Pricing in Social Networks *

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October 11, 2011

Abstract

We analyze the problem of optimal monopoly pricing in social networks in order to characterize the influence of the network topology on the pricing rule. It is shown that this influence depends on the type of providers (local versus global monopoly) and of externalities (consumption versus price). We identify two situations where the monopolist does not discriminate across nodes in the network (global monopoly with consumption externalities and local monopoly with price externalities) and characterize the relevant centrality index used to discriminate among nodes in the other situations. We also analyze the robustness of the analysis with respect to changes in demand, and the introduction of bargaining between the monopolist and the consumer.

`JEL Classification Numbers: D85, D43, C69`

`Keywords: Social Networks, Monopoly Pricing, Network Externalities, Reference Price, Centrality Measures`

*We dedicate this paper to the memory of Toni Calvó-Armengol, a gifted network theorist and a wonderful friend. We thank Coralio Ballester, Sebastian Bervoets, Yann Bramoullé, Geoffroy de Clippel, Jan Eeckhout, Matt Elliott, Sanjeev Goyal, Matt Jackson and Michael Königfor their helpful comments.

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1 Introduction

This paper analyzes the optimal pricing strategy of a monopoly in a social network. Our objective is to understand how discriminatory prices reflect (or not) the centrality of consumers in the social network. Marketing techniques to discriminate among consumers based on their social connections have long been in use. When selling new products or creating an installed base for products with network externalities, it is not uncommon for firms to offer "referral bonuses" – discounts or cash to consumers who bring new friends into the network. In doing so, the firm rewards agents with a large number of friends, and price discriminates according to the consumer's number of neighbors, or degree centrality. In a more systematic fashion, following MCI in 1990, telecommunication companies have introduced "friends and family plans" as a way to discriminate among consumers based on their number of friends and pattern of calls (Shi, 2003).

With the spectacular emergence of online social networks like Facebook, Orkut and MySpace, new possibilities for large scale social network based discriminatory pricing have emerged. Due to a combination of privacy and technical reasons, this possibility has not yet been exploited, and most of the monetization of online social networks stems from targeted advertising using data on consumer characteristics rather than their social connections. However, the discrepancy between the current revenue of Facebook (between 1.2$ and 2$ billion in 2010) and its value (82.9$ billion reported as of January 29, 2011)(Levy, 2011) suggests that new marketing opportunities based on social network data will likely be exploited in the near future. In fact, the agreement between Facebook and the group buying platform Groupon which allows consumers to sign up on Groupon on their Facebook page points in that direction. Groupon may exploit the social network of Facebook to attract new customers, offering deals and coupons to consumers who bring in new friends, thereby discriminating in favor of consumers with higher degree centrality in the network.

While the current social-network based price discrimination strategies only make use of the consumer's number of neighbors, it is very likely that more detailed data on social networks will soon be used in pricing and marketing strategies (Arthur et al. (2009), Hartline et al. (2008)). An important issue is to understand whether the number of neighbors is always the relevant measure of centrality that should be used for price discrimination. Even in case where this characteristic is relevant, it is necessary to assess its actual influence (positive or negative) on the prices that should be offered. In this paper, we consider price discrimination based on the entire social network, where each agent receives a price associated to her nodal characteristic. We consider two channels through which social networks influence a consumer's demand. In the first model of *local network externalities*, consumers benefit from the consumption of the same good by their direct neighbors. This model captures situations where agents receive discounts if they call friends who subscribe to the same network, share a common software with their colleagues or co-authors, or need to reach a critical mass of consumers to obtain a deal or launch a project. In the second model of *aspiration based reference price*, consumers construct a reference price for the good based on the price charged to their direct neighbors, and experience a positive utility if the price they receive is below their reference price. This model is applicable to situations where firms use discriminatory pricing that lacks transparency, like airline pricing and negotiated pricing.

In both models, our objective is to understand which measure of centrality is relevant to rank prices charged at different nodes. Are prices increasing or decreasing in the number of neighbors that a consumer has? Is the structure of the network at distance two (the number of neighbors of neighbors) a relevant information for optimal monopoly pricing? When does the monopoly charge uniform prices across nodes? To answer these questions, we consider a linear model, where consumers pick a random valuation for the object according to a uniform distribution. In the model of local network externalities, a consumer's utility is positively affected by the consumption of her direct neighbors ; in the model of aspiration-based price reference, a consumer's utility is positively affected by the average price charged to her direct neighbors. Using the analysis pioneered by Ballester, Calvó-Armengol and Zenou (2006), we characterize the demand of every consumer as a function of her centrality in the network. We then consider two different market structures: one where a single monopoly serves all the consumers in the network and internalizes the externalities across nodes, and one where each consumer is served by a different local monopoly.

In the local network externalities model, we first obtain a *network irrelevance result*: when a single monopoly serves all the nodes, it will optimally choose a uniform price in the network. This striking result can be explained as follows. There are two countervailing effects of the centrality of a node on the optimal price. On the one hand, a more central node generates more positive externalities on its neighbors and hence should be subsidized (the classical effect by which more central agents receive lower prices) ; on the other hand, more central agents benefit more from the object, and have a higher valuation which can be captured by the monopolist. In the linear model, these two effects are exactly balanced, giving rise to a uniform pricing strategy. (However, with general distribution functions, the irrelevance network no longer holds and one of the two effects dominates the other.) When different nodes are served by different firms, the positive externality of the consumption of the central node on the other nodes is no longer internalized. The first effect vanishes, and only the second effect remains, so that more central agents with higher valuations are charged higher prices.

In the aspiration-based price reference model, we obtain a second *network irrelevance result*, this time when every node is served by a different firm. This irrelevance result, which is robust to changes in the model, stems from the following observation. If all other firms charge the optimal monopoly price, a local monopoly cannot benefit from charging a different price. When all nodes are served by a single monopolist, this reasoning fails as the monopolist may want to increase the price at some node in order to increase demand at the neighboring nodes. For example, in a star, the monopoly has an obvious incentive to charge a high price at the hub in order to increase demand at peripheral nodes.

We finally discuss two extensions of the model. In the first extension, we consider general demand schedules and analyze the robustness of our results. In the second extension, we compute the consumer surplus accruing at each node. This enables us to analyze the agent's incentives to form links in the social network and the formation of prices as a result of a bargaining process between the monopoly and the consumer.

We now discuss briefly the related literature. The model of local network externalities finds its origin in the seminal work of Farrell and Saloner (1985) and Katz and Shapiro (1985) on network externalities. These early papers eschew the "network" dimension of network externalities and implicitly assume that consumers are affected by the global consumption of all other consumers. Models of local network externalities which explicitly take into account the graph theoretic structure of social networks have been proposed by Jullien (2001), Sundarajan (2006), Saaskhilati (2007) and Banerji and Dutta (2009). Jullien (2001) and Banerji and Dutta (2009) analyze competition between two price-setting firms. While Banerji and Dutta (2009) consider uniform prices, Jullien (2001) allows for discriminatory pricing at different nodes, and provides partial results suggesting that firms set lower prices at nodes with higher degree. Sundarajan (2006) studies monopoly pricing in a model where consumers make a deterministic choice between adopting the new product or not. Ghiglino and Goyal (2010) focus instead on a model of conspicuous consumption, where agents compare their consumption with that of their neighbors and suffer a negative consumption externality. In the same linear model as the one we consider, they characterize the competitive equilibrium prices and allocations and show that identical consumers located in asymmetric positions in the network choose to trade and end up at different equilibrium allocations. Finally, in a work which is independent from ours, Saaskhilati (2007) studies uniform monopoly pricing on social networks. His main focus is not on discriminatory pricing but on the relation between the network topology and the uniform price charged by the monopoly, and he computes optimal prices and consumer surplus for some specific network structures like symmetric networks and stars.

