Prioritizing Diversity in School Choice*

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Abstract

Promoting diversity in schools has recently emerged as an important policy goal. Typically school choice programs take into account student preferences and allocate scarce schools on the basis of priorities, using stability as the solution concept. Therefore a notion of prioritizing diversity is essential. We introduce a rich class of priorities which capture intuitive notions of diversity. *Substitutable priorities with ties* not only ensure existence of stable assignments, but also allow students of same types to be treated equally. Moreover we describe an algorithm which finds an optimal stable assignment.

1 Introduction

School choice has been a fruitful policy application of matching theory. In many school districts, school assignment takes into account student preferences. Overdemanded schools are allocated on the basis of priorities, e.g., those who live within the walk-zone of the school have higher priority for that school. Since Abdulkadiroğlu and Sönmez (2003), a growing literature has incorporated such concerns into tractable models with great success not only in theory, but also in applications. On the other hand, the mechanism design approach

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to school choice has paid relatively little attention to composition of school populations, while promoting diversity is becoming increasingly topical as a policy issue. For example, in December 2011, US Department of Justice announced[1] that

> "the US Departments of Justice and Education released two new guidance documents – one for school districts[2] and one for colleges and universities – detailing the flexibility that the Supreme Court has provided to educational institutions to promote diversity and, in the case of elementary and secondary schools, reduce racial isolation among students within the confines of the law.
>
> "The guidance makes clear that educators may permissibly consider the race of students in carefully constructed plans to promote diversity or, in K-12 education, to reduce racial isolation. It recognizes the learning benefits to students when campuses and schools include students of diverse backgrounds."

This perspective raises various questions for both policy makers and economists. At a practical level, definition of diversity, what it means to promote it, and the policies to achieve related goals come with numerous ethical and legal challenges. From a theoretical perspective, no obvious formalization of diversity stands out. Vast majority of the school choice literature relies on priorities which rank individual students. For example between two students applying to the same school, the one who has a sibling already registered at that school would have the priority. However, unlike comparisons of individuals based on whether one has a sibling or not, or whether she lives within the walk-zone or not, diversity is a property of groups. Usually it is not possible to extend rankings over individuals to orders over sets to have an intuitive comparison in terms of diversity. This necessitates an approach to priorities distinctly more general than what most of the school choice literature has used.

We introduce a class of choice rules, *substitutable priorities with ties*, to guide which students a school should admit from any given set of applicants. These choice rules generalize *substitutability* to allow indifferences (ties) such they are able that capture various natural notions of diversity while accommodating equal treatment of students. Our interpretation of *respecting priorities* is, as is commonly done in theory and practice, via *stability* with respect to school priorities and student preferences. We show that stable assignments exist, and we

[1] See http://www.justice.gov/opa/pr/2011/December/11-ag-1569.html – accessed on 10 October 2012.

[2] "Guidance on the voluntary use of race to achieve diversity and avoid racial isolation in elementary and secondary schools." See http://www.ed.gov/news/press-releases/new-guidance-supports-voluntary-efforts-promote-diversity-and-reduce-racial-isol – accessed on 10 October 2012.

describe a *modified deferred acceptance algorithm* (in the spirit of Gale and Shapley, 1962; Kelso and Crawford, 1982; and Hatfield and Milgrom, 2005) to find one. Then to find an optimal assignment, we describe *stable improvement algorithm* in the spirit of Erdil and Ergin (2008). While our notions and constructions are reminiscent of these earlier approaches, we highlight why they are not readily generalizable to our environment.

In our main application, we compare sets with respect to their distance from what a policy maker might consider an ideal composition or distribution of types at a school. That is, given an exogenously fixed distribution of types, we define measures of distance between an arbitrary set and this *target* distribution. Between two competing sets, the one that is closer to that target distribution is deemed more diverse. Consider, for example, a school with two available seats, and six applicants consisting of two Asian, two black, and two white students. Prioritizing diversity, but otherwise treating everyone equally implies any mixed pair of students should be of high priority, whereas any other pair would be of low priority. In other words, sets can be compared according to their distance from the target distribution $(1, 1, 1)$. As we explain in Section 2 such a priority order does not fit into any model previously studied in the literature,[3] but it is *substitutable with ties*. Going further, in Section 5, we show how our model captures a large class of choice rules which prioritize diversity in a natural way.

Ours is not the first study in designing school choice mechanisms with a view towards diversity. The literature discusses a number of affirmative action policies, such as controlled school choice or majority quotas. Some interpretations of promoting diversity might lead to inefficiencies without contributing towards diversity. For example, insisting on a rigid notion of diversity and ruling out all assignments that violate such a notion might lead to wasteful assignments as in Abdulkadiroglu (2005). Or as Kojima (2010) points out, using majority quotas (upper limits on how many non-minority students are allocated seats) might have the opposite effect to policy maker's intentions: even the minority students, who are meant to benefit from this *affirmative action* policy, might end up worse off compared with an allocation mechanism which relaxes quota requirements. These issues are alleviated by instead using minority quotas as in Hafalir, Yenmez, Yildirim (2012), Ehlers et al. (2012), or slot specific priorities as in Kominers and Sönmez (2012). Echenique and Yenmez (2013) axiomatize an approach very similar to what we have called *prioritizing a target distribution*. They show

[3] Within the framework of Section 5, this example would correspond to having non-integer target distribution $(2/3, 2/3, 2/3)$, which is equivalent to allowing the total number of seats to be different from the sum of quotas, and setting the target distribution as $(1, 1, 1)$, where the number of seats is two.

that priority rules that satisfy a number of appealing properties have to be of a particular form which involves ranking sets in terms of their *distance* to an ideal distribution of types.

2 Going beyond responsive priorities

The initial mechanism design approach to school choice (Abdulkadiroğlu and Sönmez, 2003) employed a matching model similar to Gale and Shapley's (1962) college admissions model. Each school is endowed with a certain number of seats and an exogenous priority order over the set of students. For a matching of students to schools to be *stable*, a student i must not be left envying another student j at school x, if i has higher priority for x than j. If these priorities can be captured by strict rankings over students, interpreting such rankings as schools' preferences brings us back to the college admissions model. Later models incorporate ties in priority orders, and begin with a weak order (i.e., a complete, transitive binary relation) on the set of students. Though the priorities are orders on individual students, effectively they act as *responsive* choice rules over the sets of students. Responsiveness is a widely studied (see, e.g., Roth and Sotomayor, 1990) property of rankings over sets. A ranking over sets is *responsive* to a preference ranking over individuals if for any two sets that differ in only one student, the set containing the student with [weakly] higher priority is ranked [weakly] higher.[4] When schools' rankings over sets of students are responsive, stability is equivalent to pairwise stability.

It is not hard to see that the priority order in our example in the introduction is not responsive to any order on students. Recall that there are six students in $N = A \cup B \cup W$, where $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and $W = \{w_1, w_2\}$. Apart from a diversity policy, they are to be "treated equally":

$$\{w_i, b_j\} \sim \{w_i, a_k\} \sim \{b_j, a_k\} \succ \{w_1, w_2\} \sim \{b_1, b_2\} \sim \{a_1, a_2\} \succ \cdots,$$

where $i, j, k \in \{1, 2\}$. If $\succsim$ were responsive to an order $\succsim'$ on $\{w_1, w_2, b_1, b_2, a_1, a_2\}$, we would have

$$\{w_1, b_1\} \succ \{w_1, w_2\} \quad \Longrightarrow \quad b_1 \succ' w_2$$

and

$$\{w_2, b_2\} \succ \{b_1, b_2\} \quad \Longrightarrow \quad w_2 \succ' b_1,$$

[4]Formally, a ranking $\succsim$ over the sets of students is said to be *responsive* to a ranking $\succsim'$ over the set of students if whenever $i \succsim' j$, we have $\{i\} \cup S \succsim \{j\} \cup S$ for any S.

hence a contradiction.

It is worth noting that the priority structure above cannot be mimicked via treating each school as a hypothetical combination of sub-schools each of prioritizing different types. Such an approach, which has been discussed as *type specific quotas* in the literature, would require each sub-school to rank students of a particular ethnicity over others. In the above example, in whatever way the two sub-schools with one seat each rank students, it will be impossible to treat students equally while prioritizing diversity.

Kelso and Crawford (1983) introduced a condition, *substitutability*, on set rankings which is significantly more permissive than responsiveness. In this much wider class of priorities, a generalized version of Gale and Shapley's Deferred Acceptance algorithm finds a stable matching, and many other attractive properties carry over from the responsive domain to substitutable domain. In order to capture a notion of promoting diversity, we will go in that same direction. But we will extend even more to also incorporate equal treatment, which necessitates set rankings to allow non-trivial indifferences, or ties. Such a generalization turns out to be more than straightforward, and requires us to reformulate the original definitions carefully, and develop novel methods to derive our results.

3 Preliminaries

Let N be a set of students, and X be a set of schools. There are q_x seats at school x, for $x \in X$. Each student can receive at most one seat, and the allocation has to *respect exogenously given priorities*, a notion formalized below.

Each school x is endowed with a *priority rule* that determines which subset of applying students receive a place at that school when there are more applicants than the seats available. Formally, a **priority rule** is a choice correspondence which associates to each subset S of the students the collection of subsets of S which may be the set of students assigned to x. That is, $\mathcal{C}_x : 2^N \to 2^{2^N}$ such that $S' \subseteq S$ for all $S' \in \mathcal{C}_x(S)$. We impose a simple *consistency* property on the choice rules we will study. Namely, (1) if a set A is chosen instead of another set B when they are both available in one instance, then the set B should not be chosen whenever A is available; (2) if both A and B are chosen in one instance, then they should both be chosen, or both be rejected whenever they are both available. Formally,

(1) If there exists S such that $A \cup B \subseteq S$ with $A \in \mathcal{C}_x(S)$ and $B \notin \mathcal{C}_x(S)$, then whenever $A \cup B \subseteq S'$, we have $B \notin \mathcal{C}_x(S')$.

(2) If there exists S such that $A, B \in \mathcal{C}_x(S)$, then whenever $A \cup B \subseteq S'$, we have $A \in \mathcal{C}_x(S')$ if and only if $B \in \mathcal{C}_x(S')$.

Given a choice rule $\mathcal{C}_x$ with the above consistency property, we can define the associated relations $\succ_x$ and $\sim_x$ as

$$\begin{aligned}&\text{If } A \in \mathcal{C}_x(S) \text{ and } B \notin \mathcal{C}_x(S) \text{ for some } S, \quad \text{then} \quad A \succ_x B,\\ &\text{if } A \in \mathcal{C}_x(S) \text{ and } B \in \mathcal{C}_x(S) \text{ for some } S, \quad \text{then} \quad A \sim_x B.\end{aligned}$$

We write $A \succsim_x B$ if $A \succ_x B$ or $A \sim_x B$.[5] Clearly the relations $\succ_x$, $\sim_x$ and $\succsim_x$ depend on $\mathcal{C}_x$, but we will suppress such dependence for notational convenience. We will call $\succsim_x$ a **priority order**.[6] A **priority structure** $\mathcal{C}$ is a vector of priority rules $(\mathcal{C}_x)_{x \in X}$.

If $S' \in \mathcal{C}_x(S)$, then we refer to S' as **a chosen set** from among the applicants S. Given $\mathcal{C}_x$ and a set S, we define the set of **definitely chosen students** as

$$DC_x(S) \quad = \bigcap_{S' \in \mathcal{C}_x(S)} S' \quad = \quad \{i \in S \mid i \in S' \text{ for all } S' \in \mathcal{C}_x(S)\}.$$

Note that $DC_x(S)$ can be empty.

Now we are ready to introduce our notion of *substitutability with ties*.

Definition 1 A priority structure is **substitutable** (with ties) if for each $x \in X$, and for each $S, T \subseteq N$ with $S \subseteq T$, the following conditions hold

(a) for each $T' \in \mathcal{C}_x(T)$, we have $T' \cap S \subseteq S'$ for some $S' \in \mathcal{C}_x(S)$, and

(b) for each $S' \in \mathcal{C}_x(S)$, we have $T' \cap S \subseteq S'$ for some $T' \in \mathcal{C}_x(T)$.[7]

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[5]We do not require $\succ_x$ nor $\succsim_x$ to be transitive nor complete.

