Long-Term Contracting with Time-Inconsistent Agents*

Daniel Gottlieb† and Xingtan Zhang‡

November 26, 2019

Abstract

We study contracts between partially naive present-biased consumers and risk neutral firms in settings with one- and two-sided commitment. Our main result is that as the number of periods grows, the welfare loss from present bias vanishes. We use the model to study two common regulatory interventions: removing commitment power from consumers and imposing limits on the fees that firms can charge. For each fixed contracting horizon, removing commitment power increases welfare when consumers are sufficiently present-biased. However, removing commitment power cannot help if the contracting horizon is long. With one-sided commitment, setting a maximum fee never benefits consumers. Overall, there are two possible interpretations of our main finding: either there is no role for regulation to correct for present bias when contractual relationships are long enough, or something must be missing from how these markets are typically modeled.

JEL: D81, D86, D91, I18

*We thank David Dillenberger, Drew Fudenberg, Faruk Gul, Michael Grubb, Botond Koszegi, Shengwu Li, George Mailath, Pietro Ortoleva, Wolfgang Pesendorfer, Philipp Strack, Dmitry Taubinsky, Jeremy Tobacman, Juuso Toikka, Edward Van Wesep, Leeat Yariv, Muhamet Yildiz, John Zhu, and seminar participants at Harvard-MIT, Princeton, UC Berkeley, University of Colorado Boulder, Wharton, SITE Psychology and Economics Workshop, and the 2019 AEAs for comments. Gottlieb is grateful for the hospitality of the Economics Department at Harvard University, where part of this research was conducted. Zhang acknowledges financial support from de Soto Capital Markets Program at University of Colorado Boulder.

†London School of Economics and Washington University in St. Louis, d.gottlieb@lse.ac.uk.

‡University of Colorado Boulder, xingtan.zhang@colorado.edu.

1 Introduction

A vast literature in behavioral economics studies markets with present-biased consumers who underestimate their bias ("partial naiveté"). An important finding from this literature is that the equilibrium is inefficient and regulation that accounts for internalities can increase welfare.[1] Models in this literature generally assume that there are only three periods, which is the minimum needed for dynamic inconsistency to play a role. But this is an unrealistic assumption since, in these models, periods are thought to be very short, typically no more than a day (O'Donoghue and Rabin, 2015).

Our paper considers a general contracting model with present-biased consumers and an arbitrary number of periods. We show that to exploit consumer naiveté firms offer contracts with two options at each point in time: a front-loaded and a back-loaded option. Consumers think they will pick the front-loaded option but pick the back-loaded one instead, effectively postponing payments to the next period. On the equilibrium path, consumers keep postponing their payments until the last period. As a result, the equilibrium has smooth consumption in all but the last period. Because the relative weight on the last period shrinks as the contracting horizon grows, the consumption path of present-biased consumers converges to the path that maximizes their long-term preferences. Therefore, the welfare loss from present bias vanishes as the contracting horizon grows.

We use our framework to study the effect of removing commitment power from consumers. One-sided commitment is prevalent in many markets (e.g., mortgages, car loans, life insurance, long-term care, and annuities).[2] Regulations that allow consumers to terminate agreements at will are often motivated by attempts to protect them. However, in standard models, removing a rational consumer's commitment power can only hurt the consumer. Our setting is a natural candidate for studying the effect of regulating commit-

[1]See, for example, Gruber and Koszegi (2001); O'Donoghue and Rabin (2003); DellaVigna and Malmendier (2004); Heidhues and Kőszegi (2010).

[2]In mortgages and other credit markets, borrowers can prepay their debt, but debtors cannot force them to repay before the contract is due. Similarly, in long-term insurance markets – such as life insurance, long-term care insurance, or annuities – policyholders are allowed to cancel their policies at all times, but firms cannot drop them.

ment power, because committing to future actions and lapsing on previous agreements are inherently inter-temporal decisions, and present bias is the most well-studied bias in intertemporal decision-making. Moreover, there is evidence that present bias is an important feature in some credit markets where regulation prevents consumers from being able to commit to long-term contracts, such as in mortgage or credit card markets.[3]

For a fixed horizon, removing commitment power helps consumers who are sufficiently present biased. This is because present-biased consumers are tempted to overborrow. Since firms would not lend to consumers who can walk away from contracts, removing commitment power restricts their access to savings, which increases welfare when consumers are sufficiently present biased. This result is in line with regulation, which often mentions consumer protection as the reason for allowing them to terminate agreements at will, although it contrasts with the intuition that the provision of commitment devices is necessarily welfare improving.

We then generalize the vanishing inefficiency result for settings with one-sided commitment. In this case, the equilibrium converges to the path that maximizes the consumer's long-term preferences subject to renegotiation proofness constraints. Since, with commitment, the equilibrium maximizes long-term preferences without these additional constraints, removing commitment power does not help when the contracting horizon is long enough.

In the third part of the paper, we turn to the particular case of constant endowments. By shutting down risk, this setting allows us to isolate the effect of shifting consumption over time. We show that controlling for impatience it is easier to sustain long-term contracts

[3] Schlafmann (2016), for example, empirically studies self-control in mortgage markets and shows that requiring higher down payments and restricting prepayment can help customers. Similarly, Ghent (2011) argues that providing access to mortgages with lower initial payments decreases savings due to time inconsistency. Gathergood and Weber (2017) study mortgage choices in the UK and find that present bias substantially raises the likelihood of choosing alternative mortgage products. And Atlas et al. (2017) study data from a sample of US households and find that present-biased individuals are more likely to choose mortgages with lower upfront costs. They also find that present-biased individuals are less likely to refinance their mortgages, which is consistent with our results from Section 4. See also Bar-Gill (2009) for a description of behavioral aspects of the subprime mortgage market. For credit card markets, see Meier and Sprenger (2010). Our equilibrium pattern of postponing repayments until the last period is generally consistent with the findings from Carter et al. (2017) on payday loans.

when consumers are time-inconsistent than when they are time-consistent. This is because, when considering a time-inconsistent consumer and a time-consistent consumer with the same "average impatience," the time-consistent consumer discounts periods further in the future by more than a time-inconsistent consumer. As a result, the time-inconsistent consumer is less hurt by front-loading payments from the periods sufficiently far in the future, which helps to support long-term contracts.

Our last result concerns the effect of limiting the fees that firms can charge. This type of policy has been popular among regulators as a way to protect consumers.[4] We show that with one-sided commitment, limiting the fee that firms can charge is never welfare improving. The intuition is that when interest rates are low so customers would like to borrow, one-sided commitment prevents them from obtaining a long-term contract. Then, imposing a maximum fee does not affect the equilibrium. The only case where a maximum fee can affect the equilibrium is when interest rates are so high that, in equilibrium, customers would like to save. However, in this case, imposing a maximum fee reduces savings, moving the equilibrium further away from the optimum.

The main message of our paper is that the inefficiency of markets with present-biased consumers crucially depends on the length of the relationship. There are two possible interpretations of our main result. If one takes the models as currently formulated as good approximations of reality, then our results suggest that there is no role for regulation that corrects for present bias as long as contractual relationships are long enough. If instead, one believes that inefficiency is a prevalent feature of real markets with present-biased consumers, then our results highlight that something must be missing from how these markets are typically modeled.

[4]For example, the Credit CARD Act of 2009 limits the amount of interest and fees that credit card companies can charge. Similarly, Dodd-Frank (Title XIV) has many provisions that limit penalties or fees in mortgage contracts. In insurance, state-level nonforfeiture laws specify minimum payments to customers who surrender their permanent life insurance policies or annuities, with each state following a slight variation of the general guidelines from the National Association of Insurance Commissioners (NAIC). And in long-term care insurance, the Department of Financial Services specifies minimum benefits that must be provided as well as minimum cash benefits that must be paid to those who lapse.

Related Literature

Our paper fits into a recent literature on contracting with behavioral agents, summarized in Kőszegi (2014) and Grubb (2015). We build on the credit card model of Heidhues and Kőszegi (2010) by considering more than two consumption periods, allowing for uncertainty, and allowing for one-sided commitment.

Our paper is also related to a literature that studies commitment contracts with time-inconsistent agents (c.f. Amador et al. (2006); Halac and Yared (2014); Galberti (2015); Bond and Sigurdsson (2017)).[5] Finally, Section 3 of our paper is related to a literature on dynamic risk-sharing with one-sided commitment. Several papers show that front-loaded payment schedules help mitigate a consumer's lack of commitment power.[6] For example, Hendel and Lizzeri (2003) theoretically and empirically examine how life insurers mitigate reclassification risk by offering front-loaded policies. Similarly, several researchers show that mortgages are front loaded to mitigate prepayment risk.[7] More recently, Handel et al. (2017) and Atal et al. (2018) show that front-loaded long-term health insurance contracts can produce substantial welfare gains by insuring policyholders against reclassification risk. The main difference between these models and ours is that consumers in our model are dynamically inconsistent.

The paper proceeds as follows. We first consider a general model with arbitrary income paths. In Section 2, we present the model with commitment. In Section 3, we introduce

[5]There are two key differences between our paper and this literature. First, this literature considers sophisticated agents, whereas our primary focus is on partially naive agents. Second, we study a different incentive aspect. This literature studies the trade-off between commitment and flexibility (agents have commitment power but, because they face an unverifiable taste shock, they value the flexibility to adjust to different taste shocks), whereas, in Section 3, we study the agent's incentive to lapse and re-contract with other firms. Our paper is also related to Bisin et al. (2015), who study the interaction between government policy and private commitments by present-biased voters and to Harris and Laibson (2001) and Cao and Werning (2018), who study the Markov equilibria in infinite-horizon problems with sophisticated consumers and show there can be multiple non-smooth equilibria. Multiplicity and non-smoothness do not arise in our setting because our model has a finite (albeit arbitrary) horizon.

[6]This literature originates with Harris and Holmstrom (1982) who present a theory of wage rigidity based on the assumption that firms can make binding contracts with workers, but workers are always allowed to switch to better jobs. See also Dionne and Doherty (1994), Pauly et al. (1995), Cochrane (1995), Krueger and Uhlig (2006), and Daily et al. (2008).

[7]See, e.g., Brueckner (1994) and Makarov and Plantin (2013).

one-sided commitment and discuss the effect of removing commitment power on welfare. Then, in Section 4, we move to the special case of a constant income. Section 5 concludes. In the supplementary appendix, we show that our results persist when the firm has the bargaining power (instead of the consumer) and when the firms do not know the consumer's naiveté parameter.

2 Model with Commitment

There is one consumer (agent) and a finite number of firms. Time is discrete and finite. To allow for arbitrary non-stationary settings, we model the stochastic environment as follows. There is a finite state space $\mathbb{S}_t$ for each $t \in \mathbb{N}$. The agent earns income $w(s_t)$ at state s_t. Let $p(s_t|s_\tau)$ denote the probability of reaching state s_t conditional on state s_τ. We say that state s_t follows state s_τ if $p(s_t|s_\tau) > 0$. A state specifies all previously realized uncertainty, so a state cannot follow two different states. We consider the T-period truncation of this setting; that is, an environment with state spaces $\mathbb{S}_t$ and conditional probabilities $p(\cdot|\cdot)$ up to period T, at which point the game ends.

Without loss of generality, we assume that no uncertainty is realized before the initial period: $\mathbb{S}_1 = \{\emptyset\}$. Let $E[\cdot|s_t]$ denote the expectation operator conditional on state s_t and let $E[\cdot]$ denote the unconditional (time-1) expectation. By taking degenerate distributions, our framework allows for deterministic income paths. Also, since the probabilities of reaching future states may depend on the current state, our framework also allows for persistent shocks, which is important to encompass environments with reclassification risk.

Firms are risk neutral and can freely save or borrow at the interest rate $R \geq 1$, so that each firm maximizes its expected discounted profits. The expected profits at state s_τ of a firm who collects state-dependent payments $\{\pi(s_t)\}_{t\geq\tau}$ are

$$E\left[\sum_{t\geq\tau} \frac{\pi(s_t)}{R^{t-\tau}} \;\middle|\; s_\tau\right].$$

The agent has quasi-hyperbolic discounting and needs a firm to transfer consumption across states.[8] At state s_τ, the agent evaluates the state-dependent consumption $\{c(s_t)\}_{t\geq\tau}$ according to

$$u(c(s_\tau)) + \beta E\left[\sum_{t>\tau} \delta^{t-s} u(c(s_t)) \;\middle|\; s_\tau\right], \tag{1}$$

where $\beta \in (0,1)$ and $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, and twice continuously differentiable. We are interested in time-inconsistent consumers who underestimate their bias – i.e., they are *partially naive* as defined by O'Donoghue and Rabin (1999). Such a consumer believes that, in all future periods, he will behave like someone with time-consistency parameter $\hat{\beta} \in (\beta, 1]$. For brevity, we will refer to a partially naive time-inconsistent consumer simply as a *time-inconsistent consumer*.[9] As a benchmark, we also consider the case of time-consistent consumers ($\hat{\beta} = \beta = 1$). Following most of the literature, we take the agent's long-run preferences as the relevant ones in our welfare calculations.[10] Therefore, consumers maximize welfare in the time-consistent benchmark but not when they are time-inconsistent.

For simplicity, we will assume that firms know the consumer's preferences. This assumption is relaxed in Appendix B, where we allow firms not to know the consumer's naiveté parameter $\hat{\beta}$. For now, we also assume that the consumer has all bargaining power and that all parties can commit to long-term contracts, so the consumer makes a take-it-or-leave-it offer of a contract in the first period, which is honored until the game ends.

Whenever the consumer is not time consistent, his ranking of consumption streams depends on when the stream is evaluated. As usual, we model the behavior of such an agent by treating his decision in each period as if it was decided by a different "self." Because the consumer is naive, each self may mispredict how his future selves will choose. We are interested in Subgame Perfect Nash Equilibria (SPNE) of this game.[11]

[8]Equivalently, the income process $w(\cdot)$ can be interpreted as the smoothest consumption that the consumer can obtain without interacting with the firms in the model. In this interpretation, $w(s_t)$ includes the amount that the individual can borrow or save from other sources at state s_t.

[9]Subsection 4.3 considers sophisticates, who fully understand their time-inconsistency ($\hat{\beta} = \beta$).

[10]See, e.g., DellaVigna and Malmendier (2004); O'Donoghue and Rabin (1999, 2001).

[11]Our game-theoretical equilibrium concept coincides with the non-strategic competitive equilibrium of

2.1 Time-Consistent Consumers

As a benchmark, we first consider a time-consistent consumer. Because parties can commit to long-term contracts, the equilibrium consumption maximizes the agent's utility in period 1,

$$E\left[\sum_{t=1}^{T}\delta^{t-1}u\left(c\left(s_t\right)\right)\right], \tag{2}$$

subject to the zero profits constraint,

$$\sum_{t=1}^{T}E\left[\frac{w\left(s_t\right)-c\left(s_t\right)}{R^{t-1}}\right]=0. \tag{3}$$

Indeed, no firm would accept a contract with negative expected profits. If profits were positive, the agent would benefit by offering a contract with slightly higher consumption. Because the objective function in (2) is strictly concave and (3) is a linear constraint, there is a unique solution. So, any SPNE of the game provides the same consumption, which solves the program above. Let W^C denote the equilibrium welfare of the time-consistent consumer, which evaluates the objective (2) at the equilibrium consumption.

2.2 Time-Inconsistent Consumers

Before presenting a general analysis of equilibrium with time-inconsistent consumers, we start with a simple illustrative example.

2.2.1 Example

There are $T = 3$ periods and no uncertainty. The agent has utility function $u(c) = \sqrt{c}$, discounts the future according to $\beta = \frac{1}{2}$ and $\delta = 1$, and is fully naive $\hat{\beta} = 1$. Let $R = 1$ and $w_1 + w_2 + w_3 = 1$.

Based on Subsection 2.1, one might think that the equilibrium consumption maximizes

Heidhues and Kőszegi (2010). We formulate the model as a game because it can be more straightforwardly generalized to settings with one-sided commitment, as we do in Section 3.

the agent's perceived utility subject to zero profits:

$$\max_{c_1,c_2,c_3} \sqrt{c_1} + \beta\left(\sqrt{c_2} + \sqrt{c_3}\right) \tag{4}$$

subject to

$$c_1 + c_2 + c_3 = 1. \tag{5}$$

We will show that this is not the case. To see this, first note that the solution to this program is $c_1 = \frac{2}{3}$, $c_2 = c_3 = \frac{1}{6}$, which gives the agent a perceived utility of $\sqrt{3/2}$.

Suppose a firm decides to offer a contract that gives a consumption in the first period of $c_1 = \frac{8}{27}$ and allows the agent to pick between two different options in the second period: a *baseline* and an *alternative* option. The baseline option provides as little consumption as possible in the second period in exchange for a high consumption in the future: $c_2(B) = 0$ and $c_3(B) = \frac{50}{27}$. The alternative option has a smoother consumption: $c_2(A) = \frac{8}{27}$ and $c_3(A) = \frac{7}{81}$.

Since the agent thinks that his future selves are perfectly patient, he believes that he will select the baseline option:

$$\sqrt{c_2(B)} + \sqrt{c_3(B)} \approx 1.36 > 0.84 \approx \sqrt{c_2(A)} + \sqrt{c_3(A)},$$

which gives him a perceived utility of $\sqrt{c_1} + \beta\left[\sqrt{c_2(B)} + \sqrt{c_3(B)}\right] = \sqrt{3/2}$, which is the same in the solution of Program (4)-(5). Therefore, the agent accepts to switch to this new contract.

However, in period 2, the agent picks the alternative option instead of the baseline, since

$$\sqrt{c_2(A)} + \beta\sqrt{c_3(A)} \approx 0.69 > 0.68 \approx \sqrt{c_2(B)} + \beta\sqrt{c_3(B)}.$$

And because the agent's actual consumption is given by the alternative option, the firm makes a profit of

$$1 - c_1 - c_2(A) - c_3(A) \approx 0.32.$$

That is, the firm profits by offering a flexible contract to the agent, which allows the firm to exploit the agent's incorrect beliefs. The firm exploits the difference in beliefs by offering a baseline option with very low consumption in period 2 ($c_2(B) = 0$) in exchange for a large future consumption ($c_3(B) = \frac{50}{27}$). And while the agent thinks that he will choose this baseline option, he ends up switching to the alternative option, which has a lower NPV but a much higher immediate consumption ($c_2(A) = \frac{8}{27},\ c_3(A) = \frac{7}{81}$).

Having shown that we cannot have an equilibrium with inflexible contracts in this intuitive example, we now characterize the equilibrium for the general case.

2.2.2 Equilibrium

Any contract that is accepted with positive probability must maximize the consumer's utility in period 1 subject to two types of constraints: zero profits, which is the same as before, and incentive constraints, which are due to consumer naiveté.

Because the consumer mispredicts his future preferences, he may disagree with the firm about the actions that his future selves will take. So we need to distinguish between what the consumer believes that he will choose and what firms believe that the consumer will choose (which we interpret as the correct beliefs). This disagreement gives rise to two sets of incentive constraints. Following Heidhues and Kőszegi (2010), we refer to them as *perceived choice constraints (PCC)* and *incentive compatibility constraints (IC)*.

PCC requires the consumer to believe that his future selves will choose the actions that maximize his perceived utility. IC requires firms to believe that the consumer's future selves will choose the actions that maximize the consumer's true utility. The option that the consumer thinks that his future selves will choose is called the *baseline* option (B). The option that firms think that the consumer's future selves will choose is called the *alternative* option (A). In principle, these options can coincide, in which case the consumer and the firms agree about which actions will be chosen. But we will show that these options are always different in equilibrium.

A time-t option history h^t is a list of options chosen by the consumer up to time t:

$h^1 = \emptyset$, $h^2 \in \{A, B\}$, $h^3 \in \{AA, AB, BA, BB\}$, etc. Since there are no actions after the last period, there is no space for disagreement at $t = T$, so that $h^T = h^{T-1}$. Figure 1 depicts the option histories when there are four periods.

An *equilibrium consumption vector* is the vector of state-dependent consumption in all option histories for all states that happen with positive probability:[12]

$$\mathbf{c} \equiv \{(c(s_1), c(s_2, h^2), c(s_3, h^3), \cdots, c(s_T, h^T)) : \; p(s_2|s_1)p(s_3|s_2)\cdots p(s_T|s_{T-1}) > 0\}.$$

A *consumption on the equilibrium path* is a vector of state-contingent consumption that happens with positive probability (using correct beliefs about the options that the consumer chooses):

$$\mathbf{c}^E \equiv \{(c(s_1), c(s_2, A), c(s_3, A, A), \cdots, c(s_T, A, ..., A)) : \; p(s_2|s_1)p(s_3|s_2)\cdots p(s_T|s_{T-1}) > 0\}.$$

Unlike the equilibrium consumption vector, the consumption on the equilibrium path only includes outcomes conditional on the consumer repeatedly picking option A.

The *equilibrium program* (P) is:

$$\max_{\{c(s_t,h^t)\}} u(c(s_1)) + \beta E\left[\sum_{t=2}^{T} \delta^{t-1} u(c(s_t, B, B, ..., B))\right], \tag{6}$$

subject to

$$\sum_{t=1}^{T} E\left[\frac{w\left(s_t\right) - c\left(s_t, A, A, ..., A\right)}{R^{t-1}}\right] = 0, \tag{Zero Profits}$$

$$u(c(s_\tau, \left(h^{\tau-1}, B\right))) + \hat{\beta} E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c\left(s_t, \left(h^{\tau-1}, B, B, ..., B\right)\right)\right) \middle| \; s_\tau\right] \tag{PCC}$$

$$\geq u(c(s_\tau, \left(h^{\tau-1}, A\right))) + \hat{\beta} E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c\left(s_t, \left(h^{\tau-1}, A, B, ..., B\right)\right)\right) \middle| \; s_\tau\right],$$

[12] Since a state of the world encodes all uncertainty realized up to that period, the distribution over future states conditional on s_t can only have full support in the trivial case where no uncertainty was realized until state s_t.