The study of optimal pricing and marketing strategies in social networks has recently received attention in the computer science literature. Following the work on influence maximization of Domingos and Richardson (2001) and Kempe, Kleinberg and Tardos (2003) which aimed at identifying influential agents in a network without any reference to price and revenue maximization, recent work by Hartline et al. (2008) and Arthur et al. (2009) compute optimal pricing strategies. They show that a simple two-price strategy (the "Influence and Exploit Marketing", where the seller chooses a set of consumers to which the product is sold for free – or at a cashback "referral bonus" –) performs very well compared to the optimal marketing strategy which is NP-hard to compute. The main difference between these approaches and ours stem from the timing of purchases. Both Hartline et al. (2008) and Arthur et al. (2009) consider sequential purchases where myopic consumers base their consumption decision on the number of consumers who have *already bought* the product. We consider instead a simultaneous consumption decision for all consumers in the network who are fully rational.

The model of aspiration based reference price has been studied in marketing (Xia, Monroe and Cox (2004), Mazumdar et al. (2005)) along lines developed in social psychology. The theory of social comparison (see Suls and Wheeler (2000) for a detailed account) posits that most outcomes (like prices and salaries) are perceived in comparison to other agents' outcomes, so that prices are deemed fair or unfair in reference to prices paid by other consumers in a similar situation. Hence, consumers construct reference prices based on what their neighbors have been charged, and evaluate the price they receive by comparison to this reference price.

The rest of the paper is organized as follows. We discuss the model of local network externalities in Section 2, and the model of aspiration based reference price in Section 3. Section 4 contains a discussion of the robustness of the analysis and an extension to bargaining over total surplus. All proofs are relegated to the Appendix.

2 Local Network Externalities

We consider a model of local network externalities, where consumers are affected by the consumption choices of their neighbors in a social network. As in Jullien (2001), Saaskhilati (2007) and Banerji and Dutta (2009), we construct a model of network externalities where consumers only care about the consumption of a subset of agents determined by an exogenous social network. We first introduce a simple linear model of demand with local network externalities (Subsection 2.1), discuss a simple example (Subsection 2.2) and characterize optimal pricing rules of a monopolist (Subsection 2.3) and of local monopolies (Subsection 2.4).

2.1 The Model

We consider a set N of consumers located along a social network, denoted g. The adjacency matrix of this network is denoted $\mathbf{G}$ with typical element $g_{ij} \in \{0,1\}$. We recall that $g_{ij} = 1$ if and only if there exists an edge in the network linking consumers i and j. We assume that the network is undirected, so that $g_{ij} = g_{ji}$. We let $d_i = \sum_j g_{ij}$ denote the *degree* of node i in the network.

Each consumer i has a unit demand for the good, and draws an intrinsic value θ_i from the uniform distribution F over $[0,1]$. We suppose that intrinsic values are independently distributed. Consumers experience *local network externalities* in the sense that their value for the good increases by the constant value $\alpha > 0$ whenever one of their neighbors consumes the good. Finally, consumers have positive linear utility for money, so that the utility of consumer i is expressed by:

$$U_i = \theta_i - p_i + \alpha \sum_j g_{ij} \Pr[j \text{ buys the good }]. \tag{1}$$

The timing of events is as follows: the monopoly chooses a price vector $(p_1, ..., p_n)$ before observing the realization of the valuation vector $(\theta_1, ..., \theta_n)$.

Each consumer learns her valuation θ_i and makes her consumption decision at the interim stage, knowing p_i and θ_i, but not the valuations θ_{-i} drawn by the other consumers. Clearly, if a consumer of type θ_i buys the good, so does any consumer of type $\theta_i' > \theta_i$. Hence, consumer i's optimal purchasing decision is characterized by a threshold value $\tilde{\theta}_i$. Furthermore, as all consumers adopt the same optimal threshold purchasing decision rules, we can compute $\tilde{\theta}_i$ using the following expression:

$$\tilde{\theta}_i = p_i - \alpha \sum_j g_{ij}(1 - F(\tilde{\theta}_j)). \tag{2}$$

Alternatively, if we let $x_i = 1 - F(\tilde{\theta}_i)$ denote the probability that consumer i buys the good, we have:

$$x_i = \begin{cases} 0 & \text{if } 1 - p_i + \alpha \sum_j g_{ij} x_j < 0 \\ 1 & \text{if } \; - p_i + \alpha \sum_j g_{ij} x_j > 0, \\ 1 - p_i + \alpha \sum_j g_{ij} x_j & \text{otherwise} \end{cases} \tag{3}$$

We now solve this system of interdependent demands in order to obtain the demand of a consumer at node i as a function of the vector of prices, $\mathbf{p} = (p_1, ..., p_n)$ charged at different nodes. This amounts to inverting the system of equations (3) under the condition that all demands x_i are contained in $[0, 1]$, and is formally equivalent to solving a linear-quadratic game as in Ballester Calvó-Armengol and Zenou (2006) and Ballester and Calvó-Armengol (2009).

Let $\lambda(\mathbf{G})$ denote the largest eigenvalue of the adjacency matrix $\mathbf{G}$. When $\alpha\lambda(\mathbf{G}) < 1$, the matrix $[\mathbf{I} - \alpha\mathbf{G}]$ is invertible and $[\mathbf{I} - \alpha\mathbf{G}]^{-1}$ is nonnegative (Debreu and Herstein, 1963). Furthermore, let $a_{ij} \geq 0$ denote the ij entry of the matrix $[\mathbf{I} - \alpha\mathbf{G}]^{-1}$. If we only consider a subset S of players with corresponding network $\mathbf{G}_S$, we let $a_{ij,S}$ denote the ij entry of the square $s \times s$ matrix $[\mathbf{I} - \alpha\mathbf{G}_S]^{-1}$. By a simple application of Theorems 1 and 2 of Ballester, Calvó-Armengol (2009), we characterize the unique system of demands in the following Proposition.