[6]Note that $\mathcal{C}_x(S) = \{S' \subseteq S \mid \; S' \succsim_x S'' \;\text{ for all }\; S'' \subseteq S\}$.

[7]It is helpful to define the **rejection correspondence** $\mathcal{R}_x$, which associates to each $S \subseteq N$, the family of subsets of S which can be rejected from among S. That is,

$$\mathcal{R}_x(S) = \{S'' \subseteq S \mid S'' = S \backslash S' \text{ for some } S' \in \mathcal{C}_x(S)\}.$$

As shown in Remark 2 in the Appendix, condition (b) of Definition 1 can be rewritten as

(b') for each $S'' \in \mathcal{R}_x(S)$, we have $S'' \subseteq T''$ for some $T'' \in \mathcal{R}_x(T)$.

It is intuitive to refer to conditions (a) and (b) as the *monotonicity* of the acceptance and rejection correspondences, respectively. Note that if $\mathcal{C}_x$ is a function for each $x \in X$, then conditions (a) and (b) in Definition 1 are equivalent.

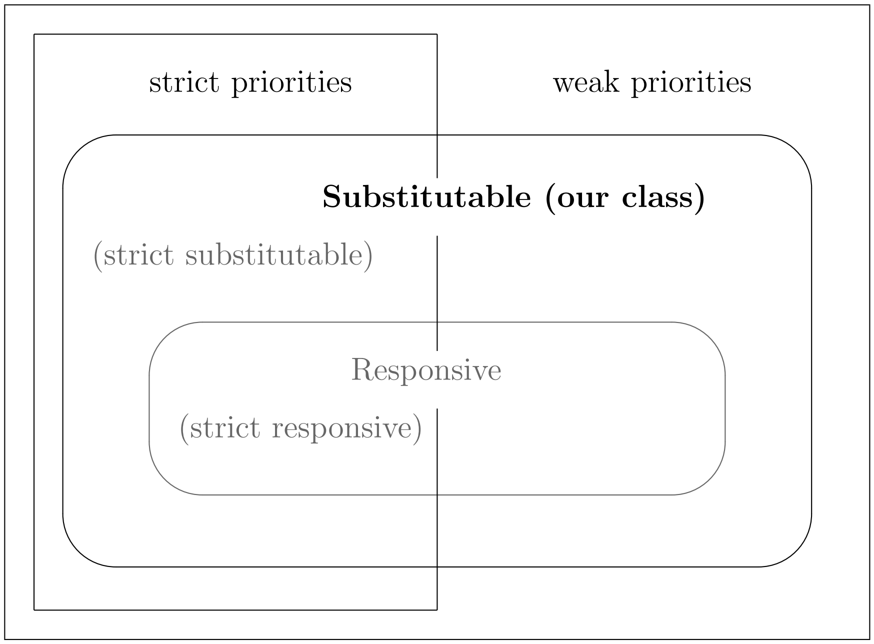

Figure 1: Our class is strictly larger than both strict substitutable priorities and responsive priorities with ties.

This definition covers various environments studied in the literature. For example, responsive preferences over sets of doctors (Roth, 1984), the school choice formulation of Abdulkadiroğlu and Sönmez (2003), and the school priorities with ties (e.g., Ehlers, 2007) are all special cases.[8] One contribution of this paper is that our generalization goes beyond, and covers natural priority structures which are not captured by any of the aforementioned models. For instance, recall the priority order specified in our motivating example

$$\{a, b\} \sim \{a', w\} \sim \{b', w'\} \quad \succ \quad \{a, a'\} \sim \{b, b'\} \sim \{w, w'\}.$$

This order is not responsive to any ranking (strict or weak) over the set of students. Nor is it strict substitutable as it involves ties. But it is substitutable in the way we defined, and therefore satisfies the attractive properties that will follow from substitutability. Moreover,

[8]Our definition is easily extended to allow different contracts between a student and a school. Each school is endowed with a priority rule (a choice correspondence over contracts) which satisfies the appropriate consistency properties, and we get a generalization of Kelso and Crawford (1982) and Hatfield and Milgrom (2005). Kelso and Crawford (1982) use formulation (a), whereas Hatfield and Milgrom (2005) use formulation (b'). While these conditions are equivalent for the settings of those papers, in our generalized environment they do not imply each other any more, as shown in Remark 3.

in Section 5 we will see that this particular example is a specific instance of a large class of priority orders which satisfy our substitutability notion.

We define an **assignment** μ to be a function $\mu : N \to X \cup N$ such that

- $\forall i \in N, \quad \mu(i) \in X \cup \{i\}$, and
- $\forall x \in X, \quad |\mu^{-1}(x)| \leq q_x$.

Each student i has a strict preference ranking R_i over $X \cup \{i\}$, where receiving i is interpreted as getting one's outside option. P_i denotes the strict part of R_i. Given a preference profile $R = (R_i)_{i \in N}$, we have a Pareto domination relation over all possible assignments.

Given students' preferences R, an assignment μ **respects priorities** $\mathcal{C}$, if

- for each $i \in N$, $\mu(i) R_i i$, and
- for each $x \in X$, for every S such that $\mu^{-1}(x) \subseteq S \subseteq \{i \in N \mid x R_i \mu(i)\}$, we have $\mu^{-1}(x) \in \mathcal{C}_x(S)$.

The definition captures the idea that no subset of students who weakly prefer x to their assigned school has higher priority than those students currently assigned to x.[9]

An assignment is called **pairwise stable** if it is not blocked by an individual student, or an individual school, or a student-school pair. That is, (1) each student i prefers her match to being unassigned; (2) each school prefers not to get rid of some of the assigned students; and (3) there is no student-school pair who are not matched, but would rather be matched. Formally,

(1) $\mu(i) R_i i$ for all $i \in N$, i.e., μ is **individually rational**,

(2) $\mu^{-1}(x) \in \mathcal{C}_x(\mu^{-1}(x))$ for all $x \in X$,

(3) there is no $(i, x) \in N \times X$ such that $x P_i \mu(i)$ and $i \in DC_x(\mu^{-1}(x) \cup \{i\})$.

[9]One can interpret the priority rules as schools' preferences, but we should note that stability does not correspond to either the *core* or the *weak core*. In order to see this, let there be a single school x, and three students 1, 2, 3. Let $\mathcal{C}_x(\{1,2,3\}) = \{\{1,2\},\{1,3\}\}$, and x be acceptable to all the students. The matching $(1x, 22)$ is not strongly blocked by any coalition, and therefore is in the weak core, but it is clearly not stable. Secondly $(1x, 2x)$ is stable, but is not in the core, because the coalition $\{x, 1, 3\}$ can block to form $(1x, 3x)$. This is in contrast with the class of strict substitutable priorities for which stability is equivalent to being in the core.

In Gale and Shapley's (1962) original formulation the property of respecting priorities was called **stability**. The term is perhaps more appropriate when schools' priorities are interpreted as their preferences. Nevertheless, staying faithful to the literature, from now on, we will call an assignment which respects priorities a **stable** assignment. Stability implies pairwise stability.[10] The following proposition says that the converse holds for substitutable priority structures.

Proposition 1 *Let $\mathcal{C}$ be a substitutable priority structure. Then an assignment is pairwise stable if and only if it respects priorities.*

Below, we show that a natural extension of Gale and Shapley's Deferred Acceptance Algorithm finds an assignment which respects priorities if the priority structure is substitutable.

Modified Deferred Acceptance Algorithm (MDA)

> **Round 1:** All students apply to their favorite schools. For each school x, if A_x^1 is the set of applicants, *an* element S_x^1 in $\mathcal{C}_x(A_x^1)$ is declared temporary winners, and the rest of the applicants, denoted $Z_x^1 = A_x^1 \backslash S_x^1$ are rejected.
>
> $\vdots$
>
> **Round t:** Those who were rejected in round $t-1$, apply to their next favorite school. For each school x, if A_x^t is the set of all students who have applied to x so far, a set of temporary winners $S_x^t \in \mathcal{C}_x(A_x^t)$ is chosen such that $Z_x^{t-1} \subseteq A_x^t \backslash S_x^t$.
>
> When every student is either matched with a school or has been rejected by all schools in his list, the algorithm ends.

Proposition 2 *Given a substitutable priority structure $\mathcal{C}$, the Modified Deferred Acceptance Algorithm returns a stable assignment.*

[10] Property (1) is part of the definition of respecting priorities. Given that for each $x \in X$, we have $\mu^{-1}(x) \in \mathcal{C}_x(\{i \mid xR_i\mu(i)\})$, it remains to verify (2) and (3). Suppose (2) did not hold. Then $\mu^{-1}(x) \notin \mathcal{C}_x(\mu^{-1}(x))$. Let $A \in \mathcal{C}_x(\mu^{-1}(x))$. Since $A \cup \mu^{-1}(x) \subseteq \mu^{-1}(x) \subseteq \{i \mid xR_i\mu(i)\}$, by consistency of priority rules, we have $\mu^{-1}(x) \notin \mathcal{C}_x(\{i \mid xR_i\mu(i)\})$, yielding a contradiction with μ respecting priorities. If (3) were not to hold, we would have $(j,x) \in N \times X$ such that $xP_j\mu(j)$ and $j \in DC_x(\mu^{-1}(x) \cup \{j\})$. Since $j \notin \mu^{-1}(x)$, we necessarily have $\mu^{-1}(x) \notin \mathcal{C}_x(\mu^{-1}(x) \cup \{j\})$. Again, by consistency, we must have $\mu^{-1}(x) \notin \mathcal{C}_x(\{i \mid xR_i\mu(i)\})$, contradicting the assumption that μ respects $\mathcal{C}$.

The above algorithm is a generalization of the student-proposing deferred acceptance algorithm to an environment which allows school priority rankings over sets of students to be substitutable with ties. The way the algorithm chooses a set of temporary winners $S_x^t \in \mathcal{C}_x(A_x^t)$ such that $Z_x^{t-1} \subseteq A_x^t \backslash S_x^t$, relies on condition (b) of substitutability, and ensures that those students that were rejected by x in a previous round are still rejected.

A priority structure $\mathcal{C}$ is **acceptant** if for each $x \in X$, for each $S \subseteq N$, and for each $S' \in \mathcal{C}_x(S)$ we have $|S'| = \min\{|S|, q_x\}$. This captures the idea that an unused school seat cannot be denied to any student. If $\mathcal{C}$ is an acceptant priority structure, and if μ is an assignment which respects $\mathcal{C}$, then μ is **non-wasteful**, i.e., $|\mu^{-1}(x)| = q_x$ whenever there exists $i \in N$ such that $xP_i\mu(i)$.

Our environment generalizes that of Ehlers (2007) and Erdil and Ergin (2008). So as in those paper, the outcome of the modified deferred acceptance algorithm is not necessarily constrained efficient, and the constrained efficient set is not necessarily a singleton. Given $\mathcal{C}$, define the **constrained efficient correspondence** $f^{\mathcal{C}}$, which assigns to each preference profile R, the set of stable assignments which are not Pareto dominated by another stable assignment. We say $f^{\mathcal{C}}$ is Pareto efficient if every $\mu \in f^{\mathcal{C}}(R)$ is Pareto efficient for every R.

4 Efficiency and constrained efficiency

Let us call a priority structure $\mathcal{C}$ **efficient** if $f^{\mathcal{C}}$ is Pareto efficient. Ergin (2002) characterizes efficient priority structures under the assumption of responsive priorities without ties. Ehlers and Erdil (2010) give a more general characterization allowing for ties in priority orders. Below, we will let the priorities be acceptant substitutable with ties, providing the most general statement in a much larger environment. This characterization result, as the ones before, confirms that $f^{\mathcal{C}}$ is Pareto efficient under very restrictive conditions.