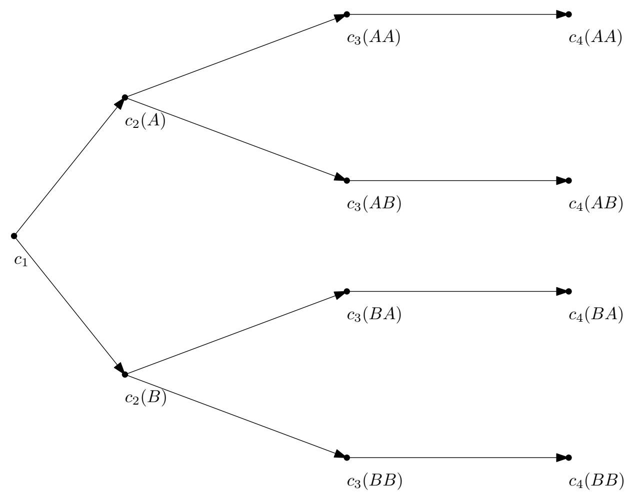

Figure 1: Option histories with $T = 4$ and no uncertainty. The consumer thinks that he will choose the baseline (B) in each node but ends up choosing the alternative (A). Ex ante, the consumer believes that his consumption will be $(c_1, c_2(B), c_3(BB), c_4(BB))$. In period 2, he deviates to $c_2(A)$, while thinking that he will receive $(c_3(AB), c_4(AB))$ in periods 3 and 4. Then, he deviates again in period 3, getting $(c_3(AA), c_4(AA))$ instead. With uncertainty, consumption also depends on the state of the world.

and

$$u(c(s_\tau, (h^{\tau-1}, A))) + \beta E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c\left(s_t, \left(h^{\tau-1}, A, B, ..., B\right)\right)\right) \middle| \, s_\tau\right] \tag{IC}$$

$$\geq u(c(s_\tau, (h^{\tau-1}, B))) + \beta E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c\left(s_t, (h^{\tau-1}, B, B, ..., B)\right)\right) \middle| \, s_\tau\right].$$

The following lemma establishes that the equilibrium program (P) characterizes the equilibrium consumption vector:

Lemma 1. ***c*** *is an equilibrium consumption vector if and only if it solves program (P).*

2.2.3 Auxiliary Program

Consider a dynamically consistent agent who differs from the one described in Subsection 2.1 in that he discounts consumption in the last period by an additional factor β. The equilibrium consumption for this agent solves the following *auxiliary program*:

$$\max_{\{c(s_t)\}} E\left[\sum_{t=1}^{T-1} \delta^{t-1} u\left(c(s_t)\right) + \beta\delta^{T-1} u\left(c(s_T)\right)\right], \tag{7}$$

subject to the zero profits constraint (3).

The following lemma establishes that the consumption on the equilibrium path for time-inconsistent agents coincides with the solution of the auxiliary program:

Lemma 2. *Suppose the consumer is time inconsistent. The consumption on the equilibrium path coincides with the solution of the auxiliary problem.*

The auxiliary program highlights that, in this model, underweighting consumption in the last period is the only distortion from time-inconsistency. To illustrate the lemma, consider the case of three periods and a constant income w. Since there is a single state of the world in each period, it can be omitted from the history. The equilibrium contract solves:

$$\max_{(c_1, c_2(A), c_2(B), c_3(A), c_3(B))} u(c_1) + \beta[\delta u(c_2(B)) + \delta^2 u(c_3(B))], \tag{8}$$

subject to

$$c_1 + \frac{c_2(A)}{R} + \frac{c_3(A)}{R^2} = w\left(1 + \frac{1}{R} + \frac{1}{R^2}\right), \tag{9}$$

$$u(c_2(B)) + \hat{\beta}\delta u(c_3(B)) \geq u(c_2(A)) + \hat{\beta}\delta u(c_3(A)), \tag{10}$$

$$u(c_2(A)) + \beta\delta u(c_3(A)) \geq u(c_2(B)) + \beta\delta u(c_3(B)), \tag{11}$$

where (9) is the zero profits constraint, (10) is the perceived choice constraint, and (11) is the incentive compatibility constraint.

Note first that incentive compatibility (11) must bind. Otherwise, we could increase $c_3(B)$ to achieve a higher utility without violating any constraint. Since (11) binds, we can rewrite the perceived choice constraint (10) as a monotonicity condition:

$$c_3(A) \leq c_3(B). \tag{12}$$

That is, because agents are more present-biased than they think they are, the contract that they actually pick (A) is more front-loaded than the contract that they think that they will pick (B). We ignore this monotonicity constraint for now and verify that it holds later.

Next, we contrast the marginal rate of substitution between $c_2(B)$ and $c_3(B)$ in the objective function (8) and the incentive compatibility constraint (11). The objective function evaluates consumption according to period-1 preferences, so the discount rate between periods 2 and 3 is δ. The incentive compatibility constraint depends on the actual choice of the time-2 self, which discounts one period by $\beta\delta < \delta$. Therefore, front-loading the baseline consumption relaxes the incentive constraint. The solution of the program must then offer a maximally front-loaded baseline contract:

$$c_2(B) = 0. \tag{13}$$

Substituting (13) in the binding constraint (11) gives

$$u(c_2(A)) + \beta\delta u(c_3(A)) = u(0) + \beta\delta u(c_3(B)), \tag{14}$$

which can be rearranged as

$$\beta\delta\left[u(c_3(B)) - u(c_3(A))\right] = u(c_2(A)) - u(0) \geq 0.$$

Therefore, the monotonicity constraint (12) is automatically satisfied at the solution and can be ignored. Substituting (13) and (14) in the objective function, we obtain the objective of the auxiliary problem (up to a constant that can be omitted from the program):

$$u(c_1) + \beta[\delta u(c_2(B)) + \delta^2 u(c_3(B))] = u(c_1) + \delta u(c_2(A)) + \beta\delta^2 u(c_3(A)) - (1-\beta)\delta u(0). \tag{15}$$

Note that, in the proof above, we used equations (13) and (14) to eliminate the baseline consumption from the equilibrium program. Substituting the consumption that solves the auxiliary program back in these two equations, we can recover the baseline consumption. Since neither the auxiliary program nor equations (13) and (14) depend on the consumer's naiveté parameter $\hat{\beta}$, it follows that, in equilibrium, both the baseline consumption and the alternative consumption are not functions of $\hat{\beta}$.

Corollary 1. *There exists an equilibrium. Moreover, any equilibrium has the same equilibrium consumption vector, which is not a function of the consumer's naiveté $\hat{\beta}$ and is a continuous function of the agent's time-inconsistency parameter $\beta \in (0, 1]$.*

Since the equilibrium consumption vector is not a function of $\hat{\beta}$, the equilibrium obtained here would also be the equilibrium if we assumed that firms did not know the consumer's naiveté $\hat{\beta}$ (see Appendix B for a formal proof).

2.2.4 Vanishing Inefficiency

We now use Lemma 2 to obtain our main result. Let W^I denote the equilibrium welfare of the time-inconsistent consumer, which evaluates the agent's consumption on the equilibrium path according to his long-term preferences (2) and recall that W^C is the welfare in the benchmark case of a time-consistent consumer. Since the time-consistent consumer maximizes welfare, the welfare loss from dynamic inconsistency is $W^C - W^I \geq 0$.

Theorem 1. *Suppose u is bounded and $\delta < 1$. Then, $lim_{T \nearrow +\infty} \left(W^C - W^I\right) = 0$.*

The theorem states that the welfare loss from dynamic inconsistency converges to zero as the contracting length grows. The assumption that u is bounded and $\delta < 1$ ensures that the discounted welfare converges (otherwise, the discounted sum may not be well-defined). Note that the theorem holds for any discount factors β and δ, so it is unrelated to the folk theorem literature from repeated games.

Recall that the only inefficiency from time inconsistency is underweighting the last period. Because the effect of the last period vanishes as the number of periods grows, the solution of the auxiliary program converges to the equilibrium consumption with time-consistent consumers as $T \nearrow +\infty$. So, even though the time-inconsistent consumer does not maximize his welfare function and has incorrect beliefs, in any equilibrium, he gets approximately the maximum welfare possible if the number of periods is large. In fact, it is precisely the fact that the agent has incorrect beliefs that causes the inefficiency to vanish, since the optimal way to exploit the agent's naiveté is to offer contracts that postpone fees until the last period.[13]

3 One-Sided Commitment

We now turn to the model in which consumers cannot commit to long-term contracts (one-sided commitment), keeping the assumption that the consumer has all bargaining power. As argued in the introduction, one-sided commitment is common in many markets. We model one-sided commitment as follows. The consumer offers a contract in each period. If a firm has accepted a contract, the consumer decides whether to keep it or replace it with a new one. If multiple firms accept a contract, the consumer picks each of them with some positive probability.[14]

3.1 Benchmark: Time-Consistent Consumers

Consider first the benchmark case of a time-consistent consumer. With one-sided commitment, the consumer switches to a new contract whenever a firm is willing to provide him terms that are better than the terms of the original contract. For the purpose of characterizing the equilibrium consumption, there is no loss of generality in restricting attention to contracts in which the consumer never lapses ("renegotiation proofness"). To see this, consider an equilibrium in which the consumer lapses in some state of the world, replacing the original contract with a contract from another firm. Since the other firm cannot lose money by offering this new contract, the old firm could have accepted a contract that substituted the terms of the old contract from this period on with the terms of the new contract, and

[13]As we illustrate in Subsection 4.3, the welfare loss from dynamic inconsistency does not vanish when the consumer is fully aware of his time consistency.

[14]In this formulation, there are no exogenous costs from walking away from a contract. This assumption is appropriate to insurance settings, where policyholders are allowed to drop coverage at no additional cost by stopping to pay their premiums (c.f., Hendel and Lizzeri (2003); Handel et al. (2017)). For credit markets, payments should be interpreted as net of collateral, so a consumer who fails to repay his debt loses the collateral. In practice, in many credit markets, failing to repay also leads to bankruptcy or reputation costs. We abstract from those here for simplicity.

the consumer would have remained with the old firm. So, to characterize the consumption that can be supported in equilibrium, we can impose *non-lapsing constraints* which require contracts to be renegotiation proof. These constraints state the consumer's outside option at each state cannot exceed the value from keeping the current contract, where the outside option corresponds to the value from the best possible contract that other firms are willing to provide.[15]

Formally, the outside option at state s_τ is defined by the recursion:

$$V^C(s_\tau) \equiv \max_{\{c(s_t)\}} u\left(c(s_\tau)\right) + E\left[\left.\sum_{t>\tau} \delta^{t-\tau} u\left(c\left(s_t\right)\right)\right| s_\tau\right], \tag{16}$$

subject to

$$\sum_{t=\tau}^{T} E\left[\left.\frac{w\left(s_t\right) - c\left(s_t\right)}{R^{t-\tau}}\right| s_\tau\right] = 0, \tag{17}$$

and

$$u\left(c(s_{\tilde{\tau}})\right) + E\left[\left.\sum_{t>\tilde{\tau}} \delta^{t-\tilde{\tau}} u\left(c\left(s_t\right)\right)\right| s_{\tilde{\tau}}\right] \geq V^C(s_{\tilde{\tau}}),\ \ \forall s_{\tilde{\tau}} \text{ with } p(s_{\tilde{\tau}}|s_\tau) > 0. \tag{18}$$

Equation (17) is the zero profits condition, whereas (18) requires the new contract itself to be renegotiation proof. The equilibrium consumption with one-sided commitment solves this program at the initial period (i.e., at state $s_1 = \emptyset$).

While one can use Program (16)-(18) to obtain the equilibrium consumption by backward induction, an easier characterization can be obtained when the consumer is time-consistent. Consider, instead, the program that replaces the non-lapsing constraints by the requirement that, at each point in time, the expected future income cannot exceed the ex-

[15] When a non-lapsing constraint binds, there are also equilibria in which the consumer lapses and recontracts with another firm. These equilibria are equivalent to the one with no lapsing in the sense that the consumer gets the same consumption and all firms make the same profits.

pected future consumption (FL):

$$\sum_{t\geq\tau} E\left[\left.\frac{w\left(s_{t}\right)-c\left(s_{t}\right)}{R^{t-\tau}}\right| s_{\tau}\right]\leq 0,\ \forall s_{\tau}. \tag{19}$$

Any contract that satisfies (17) and (19) is front loaded in the sense that, at each point in time, accumulated profits cannot be negative. In a front-loaded contract, the consumer initially overpays to the firm and is repaid later. This overpayment discourages the consumer from switching contracts.

In general, (19) is a relaxation of the non-lapsing constraints: if the continuation contract gave positive expected profits at some state, a consumer would be able to increase his utility by replacing it with another contract that gives zero profits. When consumers are dynamically consistent, however, maximizing (16) subject to either (17) and (18) or (17) and (19) gives the same solutions. Suppose a solution to this latter program did not satisfy the non-lapsing constraints. Then, there would exist a continuation contract that gives zero profits while increasing the consumer's continuation utility. Substituting the original continuation contract by this new one would then increase the consumer's utility while giving non-negative profits at $t = 1$, contradicting the optimality of the original contract.

Following the same approach as in Corollary 1, we find that any SPNE must have the same equilibrium consumption.

3.2 Time-Inconsistent Consumers

We now turn to the more interesting case of a time-inconsistent consumer. As with the time-consistent consumer, the equilibrium with one-sided commitment must satisfy non-lapsing constraints. Yet, because the consumer and the firms may disagree about the actions that will be chosen, we now need to distinguish between non-lapsing constraints according to the beliefs of the consumer and the beliefs of the firms. Equilibrium requires both of them to hold. To write down these constraints, we first define the outside options recursively.

The outside option at state s_τ according to the beliefs of firms is the highest utility that the consumer can actually obtain at that state. So, this "actual outside option" is the highest expected utility possible among contracts that are renegotiation proof and leave zero profits to the firm:

$$V(s_\tau) \equiv \max_{\{c(s_t,h_\tau^t)\}_{t\geq\tau}} u\left(c(s_\tau,h_\tau^\tau)\right) + \beta E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c(s_t,(h_\tau^\tau,B,...,B))\right)\middle| s_\tau\right],$$

subject to (PCC), (IC), the zero profits constraint

$$E\left[\sum_{t\geq\tau} \frac{w(s_t)-c(s_t,(h_\tau^\tau,A,A,...,A))}{R^{t-\tau}}\middle| s_\tau\right] = 0, \tag{20}$$

and the non-lapsing constraints

$$u\left(c_\tau(s_{\tilde\tau},(h_\tau^{\tilde\tau-1},A))\right) + \beta E\left[\sum_{t>\tilde\tau} \delta^{t-\tilde\tau} u\left(c(s_t,(h_\tau^{\tilde\tau-1},A,B,...,B))\right)\middle| s_{\tilde\tau}\right] \geq V(s_{\tilde\tau}), \tag{21}$$

$$u\left(c_\tau(s_{\tilde\tau},(h_\tau^{\tilde\tau-1},B))\right) + \hat\beta E\left[\sum_{t>\tilde\tau} \delta^{t-\tilde\tau} u\left(c(s_t,(h_\tau^{\tilde\tau-1},B,B,...,B))\right)\middle| s_{\tilde\tau}\right] \geq \hat V(s_{\tilde\tau}), \tag{22}$$

for all $s_{\tilde\tau}$ following s_τ and all $h_\tau^{\tilde\tau}$ that are continuation histories of h^τ, where $\hat V$ is the "perceived outside option," which we define next.

The outside option at state s_τ given an option history h^τ according to the consumer's beliefs is the highest utility that the consumer believes that he would be able to obtain at that state. This perceived outside option is the highest perceived utility possible among contracts that are renegotiation proof and leave zero profits to the firm:

$$\hat V(s_\tau) \equiv \max_{\{c(s_t,h_\tau^t)\}_{t\geq\tau}} u\left(c_\tau(s_\tau,h_\tau^\tau)\right) + \hat\beta E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c(s_t,(h_\tau^\tau,B,B,...,B))\right)\middle| s_\tau\right],$$

subject to (PCC), (IC), the zero profits constraint (20), and the non-lapsing constraints (21) and (22).

The *equilibrium program with one-sided commitment* (P1) adds the non-lapsing constraints to the program with two-sided commitment (P):

$$\max_{c(s_t,h^t)} u(c(s_1)) + \beta E\left[\sum_{t=1}^{T} \delta^{t-1} u(c(s_t,(B,B,...,B)))\right],$$

subject to (Zero Profits), (PCC), (IC), and the non-lapsing constraints:

$$u\left(c(s_\tau,(h^{\tau-1},A))\right) + \beta E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c(s_t,(h^{\tau-1},A,B,...,B))\right)\middle| s_\tau\right] \geq V(s_\tau),\ \ \forall s_\tau, \tag{NL}$$

and

$$u\left(c(s_\tau,(h^{\tau-1},B))\right) + \hat{\beta} E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c(s_t,(h^{\tau-1},B,B,...,B))\right)\middle| s_\tau\right] \geq \hat{V}(s_\tau),\ \ \forall s_\tau. \tag{PNL}$$

The need to impose constraint (PNL), which is associated with histories that are not reached in equilibrium, may seem counter-intuitive. But equilibrium requires the consumer to pick his optimal actions given how he thinks his future selves will behave. If (PNL) did not hold, the consumer would not expect his future selves to choose the baseline option, and so the consumption vector that solves the program would not correspond to an equilibrium consumption.

Lemma 3. ***c*** *is an equilibrium consumption vector of the model with one-sided commitment if and only if it solves program (P1).*

Auxiliary Program and Vanishing Inefficiency

As in the model with two-sided commitment, it is helpful to consider an auxiliary program corresponding to the equilibrium with a dynamically consistent agent who discounts the last period by an additional factor β. Since this agent is dynamically consistent, as shown in Subsection 2.1, we can replace the non-lapsing constraints by front-loading constraints. We refer to the maximization of (7) subject to the zero profits (3) and front-loading (19) constraints as the *auxiliary program with one-sided commitment*, which has a unique solution. The following lemma establishes that the solution of the auxiliary program coincides with the equilibrium with time-inconsistent agents:

Lemma 4. *Suppose the consumer is time inconsistent and has no commitment. The consumption on the equilibrium path coincides with the solution of the auxiliary problem with one-sided commitment.*

As in the model with commitment, underweighting consumption in the last period is the only distortion from time-inconsistency. To illustrate the lemma, consider again the case of three periods and a constant income w, so we can omit the single state of the world from the history. The equilibrium contract solves:

$$\max_{(c_1,c_2(A),c_2(B),c_3(A),c_3(B))} u(c_1) + \beta[\delta u(c_2(B)) + \delta^2 u(c_3(B))], \tag{23}$$

subject to

$$c_1 + \frac{c_2(A)}{R} + \frac{c_3(A)}{R^2} = w\left(1 + \frac{1}{R} + \frac{1}{R^2}\right), \tag{24}$$

$$u(c_2(B)) + \hat{\beta}\delta u(c_3(B)) \geq u(c_2(A)) + \hat{\beta}\delta u(c_3(A)), \tag{25}$$

$$u(c_2(A)) + \beta\delta u(c_3(A)) \geq u(c_2(B)) + \beta\delta u(c_3(B)), \tag{26}$$

$$u(c_2(B)) + \hat{\beta}\delta u(c_3(B)) \geq \hat{V_2^N}, \tag{27}$$

$$c_3(B) \geq w, \tag{28}$$

$$u(c_2(A)) + \beta\delta u(c_3(A)) \geq V_2^N, \tag{29}$$

$$c_3(A) \geq w, \tag{30}$$

where V_2^N is the actual outside option at time 2 and $\hat{V_2^N}$ is the perceived outside option at time 2. Equation (25) is the perceived choice constraint, (26) is the incentive compatibility constraint, and (27) - (30) are the non-lapsing constraints.

As in the model with two-sided commitment, the incentive compatibility constraint (26) must bind. Otherwise, we would be able to increase $c_3(B)$, achieving a higher utility without violating any constraint. Then, we can again rewrite the perceived choice constraint (25) as the monotonicity constraint (12), which will be ignored for now and verified later.

Next, we compare the marginal rate of substitution between the $c_2(B)$ and $c_3(B)$ in the objective function (23), incentive compatibility constraint (26), and non-lapsing constraint (27). The incentive constraint depends on the actual choice of the time-2 self, which discounts one period by $\beta\delta$. The non-lapsing constraint depends on the agent's prediction of his future self's choice, so it discounts one period by $\hat{\beta}\delta > \beta\delta$. And the objective function is evaluated in period 1, so the discount rate between periods 2 and 3 is $\delta > \hat{\beta}\delta$. Therefore, front-loading the baseline consumption by an amount that preserves the incentive constraint weakens the non-lapsing constraint and increases the agent's utility, so the solution of the program must offer a maximally front-loaded baseline contract: $c_2(B) = 0$.

As before, we can substitute $c_2(B) = 0$ in the (binding) incentive compatibility constraint (26), obtaining

$$u(c_2(A)) + \beta\delta u(c_3(A)) = u(0) + \beta\delta u(c_3(B)). \tag{31}$$

Notice that we can rewrite (31) as

$$\beta\delta\left[u(c_3(B)) - u(c_3(A))\right] = u(c_2(A)) - u(0) \geq 0,$$

which shows that the monotonicity constraint holds. Substituting $c_2(B) = 0$ and (31) in the objective function, we obtain the objective of the auxiliary problem (up to a constant):

$$u(c_1) + \beta[\delta u(c_2(B)) + \delta^2 u(c_3(B))] = u(c_1) + \delta u(c_2(A)) + \beta\delta^2 u(c_3(A)) - (1-\beta)\delta u(0). \tag{32}$$

Next, we verify that the non-lapsing constraints of the baseline consumption (27) and (28) can be ignored. Intuitively, because the baseline consumption is more front-loaded than the actual consumption, whenever the actual consumption is front-loaded enough to prevent agents from lapsing, the baseline consumption will also satisfy the non-lapsing constraints. For equation (28), we have

$$c_3(B) \geq c_3(A) \geq w,$$

where the first inequality follows from the monotonicity condition, and the second inequality follows from (30). For (27), the result follows from the fact that $c_2(B) = 0$.

Therefore, we can simplify the original program (23)-(30) as:

$$\max_{(c_1, c_2(A), c_3(A))} u(c_1) + \delta u(c_2(A)) + \beta\delta^2 u(c_3(A)), \tag{33}$$

subject to

$$c_1 + \frac{c_2(A)}{R} + \frac{c_3(A)}{R^2} = w\left(1 + \frac{1}{R} + \frac{1}{R^2}\right), \tag{34}$$

$$u(c_2(A)) + \beta\delta u(c_3(A)) \geq V_2^N, \tag{35}$$

$$c_3(A) \geq w. \tag{36}$$

But notice that this is the equilibrium program associated with a dynamically consistent agent who under-weights consumption in the last period by an additional term β. As argued in Section 3.1, dynamic consistency allows us to replace the non-lapsing constraint (35) by a front loading constraint (FL):

$$c_2(A) + \frac{c_3(A)}{R} \geq w\left(1 + \frac{1}{R}\right). \tag{37}$$

As in the model with two-sided commitment, we can recover the baseline consumption using $c_2(B) = 0$ and (31). Then, since neither the auxiliary program nor these equations depend on $\hat{\beta}$, it follows that, in equilibrium, both the baseline consumption and the alternative consumption are not functions of the consumer's naiveté. In particular, as we show in Appendix B, the equilibrium contracts obtained here coincide with the equilibrium contracts when firms do not know the consumer's naiveté $\hat{\beta}$.