Proposition 2.1 *If $\alpha\lambda(\mathbf{G}) < 1$, for any vector of prices $\mathbf{p} = (p_1, ..., p_n)$, there exists a unique system of demands satisfying equation (3). In this solution, the set of consumers is partitioned into three sets S_0, S_1 and $S = N \setminus (S_0 \cup S_1)$ such that:*

$$x_i = \begin{cases} 0 & \textit{if } i \in S_0 \\ 1 & \textit{if } i \in S_1 \\ \sum_{j \in S} a_{ij,S}(1 - p_j + \sum_{k \in S_1} g_{jk}) & \textit{otherwise} \end{cases} \tag{4}$$

Proposition 2.1 shows that, if externalities are not too large, the interdependence between consumer demands at different points in the network results in a *unique* system of demands. For an arbitrary price vector, $\mathbf{p}$, as demands must belong to the bounded interval $[0, 1]$, the description of equilibrium demands involves a partition of the set of nodes into (i) nodes with zero demand, (ii) nodes where consumers buy with probability one and (iii) nodes where consumers buy with a probability $x_i \in (0, 1)$. For consumers at these last nodes, the coefficients of the demand system $\frac{\partial x_i}{\partial p_j} = a_{ij}$ are exactly the entries of the matrix $[\mathbf{I} - \alpha \mathbf{G}_S]^{-1}$, which can be interpreted as the Katz-Bonacich notion of network centrality (Katz (1953), Bonacich (1987)):

Definition 2.2 *Given a network* $\mathbf{G}$ *and a positive scalar* α*, the Katz-Bonacich centrality of agent* i *in the network is given by:*

$$\mathbf{b}_i(\mathbf{G}, \alpha) = [\mathbf{I} - \alpha \mathbf{G}]^{-1}\mathbf{1} = (\sum_j a_{ij})$$

We can use the power series expansion $[\mathbf{I} - \alpha \mathbf{G}]^{-1} = \sum_{k=0}^{\infty} \alpha^k \mathbf{G}^k$, to rewrite $a_{ij} = \sum \alpha_k \mu_{ij}^k$, where μ_{ij}^k, the ij entry of the matrix $\mathbf{G}^k$, counts the number of paths of length k between i and j. Following this interpretation, the Katz-Bonacich coefficient $a_{ij,S}$ measures the sum of discounted paths from i to j in the subgraph g_S formed by consumers in S.

2.2 When does centrality matter? An example

In order to understand when and how the position of a consumer in the network affects the optimal price, we consider an example with 4 consumers located along the following network.

Suppose that $\alpha = 0.1$. The demand system for this example is given by:

$$\begin{aligned}
x_1 &= 1.13662 - 1.01033p1 - 0.10333p2 - 0.0229621p3, \\
x_2 &= 1.36625 - 0.10333p_1 - 1.0333p_2 - 0.229621p_3, \\
x_3 &= 1.26292 - 0.0114811p_1 - 0.114811p_2 - 1.13662p_3, \\
x_4 &= 1.26292 - 0.0114811p_1 - 0.114811p_2 - 1.13662p_3.
\end{aligned}$$

If a single monopolist with zero marginal cost chooses prices p_1, p_2 and p_3 to maximize $\pi = p_1x_1 + p_2x_2 + 2p_3x_3$, the optimal solution is $p_1^* = p_2^* = p_3^* = 0.5$. If each node is served by a different firm, the equilibrium price vector is given by: $p_1^* = 0.527065, p_2^* = 0.576561, p_3^* = p_4^* = 0.523774$. Hence,

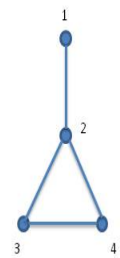

Figure 1: A four-consumer network

the ranking of prices reflects the degree centrality of the nodes. The more central node (node 2) is charged the highest price, followed by nodes 3 and 4 and finally node 1. We now show that this example illustrates a general result on the relation between prices and node centrality.

2.3 Optimal monopoly pricing

Suppose that a single monopolist chooses the vector of prices $\mathbf{p}$ in order to maximize profit. We assume that the monopoly produces each unit of the good at a constant cost $c < 1$, so that her expected profit is given by:

$$\Pi = \sum_{i \in N} x_i p_i - c x_i.$$

The following Proposition characterizes the optimal pricing strategy of the monopoly, assuming that the condition $\alpha\lambda(\mathbf{G}) < 1$ is satisfied.

Proposition 2.3 *In the model with positive consumption externalities, the optimal pricing strategy of the monopoly is to charge a uniform price* $p^* = \frac{1+c}{2}$ *at each node. Given this pricing strategy, the expected demand of a consumer* i *is given by* $x_i = \frac{1-c}{2} \sum_j a_{ij}$*, which is proportional to the Katz-Bonacich centrality measure of consumer* i *in graph* g*.*

The striking result of Proposition 2.3 is that the monopoly does not exploit differences in consumer's centralities to charge discriminatory prices, but charges instead a uniform monopoly price at each node. She lets demand adjust at each node in the network according to consumer's centralities, with consumers with higher levels of Katz-Bonacich centrality having a higher probability of purchasing the good.

The network irrelevance result of Proposition 2.3 is supported by the following intuition. When choosing the price at node i, the monopoly balances two effects: a price increase at node i raises profit at that node, but also reduces demand and profits at all other nodes in the network. In the linear model we analyze, this trade-off, measured by a positive effect $\sum a_{ij}(1-p_j)$ and a negative effect $\sum a_{ji}(c-p_j)$, is independent of a node's centrality. Hence, the monopoly faces the same trade-off at every node and optimally chooses a uniform pricing rule. However, the exact balance which leads to uniform pricing is a knife-edge result, which relies on the assumptions on the distribution function F.

Finally, consider the surplus of consumer i, measured ex ante before the consumer learns her valuation. With a linear demand system, it is easy to see that

$$CS_i = \frac{1}{2}x_i^2,$$

The consumer surplus at node i is monotonically increasing in the demand at node i, and hence directly related to the Katz-Bonacich centrality. Consumers with higher Katz-Bonacich centrality receive a larger surplus. This result also shows that if consumers endogenously choose whether to form social links, they will aim at maximizing their Katz-Bonacich centrality measure. When the linking cost is small, this will result in the formation of the complete network, as the Katz-Bonacich centrality of all players is increasing in the number of links in the social network (see Ballester Calvó-Armengol and Zenou (2006), Theorem 2 p. 1409.) For higher values of the linking cost, players' incentives to form links depend on the marginal effect of the addition of a link on the Katz-Bonacich centrality measure, which is difficult to compute for general networks. For example, Ballester Calvó-Armengol and Zenou (2004) study a model of endogenous formation of criminal networks based on the marginal effect of the addition of a new link on the Katz-Bonacich centrality measure, but do not provide any characterization of equilibrium network structures.

2.4 Pricing with local monopolies

We now suppose that each consumer is served by a different firm, and consider the pricing game played by n local monopolies with profit functions:

$$\Pi_i = x_i(p_i - c).$$

Each monopoly chooses its price p_i to maximize its own profit, taking as given the price p_j chosen at all other nodes in the social network. The following Proposition characterizes the equilibrium prices as a function of the externalities parameter α:

Proposition 2.4 *In the model with local monopolies, there exists $\overline{\alpha} > 0$ such that, for all $\alpha < \overline{\alpha}$, there is a unique equilibrium price vector $\mathbf{p}^*$ which satisfies:*

$$\mathbf{p}^* = c\mathbf{1} + \frac{1-c}{2}\mathbf{1} + \alpha\frac{1-c}{4}\mathbf{G}\mathbf{1} + \alpha^2\frac{1-c}{8}(\mathbf{G}^2\mathbf{1} - \mathbf{G}\mathbf{1}) + \mathcal{O}(\alpha^3).$$