Definition 2 Given a priority structure $\mathcal{C}$, a **weak cycle** is constituted of distinct $i, j, k \in N$, and $x, y \in X$ such that there exist $S_x, S_y \subseteq N \backslash \{i, j, k\}$ with $S_x \cap S_y = \varnothing$ satisfying

$$\textbf{(C)} \quad \begin{array}{l} j \notin DC_x(S_x \cup \{i, j\}) \\ j \in DC_x(S_x \cup \{k, j\}) \\ k \notin DC_x(S_x \cup \{i, k\}) \\ i \notin DC_y(S_y \cup \{k, i\}) \end{array}$$

$$\textbf{(S)} \quad |S_x| = q_x - 1 \text{ and } |S_y| = q_y - 1.$$

If $\mathcal{C}$ does not have any weak cycle, then it is called **strongly acyclic**.

Proposition 3 *Let $\mathcal{C}$ be an acceptant substitutable priority structure. $f^{\mathcal{C}}$ is efficient if and only if $\mathcal{C}$ is strongly acyclic.*

Ergin's (2002) characterization of efficient priority structures imposes an acyclicity condition which can be interpreted as schools having "sufficiently similar" priority rankings. It turns out that in our more general setting, such interpretation is not valid. In fact, schools can have identical priority rankings, while constrained efficiency is still short of efficiency.

Example 1 In our motivating example, we had two schools x, y, each with two seats, and six students a, a', b, b', w, w' . Both schools had the same priority rule which lead to the following priority order

$$\{a, b\} \sim \{a', w\} \sim \{b', w'\} \quad \succ \quad \{a, a'\} \sim \{b, b'\} \sim \{w, w'\}.$$

In order to verify that this priority structure has a *weak cycle* in our sense, we need to check conditions (S) and (C). Letting $S_x = \{a'\}$ and $S_y = \{b'\}$, we can see that (S) holds. As for condition (C), note that

$$\begin{aligned}
&\mathcal{C}_x(\{a', \underbrace{w}_{i}, \underbrace{b}_{j}\}) = \{\{a', w\}, \{a', b\}, \{w, b\}\} && \Rightarrow \quad b \notin DC_x(S_x \cup \{w, b\}) \\
&\mathcal{C}_x(\{a', b, \underbrace{a}_{k}\}) = \{\{b, a\}, \{b, a'\}\} && \Rightarrow \quad b \in DC_x(S_x \cup \{a, b\}) \\
&\mathcal{C}_x(\{a', w, a\}) = \{\{w, a\}, \{w, a'\}\} && \Rightarrow \quad a \notin DC_x(S_x \cup \{w, a\}) \\
&\mathcal{C}_y(\{b', w, a\}) = \{\{w, b'\}, \{w, a\}, \{b', a\}\} && \Rightarrow \quad w \notin DC_y(S_y \cup \{a, w\}).
\end{aligned}$$

Thus, we have a weak cycle, and by Proposition 3, $f^{\mathcal{C}}$ is not efficient. The following preference profile and the assignment explicitly show an instance of such inefficiency:

R_a	$R_{a'}$	R_b	$R_{b'}$	R_w	$R_{w'}$
x	x	x	y	y	
y				x	

$$\mu = \begin{pmatrix} a & a' & b & b' & w & w' \\ y & x & b & y & x & w' \end{pmatrix}$$

μ is constrained efficient, but not efficient, because there is only one possible Pareto improvement over μ which is achieved by letting a and w swap their schools, but the resulting assignment would not respect the priority structure $\mathcal{C}$. $\Diamond$

In the absence of ties, each priority rule $\mathcal{C}_x$ is singleton-valued. We know in that case that $f^{\mathcal{C}}$ is singleton-valued and reached by the deferred acceptance algorithm. On the other hand, when there are ties, the constrained efficient correspondence is not necessarily singleton-valued. Moreover, arbitrarily breaking the ties as we execute the deferred acceptance algorithm may lead to constrained inefficiency. In the case of responsive priorities the *stable improvement cycles algorithm* by Erdil and Ergin (2008) reaches a constrained efficient assignment. We explore whether such cycles can be used to solve the similar problem when priorities are acceptant and substitutable.

A special case of our environment is that of responsive priorities with ties. Motivated by the fact that an arbitrary resolution of ties in implementing the DA algorithm may lead to an assignment which is not constrained efficient, Erdil and Ergin (2008) explored stability preserving Pareto improvements. A *stable improvement cycle* is a cycle of distinct schools such that for any edge $x \to y$, there is a student i_x matched with x, who would like to be matched with y instead, and is one of the highest y-priority students among those who would like to move to y. They show that if a stable assignment is not constrained efficient, then it must admit a stable improvement cycle, and therefore by simply searching for stable improving cycles and implementing them successively, one can reach a constrained efficient assignment. When we more from responsive to substitutable priorities, their definition does *not* capture all the improvement cycles that preserve stability. That is, it is possible that a stable matching μ is Pareto dominated by another stable matching ν even though μ does not admit a stable improvement cycle in their sense.[11]

Given a stable assignment μ, who could be replacing, without violating stability, j's position at $\mu(j)$ if j were to disappear? It must be that when such a student ℓ replaces j, the new set of students must be a chosen set in the face of those who would like to be replacing j at $\mu(j)$. To formalize this idea in general, let $\mathcal{E}_j^\mu$ stand for the set of students who envy j at assignment μ:

$$\mathcal{E}_j^\mu = \{i \mid \mu(j) P_i \mu(i)\}.$$

Then, the set of **students who can replace student j at μ** is defined as

$$E_j^\mu = \{i \in \mathcal{E}_j^\mu \mid \{i\} \cup \mu^{-1}(\mu(j)) \setminus \{j\} \in \mathcal{C}_{\mu(j)}(\mathcal{E}_j^\mu \cup \mu^{-1}(\mu(j)) \setminus \{j\})\}$$

Note that $j \notin E_j^\mu$, and E_j^μ is not necessarily a singleton.

Given a priority structure $\mathcal{C}$, a preference profile R and a stable assignment μ, a **stable student improving cycle (SSIC)** consists of distinct students $i_0, i_1, \ldots, i_{n-1}, i_n = i_0$ such that $i_\ell \in E^\mu_{i_{\ell+1}}$ for all $\ell = 0, \ldots, n-1$. We denote such a cycle by $i_0 \to i_1 \to \cdots \to i_{n-1} \to i_0$.

[11]See Remark 5 in Appendix B.

Proposition 4 *Given an acceptant and substitutable priority structure $\mathcal{C}$, if a stable assignment does not admit an SSIC then it is constrained efficient.*

Without further assumptions on the priorities, the converse does not necessarily hold. That is, a constrained efficient assignment might admit an SSIC.[12] Now we formulate a condition on the priority structure which ensures that constrained efficiency rules out stable student improvement cycles. Given an acceptant substitutable priority structure $\mathcal{C}$, define a weak form of "equal treatment of equals" as follows: A priority structure satisfies **equal treatment of equal students** if given $\{i, j\} \cup S \subseteq T$, and $\{i, j\} \cup S' \subseteq T'$ such that $T \subseteq T'$, $|S| = q_x - 1$ and $|S'| = q_x - 1$,

$$S \cup \{i\}, S \cup \{j\} \in \mathcal{C}_x(T) \text{ and } S' \cup \{i\} \in \mathcal{C}_x(T') \Longrightarrow S' \cup \{j\} \in \mathcal{C}_x(T'). \qquad (\textbf{ETE})$$

Which students are to be treated equally can change from one school to the other. However, the critical requirement is that if a student can replace another given some set of applicants, then she could replace the same student given any larger set of applicants.

Proposition 5 *Let $\mathcal{C}$ be an acceptant and substitutable priority structure which satisfies ETE. Suppose that the stable assignment μ admits an SSIC, and let the shortest SSIC be*

$$i_0 \to i_1 \to \cdots \to i_{n-1} \to i_0.$$

If the assignment ν is obtained by carrying out this cycle, i.e., if

$$\nu(i) = \begin{cases} \mu(i_{\ell+1}) & \text{if } i = i_\ell \\ \mu(i) & \text{otherwise} \end{cases},$$

then ν is stable.

Corollary 1 *Suppose that an acceptant and substitutable priority structure $\mathcal{C}$ satisfies ETE. Then a stable assignment is constrained efficient if and only if it does not admit a stable student improving cycle.*

The above proposition leads to an algorithm which always returns a constrained efficient assignment. Starting from a stable assignment μ, one needs to construct a graph whose set of vertices is the set of students. For any pair (i, j) of vertices, there will be an edge from i to j if and only if $i \in E_j^\mu$. If this graph does not have a cycle which preserves stability, then μ is constrained efficient. Otherwise, we can let the cycle lead to a Pareto improving cyclic trade which would preserve stability.

[12] Remark 4 in Appendix B gives such an example.

Modified Stable Improvement Cycles Algorithm (MSIC)

Step 0:

Run the *MDA* to obtain an initial matching μ^0.

Step t $\geq$ 1:

$(t.a)$ Given μ^{t-1}, let the students stand for the vertices of a directed graph, where for each pair of students i and j, there is an edge $i \longrightarrow j$ if and only if $i \in E_j^{\mu^{t-1}}$.

$(t.b)$ If there are any stable student improving cycles in this directed graph, *select* a shortest one, and carry out this cycle to obtain μ^t, and go to step $(t+1.a)$. If there is no such cycle, then return μ^{t-1} as the outcome of the algorithm.

This algorithm will return a student optimal stable assignment, but when there are more than one such assignments, the particular outcome will depend on the selections in running the MDA in Step 0, and the specification of the cycle search in later steps.

The algorithm ensures a constrained efficient outcome. Whenever the temporary assignment μ^t is not constrained efficient, we know from Corollary 1 that it must admit an SSIC. The Floyd-Warshall algorithm[13] is a computationally efficient way to find a shortest cycle, and thanks to Proposition 5, we can carry out this SSIC to reach a stable assignment μ^{t+1} which improves upon μ^t.

Furthermore, if $\mathcal{C}$ is acyclical, then we know from Proposition 3 that any constrained efficient assignment is Pareto efficient. Thus we have

Corollary 2 *If $\mathcal{C}$ is acyclical, then the above algorithm is Pareto efficient.*

When responsive priorities involve ties, Erdil and Ergin (2008) show that there is no strategy-proof mechanism which always returns a constrained efficient matching. Since our environment subsumes theirs, the impossibility result extends to this more general setting. In particular, however the selections in the above algorithm are made (deterministic or random), any mechanism which returns a single matching (or a randomization over matchings) will fail strategy-proofness. Instead of focusing a singleton-valued selection from the set of constrained efficient matchings, we will then focus on the whole correspondence $f^{\mathcal{C}}$. While there is not a standard concept of incentive compatibility for social choice correspondences, Jackson (1992) defines the following notion of *strategy-resistance* which boils down to strategy-proofness for

[13] See Cormen, Leiserson, Rivest, and Stein (2001) for an exposition.

social choice functions: by unilaterally misrepresenting her preferences, no agent can get a match better than every matching obtained when she is truthful. Formally speaking, a correspondence F is **strategy-resistant** if there exists no i, R, R'_i, and no $\nu \in F(R'_i, R_{-i})$ such that $\nu(i) P_i \mu(i)$ for all $\mu \in F(R)$.

Proposition 6 *If $\mathcal{C}$ is acceptant and substitutable, then $f^{\mathcal{C}}$ is strategy-resistant.*

5 Application: a model of prioritizing diversity

Admissions policies often include guidelines or rules to promote diversity.[14] An intuitive way to think about diversity is to take some distribution of types in a population as the *ideal* or the *target* distribution, and compare distributions in terms of their *distance* from that specified distribution. We show that our model is able to incorporate a concern for diversity capturing this intuitive approach. Moreover, other exogenously given priorities can be accommodated in addition to diversity.