Corollary 2. *Consider the model with one-sided commitment. There exists an SPNE. Moreover, any SPNE has the same equilibrium consumption vector, which is not a function of the consumer's naiveté* $\hat{\beta}$, *and is a continuous function of the agent's time-inconsistency parameter* $\beta \in (0, 1]$.

As in the case of two-sided commitment, we can now use Lemma 4 to show that the welfare loss from dynamic inconsistency vanishes as the contracting length grows. Let W_1^C and W_1^I denote the equilibrium welfare of the time-consistent consumer (Subsection 3.1) and time-inconsistent consumer (Subsection 3.2), respectively.

Theorem 2. *Suppose* u *is bounded and* $\delta < 1$. *Then,* $lim_{T \nearrow +\infty} \left(W_1^C - W_1^I \right) = 0$.

3.3 Removing Commitment Power

We now turn to the welfare effect of removing commitment power. We first show that, for a fixed contract length, removing commitment power can make the consumer better off. Recall that, on the equilibrium path, a time-inconsistent agent gets the same consumption as a dynamically consistent consumer who under-weights the last period by an additional factor β. Commitment power allows him to smooth consumption in the first $T-1$ periods (where the objective function coincides with the welfare function) while leaving too little consumption for the last period. This distortion in the last period is large when the consumer is sufficiently time inconsistent (β is low), in which case the last period consumption is close to zero. So, if consuming zero in the last period hurts the agent enough and β is low, the agent is better off without commitment.

To formalize this argument, let $\mathcal{V}_S$ denote the agent's welfare from smoothing consumption perfectly in the first $T-1$ periods and consuming zero in the last period:

$$\mathcal{V}_S \equiv \max_{\{c(s_t)\}} \sum_{t=1}^{T-1} E\left[\delta^{t-1} u\left(c(s_t)\right)\right] + \delta^{T-1} u(0),$$

subject to

$$\sum_{t=1}^{T-1} E\left[\frac{c(s_t)}{R^{t-1}}\right] \leq \sum_{t=1}^{T} E\left[\frac{w(s_t)}{R^{t-1}}\right]$$

Let $\mathcal{V}_{NS}$ denote the agent's welfare from consuming the endowment in each state:

$$\mathcal{V}_{NS} \equiv \sum_{t=1}^{T} E\left[\delta^{t-1} u\left(w(s_t)\right)\right].$$

Proposition 1. *Suppose agents are time inconsistent and $\mathcal{V}_{NS} > \mathcal{V}_S$. There exists $\bar{\beta} > 0$ such that if $\beta < \bar{\beta}$, the welfare with one-sided commitment is greater than the welfare with two-sided commitment.*

Notice that for generic endowment paths, the condition that $\mathcal{V}_{NS} > \mathcal{V}_S$ fails when T is large enough. So, as the contracting length grows, it becomes increasingly hard to satisfy the conditions for the time-inconsistent consumer to obtain higher welfare without commitment. In fact, by Theorems 1 and 2, if the contracting length is large enough, removing commitment power cannot increase welfare. To see this, recall that, with two-sided commitment, a dynamically consistent consumer maximizes welfare subject to zero profits. Removing commitment power is equivalent to introducing front-loading constraints, so the welfare with one-sided commitment cannot be higher. But, since the welfare of time-inconsistent consumers converges to the welfare of dynamically consistent consumers, the same must be true when the consumer is dynamically inconsistent.

4 Constant Income

In the general model considered so far, contracting played two distinct roles: shifting consumption over time and to insuring against risk. We now focus on the intertemporal aspect by shutting down the risk-sharing channel. This simplification allows us to derive clearer implications of one-sided commitment and contrast them with existing results since this model with $T = 3$ is analogous to the one from Heidhues and Kőszegi (2010).[16]

Formally, we assume that the agent gets a constant income of $w > 0$ in each period. We say that the *market breaks down* if the agent gets the same consumption as the endowment along the equilibrium path: $c_t^E = w$ for all t. If the market does not break down, we say that the equilibrium features a *long-term contract*.

4.1 Equilibrium Contracts

We start with the benchmark case of a time-consistent agent:

Lemma 5. *Suppose the agent is time consistent and has no commitment power. Then, the market breaks down if $R \leq \frac{1}{\delta}$, and the equilibrium features a long-term contract if $R > \frac{1}{\delta}$.*

The lemma shows that long-term contracts can be supported if and only if the interest rate is high enough that, in the absence of commitment issues, the consumer would save ($R > \frac{1}{\delta}$). In this case, the agent makes an irreversible up-front payment that prevents him from lapsing in future periods. If the consumer would prefer to borrow at the prevailing interest rate ($R \leq \frac{1}{\delta}$), the market breaks down since he cannot commit not to drop the contract.

We now turn to time-inconsistent agents. Our next result shows that the conditions for long-term contracting with time-consistent and time-inconsistent agents coincide:

Lemma 6. *Consider a time-inconsistent agent in the one-sided commitment environment. The market breaks down if and only if $R \leq \frac{1}{\delta}$.*

Recall that the time-inconsistent agent behaves like a time-consistent agent except for the last period. Therefore, if $R > \frac{1}{\delta}$, then a long-term contract is provided and, except for the last period the equilibrium consumption is increasing over time. By the non-lapsing constraint, consumption in the last period cannot be lower than the endowment w. In some

[16]Heidhues and Kőszegi (2010) assume that there is no consumption in the first period. Introducing an initial period with no consumption would not affect the equilibrium with one-sided commitment since any contract accepted in that period would be renegotiated in the first consumption period if it did not solve our equilibrium program.

cases, it is equal to w, leaving the last-period self indifferent between lapsing or remaining with the original contract. However, if $R > \frac{1}{\beta\delta}$ even the last-period self prefers not to lapse.

Because the condition for long-term contracts to be provided is the same for time-consistent and time-inconsistent consumers, one might be tempted to conclude that dynamic inconsistency neither helps nor hurts the ability to provide long-term contracts. But this interpretation is not warranted, as holding δ fixed also makes time-inconsistent agents more impatient than time-consistent agents (because of the additional discount factor β). To allow us to compare time-consistent and time-inconsistent agents while holding their "average impatience" fixed, we introduce the notion of "weighted impatience."

Formally, consider a vector of weights $\alpha = (\alpha_1, \cdots, \alpha_T)$, where $\alpha_i > 0$ and $\sum \alpha_i = 1$. An agent's *α-weighted measure of impatience* is:

$$\alpha_1 + \alpha_2 \beta\delta + \cdots + \alpha_T \beta\delta^{T-1}. \tag{38}$$

That is, if a time-consistent agent has discount parameter δ_C and a time-inconsistent agent has discount parameters (δ_I, β), they have the same α-weighted impatience if:

$$\alpha_1 + \alpha_2 \delta_C + \cdots + \alpha_T \delta_C^{T-1} = \alpha_1 + \alpha_2 \beta\delta_I + \cdots + \alpha_T \beta\delta_I^{T-1}. \tag{39}$$

Intuitively, both agents discount the stream of utils α in the same way. Of course, they still discount other streams differently.[17] Simple algebraic manipulations show that, for any fixed vector of weights, $\beta\delta_I < \delta_C$ and $\beta\delta_I^{T-1} > \delta_C^{T-1}$. That is, because both agents have the same average impatience, present-biased individuals discount the immediate future by more and later periods by less than time-consistent individuals. Since $\frac{1}{\delta_I} < \frac{1}{\delta_C}$, Lemmas 5 and 6 imply that it is easier to sustain long-term contracting with time-inconsistent than with time-consistent consumers.

Proposition 2. *Fix a vector of weights α and consider a time-consistent and a time-*

[17] One example of α-weighted impatience is the effective discount factor introduced by Chade et al. (2008), which corresponds to an α-weighted impatience with uniform weights ($\alpha_i = \frac{1}{T}$ for all i).

inconsistent consumer with the same α-weighted impatience. If the market breaks down for the time-inconsistent consumer, it also breaks down for time-consistent consumer.

Controlling for impatience is key for the predictions of the model. Time-consistent and time-inconsistent agents with the same "long-term discount factor" δ are equally likely to obtain long-term contracts (Lemmas 5 and 6). However, controlling for impatience, time-inconsistent agents are *more* likely to obtain long-term contracts (Proposition 2). This prediction is consistent with the empirical results from Atlas et al. (2017), who find that present-biased individuals are less likely to refinance their mortgages.[18]

4.2 Maximum Fees

We now consider the effect of imposing a maximum fee in each period. As before, it is convenient to write contracts in terms of the agent's consumption. The fee paid to the firm in each state corresponds to the difference between the endowment and the consumption: $w - c(h^t)$. Therefore, specifying a maximum fee is equivalent to mandating a consumption floor $\underline{c}$.[19]

If the agent is time consistent, imposing a consumption floor introduces additional constraints in the agent's welfare maximization program, which cannot increase welfare. Suppose, instead, that the agent is time inconsistent. Recall that the equilibrium contract solves the auxiliary program, which coincides with the equilibrium program of a dynamically consistent agent that discounts the last period more heavily than a time-consistent agent. By the non-lapsing constraint, the consumption in the last period cannot be lower than the agent's income, so the consumption floor never binds in the last period. If $R\delta \leq 1$, the agent would like to borrow, so the market breaks down and the consumption floor is not binding in any

[18] Note that our results are true even though there are no immediate transaction costs in refinancing. Introducing those costs would further accentuate our results since time-inconsistent agents are more averse to immediate costs.

[19] Any contract would give negative profits if the consumption floor exceeded the agent's endowment (or, equivalently, if the maximum fee was negative). Then, no firm would offer any contract that the agent would pick, and the agent would consume his endowment in each period. Therefore, there is no loss of generality in considering consumption floors that do not exceed the agent's endowment, $\underline{c} \leq w$, or, equivalently, non-negative maximum fees.

period. If $R\delta > 1$, the agent prefers to save in the initial periods, and consumption is increasing along the first $T-1$ periods. Therefore, whenever the consumption floor binds, it must bind in the initial periods, reducing saving. Since a time-inconsistent agent already under-saves relative to the welfare-maximizing amount, this policy hurts them whenever it is binding:[20]

Proposition 3. *Suppose the agent is time inconsistent and has no commitment power. Then, mandating a minimum consumption weakly decreases welfare.*

4.3 Sophisticated Consumers

The main focus of our paper is on consumers who underestimate their present bias. However, as a benchmark, we now consider a present-biased consumer who is perfectly aware of his bias ($\beta = \hat{\beta} < 1$).

As in the case of time-consistent consumers, in the equilibrium with two-sided commitment, any contract that is accepted by a firm must maximize the utility of the period-1 self subject to the zero profits constraint. Recall that the period-1 self discounts all future periods by the additional term β. So introducing any small amount of naiveté discontinuously shifts the equilibrium consumption from the one in which an additional discount β is applied to all future periods to the one that solves the auxiliary program, in which this additional discount only applies to the last period.[21] In particular, unlike with partially naive consumers, the consumption path of a sophisticated consumer does not converge to the one that maximizes welfare as the number of periods grows.

With one-sided commitment, the equilibrium consumption must also satisfy the non-lapsing constraints, defined recursively as in (16)-(18). But, with sophistication, the front-

[20] One-sided commitment and naiveté are important for this result. With two-sided commitment, imposing a maximum fee sometimes helps time-inconsistent consumers (Heidhues and Kőszegi, 2010). But, as described in footnote 4, settings with this type of policy often have one-sided commitment. Moreover, as we show in the online appendix, a maximum fee has an ambiguous effect on welfare when consumers are sophisticated.

[21] The discontinuity of the equilibrium consumption in the agent's naiveté was previously shown by Heidhues and Kőszegi (2010).

loading constraints are not sufficient to ensure that the consumer will not lapse. The more time inconsistent the consumer, the higher the front load required to prevent lapses. The next lemma characterizes the conditions for the market to break down with sophisticated consumers:

Lemma 7. *Consider a time-inconsistent sophisticated agent in the one-sided commitment environment. There exists* $r_T(\beta, \delta) > \frac{1}{\delta}$ *such that the market breaks down if and only if* $R \leq r_T(\beta, \delta)$. *Moreover:*

1. $r_T(\beta, \delta)$ *is decreasing in* β *and in* T*, and*

2. $\lim_{\beta \nearrow 1} r_T(\beta, \delta) = \lim_{T \nearrow \infty} r_T(\beta, \delta) = \frac{1}{\delta}$.

To understand the existence of the cutoff $r_T(\beta, \delta)$, note that when the interest rate is low, the consumer would like to borrow. However, because he cannot commit to repay his debt, no firm would lend him money. So, in equilibrium, the agent consumes his endowment in each period. Claim 1 states that market breaks down "less often" when agents have a higher time-consistency parameter β and when there are more periods T. As argued previously, the amount of front-loaded payments needed to support long-term contracting is decreasing in the agent's time-consistency parameter β. Moreover, with a longer horizon, firms have more instruments to avoid the market from breaking down. Claim 2 states that, as agents become close to time consistent or as the number of periods goes to infinity, the cutoff for the market to break down approaches the cutoff with time-consistent agents.[22]

Comparing the conditions from Lemmas 6 and 7, we find that naiveté helps the provision of long-term contracts. The intuition is as follows. Front-loaded payments are crucial to sustain long-term contracts. Sophisticates fully understand how front-loaded payments will hurt their future selves. Naive agents, however, believe that their future selves will

[22] It is straightforward to show that the equilibrium contract of a sophisticated agent is continuous in β, so that "almost time-consistent" sophisticated agents get approximately the same contract as time-consistent agents do. However, although the cutoff for a sophisticated agent converges to the cutoff of time-consistent agents as the horizon grows, it is not true that sophisticated agents with "sufficiently long" horizons get approximately the same consumption as time-consistent agents.

be less hurt by front-loaded contracts than they actually will, making it easier to convince them to accept a contract and to keep them by shifting payments into the future.

We conclude by extending Proposition 2 to incorporate sophisticated time-inconsistent agents:

Proposition 4. *Fix a vector of weights α and consider time-consistent, sophisticated, and naive time-inconsistent agents with the same α-weighted impatience.*

1. *If the market breaks down for naive agents, it breaks down for sophisticates.*

2. *If the market breaks down for sophisticates, it breaks down for time-consistent agents.*

Proposition 4 shows that it is easier to sustain long-term contracting with sophisticates than with time-consistent agents. The reason is that, while present-biased individuals discount the immediate future by more than time-consistent individuals, they discount later periods by less (holding their weighted impatience constant), which makes them more willing to save further into the future. Therefore, sophisticates have a higher demand for illiquid saving, which relaxes the non-lapsing constraints.

5 Conclusion

In this paper, we study contractual relationships between time-inconsistent consumers and risk neutral firms. We show that the welfare loss from time-inconsistency vanishes as the number of periods grows. We also study the effect of removing commitment power from consumers. For each fixed contracting horizon, removing commitment power increases welfare when consumers are sufficiently time-inconsistent. However, removing commitment power does not help if the contracting horizon is long.

Our results suggest that enforcing long-term contracts may be enough to ensure efficiency with naive consumers. With sophisticated agents, the equilibrium consumption does not converge to the one that maximizes their long-term preferences. Therefore, when the contracting horizon is long enough, making individuals aware of their dynamic inconsistency hurts them. This finding contrasts with a general intuition that educating behavioral individuals about their biases would increase their welfare.

The equilibrium consumption does not depend on the degree of naiveté as long as the consumer is still partially naive ($\hat{\beta} > \beta$). However, it jumps discontinuously at the point at which the consumer becomes sophisticated ($\hat{\beta} = \beta$). This discontinuity in the perceived time inconsistency $\hat{\beta}$ contrasts with the continuity in the actual time inconsistency β. The equilibrium consumption is continuous in β both for partially naive and for sophisticated consumers. In particular, "almost time-consistent" agents get approximately the same consumption as time-consistent agents. However, "almost sophisticated" agents get the same consumption as any other naive agent, which is bounded away from the consumption of a sophisticated consumer.

The paper focuses on one particular deviation from rationality – dynamic inconsistency – for two reasons. First, dropping a long-term contract is fundamentally an intertemporal decision, and dynamic inconsistency is the most well-studied bias in intertemporal decision-making. Second, there is evidence that this bias is important in credit markets where consumers are allowed to leave previous agreements, such as mortgages or credit cards. Still, policies that remove commitment power may also be relevant in settings with other biases. For example, many countries have regulations regarding cooling-off periods, during which firms must allow consumers to return products. Cooling-off periods may be an effective policy for consumers who suffer from projection bias – that is, they mispredict their future tastes, overestimating how much it will resemble their current tastes.

More generally, removing commitment power is a particularly weak type of paternalistic intervention. Instead of having a regulator decide which policy to ban, this decision is made by one's future selves. The study of the regulation of commitment power in settings with other biases is left for future work.

Appendix. Proofs

Proof of Lemma 1. The proof is similar to the proof of Lemma 3 (one-sided commitment) and is therefore omitted. □

Proof of Lemmas 2 and 4. Consider one-sided commitment case first (Lemma 2). We claim that all IC constraints are binding. To simplify the exposition, we focus on the case of $T = 4$ and no uncertainty here (the proof for arbitrary T is analogous, except for additional notation; the proof for stochastic income is presented in the Online Appendix). There are two ICs:

$$u(c_2(A)) + \beta[\delta u(c_3(AB)) + \delta^2 u(c_4(AB))] \geq u(c_2(B)) + \beta[\delta u(c_3(BB)) + \delta^2 u(c_4(BB))], \tag{A1}$$

$$u(c_3(AA)) + \beta\delta u(c_4(AA)) \geq u(c_3(AB)) + \beta\delta u(c_4(AB)). \tag{A2}$$

First, notice that these two ICs give upper bounds on $c_4(BB)$ and $c_4(AB)$. Since no other constraints restrict $c_4(BB)$ and $c_4(AB)$ from above, (A1) must be binding at an optimum (otherwise, we can raise $c_4(BB)$, giving the agent a higher utility). Substitute the binding (A1) in the objective to eliminate $c_4(BB)$:

$$\begin{aligned}
&u(c_1) + \beta[\delta u(c_2(B)) + \delta^2 u(c_3(BB)) + \delta^3 u(c_4(BB))] \\
&= u(c_1) + \delta u(c_2(A)) + \beta[\delta^2 u(c_3(AB)) + \delta^3 u(c_4(AB))] + (\beta - 1)\delta u(c_2(B)).
\end{aligned}$$

By the same argument, (A2) must bind (otherwise, we can raise $c_4(AB)$, increasing the agent's utility). Substituting the binding (A2) in the objective, gives:

$$u(c_1) + \delta u(c_2) + \delta^2 u(c_3) + \beta\delta^3 u(c_4) + (\beta - 1)[\delta u(c_2(B)) + \delta^2 u(c_3(AB))].$$

Since $\beta < 1$, we should pick $c_2(B)$ and $c_3(AB)$ as small as possible subject to the constraints. We now show that all the perceived non-lapsing constraints hold if we set these values at their lowest possible values: $c_2(B) = c_3(AB) = 0$.

Let the contract $\hat{c}$ denote the maximizer to the perceived outside option program, $\hat{V}_2$. Suppose $\hat{V}_2 = u(\hat{c}_2) + \hat{\beta}(\delta u(\hat{c}_3) + \delta^2 u(\hat{c}_4))$. We obtain

$$\begin{aligned}
&u(c_2(B)) + \hat{\beta}(\delta u(c_3(BB)) + \delta^2 u(c_4(BB)) \\
&= u(0) + \frac{\hat{\beta}}{\beta}\beta(\delta u(c_3(BB)) + \delta^2 u(c_4(BB)) \\
&= u(0) + \frac{\hat{\beta}}{\beta}(u(c_2(A)) + \beta(\delta u(c_3(AB)) + \delta^2 u(c_4(AB))) - u(0)) \\
&= (1 - \frac{\hat{\beta}}{\beta})u(0) + \frac{\hat{\beta}}{\beta}(u(c_2(A)) + \beta(\delta u(c_3(AB)) + \delta^2 u(c_4(AB)))),
\end{aligned}$$

where the first equality follows from $c_2(B) = 0$ and the second uses the binding IC constraint. From the non-lapsing constraint at time 2, we know that $u(c_2(A)) + \beta(\delta u(c_3(AB)) + \delta^2 u(c_4(AB) \geq V_2$. Since V_2 is the best possible outside option at time 2, in particular, it is greater than or equal to the utility provided by the contract $\hat{c}$, implying

$$\begin{aligned}
&u(c_2(B)) + \hat{\beta}(\delta u(c_3(BB)) + \delta^2 u(c_4(BB)) \\
&\geq (1 - \frac{\hat{\beta}}{\beta})u(0) + \frac{\hat{\beta}}{\beta}V_2 \\
&\geq (1 - \frac{\hat{\beta}}{\beta})u(0) + \frac{\hat{\beta}}{\beta}\left[u(\hat{c}_2) + \beta(\delta u(\hat{c}_3) + \delta^2 u(\hat{c}_4)\right] \\
&= (1 - \frac{\hat{\beta}}{\beta})u(0) + (\frac{\hat{\beta}}{\beta} - 1)u(\hat{c}_2) + \left[u(\hat{c}_2) + \hat{\beta}(\delta u(\hat{c}_3) + \delta^2 u(\hat{c}_4)\right] \\
&\geq u(\hat{c}_2) + \hat{\beta}(\delta u(\hat{c}_3) + \delta^2 u(\hat{c}_4)) = \hat{V}_2,
\end{aligned}$$

where the first line follows from the non-lapsing constraint at time 2, the second uses the revealed preference, and the last line uses $\hat{c}_2 \geq 0$ and $\hat{\beta} \geq \beta$. This shows that the perceived non-lapsing constraints hold.

