Behavioral Competitive Equilibrium and Extreme Prices*

Faruk Gul†

Wolfgang Pesendorfer‡

Tomasz Strzalecki§

November 29, 2012

Abstract

We introduce a notion of *behavioral competitive equilibrium* (BCE), to study households' inability to tailor their consumption to the state of the economy. Our notion is motivated by limited cognitive ability (in particular attention, memory, and complexity) and it maintains the complete market structure of competitive equilibrium. Compared to standard competitive equilibrium, our concept yields riskier allocations and more extreme prices (both for consumption and for assets). Thus, limited cognitive ability can produce market data that are usually attributed to heightened degree of risk aversion. We provide a tractable model that is suitable for general equilibrium analysis as well as asset pricing in dynamic environments.

*This research was supported by grants from the National Science Foundation SES-0010394 and SES-1123729.

†Princeton University. Email: fgul@princeton.edu

‡Princeton University. Email: pesendor@princeton.edu

§Harvard University. Email: tomasz_strzalecki@harvard.edu

1 Introduction

Many of the empirical puzzles in macroeconomics and finance arise from researchers' inability to reconcile the levels of risk aversion implied by equilibrium models with levels that are observed in other contexts or are reasonable a priori. More specifically, many of these puzzles amount to observing that the degree of risk aversion required to accommodate typical equilibrium outcomes is implausibly high. For example, the equity premium puzzle (Mehra and Prescott, 1985) is the observation that stock returns are too large given reasonable levels of risk aversion and observed variability of aggregate consumption and dividends. Similarly, the equity home bias puzzle (French and Poterba, 1991) refers to the comparatively much higher risk aversion that investors exhibit toward foreign assets by forgoing the diversification opportunities that these markets offer.

We define a notion of competitive equilibrium, *Behavioral Competitive Equilibrium* (BCE), that enables the modeler to limit households' ability to process and react to information. We compare BCE prices to prices in a standard competitive equilibrium (SCE), and show that, holding risk aversion levels equal, BCE prices exhibit a much greater range than SCE prices. Since in SCE the price variation is a function of the representative agent's risk aversion, this implies that from the modeling viewpoint behavioral limitations can be thought of as a substitute for risk aversion: assuming behavioral limitations leads to aggregate prices that are consistent with a choice problem unaffected by such limitations, but with higher risk aversion.

In the most general terms, we identify behavioral limitations with the cost agents must incur when tailoring their *actions* to their *circumstances.* If the circumstance is a history of past actions or events, we interpret the cost as a memory cost: for an event that is too costly to remember, the agent chooses the same action whether or not it happened. If the circumstance is a future contingency, we interpret the cost as a complexity cost: the more finely a given plan depends on future events, the more complex and hence the more costly it is. Finally, if the circumstance is a current variable, we interpret the cost as an information processing or a communication cost: if paying attention to a current event is too costly, the agent chooses the same action whether or not it happened.

We consider both a static and a dynamic exchange economy. We identify *circumstance* with the realized price and *action* with a consumption level. Hence, within the classification above, our static model is one of limited information processing ability while or dynamic model imposes both information processing cost and limited memory. For our agents, adjusting their consumptions to price changes is costly. We adopt the following simple cost function: households can only choose *crude consumption plans*; that is, plans that restrict them to at most k distinct actions. Hence, all crude plans have zero cost and all others are infinitely costly.[1]

Consider, for example, a household who can distinguish between just two broad categories of economic events. This household may partition circumstances into "good times" and "bad times" and choose one consumption level for each event. Thus, the household makes two decisions: how to define good times/bad times *and* how much to consume in each contingency. While the second decision is standard, the first one is novel—this is our device for modeling how limited information processing or attention allocation responds to economic incentives. Note that in a stylized model such as ours the mathematical formulation of the behavioral household optimization problem may be more complicated than the standard household decision problem. Optimally partitioning the state space may seem like a difficult task. However, our model is not meant as a description of the household's reasoning process. Rather, we aim to model a household who, while unable to react to all price changes, responds to incentives when allocating cognitive resources.

To simplify the exposition, we assume that there is a single physical good. We also assume a continuum of households each with the same state independent CRRA utility function. The two-period model has a planning period and a consumption period. Households learn the state after the planning period and before the consumption period. In the planning period, each household chooses a crude (state-contingent) consumption plan. We show that a BCE exists and is Pareto optimal (given the restriction to crude consumption plans). Notice that a single crude consumption plan cannot distinguish among all states. Therefore, if all consumers were to choose the same crude consumption plan, markets would not clear. For example, if households

[1] In the concluding section, we consider alternative formulations of action-circumstances and alternative cost functions and discuss how our results may change with these formulations.

can only choose two consumption levels ("good times" and "bad times") then, in equilibrium, all households cannot classify states into the same two categories. It follows that in a BCE ex ante identical consumers may and sometimes must choose distinct plans, as if they did, the aggregate demand would take only two values. Therefore, consumption is more risky in a BCE than in a standard competitive equilibrium.

To study BCE prices, we fix the households' utility function and consider a sequence of economies that converges to an economy with a non-atomic endowment distribution. We show that, in the limit, BCE prices in states near the lowest or the highest possible endowment realization are *extreme*; the price of consumption converges to infinity when the endowment is at or near the lower bound of the distribution while the price converges to zero when the endowment is near or at the upper bound of the distribution. To see why this result is suggestive of very high risk aversion, consider a standard economy in which the aggregate endowment is distributed uniformly on the interval $[1, 2]$. Suppose every household has the same CRRA utility function u. Then, irrespective of the distribution of individual endowment, this economy will behave as if there is a representative agent: it follows from the first order conditions that equilibrium prices will satisfy

$$u'(x) = \lambda \cdot p(x)$$

for some constant λ. Differentiating both sides yields

$$u''(x) = \lambda \cdot p'(x).$$

A straightforward manipulation of these two equality reveals that

$$-\frac{xu''(x)}{u'(x)} = -\frac{xp'(x)}{p(x)}.$$

Clearly, the left-hand side of the above equation is the coefficient of relative risk aversion while the right-hand side is the reciprocal of the elasticity of demand. Hence, the aggregate demand will have constant elasticity equal to the reciprocal of the coefficient of risk aversion. This means that the range of observed prices will be an increasing function of the representative households coefficient of relative risk aversion. An infinite range, which is what a BCE implies, is suggestive of an infinite coefficient of relative risk aversion.

We also consider a dynamic economy in which a finite state Markov chain describes the transitions of the endowment. We show that a stationary BCE exists, which ensures that the equilibrium of the dynamic economy can be mapped to an equilibrium of an appropriately defined two-period model. This allows us to describe equilibrium allocations and prices in the dynamic economy in terms of the prices and allocations of a two-period model. As in the two-period case, we fix the utility function and consider a sequence of economies converging to a limit economy with a non-atomic invariant distribution of endowments. The extreme consumption prices of the two-period model imply extreme asset prices in the dynamic economy. Specifically, consider an asset that pays off a share of the endowment in all future periods. Because of stationarity, the price of this asset is a random variable that depends only on the current state of the economy (the current endowment). When the realized endowment is near or at the lower bound of possible endowments, the price of the asset is essentially zero. Conversely, as we approach the upper bound of possible endowment realizations, the price of this asset converges to infinity.

Our results show how behavioral constraints can create extreme prices in a competitive model. The intuition behind these results is as follows. The households' behavioral constraints imply that paying attention to very rare events is costly. On the other hand, market clearing and the fact that the aggregate endowment variability mean that at least some agents distinguish between these unlikely events. Thus, prices must incentivize a small fraction of households to absorb the aggregate outcome fluctuation in such rare events. Getting a small fraction of households to bear the aggregate risk requires extreme prices near unusually low or unusually high endowment realizations.

1.1 Relation to Literature

The game theory literature has developed strategic analogs of behavioral equilibrium. Neyman (1985), Rubinstein (1986), and Abreu and Rubinstein (1988) limit players' strategies in a repeated game to those implementable by finite state automata. Our approach is closest to

Neyman (1985) who studies Nash equilibria of a game in which the number of states in the automaton is bounded. Abreu and Rubinstein (1988) also study Nash equilibria, but with a different cost function. Rubinstein (1986) examines a lexicographic cost of complexity and imposes a version of subgame perfection which precludes agents from adopting a different automaton later in the game. In Dow (1991)'s search model, limited memory is modeled as a partition of the set of all histories and the agent is assumed to choose an optimal partition. Piccione and Rubinstein (1997) examine the relation between limited memory (i.e., imperfect recall) and time consistency. Wilson (2002) analyzes long-run inference and shows that the optimal use of a limited memory can lead to many well-studied behavioral biases. Mullainathan (2002) studies a model of coarse categorization and its implications for the predictability of asset returns and their trade volume. Masatlioglu, Nakajima, and Ozbay (2012) conduct a revealed preference analysis of a model where the agent chooses subject to an endogenous attention constraint. Jehiel (2005) and Jehiel and Samet (2007) constrain players to respond the same way in similar situations by bundling nodes into analogy classes. Despite the differences in modeling details, all these papers, including ours, restrict agents' ability to tailor their behavior to their environment by imposing additional constraints. We have adopted a simple form of this restriction to obtain a tractable competitive equilibrium model and analyze the effect of behavioral limitation on equilibrium prices.

Sims (2003) assumes agents allocate their attention optimally subject to an information-theoretic constraint. Our approach differs from Sims' approach in the formulation of the cost function; we limit the number of possible signal values while Sims (2002) formulates an entropy based constraint. Woodford (2011) modifies Sims' cost function to address observed anomalies of consumer choice behavior. Mankiw and Reis (2002) study a model that has only a fraction of agents getting new information each period. Hong, Stein, and Yu (2007) study a model in which agents are restricted to a small class of forecasting models that does not include the true model of the world. These papers study how information processing frictions impact asset prices and responses to monetary policy.

There are a number of papers that use rigidities in consumption to close the gap between the level of risk aversion needed to rationalize data and plausible levels of risk aversion. Grossman and Laroque (1990) distinguish liquid and illiquid consumption and assume that agents incur transaction costs when they sell an illiquid good. Lynch (1996) and Gabaix and Laibson (2002) study a model in which only a fraction of agents can make adjustments at a given time. Chetty and Szeidl (2010) focus on the extent to which consumption rigidities reduce stock market participation.[2] If some households cannot adjust their consumption the remaining subset must absorb all the changes in aggregate consumption. As a result, observed prices (or interest rates) are consistent with a lower elasticity of intertemporal substitution or, equivalently, lower relative risk aversion, than in a model in which all households are unrestricted. Unlike Lynch (1996) and Gabaix and Laibson (2002), we do not exogenously fix the fraction of households that can respond to a particular increase in aggregate output. Rather, we make adjustments costly and let households respond optimally. As we discuss in the next section, the fact that households choose their partitions optimally enhances the impact of their behavioral limitation on equilibrium prices.

2 BCE in a Static Economy

There is a unit mass of households with identical utility functions but different stochastic endowments of the single physical good. Aggregate endowment depends on the realization of the state $i \in N = \{1, \ldots, n\}$. Let π_i denote the probability of the uncertain state i, s_i the aggregate endowment in state i and $K(s)$ the finite support of s. Let a be the smallest and b the largest aggregate endowment realizations. We assume that $0 < a < b$. For any set X, let $|X|$ denote its cardinality; hence $|K(s)|$ is the number of distinct aggregate endowment realizations.

A standard consumption plan is a vector in $\mathcal{C} = \mathbb{R}^n_+$ and a household's utility of the consumption plan c is

$$U(c) = \sum_{i \in N} u(c_i)\pi_i$$

[2]See also Guvenen (2009), who studies limited stock market participation in a model with heterogeneity in the elasticity of intertemporal substitution in consumption.

where u is any strictly concave CRRA utility index. That is,

$$u(c_i) = \begin{cases} c_i^{1-\rho}/(1-\rho) & \text{if} \quad \rho \neq 1 \\ \ln c_i & \text{if} \quad \rho = 1 \end{cases}$$

for some $\rho > 0$.

In a BCE, households are restricted to *crude consumption plans,* which limit the number of distinct consumption levels that a household can choose to k. To avoid trivial cases, we assume throughout that $1 < k < |K(s)|$.

Definition 1. The consumption plan $c \in \mathcal{C}$ is *crude* if $|\{c_i \,|\, i \in N\}| \leq k$.

Let $\mathcal{C}_k$ be the set of all crude consumption plans. Clearly, $\mathcal{C}_k \subset \mathcal{C}_{k+1}$ for all k. Note that $\mathcal{C}_k$ is not convex since a convex combination of $c, c' \in \mathcal{C}_k$ would typically require more than k consumption levels. This non-convexity implies that, in equilibrium, identical households will might and typically will chose different consumption plans. For example, when $k = 2$ and $K(s) = 3$, some households will consume the same amount in the first two states while others consume equal quantities in the last two states.

An element p in the $n-1$-dimensional simplex $\Delta(N)$ is a (normalized) price. An endowment ω is an element of $\mathbb{R}^N_+$. At price p, the wealth of a household with endowment ω is $p \cdot \omega = \sum_{i \in N} \omega_i \cdot p_i$. That household's budget set is

$$B_k(p, p \cdot \omega) = \Big\{ c \in \mathcal{C}_k : p \cdot c \leq p \cdot \omega \Big\}$$

We suppress the endowment distribution and call $E = \{u, k, \pi, s\}$, the *economy.* A *representative household* is one with endowment s (and wealth $p \cdot s$). We call an economy a representative economy if every household is a representative household.

For any set X, let $\Delta(X)$ denote the set of simple probabilities on X; that is $\Delta(X)$ is the set of all functions $\mu \in [0,1]^X$ such that $K(\mu) = \{x : \mu(x) > 0\}$ is finite and $\sum_{x \in X} \mu(x) = 1$. For any $Y \subset X$, we let $\mu(Y)$ denote $\sum_{x \in Y} \mu(x)$. We call $K(\mu)$ the support of μ. When X is finite, we identify $\Delta(X)$ with the $|X| - 1$ dimensional simplex.

An *allocation* $\mu \in \Delta(\mathcal{C})$ is a probability on consumption plans; $\mu(c)$ represents the fraction of households who choose plan $c \in \mathcal{C}$. An allocation is crude if $K(\mu) \subset \mathcal{C}_k$. The allocation μ is *feasible* in E if

$$\sigma_i(\mu) := \sum_{c \in K(\mu)} c_i \cdot \mu(c) \leq s_i$$

for all $i \in N$. For any $\mathcal{C}' \subset \mathcal{C}$, let $M(\mathcal{C}')$ be the set of all feasible allocations such that $K(\mu) \subset \mathcal{C}'$. Hence, $M(\mathcal{C}_k)$ is the set of feasible allocations for our behavioral economy. Henceforth, μ is feasible means $\mu \in M(\mathcal{C}_k)$.

The crude consumption plan $c \in B_k(p, w)$ is *optimal at prices* p and wealth $w > 0$ if $U(c) \geq U(c')$ for all $c' \in B_k(p, w)$. Because we assume CRRA utility, optimal consumption plans are homogenous in the household's wealth; a crude consumption c is optimal for a household with wealth w at price p if and only if $\frac{p \cdot s}{w} c$ is optimal for the representative agent at price p. Therefore, aggregate demand is independent of the wealth distribution and so are BCE equilibrium prices. Moreover, we can convert the BCE allocation of the representative economy into a BCE of E with the same aggregate endowment by evaluating the wealth of each household at the equilibrium price and distributing the allocation among households in proportion to their wealth. Finally, we can rescale any BCE allocation for E to derive a BCE allocation for the representative economy with the same aggregate endowment.

Henceforth, we describe the set of BCE in terms of its representative economy counterpart and appeal to these observations to justify suppressing the wealth distribution.

Definition 2. The price-allocation pair (p, μ) is a *BCE* of E if μ is feasible for E and if $\mu(c) > 0$ implies c is optimal for the representative agent at prices p.

We call p (μ) a BCE price (allocation) if (p, μ) is a BCE for some μ (p). Two consumption plans c, c' are *conformable* if $c_i = c_j$ if and only if $c'_i = c'_j$; i.e., two consumption plans are conformable if they induce the same partition of N. We write $c \sim c'$ if c and c' are conformable.

Definition 3. An allocation is *simple* if $\mu(c) \cdot \mu(c') > 0$ and $c \sim c'$ implies $c = c'$. An allocation is *fair* if $\mu(c) \cdot \mu(c') > 0$ implies $U(c) = U(c')$.

A fair allocation yields the same utility for each consumption plan in its support. In a simple allocation each partition of N has at most one consumption plan associated with it.