Proposition 2.4 shows that local monopolies charge different prices at different nodes, so that the uniform pricing rule is no longer valid. Intuitively, given the prices chosen by other firms, a local monopoly at a more central node faces a higher demand, and hence has an incentive to choose a higher price. In order to understand what is the relevant centrality index to rank the prices chosen at different nodes in the social network, we consider an approximation around the point $\alpha = 0$, when externalities are small. Rewriting the characterization of equilibrium prices at a particular node i, we have:

$$p_i^* = c + \frac{1-c}{2} + \alpha\frac{1-c}{4}d_i + \alpha^2\frac{1-c}{8}\sum_j g_{ij}(d_j - 1) + \mathcal{O}(\alpha^3). \qquad (5)$$

Hence, at the first order, the relevant measure is the degree of the node: nodes with higher degree face higher prices. At the second order, if two consumers have the same number of neighbors, prices will be higher for the consumer who has the largest distance-two neighborhood. Clearly, equation

(5) only offers an approximation of equilibrium prices around the point $\alpha = 0$. In order to judge the accuracy of the approximation, we ran a sensitivity analysis, by generating random networks and computing, for each network, the threshold value of α for which the ranking of equilibrium prices in our approximation coincides with the exact ranking of equilibrium prices.[1] The results are given in the following table, which lists for different numbers of agents ($n = 6, 7, 8, 9, 10, 15$ and 20), the minimal, maximal and mean values of the threshold value of α over 1000 randomly generated networks.

n	6	7	8	9	10	15	20
$\overline{\alpha}_{min}$	0.19	0.14	0.01	0.01	0.005	0.01	0.01
$\overline{\alpha}_{max}$	1	1	0.38	0.38	0.305	0.15	0.11
$\overline{\alpha}_{mean}$	0.301	0.248	0.213	0.188	0.160	0.108	0.082

Table 1: Simulations for price rankings

As expected, the threshold value of α decreases with the number of agents, but remains surprisingly high, showing that the approximation is reasonably accurate in order to compare equilibrium prices charged at different nodes.

Using the first order condition for profit maximization, we can compute the demand at node i as

$$x_i = a_{ii}(p_i^* - c),$$

where a_{ii} is the Katz-Bonacich coefficient measuring the discounted number of paths from node i to node i. Because the number of paths from i to i of length 1 is zero and of length 2 is equal to the degree of node i, $a_{ii} = 1 + \alpha^2 d_i + \mathcal{O}(\alpha^3)$. By Proposition 2.4, when α is sufficiently small

$$(1 - \tilde{\theta}_i(\mathbf{p}^*)) = \frac{1-c}{2} + \alpha\frac{1-c}{4}d_i + \alpha^2(\frac{1-c}{2}d_i + \frac{1-c}{8}\sum_j g_{ij}(d_j - 1)) + \mathcal{O}(\alpha^3).$$

Hence, for low values of the externality parameter, the ranking of consumer surplus coincides with the ranking of prices, with consumers with higher degree centrality benefiting from a larger surplus. For small linking costs, consumers thus always have an incentive to form additional social links, and the complete social network is formed in equilibrium. For higher values of the linking cost, the characterization of the endogenous network structure is a complex problem that we leave for future research.

[1]We are immensely grateful to Sebastian Bervoets who wrote the computer program and ran the simulations.

3 Aspiration Based Reference Price

We now consider a model where externalities do not result from consumption but from prices. Following the literature on social comparisons, we assume that agents compare the price they receive with the prices received by their neighbors, and enjoy positive utility if they receive a lower price than the prices in their neighborhood. We start by introducing a linear model of price externalities (Subsection 3.1), continue the discussion of the example introduced in the previous section (Subsection 3.2) and characterize the optimal pricing of a global monopolist (Subsection 3.3) and of local monopolies (Subsection 3.4).

3.1 The model

We assume that utilities are defined over the average price charged to a consumer's neighbor:

$$U_i = \theta_i - p_i + \alpha \frac{1}{d_i} \sum_j g_{ij} p_j. \tag{6}$$

where θ_i is a taste parameter uniformly distributed on $[0, 1]$. As in the case of local network externalities, prices are announced before consumers learn their random valuation, and a consumer located at node i buys the good if and only if

$$\theta_i \geq p_i - \alpha \frac{1}{d_i} \sum_j g_{ij} p_j. \tag{7}$$

Notice that in the model of aspiration based reference price, a consumer's decision is independent of the consumption choices of other consumers, so agents do not need to learn the valuations of their neighbors. Furthermore, as opposed to the case of consumption externalities, we do not need to invert the demands to obtain the system of demands as a function of the price vector. Instead, the demand at node i is directly given by:

$$x_i = \begin{cases} 0 & \text{if } 1 - p_i + \frac{\alpha}{d_i} \sum_j g_{ij} p_j < 0, \\ 1 & \text{if } 1 - p_i + \frac{\alpha}{d_i} \sum_j g_{ij} p_j > 1, \\ 1 - p_i + \frac{\alpha}{d_i} \sum_j g_{ij} p_j & \text{otherwise} \end{cases}$$

We denote by $\mathbf{G}'$ the row-stochastic matrix with typical element $g'_{ij} = \frac{g_{ij}}{d_i}$.

3.2 When and how does centrality matter? The example continued

We return to the example of Section 3, and consider the demand system:

$$\begin{aligned} x_1 &= 1 - p_1 + 0.1 * p_2, \\ x_2 &= 1 - p_2 + 0.033 p_1 + 0.066 p_3, \\ x_3 &= 1 - p_3 + 0.05 p_2 + 0.05 p_3. \\ x_4 &= 1 - p_3 + 0.05 p_2 + 0.05 p_3. \end{aligned}$$

The global monopolist chooses prices p_1, p_2 and p_3 to maximize $p_1 x_1 + p_2 x_2 + 2 p_3 x_3$, yielding the optimal prices $p_1^* = 0.5387, p_2^* = 0.5807, p_3^* p_4^* = 0.5376$. We observe that this price ranking does not reflect degree centrality any longer, as the price charged to node 1 is higher than the prices charged at nodes 3 and 4. However, node 2, which is the "hub" of the network receives the highest price, a result that we will generalize below. If each node is served by a different firm, the unique equilibrium prices are uniform: each consumer is charged a price $p^* = 0.5263$.

3.3 Optimal monopoly pricing

The following Proposition characterizes the unique optimal price chosen by the monopoly in the price externalities model.

Proposition 3.1 *In the model of aspiration based reference price, if* $\alpha\lambda(\mathbf{G}' + \mathbf{G}'^T) < 2$*, there exists a unique optimal price vector* $\mathbf{p}^*$ *which satisfies:*

$$\mathbf{p}^* = (1+c) \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \alpha^k (\mathbf{G}' + \mathbf{G}'^T)^k \mathbf{1}.$$

Proposition 3.1 shows that in the model with average price externalities, the monopoly charges discriminatory prices on the social network. Each price is proportional to the Katz-Bonacich centrality of node i with respect to the matrix $\mathbf{G}' + \mathbf{G}'^T$. In order to gain additional intuition about the relevant centrality index to rank monopoly prices, we compute a first-order approximation of the prices around the point $\alpha = 0$,

$$p_i = \frac{1+c}{2} + \frac{1+c}{4} \alpha (1 + \sum_j g_{ij} \frac{1}{d_j}) + \mathcal{O}(\alpha^2).$$

The relevant centrality measure is the sum of inverse degrees of a node's neighbor. The monopoly has an incentive to charge higher prices at nodes which have higher centrality, as measured by the sum of the inverse degrees of a node's neighbors. A network of particular interest is the star, where the hub has a very high centrality measure and hence receives a higher price than the peripheral agents. This ranking of prices in the star is very intuitive: by raising the price in the hub, the monopoly is able to increase demand at *all* peripheral nodes, whereas an increase in the price of the peripheral node only increases demand at the hub. Hence the indirect positive effect of a price increase is higher for the hub than for a peripheral agent, implying that the optimal price will be higher at the hub.