We next verify that all the perceived choice constraints hold. Notice that

$$\begin{aligned}
u(c_3(AB)) + \hat{\beta}\delta u(c_4(AB)) &= u(0) + \hat{\beta}\delta u(c_4(AB)) \\
&= (1 - \frac{\hat{\beta}}{\beta})u(0) + \frac{\hat{\beta}}{\beta}\left(u(c_3(AA)) + \beta\delta u(c_4(AA))\right) \\
&\geq u(c_3(AA)) + \hat{\beta}\delta u(c_4(AA)), \qquad \text{(A3)}
\end{aligned}$$

and

$$\begin{aligned}
& u(c_2(B)) + \hat{\beta}[\delta u(c_3(BB)) + \delta^2 u(c_4(BB))] \\
&= u(0) + \hat{\beta}[\delta u(c_3(BB)) + \delta^2 u(c_4(BB))] \\
&= (1 - \frac{\hat{\beta}}{\beta})u(0) + \frac{\hat{\beta}}{\beta}\left[u(c_2(A)) + \beta[\delta u(c_3(AB)) + \delta^2 u(c_4(AB))]\right] \\
&\geq u(c_2(A)) + \hat{\beta}[\delta u(c_3(AB)) + \delta^2 u(c_4(AB))]. \qquad \text{(A4)}
\end{aligned}$$

So the perceived choice constraints hold.

So far, we have shown that $c_2(B) = c_3(AB) = 0$ under the equilibrium contract. We also showed that we can disregard the perceived choice constraints and perceived non-lapsing constraints. Recall that c_t^E denotes the consumption on the equilibrium path at time t. Substituting the binding ICs, the non-lapsing constraints on the equilibrium path can be simplified to $u(c_t^E) + \delta u(c_{t+1}^E) + \cdots + \beta\delta^{t-4}u(c_4^E) \geq V_t$.

Therefore, the original program reduces to the auxiliary program:

$$\max_{(c_1,\cdots,c_4)} u(c_1) + \delta u(c_2) + \delta^2 u(c_3) + \beta\delta^3 u(c_4), \qquad \text{(A5)}$$

subject to

$$\sum_{t=1}^{4} \frac{c_t}{R^{t-1}} = \sum_{t=1}^{4} \frac{w}{R^{t-1}}, \qquad \text{(A6)}$$

$$u(c_t) + \delta u(c_{t+1}) + \cdots + \beta\delta^{T-t}u(c_4) \geq V_t^A, \forall 2 \leq t \leq 4. \qquad \text{(A7)}$$

With two-sided commitment, the same arguments given above go through except that we can omit the non-lapsing constraints (A7). In all, the consumption on the equilibrium path coincides with the solution of the auxiliary problems. □

Proof of Theorems 1 and 2. We consider the case with one-sided commitment (Theorem 2). We omit the proof for the two-sided commitment case (Theorem 1), which is similar. We need to show that $\lim_{T\nearrow+\infty}(W_1^C - W_1^I) = 0$.

For each parameter β, let $V^A(\beta)$ denote the maximum value attained by the solution of the auxiliary program with one-sided commitment (Lemma 2). Notice that the feasible set is independent of β. When $\beta = 1$, the auxiliary program becomes a time-consistent agent's program, so that $V^A(1) = W_1^C$. We have also $\lim_{T\nearrow\infty}(W_1^I - V^A(\beta)) = \lim_{T\nearrow\infty}(1 - \beta)E\delta^{T-1}u(c(s_T)) = 0$. Since the objective function is linear in β, it follows from the Envelope Theorem that $\frac{\partial V^A(\beta)}{\partial\beta} = E\delta^{T-1}u\left(c(s_T)\right) \geq \delta^{T-1}u\left(0\right)$. Applying Lagrange's Mean Value Theorem gives

$$V^A(1) - V^A(\beta) = \frac{\partial V^A(\beta)}{\partial\beta}|_{\beta=\beta'}(1-\beta) \geq \delta^{T-1}u(0)(1-\beta),$$
$$V^A(\beta) - V^A(0) = \frac{\partial V^A(\beta)}{\partial\beta}|_{\beta=\beta''}\beta \geq \delta^{T-1}u(0)\beta,$$

where $\beta' \in (\beta, 1), \beta'' \in (0, \beta)$. Taking T to infinity leads to

$$\lim_{T\nearrow\infty} V^A(1) \geq \lim_{T\nearrow\infty} V^A(\beta) \geq \lim_{T\nearrow\infty} V^A(0).$$

To obtain the theorem, it suffices to show that:

$$\lim_{T\nearrow+\infty}[V^A(1) - V^A(0)] \leq 0.$$

Consider the auxiliary program with one-sided commitment when $\beta = 0$, which attains maximum value $V^A(0)$. Let $\mathbf{c^0} \equiv \{c^0(s_t) : s_t \in S_t(s_1), 1 \leq t \leq T\}$ denote a solution to this program. Since the objective function does not depend on $c(s_T)$ when $\beta = 0$, the solution has the lowest possible value for $c(s_T)$ that still satisfies the constraints: $\mathbf{c}^0(s_T) = w(s_T)$. Substituting this equality back, we obtain the same program that determines the consumption of a time-consistent agent with a contracting horizon consisting of the first $(T-1)$ periods.

Let c_C^1 denote the equilibrium consumption of a time-consistent agent. Since c_C^1 is in the feasible set, income cannot exceed consumption for any last-period state: $c_C^1(s_T) \geq w(s_T)$. Therefore, by revealed preference ($V^A(0)$ maximizes expected utility in the first $T-1$ periods and uses weakly higher resources), we must have

$$\begin{aligned} V^A(0) &= \textstyle\sum_{t=1}^{T-1} \sum_{s_t \in \mathbb{S}_t(s_1)} \delta^{t-1} p(s_t|s_1) u\left(c^0(s_t)\right) \\ &\geq \textstyle\sum_{t=1}^{T-1} \sum_{s_t \in \mathbb{S}_t(s_1)} \delta^{t-1} p(s_t|s_1) u\left(c_C^1(s_t)\right) \\ &= V^A(1) - \delta^{T-1} \textstyle\sum_{s_T \in \mathbb{S}_T(s_1)} p(s_T|s_1) u\left(c_C^1(s_T)\right), \end{aligned}$$

where the first line uses the definition of $V^A(0)$, the second line uses revealed preference, and the third line uses the definition of $V^A(1)$. Since $\delta < 1$ and u is bounded, we have

$$\lim_{T \nearrow +\infty} \delta^{T-1} \sum_{s_T \in \mathbb{S}_T(s_1)} p(s_T|s_1) u\left(c_C^1(s_T)\right) = 0,$$

which establishes that $\lim_{T \nearrow +\infty} [V^A(1) - V^A(0)] \leq 0$. □

Proof of Lemma 3. We show that the Subgame Perfect Nash Equilibrium (SPNE) of the game is outcome-equivalent to the solution of the maximization programs (P) and (P1).

Suppose there is one-sided commitment. We are interested in the SPNE of the game played by the firms and by the different selves of the consumer. Suppose the t-period self of the consumer offers a contract $\mathcal{C}_t'$. Specifically, a contract at time t, $\mathcal{C}_t'$, specifies consumption on each possible state in each future time $\tau \geq t$. Denote the set of possible states by $K_{t,\tau}$, in which the first subscript corresponds to the time in which the contract is offered and the second subscript corresponds to the decision-making time τ. The contract specifies consumption for each different income states, so the contracting space must be greater than the space of income states. In addition, SPNE imposes no restrictions on $K_{t,\tau}$, i.e., $K_{t,\tau}$

can be arbitrary. To keep analysis tractable, we assume that $K_{t,\tau}$ has a product structure and only depends on decision making time τ. Otherwise, we can always add more states that are never reached so that it has a product structure and the resulting equilibrium is outcome-equivalent to the original equilibrium. Specifically, we write $K_{t,\tau} = \mathbb{S}_\tau \times H_\tau$, in which H_τ consists of all the possible income-independent messages/actions that the agent can send at time τ. The income-independent messages can be arbitrary. One of the reasons that an income-independent message can arise is from the consumer's different beliefs. Since we allow any contracts, we can not impose what types of income-independent messages the consumer can send. For simplicity, we call H_τ the income-independent history. Without loss of generality, $H_1 = \emptyset$. Denote h_t a generic element in H_t. We call h_t an income-independent message. Denote $H_\tau(h_t)$ the states that can be reached at time τ from an earlier history $h_t \in H_t$ for $\tau > t$.

Fix a contract, we next write down the agent's strategy profile. Consider an agent who makes a decision at time τ. Suppose the income-independent messages that has been reached is $h_{\tau-1}$, which is an element in $H_{\tau-1}$. At time τ, the agent learns the income state, i.e., s_τ is realized. The agent needs to decide which message $a_\tau \in \Delta(H_\tau(h_{\tau-1}))$ to send, where $\Delta(\cdot)$ represents the set of lotteries. If there is one-sided commitment, the agent also needs to decide whether he will lapse or not, in which case, the strategy can be summarized by a pair (d_τ, a_τ), where $d_\tau \in \Delta(\{0,1\})$. If $d_\tau = 1$ with probability 1, then the agent stays, otherwise the contract is lapsed with a positive probability.

As described in the body of the paper, the SPNE is solved by treating the agent's decisions in each period as if it were taken by a different player (i.e., a different "self"). The main claim is that for any SPNE, the equilibrium contract must solve the program (P1) subjects to (Zero Profits), (PCC), (IC), (NL), and (PNL).

For the ease of exposition, we say that two SPNE are *equivalent* if all selves of the consumers have same actual and perceived consumption. We will establish the result through two separate claims:

Claim 1. *Fix an SPNE. There exists an equivalent SPNE in which the agent never lapses*

($d_\tau = 1, \forall \tau$).

Proof. Consider an SPNE in which the agent lapses in some period $d_\tau = 0$ with a positive probability, replacing it with a contract $\mathcal{C}''_\tau$ from another firm. Since the other firm cannot lose money by offering this new contract, the old firm could have accepted a contract that substituted the terms of the old contract from this period on with the terms of the new contract, and the agent would have accepted to remain with the old firm. The constructed new contracts together with the agent's optimal decision forms an SPNE that is equivalent to the original SPNE. □

Claim 2. *Fix an SPNE. There is an equivalent SPNE that offers two options following any history: $\#|H_t(h_{t-1})| \leq 2$, for all $h_{t-1} \in H_{t-1},\ t \geq 2$.*

Proof. From the previous claim, we can restrict attention for SPNE in which the agent never lapses. Suppose $t_1 < t_2 < t_3$. Note that self t_1's prediction about self t_3's decision coincides with self t_2's prediction about self t_3's decision. Restricting $H_t(h_{t-1})$ to two messages – one that the agent will choose and another one that the agent thinks that he will choose – does not affect the actual consumption or the perceived consumption. Put differently, if $H_t(h_{t-1})$ has at least three messages, then there is at least one of them that the agent never sends and the agent never believes other selves would send. Therefore, we can restrict the income-independent message space to be at most two: one that the agent actually choose, and one that the agent thought he would choose. □

Given these two claims, a contract offered by self t, $\mathcal{C}'_t$, must maximize the agent's utility subject to the zero profits, incentive compatibility, perceived choice, and non-lapsing constraints. In other words, the contracts $\{\mathcal{C}'_t\}_{t=1,\cdots,T}$ must form a competitive equilibrium with one-sided commitment. In addition, this competitive equilibrium gives the agent exactly same actual consumption and perceived consumption, concluding the proof of Lemma 3. □

Proof of Corollaries 1 and 2. Consider the case of one-sided commitment (Corollary 2). The case with two-sided commitment (Corollary 1) is analogous. We can focus on the auxiliary program. Let $x(s_t) \equiv u(c(s_t))$ denote the agent's utility from the consumption he gets in state s_t. We study the dual program:

$$\max_{\{x(s_t)\}} \sum_{t=1}^{T} \sum_{s_t \in \mathbb{S}_t} p(s_t|s_1) \frac{w(s_t) - u^{-1}(x(s_t))}{R^{t-1}}, \tag{A8}$$

subject to

$$\sum_{t=1}^{T-1} \sum_{s_t \in \mathbb{S}_t} \delta^{t-1} p(s_t|s_1) x(s_t) + \beta \sum_{s_T \in \mathbb{S}_T} \delta^{T-1} p(s_T|s_1) x(s_T) \geq \underline{u}, \tag{A9}$$

and

$$\sum_{t \geq \tilde{\tau}} \sum_{s_t \in \mathbb{S}_t} \delta^{t-\tilde{\tau}} p(s_t|s_{\tilde{\tau}}) x(s_t) + \beta \sum_{s_T \in \mathbb{S}_T} \delta^{T-\tilde{\tau}} p(s_T|s_{\tilde{\tau}}) x(s_T) \geq V^A(s_{\tilde{\tau}}) \;\; \forall s_{\tilde{\tau}} \in \mathbb{S}_{\tilde{\tau}}(s_\tau), \forall \tau, \tag{A10}$$

This program corresponds to the maximization of a strictly concave function over a convex set, so that, by the Theorem of the Maximum, the solution is unique. Moreover, the consumption on the equilibrium path is continuous in $\beta \in (0, 1]$. Finally, the program does not involve $\hat{\beta}$, so the equilibrium consumption vector is not a function of the consumer's naiveté. □

Proof of Proposition 1. First, the welfare with two-sided commitment approaches to $\mathcal{V}_S$ as β approaches to zero. It suffices to show that the welfare with one-sided commitment is bounded below by $\mathcal{V}_{NS}$. In the remainder of the proof, we will therefore focus on the equilibrium with one-sided commitment.

We claim that for β close to zero, the equilibrium consumption equals the endowment in all last-period states: $c(s_T) = w(s_T), \forall s_T \in \mathbb{S}_T(s_1)$. To see this, consider a perturbation that shifts consumption from a state in the last period to the preceding state, that is, it increases $c(s_{T-1})$ by $\epsilon > 0$ and reduces $c(s_T)$ by $\frac{\epsilon R}{p(s_T|s_{T-1})}$ for some $s_T \in \mathbb{S}_T$ with $p(s_T|s_{T-1}) > 0$. Let W_{s_T} denote the future value of all income up to state s_T. The amount W_{s_T} is how much the agent would be able to consume at state s_T if he saves all his income from all periods for the last one. It therefore gives an upper bound on how much the agent can consume in the last period. Since there are finitely many states and $W_{s_T} < \infty$ for all s_T, we can take the uniform bound $W \equiv \max_{s_T} W_{s_T}$. This perturbation affects the LHS of the non-lapsing constraint at state s_t by

$$\begin{aligned} & p(s_{T-1}|s_t)\left[u'(c(s_{T-1})) - \beta R\delta u'(c(s_T))\right]\delta^{T-1-t}\epsilon \\ & > p(s_{T-1}|s_t)\left[u'(0) - \beta R\delta u'(W_{s_T})\right]\delta^{T-1-t}\epsilon, \end{aligned}$$

which is positive whenever

$$\frac{u'(0)}{R\delta u'(W)} > \beta. \tag{A11}$$

The perturbation has exactly the same effect on the objective function (scaled down by δ^t and multiplied by the probability of reaching state s_{T-1}). Thus, as long as β satisfies (A11), the equilibrium will have the smallest consumption possible in the last period, which is determined by the non-lapsing constraint.

Substituting $c(s_T) = w(s_T)$ in the auxiliary program, it becomes analogous to the program of a time-consistent agent except that the contracting problem ends at period $T-1$ instead of period T:

$$\max_{\{c(s_t)\}} \sum_{t=1}^{T-1} \sum_{s_t \in \mathbb{S}_t(s_1)} \delta^{t-1} p(s_t|s_1) u\left(c(s_t)\right),$$

subject to

$$\sum_{t=1}^{T-1} \sum_{s_t \in \mathbb{S}_t(s_1)} p(s_t|s_1)\frac{w(s_t) - c(s_t)}{R^{t-1}} = 0,$$

and

$$\sum_{t=\tilde{\tau}}^{T-1} \sum_{s_t \in \mathbb{S}_t(s_{\tilde{\tau}})} \delta^{t-\tilde{\tau}} p(s_t|s_{\tilde{\tau}}) u\left(c(s_t)\right) \geq (V')^C(s_{\tilde{\tau}}) \text{ for all } s_{\tilde{\tau}},$$

for all $\tilde{\tau} = 2, ..., T$, where $(V')^C(s_{\tilde{\tau}})$ denotes the outside option for the time-consistent agent in this $(T-1)$-period economy.

It is straightforward to verify that $(V')^C(s_1)$ is bounded below by the utility from consuming the endowment in all states. If the endowment already satisfies the non-lapsing constraints, then the result follows from revealed preference because the endowment also satisfies zero profits. If the endowment does not satisfy the non-lapsing constraints, any renegotiation of the endowment satisfies the zero-profits condition and gives the time-consistent agent a strictly higher utility conditional on that state. So, replacing the endowment by the solution of the continuation program in all states where the non-lapsing constraints are violated leads to a profile of consumption that satisfies the constraints and gives a utility greater than the utility of consuming the endowment in each period. It thus follows by revealed preference that the solution of the program also gives a higher utility than consuming the endowment in all states.

Since the solution of a naive agent coincides with the solution of this auxiliary program, their welfare is also bounded below by the welfare from consuming their endowment in all periods $\mathcal{V}_{NS}$ when (A11) holds. Therefore, by continuity, if $\mathcal{V}_{NS} > \mathcal{V}_S$, there exists $\bar{\beta}_N$ such that if $\beta < \bar{\beta}_N$, the welfare with one-sided commitment dominates the welfare with two-sided commitment. □

Proof of Lemma 5. The equilibrium contract solves:

$$\max_{\{c_t\}} \sum_{t=1}^{T} \delta^{t-1} u(c_t), \tag{A12}$$

subject to

$$\sum_{t=1}^{T} \frac{c_t}{R^{t-1}} = \sum_{t=1}^{T} \frac{w}{R^{t-1}}, \tag{A13}$$

$$\sum_{t=\tau}^{T} \frac{c_t}{R^{t-\tau}} \geq \sum_{t=\tau}^{T} \frac{w}{R^{t-\tau}}, \forall 2 \leq \tau \leq T. \tag{A14}$$

Since the objective function is strictly concave and the set of feasible contracts is convex, the solution is unique. The Lagrangian is

$$\mathcal{L} = \sum_{t=1}^{T} \delta^{t-1} u(c_t) - \sum_{\tau=1}^{T} \lambda_\tau \left(\sum_{t=\tau}^{T} \frac{c_t}{R^{t-1}} - \sum_{t=\tau}^{T} \frac{w}{R^{t-1}} \right), \tag{A15}$$

where $\lambda_\tau \geq 0$. Then $\delta^{t-1} u'(c_t) = \frac{\sum_{\tau=1}^{t} \lambda_\tau}{R^{t-1}}$, or equivalently, $u'(c_t) = \frac{\sum_{\tau=1}^{t} \lambda_\tau}{(\delta R)^{t-1}}$.

First, consider $\delta R \leq 1$. Then $u'(c_1) \leq u'(c_2) \leq \cdots \leq u'(c_T)$, therefore, $c_1 \geq c_2 \geq \cdots \geq c_T$. From the zero profit condition, we then would have $c_T \leq w$. We also must have $c_T \geq w$ to prevent the agent to leave the contract in the last period. So it must be the case that $c_T = w$. Now we have $c_1 \geq \cdots \geq c_{T-1} \geq w$. From the zero profit condition $c_{T-1} \leq w$. By the non-lapsing condition, we need to have $c_{T-1} \geq w$. Similarly we conclude $c_{T-1} = w$. Using the same argument, we find $c_1 = \cdots = c_T = w$. In other words, the market breaks down.

Second, consider $\delta R > 1$. We can solve the problem with the zero-profit condition and then verify that the constraint (A14) holds automatically. Solving the problem with only the zero-profit condition gives $c_1 < c_2 < \cdots < c_T$. Notice that $c_1 < w$ because otherwise, $w \leq c_1 < c_2 < \cdots < c_T$, contradicted to the zero profit condition. Similarly we have $c_T > w$. Let ξ be the smallest index such that $c_1 < \cdots < c_\xi < w \leq c_{\xi+1} < \cdots < c_T$. It is clear that (A14) holds strictly for $\tau \geq \xi$+1. Now consider $\tau \leq \xi$, we have $\sum_{t=\tau}^{T} \frac{c_t}{R^{t-1}} = \sum_{t=1}^{T} \frac{c_t}{R^{t-1}} - \sum_{t=1}^{\tau-1} \frac{c_t}{R^{t-1}} = \sum_{t=1}^{T} \frac{w}{R^{t-1}} - \sum_{t=1}^{\tau-1} \frac{c_t}{R^{t-1}} > \sum_{t=1}^{T} \frac{w}{R^{t-1}} - \sum_{t=1}^{\tau-1} \frac{w}{R^{t-1}} = \sum_{t=\tau}^{T} \frac{w}{R^{t-1}}$. So the constraint (A14) holds strictly for $\tau \leq \xi$. In all, the equilibrium contract features growing consumption and the long-term contract is supported in this market. □

Proof of Lemma 6. We can rewrite the program as

$$\max_{c_t} u(c_1) + \delta u(c_2) + \cdots + \delta^{T-2} u(c_{T-1}) + \beta \delta^{T-1} u(c_T), \tag{A16}$$

subject to $\sum_{t=1}^{T} \frac{c_t}{R^{t-1}} = \sum_{t=1}^{T} \frac{w}{R^{t-1}}$ and $\sum_{t=\tau}^{T} \frac{c_t}{R^{t-1}} \geq \sum_{t=\tau}^{T} \frac{w}{R^{t-1}}, \forall 2 \leq \tau \leq T$.

This program has a concave objective function and the feasible set is a non-empty linear set, so there exists a solution. Since the program does not depend on β, it is clear that all equilibria have the same consumption on the equilibrium path. By Theorem of Maximum, the consumption on the equilibrium path is a continuous function of $\beta \in (0,1]$. We note that it may not be right-continuous at $\beta = 0$ because the step showing binding incentive constraints in the proof of Lemma 2 requires $\beta > 0$.

The Lagrangian is

$$\mathcal{L} = \sum_{t=1}^{T-1} \delta^{t-1} u(c_t) + \beta \delta^{T-1} u(c_T) - \sum_{\tau=1}^{T} \lambda_\tau \left(\sum_{t=\tau}^{T} \frac{c_t}{R^{t-1}} - \sum_{t=\tau}^{T} \frac{w}{R^{t-1}} \right),$$

where $\lambda_\tau \geq 0$. If $1 \leq t \leq T-1$, we have $\delta^{t-1} u'(c_t) = \frac{\sum_{\tau=1}^{t} \lambda_\tau}{R^{t-1}}$, or equivalently,$u'(c_t) = \frac{\sum_{\tau=1}^{t} \lambda_\tau}{(\delta R)^{t-1}}$. If $t = T$, $u'(c_t) = \frac{\sum_{\tau=1}^{t} \lambda_\tau}{\beta(\delta R)^{t-1}} > \frac{\sum_{\tau=1}^{t} \lambda_\tau}{(\delta R)^{t-1}}$.