Thus, if μ is simple, the cardinality of $K(\mu)$ is at most equal to the number of partitions of N with k or fewer elements. There are two other properties of allocations, monotonicity and measurability, that play a role in our characterizations of BCE equilibria. Monotonicity requires that consumption is weakly increasing in the aggregate endowment. Measurability means that the allocation remains feasible if states with identical endowments are combined into a single state.

Definition 4. The plan c is *monotone* if $c_i \geq c_j$ whenever $s_i > s_j$. The plan c is *measurable* if $c_i = c_j$ whenever $s_i = s_j$. The allocation μ is *monotone/measurable* if all $c \in K(\mu)$ are monotone/measurable.

The mean utility, $W(\mu)$, of allocation μ is

$$W(\mu) = \sum_c U(c) \cdot \mu(c)$$

We say that $\mu \in M(\mathcal{C}_k)$ solves the *planner's problem* if $W(\mu) \geq W(\mu')$ for all $\mu' \in M(\mathcal{C}_k)$. The main technical result of this section is Lemma 1, below, which enables us to relate the solutions to the planner's problem to BCE equilibria of the representative economy in Theorem 1. In an economy without our behavioral constraints, simplicity, fairness, monotonicity and measurability of solutions to the planner's problem would follow immediately from the strict concavity of the household utility function. The argument for simplicity is unaffected by behavioral constraints. But none of the remaining properties hold for a general strictly concave utility function given the crudeness constraint. Lemma 1 shows that they do hold with a strictly concave CRRA utility function.

Lemma 1. *(i) There is a solution to the planner's problem (ii) Every solution to the planner's problem is simple, fair, monotone, and measurable.*

Household preferences in our model satisfy local non-satiation and, therefore, by the first welfare theorem, BCE allocations are Pareto efficient. (Of course, Pareto efficiency is defined relative to allocations in $M(\mathcal{C}_k)$.) In a representative BCE, allocations must be fair and, by Lemma 1, the planner's problem has a fair solution. Thus, if a BCE allocation did not solve the planner's problem there would exist a fair allocation that yields higher household utility, contradicting the Pareto efficiency of BCEs. Hence, every BCE allocation must solve the planner's problem. Theorem 1, below, establishes the converse. Theorem 1 and Lemma 1 together establish the existence and Pareto-efficiency of BCE. They also show that every BCE allocation is simple, fair, monotone, and measurable.

Theorem 1. *An allocation solves the planner's problem if and only if it is a BCE allocation for the representative economy.*

Existence of a BCE and its Pareto-efficiency relies neither on CRRA preferences nor on the correspondence between solutions to the planner's problem and equilibria; it is possible to establish existence (and Pareto-efficiency) using standard fixed-point arguments and relying only on the concavity of the household utility function. Such a proof would not yield the monotonicity and measurability of equilibrium allocations and the related monotonicity of BCE prices. Indeed, without CRRA utility, it is possible to construct examples of BCE that do not satisfy these properties as we illustrate in the following section.

For any $r \in \{s_i : i \in N\}$, let $p(r) = \sum_{\{i:s_i=r\}} p_i$ and $\pi(r) = \sum_{\{i:s_i=r\}} \pi_i$. Two prices $p, \hat{p}$ are *equivalent* if $p(r) = \hat{p}(r)$ for all r. In a *pure endowment economy* the realized endowment resolves all uncertainty; that is, s is one-to-one. In that case, $p(s_i) = p_i$ and $\pi(s_i) = \pi_i$ for all i. Hence, for a pure endowment economy, $p, \hat{p}$ are equivalent if and only if $p = \hat{p}$.

Definition 5. The price p is *monotone* if $r > \hat{r}$ implies $p(r)/\pi(r) \leq p(\hat{r})/\pi(\hat{r})$.

We say that the BCE price is *essentially unique* if and only if all BCE prices are equivalent. Theorem 2 below shows that the BCE price is essentially unique and monotone. Hence, a pure endowment economy has a unique BCE price.

Theorem 2. *The BCE price of any economy is essentially unique and monotone.*

Next, we illustrate Theorems 1 and 2 in a simple example. There are four equally likely states with endowments between 1 and 2. Specifically,

$$s_1 = 1, s_2 = 4/3, s_3 = 5/3, s_4 = 2$$

The utility function is logarithmic and households must choose 2−crude plans. As a benchmark, first consider a standard competitive equilibrium without the crudeness constraint. In that case, the representative household consumes her endowment and the prices are given by p^* where

$$p_1^* = .35; p_2^* = .26, p_3^* = .21, p_4^* = .18$$

Next, we describe the BCE with $k = 2$, that is, all households are restricted to 2−crude plans. Lemma 1 implies that equilibrium consumption plans are characterized by a cutoff state; that is, each household chooses a state $j \in \{1, 2, 3, 4\}$ such that states $i \leq j$ are associated with low consumption ("bad times") and states $i > j$ are associated with high consumption ("good times"). There are three distinct plans chosen by consumers; around 39% of consumers identify states $2, 3$ and 4 as "good times" and single out state 1 for a low consumption; consumption levels in that plan are $c_1 = .85, c_2 = c_3 = c_4 = 1.7$. Around 36% of households choose identify states 1 and 2 as bad times and states 3 and 4 as good times; the corresponding consumption levels are $c_1 = c_2 = 1.04$ and $c_3 = c_4 = 1.96$. Finally, the remaining 25% of identify states $1, 2$ and 3 as bad times and single out state 4 for high consumption; their consumption levels are $c_1 = c_2 = c_3 = 1.18$ and $c_4 = 2.56$. The equilibrium price is p where

$$p_1 = .4, p_2 = .25, p_3 = .21, p_4 = .133$$

Notice that the largest difference between the equilibrium price in a standard economy and the BCE price is in the states with the highest and lowest endowments. The ratio of p_1^* and p_4^* is equal to 2, the ratio of the aggregate endowment in those two states. By contrast, the ratio of p_1 and p_4 is 3. As we will show in the next section, this is no accident. In any BCE with many states, the prices in states with endowments near the upper or lower bounds exhibit the greatest departure from standard equilibrium prices.

Monotonicity of BCE consumption plans is a key result of this section that will be used to derive pricing results in the subsequent sections. Next, we illustrate how those results can fail if the utility function is not CRRA. The example is identical to the one above but with a different utility function. The utility index is u such that

$$u(z) = \begin{cases} 2z & \text{if } z \leq 1 \\ 1+z & \text{if } z \in [1,2] \\ 2+z/2 & \text{if } z > 2 \end{cases}$$

For this utility function the unique BCE allocation consists of two consumption plans; 2/3 of consumers choose the plan $c_1 = c_2 = 1, c_3 = c_4 = 2$ while 1/3 of consumers choose the plan $c_1 = c_3 = 1$ and $c_2 = c_4 = 2$. The BCE price is p such that $p_i = 1/4$ for all i. Notice that the second consumption plan is not monotone and hence the example shows that with general risk averse utility functions monotonicity may fail.[3]

3 Consumption Risk and Price Variation in a BCE

In this section, we compare the BCE for economy E to the standard equilibrium for E. In particular, we show that consumption is riskier and prices are more extreme in a BCE than in a standard competitive equilibrium (SCE).

We first establish that BCE yields greater consumption risk. Given π and $z \in \mathbb{R}^N$, we let F_z denote the cumulative distribution of the random variable z; that is, $F_z(x) = \sum_{i:z_i \leq x} \pi_i$. Hence, F_s is the cdf of the endowment and F_c is the cdf of consumption associated with the plan c. Then, G_μ, the cdf of consumption given the allocation μ is

$$G_\mu(x) = \sum_c F_c(x) \cdot \mu(c).$$

Consider a BCE equilibrium (p, μ) for the representative economy. We can think of the (random) consumption as the result of a two stage lottery; the first stage reveals the state $i \in N$ and the second stage reveals the consumption in state i. Since $k < |K(s)|$, consuming the endowment

[3]The utility function in the example is not strictly concave. However, it is straightforward to show that a strictly concave approximation of the utility function in this example would also lead to non-monotone equilibrium consumption plans.

in every state is not feasible and there are $c \in K(\mu)$ with $c_i \neq s_i$ for some $i \in N$. Moreover, $s_i \geq \sigma_i(\mu)$ for all i since μ is feasible and hence the expected value of consumption in state i is less than or equal to s_i, the SCE consumption in state i. This implies that SCE consumption second order stochastically dominates BCE consumption and hence welfare (i.e., mean utility) in a BCE is strictly less than welfare in a SCE. Put differently, the fact that s is not in $\mathcal{C}_k$, by itself, ensures that the representative household bears greater consumption risk in a BCE than in a SCE.

More generally, when different households have different endowments, a BCE may result in less income inequality than an SCE. In this case, it is difficult to conclude that ex ante welfare will be lower in a BCE than in an SCE; prices will not be the same in the two different types of equilibrium and given a fixed endowment distribution, different prices will imply different wealth distributions. A tighter wealth distribution will imply higher ex ante welfare, higher possibly than in a SCE.

We now show that for any fixed range of the endowment realization $[a, b]$, the equilibrium price $p \in \Delta(N)$ in a BCE (normalized by the probability of the state) can be arbitrarily large or arbitrarily close to zero. Thus, BCE exhibits *extreme prices*. Specifically, extreme prices emerge when the endowment has many possible realizations and F_s approaches a nonatomic, continuous distribution. The next subsection defines this notion precisely and establishes the main result.

3.1 Convergent Economies and Extreme Prices

Let E^n be a pure endowment economy with $n \geq k+1$ states and order states so that $s_i < s_j$ if $i < j$. We define a sequence of economies converging to a limit economy with a continuous endowment distribution.

Definition 6. A sequence of economies $\{E^n\} = \{(u, k, \pi^n, s^n)\}$ is *convergent* if s^n converges in distribution to a random variable with a continuous and strictly positive density on $[a, b]$.

Let p^n be the equilibrium price of E^n. The *pricing kernel* $\kappa^n \in \mathbb{R}^n_+$ is defined as the equilibrium price normalized by the probability of the state:

$$\kappa_i^n = \frac{p_i^n}{\pi_i^n} \tag{1}$$

Theorem 3, below, characterizes $\{p^n\}$, the equilibrium price sequence, and $\{\kappa^n\}$, the corresponding sequence of price kernels, for a convergent sequence of economies. For any real-valued function X on $\{1, \ldots, n\}$ and $A \subset \mathbb{R}$, let $\Pr(X \in A)$ denote the probability that X takes a value in A; that is,

$$\Pr(X \in A) := \sum_{\{i : X_i \in A\}} \pi_i$$

Any sequence $\{x^n\} \subset X$ for some compact set X has a convergent subsequence. In the statements below, we use this fact and write $\lim x^n$ to denote any such limit.

Theorem 3. *For any convergent sequence of economies,* $\lim p_1^n > 0$ *and* $\lim \Pr(\kappa^n < \epsilon) > 0$ *for all* $\epsilon > 0$*. Furthermore, if* $\rho \geq 1$*, then* $\lim \Pr(\kappa^n > K) > 0$ *for all* K*; if* $\rho > 1$*, then* $\lim \Pr(p^n = 0) > 0$.

For the state with the lowest endowment, Theorem 3 establishes that the limit price is greater than zero even though the limit probability of that state is zero. Thus, consumption in the lowest endowment state is extremely expensive. Clearly, this implies that the pricing kernel in state 1 converges to infinity. However, since the probability of state 1 converges to zero, this leaves open the question of whether there is a positive limit probability of an arbitrarily high pricing kernel. Theorem 3 shows that this is the case if the household is sufficiently risk averse, with a parameter of relative risk aversion greater or equal to 1.

Theorem 3 also establishes that there is a positive limit probability that the pricing kernel is arbitrarily close to zero. By Theorem 2 this occurs when the endowment realization is near its upper bound b. Finally, if relative risk aversion is above 1, a stronger result is true: the limit price is zero in a nontrivial interval of states near the highest endowment realization.

To prove Theorem 3, we first establish the following dominance lemma (Lemma 16). Let p^n be the equilibrium price for $E^n = (u, k, \pi^n, s^n)$ and let L_{k-1}^n be the maximal utility of a household who must choose $k-1$-crude plans at price p^n and with the wealth of the representative household. It is clear that $k-1$-crude plans can do no better than $k-$crude plans and therefore L^n_{k-1} can be no greater than the equilibrium utility of a representative household. The dominance lemma shows that L^n_{k-1} is uniformly bounded away from the equilibrium utility for all n; hence $k-1-$crude plans do uniformly worse than $k-$crude plans at the equilibrium price.

To sketch the argument for the first part of Theorem 3, assume that the equilibrium price in state 1 converges to zero. In equilibrium, some households must choose a lower consumption in state 1 than in all other states because aggregate consumption is lower in state 1 than in all other states and because all equilibrium plans are monotone. An alternative plan for these consumers would be to set consumption in state 1 equal to consumption in state 2 while reducing all consumption a bit so as to satisfy the budget constraint. If the price in state 1 goes to zero then this plan yields essentially the same utility as the original plan. But since the new plan is $k-1$ crude we have established a contradiction to the dominance lemma. Hence, the price in state 1 must stay bounded away from zero.

A similar application of the dominance lemma shows that consumption in the highest endowment states must be very cheap so that consumers find it worthwhile to single them out; so cheap that the probability-weighted utility in those states stays bounded away from zero. As a consequence, the pricing kernel must be close to zero. For the final part of Theorem 3 note that utility is bounded above if $\rho > 1$ and, therefore, consumers are unwilling to single out very unlikely low-price events no matter how low the price. In that case, part of the aggregate endowment near b is not consumed and prices are zero.

3.2 Limit Prices

When characterizing equilibrium prices with many possible endowment realizations, it is easier to work with *cumulative prices*: a nondecreasing, right-continuous function $H : \mathbb{R} \to [0,1]$ is a *cumulative price* if $H(x) = 0$ for $x < 0$ and $H(1) = 1$. Given any E^n and $x \in [\pi^n_1, 1]$, let $\jmath^n_x = \max\{j : \sum_{i \leq j} \pi^n_i \leq x\}$. If $x \in [0, \pi^n_1]$, we let $\jmath^n_x = 1$. Then, for any price p^n of E^n, define its cumulative, P^n as follows:

$$P^n(x) := \sum_{i \leq \jmath_x^n} p_i^n \tag{2}$$

Note that $P^n\left(F_{s^n}(s_i^n)\right)$ is the cost of one unit of consumption in the event $\{1, \ldots, i\}$. In our discrete economies, the BCE cumulative price is necessarily a step function.

Let $\mathcal{H}$ be the set of all cumulatives. For any sequence $H^n \in \mathcal{H}$, we say that $H \in \mathcal{H}$ is its limit if the restriction of H to the unit interval is continuous and $\lim_n H^n(x) = H(x)$ for all $x \in (0,1]$. We say that $P \in \mathcal{H}$ is a *limit BCE price* for E^n if E^n has a subsequence of BCE cumulative prices that converge to P.

Lemma 2. *Every convergent E^n has a concave limit BCE price P.*

Below, we pass to the relevant subsequence of prices for some convergent sequence E^n and refer to a limit BCE price simply as the limit BCE price P. As a benchmark, consider the limit price of standard economies. The unique SCE price $p \in \Delta(N)$ is

$$p_i^* = \frac{\pi_i u'(s_i)}{\sum_{j=1}^n \pi_j u'(s_j)}$$

Since $s_i \in [a,b]$ it follows that for all $i \in N$ the pricing kernel κ^* of a standard economy satisfies:

$$\kappa_i^* = \frac{p_i^*}{\pi_i} \in \left[\frac{u'(b)}{u'(a)}, \frac{u'(a)}{u'(b)}\right]$$

Thus, the SCE pricing kernel is bounded away from zero and infinity. Moreover, if $b - a$ is small then $u'(b)/u'(a)$ is close to 1 and therefore the range of the pricing kernel is small. It is straightforward to show that there is a unique equilibrium limit price P^* along for any convergent sequence of SCE prices. This limit price is

$$P^*(x) = \frac{\int_0^x u'(F^{-1}(r))dr}{\int_0^1 u'(F^{-1}(r))dr}$$

where F is the cdf of the limit endowment. Hence, P^* is continuous and differentiable on $[0,1]$ with a derivative equal to the limit pricing kernel and uniformly bounded away from zero and infinity. If the variation in limit endowment, $b - a$, is small then this derivative is nearly constant and hence bounded above and away from 0. The following theorem show that limit BCE prices of convergent sequence have neither of these properties.

Theorem 4. *If P is a limit BCE price for some convergent E^n, then $P(0) > 0$ and $P'(1) = 0$. If $\rho \geq 1$ then, in addition, $P'(0) = \infty$ and if $\rho > 1$ then $P(x) = 1$ for some $x < 1$.*

Theorem 4 translates Theorem 3 into properties of the limit price: $P(0) > 0$ follows from the fact that $\lim p_1^n > 0$ and $P'(0) = \infty$ for $\rho \geq 1$ follow from the fact that the pricing kernel is arbitrarily large in the limit; $P'(1) = 0$ follows from the fact that the pricing kernel is arbitrarily close to zero; the final part of the theorem follows from the fact that the limit price is zero with positive probability if $\rho > 1$.