Using the first order condition for profit maximization, demand at node i is given by:

$$x_i = p_i^* - c - \alpha \sum \frac{g_{ij}}{d_j}(p_j^* - c).$$

Hence agents with many neighbors of low degree (like the hub of the star) receive a smaller surplus than agents with few neighbors of high degree (like the peripheral agents in the star), reflecting the intuition that agents at the hub of the star are harmed by their location in the network, as they pay a higher price than their neighbors. This result suggests that in a model of endogenous network formation, agents would try to avoid occupying central positions, resulting in the construction of symmetric social networks.

3.4 Pricing with local monopolies

We now consider a situation where each node is served by a different local monopoly. We obtain a new network irrelevance result:

Proposition 3.2 *In the model of aspiration based reference price, there is a unique symmetric equilibrium where all local monopolies charge the same price $p^* = \frac{1+c}{2-\alpha}$. The expected consumption is then the same at every node.*

In order to understand Proposition 3.2, notice that if all other nodes charge the price p^* which maximizes $xp(1-\alpha)$, it is a best response for node i to choose the same monopoly price. Hence, prices are uniform across the network and equal to the monopoly price. We remark that this network irrelevance result, as opposed to that of the previous section, is a *robust result* which, as shown in Section 4, holds irrespective of the specific assumptions on the distribution function F. Furthermore, it implies that consumers at every node have the same demand, so that consumer surplus is independent of a node's centrality. In fact, consumers have an incentive to connect to one other node in order to benefit from the positive social comparison effect, but no incentive to form any additional link.

4 Extensions

In this Section, we discuss two extensions of the model: optimal monopoly pricing with general distributions (Subsection 4.1) and bargaining between the monopolist and consumers on the division of the surplus (Subsection 4.2).

4.1 General Distribution Functions

In the analysis so far, we have restricted attention to linear demands generated by a uniform distribution of valuations. This restriction is motivated by tractability considerations. With linear demands, optimal prices are characterized as the solution to a system of linear equations, allowing for a study of the relation between prices and node centrality in arbitrary network topologies. Furthermore, with uniform distributions, the marginal effect of a change in prices on demand is independent of the price level, eliminating the complexity which would result from the curvature of the demand function. However, we are aware of the fact that the assumption of uniform distribution is restrictive, and we discuss in this extension partial results obtained under general distribution functions.

We consider a general distribution of valuations F over a compact interval $[\underline{\theta}, \overline{\theta}] \subset \Re_+$ with continuous and bounded density f. Assume that F satisfies the *monotone hazard rate condition*: $\frac{f(\theta)}{1-F(\theta)}$ is monotonically increasing in θ.

4.1.1 Local network externalities

We focus attention to optimal prices such that demand at every node is interior. To this end, we need to impose additional assumptions on the distribution function and the externalities parameter α. Define:

$$\begin{aligned}\Phi(\theta) &\equiv \frac{1-F(\theta)+\theta}{2f(\theta)},\\ \Psi(\theta) &\equiv \frac{1-F(\theta)}{f(\theta)}.\end{aligned}$$

By the monotone hazard rate property, Ψ is a decreasing function. Notice that Φ may either be increasing or decreasing, and is a constant function if and only if F is the uniform distribution. We will assume however that $\Phi'(\theta) \leq 1$ so that $\Phi(\theta) - \theta$ is a decreasing function. To guarantee the existence of an interior demand equilibrium, we let $\phi(\underline{\theta}) - \underline{\theta} \geq \alpha(n-1) \geq 0 \geq \Phi(\overline{\theta}) - \overline{\theta}$ and $\Psi(\underline{\theta}) - \underline{\theta} \geq \alpha(n-1) \geq 0 \geq \Psi(\overline{\theta}) - \overline{\theta}$. Under these assumptions, we can prove:

Proposition 4.1 *There exists an interior optimal pricing strategy for the monopolist which is characterized by the solution to the system of equations:*

$$\tilde{\theta}_i = \Phi(\tilde{\theta}_i) - \alpha \sum g_{ij}(1 - F(\tilde{\theta}_j)).$$

There exists an interior equilibrium price vector for the local monopolists which is characterized by the solution to the system of equations:

$$\tilde{\theta}_i = \Psi(\tilde{\theta}_i) - \alpha \sum g_{ij}(1 - F(\tilde{\theta}_j)).$$

This characterization of equilibrium enables us to study the robustness of Propositions 2.3 and 2.4 with respect to changes in the distribution. We first consider the case of a single monopolist. The network irrelevance result of Proposition 2.3 does not hold with a general distribution F. The discussion of optimal pricing rules differs when $\Phi(\theta)$ is increasing and $\Phi(\theta)$ is decreasing. When $\Phi(\theta)$ is decreasing, the optimal pricing rule leads the monopoly to charge a higher price at more central nodes:

Corollary 4.2 *Suppose that $\Phi(\theta)$ is decreasing. For any two nodes i and j such that $g_{ik} = 1 \Rightarrow g_{jk} = 1$, $p_j > p_i$.*

When $\Phi(\theta)$ is increasing, there is no general result on the relation between node centrality and prices. However, as the following example shows, more central nodes may experience either higher or lower prices, depending on the shape of the distribution F.

Example 4.3 *Suppose that $n = 3$ and $g_{12} = g_{23} = 1$, $g_{13} = 0$. Let $F(\theta) = \theta^2(\frac{3\beta}{2} - \frac{1}{2}) + \frac{3}{2}(1 - \beta)$ for $\beta, \theta \in [0, 1]$*

It is easy to check that the distribution function F satisfies the assumption $\Phi'(\theta) < 1$. Notice that, for $\beta < \frac{1}{3}$, the function $\Phi(\theta)$ is increasing, for $\beta > \frac{1}{3}$, the function $\Phi(\theta)$ is decreasing, and for $\beta = \frac{1}{3}$, the distribution F is uniform. Let p be the optimal price charged at the peripheral nodes 1 and 3 and q

the price charged at the central node 2. The following table lists the optimal prices for different values of β with $\alpha = 0.1$:

β	0	0.2	0.4	0.6	0.8	1.0
p	0.408	0.457	0.517	0.555	0.580	0.598
q	0.397	0.448	0.522	0.572	0.608	0.636

Table 2: Optimal prices for different distributions

Table 2 shows that the ranking of prices at the central and peripheral node varies with β: when the distribution puts more weight on low values of θ (lower value of β), the central node has a lower price than the peripheral node ; this result is reversed when the distribution puts more weight on high values of θ.

By contrast, the result of Proposition 2.4, showing that more central nodes are charged higher prices by local monopolists, remains valid for general distributions of θ:

Corollary 4.4 *For any two nodes i and j such that $g_{ik} = 1 \Rightarrow g_{jk} = 1$, $p_j > p_i$.*

Corollary 4.4 reflects the robust intuition that a local monopoly which does not internalize the effect of a change in price at the demand at other nodes, always prefers to charge higher prices at more central nodes which have a higher demand.