Let there be an exogenous priority ranking over the set of students. Denoting this weak order by $\succsim^{exo} \in N \times N$, its associated linear order is denoted by $\succ^{exo}$, whereas indifference is denoted by $\sim^{exo}$. Let $T = \{\tau_1, \ldots, \tau_m\}$ be the set of types, and $\tau : N \to T$ be a function such that $\tau(i)$ indicates student i's type. For every school x, there are type-specific quotas $k_x^{\mathcal{T}} = (k_x^{\tau_1}, \ldots, k_x^{\tau_m})$ such that $1 \leq k_x^{\tau} \leq q_x$, and $\sum_\tau k_x^\tau = q_x$. We can interpret $k_x^{\mathcal{T}}$ as the *target distribution*[15] of types at school x. For an acceptant priority order, it is sufficient to specify rankings over q_x-subsets of students. For any q_x-subset S, let us define its *distance* from the target distribution $k_x^{\mathcal{T}}$ as

$$d_x^k(S) = d(k_x^{\mathcal{T}}, S) = \sum_{\tau \in T} ||S_\tau| - k_x^\tau|,$$

where S_τ denotes $\{s \in S : \tau(s) = \tau\}$. While d_x^k depends on the target distribution k_x^T, we will suppress such dependence in our notation when it leads to no confusion. Note that if $|S| = q_x$, then $d_x(S)$ is even.

[14]In some cases in order to promote the welfare of an underprivileged or an underrepresented group, one might set quotas that favor members of such a group. Such policies, sometimes called *affirmative action* or *controlled school choice*, might turn out to be counterproductive as illustrated by Kojima (2012). Some seemingly natural ways of granting favorable treatment to a specific group might lead to each member of such a group ending up worse off compared with the assignment which ignores such treatment.

[15]It is possible to work with target distributions that have non-integer entries, or whose entries add up to a number larger than the total number of seats available.

Definition 3 *A priority order* $\succsim_x$ **prioritizes diversity (PD)** *if for every* S, S' *such that* $|S| = |S'| = q_x$,

$$d_x(S) < d_x(S') \quad \Rightarrow \quad S \succ_x S'.$$

The distance function determines how far a set is from the target distribution of types at a school. Diversity concerns do not help choose between sets that are equidistant from the target distribution. If there is an exogenous priority order over students, one can use such priorities to rank sets that are considered equally diverse in the sense specified by the distance function:

Definition 4 *Given an exogenous priority ranking* $\succsim^{exo}$ *over the set of students, a priority order* $\succsim_x$ *is* **diversity constrained responsive (DCR)** *to* $\succsim^{exo}$ *if whenever* $|T \cup \{s'\}| = |T \cup \{s''\}| = q_x$ *and* $d_x(T \cup \{s'\}) = d_x(T \cup \{s''\})$, *we have*

$$T \cup \{s'\} \succsim_x T \cup \{s''\} \quad \Leftrightarrow \quad s' \succsim^{exo} s''.$$

In this formulation, diversity is the leading priority. Other priorities (captured by $\succsim^{exo}$) are used to break ties between sets of *equal diversity index*, i.e., sets that are of equal distance to the target distribution.

Much of the literature on school choice is on priorities responsive to some exogenous order over the set of students. For example, other things being equal, a student who lives in the walk-zone of the school is of higher priority than one that lives outside that walk-zone. We now give an example, where such priorities can be handled together with concerns for diversity.

Example 2 (Racial Balance & Walk-Zone) Suppose there is a school with three seats and the following target distribution: $(q^a, q^b, q^w) = (1, 1, 1)$. The exogenous priority order on the students prioritizes the students w, w', b who live in the school's walk-zone:

$$w \sim^{exo} w' \sim^{exo} b \quad \succ^{exo} \quad b' \sim^{exo} a.$$

The following order prioritizes racial diversity, and uses walk-zone priorities as a secondary criterion:

$$\begin{aligned}
&\{a, b, w\} \sim \{a, b, w'\} \quad \succ \\
&\{a, b', w\} \sim \{a, b, w'\} \quad \succ \\
&\{b, w, w'\} \quad \succ \\
&\{b, a, w'\} \sim \{b, b', w\} \sim \{a, w, w'\} \quad \succ \\
&\{a, b, b'\} \quad \succ \cdots
\end{aligned}$$

Again, to our knowledge, this sort of priorities are not captured by any of the models in the literature. $\diamondsuit$

Proposition 7 *Let $\succsim^{exo}$ be a weak order on students. If an acceptant priority rule $\succsim$ prioritizes diversity (PD) and is diversity constrained responsive (DCR) to $\succsim^{exo}$, then it is substitutable.*

Therefore, when the school allocation mechanism respects priorities which promote diversity in the sense we formalized above, the MDA always returns a stable assignment. Furthermore, stable assignments which are not optimal within the set of stable assignments can be improved via trading cycles:

Proposition 8 *Given a priority structure which satisfies PD and DCR, a stable assignment is constrained efficient if and only if it does not admit a stable student improving cycle.*

Unlike in Proposition 4 in Section **??**, where we deal with substitutable priorities in general, we do not require the condition ETE for the above proposition to hold.

Remark 1 This formulation brings to mind other possibilities for the distance we might want to use. Consider, for example, the following distance function:

$$\tilde{d}_x(S) = d(q_x, S) = \sum_{\tau \in T} ||S_\tau| - q_x^\tau|^k.$$

The proof of Proposition 7 can be modified to show that when $k > 1$, the above distance function leads to a substitutable priority rule. On the other hand, if $k < 1$, this is not necessarily the case.[16] More generally, it is the convexity of the distance function which would ensure substitutability.

The target composition of types might specify, for each type τ, a range $[\underline{q}_x^\tau, \bar{q}_x^\tau]$ of numbers instead of a single number. Then our formulation would also incorporate the soft bounds approach introduced by Ehlers, Hafalir, Yenmez and Yildirim (2011). To put it formally,

[16]In order to see this, let $q^T = (q^a, q^b, q^c, q^d) = (2, 2, 4, 2)$ be the target distribution of types a, b, c, d. Let $T = \{a_1, \ldots, a_4, b_1, \ldots, b_4, d_1, \ldots d_6\}$ and $S = T \backslash \{d_4, d_5, d_6\}$. If $k = 1/2$ in the above distance function, any T' in $\mathcal{C}(T)$ would consist of two students of type a and b each, and six students of type d. In particular $\{d_1, d_2, d_3\} \subset T'$ for every $T' \in \mathcal{C}(T)$. However, every $S' \in \mathcal{C}(S)$ consists of four students of type a and b each, and only two students of type d. So at least one of d_1, d_2, d_3 would be excluded from every chosen set, and therefore $T' \cap S \nsubseteq S'$ for any $S' \in \mathcal{C}(S)$.

suppose that for each school x and type $\tau \in T$, we have quotas $\underline{q}_x \leq \bar{q}^x$ such that $\sum_t \underline{q}_x^\tau \leq q_x \leq \sum_t \bar{q}_x^\tau$. Denote the distance between an integer m and an interval $[a, b]$ as $\delta(m, [a, b])$:

$$\delta(m, [a, b]) = \begin{cases} a - m & \text{if } m < a, \\ 0 & \text{if } a \leq m \leq b, \\ m - b & \text{if } b < m. \end{cases}$$

Then we can set a distance function $\hat{d}$ as

$$\hat{d}_x(S) = d(q_x, S) = \sum_{\tau \in T} (\delta(|S_\tau|, [\underline{q}_x^\tau, \bar{q}_x^\tau]))^k,$$

for some $k \geq 1$. If $k = 1$ in prioritizing diversity, and if each school has a strict exogenous priority order of its own, then we get the design proposed by Ehlers, Hafalir, Yenmez and Yildirim (2011). $\diamondsuit$

6 Conclusion

In this paper, we have developed a general class of substitutable priority rankings which allow indifferences. Respecting priorities, in this model, means that the assignment is stable with respect to preferences and the priority structure. Stable assignments exist, and a modified version of the celebrated deferred acceptance algorithm finds one. The outcome is not necessarily optimal from students' perspective, and for a reasonable subclass of substitutable priority rankings, we describe an algorithm that finds an optimal stable assignment. We show that along with standard priorities like walk-zone or sibling, a seemingly complex but practical concern for diversity is well captured by our model.

A Appendix: Proofs

Proof of Proposition 1

Denote $U = \mu^{-1}(x)$. If μ is pairwise stable, then it must be that for any ℓ with $xP_\ell\mu(\ell)$, we have $U \in \mathcal{C}_x(U \cup \{\ell\})$. Since students' preferences over schools are strict, we can write this in a seemingly stronger way: for any ℓ with $xR_\ell\mu(\ell)$, we have $U \in \mathcal{C}_x(U \cup \{\ell\})$. We would like to show that if $S \subseteq N$ such that $xR_i\mu(i)$ for all $i \in S$, then $U \in \mathcal{C}_x(U \cup S)$. In order to conclude via induction on $|S|$, it is sufficient to show that

$$[U \in \mathcal{C}_x(U \cup S) \text{ and } U \in \mathcal{C}_x(U \cup \{k\})] \quad \Rightarrow \quad U \in \mathcal{C}_x(U \cup S \cup \{k\}).$$

Now, suppose for a contradiction that $U \notin \mathcal{C}_x(U \cup S \cup \{k\})$. Then for any $T \in \mathcal{C}_x(U \cup S \cup \{k\})$, we know that T is chosen instead of U. On the other hand, U is chosen from among $U \cup S$, thus T cannot be a subset of $U \cup S$, i.e., $T \nsubseteq U \cup S$. Since $T \subseteq U \cup S \cup \{k\}$, we conclude that $k \in T$. On the other hand, $U \in \mathcal{C}_x(U \cup \{k\})$ implies that $\{k\} \in \mathcal{R}_x(U \cup \{k\})$, which implies, due to substitutability, $\{k\} \subseteq (U \cup S \cup \{k\}) \backslash T$ for some $T \in \mathcal{C}_x(U \cup S \cup \{k\})$. But then $k \notin T$ yielding the desired contradiction. □

Proof of Proposition 2

A_x^t is the set of students who have applied to school x in some round $k \leq t$. Hence

$$A_x^1 \subseteq A_x^2 \subseteq \cdots$$

The algorithm requires that those students rejected in rounds $k \leq t-1$ would still be rejected if they were considered to be among the applicants in round t. This can be ensured thanks to condition (b) of substitutability, because $Z_x^{t-1} = A_x^{t-1} \backslash S_x'$ for some $S_x' \in \mathcal{C}_x(A_x^{t-1})$ and $A_x^{t-1} \subseteq A_x^t$ together imply that there exist $Z_x^t = A_x^t \backslash S_x''$ such that $Z_x^t \supseteq Z_x^{t-1}$ for some $S_x'' \in \mathcal{C}_x(A_x^t)$.

In order to see that the algorithm indeed ends, note that at any round if a student is not matched, then she applies to her next favorite school in the following round. Therefore, she either exhausts all her acceptable schools by going down all the way to the end of her preference list, or ends up being matched with some school.

Suppose that the algorithm ends at round m, and μ is the matching obtained at the end. Students only apply to schools they find acceptable, so a student i would only be matched with a school x where $xR_i i$. Therefore μ is individually rational. Secondly, since $A_x^m \backslash \mu^{-1}(x) = Z_x^m \in \mathcal{R}_x(A_x^m)$, we have

$$\mu^{-1}(x) \in \mathcal{C}_x(A_x^m).$$

Those who weakly prefer x to their match under μ are either matched with x, or have applied to x at some round of the algorithm. Thus, $A_x^m = \{i \mid xR_i\mu(i)\}$. Now, since $\mu^{-1}(x) \in \mathcal{C}_x(\{i \mid xR_i\mu(i)\})$, we conclude from condition (a) of substitutability that $\mu^{-1}(x) \in \mathcal{C}_x(S)$ for any S such that $\mu^{-1}(x) \subseteq S \subseteq \{i \mid xR_i\mu(i)\})$. Hence μ is stable.

□

Proof of Proposition 3

The main part in proving the proposition is ($\Leftarrow$), i.e., showing that a strongly acyclic $\mathcal{C}$ leads to efficient $f^{\mathcal{C}}$. We will prove this part in two steps.