We can relate the quantity $P(0)$ to the value of relaxing the crudeness constraint. Suppose, at a cost τ, the household can relax the crudeness constraint from k to $k+1$. Thus, after this trade the household has smaller wealth (by τ) but can choose a plan in $\mathcal{C}_{k+1}$. In equilibrium, some households must consume at least a in the lowest state and, therefore, if $\tau < a \cdot P(0)$ (where a is the lowest endowment realization) then the household is sure to benefit from this trade. Consider a new plan that isolates state 1 and consumes $\epsilon > 0$ in that state. In the limit, this plan relaxes the budget constraint by $P(0)(a - \epsilon) > \tau$ for ϵ sufficiently small. The utility of this plan is only slightly lower than the original utility and their difference vanishes as n goes to infinity because the probability of state 1 goes to zero. Thus, the magnitude of $aP(0)$ is bounded above by shadow price of the crudeness constraint. It follows that $P(0)$ converges to zero as k goes to infinity: the equivalence of equilibrium allocations and solutions to the planner's problem established in Theorem 1 ensures that the value of an additional partition element must go to zero eventually. Since $aP(0)$ is smaller than the shadow price of additional partition elements it follows that $P(0)$ must converge to zero as k goes to infinity.

3.3 An Example

Let $E^n = (u, k, \pi^n, s^n)$ be a convergent sequence of pure endowment economies such that $u(x) = \ln x$ and suppose that the limit endowment is uniformly distributed on $[a, b]$. Let $k = 2$, i.e., each household is confined to at most two different consumption levels. Theorem 1 and

Lemma 1 together imply that for each n, a BCE (p^n, μ^n) exists and involves households choosing monotone consumption plans. For $k = 2$, monotonicity means that for each household, there is a cutoff state j and two consumption levels $x \leq y$ such that the household consumes x if the state is lower than j and consumes y otherwise. Since the utility function is unbounded, the price must be strictly greater than zero in every state, which implies that the feasibility constraint holds with equality. This, in turn, implies that for every j, there is a consumption plan in the support of μ^n with cutoff j; otherwise aggregate consumption would be the same in two consecutive states and since aggregate endowment is strictly increasing, feasibility would not be satisfied with equality.

Figure 1 depicts the BCE limit price (P—solid line) and the standard limit price (P^*—dashed line). By Lemma 2, the limit cumulative price P is concave. By Theorem 4, $P(0) > 0$; that is, the limit price has a mass point at 0.

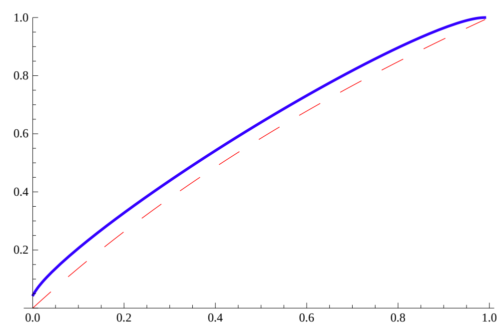

Figure 1: P and P^*

Figure 2 shows the derivatives of the limit BCE price (P'—solid line) and the standard limit price ($[P^*]'$—dashed line). By Theorem 3, P' tends to zero at 1 (as the endowment converges to b) and to infinity at 0 (as the endowment converges to a).

4 BCE in a Dynamic Economy

In this section, we extend our analysis to a dynamic Lucas-tree economy. We show that there is a one-to-one correspondence between the stationary BCE of our dynamic economy and the BCE of a corresponding static economy. This correspondence enables us to relate the extreme

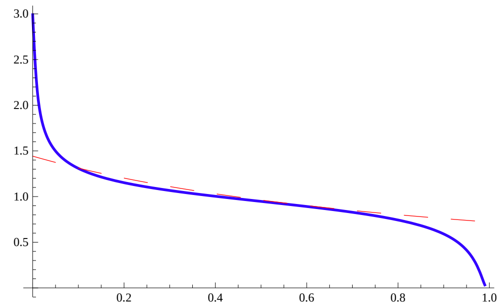

Figure 2: P' and $[P^*]'$

consumption prices analyzed in Theorem 3 to extreme asset prices.

As in the static economy, $N = \{1, \ldots, n\}$ is a finite set of states and each state $i \in N$ implies a dividend (endowment) realization $s_i \in [a, b]$. A t-period history h is a vector $(i_1, \ldots, i_t) \in N^t$; we write hi for the history (h, i). We call H^t the set of all t-period histories and $H = \bigcup_{t \geq 1} N^t$ the set of all histories. Given any history $h = (i_1, \ldots, i_t)$, we let $\jmath(h) = i_t$ and let $H_i^t = \{h \in N^t : \jmath(h) = i\}$ be the set of all t-period histories that end in i.

A matrix of transition probabilities, Φ, describes the evolution of the state; Φ_{ij} is the probability that the state at date $t + 1$ is j given that it is i on date t. We assume that Φ has a stationary distribution π; that is,

$$\pi = \pi \cdot \Phi \tag{3}$$

The initial state (the period 1 history) is drawn from the stationary distribution π. Therefore, the probability of history $h = (i_1, \ldots, i_t) \in H^t$ is

$$\lambda_h = \pi_{i_1} \cdot \Phi_{i_1 i_2} \cdots \Phi_{i_{t-1} i_t} \tag{4}$$

Households choose a consumption plan before learning the initial state. The assumption that the initial state is chosen according to the invariant distribution means that we can ignore transitory effects of the initial condition. As we show below, the economy has a stationary equilibrium allocation and stationary equilibrium prices. Moreover, we can map the dynamic economy to the two-period economy analyzed in the previous section.

A function $d \in \mathbb{R}_+^H$ is a (dynamic) consumption plan and $\mathcal{D}$ is the set of all consumption plans. The definition of crude consumption plans mirrors the corresponding definition for the static economy:

Definition 7. The consumption plan $d \in \mathcal{D}$ is *crude* if $|\{d_h \,|\, h \in H\}| \leq k$.

Let $\mathcal{D}_k$ be the set of all *crude* consumption plans. As in the static economy, we assume that each household chooses its crude consumption plan before the initial state is realized. Hence, each household is restricted to at most k different levels of consumption throughout its entire lifetime.[4]

The household's utility from the consumption plan d is

$$V(d) = (1-\beta) \sum_{t \geq 1} \sum_{h \in N^t} u(d_h) \beta^{t-1} \lambda_h \tag{5}$$

where $\beta \in (0,1)$ is the discount factor. The sextuple $E^* = (u, \beta, k, \pi, s, \Phi)$ is a dynamic economy.

An *allocation* is a probability distribution on $\mathcal{D}$. It is crude if its support is contained in the set of crude consumption plans. Thus, the set of dynamic allocations is $\Delta(\mathcal{D})$ and the allocation $\nu \in \Delta(\mathcal{D})$ is crude if $K(\nu) \subset \mathcal{D}_k$. The allocation ν is *feasible* in E^* if

$$\sigma_h^*(\nu) := \sum_{d \in K(\nu)} d_h \cdot \nu(d) \leq s_{\jmath(h)}$$

for all $h \in H$. For any $\mathcal{D}' \subset \mathcal{D}$, let $M^*(\mathcal{D}')$ be the set of all feasible allocations ν such that $\nu(\mathcal{D}') = 1$. Hence, $M^*(\mathcal{D}_k)$ is the set of feasible crude allocations for E^*. Henceforth, ν is feasible means $\nu \in M^*(\mathcal{D}_k)$.

A function $q \in \mathbb{R}_+^H$ is a (dynamic) *price* if $\sum_H q_h = 1$. The representative household's budget is

$$B_k^*(q) = \left\{ d \in \mathcal{D}_k : \sum_{h \in H} q_h [d_h - s_{\jmath(h)}] \leq 0 \right\} \tag{6}$$

The crude consumption plan $d \in B_k^*(p)$ is *optimal at prices* q if $V(d) \geq V(d')$ for all $d' \in B_k^*(q)$.

[4] An alternative version of the model in which households face a weaker constraint and are allowed to choose k different consumption levels *each* period is also possible. If the state process is iid, the alternative model would have exactly the same set of equilibria as our current model. If the Markov process is not iid, then the alternative model would be more difficult to analyze but we conjecture that extreme asset prices would emerge in that version as well.

Definition 8. The price-allocation pair (q, ν) is a *BCE* of E^* if ν is feasible for E^* and if $\nu(d) > 0$, then d is optimal at prices q.

Fix a dynamic economy $E^* = (u, \beta, k, \pi, s, \Phi)$ and consider the static economy $E = (u, k, \pi, s)$. The two economies share the same utility function and crudeness constraint. In both economies the initial endowment is chosen according to the distribution π. In the dynamic economy, the endowment evolves according to a Markov process while in the static economy the endowment stays fixed. Since π is the stationary distribution of the Markov process with transition matrix Φ it follows that

$$\sum_{h \in H_i^t} \lambda_h = \pi_i \tag{7}$$

for all $t \geq 1$. Hence, the ex ante probability of state i in period t is π_i for every t. Since $(1-\beta)\sum_t \sum_{h \in H_i^t} \beta^{t-1}\lambda_h = \pi_i$, the dynamic economy can be thought of as a version of the static economy in which each state i is split into many identical states corresponding to the branches of the event tree that end with i. We refer to $E = (u, k, \pi, s)$ as the *static economy for* $E^* = (u, \beta, k, \pi, s, \Phi)$. The measurability of BCE consumption plans in E will yield the stationarity of BCE consumption plans in E^*.

A consumption plan is stationary if consumption depends only on the current state. We can associate stationary consumption plans of the dynamic economy E^* with consumption plans of the corresponding static economy E. Formally, the plan d is *stationary* if there exists a consumption plan, c, for the static economy E such that $d_h = c_{\jmath(h)}$ for all $h \in H$. Let $\bar{\mathcal{D}}$ denote the set of stationary plans and let $T_1 : \mathcal{C} \to \bar{\mathcal{D}}$ be the above one-to-one mapping between static consumption plans and stationary dynamic plans. Thus, $d = T_1(c)$ is the dynamic consumption plan in which a household consumes c_i whenever the state i occurs. The set of stationary allocations is $\Delta(\bar{\mathcal{D}})$. Let $T_3 : \Delta(\mathcal{C}) \to \Delta(\bar{\mathcal{D}})$ be the one-to-one mapping between allocations in the static economy and stationary allocations in the dynamic economy defined by $\nu(d) = \mu(T_1^{-1}(d))$ for $d \in \bar{\mathcal{D}}$ and $\nu(d) = 0$ for $d \notin \bar{\mathcal{D}}$.

A price is stationary if the price after history h depends only on the current state $\jmath(h)$ and on the discounted probability of history h appropriately normalized. More precisely, q is *stationary* if there is a static price p (and a corresponding pricing kernel κ) such that for all

$t \geq 1$ and all $h \in N^t$

$$q_h = \lambda_h(1-\beta)\beta^{t-1}\tfrac{p_{\jmath(h)}}{\pi_{\jmath(h)}} \tag{8}$$
$$= \lambda_h(1-\beta)\beta^{t-1}\kappa_{\jmath(h)}$$

Equations (7) and (8) imply

$$\sum_{t=1}^{\infty}\sum_{h\in H_i^t} q_h = p_i \tag{9}$$

for all i and hence $\sum_{h\in H} q_h = \sum_{i=1}^{n} p_i = 1$. Hence, for each static price p there is a corresponding stationary dynamic price and, conversely, each stationary dynamic price can be mapped to a static price.

Let $\bar{Q} \subset \Delta(H)$ be the set of stationary prices. Let $T_2 : \Delta(N) \to \bar{Q}$ be the one-to-one mapping between prices in the static economy and stationary prices in the dynamic economy defined above. To summarize: $T_1 : \mathcal{C} \xrightarrow{1-1} \bar{\mathcal{D}}$ is the mapping that identifies a the unique stationary consumption plan associated with each static consumption plan, $T_2 : \Delta(N) \xrightarrow{1-1} \bar{Q}$ defines the unique stationary price associated with each static price and $T_3 : \Delta(\mathcal{C}) \xrightarrow{1-1} \Delta(\bar{\mathcal{D}})$ defines the unique stationary allocation associated with each static allocation.

Equation (7) implies that

$$V(T_1(c)) = (1-\beta)\sum_{t\geq 1}\sum_{h\in N^t} u(c_{\jmath(h)})\beta^{t-1}\lambda_h = U(c) \tag{10}$$

and Equation (9) implies that $c \in B_k(p)$ if and only if $T_1(c) \in B_k^*(T_2(p)) \cap \bar{\mathcal{D}}$. Finally, note that $\mu \in \Delta(\mathcal{C})$ is feasible in E if and only if $T_3(\mu) \in \Delta(\bar{\mathcal{D}})$ is feasible in E^*.

Theorem 5 below connects BCEs of the dynamic economy and BCEs of the corresponding static economy. An equilibrium allocation of the static economy yields a stationary equilibrium allocation of the corresponding dynamic economy an equilibrium price of the static economy yields a stationary equilibrium price of the dynamic economy.

Theorem 5. *(i) If (p,μ) is a BCE of E, then $(T_2(p), T_3(\mu))$ is a BCE of E^*. (ii) If ν is a BCE allocation for E^* then $\nu \in \Delta(\bar{\mathcal{D}})$ and $T_3^{-1}(\nu)$ is a BCE allocation for E.*

Theorem 5 leaves open the possibility of a non-stationary BCE price (supporting a stationary BCE allocation). Theorem 5 relies on the assumption that the initial state is chosen according to the stationary distribution π. Without this assumption, there might still be an analogue of Theorem 5 but the mapping between the dynamic and the static economy would be more complicated.

5 Extreme Asset Prices and the Safe Haven Premium

Recall that a static pure endowment static economy $E = (u, k, \pi, s)$ has a unique BCE price. In this section, we relate this price p to asset prices in the dynamic economy. To simplify the exposition, we restrict ourselves to iid transitions: $\Phi_{ij} = \pi_j$ for all i, j and refer to a dynamic economy with constant transition probabilities as an *iid economy*. In addition, we assume $\rho = 1$ so that $u(z) = \ln z$. These restrictions are made for expositional ease. All results below can be extended to arbitrary Markov transitions provided that all ratios of transition probabilities stay bounded. With $\rho > 1$ the equilibrium price of consumption may be zero; this would allow us to strengthen some of the results below but at the cost of a more cumbersome notation. With $\rho < 1$ we would need to slightly weaken the result on extremely low asset prices.

Consider an asset $z = (z_1, \dots, z_n)$ in zero net supply that delivers z_i units of the consumption good next period in state i. Let $r_h(z)$ be the BCE price of this asset in terms of current period consumption after history h. Recall that $\jmath(h) \in N$ is the last state of history h. A standard no-arbitrage argument implies that

$$r_h(z) = \frac{\sum_{i \in N} q_{hi} z_i}{q_h} \tag{11}$$

where the numerator is the expected value of the return z after history h and the denominator is the price of consumption after history h. Since q is stationary, the price of the asset depends only on $\jmath(h)$ and therefore $r_h(z) = r_{\jmath(h)}(z)$. Formulas (8) and (11) imply that

$$r_h(z) = r_{\jmath(h)}(z) = \beta \frac{E[\kappa z]}{\kappa_{\jmath(h)}} \tag{12}$$

where $E[\xi]$ is the expectation of the random variable $\xi : N \to \mathbb{R}$ with respect to π. Since $\kappa_i > 0$ for all $i \in N$ the asset price is well defined and since r_h depends only on the most recent state we write $r(z) = (r_1(z), \ldots, r_n(z))$ for the vector of asset prices.

As in Theorem 3, we consider a sequence of economies that converges to a limit economy with a continuous distribution of endowments. Formally, we say that a sequence of iid-economies $\{E^{*n}\} = \{(u, \beta, k, \pi^n, s^n)\}$ is *convergent* if the corresponding sequence of static economies $\{E^n\} = \{(u, k, \pi^n, s^n)\}$ is convergent. Consider a sequence of asset returns $\{z^n\}$ with $z^n \in \mathbb{R}^n$. We say that $\{z^n\}$ is bounded if there are $0 < \gamma_1 < \gamma_2 < \infty$ such that $z_i^n \in [\gamma_1, \gamma_2]$ for all i and n.

Theorem 6 shows that limit equilibrium asset prices are extremely high if the endowment is near its upper bound and extremely low if the endowment is near its lower bound.