If $\delta R \leq 1$, using the same argument in the proof of Lemma 5, we have $c_1 = \cdots = c_T = w$. Now if $\delta R > 1$, consider the problem with the same objective function and the zero profit condition and the last period non-lapsing constraint:

$$\tilde{c} = (\tilde{c}_1, \cdots, \tilde{c}_T) = arg \max_{c_t} u(c_1) + \delta u(c_2) + \cdots + \delta^{T-2} u(c_{T-1}) + \beta \delta^{T-1} u(c_T),$$

subject to $\sum_{t=1}^{T} \frac{c_t}{R^{t-1}} = \sum_{t=1}^{T} \frac{w}{R^{t-1}}$ and $c_T \geq w$. Applying Lagrangian condition gives $u'(\tilde{c}_1) = \cdots = (\delta R)^{T-2} u'(\tilde{c}_{T-1}) \leq \beta (\delta R)^{T-1} u'(\tilde{c}_T)$. Since $\delta R > 1$, we have $\tilde{c}_1 < \tilde{c}_2 < \cdots < \tilde{c}_{T-1}$. We next verify that $\tilde{c}$ satisfies all the non-lapsing constraints: $\sum_{t=\tau}^{T} \frac{c_t}{R^{t-1}} \geq \sum_{t=\tau}^{T} \frac{w}{R^{t-1}}, \forall 2 \leq \tau \leq T$, in which case, $\tilde{c}$ would be the optimal solution for the original problem and the long-term contract is supported in this market. To see that, let ξ be the largest index such that $\tilde{c}_\xi < w$. If $\tau \geq \xi + 1$, $\sum_{t=\tau}^{T} \frac{\tilde{c}_t}{R^{t-1}} \geq \sum_{t=\tau}^{T} \frac{w}{R^{t-1}}$. If $\tau \leq \xi$, we have $\sum_{t=\tau}^{T} \frac{\tilde{c}_t}{R^{t-1}} = \sum_{t=1}^{T} \frac{\tilde{c}_t}{R^{t-1}} - \sum_{t=1}^{\tau-1} \frac{\tilde{c}_t}{R^{t-1}} = \sum_{t=1}^{T} \frac{w}{R^{t-1}} - \sum_{t=1}^{\tau-1} \frac{\tilde{c}_t}{R^{t-1}} > \sum_{t=1}^{T} \frac{w}{R^{t-1}} - \sum_{t=1}^{\tau-1} \frac{w}{R^{t-1}} = \sum_{t=\tau}^{T} \frac{w}{R^{t-1}}$.

In summary, the market breaks down if and only if $R \leq \frac{1}{\delta}$. □

Proof of Proposition 2. Presented in the text. □

Proof of Proposition 3. If $\delta R \leq 1$, the market breaks down for naifs, so welfare is unchanged with mandating a minimum consumption. Now suppose $\delta R > 1$. Denote

$(c_1^1, c_2^1, \cdots, c_T^1)$ the maximizer to the program V_1^A, and $(c_1^2, c_2^2, \cdots, c_T^2)$ the maximizer to the program with the mandate, denoted as V_1^{PA}.

We first claim that $c_T^1 \geq c_T^2$. We know that $c_1^1 < c_2^1 < \cdots < c_{T-1}^1$ and $c_T^1 \geq w$. If $c_1^1 \geq \underline{c}$, then the welfare is unchanged with the mandate since none of the consumption is affected. If $c_1^1 < \underline{c}$, then c_1^2 would hit the lowest possible consumption level, $\underline{c}$. So $c_1^1 < c_1^2$. Assume that k is the largest index such that $c_k^1 < \underline{c}$. Since $\sum_{t=1}^T \frac{c_t^1}{R^{t-1}} = \sum_{t=1}^T \frac{c_t^2}{R^{t-1}}$, it is clear that $\sum_{t=k+1}^T \frac{c_t^1}{R^{t-1}} > \sum_{t=k+1}^T \frac{c_t^2}{R^{t-1}}$. Since the auxiliary program is dynamically consistent, both $(c_{k+1}^1, \cdots, c_T^1)$ and $(c_{k+1}^2, \cdots, c_T^2)$ maximize the time $(k+1)$ auxiliary program but subject to different resource constraints. More resources must lead to a weakly higher last period consumption, implying $c_T^1 \geq c_T^2$. Note that

$$
\begin{aligned}
& u(c_1^1) + \delta u(c_2^1) + \cdots + \delta^{T-1} u(c_T^1) \\
& = u(c_1^1) + \delta u(c_2^1) + \cdots + \beta \delta^{T-1} u(c_T^1) + (1-\beta)\delta^{T-1} u(c_T^1) \\
& \geq u(c_1^2) + \delta u(c_2^2) + \cdots + \beta \delta^{T-1} u(c_T^2) + (1-\beta)\delta^{T-1} u(c_T^1) \\
& \geq u(c_1^2) + \delta u(c_2^2) + \cdots + \beta \delta^{T-1} u(c_T^2) + (1-\beta)\delta^{T-1} u(c_T^2) \\
& = u(c_1^2) + \delta u(c_2^2) + \cdots + \delta^{T-1} u(c_T^2), \qquad \text{(A17)}
\end{aligned}
$$

where the first inequality is from the fact that $(c_1^1, c_2^1, \cdots, c_T^1)$ maximizes the program V_1^A, and the second inequality follows from that $c_T^1 \geq c_T^2$. So the mandate weakly decreases welfare. □

Proof of Lemma 7. The existence and uniqueness of the equilibrium consumption follows from the same argument as in the time-consistent case (rewriting in terms of the utility of consumption, we obtain a program that corresponds to the maximization of a strictly concave function subject to linear constraints). Let λ_1 denote the Lagrangian multiplier associated with the zero-profit constraint, and let λ_τ denote the Lagrangian multiplier associated with the non-lapsing constraints. Let $r \equiv \delta R$ and $x_i \equiv \lambda_{i+1} R^{i+1}$ for $i = 1, ..., T-1$.

The Lagrangian optimality conditions imply

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \beta r & 1 & 0 & \cdots & 0 \\ \beta r^2 & \beta & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \beta r^{T-2} & \beta r^{T-3} & & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{T-1} \end{pmatrix} = \begin{pmatrix} 1-\beta r \\ 1-\beta r^2 \\ 1-\beta r^3 \\ \vdots \\ 1-\beta r^{T-1} \end{pmatrix}.$$

We first examine the conditions for the market to break down: $c_1 = \cdots = c_T = w$. We need to have $\lambda_\tau \geq 0, \forall \tau \geq 2$, or equivalently, $x_i \geq 0$ for all $1 \leq i \leq T-1$. Inverting the lower-triangular matrix, we can find that the necessary and sufficient condition for $x_i \geq 0\ \forall 1 \leq i \leq T-1$ is:

$$1 \geq \beta r + \beta(1-\beta)r^2 + \cdots + \beta(1-\beta)^{T-2} r^{T-1}. \tag{A18}$$

The right-hand-side is an increasing function of r. If $r = 0$, LHS>RHS. If $r \to \infty$, LHS<RHS. So there exists a unique $r_T(\beta)$ such that the (A18) becomes an equality. Since the RHS evaluated at $r = 1$ is strictly less than 1: $\sum_{t=0}^{T-2} \beta(1-\beta)^t < \sum_{t=0}^{\infty} \beta(1-\beta)^t = \beta \frac{1}{1-(1-\beta)} = 1$, we must have $r_T(\beta) > 1$. Let $r_T(\beta,\delta) := \frac{r_T(\beta)}{\delta} > \frac{1}{\delta}$. The market breaks down when $R \leq r_T(\beta,\delta)$.

We now turn to the properties of $r_T(\beta,\delta)$. We first show that $r_T(\beta,\delta)$ is decreasing in β. Recall that $r_T(\beta,\delta) = \frac{r_T(\beta)}{\delta}$, where $r_T(\beta)$ solves the equation (A18). It is sufficient to show that $r_T(\beta)$ is decreasing in β. The right hand side of (A18) is a geometric series, implying

$$1 = \beta r \frac{1-(1-\beta)^{T-1} r^{T-1}}{1-(1-\beta)r}. \tag{A19}$$

Rearranging terms leads to

$$1 - r + \beta(1-\beta)^{T-1} r^T = 0. \tag{A20}$$

Taking derivative with respect to β, we obtain

$$r_T'(\beta) = \frac{(1-T\beta)(1-\beta)^{T-2}r_T^T(\beta)}{1-\beta(1-\beta)^{T-1}r_T^{T-1}T} = \frac{(1-T\beta)(1-\beta)^{T-2}r_T^{T+1}(\beta)}{T-(T-1)r_T(\beta)}. \tag{A21}$$

We can rewrite equation (A20) as

$$r^T = \frac{1}{\beta(1-\beta)^{T-1}}(r-1). \tag{A22}$$

This equation has two real positive roots, one of which is $\frac{1}{1-\beta}$ and the other one is $r_T(\beta)$. The left-hand-side of equation (A22) is a line with a slope $\frac{1}{\beta(1-\beta)^{T-1}}$. We can verify that when $\beta = \frac{1}{T}$. The LHS and the RHS of equation (A22) are tangent at $r = \frac{T}{T-1}$. Thus, we have $r_T(\beta) > \frac{T}{T-1}$ if $\beta < \frac{1}{T}$ and $r_T(\beta) < \frac{T}{T-1}$ if $\beta > \frac{1}{T}$. Finally, from equation (A21), we know that $r'(\beta) < 0$. So $r_T(\beta, \delta)$ is a decreasing function of β.

We then show that $r_T(\beta, \delta)$ is decreasing in T. Suppose the interest rate is such that the market breaks down with T periods, i.e., $(w, w, \cdots, w)$ is the solution to the sophisticates' program. Consider the non-lapsing constraint at time 2. First, since $(w, \cdots, w)$ must satisfy the constraints, $u(w) + \beta\sum_{i=2}^{T}\delta^{i-1}u(w) \geq V_2^S$. On the other hand, by definition, $V_2^S \geq u(w) + \beta\sum_{i=2}^{T}\delta^{i-1}u(w)$. So $V_2^S = u(w) + \beta\sum_{i=2}^{T}\delta^{i-1}u(w)$, implying the market breaks down with $(T-1)$ period. So we have shown that if the interest rate is such that the market breaks down with T periods, the market also breaks down with $(T-1)$ periods. Put differently, the cutoff $r_T(\beta, \delta)$ must be decreasing in T.

Finally, we show the limiting results. As $\beta \to 1$, the right-hand-side of equation (A18) becomes βr. So, $\lim_{\beta\nearrow 1} r_T(\beta) = 1$. It then follows that $\lim_{\beta\nearrow 1} r_T(\beta, \delta) = \frac{\lim_{\beta\nearrow 1} r_T(\beta)}{\delta} = \frac{1}{\delta}$. As $T \to +\infty$, the right-hand-side of equation (A18) becomes $\frac{\beta r}{1-(1-\beta)r}$. Solving $\frac{\beta r}{1-(1-\beta)r} = 1$ gives $r = 1$. Thus, $\lim_{T\nearrow\infty} r_T(\beta, \delta) = \frac{\lim_{T\nearrow\infty} r_T(\beta)}{\delta} = \frac{1}{\delta}$. □

Proof of Proposition 4. We compare when the market breaks down for each type of agents while fixing an α-weighted impatience. Recall the conditions for market breakdown: (1)

$R \leq \frac{1}{\delta_C}$ for a time-consistent agent; (2) $R \leq r_T(\beta, \delta_I)$ for a sophisticate; and (3) $R \leq \frac{1}{\delta_I}$ for a partial naif. Since $\frac{1}{\delta_I} < \frac{1}{\delta_C}$ and $\frac{1}{\delta_I} < r_T(\beta, \delta_I)$, it is easier to sustain long-term contracting with naifs than with both sophisticates and time-consistent consumers.

We next show that it is easier to sustain long-term contracting with sophisticates than with time-consistent agents. Recall that the equilibrium contract for a sophisticate maximizes the utility of the period-1 self subject to zero profits and non-lapsing constraints. Starting from $c_1 = c_2 = ... = c_T = w$, suppose we shift consumption from period 1 to period T by $\epsilon > 0$: $c_1 = w - \epsilon, \;\; c_N = w + \epsilon R^{T-1}$. This transfer keeps the non-lapsing and zero profits constraints satisfied and changes the agent's utility by $\left(\beta \delta_I^{T-1} R^{T-1} - 1\right) u'(w)\epsilon$. For the market to break down, shifting consumption to the last period cannot increase the agent's utility, so we must have

$$\beta \delta_I^{T-1} R^{T-1} \leq 1. \tag{A23}$$

Using the fact that the time-inconsistent agent discounts the last period by less ($\beta \delta_I^{T-1} \geq \delta_C^{T-1}$), we find that (A23) implies $\delta_C R \leq 1$. That is, whenever the market breaks down for sophisticates, it also breaks down for time-consistent agents.

References

Amador, M., I. Werning, and G.-M. Angeletos (2006). Commitment vs. flexibility. *Econometrica 74*(2), 365–396.

Atal, J. P., H. Fang, M. Karlsson, and N. R. Ziebarth (2018). Long-term health insurance: Theory meets evidence. *Working Paper*.

Atlas, S. A., E. J. Johnson, and J. W. Payne (2017). Time preferences and mortgage choice. *Journal of Marketing Research 54*(3), 415–429.

Bar-Gill, O. (2009). The law, economics and psychology of subprime mortgage contracts. *Cornell Law Review 94*(5), 1073–1152.

Bisin, A., A. Lizzeri, and L. Yariv (2015). Government policy with time inconsistent voters. *American Economic Review 105*(6), 1711–37.

Bond, P. and G. Sigurdsson (2017). Commitment contracts. *The Review of Economic Studies 85*(1), 194–222.

Brueckner, J. K. (1994). Borrower mobility, adverse selection, and mortgage points. *Journal of Financial Intermediation 3*(4), 416–441.

Cao, D. and I. Werning (2018). Saving and dissaving with hyperbolic discounting. *Econometrica 86*(3), 805–857.

Carter, S. P., K. Liu, P. M. Skiba, and J. Sydnor (2017). Time to repay or time to delay? the effect of having more time before a payday loan is due. *Working Paper*.

Chade, H., P. Prokopovych, and L. Smith (2008). Repeated games with present-biased preferences. *Journal of Economic Theory 139*(1), 157–175.

Cochrane, J. H. (1995). Time-consistent health insurance. *Journal of Political Economy 103*(3), 445–473.

Daily, G., I. Hendel, and A. Lizzeri (2008). Does the secondary life insurance market threaten dynamic insurance? *American Economic Review 98*(2), 151–156.

DellaVigna, S. and U. Malmendier (2004). Contract design and self-control: Theory and evidence. *The Quarterly Journal of Economics 119*(2), 353–402.

Dionne, G. and N. A. Doherty (1994). Adverse selection, commitment, and renegotiation: Extension to and evidence from insurance markets. *Journal of Political Economy 102*(2), 209–235.

Galberti, S. (2015). Commitment, flexibility, and optimal screening of time inconsistency. *Econometrica 83*(4), 1425–1465.

Gathergood, J. and J. Weber (2017). Financial literacy, present bias and alternative mortgage products. *Journal of Banking & Finance 78*, 58–83.

Ghent, A. (2011). Subprime mortgages, mortgage choice, and hyperbolic discounting. *Working paper, Zicklin School of Business*.

Grubb, M. D. (2015). Overconfident consumers in the marketplace. *The Journal of Economic Perspectives 29*(4), 9–35.

Gruber, J. and B. Koszegi (2001). Is addiction "rational"? theory and evidence. *Quarterly Journal of Economics 116*(4), 1261–1303.

Halac, M. and P. Yared (2014). Fiscal rules and discretion under persistent shocks. *Econometrica 82*(5), 1557–1614.

Handel, B., I. Hendel, and M. D. Whinston (2017). The welfare effects of long-term health insurance contracts. *Working paper*.

Harris, C. and D. Laibson (2001). Dynamic choices of hyperbolic consumers. *Econometrica 69*(4), 935–957.

Heidhues, P. and B. Kőszegi (2010). Exploiting naivete about self-control in the credit market. *The American Economic Review 100*(5), 2279–2303.

Hendel, I. and A. Lizzeri (2003). The role of commitment in dynamic contracts: Evidence from life insurance. *Quarterly Journal of Economics 118*(1), 299–327.

Kőszegi, B. (2014). Behavioral contract theory. *Journal of Economic Literature 52*(4), 1075–1118.

Krueger, D. and H. Uhlig (2006). Competitive risk sharing contracts with one-sided commitment. *Journal of Monetary Economics 53*(7), 1661–1691.

Makarov, I. and G. Plantin (2013). Equilibrium subprime lending. *Journal of Finance 68*(3), 849–879.

Meier, S. and C. Sprenger (2010). Present-biased preferences and credit card borrowing. *American Economic Journal: Applied Economics 2*(1), 193–210.

Myerson, R. B. (1983). Mechanism design by an informed principal. *Econometrica 51*(6), 1767–1797.

O'Donoghue, T. and M. Rabin (1999). Doing it now or later. *American Economic Review 89*(1), 103–124.

O'Donoghue, T. and M. Rabin (2001). Choice and procrastination. *Quarterly Journal of Economics 116*(1), 121–160.

O'Donoghue, T. and M. Rabin (2003). Studying optimal paternalism, illustrated by a model of sin taxes. *American Economic Review 93*(2), 186–191.

O'Donoghue, T. and M. Rabin (2015). Present bias: Lessons learned and to be learned. *American Economic Review 105*(5), 273–79.

Pauly, M. V., H. Kunreuther, and R. Hirth (1995). Guaranteed renewability in insurance. *Journal of Risk and Uncertainty 10*(2), 143–156.

Schlafmann, K. (2016). Housing, mortgages, and self control. *Working paper, Stockholm University*.

Yilankaya, O. (1999). A note on the seller's optimal mechanism in bilateral trade with two-sided incomplete information. *Journal of Economic Theory 87*(1), 267–271.

Supplementary Appendix - Not For Publication

A. Proof of Lemma 2 for General Income Distributions

In this appendix, we show that a naive agent's program is equivalent to the auxiliary program for general income distribution. We show the results for one-sided commitment. If there is two-sided commitment, we can simply ignore the non-lapsing constraints in the proof.

Recall that the naive agent's program is

$$\max_{c(s_t,h^t)} u(c(s_1)) + \beta E\left[\sum_{t=2}^{T} \delta^{t-1} u(c(s_t,(B,B,...,B)))\right],$$

subject to

$$\sum_{t=1}^{T} E\left[\frac{w(s_t) - c(s_t,(A,A,...,A))}{R^{t-1}}\right] = 0, \quad \text{(Zero Profits)}$$

$$u(c(s_\tau,(h^{\tau-1},B))) + \hat{\beta} E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c\left(s_t,\left(h^{\tau-1},B,B,...,B\right)\right)\right)\middle| s_\tau\right] \quad \text{(PCC)}$$

$$\geq u(c(s_\tau,(h^{\tau-1},A))) + \hat{\beta} E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c\left(s_t,\left(h^{\tau-1},A,B,...,B\right)\right)\right)\middle| s_\tau\right],$$

and

$$u(c(s_\tau,(h^{\tau-1},A))) + \beta E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c\left(s_t,\left(h^{\tau-1},A,B,...,B\right)\right)\right)\middle| s_\tau\right] \quad \text{(IC)}$$

$$\geq u(c(s_\tau,(h^{\tau-1},B))) + \beta E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c\left(s_t,\left(h^{\tau-1},B,B,...,B\right)\right)\right)\middle| s_\tau\right],$$

and non-lapsing constraints:

$$u\left(c(s_\tau,(h^{\tau-1},A))\right) + \beta E\left[\sum_{t>\tau} \delta^{t-\tau} u\left(c(s_t,(h^{\tau-1},A,B,...,B))\right)\middle| s_\tau\right] \geq V(s_\tau), \ \ \forall s_\tau, \quad \text{(NL)}$$

and

$$u\left(c(s_\tau,(h^{\tau-1},B))\right)+\hat{\beta}E\left[\sum_{t>\tau}\delta^{t-\tau}u\left(c(s_t,(h^{\tau-1},B,B,...,B))\right)\middle|s_\tau\right]\geq\hat{V}(s_\tau),\ \ \forall s_\tau. \tag{PNL}$$

We first note that the incentive compatibility constraints (IC) must be binding on the equilibrium path, because otherwise we can increase $c(s_T,h^\tau,B,B,...,B)$ without affecting all other constraints while weakly increase the agent's perceived utility. Given incentive constraints are binding, we can simplify (PCC) as

$$u(c(s_\tau,(h^{\tau-1},B)))\leq u(c(s_\tau,(h^{\tau-1},A))). \tag{sA1}$$

Substituting the binding IC constraints in the objective gives

$$E\sum_{t=1}^{T-1}\delta^{t-1}u(c(s_t,A,\cdots,A))+\beta\delta^{T-1}u(c(s_T,A,\cdots,A))+(\beta-1)\delta^{t-1}u(c(s_t,A,\cdots,A,B)).$$

Since $\beta<1$, we want to choose $c(s_t,A,\cdots,A,B)$ as small as possible (subject to the constraints). We now show that under the optimal contract, $c(s_t,A,\cdots,A,B)=0$. We need to verify that setting $c(s_t,A,\cdots,A,B)=0$ would not violate all other constraints. First, (PCC) holds because (sA1) holds.