4.1.2 Aspiration based reference prices

In the model of aspiration based reference prices, Proposition 3.1 shows that the relevant centrality index is the sum of inverse degrees of a node's neighbor. This Proposition cannot easily be extended to general distributions. However, we can use the same intuition as in Proposition 3.1 to compare prices in the specific context of a star:

Remark 4.5 *Suppose that the network g is a star. In the model of aspiration based reference price with general distributions, the single monopolist always charges a higher price to the hub than to the peripheral agents.*

As opposed to the network irrelevance result of the model of local network externalities, the network irrelevance result of Proposition 3.2 is robust to changes in the demand function. In fact, define the monopoly price p^* as the solution to the equation

$$\frac{1 - F(p(1-\alpha))}{f(p(1-\alpha))} = p - c \tag{8}$$

Remark 4.6 *In the model of aspiration based reference price with general distributions, there is always an equilibrium where the local monopolies all charge the monopoly price p^*.*

4.2 Bargaining

In the previous sections, it was implicitly assumed that suppliers had all the bargaining power. In the present subsection, we would like to get some intuition on the influence of consumers' bargaining power.

4.2.1 Local network externalities

Suppose that the price at node i is negotiated between the monopolist and the consumer at that node. In order to avoid any difficulties, we suppose that bargaining takes place before the valuation of the consumer is known, when both parties have symmetric information. In that case, the total surplus to be shared is given by

$$CS_i = (x_i(c))^2.$$

where demand is computed as the minimal value for which the trading surplus between the monopolist and the consumer is positive. As c is uniform across nodes, as in the case of uniform monopoly pricing, demand at node i is proportional to the node's Katz-Bonacich centrality. If we assume that the monopolist has the same bargaining power at all nodes, the surplus accruing to the monopolist is thus monotonically increasing in the Katz-Bonacich centrality measure at that node. Hence, when prices are formed by bargaining between the monopolist and the consumers, the price at node i reflects the Katz-Bonacich centrality of that node.

4.2.2 Aspiration based reference price

If the monopolist and the consumer bargain before the valuation is known, the relevant price to compare is the "participation fee" T_i which is paid by the consumer before she learns her valuation. The total surplus of a consumer at node i is given by:

$$CS_i = \frac{1}{2}(1 + \alpha \frac{1}{d_i} \sum_j g_{ij} T_j - c)^2.$$

If, at each node, the monopolist receives an equal share γ of the surplus, then for low values of α, there exists an equilibrium where the monopolists charge the same price T at each node with

$$T = \frac{\gamma}{2}(1 + \alpha T - c)^2.$$

Hence, for low values of externalities, there exists a price T resulting from the bargaining between the monopolist and the consumers at every node, which is independent of the centrality of the consumer in the social network.

5 Conclusions

This paper contributes to an emerging literature which tries to understand how a monopolist optimally discriminates in a social networks according to consumer's centrality. As opposed to some recent contributions in computer science, which focus on sequential consumption decisions among myopic consumers, we consider simultaneous consumption choices among perfectly rational agents. We show that in a model of local network externalities where consumers are positively affected by the consumption of their neighbors, a single monopolist does not discriminate across the network. Local monopolies charge higher prices at nodes which have a higher degree centrality. When consumers compare the price they receive with the average price in their social neighborhood, a single monopolist has an incentive to charge a higher price to a node which has many neighbors of small degree, like the hub of a star. Local monopolies do not internalize the price externalities and in equilibrium charge a uniform price across the network.

We would like to mention two open problems that deserve further study. First, the study of endogenous formation of the social network requires a detailed analysis of the marginal value of additional links that we would like to undertake in future research. Second, as in any model of price discrimination, consumers located at different nodes in the social network end up paying different prices for the good, and could resell the good to one another. The study of models of resale along social networks is obviously an important area for future research.

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7 Proofs

Proof of Proposition 2.1: Because x_i is increasing in x_j, the system of equations (3) exhibits complementarities, and the analysis of Corollary 1 in Ballester and Calvó-Armengol (2009) applies. However, Ballester and Calvó-Armengol (2009) only consider a situation with positivity constraints

$x_i \geq 0$, and we need to adapt their argument to take care of the additional constraints $x_i \leq 1$. First note that, once S_0 and S_1 are fixed, by Theorem 2 in Ballester and Calvó-Armengol (2009), the demand of consumers in S is uniquely given by the Katz-Bonacich centrality measure restricted to S, as stated in the Proposition. The only statement to prove is that the partition of the set N of consumers into the three sets S_0, S_1 and S is unique. Suppose by contradiction that there exist two systems of demands characterized by two different partitions S_0, S_1, S and S'_0, S'_1, S' with corresponding demands $\mathbf{x}^*$ and $\mathbf{x}'^*$. Transform the problem by increasing prices p_i for all $i \in S_1$ to the point where:

$$p_i = \alpha \sum g_{ij} x_j.$$

Let $\mathbf{p}'$ denote the new price vector. Because none of the demand vectors have been changed, and the constraints are still satisfied, the initial demand system $\mathbf{x}^*$ remains a solution to the new problem. Furthermore, for the new vector of prices $\mathbf{p}'$, the constraint $x_i \leq 1$ become irrelevant, and $\mathbf{x}^*$ is a solution to the classical linear complementarity problem studied by Ballester and Calvó-Armengol (2009). Hence, the system of demands $\mathbf{x}^*$ is the *unique solution* to the new problem with prices $\mathbf{p}'$. Next notice that $\mathbf{p}' > \mathbf{p}$, (all prices in $\mathbf{p}'$ are at least as large as prices in $\mathbf{p}$ and some prices are strictly higher). Consider the solution $\mathbf{x}'^*$ to the initial problem with prices $\mathbf{p}$, and let $\mathbf{p}$ increase incrementally towards $\mathbf{p}'$. Because the matrix $[\mathbf{I} - \alpha \mathbf{G}]^{-1}$ is nonnegative, this increase in price will initially result in a reduction of $x_i'^*$ for all $i \in S$, and possibly the move of some consumers from S_1 to S. By successive steps, we observe that this increase of prices from $\mathbf{p}$ to $\mathbf{p}'$ will necessarily result in a reduction of all quantities $\mathbf{x}_i'^*$. This shows in particular, that if $x_i'^* = 0$ for the price system $\mathbf{p}$, then $x_i'^* = 0$ for the price system $\mathbf{p}'$. As S_0 is the unique set of consumers with zero demand at prices $\mathbf{p}'$, we conclude that $S'_0 \subseteq S_0$. Clearly, we can repeat the same argument interchanging the roles of S_0 and S'_0, in order to obtain $S_0 \subseteq S'_0$, establishing that $S_0 = S'_0$, and concluding the proof of the Proposition.