Theorem 6. *Let $\{z^n\}$ be a bounded sequence of asset returns, let $\{E^{*n}\}$ be a convergent sequence of iid economies and let $r^n(z^n)$ be the equilibrium asset price of z^n in E^{*n}. Then,* $\lim \mathit{Prob}\ (r^n(z^n) < \epsilon) > 0$ *for all* $\epsilon > 0$ *and* $\lim \mathit{Prob}\ (r^n(z^n) > K) > 0$ *for all* K*.*

Theorem 6 is a corollary of Theorem 3 and Equation (12) above. Theorem 3 can be used to derive another asset pricing implication that relates *riskfree* and *nearly-riskfree* asset prices. Let $e^n = (1, \ldots, 1)$ be a risk-free asset and let $e^{n\epsilon}$ be the following nearly-riskfree asset:

$$e_i^{n\epsilon} = \begin{cases} 1 & \text{if } i \geq \jmath_\epsilon^n \\ 0 & \text{if } i < \jmath_\epsilon^n \end{cases}$$

Recall that $\jmath_x^n = \max\{j : \sum_{i \leq j} \pi_i \leq x\}$ and hence $e^{n\epsilon}$ yields 1 in all but the $\epsilon-$fraction of states with the lowest endowment. Let $\{E^{*n}\}$ be a convergent sequence of iid-economies and let $r^n(e^n)$ and $r^n(e^{n\epsilon})$ be the equilibrium asset price of e^n and $e^{n\epsilon}$ in E^{*n}.

Theorem 7. *There is $\delta > 0$ such that* $\lim \frac{r_i^n(e^{n\epsilon})}{r_i^n(e^n)} \leq 1 - \delta$ *for all* $\epsilon > 0$.

Theorem 7 shows that the risk-free premium does not converge to zero as the returns of the nearly riskfree asset converges in distribution to the returns of the riskfree asset. This safe-haven premium comes about because the price of consumption in the lowest endowment state is bounded away from zero and, therefore, the risk-free asset is always more costly than the nearly risk free asset.

6 Conclusion

In this paper, we analyze the implication of crude consumption plans on equilibrium asset prices in a standard Lucas-style endowment economy. We assumed that agents' consumption plans must be k-crude, that is, may take on at most k distinct values. We show that the crudeness constraint leads to extreme and volatile prices when the endowment realization is near its upper or lower bound.

In settings where agents are cognitively constrained, the market mechanism acquires a new function; allocate agents' scarce attention. For markets to clear, the equilibrium prices have to accentuate the relevant events to attract the households' attention. Since it is particularly costly to pay attention to tail events, the price variation in those events has to be large enough to make them salient.

In our formulation, the household's consumption decision is crude, i.e., does not depend on all the details of the underlying economic conditions. On the other hand, the household financing decisions are unconstrained and typically a household's net trades will depend on the exact state of the world. An alternative model would impose crudeness constraints on the financial transactions. For example, assuming household's *net trades* are 2-crude amounts to assuming that states are partitioned into "borrowing states" and "lending states" and the household borrows some fixed amount x whenever she finds herself in a borrowing state and lends a fixed amount y if she finds herself in a lending state. More generally, assuming k-crude net trades would allow us to capture households that are constrained in their financial transactions.

There is a modeling tradeoff between the two types of constraints. If net trades must satisfy a crudeness-constraint then – for a generic choice of the endowment – the consumption choice will not be crude. Which constraint is appropriate depends on the particular application; specifically, it depends on what is the household's *active decision*, i.e., the decision that is the focus of the analysis, and which is the *residual decision*, i.e., the decision implied by the active decision and the budget constraint. Our model describes the allocation of real resources and in this context the focus of the household's decision is the consumption choice.

A Appendix: Proofs

If $K(\mu) \subset \{c^1, \ldots, c^m\}$, we write $\mu = (\mathbf{a}, \mathbf{c})$ where $\mathbf{a} = (\alpha^1, \ldots, \alpha^m)$, $\mathbf{c} = (c^1, \ldots, c^m)$ and $\mu(c^l) = \alpha^l$ for all l. It will be understood that $\mathbf{a} = (\alpha^1, \ldots, \alpha^m)$, $\hat{\mathbf{a}} = (\hat{\alpha}^1, \ldots, \hat{\alpha}^m)$, and so forth. We follow the same convention with $\mathbf{c}, \hat{\mathbf{c}}$ etc. If $\{c^1, \ldots, c^m\}$ contains exactly one representative from each equivalence class of $\sim$, we say that $\mu = (\mathbf{a}, \mathbf{c})$ is in simple form. Thus, μ can be expressed in simple form if and only if it is simple. We write δ_c for the allocation μ where all households consume $c \in \mathcal{C}_k$.

A.1 Proof of Lemma 1

Lemma 3. *If μ is feasible and not simple, then there is a simple and feasible μ' such that $W(\mu') > W(\mu)$.*

Proof. Let $\mu = (\mathbf{a}, \mathbf{c})$. If μ is not simple, there is $c, c' \in K(\mu)$ such that $c \sim c'$. Let $c^* = \gamma \cdot c + (1-\gamma)c'$ where $\gamma = \frac{\mu(c)}{\mu(c)+\mu(c')}$ and let μ^* be the allocation derived from μ by replacing c, c' with ($\mu(c) + \mu(c')$ probability of) c^*. Since, $c \sim c'$ are crude, so is c^* and μ^*. Since u is strictly concave, $W(\mu^*) > W(\mu)$. Note that $|K(\mu^*)| < |K(\mu)|$. If μ^* is simple, we are done. Otherwise, repeat the above argument. Since $K(\mu)$ is finite, this process must terminate with a simple allocation. □

Lemma 4. *If μ is feasible, simple but not fair, then there is a feasible, simple and fair μ' such that $W(\mu') > W(\mu)$ and $|K(\mu')| \leq |K(\mu)|$.*

Proof. Let $\mu = (\mathbf{a}, \mathbf{c})$, let x^l be the certainty equivalent of c^l and $\bar{x}^l$ be the corresponding constant consumption plan; that is, $u(x^l) = U(c^l)$ and $\bar{x}^l_i = x^l$ for all i, l. Also, let $x = \sum_{l=1}^m \alpha^l c^l$ and let $\bar{x}$ be the corresponding constant consumption plan. Let $\hat{\mu} = (\hat{\mathbf{a}}, \hat{\mathbf{c}})$ such that $\hat{\alpha}^l = \frac{\alpha^l x^l}{x}$ and $\hat{c}^l = \frac{x c^l}{x^l}$ for all l. Finally, let $\bar{\mu} = (\bar{\mathbf{a}}, \bar{\mathbf{c}})$ such that $\bar{\alpha}^l = \alpha^l$ and $\bar{c}^l = \bar{x}^l$ for all l. Since u is strictly concave and μ is not fair, $W(\delta_{\bar{x}}) > W(\bar{\mu})$. Since u is CRRA,

$$u^{-1}(U(\hat{c}^l)) = \frac{x}{x^l} u^{-1}(U(c^l)) = \frac{x}{x^l} x^l = x;$$

hence, $W(\hat{\mu}) = W(\delta_{\bar{x}})$. By definition, $W(\bar{\mu}) = W(\mu)$. Hence, $W(\hat{\mu}) > W(\mu)$. By construction $\hat{\mu}$ is fair. It is easy to verify that $\sum_l \hat{c}_i^l \hat{\alpha}^l = \sum_l c_i^l \alpha^l$ for all $i \in N$ and hence $\hat{\mu}$ is feasible. Clearly, $|K(\hat{\mu})| \leq |K(\mu)|$. □

Lemma 5. *A solution to the planner's problem exists and every solution to the planner's problem is simple and fair.*

Proof. The allocation δ_c such that $c_i = \min_i s_i$ for all i is feasible. Thus, $M(\mathcal{C}_k)$ is non-empty. Since δ_s second order stochastically dominates any feasible μ it follows that $W(\mu) < W(\delta_s)$ for every feasible $\mu \in M(\mathcal{C}_k)$. Hence,

$$W_k = \sup_{\mu \in M(\mathcal{C}_k)} W(\mu)$$

is well-defined. By Lemmas 3 and 4, there exists a sequence of feasible, simple, and fair allocations $\mu^t = (\mathbf{a}^t, \mathbf{c}^t)$ such that $W(\mu^t) \geq W_k - 1/t$ and $\mathbf{a}^t \in \mathbb{R}_+^m$ for all t, where m is the cardinality of the set of all partitions of N with k or fewer elements.

By passing to a subsequence, $\mathbf{a}^t = (\alpha^{1t}, \ldots, \alpha^{mt})$ converges to some $\mathbf{a} \in \Delta(\mathbb{R}_+^m)$. If c^{lt} is unbounded for some l, we must have $\alpha^l = 0$. Let $A \subset N$ be the set of l such that $\alpha^l \neq 0$. Then, $A \neq \emptyset$ and c^{lt} is bounded for all $l \in A$. Hence, there exists a subsequence of μ^t along which c^{lt} converges to some $c^l \in \mathcal{C}_k$ for every $l \in A$.

Let $\mu = (\mathbf{a}, \mathbf{c})$ where $\mathbf{a} = (\alpha^l)_{l \in A}$ and $\mathbf{c} = (c^l)_{l \in A}$. Since $\lim W(\mu^t) = W_k$ and each μ^t is fair, $U(c^{lt}) = W(\mu^t)$. So, by the continuity of u, we have $U(c^l) = W_k$ for all $l \in A$ and therefore $W(\mu) = W_k$. Finally, $\sum_{l \in A} \alpha^{lt} c_i^{lt} \leq \sum_{l \in A} \alpha^{lt} c_i^{lt} \leq s_i$ for all i, l, t and so $\sum_{l \in A} \alpha^l c_i^l \leq \sum_{l \in A} \alpha^l c_i^{lm} \leq s_i$ for all $i, l,$. Hence μ is feasible and therefore μ solves the planner's problem. Then, Lemmas 3 and 4 imply that μ must be simple and fair. □

Lemma 6. *Let $E = (u, k, \pi, s)$, $\hat{E} = (u, k, \hat{\pi}, \hat{s})$ be such that $F_s = F_{\hat{s}}$ and let W_k and $\hat{W}_k$ be the maximal mean utility attainable in E and $\hat{E}$ respectively. Then, $W_k = \hat{W}_k$.*

Proof. Without loss of generality, we assume that E is a pure endowment economy; that is, $N = \{1, \ldots, n\}$, s is one-to-one, $\hat{N} = \{ij : i \in N, j \in N_i\}$ for some collection of N_i's such that $N_i = \{1, \ldots, n_i\}$ for all i, $s_i = s_{ij}$ for all i and $j \in N_i$ and $\sum_{j \in N_j} \hat{\pi}_{ij} = \pi_i$. We will show that *(i)* for any feasible allocation in E, there is a feasible allocation in $\hat{E}$ that achieves the same mean utility and *(ii)* vice versa. Then, 5 ensures that $W_k = \hat{W}_k$.

(i) For any feasible allocation $\mu = (\mathbf{a}, \mathbf{c})$ in E define the allocation $\hat{\mu} = (\hat{\mathbf{a}}, \mathbf{c})$ for the economy $\hat{E}$ as follows: $\hat{c}_i^l = c_{ij}^l$ for all i and $j \in N_i$. Clearly, $\hat{W}(\hat{\mu}) = W(\mu)$ and $\hat{\mu}$ is feasible (for $\hat{E}$).

(ii) We will first show that given any consumption $\hat{c}$ for $\hat{E}$, there is an allocation, $\mu = (\mathbf{a}, \mathbf{c})$, for E such that $W(\mu) = \hat{U}(\hat{c})$ and

$$\sigma_i(\mu) = \frac{1}{\pi_i} \sum_{j \in N_i} \pi_{ij} \hat{c}_{ij}$$

for all $i \in N$. Let $L = \{(j_1, \ldots, j_n) : j_i \in N_i \forall i \in N\}$ and let $\gamma^{ij} = \frac{\hat{\pi}^{ij}}{\pi_i}$ for all $i \in N$ and $j \in N_i$. For each $l = (j_1, \ldots, j_n) \in L$, let $\alpha^l = \gamma^{1j_1} \cdot \gamma^{2j_2} \cdots \gamma^{nj_n}$ and $c_i^l = \hat{c}_{ij_i}$. Since $\hat{c}$ is crude and $\{c_i^l : i \in N\} \subset \{\hat{c}_{ij_i} : i \in N, j \in N_i\}$, each c^l is crude. Verifying that $W(\mu) = U(\hat{c})$ and that the equation displayed above holds for all $i \in N$ is straightforward.

To conclude, let $\hat{\mu} = (\hat{\mathbf{a}}, \hat{\mathbf{c}})$ be a feasible allocation in $\hat{E}$. For each $\hat{c}^l \in K(\hat{\mu})$, choose μ^l as described above. Let $\mu = \sum_l \hat{\alpha}^l \cdot \mu^l$. Then, μ is feasible since $\sigma_i(\mu) \leq \max_{j \in N_i} \sigma_{ij}(\hat{\mu}) \leq s_i$ for all $i \in N$. Since $W(\mu^l) = \hat{U}(\hat{c}^l)$ for all l, we have $W(\mu) = \hat{W}(\hat{\mu})$ as desired. □

To conclude the proof of Lemma 1, we will show that if $\mu = (\mathbf{a}, \mathbf{c})$ is a solution to the planner's problem, then it is also monotone and measurable. Assume μ is not measurable; that is, $c_j^l > c_k^l$ and $s_j = s_k$ some l with $\alpha^l > 0$. Then, let $\hat{\mu} = (\hat{\mathbf{a}}, \hat{\mathbf{c}})$ where $\hat{\mathbf{a}} = (\alpha^1, \ldots, \alpha^{l-1}, \alpha^{l_j}, \alpha^{l_k}, \alpha^{l+1}, \ldots, \alpha^m)$, $\hat{\mathbf{c}} = (c^1, \ldots, c^{l-1}, c^{l_j}, c^{l_k}, c^{l+1}, \ldots, c^m)$ and $\alpha^{l_j} = \frac{\alpha^l \pi_j}{\pi_j + \pi_k}$, $\alpha^{l_k} = \alpha^l - \alpha^{l_k}$, $c_i^{l_j} = c_i^{l_k} = c_i^l$ for all $i \neq j, k$, $c_j^{l_j} = c_k^{l_j} = c_j^l$ and $c_j^{l_k} = c_k^{l_k} = c_k^l$. Then, $\hat{\mu}$ achieves the same mean utility as μ and is feasible but not simple contradicting Lemma 4.

To prove that μ satisfies monotonicity, we note that if $s_i \geq s_j$, then $\sigma_i(\mu) \geq \sigma_j(\mu)$. If not, $\sigma_i(\mu) < s_i$ and there must be some l such that $c_i^l < c_j^l$. Then, let $\hat{c}_t = c_t^l$ for $t \neq i$ and $\hat{c}_i = c_j^l$. Note that $\hat{c}$ is crude and yields a strictly higher utility than c^l. Since $\sigma_i(\mu) < s_i$, mean utility can be increased by replacing c^l with $\hat{c}$ for a small fraction of households, contradiction the optimality of μ.

So, assume μ fails monotonicity: then, $c_i^l > c_j^l$ for some i, j such that $s_i < s_j$. Without loss of generality, we assume $i = 1, j = 2$ and $l = 1$. We showed in the previous paragraph that $\sigma_2(\mu) \geq \sigma_1(\mu)$, so we must have c^o such that $c_2^o > c_1^o$; again without loss of generality, we assume $o = 2$ and that $\pi_i \geq \pi_j$ (an obvious adjustment is needed if this last inequality is reversed). To summarize: $c_1^1 > c_2^1; c_1^2 < c_2^2$ and $\pi_1 \geq \pi_2$.

Construct a new state space $\hat{N} = N \cup \{n+1\}$ and $\hat{s} \in [a,b]^{n+1}$ such that $\hat{s}_t = s_t$ for all $t \in N$ and $\hat{s}_{n+1} = s_1$. Also, set $\hat{\pi}_t = \pi_t$ for all $t \neq 1, n+1$, $\hat{\pi}_1 = \pi_2$ and $\hat{\pi}_{n+1} = \pi_1 - \hat{\pi}_2$. If $\pi_{n+1} = 0$, then ignore state $n+1$ in the argument below.

Next we construct a feasible allocation $\hat{\mu} = (\hat{\mathbf{a}}, \hat{c})$ for $\hat{E}$ such that (i) $\hat{W}(\hat{\mu}) = W(\mu)$ and (ii) $\hat{\mu}$ is not fair; that is, there is i, j such that $U(c_i) \neq U(c_j)$. By Lemma 6, $\hat{W}(\hat{\mu}) = W(\mu)$ implies that $\hat{\mu}$ solves the planner's problem for E. Since $\hat{\mu}$ is not fair this contradicts Lemma 5 and therefore completes the proof.