We then verify that the perceived non-lapsing constraints hold if actual non-lapsing constraints (NL) hold. Suppose $\{\hat{c}(s_t,h_\tau^t):t\geq\tau\}$ solves the perceived outside option program $\hat{V}(s_\tau)$. So we have

$$\hat{V}(s_\tau)=u(\hat{c}(s_\tau,h_\tau^\tau))+\hat{\beta}E[\delta^{t-\tau}u(\hat{c}(s_t,(h_\tau^\tau,B,\cdots,B)))|s_\tau]. \tag{sA2}$$

We next verify the perceived non-lapsing constraint at $(s_\tau,(h^{\tau-1},B))=(s_\tau,(A,\cdots,A,B))$. Other perceived non-lapsing constraints can be similarly verified. Note that

$$u(c(s_\tau,(h^{\tau-1},B)))+\hat{\beta}E\left[\sum_{t>\tau}\delta^{t-\tau}u(c(s_t,(h^{\tau-1},B,\cdots,B)))\middle|s_\tau\right]$$

$$= \left(1 - \frac{\hat{\beta}}{\beta}\right) u(0)) + \frac{\hat{\beta}}{\beta} u(c(s_\tau, (h^{\tau-1}, B))) + \hat{\beta} E\left[\sum_{t>\tau} \delta^{t-\tau} u(c(s_t, (h^{\tau-1}, B, \cdots, B))) \middle| s_\tau\right] \tag{sA3}$$

$$= \left(1 - \frac{\hat{\beta}}{\beta}\right) u(0) + \frac{\hat{\beta}}{\beta}\left(u(c(s_\tau, (h^{\tau-1}, A))) + \beta E\left[\sum_{t>\tau} \delta^{t-\tau} u(c(s_t, (h^{\tau-1}, A, B, \cdots, B))) \middle| s_\tau\right]\right) \tag{sA4}$$

$$\geq \left(1 - \frac{\hat{\beta}}{\beta}\right) u(0) + \frac{\hat{\beta}}{\beta} V(s_\tau) \tag{sA5}$$

$$\geq \left(1 - \frac{\hat{\beta}}{\beta}\right) u(0) + \frac{\hat{\beta}}{\beta}\left(u(\hat{c}(s_\tau, h_\tau^\tau)) + \beta E\left[\sum_{t>\tau} \delta^{t-\tau} u(\hat{c}(s_t, h_\tau^\tau, B, \cdots, B)) \middle| s_\tau\right]\right) \tag{sA6}$$

$$= \left(1 - \frac{\hat{\beta}}{\beta}\right) u(0) + \frac{\hat{\beta}}{\beta} u(\hat{c}(s_\tau, h_\tau^\tau)) + \hat{\beta} E\left[\sum_{t>\tau} \delta^{t-\tau} u(\hat{c}(s_t, h_\tau^\tau, B, \cdots, B)) \middle| s_\tau\right] \tag{sA7}$$

$$= \left(1 - \frac{\hat{\beta}}{\beta}\right) u(0) + (\frac{\hat{\beta}}{\beta} - 1) u(\hat{c}(s_\tau, h_\tau^\tau)) + \hat{V}(s_\tau) \tag{sA8}$$

$$\geq \hat{V}(s_\tau), \tag{sA9}$$

where (sA3) follows from $c(s_\tau, (h^{\tau-1}, B)) = 0$, (sA4) from (IC), (sA5) from the actual non-lapsing constraints (NL), (sA6) follows from a revealed preference argument since $\hat{c}$ is also feasible in program $V(s_\tau)$, (sA7) follows from simple algebra, (sA8) uses (sA2), and (sA9) follows from $\hat{c}(s_\tau, h_\tau^\tau) \geq 0$.

By the previous argument, the perceived choice constraints and the perceived non-lapsing constraints can be ignored, so the program reduces to:

$$\max E \sum_{t=1}^{T-1} \delta^{t-1} u(c(s_t, (A, \cdots, A))) + \beta \delta^{T-1} u(c(s_T, (A, \cdots, A))),$$

subject to the zero-profit condition and the actual non-lapsing constraints (NL). Since the objective is the same as the utility of a dynamically consistent consumer, we can replace the non-lapsing constraints by front-loading constraints (FL). So $c^{1E} = c^{1A}$. □

B. Market Power

Throughout the paper, we assumed that the consumer had all bargaining power. We now consider the case in which a firm has all the bargaining power. Since the firm can always commit to a contract, there is no loss of generality in assuming that the firm makes a take-it-or-leave it offer to the consumer, which happens at time 1. We assume that $R > 1$ to ensure that the firm's discounted profit converges as the contracting horizon goes to infinity. Let $\underline{U}$ denote the consumer's outside option ("reservation utility").

As before, our main focus is on (partially naive) time-inconsistent consumers. As a benchmark, we also consider time-consistent consumers. Let $W_T^C(\underline{U})$ and $W_T^I(\underline{U})$ denote the welfare of time-consistent and time-inconsistent consumers, respectively. Let $V_T^C(\underline{U})$ and $V_T^I(\underline{U})$ denote the firm's profit when the consumer is time consistent and time inconsistent, respectively. We will omit the subscript T for notational simplicity.

We now show that when the firm has the bargaining power, the inefficiency also vanishes as the horizon grows. However, unlike in the case where bargaining power is on the consumer's side, the equilibrium converges to a different point on the Pareto frontier.

Proposition 5. *Suppose u is bounded, $\delta < 1$, and $R > 1$. Then,*

$$\lim_{T \nearrow \infty} (W^C(\underline{U}') - W^I(\underline{U})) = 0, \quad \lim_{T \nearrow \infty} (V^C(\underline{U}') - V^I(\underline{U})) = 0,$$

where $\underline{U}' \equiv \underline{U} + (1-\beta)\frac{\delta}{1-\delta}u(0)$.

Proof. For simplicity, we will only present the proof for the case with a constant deterministic income. The general proof follows the same steps as the proof of Theorem 1. The equilibrium profit with time-consistent agents solves:

$$V_T^C(\underline{U}) := \max_{\{c_t\}} \sum_{t=1}^{T} \frac{w - c_t}{R^{t-1}}, \tag{sB1}$$

subject to

$$\sum_{t=1}^{T} \delta^{t-1} u(c_t) \geq \underline{U}. \tag{sB2}$$

With time-inconsistent agents, the equilibrium profits are determined by:

$$V^I_{T,\beta,\hat{\beta}}(\underline{U}) := \max_{\{c_t\}} \sum_{t=1}^{T} \frac{w - c_t(A \cdots A)}{R^{t-1}}, \tag{sB3}$$

subject to (IC), (PCC), and

$$u(c_1) + \beta \sum_{t=2}^{T} \delta^{t-1} u(c_t(B \cdots B)) \geq \underline{U}. \tag{sB4}$$

Consider the following auxiliary program:

$$V^A_{T,\beta}(\underline{U}') := \max_{\{c_t\}} \sum_{t=1}^{T} \frac{w - c_t}{R^{t-1}}, \tag{sB5}$$

subject to

$$\sum_{t=1}^{T-1} \delta^{t-1} u(c_t) + \beta \delta^{T-1} u(c_T) \geq \underline{U}'. \tag{sB6}$$

We show that the equilibrium consumption must solve the auxiliary program for

$$\underline{U}' = \underline{U} + (1 - \beta) u(0)(\delta + \cdots + \delta^{T-2}).$$

For simplicity, we present the proof for $T = 3$. The proof for general T is similar and is therefore omitted. The equilibrium consumption for time-inconsistent agents solves the following program:

$$\max_{(c_1, c_2(A), c_2(B), c_3(A), c_3(B))} w - c_1 + \frac{w - c_2(A)}{R} + \frac{w - c_3(A)}{R^2}, \tag{sB7}$$

subject to

$$u(c_1) + \beta[\delta u(c_2(B)) + \delta^2 u(c_3(B))] \geq \underline{U}, \tag{sB8}$$

$$u(c_2(B)) + \hat{\beta}\delta u(c_3(B)) \geq u(c_2(A)) + \hat{\beta}\delta u(c_3(A)), \tag{sB9}$$

$$u(c_2(A)) + \beta\delta u(c_3(A)) \geq u(c_2(B)) + \beta\delta u(c_3(B)). \tag{sB10}$$

By the same argument as in the main text, it follows that the IC constraint (sB10) must bind and $c_2(B) = 0$. Using (sB10) to substitute for $c_3(B)$, the participation constraint (sB8) becomes

$$u(c_1) + \beta\delta u(c_2(B)) + \beta\delta^2 u(c_3(B)) \tag{sB11}$$

$$= u(c_1) + \beta\delta u(c_2(B)) + \delta(u(c_2(A) + \beta\delta u(c_3(A)) - u(c_2(B)) \tag{sB12}$$

$$= u(c_1) + \delta u(c_2(A) + \beta\delta^2 u(c_3(A)) - \delta(1-\beta)u(c_2(B)) \tag{sB13}$$

$$= u(c_1) + \delta u(c_2(A) + \beta\delta^2 u(c_3(A)) - \delta(1-\beta)u(0). \tag{sB14}$$

We can verify that (sB9) holds as long as the IC binds and $c_2(B) = 0$. Thus, the equilibrium consumption for time-inconsistent consumers solves:

$$\max_{(c_1,c_2,c_3)} w - c_1 + \frac{w - c_2}{R} + \frac{w - c_3}{R^2}, \tag{sB15}$$

subject to

$$u(c_1) + \delta u(c_2) + \beta\delta^2 u(c_3) \geq \underline{U} + \delta(1-\beta)u(0), \tag{sB16}$$

This is the same program as the program with a dynamically consistent agent who discounts the last period by an extra β.

We now obtain the convergence result. Note that the participation constraints must be binding both in the auxiliary program and in the program for time-consistent consumers. So $W^C = \underline{U}$ and $\sum_{t=1}^{T-1} \delta^{t-1} u(c_t^A(\beta, T)) + \beta\delta^{T-1} u(c_T^A(\beta, T)) = \underline{U}$, where

$c^A(\beta, T, \underline{U}) := (c_1^A(\beta, T, \underline{U}), \cdots, c_T^A(\beta, T, \underline{U}))$ denotes the equilibrium consumption in the auxiliary program. Omitting the dependence of c^A on β, T, and $\underline{U}$ for notational simplicity, we have:

$$\begin{aligned}
W^A_{\beta,T}(\underline{U}) &= \sum_{t=1}^{T} \delta^{t-1} u(c_t^A) \\
&= \sum_{t=1}^{T-1} \delta^{t-1} u(c_t^A) + \beta\delta^{T-1} u(c_T^A) + (1-\beta)\delta^{T-1} u(c_T^A) \\
&= \underline{U} + (1-\beta)\delta^{T-1} u(c_T^A) \\
&= W^C + (1-\beta)\delta^{T-1} u(c_T^A).
\end{aligned}$$

So $\lim_{T\nearrow\infty} |W^C - W^A| = \lim_{T\nearrow\infty} (1-\beta)\delta^{T-1} u(c_T^A) = 0$ (since u is bounded and $\delta < 1$).

We now turn to the firm's profit. Let λ denote the Lagrangian multiplier with the constraint (sB6). FOC gives

$$\lambda\delta^{t-1} u'(c_t^A) = \frac{1}{R^{t-1}}, \forall t = 1, \cdots, T-1,$$

and

$$\lambda\beta\delta^{T-1} u'(c_T^A) = \frac{1}{R^{T-1}}.$$

Differentiating w.r.t β on the binding participation constraint gives

$$\sum_{t=1}^{T-1} \delta^{t-1} u'(c_t^A) \frac{\partial c_t^A}{\partial \beta} + \beta\delta^{T-1} u'(c_T^A) \frac{\partial c_T^A}{\partial \beta} + \delta^{T-1} u(c_T^A) = 0.$$

Now,

$$
\begin{aligned}
\frac{\partial V^A(\beta)}{\partial\beta} &= \sum_{t=1}^{T} \frac{-\frac{\partial c_t^A}{\partial\beta}}{R^{t-1}} \\
&= -\sum_{t=1}^{T-1} \lambda\delta^{t-1}u'(c_t^A)\frac{\partial c_t^A}{\partial\beta} - \lambda\beta\delta^{T-1}u'(c_T^A)\frac{\partial c_T^A}{\partial\beta} \\
&= \lambda\delta^{T-1}u(c_T^A) \\
&\geq \lambda\delta^{T-1}u(0),
\end{aligned}
$$

where the inequality comes from $c_T^A \geq 0$. Applying Lagrange's Mean Value Theorem gives

$$
\begin{aligned}
V^A(1) - V^A(\beta) &= \frac{\partial V^A(\beta)}{\partial\beta}|_{\beta=\beta'}(1-\beta) \geq \lambda\delta^{T-1}u(0)(1-\beta), \\
V^A(\beta) - V^A(0) &= \frac{\partial V^A(\beta)}{\partial\beta}|_{\beta=\beta''}\beta \geq \lambda\delta^{T-1}u(0)\beta,
\end{aligned}
$$

where $\beta' \in (\beta, 1), \beta'' \in (0, \beta)$. Since λ is bounded because $\lambda = \frac{1}{u'(c_1^A)} \leq \frac{1}{u'\left(w\sum_{t=1}^{T}\frac{1}{R^{t-1}}\right)} \leq \frac{1}{u'\left(\frac{w}{1-\frac{1}{R}}\right)}$, sending T to infinity leads to

$$
\lim_{T\nearrow\infty} V^A(1) \geq \lim_{T\nearrow\infty} V^A(\beta)) \geq \lim_{T\nearrow\infty} V^A(0).
$$

In order to show that $\lim_{T\nearrow\infty}(V^A(1) - V^A(\beta)) = 0$, it is sufficient to show that $\lim_{T\nearrow\infty}(V^A(1) - V^A(0)) = 0$.

We write the program for $V^A(0)$:

$$
V_T^A(0) := \max_{\{c_t\}} \sum_{t=1}^{T} \frac{w - c_t}{R^{t-1}}, \tag{sB17}
$$

subject to

$$
\sum_{t=1}^{T-1} \delta^{t-1}u(c_t) \geq \underline{U}. \tag{sB18}
$$

It is immediate that c^T must be equal to 0. Then the program reduces to

$$V_T^A(0) := \max_{\{c_t\}} \sum_{t=1}^{T-1} \frac{w - c_t}{R^{t-1}} + \frac{w}{R^{T-1}}, \tag{sB19}$$

subject to

$$\sum_{t=1}^{T-1} \delta^{t-1} u(c_t) \geq \underline{U}. \tag{sB20}$$

Then $V_T^A(0) = V_{T-1}^A(1) + \frac{w}{R^{T-1}}$. It follows that $\lim_{T\nearrow\infty}(V_T^A(1) - V_T^A(0)) = \lim_{T\nearrow\infty}(V_T^A(1) - V_{T-1}^A(1) - \frac{w}{R^{T-1}}) = \lim_{T\nearrow\infty}(V_T^A(1) - V_{T-1}^A(1)) - \lim_{T\nearrow\infty}\frac{w}{R^{T-1}} = \lim_{T\nearrow\infty}(V_T^A(1) - V_{T-1}^A(1)) = 0$. $\square$

C. Consumer Heterogeneity

In the main text, we assumed that firms knew the consumer's preferences. We now consider how the results generalize when the consumer has private information about his preference parameters.

C.1 Heterogeneous Naive Consumers

We consider a group of heterogeneous naive consumers who are indistinguishable by firms. We assume there are I types of consumers. With probability q_i the consumer is a naive consumer represented by a pair of time-inconsistency and naiveté parameters $(\beta_i, \hat{\beta}_i)$, where $\hat{\beta}_i > \beta_i$.

Assumption 1. The minimal naiveté parameter is greater than the maximal time-inconsistency parameter: $\min\{\hat{\beta}_i\} > \max\{\beta_i\}$, or equivalently $\hat{\beta}_j > \beta_i, \forall i, j$.

Under Assumption 1, we show that, as in the case of private information about the naiveté parameter, the results in the text remain unchanged.

Formally, suppose the consumer has a constant income w and both parties can commit to long-term contracts. Because of two-sided commitment, firms offer contract at time 1 and consumers choose which contract to accept. As in the text, the equilibrium of the game coincides with a competitive equilibrium. In a competitive equilibrium, agents choose contracts to maximize utility, no contract in the equilibrium set makes negative expected profits, and there is no contract outside the equilibrium set that, if offered, would make a positive profit.[23]

There are two types of consumers, with time-inconsistency parameters β_1 and β_2. Both types of consumers are partially naive with the same naiveté parameter: $\hat{\beta} > \beta_2 > \beta_1$. The firm believes that the consumer's time-inconsistency parameter equals β_1 with probability q and equals β_2 with probability $1 - q$.

Let $\mathcal{C}^i$ denote the full-information contract for type β_i, that is, the solution of program (6) when the consumer's time-inconsistency parameter equals β_i. Let $\mathbf{c}^i$ denote the consumption on the equilibrium path in $\mathcal{C}^i$, that is, the solution to the auxiliary program (7). Our key result shows that in any competitive equilibrium, each type β_i gets contract $\mathcal{C}^i$, so that the equilibrium is fully separating with each type getting the same contract as in the model with symmetric information.

The following lemma will be useful in establishing our main result:

Lemma 8. *There is no competitive equilibrium in which both types get the same equilibrium consumption.*

Proof. We argue by contradiction. Suppose the β_i consumer gets $\overline{\mathcal{C}}^i$. Suppose both types of consumers get the same equilibrium consumption. Since the equilibrium path consumption in $\mathcal{C}^1$ is different from the equilibrium path consumption in $\mathcal{C}^2$, we conclude at least one type of consumers does not get the full-information contract. Without loss, we assume it is the β_1 consumer. Consider a contract $\overline{\mathcal{C}}$, in which the terms after first period are exactly described as in $\mathcal{C}^1$ and the first period consumption is equal to $\mathbf{c}_1^1 - \epsilon$. Here ϵ is a small positive

[23]With one-sided commitment and private information about β, the equivalence result about the equilibrium of the game and the competitive equilibrium does not automatically generalize. Characterizing the equilibrium requires solving an informed principal problem, which is beyond the scope of this paper.

number. Recall that $\mathcal{C}^1$ gives the β_1 consumer the highest possible perceived utility, and the equilibrium path consumption is uniquely determined. Given that the β_1 consumer's equilibrium consumption is different from $\mathbf{c}^1$, we can find a $\epsilon > 0$ small enough such that $\overline{\mathcal{C}}$, if offered, is accepted by the β_1 consumer. Note that firms make a positive profit because $\epsilon > 0$, contradicted to the definition of competitive equilibrium. □

Lemma 9. *There exists a competitive equilibrium where both types of consumers get their full-information contract, $\mathcal{C}^i$.*

Proof. We directly show that $\{\mathcal{C}_1, \mathcal{C}_2\}$ is indeed a competitive equilibrium. In this equilibrium, the β_i consumer optimally chooses $\mathcal{C}^i$, which is the full-information contract. Note that both contracts give firms zero profits. We need to show that there is no contract outside the equilibrium set $\{\mathcal{C}_1, \mathcal{C}_2\}$ that, if offered, would make a positive profit. We argue by contraction. Suppose a firm could offer $\overline{\mathcal{C}}$ and make a positive profit. Suppose $\overline{\mathcal{C}}$ is accepted by the β_1 consumer with a positive probability. By lowering the first period consumption in $\overline{\mathcal{C}}$, we can assume without loss of generality that the β_1 consumer strictly prefers $\overline{\mathcal{C}}$ over $\mathcal{C}^1$ and the firm still makes a positive profit, contradicted to that that $\mathcal{C}^1$ gives the β_1 consumer the highest possible perceived utility subject to the zero-profit condition. □

Let $W^{I,i}$ denote the equilibrium welfare of the β_i consumer, which evaluates the agent's consumption on the equilibrium path according to his long-term preferences (2) and recall that W^C is the welfare in the benchmark case of a time-consistent consumer. Since the time-consistent consumer maximizes welfare, the welfare loss from dynamic inconsistency is $W^C - W^{I,i} \geq 0$.

Proposition 6. *Suppose u is bounded and $\delta < 1$. Then, in any competitive equilibrium, $lim_{T \nearrow +\infty} \left(W^C - W^{I,i}\right) = 0$.*

Proof. We proceed by showing that the outcome associated with a competitive equilibrium is unique and given by the outcome in Lemma 9. Since we know from Lemma 8 that a pooling outcome can not arise, we need to rule out semi-separating equilibria where one type plays a mixed strategy. We argue by contradiction. Suppose the β_1 consumer would accept a contract with a positive probability and the contract gives the β_1 consumer a stream of equilibrium consumption that is different from the equilibrium oath consumption in the full-information contract $\mathcal{C}_1$. Consider a contract $\overline{\mathcal{C}}$, in which the terms after first period are exactly described as in $\mathcal{C}^1$ and the first period consumption is equal to $\mathbf{c}_1^1 - \epsilon$. Similar to the proof of Lemma 8, this contract will be accepted with a positive probability and give firms a positive profit, contradicted to the definition of competitive equilibrium.

As a result, in any competitive equilibrium, the β_i consumer's equilibrium consumption is exactly given by the equilibrium path consumption in the full-information contract $\mathcal{C}_i$. Due to Theorem 1, $lim_{T \nearrow +\infty} \left(W^C - W^{I,i} \right) = 0$. □

C.2 Both Sophisticated and Naive Consumers

We now consider the possibility that consumers may be sophisticated as well as being naive. For the ease of exposition, we assume there are two types of consumers, either the consumer is a $(\beta, \hat{\beta})$ naive agent or the consumer is a β^S sophisticated agent. Firms do not know consumers' types. The probability of a naif is q, and the probability of a sophisticate is $1 - q$.

Denote $\mathcal{C}^S$ the full-information contract for the sophisticate, that is, the contract solves a sophisticate's problem in the case that firms know that the consumer is sophisticated. Denote $\mathcal{C}^N$ the full-information contract for the naif, that is, the contract that solves the naive consumer's problem in the case that firms know that the consumer is naive. Denote $\mathbf{c}^N$ the consumption on the equilibrium path in $\mathcal{C}^N$, i.e., the solution to the auxiliary program (7).

We will show that the results are dramatically different, depending on whether the sophisticate or the naif has a higher time-inconsistency parameter. If the naif's time-inconsistency parameter is greater than the sophisticate's, we will restore the vanishing inefficiency result as the contracting horizon is long enough. However, if the sophisticate's time-inconsistency parameter is greater than the naif's, then the naif's contract is inefficient in general.

We first present the result when the naif's time-inconsistency parameter is greater than the sophisticate's.

Proposition 7. *Suppose $\beta^S \leq \beta$. In any competitive equilibrium, both types of consumers get their full-information contract. As a result, there is no welfare loss for the naif as the contracting horizon is long enough.*

Proof. Using the same potential deviation as in the proof of Lemma 8, we can show that a competitive equilibrium where both types of consumers get the same equilibrium consumption can not arise. Similarly, we can also rule out semi-separating equilibrium. We next show that $\{\mathcal{C}_S, \mathcal{C}_N\}$ is a competitive equilibrium. In this equilibrium, the sophisticate chooses $\mathcal{C}_S$ and the naif chooses $\mathcal{C}_N$. We need to verify that the they don't have an incentive to deviate. First, the naif consumer strictly prefer $\mathcal{C}_N$ over $\mathcal{C}_S$, because $\mathcal{C}_S$ is a contract that satisfies the constraints in the naif's program (6) in the main text. Second, we show that the sophisticate strictly prefer $\mathcal{C}_S$ over $\mathcal{C}_N$. Suppose the sophisticate chooses the contract $\mathcal{C}_N$. Since $\beta^S \leq \beta$, the sophisticate perceive that the consumption stream is given by consumption stream in the alternative options. Since the consumption stream in the alternative options must satisfy the zero-profit condition, the sophisticate would strictly prefer his full-information contract.