Given a priority structure $\mathcal{C}$, a **generalized weak cycle of size** n is constituted of distinct schools $x_0, x_1, \ldots, x_{n-1} \in X$ and distinct students $j, i_0, i_1, \ldots, i_{n-1} \in N$ with $n \geq 2$ such that

(1) $x_\ell \neq x_{\ell+1}$ for $\ell \in \{0, 1, \ldots, n-1\}$ (with $x_n = x_0$),

(2) there exist mutually disjoint sets of students $S_{x_0}, \ldots, S_{x_{n-1}} \subseteq N \setminus \{j, i_0, i_1, \ldots, i_{n-1}\}$ such that

$$\textbf{(C)} \quad \begin{aligned} & j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}) \\ & j \in DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}) \\ & i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\ & i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\ & \quad \vdots \\ & i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}) \\ & i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\}) \end{aligned}$$

$$\textbf{(S)} \quad |S_{x_\ell}| = q_{x_\ell} - 1 \text{ for } \ell = 0, 1, \ldots, n-1.$$

Step 1: If there exists a Pareto inefficient assignment $\mu \in f^{\mathcal{C}}(R)$, then $\mathcal{C}$ has a generalized weak cycle.

Proof of Step 1: Suppose that $\mu \in f^{\mathcal{C}}(R)$ is not Pareto efficient. Of all the Pareto improvements over μ, let ν be one which has the least number of students improving over μ. Denote by N' the set of students who are better off under ν compared with μ:

$$N' = \{i \mid \nu(i) P_i \mu(i)\}.$$

Denote by $\mathcal{E}_j^\mu$ the set of students who envy the student j under μ:

$$\mathcal{E}_j^\mu = \{\ell \in N \mid \mu(j) P_\ell \mu(\ell)\}.$$

Set $\mathcal{E}'_j$ to be the set of students in N' who envy j. That is,

$$\mathcal{E}'_j = \mathcal{E}_j^\mu \cap N' = \{\ell \in N' \mid \mu(j) P_\ell \mu(\ell)\}.$$

Since $\mathcal{C}$ is acceptant, μ must be non-wasteful, and therefore the "reshuffling lemma" applies, i.e., any Pareto improvement over μ is due to reshuffling of already assigned objects between their recipients. Therefore if $j \in N'$, that is, if j is part of an improvement, she must receive someone else's object, whereas her object must be reassigned to another person who necessarily is also better off. In other words we have $\mu(j) \in \nu(N')$. In particular, $\mu(j)$ is desired by some student in N' under μ, and hence $\mathcal{E}'_j$ is nonempty. Because μ respects priorities, we have

$$\mu^{-1}(\mu(j)) \in \mathcal{C}_{\mu(j)}(\mathcal{E}^{\mu}_j \cup \mu^{-1}(\mu(j))).$$

Furthermore, $\mathcal{E}'_j \subseteq \mathcal{E}^{\mu}_j$ and $\mathcal{C}$ being substitutable imply that

$$\mu^{-1}(\mu(j)) \in \mathcal{C}_{\mu(j)}(\mathcal{E}'_j \cup \mu^{-1}(\mu(j))).$$

Removing j from the choice set, we conclude, again using substitutability, that $\mu^{-1}(\mu(j))\backslash\{j\}$ is a subset of a chosen element from $\mathcal{E}'_j \cup \mu^{-1}(\mu(j))\backslash\{j\}$. In other words

$$\mu^{-1}(\mu(j))\backslash\{j\} \subseteq S' \quad \text{for some} \quad S' \in \mathcal{C}_{\mu(j)}(\mathcal{E}'_j \cup \mu^{-1}(\mu(j))\backslash\{j\}).$$

Any such S' has exactly one element from $\mathcal{E}'_j$, and let E'_j be the set of those elements:

$$E'_j = \left\{ \ell \;\middle|\; \begin{array}{c} \ell \in \mathcal{E}'_j, \;\text{ and }\; (\mu^{-1}(\mu(j))\backslash\{j\}) \cup \{\ell\} = S' \\ \text{for some } S' \in \mathcal{C}_{\mu(j)}(\mathcal{E}'_j \cup \mu^{-1}(\mu(j))\backslash\{j\}) \end{array} \right\}$$

Thus, E'_j is a nonempty subset of N' for each $j \in N'$. Consider a directed graph whose set of vertices is N'. For each $i \in E'_j$, let there be a directed edge from i to j. Therefore, every vertex in this graph has an incoming edge, and since it is a finite graph, there must be a cycle.

Let the shortest cycle in this graph consist of students $i_0, i_1, \ldots, i_{n-1}, i_n = i_0$, where $n \geq 2$, and there is an edge from i_ℓ to $i_{\ell+1}$ for $\ell = 0, 1, \ldots, n-1$. Denoting $\mu(i_\ell) = x_\ell$, since i_ℓ envy $i_{\ell+1}$, we have $x_\ell \neq x_{\ell+1}$ for each ℓ. In fact, these schools $x_0, \ldots, x_{n-1}$ must be distinct, for otherwise we would have a shorter cycle, which would give a Pareto improvement over μ, involving a smaller number of students improving. To be more precise, if $x_0 = x_k$ for some $k \leq n-1$, then the cyclic trade which allows i_ℓ take $x_{\ell+1}$ for $\ell = 0, \ldots, k-1$, and letting i_k take x_0 would lead to a Pareto improvement over μ. Since $k < n$, this would contradict with the assumption that ν was the "smallest" improvement over μ. Since $\mu(i_\ell) = x_\ell$, the students $i_0, \ldots, i_{n-1}$ are necessarily distinct.

The fact that μ respects acceptant priorities implies that it is non-wasteful. Since each x_ℓ is desired by some student at assignment μ, all seats at these schools must be assigned under

μ. Denoting $S_{x_\ell} = \mu^{-1}(x_\ell)\backslash\{i_\ell\}$, we know that $S_{x_0}, \ldots, S_{x_{n-1}}$ are mutually disjoint subsets of $N \setminus \{i_0, i_1, \ldots, i_{n-1}\}$, because $x_0, x_1, \ldots, x_{n-1}$ are distinct schools. Moreover we have

(1) $|S_{x_\ell}| = q_{x_\ell} - 1$,

(2) $$\begin{aligned} & i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\ & i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\ & \quad \vdots \\ & i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}) \\ & i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\}) \end{aligned} \tag{*}$$

because otherwise, if student i_ℓ were to be in $DC_{x_{\ell+1}}(S_{x_{\ell+1}} \cup \{i_{\ell+1}, i_\ell\})$ for some ℓ, then we would have $S_{x_{\ell+1}} \cup \{i_{\ell+1}\} \notin \mathcal{C}_{x_{\ell+1}}(S_{x_{\ell+1}} \cup \{i_{\ell+1}, i_\ell\})$, contradicting stability of μ.

Let ω be the assignment derived from μ by letting the students $i_0, i_1, \ldots, i_{n-1}$ exchange their schools along the improvement cycle suggested above. In other words,

$$\omega(i) = \begin{cases} \mu(i) & i \neq i_\ell \\ \mu(i_{\ell+1}) & i = i_\ell \end{cases}$$

ω Pareto dominates μ, whereas μ is constrained efficient, so ω must not be stable. Therefore the cyclic trade letting i_ℓ take $\mu(i_{\ell+1})$ for $\ell = 0, 1, \ldots, n-1, n \equiv 0$ cannot be respecting priorities. All school priorities are acceptant, and the new matching ω is clearly individually rational. Therefore we know from Proposition 1 that there must be a blocking pair involving one of these schools. Suppose that j and x_0 form a blocking pair for ω, i.e., $\omega^{-1}(x_0) \notin \mathcal{C}_x(\omega^{-1}(x_0) \cup \{j\})$. Then $x_0 P_j \omega(j)$ and

$$j \in DC_{x_0}(\omega^{-1}(x_0) \cup \{j\}) = DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}). \tag{**}$$

First, note that $j \neq i_{n-1}$, because $\omega(i_{n-1}) = x_0 P_j \omega(j)$. Secondly, $j \neq i_0$, because $\omega(i_0) P_{i_0} \mu(i_0) = x_0$, while $x_0 P_j \omega(j)$. And lastly if $j = i_k$ for some $k \in \{1, \ldots, n-2\}$, then we have an envy cycle

$$i_0 \to i_1 \to \cdots \to i_k \to i_0$$

which would allow a Pareto improvement involving only $k+1 \leq n-1$ students, contradicting our earlier choice of a smallest Pareto improvement over μ. Thus $j \notin \{i_0, \ldots, i_{n-1}\}$.

Furthermore, stability of μ implies

$$j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}). \tag{***}$$

Thus, combining (*), (**), and (***), we have a generalized weak cycle

$$
\begin{aligned}
& j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}) \\
& j \in DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}) \\
& i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\
& i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\
& \quad \vdots \\
& i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}) \\
& i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\})
\end{aligned}
$$

with $|S_{x_\ell}| = q_{x_\ell} - 1$ for $\ell = 0, 1, \ldots, n-1$.

Step 2: If $\mathcal{C}$ has a generalized weak cycle, then it has a weak cycle.

Proof of Step 2: Suppose that $\mathcal{C}$ has a generalized weak cycle and let the size of its shortest generalized weak cycle be n. We will show that $n = 2$, which will prove step 2, since a weak cycle is a generalized weak cycle of size 2. Suppose that $x_0, x_1, \ldots, x_{n-1} \in X$; $j, i_0, i_1, \ldots, i_{n-1} \in N$ and $S_{x_0}, \ldots, S_{x_{n-1}} \subseteq N \setminus \{j, i_0, \ldots, i_{n-1}\}$ form a shortest generalized weak cycle. We will assume that it is of size $n \geq 3$, and reach a contradiction.

Let us look at the the set of definitely chosen students from $S_{x_1} \cup \{i_0, i_2\}$ according to the priorities of x_1. Is i_0 in this set or not?

If so, i.e., if $i_0 \in DC_{x_1}(S_{x_1} \cup \{i_0, i_2\})$, then

$$
\begin{aligned}
& i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\}) \\
& i_0 \in DC_{x_1}(S_{x_1} \cup \{i_0, i_2\}) \\
& i_2 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_2\}) \\
& i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}),
\end{aligned}
$$

which is a weak cycle, i.e., a generalized weak cycle of length 2, contradicting with our assumption of shortest cycle being of length at least 3.

If on the other hand, $i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_0, i_2\})$, then we get the following generalized weak cycle

$$
\begin{aligned}
& j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}) \\
& j \in DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}) \\
& i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\
& i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\
& \quad \vdots \\
& i_2 \notin DC_{x_3}(S_{x_3} \cup \{i_3, i_2\}) \\
& i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_2, i_0\})
\end{aligned}
$$

with $|S_{x_\ell}| = q_{x_\ell} - 1$ for $\ell = 0, 1, 3, \dots, n-1$. This cycle is shorter than the one we started with, because it does not have x_2, hence yields the desired contradiction to our original cycle being the shortest.

$\underline{\Longrightarrow}$**:** Let N, X, and q and $\mathcal{C}$ be given. Assume that $\mathcal{C}$ has a weak cycle. Let $i, j, k \in N$, and $x, y \in X$ such that there exist $S_x, S_y \subseteq N \backslash \{i, j, k\}$ with $S_x \cap S_y = \varnothing$ satisfying

$$
\begin{aligned}
& j \notin DC_x(S_x \cup \{i, j\}) \\
& j \in DC_x(S_x \cup \{k, j\}) \\
& k \notin DC_x(S_x \cup \{k, i\}) \\
& i \notin DC_y(S_y \cup \{k, i\})
\end{aligned}
$$

with $|S_x| = q_x - 1$ and $|S_y| = q_y - 1$.