The allocation $(\hat{\mathbf{a}}, \hat{\mathbf{c}})$ has $m+4$ distinct consumption plans; for $1 \leq l \leq m$

$$\hat{c}_i^l = \begin{cases} c_i^l & \text{if } 1 \leq i \leq n \\ c_1^l & \text{if } i = n+1 \end{cases}$$

for $l = m+1, m+2$

$$\hat{c}_i^{m+1} = \begin{cases} c_1^1 & \text{if } i = 1,2 \\ c_i^1 & \text{if } 2 \leq i \leq n \\ c_1^1 & \text{if } i = n+1 \end{cases}; \quad \hat{c}_i^{m+2} = \begin{cases} c_2^1 & \text{if } i = 1,2 \\ c_i^1 & \text{if } 2 \leq i \leq n \\ c_1^1 & \text{if } i = n+1 \end{cases}$$

and for $l = m+3, m+4$

$$\hat{c}_i^{m+3} = \begin{cases} c_1^2 & \text{if } i = 1,2 \\ c_i^2 & \text{if } 2 \leq i \leq n \\ c_1^2 & \text{if } i = n+1 \end{cases}; \quad \hat{c}_i^{m+4} = \begin{cases} c_2^2 & \text{if } i = 1,2 \\ c_i^2 & \text{if } 2 \leq i \leq n \\ c_1^2 & \text{if } i = n+1 \end{cases}$$

Define

$$\gamma := \frac{c_2^2 - c_1^2}{c_2^2 - c_1^2 + c_1^1 - c_2^1}$$

and note that $0 < \gamma < 1$. Choose $0 < \epsilon < \min\{\alpha^1, \alpha^2\}$ and let

$$\hat{\alpha}^l = \begin{cases} \alpha^l & \text{if } 3 \leq l \leq m \\ \alpha^1 - \epsilon(1-\gamma) & \text{if } l = 1 \\ \alpha^2 - \epsilon\gamma & \text{if } l = 2 \\ \epsilon\gamma/2 & \text{if } l = m+1, m+2 \\ \epsilon\gamma/2 & \text{if } l = m+3, m+4 \end{cases}$$

It is straightforward to check that the allocation is feasible. Moreover, since $\hat{\pi}_1 = \hat{\pi}_2$ it follows that $\hat{W}(\hat{\mu}) = W(\mu)$. Since $U(\hat{c}_{m+1}) > U(\hat{c}_{m+2})$ it follows that $\hat{\mu}$ is not fair as desired. □

A.2 Proof of Theorem 1

Let $Z = \mathbb{R}^N_{++}$. Then, for all $z \in Z$, let $M^z(\mathcal{C}')$ be the set of all allocations with support contained in $\mathcal{C}'$ that are feasible for the economy $E = (u, k, \pi, z)$. Let

$$W_k(z) = \max_{\mu \in M^z(\mathcal{C}_k)} W(\mu).$$

Hence, W_k is the planner's value as a function of the endowment. Let $Z^s = \{z \in Z : W_k(z) > W_k(s)\}$.

Clearly, $W_k(z) > W_k(y)$ whenever $z_i > y_i$ for all $i \in N$ since we can take the optimal allocation for y and increase every consumption in every state by a small constant amount. Hence, Z^s is nonempty.

Suppose $|z_i - y_i| < \epsilon$ for all i. Let $y_i^+ = \max\{y_i, z_i\}$, $y_i^- = \min\{y_i, z_i\}$ for all i and let $\mu = (\mathbf{a}, \mathbf{c})$ be optimal for (u, π, y^+). Then, $(\mathbf{a}, (1 - \frac{\epsilon}{a})\mathbf{c})$ is feasible for (u, π, y^-). Since $W(\mathbf{a}, (1 - \frac{\epsilon}{a})\mathbf{c})$ is continuous in ϵ at $\epsilon = 0$ and $W_k(y)$ is nondecreasing in each coordinate, for $\epsilon' > 0$, there exists $\epsilon > 0$ such that

$$|W_k(y)) - W_k(z))| \leq |W_k(y^+)) - W_k(y^-))| < \epsilon'$$

proving that W is continuous at y and hence Z^y is open.

We note that since W is a concave function of μ, W_k is a concave function of z and hence the set Z^s is convex. To see that, fix $z^1, z^2 \in Z^s$ and choose $\mu^i \in M^{z^i}(\mathcal{C}_k)$ such that $W(\mu^i) = W_k(z^i)$ for $i = 1, 2$. By Lemma 1, such μ^i exist. Clearly, $\gamma\mu^1 + (1-\gamma)\mu^2 \in M^{\hat{z}}(\mathcal{C}_k)$ for $\hat{z} = \gamma z^1 + (1-\gamma)z^2$ and hence $W_k(\gamma z^1 + (1-\gamma)z^2) \geq W(\gamma\mu^1 + (1-\gamma)\mu^2) = \gamma W(\mu^1) + (1-\gamma)W(\mu_2) = \gamma W_k(z^1) + (1-\gamma)W_k(z^2)$.

Since Z^s is nonempty, open, and convex, and $s \notin Z^s$, by the separating hyperplane theorem, there exists $p \in \mathbb{R}^n$ such that $p_i \neq 0$ for some i and $\sum_i p_i \cdot z_i > \sum_i p_i \cdot s_i$ for all $z \in Z^s$. Since W_k is nondecreasing in each coordinate, we must have $p_i \geq 0$ for every $i \in N$ and hence we can normalize p to ensure that $p \in \Delta(N)$.

Let $\mu = (\mathbf{a}, \mathbf{c})$ be a solution to the planner's problem, where $\alpha^l > 0$ for all l. The argument establishing that each c^l must maximize U given budget $B(p)$ is standard and omitted, as is the proof of the following lemma:

Lemma 7. *If (p, μ) is a BCE, then μ is Pareto-efficient.*

Finally, to see that if (p, μ) is a BCE, then μ must be a solution to the planner's problem, note that since every household has the same endowment, μ must be fair. But then, if μ did not solve the planner's problem, the solution to the planner's problem would Pareto-dominate it, contradicting Lemma 7. □

A.3 Proof of Theorem 2

Lemma 8. *The BCE price of a pure endowment economy is unique.*

Proof. First, we show that for all c in the support of μ, $c_i > 0$ for all i. For any c in the support of μ, let $A = \{i : c_i = 0\}$. If $A \neq \emptyset$, household optimality implies $\sum_{i \in A} p_i = 1$ and $\sum_{i \in N \setminus A} p_i = 0$; otherwise consumption can be raised by ϵ on the set A and lowered by $\frac{\epsilon \sum_{i \in A} p_i}{\sum_{i \in N \setminus A} p_i}$ on the set $N \setminus A$ resulting in an overall increase of utility for small ϵ. It follows that c costs the same as $2c$ and since $c_i > 0$ for some i and u is strictly increasing, c cannot be optimal if $A \neq \emptyset$.

Since E is a pure endowment economy, assume without loss of generality that $s_i < s_{i+1}$. For any μ, let $I(\mu) = \{i < n : c_i < c_{i+1}$ for some $c \in K(\mu)\}$. Since every BCE allocation solves the planner's problem and $k > 1$ (i.e., household can have at least two distinct consumption levels), $I(\mu) \neq \emptyset$. Hence, for any competitive allocation μ, let $J(\mu) = \max I(\mu)$. Let (μ^l, p^l) for $l = 1, 2$ be two BCE.

We claim that $i \notin I(\mu^l)$ implies $i+1 \notin I(\mu^l)$. To see why this is the case, note that if $i \notin I(\mu^l)$, then $\sigma_i(\mu^l) = \sigma_{i+1}(\mu^l)$ and therefore $\sigma_{i+1}(\mu^l) < s_{i+1}$ and hence $p_{i+1} = 0$. Then, if $c_{i+2} > c_{i+1}$ for any $c \in K(\mu^l)$, define $\hat{c}_j = c_j$ for $j \leq i$, $\hat{c}_j = c_j$ for $j \geq i+2$ and $\hat{c}_j = c_{i+1}$ and note that $\hat{c}$ is crude, costs the same as c but yields strictly higher utility, contradicting the fact that μ^l is a BCE allocation.

Next, we claim that $J(\mu^1) = J(\mu^2)$. If not, assume without loss of generality that $J(\mu^1) > J(\mu^2)$. Define $\hat{s}_j = s_j$ for all $j \leq J(\mu^2)$ and $\hat{s}_j = s_j + 1$ for all $j \geq J(\mu^2) + 1$. Then, since we established in the preceding paragraph that $p_j = 0$ for all $j > J(\mu^2)$, we conclude that (p^2, μ^2) is a BCE for the economy with endowment $\hat{s}$. Therefore, by Theorem 1, $W_k(s) = W_k(\hat{s})$. But, since $i := J(\mu^2) < J(\mu^1)$, the previous claim implies $i \in I(\mu^1)$. Hence, there exist $c \in K(\mu^1)$ such that $c_i < c_{i+1}$. Since c is monotone (by Lemma 1), $\hat{c}$ defined by $\hat{c}_j = c_j$ for all $j \leq i$ and $\hat{c}_j = c_j + 1$ for all $j \geq k+1$ is crude. Let $\hat{\mu}$ be the allocation derived from μ^1 by replacing c with $\hat{c}$. Note that $\hat{\mu}$ yields strictly higher mean utility than μ^1 and is feasible for the economy with endowment $\hat{s}$, contradicting $W_k(s) = W_k(\hat{s})$.

Note that if $J(\mu^1) = J(\mu^2) = 1$, then $p_1^1 = p_1^2 = 1$ and hence $p^1 = p^2$ as desired. So, henceforth we assume $J(\mu^1) = J(\mu^2) > 1$. By Theorem 1, both μ^1, μ^2 solve the planner's problem. Then, the linearity of W ensures that $\mu = .5\mu^1 + .5\mu^2$ also solves the planner's problem and hence by Theorem 1, there exists some p such that (p, μ) is a BCE. Then, the previous claim ensures that $J := J(\mu) = J(\mu^1) = J(\mu^2) > 1$.

For any c such that $c_j > 0$ for all j and for any $i = 1, \ldots, n-1$, define

$$MRS_i(c) = \frac{\sum_{\{j \leq i\}} \pi_j u'(c_j)}{\sum_{\{j > i\}} \pi_j u'(c_j)}$$

For the price p defined above, define $q \in \mathbb{R}^n$ such that $q_i = \sum_{j \leq i} p_i$. Define q^1, q^2 in an analogous fashion. For any $i \leq J$, pick $c^i \in K(\mu^1)$ such that $c^i_i < c^i_{i+1}$. Note that one possible crude plan can be constructed by changing consumption in all states $j \leq i$ by ϵ and in all states $j > i$ by ϵ' in a budget neutral manner. The optimality of c^i ensures that this alternative plan cannot increase utility which means:

$$q^1_i = MRS_i(c^i)(1 - q^1_i)$$

for all $i \leq J$. But since $K(\mu^1) \subset K(\mu)$, the equations above also hold for q proving that $q_j = q^1_j$ for all j and hence $p^1 = p$. A symmetric argument ensures that $p^2 = p$. □

Let $E = (u, k, \pi, s)$, $\hat{E} = (u, k, \hat{\pi}, \hat{s})$ be two static economies. We say that $\hat{E}$ is noisier than E if there is a function $g : \hat{N} \to N$ such that $\hat{s}_j = s_i$ whenever $g(j) = i$ and $\sum_{j:g(j)=i} \hat{\pi}_j = \pi_i$ for all i. We call such a g is a *homomorphism.* The homomorphism g is rational (uniform) if $i = g(j)$ implies $\frac{\hat{\pi}_j}{\pi_i}$ is a rational number ($g(j) = g(j')$ implies $\hat{\pi}_j = \hat{\pi}_{j'}$). Clearly, if g is uniform, then it is rational. When $\hat{E}$ is noisier than E, we write $[\hat{E}\|E]$; if there exists a rational (uniform) homomorphism from $\hat{E}$ to E, then we say $[\hat{E}\|E]$ is rational (uniform).

For any consumption c, price p and allocation $\mu = (\mathbf{a}, \mathbf{c})$ in E, define the corresponding consumption $\hat{c}$, price $\hat{p}$ and allocation $\hat{\mu} = (\mathbf{a}, \hat{\mathbf{c}})$ for $\hat{E}$ as follows: $\hat{c}_j = c_{g(j)}$; $\hat{p}_j = \frac{\hat{\pi}_j p_{g(j)}}{\pi_{g(j)}}$ for all j; $\hat{\mathbf{c}}$ is such that $\hat{c}^l = \theta_1(c^l)$ for all l. For any c let $\theta_1(c)$ be the $\hat{c}$ defined above; let $\theta_2(p)$ be the $\hat{p}$ defined above and for $\theta_3(\mu)$ be the $\hat{\mu}$ defined above. Let $\Theta^g = (\theta_1, \theta_2, \theta_3)$. Also, let $D(p)$ be the set of solutions to a households utility maximization problems at price p in the economy E. We use $\hat{D}$, $\bar{D}$ etc. for $\hat{E}$, $\bar{E}$ etc.

Lemma 9. *If $[\hat{E}\|E]$ is rational and $c \in D(p)$, then $\theta_1(c) \in \hat{D}(\theta_2(p))$.*

Proof. The following assertions are easy to verify: (1) $U(c) = \hat{U}(\theta_1(c))$ and (2) $c \in B(p)$ implies $\theta_1(c) \in \hat{B}(\hat{p})$.

Since $[\hat{E}\|E]$ is rational, there exists $\bar{E}$ such that $[\bar{E}\|\hat{E}]$ and $[\bar{E}\|E]$ are uniform. Let g be a $[\hat{E}\|E]$-homomorphism and $\Theta^g = (\theta_1, \theta_2, \theta_3)$; let $\hat{g}$ be a uniform $[\bar{E}\|\hat{E}]$-homomorphism and $\Theta^{\hat{g}} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$. Then, $\bar{g} = g \circ \hat{g}$ is a uniform $[\bar{E}\|E]$-homomorphism and let $\Theta^{\bar{g}} = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$, $\bar{\theta}_l = \hat{\theta}_l \circ \theta_l$ for $l = 1, 2, 3$. The two assertions above imply that we are done if we can show $\bar{c} \in D(\bar{\theta}_2(p))$ implies $\bar{c} = \bar{\theta}_1(c)$ for some $c \in D(p)$.

Suppose not and assume without loss of generality that for some $\bar{c} \in \bar{D}(\bar{\theta}_2(p))$, $\bar{c}_1 \neq \bar{c}_2$ despite $\bar{g}(1) = \bar{g}(2)$; that is, despite $\bar{p}_1 = \bar{p}_2$. Consider the endowment $\tilde{s}$ such that $\tilde{s}_1 = \tilde{s}_2 = (\bar{c}_1 + \bar{c}_2)/2$ and $\tilde{s}_i = \bar{c}_i$ for all $i > 2$. Let $c'_1 = c_2$ and $c'_2 = c_1$, $c'_i = \hat{c}_i$ for all $i > 2$. Note that $\bar{U}(c') = \bar{U}(\bar{c})$ and therefore $c' \in \bar{D}(\bar{p})$. Hence, $\mu = 5\delta_{\bar{c}} + .5\delta_{c'}$ is a BCE for the economy $(u, \bar{\pi}, \tilde{s})$. Therefore, μ is a solution to the planner's problem (by Theorem 1). But μ is not measurable, contradicting Lemma 1. □

Lemma 10. *For any $E, \hat{E}$ such that $[\hat{E} \| E]$, $c \in D(p)$ implies $\theta_1(c) \in \hat{D}(\theta_2(p))$.*

Proof. Let $\hat{E} = (u, \hat{\pi}, \hat{s})$ where $\hat{\pi} \in \Delta(\hat{N})$ and let g be the $[\hat{E} \| E]$-homomorphism. We can construct a sequence $\hat{\pi}^m$ converging to $\hat{\pi}$ such that for all j, $\frac{\hat{\pi}_j}{\pi_{g(j)}}$ is a rational number. Then, g is a rational homomorphism from $E^m = (u, \hat{\pi}^m, \hat{s})$ to E. Let $c \in D(p)$ and $\hat{c} = \theta_1(c)$, $\hat{p} = \theta_2(p)$. Then, by Lemma 9,

$$\sum_{i \in \hat{N}} u(\hat{c}_i) \cdot \hat{\pi}_i^m \geq \sum_{i \in \hat{N}} u(c'_i) \cdot \hat{\pi}_i^m$$

for all $c' \in B(\hat{p})$ and therefore $\sum_{i \in \hat{N}} u(\hat{c}_i) \hat{\pi}_i \geq \sum_{i \in \hat{N}} u(c'_i) \hat{\pi}_i$ for all $c' \in B(\hat{p})$ as desired. □

Lemma 11. *The BCE price of a pure endowment economy is monotone.*

Proof. Suppose (p, μ) is the BCE of some pure endowment economy $E = (u, k, \pi, s)$ and $\frac{p_i}{\pi_i} > \frac{p_j}{\pi_j}$ for some $s_i > s_j$. Assume $\pi_i > \pi_j$ (if the opposite inequality holds, a symmetric argument applies; in case of equality ignore s_{n+1}). Let $\hat{s}_{n+1} = s_i$ and $\hat{\pi}_{n+1} = \pi_i - \pi_j$, $\hat{s}_l = s_l$ for all $l \leq n$, $\hat{\pi}_i = \pi_j$ and $\hat{\pi}_l = \pi_l$ for all $l \neq i, n+1$. Then, $[\hat{E} \| E]$ and hence, by Lemma 10, $(\hat{p}, \hat{\mu}) := (\theta_2(p), \theta_3(\mu))$ is a BCE of $\hat{E}$. By assumption $\hat{p}_i > \hat{p}_j$ and $\hat{\pi}_i = \hat{\pi}_j$ which means no household will consume more in state i than in state j. Since $s_i > s_j$, this implies $\hat{p}_i = 0$, a contradiction. □

Lemma 12. *The BCE price of any economy is essentially unique and monotone.*

Proof. Let $\hat{E} = (u, k, \hat{\pi}, \hat{s})$ be any economy and let $E = (u, k, \pi, s)$ be the corresponding pure endowment economy; that is $[\hat{E} \| E]$. For any plan measurable plan $\hat{c}^l$ for $\hat{E}$, define the plan c^l for E as follows $c_i^l = \hat{c}_j$ for some j such that $s_i = g(\hat{s}_j)$. Then, define μ^l in the obvious way: $\mu^l(c^1) = \hat{\mu}(\hat{c})$. Given a price $\hat{p}$ for E, define p^l for E as follows: $p_i^1 = \sum_{j: g(j) = i} \hat{p}_j$ for all i.