We next verify that there is no contract outside the equilibrium set $\{\mathcal{C}_S, \mathcal{C}_N\}$ that, if offered, would make a positive profit. We argue by contraction. Suppose a firm could offer $\overline{\mathcal{C}}$ and make a positive profit. Similar to the proof of Lemma 9, there is a contradiction if the naif accepts $\overline{\mathcal{C}}$ with a positive probability. Suppose the naif accepts $\overline{\mathcal{C}}$ with a positive probability and firms make a positive profit. By lowering the first period consumption in $\overline{\mathcal{C}}$, we can assume without loss of generality that the sophisticate strictly prefers $\overline{\mathcal{C}}$ over $\mathcal{C}^S$ and the firm still makes a positive profit, contradicted to that that $\mathcal{C}^S$ gives the sophisticated consumer the highest possible utility subject to the zero-profit condition. □

Now we consider the case that the sophisticate's time-inconsistency parameter is greater than the naif's time-inconsistency parameter: $\beta^S > \beta$.

Lemma 10. *Suppose $\beta^S > \beta$. The competitive equilibrium must give a separating outcome for the sophisticate and the naif. In addition, the sophisticated consumer gets the full-information contract, $\mathcal{C}^S$.*

Proof. Using the same potential deviation as in the proof of Lemma 8, we can show that a competitive equilibrium where both types of consumers get the same equilibrium consumption can not arise. Similarly, we can also rule out semi-separating equilibrium.

We next show that the sophisticated consumer gets the full-information contract in any competitive equilibrium. We argue by contraction. Suppose the sophisticate's equilibrium consumption is different from the consumption in the full-information contract. Consider a contract $\overline{\mathcal{C}}$, in which the terms after first period are exactly described as in $\mathcal{C}^S$ and the first period consumption is ϵ lower, where ϵ is a small positive number. Recall that $\mathcal{C}^N$ gives the sophisticate the highest possible utility, and the steam of equilibrium consumption is uniquely determined. Given that the sophisticate's equilibrium consumption is different from the stream in the full-information contract, we can find a $\epsilon > 0$ small enough such that $\overline{\mathcal{C}}$, if offered, is accepted by the sophisticate. Note that firms make a positive profit because $\epsilon > 0$, contradicted to the definition of competitive equilibrium. □

As a result of the previous lemma, we are left to determine the contract for the naif. Denote the contract by $\overline{\mathcal{C}}^N$. We need to make sure that the sophisticate prefers to take $\mathcal{C}^S$ over $\overline{\mathcal{C}}^N$. Denote the sophisticate's equilibrium consumption stream $(c_1^S, c_2^S, \cdots, c_T^S)$. Let U^S denote the sophisticate's perceived utility:

$$U^S = u(c_1^S) + \beta\delta u(c_2^S) + \beta\delta^2 u(c_3^S) + \cdots + \beta\delta^{T-1} u(c_T^S).$$

Lemma 11. *Suppose $u(0) = 0$ and $\beta^S > \beta$. Denote the naif's equilibrium consumption by $(c_1^N, c_2^N, \cdots, c_T^N)$. The first period consumption c_1^N is given by the maximum root of $u(c_1) + \frac{\beta^S}{\beta}\delta h(c_1) = U^S$, where $h(\cdot)$ is given by the following program*

$$h(c_1) = \max u(c_2) + \beta\delta u(c_3) + \cdots + \beta\delta^{T-2} u(c_T),$$

subject to

$$\sum_{t=1}^{T} \frac{c_t - w}{R^{t-1}} = 0. \tag{sC1}$$

$(c_2^N, \cdots, c_T^N)$ *solves the program* $h(c_1^N)$.

Proof. We prove the lemma by writing down the naif's program concretely when $T = 3$. Formally, the program for the naif is

$$\max_{(c_1, c_2(A), c_2(B), c_3(A), c_3(B))} u(c_1) + \beta\delta u(c_2(B)) + \beta\delta^2 u(c_3(B)),$$

subject to

$$c_1 + \frac{c_2(A)}{R} + \frac{c_3(A)}{R^2} = w(1 + \frac{1}{R} + \frac{1}{R^2}),$$
$$u(c_2(B)) + \hat{\beta}\delta u(c_3(B)) \geq u(c_2(A)) + \hat{\beta}\delta u(c_3(A)),$$
$$u(c_2(A)) + \beta\delta u(c_3(A)) \geq u(c_2(B)) + \beta\delta u(c_3(B)),$$
$$u(c_1) + \beta^S \delta u(c_2(B)) + \beta^S \delta^2 u(c_3(B)) \leq u(c_1^S) + \beta^S \delta u(c_2^S) + \beta^S \delta^2 u(c_3^S).$$

The last constraint says that the sophisticate prefers $\mathcal{C}^S$ over the contract for the naif. It can be shown that the full-information contract $\mathcal{C}^N$ is not feasible because it violates the last constraint. So the last constraint must bind. We also have $u(c_2(A)) + \beta\delta u(c_3(A)) = u(c_2(B)) + \beta\delta u(c_3(B))$, because otherwise firms can offer $(c_1, c_2(A), c_2(B), c_3(A) - \epsilon, c_3(B))$ and make a profit. Similar to the main text, the perceived time 2 consumption must be zero: $c_2(B) = 0$, because otherwise firms can decrease $c_2(B)$ and increase $c_3(B)$ to make zero profit condition slack. Specifically, consider the following perturbation $(u(c_1), u(c_2(A)) - (1 - \beta)\epsilon, u(c_2(B)) - \epsilon, u(c_3(A)), u(c_3(B)) + \frac{\epsilon}{\delta})$.

Using $c_2(B) = 0$ to simplify the program leads to

$$\max u(c_1) + \delta u(c_2) + \beta\delta^2 u(c_3),$$

subject to

$$c_1 + \frac{c_2}{R} + \frac{c_3}{R^2} = w(1 + \frac{1}{R} + \frac{1}{R^2}),$$
$$u(c_1) + \frac{\beta^S}{\beta}\delta u(c_2) + \beta^S \delta^2 u(c_3) = U^S.$$

Since $\beta^S > \beta$, together with the last constraint, the objective function is equivalent to

$$\max c_1,$$

subject to

$$c_1 + \frac{c_2}{R} + \frac{c_3}{R^2} = w(1 + \frac{1}{R} + \frac{1}{R^2}),$$
$$u(c_1) + \frac{\beta^S}{\beta}\delta u(c_2) + \beta^S \delta^2 u(c_3) = U^S.$$

To solve this problem, we first fix c_1 and consider the following program

$$h(c_1) = \max u(c_2) + \beta\delta u(c_3)$$

subject to

$$\frac{c_2}{R} + \frac{c_3}{R^2} = W - c_1. \tag{sC2}$$

In the $h(\cdot)$ program, FOC implies that $u'(c_2) = \beta\delta R u'(c_3)$. Together with (sC2), we can solve c_2 and c_3 as a function of c_1. Given the value of $h(c_1)$, then c_1 is given by the maximal root of the equation

$$u(c_1) + \frac{\beta^S}{\beta}\delta h(c_1) = U^S.$$

Once we solve c_1, we can find c_2 and c_3 from the program $h(c_1)$. From there, we can solve the contract for the naif. □

Equipped with the lemma, using numerical analysis, we can show that the naif's contract is inefficient when the utility is CARA.

D. General Discount Functions

We focused on quasi-hyperbolic discounting, which is the canonical model of present bias. This appendix generalizes our approach to arbitrary present-biased preferences, establishing the equivalence between the equilibrium and the solution of an auxiliary program. To simplify exposition, we assume that *(a)* income is constant w, and *(b)* consumers can commit to long-term contracts (two-sided commitment).

To allow for arbitrary discounting, we assume that at time $\tau \in \{1, 2, ...\}$, the agent evaluates a consumption stream $\{c_t\}_{t \geq \tau}$ according to

$$u(c_\tau) + \sum_{t > \tau} D_{t-\tau} u(c_t), \tag{sD1}$$

where the discount factor $D_t \in (0, 1)$ is decreasing in t. It is well known that preferences represented by (sD1) are time consistent if and only if $D_t = D_1^t$. We assume, instead, that preferences are present biased:

$$D_{i+j} > D_i D_j \tag{sD2}$$

for all i, j. This inequality states that the individual becomes more impatient as a period approaches. It is straightforward to verify that this inequality holds under both quasi-hyperbolic and under hyperbolic discounting.

The agent can be naive or sophisticated. A *naif* has discount factor $(D_1, \cdots, D_{T-1})$ but believes that, in the future, he will behave like an agent with discount factor $(\hat{D}_1, \cdots, \hat{D}_{T-2})$ where $\hat{D}_i > D_i$. A *sophisticate* perfectly knows his time-consistency parameter: $\hat{D}_t = D_t$ for all t. We assume that a naive agent overestimates the patience of his future selves:

$$\frac{\hat{D}_{i+1}}{\hat{D}_i} \geq \frac{D_{i+1}}{D_i},$$

with strict inequality for at least one i. Note that, with quasi-hyperbolic discounting, this inequality becomes the usual condition: $\hat{\beta} \geq \beta$.

The proposition below characterizes the agent's consumption on the equilibrium path in terms of a simpler auxiliary program:

Proposition 8. *The consumption of a naive agent on the equilibrium path coincides with the equilibrium of an agent with utility function:*

$$u(c_1) + \frac{D_{T-1}}{D_{T-2}}u(c_2) + \frac{D_{T-1}}{D_{T-3}}u(c_3) + \cdots + \frac{D_{T-1}}{D_1}u(c_{T-1}) + D_{T-1}u(c_T). \tag{sD3}$$

Proof. To simplify notation, we present the proof for $T = 4$. In that case, the equilibrium consumption of naive agent solves:

$$\max_{(c)} u(c_1) + D_1 u(c_2(B)) + D_2 u(c_3(BB)) + D_3 u(c_4(BB)), \tag{sD4}$$

subject to

$$c_1 + \frac{c_2(A)}{R} + \frac{c_3(AA)}{R^2} + \frac{c_4(AA)}{R^3} = w(1 + \frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3}),$$
$$u(c_2(B)) + \hat{D}_1 u(c_3(BB)) + \hat{D}_2 u(c_4(BB)) \geq u(c_2(A)) + \hat{D}_1 u(c_3(AB)) + \hat{D}_2 u(c_4(AB)),$$
$$u(c_2(A)) + D_1 u(c_3(AB)) + D_2 u(c_4(AB)) \geq u(c_2(B)) + D_1 u(c_3(BB)) + D_2 u(c_4(BB)),$$
$$u(c_3(AB)) + \hat{D}_1 u(c_4(AB)) \geq u(c_3(AA)) + \hat{D}_1 u(c_4(AA)),$$
$$u(c_3(AA)) + D_1 u(c_4(AA)) \geq u(c_3(AB)) + D_1 u(c_4(AB)).$$

The perturbation $c_2(B) - \epsilon$ and $c_4(BB) + \frac{1}{D_2}\epsilon$, where $\epsilon > 0$, does not affect the IC but improves the objective function by $-D_1 + \frac{D_3}{D_2}$, which is positive because of $D_3 > D_2 D_1$. Thus, the solution entails $c_2(B) = 0$.

Similarly, the perturbation $c_3(BB) - \frac{1}{D_1}\epsilon$ and $c_4(BB) + \frac{1}{D_2}\epsilon$ for $\epsilon > 0$ does not affect PPC because $\frac{\hat{D}_2}{D_2} \geq \frac{\hat{D}_1}{D_1}$. It also does not affect the IC. However, it improves the objective function by $-\frac{D_2}{D_1} + \frac{D_3}{D_2} > 0$, which holds because of $D_3 D_1 > D_2^2$. Thus, $c_3(BB) = 0$.

Substituting $c_2(B) = c_3(BB) = 0$ in the objective function, gives:

$$u(c_1) + \frac{D_3}{D_2}u(c_2(A)) + \frac{D_3 D_1}{D_2}u(c_3(AB)) + D_3 u(c_4(AB)).$$

We now show that $c_3(AB) = 0$. Consider the following perturbation $c_3(AB) - \epsilon$ and $c_4(AB) + \frac{1}{D_1}\epsilon$. This does not change the IC, but improves the objective function by $-\frac{D_3 D_1}{D_2} + \frac{D_3}{D_1} > 0$, which holds because of $D_2 > D_1 D_1$. Substituting $c_3(AB) = 0$, we obtain the objective function in our auxiliary program (up to a constant):

$$u(c_1) + \frac{D_3}{D_2}u(c_2(A)) + \frac{D_3}{D_1}u(c_3(AA)) + D_3 u(c_4(AA)).$$

Therefore, the sophisticated agent's consumption on the equilibrium path maximizes (sD1) subject to zero profits, whereas the naive agent's consumption on the equilibrium path maximizes (sD3) subject to zero profits. □

We now use Proposition 8 to obtain two additional results. First, we show that, with general discount functions, a naif always saves more initially than a sophisticate. This is a weaker version of Theorems 1 and 2 that holds much more generally. We then turn to the case of hyperbolic discounting, and show that, as in Theorems 1 and 2, the welfare of naifs converges to the welfare of time consistent agents.

Naifs save more in the first period. Given a vector $x = (x_2, \cdots, x_T)$, consider the following program:

$$\max_{(c_1,\cdots,c_T)} u(c_1) + x_2 u(c_2) + x_3 u(c_3) + \cdots x_T u(c_T) \tag{sD5}$$

subject to

$$\sum_{t=1}^{T} \frac{c_t}{R^{t-1}} = W,$$

where $W = \sum_{t=1}^{T} \frac{w}{R^{t-1}}$. In naif's program corresponds to the previous program with vector

$$x^N = \left(\frac{D_{T-1}}{D_{T-2}}, \frac{D_{T-1}}{D_{T-3}}, \cdots, \frac{D_{T-1}}{D_1}, D_{T-1} \right),$$

whereas the sophisticate's program takes vector

$$x^S = (D_1, D_2, \cdots, D_{T-1}).$$

Recall from equation (sD2) that $x_t^N > x_t^S, \forall t$.

The first-order-condition of Program (sD5) can be written as

$$u'(c_1) = Rx_2 u'(c_2) = \cdots = R^{T-1} x_T u'(c_T).$$

Suppose the naif consumes weakly more than the sophisticate in the first period, i.e., $c_1^N \geq c_1^S$. Note that for any $t = 2, \cdots, T$,

$$R^{t-1} x_t^S u'(c_t^S) = u'(c_1^S) \geq u'(c_1^N) = R^{t-1} x_t^N u'(c_t^N) > R^{t-1} x_t^S u'(c_t^N),$$

it follows that $u'(c_t^S) > u'(c_t^N)$, i.e., $c_t^N > c_t^S, \forall t = 2, \cdots, T$. Together with $c_1^N \geq c_1^S$, this contradicts to the zero-profit condition. As a result, naifs must consume less than the sophisticate in the first period (i.e., they save more than sophisticates in the first period).

Hyperbolic discounting. We now move to the case of hyperbolic discounting, where $D_t = \frac{1}{1+kt}$. The agent evaluates a consumption stream $\{c_t\}_{t \geq \tau}$ according to

$$u(c_\tau) + \sum_{t > \tau} \frac{1}{1 + k(t - \tau)} u(c_t). \tag{sD6}$$

The agent is *time-consistent* if $k = 0$ and *time-inconsistent* if $k > 0$. A time-inconsistent agent can be either naive or sophisticated. A *naif* has true time-consistency parameter k but believes that, in the future, he will behave like an agent with time-consistency parameter $\hat{k} \in (0, k)$. A *sophisticate* perfectly knows his time-consistency parameter: $\hat{k} = k$.

Applying Proposition 8, a naif's consumption stream is given by the following program

$$\max_{(c_1,\cdots,c_T)} u(c_1) + \frac{1+(T-2)k}{1+(T-1)k}u(c_2) + \frac{1+(T-3)k}{1+(T-1)k}u(c_3) + \cdots + \frac{1}{1+(T-1)k}u(c_T), \tag{sD7}$$

subject to the zero-profit condition

$$\sum_{t=1}^{T} \frac{c_t}{R^{t-1}} = W,$$

where $W = \sum_{t=1}^{T} \frac{w}{R^{t-1}}$. Note that for any fixed k, $\lim_{T\to\infty} \frac{D_{T-1}}{D_{T-i}} = 1$.

Suppose $(c_1^I, \cdots, c_T^I)$ is the solution to the above program, where the supercript I stands for "time-inconsistent". Denote $(c_1^C, \cdots, c_T^C)$ the time-consistent agent's consumption stream, which is given by

$$\max u(c_1) + u(c_2) + \cdots + u(c_T) \tag{sD8}$$

subject to the zero-profit condition

The welfare is defined as the time consistent agent's utility

$$W_T(c) = u(c_1) + u(c_2) + \cdots + u(c_T).$$

We write $W^I = W_T(c^I)$ and $W^C = W_T(c^C)$ (we suppress the subscript when it does not create confusion).

As the number of periods grows, the welfare defined by the series above is generally divergent.[24] We therefore use the *limit of means* criterion to evaluate welfare. We make the following technical assumption:[25]

[24]For example, $W^C \geq u(w) + \cdots + u(w) = Tu(w)$, which diverges as $T \to \infty$.

[25]A possible alternative would be to assume exponential-hyperbolic preferences, where $D_t = \frac{\delta^{t-1}}{1+kt}$. In that case, the welfare criterion is

$$W_{T,\delta}(c) = \sum_{t=1}^{T} \delta^{t-1} u(c_t),$$

which is well defined if we assume bounded utility and $\delta < 1$. In that case, we do not need Assumption 2 to

Assumption 2. The utility function $u(\cdot)$ satisfies

$$\limsup_{\xi \to 0} \frac{\xi^2 (\log \xi)^2}{|u''((u')^{-1}(\xi))|} < +\infty.$$

Assumption 2 says that the second derivative $u''(\cdot)$ does not go to zero too quickly. It is straightforward to verify that the class of CARA preferences satisfies Assumption 2. Since we focus on bounded utility functions, we can also use our framework with CRRA preferences when risk aversion exceeds one. It can be shown that this class of preferences also satisfies Assumption 2.

We now present the vanishing-inefficiency results for the agents with hyperbolic discounting:

Theorem 3. *Suppose Assumption 2 holds and the utility function $u(\cdot)$ is bounded. Then,*

$$\lim_{T \nearrow \infty} \frac{W^I - W^C}{T} = 0.$$

Proof. Since c^C maximizes the welfare function $W_T(c)$, it immediately follows that $W^I \leq W^C, \forall T$. Thus,

$$\limsup_{T \nearrow \infty} \frac{W^I - W^C}{T} \leq 0. \tag{sD9}$$

Denote $d_{t,T} = \frac{D_{T-1}}{D_{T-t}}, \forall t = 1, \cdots, T$, where we adopt the notation that $D_0 = 1$. The objective function in the naif's auxiliary program becomes

$$\sum_{t=1}^{T} d_{t,T} u(c_t). \tag{sD10}$$

---

show that the welfare of naifs converges to the welfare of time-consistent agents:

$$\lim_{T \nearrow \infty} \left(W_{T,\delta}(c^N) - W_{T,\delta}(c^C) \right) = 0.$$

The proof is similar to the proof of Theorem 3. The main step is that (sD11) becomes

$$W_{T,\delta}(c^N) - W_{T,\delta}(c^C) \geq \sum_{t=1}^{T} \delta^{t-1} \frac{k(t-1)}{1 + k(T-1)} [u(c_t^N) - u(c_t^C)] \geq -2k \max\{u(\cdot)\} \frac{\delta}{(1-\delta)^2 (1 + k(T-1))},$$

which approaches to zero as $T \to \infty$.

It follows that

$$W^I = \sum_{t=1}^{T} u(c_t^N) = \sum_{t=1}^{T} [d_{t,T} u(c_t^N) + (1 - d_{t,T}) u(c_t^N)]$$
$$\geq \sum_{t=1}^{T} [d_{t,T} u(c_t^C) + (1 - d_{t,T}) u(c_t^N)],$$

where the last step comes from the fact that c^N maximizes (sD10) and that c^C is feasible. Rearranging,

$$\begin{aligned} W^I &\geq \sum_{t=1}^{T} \left[u(c_t^C) + (1 - d_{t,T})[u(c_t^N) - u(c_t^C)] \right] \\ &= \sum_{t=1}^{T} \left[u(c_t^C) + \frac{k(t-1)}{1 + k(T-1)} [u(c_t^N) - u(c_t^C)] \right] \\ &= W^C + \sum_{t=1}^{T} \frac{k(t-1)}{1 + k(T-1)} [u(c_t^N) - u(c_t^C)]. \end{aligned} \tag{sD11}$$

We next show a series of lemmas to bound the second term. Denote λ^N the Lagrangian multiplier on the zero-profit condition in the naif's program, and λ^C the Lagrangian multiplier in the time-consistent agent's program.

Lemma 12. *There exist* $\underline{\lambda} > 0$ *and* $\overline{\lambda} > 0$ *such that*

$$\underline{\lambda} \leq \min(\lambda^N, \lambda^C) \leq \max(\lambda^N, \lambda^C) \leq \overline{\lambda}.$$

Proof. From the first-order-condition, we know that

$$\lambda^N = u'(c_1^N), \lambda^C = u'(c_1^C).$$

Note that the first period consumption must be between 0 and $\sum_{t=1}^{\infty} \frac{w}{R^{t-1}} = \frac{w}{1-R}$. The lemma follows immediately by letting $\overline{\lambda} = u'(0)$ and $\underline{\lambda} = u'\left(\frac{w}{1-R}\right)$. □

Lemma 13. *There exists a constant* $A > 0$ *such that* $|t(u(c_t^N) - u(c_t^C))| < A, \forall t, \forall T$.