Consider the preference profile R where students in S_x and S_y, respectively, rank x and y as their top choice, and the preferences of i, j, and k are such that $yP_ixP_iiP_i\cdots$, $xP_jjP_j\cdots$, and $xP_kyP_kkP_k\cdots$. Finally, let students outside $S_x \cup S_y \cup \{i, j, k\}$ prefer not to be assigned to any school. Consider the assignment μ such that for each $\ell \in S_x \cup \{i\}$ one has $\mu(\ell) = x$, and for each $\ell \in S_y \cup \{k\}$ one has $\mu(\ell) = y$. Now the only candidates for blocking pairs are $(j, x), (k, x)$, and (i, y). However, the weak cycle conditions are such that $j \notin DC_x(S_x \cup \{i, j\})$ $k \notin DC_x(S_x \cup \{k, i\})$, and $i \notin DC_y(S_y \cup \{k, i\})$, ensuring that μ respects priorities $\mathcal{C}$. Moreover, there is only one assignment that Pareto dominates μ, namely the assignment ν obtained from μ by letting i and k trade their assigned schools. Since $j \in DC_x(S_x \cup \{j, k\})$, $xP_j\nu(j)$ and $\nu^{-1}(x) = S_x \cup \{k\}$, the assignment ν does not respect $\mathcal{C}$. Thus μ is constrained efficient, but not Pareto efficient. $\square$

Proof of Proposition 4

We will now show that if a stable assignment μ is Pareto dominated by another stable assignment ν, then μ must admit an SSIC. From this, it will follow that if μ does not admit an SSIC, then it must be constrained efficient.

Let $N' = \{i \in N \mid \mu(i) \neq \nu(i)\}$ and $X' = \{\nu(i) \mid i \in N'\}$. For any $i \in N'$, we know by the reshuffling lemma that $\mu(i) \in X'$. Let $i \in N'$, and $\mu(i) = x$. Denote

$$D_x^\mu = \{j \in N \mid xP_j\mu(j)\}, \quad D_x' = \{j \in N' \mid xP_j\mu(j)\}, \quad D_x'' = \{j \in N \backslash N' \mid xP_j\mu(j)\}$$

and set[17]

$$\bar{D}_x = D_x' \sqcup D_x'' \sqcup \mu^{-1}(x) = D_x^\mu \sqcup \mu^{-1}(x).$$

Stability of μ implies that

$$\mu^{-1}(x) \in \mathcal{C}_x(\bar{D}_x)$$

Moreover, stability of ν implies that

$$D_x'' \subseteq T'' \text{ for some } T'' \in \mathcal{R}_x(D_x^\nu \sqcup \nu^{-1}(x)). \tag{$\star$}$$

ν Pareto dominates μ, so those who desire x at ν, desire x at μ as well. Therefore $D_x^\nu = \{j \in N \mid xP_j\nu(j)\} \subseteq D_x^\mu$. Moreover, if $j \in \nu^{-1}(x)$, then either $j \in \mu^{-1}(x)$ or $j \in D_x'$. And finally, since $\mu(i) = x$ and $i \in N'$, we know that $i \notin \nu^{-1}(x)$, and $\nu(i)P_ix$. Therefore $i \notin D_x^\nu$. Thus

$$D_x^\nu \sqcup \nu^{-1}(x) \subseteq D_x^\mu \cup \nu^{-1}(x) \subseteq \bar{D}_x \backslash \{i\}. \tag{$\star\star$}$$

Now we conclude by using $(\star), (\star\star)$, and substitutability that

$$D_x'' \subseteq T' \text{ for some } T' \in \mathcal{R}_x(\bar{D}_x \backslash \{i\}).$$

Denoting

$$S' = (\bar{D}_x \backslash \{i\}) \backslash T',$$

we have

$$S' \in \mathcal{C}_x(\bar{D}_x \backslash \{i\}) \quad \text{and} \quad S' \cap D_x'' = \varnothing.$$

Note that

$$\bar{D}_x \backslash \{i\} = D_x' \sqcup D_x'' \sqcup [\mu^{-1}(x) \backslash \{i\}],$$

[17]We use $\sqcup$ to denote "disjoint union" throughout the Appendix.

and $|\mu^{-1}(x)\backslash\{i\}| \leq q_x - 1$. Since $\mathcal{C}$ is acceptant, and $|\bar{D}_x\backslash\{i\}| \geq q_x$, we must have $|S'| \geq q_x$. Because of $|\mu^{-1}(x)\backslash\{i\}| \leq q_x - 1$ and that $S' \cap D''_x = \varnothing$, we have

$$S' \cap D'_x \neq \varnothing.$$

Hence, there exists $i' \in D'_x$ such that $\{i'\} \cup [\mu_x^{-1}\backslash\{i\}] \in \mathcal{C}_x(\bar{D}_x\backslash\{i\})$, i.e.,

$$i' \in E_i^{\mu}.$$

Now construct a directed graph with N' being its set of vertices. For any $i \in N'$, the above argument shows that there is $i' \in N'$ such that $i' \in E_i^{\mu}$, so draw an edge $i' \to i$. Since this is a finite graph with every vertex having an incoming edge, there must be cycle. By construction, this is an SSIC. □

Proof of Proposition 5

Denote the assignment obtained by carrying out this SSIC by ν, i.e., define matching ν as

$$\nu(j) = \begin{cases} \mu(i_{\ell+1}) & \text{if } j = i_\ell \\ \mu(j) & \text{otherwise} \end{cases}$$

Case 1: If the schools $\mu(i_0), \mu(i_1), \ldots, \mu(i_{n-1})$ are distinct, then one straightforwardly verifies that ν is stable.

Case 2: Now consider the case in which the schools $\mu(i_0), \mu(i_1), \ldots, \mu(i_{n-1})$ are not distinct.

Suppose for a contradiction that ν is not stable. So by Proposition 1 it must admit a blocking pair (j, x), with $j \in N$, and $x \in X$. That is,

$$j \in DC_x(\nu^{-1}(x) \cup \{j\}) \quad \text{and} \quad xP_j\nu(j).$$

Note that $\mu(j) \neq x$, and such a school x must appear more than once in the SSIC, for otherwise $\nu^{-1}(x) = [\mu^{-1}(x)\backslash\{i_{\ell+1}\}] \cup \{i_\ell\}$ and $i_\ell \in E_{i_{\ell+1}}$, and hence $j \notin DC_x((\mu^{-1}(x)\backslash\{i_{\ell+1}\}) \cup \{i_\ell, j\})$, contradicting with (j, x) being a blocking pair.

Suppose that the school x is involved in moves $i_{k^t} \to i_{k^t+1}$ for $t = 1, \ldots, m$, such that the SSIC looks like:

$$i_0 \to \cdots \to i_{k^1} \to i_{k^1+1} \to \cdots \to i_{k^2} \to i_{k^2+1} \to \cdots \to i_{k^m} \to i_{k^m+1} \to \cdots \to i_{n-1},$$

where $k^t \in \{0, \ldots, n-1\}$ and $\mu(i_{k^t+1}) = x$ for all $t \in \{1, 2, \ldots, m\}$.

Since (j,x) is a blocking pair for ν, we have $xP_j\nu(j)$ and $j \in DC_x(\nu^{-1}(x)\cup\{j\})$. Thus $xP_j\nu(j)R_j\mu(j)$, and $j \in \mathcal{E}^{\mu}_{k^t+1}$ for all t.

The definition of SSIC and substitutability implies that for each $t \in \{1,\ldots,m\}$ there exists A_t such that

$$[\mu^{-1}(x)\backslash\{i_{k^1+1},\ldots i_{k^m+1}\}]\cup\{i_{k^t}\} \;\subseteq\; A_t \in \mathcal{C}_x(\nu^{-1}(x)\cup\{j\}),$$

because $i_{k^t} \in E^{\mu}_{i_{k^t+1}}$ for all $t \in \{1,2,\ldots,m\}$.

Note that j is in A_t, because $j \in DC_x(\nu^{-1}(x)\cup\{j\})$.

Thus we get

$$[\mu^{-1}(x)\backslash\{i_{k^1+1},\ldots,i_{k^m+1}\}]\cup\{i_{k^t}\}\cup\{j\} \;\subseteq\; A_t \in \mathcal{C}_x(\nu^{-1}(x)\cup\{j\}).$$

Let us write A_t as the disjoint union

$$A_t = B_t \sqcup [\mu^{-1}(x)\backslash\{i_{k^1+1},\ldots,i_{k^m+1}\}].$$

So for all $t \in \{1,\ldots,m\}$:

$$\{i_{k^t}, j\} \subseteq B_t \subseteq \{i_{k^1},\ldots,i_{k^m},j\}, \qquad \text{and} \qquad |B_t| = m.$$

There must exist t,t' such that $B_t \neq B_{t'}$, for otherwise $\{i_{k^1},\ldots,i_{k^m},j\} \subseteq B_t$ contradicting with $|B_t| = m$. Let the symmetric difference of B_t and $B_{t'}$ be $\{i_{k^r}, i_{k^s}\}$, where $r < s$, so that

$$B_t = \tilde{B}\cup\{i_{k^r}\} \qquad \text{and} \qquad B_{t'} = \tilde{B}\cup\{i_{k^s}\},$$

and hence

$$A_t = \tilde{A}\cup\{i_{k^r}\} \qquad \text{and} \qquad A_{t'} = \tilde{A}\cup\{i_{k^s}\},$$

where $\tilde{A} = \tilde{B}\sqcup[\mu^{-1}(x)\backslash\{i_{k^1},\ldots,i_{k^m}\}]$.

Since

$$\begin{aligned}
&A_t, A_{t'} \in \mathcal{C}_x(\nu^{-1}(x)\cup\{j\}),\\
&\nu^{-1}(x)\cup\{j\} \subseteq \mathcal{E}^{\mu}_{i_{k^s+1}} \cup [\mu^{-1}(x)\backslash\{i_{k^s+1}\}], \text{ and}\\
&[\mu^{-1}(x)\backslash\{i_{k^s+1}\}]\cup\{i_{k^s}\} \in \mathcal{C}_x(\mathcal{E}^{\mu}_{i_{k^s+1}} \cup [\mu^{-1}(x)\backslash\{i_{k^s+1}\}]),
\end{aligned}$$

ETE[18] implies that $[\mu^{-1}(x)\backslash\{i_{k^s+1}\}]\cup\{i_{k^r}\} \in \mathcal{C}_x(\mathcal{E}^{\mu}_{i_{k^s+1}} \cup [\mu^{-1}(x)\backslash\{i_{k^s+1}\}])$, and therefore $i_{k^r} \in E^{\mu}_{i_{k^s+1}}$. Hence there is a shorter SSIC which looks like

$$i_0 \to \cdots \to i_{k^1} \to i_{k^1+1} \to \cdots \to i_{k^r} \to i_{k^s+1} \to \cdots \to i_{k^m} \to i_{k^m+1} \to \cdots \to i_{n-1},$$

[18] Recall that ETE requires that if i_{k^r} can substitute i_{k^s} to complement some set A, then she can substitute him to complement any other set B from any larger set of applicants.

contradicting with the initial assumption that the original SSIC was the shortest such cycle. $\square$

Proof of Proposition 6

Assume the contrary, and let $\nu \in f^{\mathcal{C}}(R_i', R_{-i})$ such that $\nu(i) P_i \mu(i)$ for all $\mu \in f^{\mathcal{C}}(R)$.

First of all, there exists an acceptant substitutable strict resolution C of $\mathcal{C}$ such $\nu = f^C(R_i', R_{-i})$.

A matching stable with respect to C and R is stable with respect to $\mathcal{C}$ and R.

Therefore, there exists $\mu \in f^{\mathcal{C}}(R)$ such that μ weakly Pareto dominates $f^C(R)$. Hence $\nu(i) P_i \mu(i) R_i f^C(R)(i)$.

This, in turn, implies that $f^C(R_i', R_{-i}) P_i f^C(R)$, i.e., f^C is not strategy-proof. But this is a contradiction, because if C is acceptant, substitutable and strict, then we know from Hatfield and Milgrom (2005) that f^C is strategy-proof. $\square$

Proof of Proposition 7

We begin with two lemmas.