Suppose there are two BCE for $\hat{E}$, $(\hat{p}^l, \hat{\mu}^l)$ for $l = 1, 2$ such that $\hat{p}^1$ and $\hat{p}^2$ are not equivalent. Then, since $\hat{\mu}^l$'s are measurable, (Lemma 1) the corresponding μ^l are well-defined allocations for E. It is easy to see that (p^l, μ^l) are BCE equilibria for E. But since $\hat{p}^1$ and $\hat{p}^2$ are not equivalent, $p^1 \neq p^2$, contradicting Lemma 8.

Finally, take BCE of E and recall that by Lemma 10, $(\hat{p}, \hat{\mu}) := (\theta_2(p), \theta_3(\mu))$. By Lemma 11, (p, μ) is monotone and therefore, so is $(\hat{p}, \hat{\mu})$. Then essential uniqueness ensures that all BCE prices of $\hat{E}$ are monotone. □

A.4 Dominance Lemma

Fix a convergent sequence of economies $\{E^n\}$; let p^n, μ^n be a BCE of $E^n = (u, k, \pi^n, s^n)$. To simplify the notation below, we will not index the consumption sets, budget sets and the utility function by n. Let $f : [a, b] \to \mathbb{R}$ be the density of the limit endowment. Any sequence $\{x^n\} \subset X$ for some compact set X has a convergent subsequence. We use this fact and write $\lim x^n$ to denote the limit of a convergent subsequence without specifying the particular subsequence.

Lemma 13. *There is $\delta > 0$ such that $U(s^n) > U(\tilde{c}) + \delta$ for all $\tilde{c} \in K(\mu^n)$ and all n.*

Proof. For $l = 0, \ldots, k+1$, let $x_l := a + (b-a)\frac{l}{k+1}$ and define $N_l := \{i \in N | s_i^n \in [x_l, x_{l+1}]\}$ for $l = 0, \ldots, k$. Let $\beta_l := \int_{x_l}^{x_{l+1}} f(z)dz = \lim \sum_{N_l} \pi_i^n$ and note that $\beta_l > 0$ for all l. Let $\mathcal{A}_l \subset \mathcal{C}_k$ be the $k-$crude plans that are constant on N_l; note that any monotone $k-$crude plan belongs to some $\mathcal{A}_l$ for some $l = 0, \ldots, k$. Define $\alpha_l^n := \mu^n(\mathcal{A}_l)$ and note that $\sum_{l=0}^k \alpha_l^n = 1$; for $i \in N_l$ let $y_l^n = \sum_{\mathcal{A}_l} \mu^n(c) c_i / \alpha_l^n$ be the average consumption of those plans in the support of μ^n that are constant on N_l. Since $U(\tilde{c}) = U(c')$ for all $\tilde{c}, c' \in K(\mu^n)$ it follows that $U(\tilde{c}) = \sum_{K(\mu^n)} \mu^n(c) U(c)$.

$$
\begin{aligned}
\sum_{\mathcal{C}_k} \mu^n(c) U(c) &= \sum_{\mathcal{C}_k} \mu^n(c) \left(\sum_{i \in N_l} \pi_i^n u(c_i) + \sum_{i \notin N_l} \pi_i^n u(c_i) \right) \\
&\leq \sum_{i \notin N_l} \pi_i^n u(s_i) + \sum_{\mathcal{C}_k} \mu^n(c) \sum_{i \in N_l} \pi_i^n u(c_i) \\
&\leq \sum_{i \notin N_l} \pi_i^n u(s_i) + \sum_{i \in N_l} \pi_i^n \left(\alpha_l^n u(y_l^n) + (1 - \alpha_l^n) u \left(\frac{s_i^n - \alpha_l^n y_l^n}{1 - \alpha_l^n} \right) \right)
\end{aligned}
$$

By concavity,

$$\sum_{i\in N_l}\pi_i^n\left(u(s_i^n)-\alpha_l^n u(y_l^n)-(1-\alpha_l^n)u\left(\frac{s_i^n-\alpha_l^n y_l^n}{1-\alpha_l^n}\right)\right)\geq 0$$

Thus, to prove the Lemma, it suffices to show that the above expression cannot converge to zero for all l. Note that $\alpha_l^n \geq 1/(k+1)$ for some l and hence we may choose l such that $\lim \alpha_l^n = \alpha > 0$. Feasibility requires that y_l^n stays bounded; hence, let $\lim y_l^n = y_l$.

First, assume $\alpha_l = 1$. In that case, $y_l \leq x_l$ since $s_i^n - \alpha_l^n y_l^n \geq 0$. Since $\lim_{\gamma\to 1}(1-\gamma)u(z/(1-\gamma)) = 0$ for all $z > 0$ it follows that

$$\lim\sum_{i\in N_l}\pi_i^n\left(u(s_i^n)-\alpha_l^n u(y_l^n)-(1-\alpha_l^n)u\left(\frac{s_i^n-\alpha_l^n y^n}{1-\alpha_l^n}\right)\right)\geq \int_{x_l}^{x_{l+1}}\left(u(z)-u(x_l)\right)f(z)dz>0$$

Second, assume $0 < \alpha_l < 1$. Then,

$$\lim\sum_{i\in N_l}\pi_i^n\left(u(s_i)-\alpha_l^n u(y_l^n)-(1-\alpha_l^n)u\left(\frac{s_i^n-\alpha_l^n y_l^n}{1-\alpha_l^n}\right)\right)= \int_{x_l}^{x_{l+1}}\left(u(z)-\alpha_l u(y_l)-(1-\alpha_l)u\left(\frac{z-\alpha_l y_l}{1-\alpha_l}\right)\right)f(z)dz>0$$

where the last inequality is an immediate consequence of the fact that u is strictly concave and $f > 0$. □

Next, we define optimal consumption plans at BCE price p^n for an arbitrary crudeness constraint $m \geq 1$. For $\rho \leq 1$ the utility function u is unbounded and, therefore, $p_i^n > 0$ for all i, n. For $\rho > 1$ the utility function is bounded above by 0 and $p_i^n = 0$ is possible. To ensure existence of optimal consumption plans for $\rho > 1$ and all m, let $\bar{\mathcal{C}}_m = \{c \in [\mathbb{R}_+ \cup \{\infty\}]^n : |\{c_1^n, \ldots, c_n^n\}| \leq m\}$ and define $u(\infty) = 0$. Let $\bar{B}_m(p^n, w) = \{c \in \bar{\mathcal{C}}_m : p^n \cdot c \leq w\}$ be the budget constraint if the budget is w and the price is p^n. Let $L_m^n : \mathbb{R}_+ \to \mathbb{R}_+$ be the indirect utility function at the BCE price p^n; that is,

$$L_m^n(w) = \max_{c\in\bar{B}_m(p^n,w)} U(c)$$

It is straightforward to show that L_m^n is well defined for all n, m and all $w > 0$. Let $w^n =$

$\sum_N p_i^n s_i^n$ be the wealth of the representative consumer and let $\bar{D}_m(w, p^n) \subset \bar{\mathcal{C}}_m$ be the optimal consumption plans with crudeness constraint m and wealth w at the BCE price p^n.

For Lemma 14 we fix $c^n \in \bar{D}_m(w^n, p^n)$ and let $A_1^n, \ldots, A_m^n$ be the partition of the state space N corresponding to the plan c^n. That is, $i, j \in A_l^n$ if and only if $c_i^n = c_j^n$ for all $i, j \in A_l^n$. Let $\beta_l^n := \sum_{A_l^n} \pi_i^n$ and $\bar{p}_m^n := \sum_{A_l^n} p_i^n$; for $l = 1, \ldots, m$ define $\beta_l = \lim \beta_l^n$ and $\bar{p}_l := \lim \sum_{A_l^n} p_i^n$.

Lemma 14. *If $\beta_l > 0$ there is $\bar{n}, \delta > 0$ such that $c_j^n > \delta$ for $j \in A_l^n$ for all $n > \bar{n}$.*

Proof. Since u is concave it follows that L_m^n is concave and hence $[L_m^n]'$, the derivative of L_m^n, is decreasing. Therefore,

$$\begin{aligned} L_m^n(w^n) &\geq L_m^n\left(\frac{w^n}{2}\right) + \frac{w^n}{2}[L_m^n]'(w^n) \\ &\geq u\left(\frac{a}{2}\right) + \frac{a}{2}u'(c_j^n)\frac{\beta_l^n}{\bar{p}_l^n} \\ &\geq u\left(\frac{a}{2}\right) + \frac{a}{2}u'(c_j^n)\beta_l^n \end{aligned}$$

Since $u(b) \geq L_m^n(w^n)$ and $\beta_l > 0$ it follows that $u'(c_j^n)$ is bounded above and, therefore, c_j^n is bounded away from zero. □

For Lemma 15 we fix $c^n \in \bar{D}_{k-1}(w^n, p^n)$ and let $A_1^n, \ldots, A_{k-1}^n$ be the partition of the state space N corresponding to the plan c^n as above. As above, define $\beta_l^n := \sum_{A_l^n} \pi_i^n$ and $\bar{p}_l^n := \sum_{A_l^n} p_i^n$ and $\beta_l = \lim \beta_l^n, \bar{p}_l = \lim p_l^n$. Note that $u(a) \leq L_m^n(w^n) \leq u(b)$ for all m and, therefore, $\lim(L_k^n(w^n) - L_{k-1}^n(w^n))$ is well defined.

Lemma 15. *If $\lim(L_k^n(w^n) - L_{k-1}^n(w^n)) = 0$ then $\lim L_{k-1}^n(w^n) \geq \int_a^b u(z)f(z)dz$.*

Proof. By definition, $U(c^n) = L_{k-1}^n(w^n)$. We define the consumption plan $\hat{c}^n$ as follows:

$$\hat{c}_i^n = \begin{cases} \frac{\sum_{A_l^n} \pi_i^n s_i^n}{\beta_l^n} & \text{if } i \in A_l^n \text{ such that } \beta_l > 0 \\ \min_{A_l^n} s_i^n & \text{if } i \in A_l^n \text{ such that } \beta_l = 0 \end{cases}$$

Note that $\lim U(\hat{c}^n) \geq \int_a^b u(z)f(z)dz$ since u is concave.

Claim 1: If $\lim(L_k^n(w^n) - L_{k-1}^n(w^n)) = 0$, then $\lim \sum p_i^n(\hat{c}_i^n - s_i^n) \leq 0$.

Proof: We will show that $\lim \sum_{A_l^n} p_i^n(\hat{c}_i^n - s_i^n) \le 0$ for all l. This is immediate if $\beta_l = 0$ or if $\bar{p}_l = 0$. Hence, assume $\beta_l > 0$ and $\bar{p}_l > 0$ and define

$$\begin{aligned}
A_{l1}^n &= \{i \in A_l^n : s_i^n \ge \hat{c}_i^n\} \\
A_{l2}^n &= \{i \in A_l^n : s_i^n < \hat{c}_i^n\} \\
\beta_{l1}^n &= \sum_{A_{l1}^n} \pi_i^n \\
\beta_{l2}^n &= \sum_{A_{l2}^n} \pi_i^n \\
\bar{p}_{l1}^n &= \sum_{A_{l1}^n} p_i^n \\
\bar{p}_{l2}^n &= \sum_{A_{l2}^n} p_i^n \\
b_l^n &= c_i^n \bar{p}_l^n \text{ for any } i \in A_l^n
\end{aligned}$$

Let $\beta_{l1}, \beta_{l2}, \bar{p}_{l2}, \bar{p}_{l2}, b_l$ be the limits of $\beta_{l1}^n, \beta_{l2}^n, \bar{p}_{l2}^n, \bar{p}_{l2}^n, b_l^n$ along an appropriate subsequence. From Lemma 14 and the fact that $\bar{p}_l > 0$ it follows that $b_l > 0$; also note that $\beta_{l1} > 0, \beta_{l2} > 0$ because $s_i^n \in [a, b]$ converges to a continuous random variable.

Let x_l, y_l solve

$$\max_{\{(x,y): \bar{p}_{l1}x + \bar{p}_{l2}y = b_l\}} \beta_{l1}u(x) + \beta_{l2}u(y)$$

Next, we show that $\frac{\bar{p}_{l1}}{\beta_{l1}} = \frac{\bar{p}_{l2}}{\beta_{l2}}$. If not, then $\frac{\bar{p}_{l2}}{\beta_{l2}} > \frac{\bar{p}_{l1}}{\beta_{l1}}$ since, by Theorem 2, $\frac{p_i}{\pi_i}$ is monotone. Let $\tilde{c}^n$ be the following consumption plan:

$$\tilde{c}_i^n = \begin{cases} c_i^n & \text{if } i \notin A_l^n \\ \frac{b_l^n}{\bar{p}_{1l}^n x_l + \bar{p}_{2l}^n y_l} x_l & \text{if } i \in A_{1l}^n \\ \frac{b_l^n}{\bar{p}_{1l}^n x_l + \bar{p}_{2l}^n y_l} y_l & \text{if } i \in A_{2l}^n \end{cases}$$

An elementary argument shows that $\frac{\bar{p}_{l2}}{\beta_{l2}} > \frac{\bar{p}_{l1}}{\beta_{l1}}$ implies $\max_{(x,y)\in C_l} \beta_{l1}u(x) + \beta_{l2}u(y) > \beta_l u\left(b_l/\bar{p}_l\right)$. This, in turn implies that $U(\tilde{c}^n) > U(c^n)$ for large n. Since $\tilde{c}^n \in B_k(p^n, w^n)$ and $U(c^n) = L_{k-1}^n(w^n)$ this contradicts the hypothesis of the lemma. Hence, $\frac{\bar{p}_{l1}}{\beta_{l1}} = \frac{\bar{p}_{l2}}{\beta_{l2}} = \frac{\bar{p}_l}{\beta_l}$.

A routine argument shows that this, in turn, implies

$$\lim \sum_{A_l^n} \left| p_i^n - \pi_i^n \frac{\bar{p}_l^n}{\beta_l^n} \right| = 0$$

Since $\sum_{A_l^n} p_i^n(\hat{c}_i^n - s_i^n) \leq b \sum_{A_l^n} \left| p_i^n - \pi_i^n \frac{\bar{p}_l^n}{\beta_l^n} \right|$, it follows that $\lim \sum_{A_l^n} p_i^n(\hat{c}_i^n - s_i^n) \leq 0$. This concludes the proof of Claim 1.