Proof of Proposition 2.3: We first note that the monopoly will never choose a price vector $\mathbf{p}$ such that the constraint $x_i \geq 0$ or $x_i \leq 1$ binds for some i. Suppose by contradiction that $i \in S_0$ and the constraint $x_i \geq 0$ is binding. If this is the case, we must have $p_i > 1$. By reducing the price p_i so that $x_i > 0$, the monopoly increases the profit made on node i (because $(p_i - c)x_i > 0$), and increases the value x_i which results in an increase in x_j for all $j \neq i$. This is a profitable deviation. Conversely, suppose that $i \in S_1$ and the constraint $x_i \leq 1$ is binding. By increasing the price so that $p_i = \alpha \sum g_{ij} x_j$, the monopoly increases the profit made on consumer

i, and does not change the profit made on any other consumer because x_i remains equal to 1. Again, this is a profitable deviation, establishing that the monopoly will always choose a vector of prices so that demands are interior. Now, if

$$\Pi = \sum_i (p_i - c) \sum_j (a_{ij}(1 - p_j)),$$

differentiating with respect to p_i we find:

$$\frac{\partial \Pi}{\partial p_i} = \sum_j (a_{ij}(1 - p_j)) - \sum_j (a_{ji}(p_j - c)).$$

Because the network is undirected, $\mathbf{G}$ is a symmetric matrix, and so is $[\mathbf{I} - \alpha \mathbf{G}]^{-1}$, so that $a_{ji} = a_{ij}$. We thus have, for all i,

$$\frac{\partial \Pi}{\partial p_i} = \sum_j a_{ij}(1 + c - 2p_j) = 0.$$

Because $a_{ij} \geq 0$ for all ij, this last system of equations has a unique solution: $p_j = \frac{1+c}{2}$ for all j.

Proof of Proposition 2.4: We first note that firm i will always choose a price p_i such that the constraints $0 \leq x_i$ and $x_i \leq 1$ are not binding : if $0 \leq x_i$ were binding, the firm could increase its profit by reducing its price so that $x_i > 0$ and if $x_i \leq 1$ were binding, the firm could increase its profit by increasing its price and selling the same quantity $x_i = 1$. Hence, we can restrict attention to choices of prices p_i which maximize:

$$\Pi = (p_i - c) \sum_j a_{ij}(1 - p_j),$$

resulting in the first order condition:

$$2a_{ii}p_i + \sum_{j \neq i} a_{ij} p_j = a_{ii}c + \sum_j a_{ij}.$$

This system can be rewritten in matrix form as

$$(\mathbf{A} + \Delta(\mathbf{A}))\mathbf{p} = c\Delta(\mathbf{A}) + \mathbf{A}\mathbf{1} \tag{9}$$

where $\Delta(\mathbf{A})$ denotes the diagonal matrix formed by picking the diagonal elements of $\mathbf{A}$, i.e. the diagonal matrix such that $d_{ii} = a_{ii}$ and $d_{ij} = 0$ for $i \neq j$. By simple algebraic manipulations, we obtain:

$$\begin{aligned}(\mathbf{I}+\mathbf{A}^{-1}\Delta(\mathbf{A}))(\mathbf{p}-c\mathbf{1}) &= (1-c)\mathbf{1},\\ (\mathbf{I}-\frac{1}{2}(\mathbf{I}-\mathbf{A}^{-1}\Delta(\mathbf{A})))(\mathbf{p}-c\mathbf{1}) &= \frac{1-c}{2}\mathbf{1}.\end{aligned}$$

Recalling that $\mathbf{A}=[\mathbf{I}-\alpha\mathbf{G}]^{-1}$,

$$\mathbf{A}^{-1}\Delta(\mathbf{A})=(\mathbf{I}-\alpha\mathbf{G})\Delta((\mathbf{I}-\alpha\mathbf{G})^{-1}). \tag{10}$$

Hence, if $\frac{1}{2}\lambda((\mathbf{I}-\alpha\mathbf{G})\Delta((\mathbf{I}-\alpha\mathbf{G})^{-1}))<1$, we can invert the matrix $(\mathbf{I}+\mathbf{A}^{-1}\Delta(\mathbf{A}))$ to obtain:

$$\mathbf{p}-c\mathbf{1}=\frac{1-c}{2}(\mathbf{I}-\frac{1}{2}(\mathbf{I}-(\mathbf{I}-\alpha\mathbf{G})\Delta((\mathbf{I}-\alpha\mathbf{G})^{-1})))^{-1}\mathbf{1}. \tag{11}$$

Now recall that

$$(\mathbf{I}-\alpha\mathbf{G})^{-1}=\sum_{k=0}^{\infty}\alpha^k\mathbf{G}^k,$$

Furthermore,

$$\Delta(\sum_{k=0}^{\infty}\alpha^k\mathbf{G}^k)=\sum_{k=0}^{\infty}\alpha^k\Delta(\mathbf{G}^k),$$

Hence

$$\mathbf{I}-(\mathbf{I}-\alpha\mathbf{G})\Delta((\mathbf{I}-\alpha\mathbf{G})^{-1})=\sum_{k=1}^{\infty}\alpha^k(\mathbf{G}(\Delta(\mathbf{G}^{k-1}))-\Delta(\mathbf{G}^k)). \tag{12}$$

This last expressions shows that there exists $\overline{\alpha}$ such that $\lambda(\mathbf{I}-(\mathbf{I}-\alpha\mathbf{G})\Delta((\mathbf{I}-\alpha\mathbf{G})^{-1})<1$ for $\alpha\leq\overline{\alpha}$. Hence, the matrix $(\mathbf{I}-\frac{1}{2}((\mathbf{I}-\alpha\mathbf{G})\Delta((\mathbf{I}-\alpha\mathbf{G})^{-1})))$ is invertible if external effects are sufficiently small. To finish the computation, note that:

$$\begin{aligned}(\mathbf{I}-\frac{1}{2}(\mathbf{I}-(\mathbf{I}-\alpha\mathbf{G})\Delta((\mathbf{I}-\alpha\mathbf{G})^{-1})))^{-1} &= \sum_{l=0}^{\infty}(\frac{1}{2})^l(\mathbf{I}-(\mathbf{I}-\alpha\mathbf{G})\Delta((\mathbf{I}-\alpha\mathbf{G})^{-1}))^l,\\ &= \sum_{l=0}^{\infty}(\frac{1}{2})^l(\sum_{k=1}^{\infty}\alpha^k(\mathbf{G}(\Delta(\mathbf{G}^{k-1}))-\Delta(\mathbf{G}^k)))^l.\end{aligned}$$

We now want to express the composition of two power series as a power series, and compute the coefficients $\mathbf{C}_m$ such that:

$$\sum_{l=0}^{\infty}(\sum_{k=1}^{\infty}\alpha^k\mathbf{G}(\Delta(\mathbf{G}^{k-1}))-\Delta(\mathbf{G}^k)))^l=\sum_{m=0}^{\infty}\alpha^m\mathbf{C}_m. \tag{13}$$

These coefficients can be obtained by using the Faà di Bruno formula on the composition of power series. [2] To express this formula, consider the composition of two power series:

$$\sum_l(\sum_k a_k\alpha_k)^l=\sum c_m\alpha^m,$$

For any integer m, let $\mathcal{P}(m)$ denote the set of all partitions of the integer m, i.e., sets of integers $k_1,..k_R$ such that $\sum_r k_r=m$. Then, the Faà di Bruno formula states that:

$$c_m=\sum_{k_1,\ldots,k_R\in\mathcal{P}(m)}a_{k_1}a_{k_2}...a_{k_R}.$$

Applying the formula, we find:

$$\mathbf{C}_m=\sum_{k_1,k_2,\ldots,k_R|\sum k_r=m}\prod(\mathbf{G}(\Delta(\mathbf{G}^{k_r-1}))-\Delta(\mathbf{G}^{k_r}). \tag{14}$$

Computing the first terms of the sequence, we find: $\mathbf{C}_0=\mathbf{I},\mathbf{C}_1=\mathbf{G},\mathbf{C}_2=\mathbf{G}^2-\Delta(\mathbf{G}^2)=\mathbf{G}^2-\Delta(\mathbf{G1})$, where the last equality is due to the fact that the diagonal elements of $\mathbf{G}^2$ are equal to the degrees of the agents. We thus find the formula in the Proposition.

