_i^-(j) \right|$. Together we get that $V_{full} = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym}) + K_{cyclic} + K_{neg}$.

Proof of Proposition 9 In both segregated and desegregated environments the externality structure is symmetric and, following Corollary 5.1, all rankings are optimal. Consider first the segregated environment. Since all rankings are optimal, a possible optimal mechanism is $v = (c, ..., c - (n_1 - 1), c, ..., c - (n_2 - 1))$. Hence, the optimal payment for the principal is $v(n_1, n_2) = n \cdot c - \sum_{l=1}^{n_1-1} l - \sum_{k=1}^{n_2-1} k = n \cdot c - \frac{n_1(n_1-1)}{2} - \frac{(n-n_1)(n-n_1-1)}{2}$. Assuming that $v(n_1, n_2)$ is continuous with n_1 then $\frac{\partial v(n_1,n_2)}{\partial n_1} = n - 2n_1$ and maximum is achieved at $n_1^* = n_2^* = \frac{n}{2}$, and the cost of incentivizing is declining with $\mid n_1 - n_2|$. In the desegregated example, a possible optimal mechanism is $v = (c, ..., c, c - n_1, ..., c - n_1)$. Therefore, the principal's payment is $v(n_1, n_2) = n \cdot c - (n - n_1) \cdot n_1$. Again, let us assume that $v(n_1, n_2)$ is continuous with n_1, in which case solving $\frac{\partial v(n_1,n_2)}{\partial n_1} = 2n_1 - n = 0$ results in the minimum payment for the principal in the desegregated environment being received at $n_1^* = n_2^* = \frac{n}{2}$, and the cost of incentivizing is increasing with $|n_1 - n_2|$. In a status environment, since group B_1 is the more esteemed group, all agents from B_1 beat all agents from B_2; therefore agents from B_1 should precede the agents from B_2 in the optimal ranking. A possible optimal ranking is $\varphi = \{i_1, ., , , i_{n_1}, j_1, ..., j_{n_2}\}$ when $i_k \in B_1$, $j_m \in B_2$ and $1 \leq k \leq n_1$, $1 \leq m \leq n_2$. Therefore, a possible optimal mechanism is $v = (c, c-1..., c-(n_1-1), c-n_1, ..., c-n_1)$. The principal's payment is $v(n_1, n_2) = n \cdot c - \sum_{l=1}^{n_1-1} l - n_2 \cdot n_1 = n \cdot c - \frac{n_1(n_1-1)}{2} - (n - n_1)n_1 = \frac{1}{2}n_1 - nn_1 + \frac{1}{2}n_1^2 + cn$. Again, assuming that $v(n_1, n_2)$ is continuous with n_1, $\frac{\partial v(n_1,n_2)}{\partial n_1} = n_1 + \frac{1}{2} - n = 0$ and the minimal payment is achieved at $n_1^* = n - \frac{1}{2}$. Note that $V(n_1 = n) = V(n_1 = n - 1)$. Therefore, the best scenario for the principal is when $n_1 = n$. Alternatively, the cost of incentivizing is decreasing with n_1. ■

Proof of Proposition 10 Since the optimal mechanism is a DAC mechanism, it is a result of the following optimization problem:

$$\max_{(j_1, j_2, ..., j_n)} [u_{j_2}(j_1) + u_{j_3}(j_1, j_2) + ... + u_{j_n}(j_1, ..., j_{n-1})]$$

Assume, without loss of generality, that the tournament ranking φ is the identity permutation; hence $\varphi(i) = i$, and $W_\varphi = u_2(1) + u_3(1, 2) + ... + u_n(1, ..., n-1)$. W_φ is the principal's revenue extraction. By contradiction, assume that there exists a different ranking denoted by σ such that $W_\varphi \leq W_\sigma$. First, assume that σ is obtained from having two *adjacent* agents i and j ($j = i + 1$) in φ trade places such that i precedes j in φ (hence i beats j) and j precedes i in σ. Therefore, $\sigma = \{1, ..., i-1, j, i, ..., n\}$. First note that all the agents that appear after j in order φ earn the same payoff in the DAC mechanism of both φ and σ. The same holds also for all the agents who appear before i in the order φ. So the cost of the DAC mechanisms with respect to φ and σ differs only in terms of the payoff of agents i and j, and we get that

$$W_\sigma = W_\varphi + A$$

when $A = [u_i(1, ..., i-1, j) - u_i(1, ..., i-1)] - [u_j(1, ..., i-1, i) - u_j(1, ..., i-1)]$. The term A compares the marginal contribution of i relative to the marginal contribution of j, given a subset $S = \{1, ..., i-1\}$. Therefore, $A < 0$, which entails $W_\sigma < W_\varphi$.

Note that since φ is the tournament ranking, agent 1 beats all agents, agent 2 beats all agents except agent 1, and so on. Now consider $\sigma = \{i_1, ..., i_n\}$ such that $\varphi \neq \sigma$. If agent 1 is not located first, by a sequence of adjacent swaps $(1, i_j)$, we move agent 1 to the top of the ranking. In each of the substitutions agent 1 beats i_j. Next, if agent 2 is not located at the second place, by a sequence of adjacent substitutions $(2, i_j)$, we move agent 2 to the second place. Again, agent 2 beats all agents i_j. The process ends in at most n stages and produces the desired order φ. As demonstrated, any adjacent substitution results in a higher extraction, and therefore $W_\sigma < W_\varphi$. Therefore, the DAC mechanism with respect to the tournament ranking is unique and optimal. ■