Lemma 1 *Let $\succsim$ be a priority order which prioritizes the target distribution q^T, and is DCR with respect to some $\succsim^{exo}$. If $|S| \geq q$, then $S' \in \mathcal{C}(S)$ if and only if S' has the following properties:*

(L1) $|S'| = q$,

(L2) If $|S_\tau| \leq q^\tau$, then $S_\tau \subseteq S'$,

(L3) If $|S_\tau| > q^\tau$, then $|S'_\tau| \geq q^\tau$, and for all $s' \in S'_\tau$ and $\hat{s} \in S_\tau \backslash S'_\tau$, we have $s' \succsim^{exo} \hat{s}$,

(L4) If $|S'_\tau| > q^\tau$, then for all $s' \in S'_\tau$ and $\hat{s} \in S \backslash S'$, we have $s' \succsim^{exo} \hat{s}$.

Proof of Lemma.
($\Rightarrow$): (L1) Let $S' \in \mathcal{C}(S)$. Since $\succsim$ is acceptant and $|S| \geq q$, $|S'| = q$ for every $S' \in \mathcal{C}(S)$.

(L2) $d(S')$ is even since $|S'| = q$. If $d(S') = 0$ then the condition trivially holds, so assume $d(S') = 2a$, where $a \geq 1$. Suppose, for a contradiction, that there is a type τ' such that $S_{\tau'} \nsubseteq S'$. Pick an agent in $S_{\tau'} \backslash S'$, denoted by s'. Since $|S'| = q$ and $\sum_\tau q^\tau = q$, there must be τ'' such that $|S'_{\tau''}| > q^{\tau''}$. Let $s'' \in S'_{\tau''}$.

Setting $S'' = [S' \backslash \{s''\}] \sqcup \{s'\}$,

$$\sum_{\tau} ||S'_{\tau}| - q^{\tau}| = ||S'_{\tau'}| - q^{\tau'}| + ||S'_{\tau''}| - q^{\tau''}| + \sum_{\tau \neq \tau', \tau''} ||S'_{\tau}| - q^{\tau}|,$$

and

$$\sum_{\tau} ||S''_{\tau}| - q^{\tau}| = ||S''_{\tau'}| - q^{\tau'}| + ||S''_{\tau''}| - q^{\tau''}| + \sum_{\tau \neq \tau', \tau''} ||S''_{\tau}| - q^{\tau}|.$$

Since

$$\begin{aligned} ||S'_{\tau'}| - q^{\tau'}| &= ||S''_{\tau'}| - q^{\tau'}| + 1 \\ ||S'_{\tau''}| - q^{\tau''}| &= ||S''_{\tau''}| - q^{\tau''}| + 1, \end{aligned}$$

we have $d(S'') = 2a - 2$, and by (PD), $S'' \succ S'$, but $S' \in \mathcal{C}(S)$, a contradiction.

(L3) By the argument above used in verifying (L2), we conclude that $|S'_{\tau}| \geq q^{\tau}$. Secondly, suppose for a contradiction that $|S_{\tau}| \geq q^{\tau}$, but there exist $\hat{s} \in S_{\tau} \backslash S'_{\tau}$ and $s' \in S'_{\tau}$ such that $\hat{s} \succ^{exo} s'$. Since s' and $\hat{s}$ are of the same type, $[S' \backslash \{s'\}] \cup \{\hat{s}\}$ and S' are equidistant from q^T. Now (DCR) implies $[S' \backslash \{s'\}] \sqcup \{\hat{s}\} \succ S'$, whereas we have $S' \in \mathcal{C}(S)$, yielding a contradiction.

(L4) suppose for a contradiction that there exist $\hat{s} \in S \backslash S'$ and $s' \in S'_{\tau}$ such that $\hat{s} \succ^{exo} s'$. Conditions (L1)–(L3) and the fact that $\hat{s} \in S \backslash S'$ imply that $|S_{\hat{\tau}}| > q^{\hat{\tau}}$, where $\tau(\hat{s}) = \hat{\tau}$. Since $|S'_{\tau}| > q^{\tau}$, we have $|S'_{\tau} \backslash \{s'\}| \geq q^{\tau}$. Therefore $[S' \backslash \{s'\}] \sqcup \{\hat{s}\}$ and S' are equidistant from q^T. Thus, by (DCR) we have $[S' \backslash \{s'\}] \sqcup \{\hat{s}\} \succ S'$, a contradicting with the fact that $S' \in \mathcal{C}(S)$. $\square$

(⇐): Suppose S' satisfies (L1)–(L4) but $S' \notin \mathcal{C}(S)$. Then there is $S'' \in \mathcal{C}(S)$ such that $S'' \succ S'$. Note that S'' satisfies (L1)–(L4) by part (⇒). Secondly, $d(S') = 2a$ if and only if $d(S'') = 2a$.

The symmetric difference of sets S' and S'' has an even number of elements, and we will prove the argument by induction on $\Delta = \frac{1}{2}(|S' \backslash S''| + |S'' \backslash S'|)$.

(Step $\Delta = 1$) Let $S' \backslash S'' = \{s'_1\}$ and $S'' \backslash S' = \{s''_1\}$. We know that $d(S') = d(S'') = 2a$, $S' \notin \mathcal{C}(S)$, and $S'' \in \mathcal{C}(S)$, therefore (DCR) implies $s''_1 \succ^{exo} s'_1$.

If $\tau(s'_1) \neq \tau(s''_1)$, then $|S''_{\tau(s'_1)}| = q^{\tau(s'_1)} - 1$, which contradicts with $s'_1 \in S''$ and (L3). Hence $\tau(s'_1) = \tau(s''_1)$. But now (L3) applied to S' implies $s'_1 \succsim^{exo} s''_1$, contradicting $s''_1 \succ^{exo} s'_1$.

(Step $\Delta = n$) Assume the conclusion holds for $\Delta < n$, and let $s'_1, \ldots, s'_n \in S' \backslash S''$, and

$s''_1, \ldots, s''_n \in S''\backslash S'$. Without loss of generality, we assume that $s'_1 \succsim^{exo} s'_2 \succsim^{exo} \cdots \succsim^{exo} s'_n$ and $s''_1 \succsim^{exo} s''_2 \succsim^{exo} \cdots \succsim^{exo} s''_n$.

Case 1: ($s'_1 \succ^{exo} s''_1$). Then $d([S''\backslash\{s''_1\}] \sqcup \{s'_1\}) = 2a$ and $[S''\backslash\{s''_1\}] \sqcup \{s'_1\} \succ S''$, a contradiction.

Case 2: ($s'_1 \sim^{exo} s''_1$). Then $S'' \sim [S''\backslash\{s''_1\}] \sqcup \{s'_1\}$. This means that $[S''\backslash\{s''_1\}] \sqcup \{s'_1\} \succ S'$ and $|([S''\backslash\{s''_1\}] \sqcup \{s'_1\}) \cap S'| = q - (n-1)$. This case reduces to $n-1$, and by assumption, the conclusion holds.

Case 3: ($s''_1 \succ^{exo} s'_1$). Then $s''_1 \in S\backslash S'$ and (L4) imply that $|S'_{\tau(s'_1)}| = q^{\tau(s'_1)}$. Note that $\tau(s'_1) \neq \tau(s''_1)$. Then

$$\begin{aligned} |S_{\tau(s'_1)} \cap S''| < q^{\tau(s'_1)} \quad &\text{if} \quad \tau(s''_i) \neq \tau(s'_1) \quad \forall i \in \{2, \ldots, n\}, \\ |S_{\tau(s'_1)} \cap S''| = q^{\tau(s'_1)} \quad &\text{if} \quad \tau(s''_i) = \tau(s'_1) \quad \exists i \in \{2, \ldots, n\}. \end{aligned}$$

Clearly, a case that $|S_{\tau(s'_1)} \cap S''| < q^{\tau(s'_1)}$ leads to a contradiction. When $|S_{\tau(s'_1)} \cap S''| = q^{\tau(s'_1)}$, since $|S_{\tau(s'_1)}| > q^{\tau(s'_1)}$, $s'_1 \succsim^{exo} s''_i$. Then $d([S''\backslash\{s''_i\}] \sqcup \{s'_1\}) = 2a$, and

$$[S''\backslash\{s''_i\}] \sqcup \{s'_1\} \succsim S''.$$

Since $S'' \in \mathcal{C}(S)$, it must be

$$[S''\backslash\{s''_i\}] \sqcup \{s'_1\} \sim S''.$$

Then $s'_1 \sim^{exo} s''_i$. Therefore $[S''\backslash\{s''_i\}] \sqcup \{s'_1\} \in \mathcal{C}(S)$ and $[S''\backslash\{s''_i\}] \sqcup \{s'_1\} \succ S'$. Notice that $|([S''\backslash\{s''_i\}] \sqcup \{s'_1\}) \cap S'| = q - (n-1)$, which reduces to $n-1$. $\square$

The following lemma says that given $\succsim$, if monotonicity of the acceptance and rejection correspondences are satisfied by pairs of sets $S \subseteq T$ with $|T\backslash S| = 1$, then $\succsim$ is substitutable.

Lemma 2 *Given a priority rule $\mathcal{C}$, suppose that for each $S \subseteq T$ with $|T\backslash S| = 1$, the following conditions hold*

(a) for each $T' \in \mathcal{C}(T)$, we have $T' \cap S \subseteq S'$ for some $S' \in \mathcal{C}(S)$, and

(b) for each $S' \in \mathcal{C}(S)$, we have $T' \cap S \subseteq S'$ for some $T' \in \mathcal{C}(T)$.

Then $\mathcal{C}$ is substitutable.

Proof of Lemma. We need to show that conditions (a) and (b) hold for arbitrary pairs of sets $S \subseteq T$. We will do induction on $k = |T \backslash S|$. We are given that the conditions hold whenever $k = 1$. Assuming that that hold for $k \leq m-1$, we need to verify them for $k = m$.

Let $|T \backslash S| = m$.

(a) Given $T' \in \mathcal{C}(T)$, pick some $t \in T \backslash T'$, and set $\tilde{T} = T \backslash \{t\}$. By the induction hypothesis, $T' \cap \tilde{T} \subseteq T''$ for some $T'' \in \mathcal{C}(\tilde{T})$. Since $T' \cap \tilde{T} = T'$ and $\mathcal{C}$ is acceptant, $T' = T''$ and $T' \in \mathcal{C}(\tilde{T})$.

If $t \notin S$, then $S \subseteq \tilde{T}$. Since $|\tilde{T} \backslash S| = m-1$, by the induction hypothesis $T' \cap S \subseteq S'$ for some $S' \in \mathcal{C}(S)$.

If, on the other hand, $t \in S$ for all $t \in T \backslash T'$, then let $\hat{T} = T \backslash \{\hat{t}\}$ for some $\hat{t} \notin S$. Such $\hat{t}$ is necessarily in T'. Again, by the induction hypothesis, $T' \backslash \{\hat{t}\} = T' \cap \hat{T} \subseteq T''$ for some $T'' \in \mathcal{C}(\hat{T})$.

$S \subseteq \hat{T}$, because $\hat{t} \notin S$. Using the induction hypothesis

$$T' \cap S = (T'' \sqcup \{\hat{t}\}) \cap S = T'' \cap S \subseteq S'' \quad \text{for some} \quad S'' \in \mathcal{C}(S).$$

(b) Given $S' \in \mathcal{C}(S)$, let $t \in T \backslash S$ and $\tilde{T} = T \backslash \{t\}$. Then $|\tilde{T} \backslash S| = k-1$, and there must exist $\tilde{T}' \in \mathcal{C}(\tilde{T})$ such that $\tilde{T}' \cap S \subseteq S'$.

$\tilde{T} \subseteq T$ and $|T \backslash \tilde{T}| = 1$, therefore there exists $T' \in \mathcal{C}(T)$ such that $T' \cap \tilde{T} \subseteq \tilde{T}'$. Intersection both sides with S, we get $S \cap T' \cap \tilde{T} \subseteq S \cap \tilde{T}'$. Since $S \cap \tilde{T} = S \cap T$, and $\tilde{T}' \cap S \subseteq S'$, we conclude $T' \cap S \subseteq S'$. $\square$

Proof of Proposition 7.