From Claim 1 it follows that $\gamma^n \hat{c}^n \in \bar{B}_{k-1}^n$ for some $\gamma^n \to 1$. Since $\lim L_{k-1}^n(w^n) \geq \lim U(\hat{c}^n) \geq \int_a^b u(z)f(z)dz$, it follows that $\lim U(\gamma^n \hat{c}^n) \geq \int_a^b u(z)f(z)dz$ as desired. □

Lemma 16. *There is $\bar{n}, \delta > 0$ (independent of n) such that $L_k^n(w^n) > L_{k-1}^n(w^n) + \delta$ for $n \geq \bar{n}$.*

Proof. Since $\bar{B}_k(p^n, w^n) \supset \bar{B}_{k-1}(p^n, w^n)$ it follows that $L_k^n(w^n) \geq L_{k-1}^n(w^n)$ for all n. By Lemma 15, if $L_k^n(w^n) - L_{k-1}^n(w^n) \to 0$ along any subsequence then $L_{k-1}^n(w^n) \geq \int_a^b u(z)f(z)dz$ which contradicts Lemma 13 since $L_k^n(w^n) = U(c^n)$ for $c^n \in K(\mu^n)$. □

A.5 Extreme Prices

Fix a convergent sequence of economies $\{E^n\}$; let p^n, μ^n be a BCE of $E^n = (u, k, \pi^n, s^n)$.

Lemma 17. *If $\sigma_i(\mu^n) < s_i$ then $p_i^n = 0$.*

Proof. If $c \in \mathcal{C}_k$ then $\gamma c \in \mathcal{C}_k$ for all $\gamma > 0$. Since U is strictly increasing it follows that $\sum_N p_i^n c_i = \sum_N p_i^n s_i^n$ for any $c \in B_k^n(p^n, w^n)$ that maximizes utility. Therefore, $\sum_N p_i^n s_i^n = \sum_{c \in K(\mu^n)} \mu^n(c) \sum_N p_i^n c_i = \sum_N p_i^n \sigma_i(\mu^n)$. Since $s_i^n \geq \sigma_i(\mu^n)$ for all i, the lemma follows. □

Lemma 18. *There is $\delta > 0$ such that $1 - \delta > p_1^n \geq \delta$ for all n*

Proof. If $p_1^n \to 1$ then $\sum_{i>1} p_i^n \to 0$ and, since $\pi_1^n \to 0$, $L_k^n(w^n) > u(b)$ for all $k \geq 2$. This violates feasibility since $s^n \leq b$. Thus, we have shown that $p_1^n \leq 1 - \delta$ for some $\delta > 0$. Since c is monotone and $\sum_N p_i^n = 1$, it follows that

$$b \geq w^n \geq \sum_{i>1} c_i^n p_i^n \geq \sum_{i>1} c_2^n p_i^n \geq c_2^n \delta$$

and therefore, $b/\delta \geq c_2^n$. Next, we show that for all n there is $c \in K(\mu^n)$ such that $c_1^n < c_2^n$. If not then $\sigma_1(\mu^n) = \sigma_2(\mu^n) \leq s_1^n < s_2^n$ and, by Lemma 17, $p_2^n = 0$. Then, by Theorem 2, $p_i^n = 0$ for all $i \geq 2$ contradicting the fact that $p_1^n \leq 1 - \delta$ for some $\delta > 0$. Thus, let $c \in K(\mu^n)$ with $c_1^n < c_2^n$. Let $\hat{c} \in \mathcal{C}_{k-1}^n$ be the following consumption plan:

$$\hat{c}_i^n = \begin{cases} c_2^n & \text{if } i = 1, 2 \\ c_i^n & \text{otherwise.} \end{cases}$$

Note that $U(\hat{c}^n) \geq U(c^n)$. Define $\gamma^n = \frac{w^n}{w^n + p_1^n(c_2^n - c_1^n)}$ and note that $\gamma^n \hat{c}^n \in B_{k-1}^n(w^n)$. If $p_1^n \to 0$ then $\gamma^n \to 1$ and therefore $U(\hat{c}^n) - U(\gamma^n \hat{c}^n) \to 0$. Since $L_k^n(w^n) - L_{k-1}^n(w^n) \leq U(c^n) - U(\gamma^n \hat{c}^n) \to 0$ this, in turn, contradicts Lemma 16. □

Lemma 19. *If $\rho > 1$ there is $\bar{n}, \beta > 0$ such that such that $n \geq \bar{n}$ and $\sum_{j \geq i} \pi_j^n < \beta$ implies $p_i^n = 0$.*

Proof. By Lemma 14 there is $\delta_1 > 0, \bar{n}$, such that for all $n > \bar{n}$, $c_i^n \geq \delta_1$ for all i with $\sum_{j \geq i} \pi_j^n \leq 1/2$. By Lemma 16 there is $\delta_2 > 0, \bar{n}$ such that $L_k^n(w^n) > L_{k-1}^n(w^n) + \delta_2$ for $n \geq \bar{n}$. Let $c_i^n < c_{i+1}^n$ and assume $\sum_{j \geq i} \pi_j^n \leq 1/2$. Let $\hat{c} \in B_{k-1}(p^n, w^n)$ be the following consumption plan:

$$\hat{c}_j^n = \begin{cases} c_i^n & \text{if } j \geq i \\ c_j^n & \text{otherwise.} \end{cases}$$

Since $c_i^n \geq \delta_1$ and $u(z) \leq 0$ for all z,

$$\begin{aligned} \delta_2 < L_k^n(w^n) - L_{k-1}^n(w^n) \leq U(c^n) - U(\hat{c}^n) &= \sum_{j \geq i} \pi_j^n (u(c_j^n) - u(\hat{c}_j^n)) \\ &\leq \sum_{j \geq i} \pi_j^n (u(c_j^n) - u(\delta_1)) \leq \sum_{j \geq i} \pi_j^n (-u(\delta_1)) \end{aligned}$$

and, therefore, $\frac{\delta_2}{-u(\delta_1)} \leq \sum_{j \geq i} \pi_j^n$ for $n \geq \bar{n}$. This implies that for all i such that $\sum_{j \geq i} \pi_j^n < \frac{\delta_2}{-u(\delta_1)}$ we have $c_i^n = c_{i+1}^n$ for all $c \in K(\mu^n)$ and, by Lemma 17, $p_j^n = 0$ for $j \geq i$. Setting $\beta = \min\left\{\frac{\delta_2}{-u(\delta_1)}, \frac{1}{2}\right\}$ thus yields the Lemma. □

For $\rho \geq 1$ let $\gamma^* = \infty$ and for $0 < \rho < 1$, let $\gamma^* = 1/(1-\rho)$.

Lemma 20. *If $\gamma < \gamma^*$ and $\{\imath^n\}$ satisfies $\lim s^n_{\imath^n} = b$ then*

$$\lim \frac{\sum_{i \geq \imath^n} p_i^n}{\left(\sum_{i \geq \imath^n} \pi_i^n\right)^\gamma} = 0$$

Proof. If $\rho > 1$ then Lemma 19 implies the result. Thus, assume $\rho \leq 1$ and note that since $\lim_{z \to \infty} u(z) = \infty$ it follows that $p_i^n > 0$ for all i. Let $\{\imath^n\}$ be a sequence that satisfies the hypothesis of the lemma. By Lemma 17 since $p^n_{\imath^n} > 0$, there is $c \in K(\mu^n)$ such that $c_{\imath^n - 1} < c_{\imath^n}$. Let c^n be such a consumption plan and let $A^n = \{\imath^n, \ldots, \jmath^n\}$ be such that $c_i^n = c^n_{\imath^n}$ if and only if $i \in A^n$. Note that there is $x > 0$ such that $c_{\imath^n - 1} \geq x > 0$ for large n. Otherwise, by monotonicity $c_i^n \to 0$ for all $i \leq \imath^n - 1$ along some subsequence. Since $\sum_{i \leq \imath^n - 1} \pi_i^n \to 1$ this contradicts Lemma 14. Let $\hat{c} \in B^n_{k-1}(p^n, w^n)$ be the following consumption plan:

$$\hat{c}_i = \begin{cases} c^n_{\imath^n - 1} & \text{if } i \in A^n \\ c_i & \text{otherwise.} \end{cases}$$

By Lemma 16 there is $\bar{n}, \delta$ such that $U(c^n) - U(\hat{c}^n) \geq \delta > 0$ for $n > \bar{n}$. This, in turn, implies $\sum_{i \in A^n} \pi_i^n (u(c_i^n) - u(x)) \geq \sum_{i \in A^n} \pi_i^n (u(c_i^n) - u(\hat{c}_i^n)) \geq \delta$. Note that $c^n_{\jmath^n} \leq w^n / (\sum_{i=\imath^n}^{\jmath^n} p_i^n)$ and hence

$$\sum_{j \in A^n} \pi_j^n \left(u\left(\frac{w^n}{\sum_{i=\imath^n}^{\jmath^n} p_i^n} \right) - u(x) \right) \geq \delta$$

Let $\beta^n := \sum_{i \in A^n} \pi_i^n$ and $\bar{p}^n := \sum_{i \in A^n} p_i^n$. Since $w^n \leq b$ this requires that

$$\beta^n u\left(\frac{b}{\bar{p}^n}\right) \geq \delta + \beta^n u(x)$$

or

$$\bar{p}^n \leq \frac{b}{u^{-1}\left(\frac{\delta + \beta^n u(x)}{\beta^n}\right)}$$

Substituting for u and setting $\delta^n := \delta + \beta^n u(x)$ we obtain

$$\bar{p}^n \leq \begin{cases} \frac{b}{((1-\rho)(\delta^n/\beta^n))^{\frac{1}{1-\rho}}} & \text{if } \rho \in (0,1) \\ \frac{b}{e^{\delta^n/\beta^n}} & \text{if } \rho = 1 \end{cases}$$

Since $\lim \beta^n = 0$ and $\lim \delta^n = \delta > 0$, a straightforward calculation shows that $\lim \frac{\bar{p}^n}{(\beta^n)^\gamma} = 0$ for $\gamma < \gamma^*$ in either case. □

Lemma 21. *If* $\{\imath^n\}$ *satisfies* $\lim s^n_{\imath^n} = b$ *then*

$$\lim \frac{p^n_{\imath^n}}{\pi^n_{\imath^n}} = 0$$

Proof. Suppose, contrary to the assertion of the lemma, there is $\imath^n$ such that $\frac{p^n_{\imath^n}}{\pi^n_{\imath^n}} \geq m > 0$ for all n. Let $\beta^n := \sum_{i \geq \imath^n} \pi^n_i$ and let $\jmath^n := \max\{j \in N : \sum_{i \geq j} \pi^n_i \geq 2\beta^n\}$. Note that $\jmath^n$ satisfies the conditions of Lemma 20 and therefore

$$\lim \frac{\sum_{i \geq \jmath^n} p^n_i}{\sum_{i \geq \jmath^n} \pi^n_i} = 0$$

But $p^n_i/\pi^n_i \geq m$ for all $\jmath^n \leq i \leq \imath^n$ and therefore

$$\lim \frac{\sum_{i \geq \jmath^n} p^n_i}{\sum_{i \geq \jmath^n} \pi^n_i} \geq \frac{m\beta^n}{2\beta^n}$$

yielding the desired contradiction. □

Lemma 22. *Assume* $\rho \geq 1$ *and let* $\{\imath^n\}$ *be such that* $\lim s^n_{\imath^n} = a$ *then* $\lim \frac{p^n_{\imath^n}}{\pi^n_{\imath^n}} = \infty$.

Proof. Assume $\{\imath^n\}$ satisfies the hypothesis of the lemma. If $p^n_{\imath^n+1} = 0$ along any subsequence then, since $\lim \sum_{j \leq \imath^n} \pi^n_j = 0$, it follows that $L^n_k(w^n) > u(b)$ violating feasibility. Thus, $p^n_{\imath^n+1} > 0$ for all n sufficiently large. By Lemma 17 this, in turn, implies that there is $c^n \in K(\mu^n)$ such that $c^n_{\imath^n} < c^n_{\imath^n+1}$. Let $A^n_1, A^n_2, \ldots, A^n_k$ be the partition of N corresponding to the plan c^n. That is, consumption c^n_i is constant for all $i \in A^n_l$ for all l and all n. Let $x^n_l = c^n_i$ for $i \in A^n_l$ and let $\bar{p}_l = \sum_{i \in A^n_l} p^n_i$ and let $\beta^n_l := \sum_{i \in A^n_l} \pi^n_i$. Let $A^n_1 = \{i \in N | c^n_i = c^n_{\imath^n}\}$. First note that $\beta^n_1 \to 0$ by construction and, therefore, by an argument identical to the one given in the proof of Lemma 18 it follows that $\bar{p}_1^n \geq \delta > 0$. Optimality of x_l^n implies that for $\theta = 1/\rho$

$$x_1^n = \left(\frac{\beta_1^n}{\bar{p}_1^n}\right)^{\theta} \frac{w^n}{\sum_{l=1}^k (\beta_l^n)^\theta (\bar{p}_l^n)^{1-\theta}}$$

If $\beta_l^n \bar{p}_l^n$ converges to zero for all l then (by a straightforward argument) $L_k^n(w^n) > u(b)$ for large n. Thus, it follows that $\beta_l^n \bar{p}_l^n$ is bounded away from zero for some l and, therefore, $\frac{w^n}{\sum_{l=1}^k (\beta_l^n)^\theta (\bar{p}_l^n)^{1-\theta}}$ is bounded. The above stated optimality condition then implies that $x_1^n \to 0$.

Assume that $\frac{p_{\imath^n}^n}{\pi_{\imath^n}^n} \leq m < \infty$ for all n. To prove the lemma will show that c^n cannot be an optimal consumption plan for n sufficiently large. Let l be such that $\beta_l^n \bar{p}_l^n \geq \delta$ for all n. By Lemma 14 and the fact that c^n satisfies the budget constraint it follows that x_l^n stays bounded away from zero and infinity. Let $\hat{c}^n$ be the following consumption plan:

$$\hat{c}_i^n = \begin{cases} c_i^n & \text{if } i \notin A_l^n \cup \{\imath^n\} \\ \frac{x_l^n \bar{p}_l^n}{\bar{p}_l^n + m\pi_{\imath^n}^n} & \text{if } i \in A_l^n \cup \{\imath^n\} \end{cases}$$

By construction, $\hat{c}^n \in B_k(p^n, w^n)$. Consider a subsequence such that $x_l^n \to x_l > 0, \beta_l^n \to \beta_l > 0$. Then,

$$\frac{U(\hat{c}^n) - U(c^n)}{\pi_{\imath^n}^n} = (u(x_l^n) - u(x_1^n)) + \frac{\beta_l^n}{\pi_{\imath^n}^n}\left(u\left(\frac{x_l^n \bar{p}_l^n}{\bar{p}_l^n + m\pi_{\imath^n}^n}\right) - u(x_l^n)\right) \tag{13}$$

It is straightforward to verify that the second term on right hand side of (13) converges to $-\frac{\beta_l}{\bar{p}_l} x_l u'(x_l) m$. Note that $\lim(u(x_l^n) - u(x_1^n)) = u(x_l) - \lim u(x_1^n) = \infty$ since $\rho \leq 1$ and therefore the expression on the right hand side of 13 converges to $+\infty$. Hence $U(\hat{c}^n) > U(c^n)$ for large n, as desired. □

Proof of Theorem 3: Lemma 18 proves that $\lim p_1^n > 0$. Next, we show that $\lim \Pr(\kappa^n > K) > 0$ for all K if $\rho \geq 1$. Without loss of generality, choose $K > 1$ and let $\jmath^n := \min\{j \in N | \kappa_j^n \leq K\}$. Note that $\jmath^n$ exists because $\sum_N p_i^n = \sum_N \pi_i^n$. By Theorem 2, $\{i \in N : \kappa_i^n > K\} = \{1, \ldots, \jmath^n - 1\}$. From Lemma 22 it follows that $\lim s_{\jmath^n}^n > a$ and, therefore, $\lim s_{\jmath^n - 1}^n > a$. Since the limit endowment has a strictly positive density on $[a, b]$ the result follows.

Next, we show that $\lim \Pr(\kappa^n < \epsilon) > 0$ for all $\epsilon > 0$. Without loss of generality, choose $\epsilon < 1$ and let $\jmath^n := \max\{j \in N | \kappa_j^n \geq \epsilon\}$. Note that $\jmath^n$ exists because $\sum_N p_i^n = \sum_N \pi_i^n$. Theorem 2 implies that $\{i \in N : \kappa_i^n < \epsilon\} = \{\jmath^n + 1, \dots, n\}$. From Lemma 21 it follows that $\lim s_{\jmath^n}^n < b$ and, therefore, $\lim s_{\jmath^n+1}^n < b$. Since the limit endowment has a strictly positive density on $[a, b]$ the result follows. Lemma 19 proves the final part of the Theorem. □

A.6 Limit Price

Fix a convergent sequence of economies $E^n = (u, k, \pi^n, s^n)$. Let μ^n be a BCE allocation for E^n, $K(\mu^n)$ be the set of consumption plans in its support. Let P^n a sequence of BCE cumulative price of E^n. For any $x \in [0, 1]$, let $\Pi^n(x) = \sum_{i \leq \jmath_x^n} \pi_i$ and let $I(x) = \max\{0, \min\{x, 1\}\}$.