Proof. From the first-order-condition, we know that

$$\frac{\lambda^N d_{t,T}}{R^{t-1}} = u'(c_t^N), \frac{\lambda^C}{R^{t-1}} = u'(c_t^C)$$

Denote $g(\cdot) = (u')^{-1}(\cdot)$. Inverting above equations to solve for c_t^N and c_t^C,

$$c_t^N = g\left(\frac{\lambda^N d_{t,T}}{R^{t-1}}\right), c_t^C = g\left(\frac{\lambda^C}{R^{t-1}}\right).$$

Note that $\frac{du(g(x))}{dx} = \frac{x}{u''(g(x))}$. Applying Lagrangian Mean Value Theorem, there exists ξ, where $\frac{\min(\lambda^C, \lambda^N d_{t,T})}{R^{t-1}} \leq \xi \leq \frac{\max(\lambda^C, \lambda^N d_{t,T})}{R^{t-1}}$, such that

$$|t(u(c_t^N) - u(c_t^C))| = t\left|u\left(g\left(\frac{\lambda^N d_{t,T}}{R^{t-1}}\right)\right) - u\left(g\left(\frac{\lambda^C}{R^{t-1}}\right)\right)\right| \tag{sD12}$$

$$= t\left|\frac{\xi}{u''(g(\xi))}\left(\frac{\lambda^N d_{t,T}}{R^{t-1}} - \frac{\lambda^C}{R^{t-1}}\right)\right|. \tag{sD13}$$

Let $x = \frac{1}{R^{t-1}}$, then

$$x\underline{\lambda} d_{t,T} \leq x \min(\lambda^C, \lambda^N d_{t,T}) \leq \xi \leq x \max(\lambda^C, \lambda^N d_{t,T}) \leq x\overline{\lambda}.$$

So $x \geq \frac{\xi}{\overline{\lambda}}$. Note that $d_{t,T} = \frac{1+k(T-t)}{1+k(T-1)} \geq \frac{1}{1+k(t-1)}$. So,

$$x \leq \frac{\xi}{\underline{\lambda} d_{t,T}} \leq \frac{\xi(1 + k(t-1))}{\underline{\lambda}} = \frac{\xi(1 - k\frac{\log(x)}{\log R})}{\underline{\lambda}} \leq \frac{\xi(1 - k\frac{\log(\xi) - \log(\overline{\lambda})}{\log R})}{\underline{\lambda}}. \tag{sD14}$$

We can rewrite (sD13) as

$$
\begin{aligned}
|t(u(c_t^N) - u(c_t^C))| &\leq \left(-\frac{\log x}{\log R} + 1\right) \left|\frac{\xi}{u''(g(\xi))}\right| \frac{2\overline{\lambda}}{R^{t-1}} \\
&= \left(-\frac{\log x}{\log R} + 1\right) \left|\frac{\xi}{u''(g(\xi))}\right| 2\overline{\lambda}x \\
&\leq 2\frac{\overline{\lambda}}{\underline{\lambda}} \left(-\frac{\log \xi - \log(\overline{\lambda})}{\log R} + 1\right) \left(1 - k\frac{\log \xi - \log \overline{\lambda}}{\log R}\right) \frac{\xi^2}{|u''(g(\xi))|} \\
&\leq constant * \frac{(\log \xi)^2 \xi^2}{|u''(g(\xi))|},
\end{aligned}
$$

where the first line comes from $t = -\frac{\log x}{\log R} + 1$ and Lemma 12, the second line comes from $x = \frac{1}{R^{t-1}}$, and the third line comes from (sD14). By Assumption 2, there exists $A > 0$ such that $|t(u(c_t^N) - u(c_t^C))| < A$. $\square$

Lemma 14. *There exists a constant $A' > 0$ such that*

$$
\sum_{t=1}^{T} \frac{k(t-1)}{1 + k(T-1)} |u(c_t^N) - u(c_t^C)| < A', \forall T
$$

Proof. Using Lemma 13, it follows that

$$
\begin{aligned}
\sum_{t=1}^{T} \frac{k(t-1)}{1 + k(T-1)} [u(c_t^N) - u(c_t^C)] &\leq \sum_{t=1}^{T} \frac{k(t-1)}{1 + k(T-1)} \frac{A}{t} \\
&\leq \sum_{t=1}^{T} \frac{k}{1 + k(T-1)} A + \sum_{t=1}^{T} \frac{-k}{1 + k(T-1)} A \frac{1}{t} \\
&\leq \frac{kAT}{1 + k(T-1)} + \frac{-k}{1 + k(T-1)} A \log(T).
\end{aligned}
$$

Note that as $T \to \infty$, the first term converges to A, and the second term converges to 0. So there exists a constant $A' > 0$ such that

$$
\sum_{t=1}^{T} \frac{k(t-1)}{1 + k(T-1)} [u(c_t^N) - u(c_t^C)] < A', \forall T.
$$

$\square$

Returning to (sD11), we have

$$\liminf_{T\nearrow\infty} \frac{W^I - W^C}{T} \geq -\liminf_{T\nearrow\infty} \frac{A'}{T} = 0. \tag{sD15}$$

Together with (sD9), it implies that $\lim_{T\nearrow\infty} \frac{W^I - W^C}{T}$ exists, and

$$\lim_{T\nearrow\infty} \frac{W^I - W^C}{T} = 0.$$

This completes the proof. □

E. Removing Commitment Power for Sophisticates

In this appendix, we show that, for a fixed contract length, removing commitment power can make the sophisticated consumer better off. The intuition for this result is that commitment power allows the consumer to borrow, and time-inconsistent consumers are tempted to over-borrow. Thus, the welfare effect of removing commitment depends whether the welfare gain from being able to borrow is outweighed by the welfare loss from over-borrowing. The over-borrowing problem is more severe if the consumer is very time inconsistent (β is small) and saving is important for welfare (δ is large). In that case, removing commitment power increases welfare.

Proposition 9. *Suppose the consumer is sophisticated. There exists $\bar{\beta} > 0$ and $\bar{\delta} < 1$ such that, if $\beta < \bar{\beta}$ and $\delta > \bar{\delta}$, the welfare with one-sided commitment is greater than the welfare with two-sided commitment.*

Proof. Fix an equilibrium with two-sided commitment. Because when $\beta = 0$, consuming in any period other than in the initial period is costly and does not increase the agent's utility, the agent consumes all expected PDV of income in the first period:

$$c(s_1) = \sum_{t=1}^{T} \sum_{s_t} p(s_t|s_1) \frac{w(s_t)}{R^{t-1}},$$

and $c(s_t) = 0$ for all $s_t \neq s_1$.

Next, fix an equilibrium with one-sided commitment. Let $x(s_t) = u(c(s_t))$. By the dual program, there exists some utility level $\underline{u}$ to the agent, for which $\{x(s_t)\}$ solves the program:

$$\max_{\{x(s_t)\}} \sum_{t=1}^{T} \sum_{s_t \in \mathbb{S}_t} p(s_t|s_1) \frac{w(s_t) - u^{-1}(x(s_t))}{R^{t-1}}, \tag{sE1}$$

subject to

$$x(s_1) + \beta \sum_{t=2}^{T} \sum_{s_t \in \mathbb{S}_t} \delta^{t-1} p(s_t|s_1) x(s_t) \geq \underline{u}, \tag{sE2}$$

and

$$x(s_{\tilde{\tau}}) + \beta \sum_{t>\tilde{\tau}} \sum_{s_t \in \mathbb{S}_t} \delta^{t-\tilde{\tau}} p(s_t|s_{\tilde{\tau}}) x(s_t) \geq V^S(s_{\tilde{\tau}}) \ \ \forall s_{\tilde{\tau} \in \mathbb{S}_\tau}, \forall \tau, \tag{sE3}$$

where the outside option $V^S(s_t)$ is the utility of the best contract that the agent can obtain by signing a new contract at state s_t:

$$V^S(s_\tau) \equiv \max_{c(s_\tau), \{c(s_t): s_t \in \mathbb{S}_t\}} u(c(s_\tau)) + \beta \sum_{t>\tau} \sum_{s_t \in \mathbb{S}_t} \delta^{t-\tau} p(s_t|s_\tau) u(c(s_t)),$$

subject to

$$\sum_{t \geq \tau} \sum_{s_t \in \mathbb{S}_t} p(s_t|s_\tau) \frac{w(s_t) - c(s_t)}{R^{t-\tau}} = 0,$$

$$u(c(s_{\tilde{\tau}})) + \beta \sum_{t>\tilde{\tau}} \sum_{s_t \in \mathbb{S}_t} \delta^{t-\tilde{\tau}} p(s_t|s_{\tilde{\tau}}) u(c(s_t)) \geq V^S(s_{\tilde{\tau}}) \ \ \forall s_{\tilde{\tau}} \in S_{\tilde{\tau}}(s_\tau),$$

for $\tilde{\tau} = \tau + 1, ..., T$.

The optimal solution is continuous in $\beta \in [0, 1]$. When $\beta = 0$, the agent would like to consume in the current period as much as possible. Applying backward induction starting from states in period $T - 1$, we find that the renegotiation proofness constraints bind in all continuation programs, so that $c(s_{T-1}) = w(s_{T-1})$ and $c(s_T) = w(s_T)$ for the outside options. Proceeding backwards, it follows that the solution of this program features $c(s_t) = w(s_t)$ in all states. That is, with $\beta = 0$, the agent would like to borrow as much as possible. But firms know that, after borrowing, the agent would prefer to drop the contract instead of repaying, so they are not willing to lend. So, in equilibrium, the agent consumes his income in all states.

Comparing the solutions under one- and two-sided commitment, we find that the agent's welfare is higher with one-sided commitment consuming the endowment in every state than consuming the expected PDV of all income right away and consuming zero in all future periods if the following condition holds:

$$\sum_{t=1}^{T}\sum_{s_t}\delta^{t-1}p(s_t|s_1)u\left(w(s_t)\right) > u\left(\sum_{t=1}^{T}\sum_{s_t}p(s_t|s_1)\frac{w(s_t)}{R^{t-1}}\right),$$

By Jensen's inequality, this condition is satisfied if $R\delta \geq 1$. Because of the continuity and $R \geq 1$, there exists $\bar{\beta}_S > 0$ and $\bar{\delta}_S \equiv \frac{1}{R} < 1$ such that, if $\beta < \bar{\beta}_S$ and $\delta > \bar{\delta}_S$, the welfare with one-sided commitment dominates the welfare with two-sided commitment. We have therefore established Proposition 9. □

F. Maximum Fee with Sophisticated Agents

In this appendix, we show that a maximum fee (i.e., a minimum consumption policy) has an ambiguous welfare effect when the agent is sophisticated. To show this ambiguity, we present two simple examples with opposite welfare effects of such a policy.

Suppose $T = 3, \beta = 0.9, R = 1.1, \delta = 1, w = 1, u(c) = \log(c + \epsilon)$, where ϵ is a very small positive number. We first solve the problem without the mandate. We can find the equilibrium contract by solving $u'(c_1) = \beta R u'(c_2) = \beta R^2 u'(c_3)$, which gives $c_1^1 = 0.977, c_2^1 = 0.9672, c_3^1 = 1.0639$. We can verify that all the constraints hold:

$$c_1^1 + \frac{c_2^1}{R} + \frac{c_3^1}{R^2} = 2.7355 = 1 + \frac{1}{R} + \frac{1}{R^2}, \tag{sF1}$$

$$u(c_2^1) + \beta u(c_3^1) = 0.0224 > 0 = V_2^S. \tag{sF2}$$

Thus, the welfare without the mandate is $W_1 = u(c_1^1) + u(c_2^1) + u(c_3^1) = 0.0053$.

Example of the mandate increasing welfare: Now consider the problem with the mandate $\underline{c} = 0.97$. Then $c_2^2 = \underline{c} = 0.97$. Solving $\max u(c_1) + \beta u(c_3)$ gives $c_1^2 = 0.9756$ and $c_3^2 = 1.0625$. We can verify that all the constraints hold:

$$c_1^2 + \frac{c_2^2}{R} + \frac{c_3^2}{R^2} = 2.7355 = 1 + \frac{1}{R} + \frac{1}{R^2}, \tag{sF3}$$

$$u(c_2^2) + \beta u(c_3^2) = 0.0241 > 0 = V_2^S. \tag{sF4}$$

The welfare with the mandate is $W_2 = u(c_1^2) + u(c_2^2) + u(c_3^2) = 0.0055$. Thus, the mandate strictly increases welfare.

Example of the mandate decreasing welfare: Assume that now $\underline{c} = 0.98$. So $c_1^2 \geq \underline{c} > c_1^1$. We can prove that the mandate strictly decreases welfare by the following:

$$\begin{aligned} u(c_1^1) + u(c_2^1) + u(c_3^1) &= \frac{1}{\beta}\left(u(c_1^1) + \beta(u(c_2^1) + u(c_3^1))\right) - \frac{1-\beta}{\beta}u(c_1^1) \\ &\geq \frac{1}{\beta}\left(u(c_1^2) + \beta(u(c_2^2) + u(c_3^2))\right) - \frac{1-\beta}{\beta}u(c_1^1) \\ &> \frac{1}{\beta}\left(u(c_1^2) + \beta(u(c_2^2) + u(c_3^2))\right) - \frac{1-\beta}{\beta}u(c_1^2) \\ &= u(c_1^2) + u(c_2^2) + u(c_3^2), \end{aligned} \tag{sF5}$$

where the first inequality comes from the fact that (c_1^1, c_2^1, c_3^1) is the solution without the mandate, and the second inequality comes from that $c_1^1 < c_1^2$.

G. Non-Exclusive Contracts

The equilibrium studied in the main text involved consumers turning down baseline options with high future payoffs for alternatives that pay much less in total but benefit the impatient with a little extra in the current period. In this appendix, we assume a consumer can keep the baseline option and borrow from a third party, i.e., there is a possibility of additional contracting. In this appendix, we will show that there is no role for firms, and the equilibrium consumption coincides with the equilibrium consumption in a consumption-saving problem.

Similar to the main text, for the purpose of characterizing the equilibrium consumption, there is no loss of generality in restricting attention to contracts in which the consumer never contracts with another principal. To see this, consider an equilibrium in which the consumer recontracts in some state of the world, signing another contract from another firm. Since the other firm cannot lose money by offering this new contract, the old firm could have accepted a contract that gives the consumers the consumption according to the terms of the new contract from this period on, and the consumer would have remained with the old firm. So, to characterize the consumption that can be supported in equilibrium, we can impose *no additional contracting constraints* which require that the consumer does not have an incentive to recontract with another firm. These constraints state the consumer's value by signing another contract at each state cannot exceed the value from keeping the current contract, where the value is calculated by the best possible contract that other firms are willing to provide.

An *equilibrium consumption vector* is the vector of state-dependent consumption in all option histories for all states that happen with positive probability:

$$\mathbf{c} \equiv \{(c(s_1), c(s_2, h^2), c(s_3, h^3), \cdots, c(s_T, h^T)) : \; p(s_2|s_1)p(s_3|s_2) \cdots p(s_T|s_{T-1}) > 0\}.$$

A *consumption on the equilibrium path* is a vector of state-contingent consumption that happens with positive probability (using correct beliefs about the options that the consumer chooses):

$$\mathbf{c}^E \equiv \{(c(s_1), c(s_2, A), c(s_3, A, A), \cdots, c(s_T, A, ..., A)) : \; p(s_2|s_1)p(s_3|s_2) \cdots p(s_T|s_{T-1}) > 0\}.$$

Unlike the equilibrium consumption vector, the consumption on the equilibrium path only includes outcomes conditional on the consumer repeatedly picking option A.

We first establish the following lemma when there is a possibility of additional contracting.

Lemma 15. *Suppose* **c** *is an equilibrium consumption vector when there is a possibility of additional contracting. Fixed* (s_t, h^t)*. Then the expected present discounted values of consumption on any path starting with* (s_t, h^t) *are same.*

Proof. To prove the lemma, we argue by contradiction. There exist two paths of consumption stream starting with (s_t, h^t) that have different expected present discounted values. Without loss of generality, assume that one path, denoted as $\hat{\mathbf{c}}$, has a higher expected present value than the other path, denoted as $\tilde{\mathbf{c}}$.

We note that since **c** is the equilibrium consumption vector, it must satisfy the *no additional contracting constraints*. Given that the present value of $\hat{\mathbf{c}}$ is higher than the present value of $\tilde{\mathbf{c}}$. There are two possibilities, either $\tilde{\mathbf{c}}$ starts with the baseline option or $\tilde{\mathbf{c}}$ starts with the alternative option. In either case, we show that the no additional contracting constraints would be violated. First, suppose $\tilde{\mathbf{c}}$ starts with the baseline option. In this case, the baseline option would not be the consumer's perceived consumption, because the consumer perceives that he has an incentive to recontract with another firm, who can give the consumer a slightly higher consumption in the baseline option. Specifically, consider another contract **c'**, which has the same term as **c** except that we increase ϵ in the consumption in the baseline option of $\tilde{\mathbf{c}}$. Similarly, if $\tilde{\mathbf{c}}$ starts with the alternative option, the consumer can recontract with another firm, who gives him a slightly higher consumption in the alternative option. □

To see the proof more clearly, we write down concretely in the case that $T = 3$. Similar to the main text, the equilibrium contract c specifies the following consumption

$$(c_1(s_1), c_2(s_2, A), c_2(s_2, B), c_3(s_3, A), c_3(s_3, B)),$$

We claim that given any state s_2, the expected present discounted value of the perceived consumption stream and the expected present discounted value of the actual consumption stream must equal:

$$c_2(s_2, B) + \frac{E[c_3(s_3, B)|s_2]}{R} = c_2(s_2, A) + \frac{E[c_3(s_3, A)|s_2]}{R}. \tag{sG1}$$

To see this, we define the following programs

$$V(c_2, c_3(s_3)) = \max_{(c_2', c_3')} u(c_2') + \beta\delta E[u(c_3'(s_3))|s_2], \quad \hat{V}(c_2, c_3) = \max_{(c_2', c_3')} u(c_2') + \hat{\beta}\delta E[u(c_3'(s_3))|s_2],$$

subject to

$$c_2' + \frac{E[c_3'(s_3)|s_2]}{R} = c_2 + \frac{E[c_3(s_3)|s_2]}{R}.$$

Note that in the above programs, the values only depend on the expected present discounted value of the consumption stream available to the consumer: $c_2 + \frac{E[c_3(s_3)|s_2]}{R}$. To show (sG1) holds, we argue by contradiction. For example, suppose the baseline option has a higher present value: $c_2(s_2, B) + \frac{E[c_3(s_3,B)|s_2]}{R} > c_2(s_2, A) + \frac{E[c_3(s_3,A)|s_2]}{R}$. Then we must have

$$V\left(c_2(s_2, B), c_3(s_3, B)\right) > V\left(c_2(s_2, A), c_3(s_3, A)\right),$$
$$\hat{V}\left(c_2(s_2, B), c_3(s_3, B)\right) > \hat{V}\left(c_2(s_2, A), c_3(s_3, A)\right).$$

In other words, with the possibility of additional contracting, if the perceived path has a higher present value, then the consumer and the firm actually agree that the consumer will take the baseline option, contrary to that the consumer and the firm disagree. Another way to put this is that the consumer will recontract with another firm at time 2, contradicted to the no additional contracting constraints.

We next establish an equivalence between our problem and a consumption-saving problem. In the consumption-saving problem, there is no firms, and the consumer can freely save or borrow at an interest rate R and he decides how much to consume and save every period. In this problem, recall that the consumer mispredicts his future consumption/saving decision, so he plays a game with future selves. The strategy space consists of the consumption decision in each period. The equilibrium notion is still the SPNE (Subgame Perfect

Nash Equilibrium). Denote the equilibrium as $\bar{\mathbf{c}}$.

Proposition 10. *The consumption on the equilibrium path in the model with the possibility of additional contracting is exactly same as the consumption in the consumption-saving problem, i.e.,*

$$\boldsymbol{c}^E = \bar{\boldsymbol{c}}.$$

Proof. It is sufficient to show that the first period consumption is same: $c_1^E = \bar{c}_1$. Once the first period consumption is determined, the rest of the program is same except now there are $(T-1)$ periods. We show the the two programs are exactly the same. For the ease of exposition, we assume the income is a constant w.

We first consider the model with the possibility of additional contracting. The equilibrium program is

$$\max_{\{c(h^t)\}} u(c_1) + \beta E\left[\sum_{t=2}^{T} \delta^{t-1} u(c_t(B, B, ..., B))\right],$$

subject to the zero-profit condition, perceived-choice constraints, incentive constraints, and no additional contracting constraints. We can replace perceived-choice constraints with monotonicity conditions that say the consumption in the baseline option is lower than the consumption in the alternative option. Lemma 15 implies that the firm makes same profit on any path starting with h^t. We call them the same-profit conditions. Together with no additional contracting constraints, incentive constraints trivially hold. So incentive constraints can be disregarded. Having established that, we focus on the problem for c_1. Note that the perceived consumption stream must also give the firm zero-profit because of Lemma 15. We write $\hat{c}_t = c_t(B, B, ..., B)$ for short. So the program for c_1 is given by

$$V_1(W_1) = \max_{c_1} u(c_1) + \beta E\left[\sum_{t=2}^{T} \delta^{t-1} u(\hat{c}_t)\right],$$

subject to

$$
\begin{aligned}
& c_1 + \sum_{t=2}^{T} \frac{\hat{c}_t}{R^{t-1}} = W_1, \\
& \hat{c}_\tau = \arg\max \hat{V}_\tau(W_\tau), \forall \tau \geq 2, \\
& W_{t+1} = R(W_t - \hat{c}_t), \forall t > 1,
\end{aligned}
$$

where $\hat{V}_\tau(\cdot)$ is recursively defined as follows,

$$
\hat{V}_\tau(W_\tau) = \max_{\hat{c}_\tau} u(\hat{c}_\tau) + \hat{\beta} E \left[\sum_{t=\tau+1}^{T} \delta^{t-1} u(\hat{c}_t) \right],
$$

subject to

$$
\begin{aligned}
& \sum_{t=\tau}^{T} \frac{\hat{c}_t}{R^{t-1}} = W_\tau, \\
& \hat{c}_{\tau'} = \arg\max \hat{V}_{\tau'}(W_{\tau'}), \forall \tau' \geq \tau + 1, \\
& W_{t+1} = R(W_t - \hat{c}_t), \forall t > \tau.
\end{aligned}
$$

This program is exactly the program for the consumption-saving problem, in which the consumer chooses to consume and save in every period. As a result, the consumption on the equilibrium path in the model with the possibility of additional contracting is exactly same as the consumption in the consumption-saving problem. □