Proof of Proposition 3.1:

We can restrict attention to choices of prices p_i which result in interior demands, and maximize:

$$\Pi=\sum_i(p_i-c)(1-p_i+\alpha\sum_j g'_{ij}p_j),$$

resulting in the first order conditions:

$$2p_i-\alpha\sum_j g'_{ij}p_j+\sum_j g'_{ji}=1+c, \tag{15}$$

or in matrix terms

[2] See Johnson (2002) for an historical account of the formula and its uses and variants.

$$(\mathbf{I} - \alpha\frac{1}{2}(\mathbf{G}' + \mathbf{G}'^T))\mathbf{p} = \frac{1+c}{2}\mathbf{1}. \tag{16}$$

If $\alpha\frac{1}{2}\lambda(\mathbf{G}' + \mathbf{G}'^T) < 1$, the matrix $(\mathbf{I} - \alpha\frac{1}{2}(\mathbf{G}' + \mathbf{G}'^T)$ is invertible, and the solution is given by

$$\mathbf{p} = [(\mathbf{I} - \alpha\frac{1}{2}(\mathbf{G}' + \mathbf{G}'^T)]^{-1}\frac{1+c}{2}\mathbf{1}. \tag{17}$$

We use the power series expansion to write:

$$[(\mathbf{I} - \alpha\frac{1}{2}(\mathbf{G}' + \mathbf{G}'^T))]^{-1} = \sum_{k=0}^{\infty} \frac{1}{2^k}\alpha^k(\mathbf{G}' + \mathbf{G}'^T)^k,$$

establishing the Claim.

Proof of Proposition 3.2: We can restrict attention to choices of prices p_i which result in interior demands, and let each firm maximize:

$$\Pi = (p_i - c)(1 - p_i + \alpha\sum_j g'_{ij}p_j),$$

resulting in the first order conditions:

$$2p_i - \alpha\sum_j g'_{ij}p_j = 1 + c, \tag{18}$$

or in matrix terms

$$(\mathbf{I} - \alpha\frac{1}{2}(\mathbf{G}'))\mathbf{p} = \frac{1+c}{2}\mathbf{1}. \tag{19}$$

Because the matrix $\mathbf{G}'$ is stochastic, $\mathbf{G}'\mathbf{1} = \mathbf{1}$, and $p^* = \frac{1+c}{2-\alpha}\mathbf{1}$ is the unique solution to equation (19)

Proof of Proposition 4.1: We focus on interior demands, characterized by:

$$\tilde{\theta}_i = p_i - \alpha\sum g_{ij}(1 - F(\tilde{\theta}_j)).$$

Rewrite the profit of the monopoly as:

$$\pi = \sum_i(\tilde{\theta}_i + \alpha\sum g_{ij}(1 - F(\tilde{\theta}_j)))(1 - F(\tilde{\theta}_i)),$$

and differentiate with respect to $\tilde{\theta}_i$ to obtain:

$$\tilde{\theta}_i = \Phi(\tilde{\theta}_i) - \alpha \sum g_{ij}(1 - F(\tilde{\theta}_j)). \tag{20}$$

The second order condition is satisfied as $\Phi(\theta) - \theta$ is decreasing. To prove existence of a solution to the system of equations, consider the function $T : [\underline{\theta}, \overline{\theta}]^n \to [\underline{\theta}, \overline{\theta}]^n$ such that $T(\theta_{-i})$ is the unique solution to equation 20. The function T is a continuous function from a compact interval of $\Re^n$ to itself and admits a fixed point by Brouwer's fixed point theorem. Similarly, rewrite the profit of a local monopoly as:

$$\pi_i = (\tilde{\theta}_i + \alpha \sum g_{ij}(1 - F(\tilde{\theta}_j)))(1 - F(\tilde{\theta}_i)),$$

and follow the same steps to show existence of an interior demand equilibrium characterized by the equation:

$$\tilde{\theta}_i = \Phi(\tilde{\theta}_i) - \alpha \sum g_{ij}(1 - F(\tilde{\theta}_j)). \tag{21}$$

Proof of Corollary 4.2: Consider the two equations:

$$\begin{aligned} \tilde{\theta}_i &= \Phi(\tilde{\theta}_i) - \alpha \sum g_{ik}(1 - F(\tilde{\theta}_k)) \\ \tilde{\theta}_j &= \Phi(\tilde{\theta}_j) - \alpha \sum g_{jk}(1 - F(\tilde{\theta}_k)) \end{aligned}$$

Taking the difference between the two equations,

$$\tilde{\theta}_i - \tilde{\theta}_j = \Phi(\tilde{\theta}_i) - \Phi(\tilde{\theta}_j) + \alpha \sum_{k|g_{ik}=0} g_{jk}(1 - F(\tilde{\theta}_k)) + \alpha g_{ij} F(\tilde{\theta}_j) - F(\tilde{\theta}_i). \tag{22}$$

which implies that $\tilde{\theta}_i > \tilde{\theta}_j$ as $\Phi(\theta)$ is decreasing and $F(\theta)$ increasing. Finally, as $\Phi(\theta)$ is decreasing, $\Phi(\theta_i) = p_i < \Phi(\theta_j) = p_j$.

Proof of Corollary 4.4: The proof is identical to the proof of Corollary 4.2, observing that $\Psi(\theta)$ is always decreasing.

Proof of Remark 4.5: Let p be the price charged to the peripheral agents and q the price charged to the hub. The profit of the monopolist is given by

$$\pi = q(1 - F(q - \alpha p)) + (n-1)p(1 - F(p - \alpha q)).$$

Because the monopolist can always choose to charge q to the peripheral nodes and p to the central node, by a revealed preference argument we must have:

$$p(1 - F(p - \alpha q)) \geq q(1 - F(q - \alpha p)).$$

Now suppose by contradiction that $p > q$, and consider a change where the monopolist increases the price of the peripheral nodes to p. By doing this, it increases the profit on the central node as $1 - F(p - \alpha q) < 1 - F(p - \alpha p)$. Furthermore, the new profit at any of the peripheral nodes is:

$$p(1 - F(p - \alpha p)) > p(1 - F(p - \alpha q)) \geq q(1 - F(q - \alpha p)),$$

so that the profit at the peripheral node also increases, contradicting the fact that (p, q) is an optimal pricing strategy.

Proof of Remark 4.6: Let all other firms choose the monopoly price p^*. Then the local monopoly chooses a price p to maximize

$$\pi_i = (p - c)(1 - F(p - \alpha p^*)).$$

Obviously, this profit function is maximized at $p = p^*$.