Lemma 23. *E^n is convergent implies $\lim \Pi^n = I$.*

Proof. Straightforward and omitted. □

Definition 9. If $H, H^n \in \mathcal{H}$ and $H(x) = \lim H^n(x)$ at every continuity point x of H, we say that H is a weak limit of H^n.

Lemma 24. *H is a weak limit of H^n, $\lim x_n = x$ and H is continuous at x implies $\lim H^n(x_n) = H(x)$.*

Proof. For every $\epsilon > 0$, there exists $\delta > 0$ such that $H(x - \delta) > H(x) - \epsilon$. Since H is nondecreasing, we can choose δ so that H is continuous at $x - \delta$. Hence, there exists $\bar{n}$ such that $H^n(x - \delta) > H(x - \delta) - \epsilon$ and $x_n \geq x - \delta$ for all $n > \bar{n}$. It follows that $H^n(x_n) > H(x) - 2\epsilon$ for all $n > \bar{n}$. Together with a symmetric argument, this yields $\hat{n}$ such that $H(x) + 2\epsilon > H^n(x_n) > H(x) - 2\epsilon$ for all $n > \hat{n}$. □

Lemma 25. *P^n has a subsequence that has a weak limit. The restriction of any such weak limit to $[0, 1]$ is continuous and concave.*

Proof. Since $\mathcal{H}$ is a tight family, Helly selection theorem (Theorem 25.9 of Billingsley, 1995) implies that there exists $P \in \mathcal{H}$ and a subsequence P^{n_m} such that P is the weak limit of P^{n_m}. To simplify the notation, we assume that this sequence is P^n itself. Hence, the remainder of this proof, we assume P^n is the cumulative of some BCE price p^n for the economy E^n and $P = \lim P^n$. To conclude the proof, we will show that P has all the desired properties.

Next, we prove that P the restriction of P to the unit interval is concave: fix $0 \leq x_1 < x_2 \leq 1$ and $\lambda \in (0, 1)$; let $x_3 := \lambda x_1 + (1 - \lambda)x_2$. First, assume x_i, for $i = 1, 2, 3$ are all continuity points of P's extension. Let $z_i^n = \Pi^n(x_i)$ for $i = 1, 2, 3$. Since p^n/π^n is monotone,

$$\frac{P^n(z_3^n) - P^n(z_1^n)}{z_3^n - z_1^n} \geq \frac{P^n(z_2^n) - P^n(z_3^n)}{z_2^n - z_3^n}$$

and therefore,

$$P^n(z_3^n) \geq \frac{z_2^n - z_3^n}{z_2^n - z_1^n} P^n(z_1^n) + \frac{z_3^n - z_1^n}{z_2^n - z_1^n} P^n(z_2^n)$$

By construction, $\frac{z_2^n - z_3^n}{z_2^n - z_1^n}$ converges to λ; since E^n is convergent z_i^n converges to x_i (Lemma 23) and hence by Lemma 24, $P^n(z_i^n)$ converges to $P^n(x_i)$ for $i = 1, 2, 3$. Therefore, $P(x_3) = \lim P^n(r_3^m) \geq \lambda \lim P(z_1^n) + (1 - \lambda) \lim P(z_2^n) = \lambda P(x_1) + (1 - \lambda)P(x_2)$ as desired.

Since P is continuous except possibly on a countable set and right-continuous, if $P(x_3) < \lambda P(x_1) + (1 - \lambda)P(x_2)$ for any x_1, x_2, x_3, we can find $y_i > x_1$ close to x_i for the y_i's are continuity points of P and $P(y_3) < \lambda P(y_1) + (1 - \lambda)P(y_2)$, contradicting the above argument.

Since the restriction of P to the unit interval is concave, it is continuous on $(0, 1)$. Since P is right-continuous, it is continuous at 0. Since P is nondecreasing, $P(x) \leq P(1)$ for all $x \in [0, 1)$; thus, $\lim_{x\uparrow 1} P(x) \leq 1$. On the other hand, by concavity, $P(x) \geq (1 - x)P(0) + xP(1) \geq x$, so $\lim_{x\uparrow 1} P(x) \geq 1$ proving the continuity of P at 1. Hence P is continuous on $[0, 1]$. Thus, $\lim P^n(x) = P(x)$ for all $x \in (0, 1]$. □

If P is a weak limit of BCE cumulative prices P^n and its restriction to $[0, 1]$ is continuous, then Lemma 15 implies that P is a limit BCE price E^n. (Set $x_n = x$ for all $n \in (0, 1]$ to verify that P^n converges to P pointwise on $(0, 1]$.) Thus we have shown that every convergent E^n has a limit price and that the restriction of P to the unit interval is concave.

Proof of Theorem 4: That $0 < P(0)$ follows from Lemma 18. Next, we prove that $P'(1) = 0$.

For each $r = 1, 2, \ldots$, let $x(r) = \frac{r-1}{r}$ and choose $n(r)$ so that for $A(r) := \left\{\jmath^{n(r)}_{x(r)} + 1, \ldots, n(r)\right\}$

$$\left|\frac{1-P\left(x(r)\right)}{1-x(r)} - \frac{1-P^{n(r)}(x(r))}{1-\Pi^{n(r)}(x(r))}\right| = \left|r(1-P\left(x(r)\right)) - \frac{\sum_{A(r)} p_i^{n(r)}}{\sum_{A(r)} \pi_i^{n(r)}}\right| \leq 1/r \tag{14}$$

The sequence $\{\jmath^{n(r)}_{x(r)}\}, r = 1, 2, \ldots$ satisfies the hypothesis of Lemma 20 and, therefore,

$$\lim \frac{\sum_{A(r)} p_i^{n(r)}}{\sum_{A(r)} \pi_i^{n(r)}} = 0$$

Inequality (14) then implies that $\lim_{r\to\infty} r(1 - P\left(\frac{r-1}{r}\right)) = 0$ and hence $P'(1) = 0$ as desired.

Next, we prove that $P'(0) = \infty$ if $\rho \geq 1$. For $r = 1, 2, \ldots$, let $y(r) = \frac{2}{r}$, $z(r) = \frac{1}{r}$ and choose $n(r)$ so that for $A(r) = \left\{\jmath^{n(r)}_{z(r)} + 1, \ldots, \jmath^{n(r)}_{y(r)}\right\}$

$$\left|r(P(z(r)) - P\left(y(r)\right)) - \frac{\sum_{A(r)} p_i^{n(r)}}{\sum_{A(r)} \pi_i^{n(r)}}\right| \leq 1/r \tag{15}$$

Since P is concave, $\frac{P(\epsilon)-P(0)}{\epsilon} \geq r(P(z(r)) - P\left(y(r)\right))$ for $\epsilon \leq 1/r$ and hence the result follows if

$$\frac{\sum_{A(r)} p_i^{n(r)}}{\sum_{A(r)} \pi_i^{n(r)}} \to \infty$$

as $r \to \infty$. Let $\imath^n = \jmath^{n(r)}_{y(r)}$ then $s^n_{\imath^n} \to a$ and therefore Lemma 22 and Theorem 2 complete the argument. Finally, note that Lemma 19 implies that, for $\rho > 1$, $P(1 - \delta) = 1$ for some $\delta > 0$. $\square$

A.7 Dynamic Economy

Let $E = (u, k, \pi, s)$ and let $E^* = (u, \beta, k, \pi, s, \Phi)$. Let $W^*(\nu) = \sum_d V(d) \cdot \nu(d)$ and let $W^*_k = \sup_{M^*(\mathcal{D}_k)} W(\nu)$. We call $\nu \in M^*(\mathcal{D}_k)$ a solution to the planner's problem in the dynamic economy if $W^*(\nu) = W^*_k$. Let $W_k = \max_{M(\mathcal{C}_k)} W(\mu)$ be the value of the planner's problem for the static economy.

Consider the static economy $E^t = (u, \pi^t, s^t)$ where $\pi^t_h = \lambda_h$, $s^t_h = s_{\jmath(h)}$ for all $h \in N^t$. Let W^t be the planner's value function for this economy. Note that $d \in \mathbb{R}^n \times \mathbb{R}^{n^2} \times \ldots$. For any $\nu \in M^*(\mathcal{D}_k)$ let ν_t be the marginal of ν on $\mathbb{R}^{n^t}$.

Lemma 26. *The allocation ν is a solution to the planner's problem in the dynamic economy if and only if $W^t(\nu_t) = W_k$ for all t.*

Proof. If ν is a feasible allocation for the dynamic economy, then ν_t is a feasible allocation for E^t. By definition, $W^*(\nu) = \sum_{t \geq 1}(1-\beta)\beta^{t-1}W^t(\nu_t)$. Then, since $F_{s^t} = F_s$, Lemma 6 implies that for any feasible ν, $W^t(\nu_t) \leq W_k$ for all t. Therefore, to conclude the proof, it suffices to show that $W^*_k \geq W_k$. Let μ be a solution to the planner's problem in the stationary economy, then the stationary allocation $T_3(\mu)$ is feasible for the dynamic economy and $W^*(T_3(\mu)) = W^t((T_3(\mu))_t) = W_k$ for all t. □

Lemma 27. *Every solution to the planner's problem in the dynamic economy is stationary.*

Proof. Suppose ν is a nonstationary solution to the planner's problem. By Lemma 26, $W^t(\nu_t) = W_k$ and since $F_{s^t} = F_s$, we conclude that ν_t solves the planner's problem for E^t. If follows that ν_t is measurable and in particular, $\nu(d) > 0$ implies $d_h = d_{h'}$ for all $h, h' \in H^t_i$ and i, t. Then, since ν is not stationary, there must be t, t', h, h', d, i such that $h \in N^t$ and $h' \in N^{t'}$ and $\jmath(h) = \jmath(h') = i$, $d_h \neq d_{h'}$ and $\nu(d) > 0$. Consider the economy $\hat{E} = (u, \hat{\pi}, \hat{s})$ where $\hat{N} = N^t \cup N^{t'}$, $s_{\hat{h}} = s_{\jmath(\hat{h})}$ for all $\hat{h} \in \hat{N}$ let $\hat{\pi}_{\hat{h}} = .5\lambda_{\hat{h}}$. Define the consumption plan $\hat{c}$ for $\hat{E}$ as follows $\hat{c}_{\hat{h}} = d_{\hat{h}}$ for all $\hat{h} \in \hat{N}$. Our choice of t, t', d ensures that $\hat{c}$ fails measurability.

Let $\hat{W}$ be the planner's value function for the economy $\hat{E}$ and note that since $F_{\hat{s}} = F_s$, by Lemma 6, $\hat{W} = W_k$. Let ν^2 be the marginal of ν on $N^t \times N^{t'}$ and note that ν^2 is an allocation for $\hat{E}$. Then, by Lemma 26, $\hat{W}(\nu^2) = .5W^t(\nu_t) + .5W^t(\nu_{t'}) = W_k$. Therefore ν^2 solves the planner's problem in $\hat{E}$. But ν^2 fails measurability since $\nu^2(\hat{c}) > 0$, contradicting Lemma 1. □

Proof of Theorem 5 Assume (p, μ) is a BCE of E but $(T_2(p), T_3(\mu))$ is not a BCE of E^*. Hence, there exists $d \in B^*(T_2(p))$ such that $V(d) > W(\mu) = U(c)$ for $c \in D(p)$. Let $X = \{d_h : h \in H\}$

and for all $i \in N$, $x \in X$, let $H_i^t(x) = \{h \in H_i^t : d_h = x\}$ and

$$\chi_{ix} = \sum_{t \geq 1} \sum_{h \in H_i^t(x)} (1-\beta)\beta^{t-1}\lambda_h.$$

Let $\hat{N} = N \times X$, $\hat{\pi}_{ix} = \chi_{ix}$ for all $i \in N$ and $x \in X$, $\hat{s}_{ix} = s_i$ for all i. Define $g(ix) = i$ and note it is a $[\hat{E}\|E]$-homomorphism. Therefore, by Lemma 10, $\theta_1(c) \in \hat{D}(\theta_2(p))$ whenever $c \in D(p)$. Define $\hat{c}_{ix} = x$ for all $ix \in \hat{N}$ and note that $U(\hat{c}) = V(d) > U(c) = U(\theta_1(c))$ for any $c \in D(p)$, contradicting the fact that $\theta_1(c) \in D(\theta_2(p))$.

Finally, let (q, ν) be any BCE for E^*. Lemma 26 implies that there is a fair solution to the planner's problem. Standard arguments ensure that ν must be Pareto-efficient; since all households have the same endowment ν must also be fair. It follows that ν must solve the planner's problem. Then, Lemma 27 establishes that ν is stationary. If $T_3^{-1}(\nu)$ is not a BCE allocation for E, then there exists an allocation μ for E such that $W(\mu) > W(T_3^{-1}(\nu))$, which implies that $W^*(T_3(\mu)) > W(\nu)$, so ν is not a solution to the planner's problem, a contradiction. □

Proof of Theorem 6: Since z^n is bounded, it follows that $\sum_N p_i^n z_i^n \in [\gamma_1, \gamma_2]$. Theorem 6 (i) then follows from Theorem 3(i) and Theorem 6 (ii) follows from Theorem 3(ii). □

Proof of Theorem 7: Since $\pi_1^n \to 0$, it follows that $e_1^{n\epsilon} = 0$ for n sufficiently large. Substituting for r^n, this implies that

$$\frac{r_i^n(e^n)}{r_i^n(e^{n\epsilon})} \geq \frac{\sum_{i \in N} p_i^n}{\sum_{i>1} p_i^n}$$

for n sufficiently large. Theorem 3(i) now implies the result. □

References

Abreu, D., and A. Rubinstein (1988): "The structure of Nash equilibrium in repeated games with finite automata," *Econometrica: Journal of the Econometric Society*, 56(6), 1259–1281.

Billingsley, P. (1995): *Probability and measure.* Wiley.

Chetty, R., and A. Szeidl (2010): "The Effect of Housing on Portfolio Choice ," *Working paper, Harvard University.*

Dow, J. (1991): "Search Decisions with Limited Memory," *Review of Economic Studies*, 58(1), 1–14.

French, K., and J. Poterba (1991): "Investor Diversification and International Equity Markets," *American Economic Review*, 81(2), 222–226.

Gabaix, X., and D. Laibson (2002): "The 6D bias and the equity-premium puzzle," *NBER macroeconomics annual*, 16, 257–312.

Grossman, S., and G. Laroque (1990): "Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods," *Econometrica*, 58(1), 25–51.

Guvenen, F. (2009): "A parsimonious macroeconomic model for asset pricing," *Econometrica*, 77(6), 1711–1750.

Hong, H., J. Stein, and J. Yu (2007): "Simple forecasts and paradigm shifts," *The Journal of Finance*, 62(3), 1207–1242.

Jehiel, P. (2005): "Analogy-based expectation equilibrium," *Journal of Economic Theory*, 123(2), 81–104.

Jehiel, P., and D. Samet (2007): "Valuation equilibrium," *Theoretical Economics*, 2(2), 163–185.

Lynch, A. (1996): "Decision frequency and synchronization across agents: Implications for aggregate consumption and equity return," *The Journal of Finance*, 51(4), 1479–1497.

Mankiw, N., and R. Reis (2002): "Sticky Information Versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve," *Quarterly Journal of Economics*, 117(4), 1295–1328.

Masatlioglu, Y., D. Nakajima, and E. Ozbay (2012): "Revealed attention," *The American Economic Review*, 102(5), 2183–2205.

Mehra, R., and E. Prescott (1985): "The equity premium: A puzzle," *Journal of monetary Economics*, 15(2), 145–161.

Mullainathan, S. (2002): "Thinking Through Categories," *mimeo*.

Neyman, A. (1985): "Bounded complexity justifies cooperation in the finitely repeated prisoners' dilemma," *Economics Letters*, 19(3), 227–229.

Piccione, M., and A. Rubinstein (1997): "On the Interpretation of Decision Problems with Imperfect Recall," *Games and Economic Behavior*, 20(1), 3–24.

Rubinstein, A. (1986): "Finite automata play the repeated prisoner's dilemma," *Journal of economic theory*, 39(1), 83–96.

Sims, C. (2003): "Implications of rational inattention," *Journal of Monetary Economics*, 50(3), 665–690.

Wilson, A. (2002): "Bounded memory and biases in information processing," *Working paper*.

Woodford, M. (2011): "Inattentive Valuation and Reference-Dependent Choice," *mimeo*.