Lemma 2 allows us to conclude that $\succsim$ is substitutable by checking conditions (a) and (b) for pairs of sets S and T such that $|T \backslash S| = 1$. Since $\succsim$ is acceptant, it suffices to prove the claim when $|S| \geq q+1$. Thus, let $\{t\} = T \backslash S$ and $|S| \geq q+1$.

Proof of condition (a): Suppose $T' \in \mathcal{C}(S \sqcup \{t\})$ and $d(T') = 2a$. If $t \notin T'$, then $T' \subseteq S$ and properties (L1)–(L4) hold. Thus $T' \in \mathcal{C}(S)$, and condition (a) is trivially satisfied.

So now suppose that $t \in T'$.

Case 1. $|T_{\tau(t)}| \leq q^{\tau(t)}$. Then $S_{\tau(t)} \subseteq T'$, and for all $\hat{s} \in S \backslash T'$,

$$d([T' \backslash \{t\}] \sqcup \{\hat{s}\}) = 2a + 2.$$

Let $\hat{s}_1 \succsim^{exo} \hat{s}_2 \succsim^{exo} \cdots \succsim^{exo} \hat{s}_n$ be the students in $S \backslash T'$.

We claim that $S' = [T' \backslash \{t\}] \sqcup \{\hat{s}_1\} \in \mathcal{C}(S)$. Clearly $|S'| = q$.

For all S'_τ with $|S'_\tau| \leq q^\tau$, $S'_\tau = T'_\tau \subseteq S'$ if $\tau \neq \tau(t)$. $S'_{\tau(t)} = T'_{\tau(t)} \backslash \{t\} \subseteq S'$.

For all S'_τ with $|S'_\tau| > q^\tau$, since $\hat{s}_1 \succsim^{exo} \hat{s}$ for all $\hat{s} \in S \backslash T'$ and

$$q^{\tau(\hat{s}_1)} \leq |T'_{\tau(\hat{s}_1)}| < |S'_{\tau(\hat{s}_1)}|,$$

(L3) and (L4) hold. By Lemma 1, $S' \in \mathcal{C}(S)$.

Case 2. $|T_{\tau(t)}| > q^\tau$. Then $|T'_{\tau(t)}| \geq q^\tau$. If $|T'_{\tau(t)}| = q^{\tau(t)}$, then there is $\hat{s} \in S \backslash T'$ such that $\tau(\hat{s}) = \tau(t)$. For these $\hat{s}$, $d([T' \backslash \{t\}] \sqcup \{\hat{s}\}) = 2a$. By (DCR), we can find $\hat{s}_1$ who is at least as good as any other agent who is in $S_{\tau(t)}$. It is easy to see that $[T' \backslash \{t\}] \sqcup \{\hat{s}_1\}$ satisfies (L1)–(L4).

Otherwise, $|T'_{\tau(t)}| > q^{\tau(t)}$, in which case

$$d([T' \backslash \{t\}] \sqcup \{\hat{s}\}) = 2a$$

for all $\hat{s} \in S \backslash T'$. If $\hat{s}_1 \succsim^{exo} \hat{s}_2 \succsim^{exo} \cdots \succsim^{exo} \hat{s}_n$ are the students in $S \backslash T'$, we conclude that $[T' \backslash \{t\}] \sqcup \{\hat{s}_1\}$ satisfies (L1)–(L4). $\Diamond$

Proof of condition (b): We will show that the rejection correspondence $\mathcal{R}$ is monotonic in the sense that for each $R' \in \mathcal{R}(S)$, we have $R' \subseteq T''$ for some $T'' \in \mathcal{R}(T)$. Condition (b) will then follow from Remark 2.

Suppose $R' \in \mathcal{R}(S)$. By definition, $R' = S \backslash S'$ for some $S' \in \mathcal{C}(S)$. We claim that there is $\hat{s} \in (S \sqcup \{t\}) \backslash R'$ such that $R' \sqcup \{\hat{s}\} \in \mathcal{R}(S \sqcup t)$.

Case 1. $|(S \sqcup \{t\})_{\tau(t)}| \leq q^{\tau(t)}$. Then $|S_{\tau(t)}| < q^{\tau(t)}$. This implies that there is τ' such that $|S'_{\tau'}| > q^{\tau'}$. Then for all $s_i \in R'_{\tau'}$, $\hat{s} \succsim^{exo} s_i$ for all $\hat{s} \in S'_{\tau'}$. Consider such types $\{\tau'_1, \ldots, \tau'_m\}$. Take $s^* \in \bigcup_{i \in \{1,\ldots,m\}} S_{\tau'_i}$ in a way that $\hat{s} \succsim^{exo} s^*$ for all $\hat{s} \in \bigcup_{i \in \{1,\ldots,m\}} S_{\tau'_i}$. We see that $R' \sqcup \{s^*\} \in \mathcal{R}(S \sqcup \{t\})$. Let $S'' = (S \sqcup \{t\}) \backslash (R' \sqcup \{s^*\})$. Since $|(S \sqcup \{t\})_{\tau(t)}| \leq q^{\tau(t)}$, $(S \sqcup \{t\})_{\tau(t)} \subseteq S''$. For $\tau(s^*)$, $|S''_{\tau(s^*)}| \geq q^{\tau(s^*)}$ and by construction, S'' satisfies other properties. Hence, $S'' \in \mathcal{C}(S \sqcup \{t\})$ if and only if $R' \sqcup \{s^*\} \in \mathcal{R}(S \sqcup \{t\})$.

Case 2. $|(S \sqcup t)_{\tau(t)}| > q^{\tau(t)}$. Then we can find $\hat{s} \in (S \sqcup \{t\})_{\tau(t)}$ such that $s_i \succsim^{exo} \hat{s}$ for all $s_i \in (S \sqcup \{t\})_{\tau(t)}$. Consider τ' such that $|S'_{\tau'}| > q^{\tau'}$ and let them be $\{\tau'_1, \ldots, \tau'_m\}$ (possibly empty). If for all such τ'_i, $s \succsim^{exo} \hat{s}$, for all $s \in S'_{\tau'_i}$ or there is no such τ_i, then $S \backslash (R' \sqcup \{\hat{s}\})$ satisfies (L1)–(L4) and we are done. Otherwise there is $s_i \in S'_{\tau'_i}$ such that $\hat{s} \succ^{exo} s_i$ for some $i \in \{1, \ldots, m\}$. Then we can find $\hat{s}_i$ such that $s_i \succsim^{exo} \hat{s}_i$ for all $s_i \in S'_{\tau'_i}$. Let s^* be such that $\hat{s}_i \succsim^{exo} s^*$ for any i. Then it also easy to see that $S'' = S \backslash (R' \sqcup \{s^*\})$ satisfies (L1)–(L4). Therefore, $R' \sqcup \{s^*\} \in \mathcal{R}(S \sqcup \{t\})$.

Hence condition (b) holds. $\square$

Proof of Proposition 8

It follows from Propositions 4 and 7 that if an assignment stable with respect to $\mathcal{C}$ does not admit an SSIC, then it is constrained efficient. For the other direction, it suffices to show that if a stable assignment admits an SSIC, then the assignment obtained by carrying out the shortest SSIC is stable.

As in the proof of Proposition 5, consider the shortest SSIC, and denote by ν the assignment obtained by carrying out this SSIC. Suppose, for a contradiction, that ν is not stable. First of all, it is straightforward to see that if the schools $\mu(i_0), \ldots, \mu(i_{n-1})$ involved in this cycle were distinct, ν would be stable. So let us assume that the schools are not distinct. Again, as in the proof of Proposition 5, if (j, x) is a blocking pair, then the school x must appear more than once in this SSIC. Let x be involved in moves $i_{k^t} \to i_{k^t+1}$ such that $\nu(i_{k^t}) = \mu(i_{k^t+1}) = x$ for $t = 1, \ldots, m$.

Claim: $\tau(i_{k^t}) \neq \tau(i_{k^u})$ for every $t, u \in \{1, \ldots, m\}$ and $t \neq u$.

Proof of claim. Assume otherwise, and thus $\tau = \tau(i_{k^t}) = \tau(i_{k^u})$ for some t and $u \neq t$. Since $i_{k^t} \in E^{\mu}_{i_{k^t+1}}$ and $i_{k^u} \in \mathcal{E}^{\mu}_{i_{k^t+1}}$,

$$[\mu^{-1}(x) \backslash \{i_{k^t+1}\}] \sqcup \{i_{k^t}\} \succsim [\mu^{-1}(x) \backslash \{i_{k^t+1}\}] \sqcup \{i_{k^u}\}.$$

Since i_{k^t} and i_{k^u} are of the same type, we have

$$d([\mu^{-1}(x) \backslash \{i_{k^t+1}\}] \sqcup \{i_{k^t}\}) = d([\mu^{-1}(x) \backslash \{i_{k^t+1}\}] \sqcup \{i_{k^u}\}),$$

which implies by (DCR) that

$$i_{k^t} \succsim^{exo} i_{k^u}.$$

Using a symmetric argument, we must have $i_{k^u} \succsim^{exo} i_{k^t}$, and therefore

$$i_{k^t} \sim^{exo} i_{k^u}.$$

Hence $i_{k^u} \in E^{\mu}_{k^t+1}$, allowing us to construct a shorter SSIC. $\Diamond$

Since (j, x) is a blocking pair, we know that $j \in DC_x(\nu^{-1}(x) \cup \{j\})$ and $x P_j \mu(j)$.

Case 1. $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(j)}| \leq q_{\tau(j)}$. Then $|(\nu^{-1}(x))_{\tau(j)}| \leq q_{\tau(j)} - 1$. On the other hand, since μ is stable and j is not in $\mu^{-1}(x)$, $|(\mu^{-1}(x))_{\tau(j)}| \geq q_{\tau(j)}$. That means at least one student of type $\tau(j)$ must have left x in this SSIC. Say i_{k^t+1} is that student. Now since $i_{k^t} \in E^{\mu}_{i_{k^t+1}}$, and $j \in \mathcal{E}^{\mu}_{i_{k^t+1}}$, we must have $\tau(i_{k^t}) = \tau(j)$. That means another student of type $\tau(j)$ replaces i_{k^t+1} in the SSIC. Therefore, there must some other student, say i_{k^u+1}, of type $\tau(j)$ who leaves x in the SSIC. By the same argument, we must have $\tau(i_{k^u}) = \tau(j)$. Thus we have $\tau(i_{k^t}) \neq \tau(i_{k^u})$, contradicting the claim.

Case 2. $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(j)}| > q_{\tau(j)}$. Since $j \in DC_x(\nu^{-1}(x) \sqcup \{j\})$, we have $j \in A$ for all $A \in \mathcal{C}_x(\nu^{-1}(x) \sqcup \{j\})$. Moreover $|A_{\tau(j)}| \geq q_{\tau(j)}$ and

$$j \succsim^{exo} \ell \text{ for all } \ \ell \in (\nu^{-1}(x) \sqcup \{j\})_\tau \backslash A_{\tau(j)}.$$

If $\ell = i_{k^t}$ for some t, then stability of μ and $j P_j \mu(j)$ imply $|(\mu^{-1}(x))_{\tau(j)}| \geq q_{\tau(j)}$. Therefore $|(\mu^{-1}(x) \sqcup \{j\})_{\tau(j)}| \geq q_{\tau(j)} + 1$. Since $\mu(i_{k^t}) = x$, we necessarily have $i_{k^t} \succsim^{exo} j$. Thus $i_{k^t} \sim^{exo} j$, and $d([\nu^{-1}(x) \backslash \{i_{k^t}\}] \sqcup \{j\}) = d(\nu^{-1}(x))$. Since $[\nu^{-1}(x) \backslash \{i_{k^t}\}] \sqcup \{j\} \in \mathcal{C}(\nu^{-1}(x) \sqcup \{j\})$, we must also have $\nu^{-1}(x) \in \mathcal{C}(\nu^{-1}(x) \sqcup \{j\})$, contradicting with $j \in DC_x(\nu^{-1}(x) \sqcup \{j\})$.

If there is no i_{k^t} such that $i_{k^t} = \ell$, then $\mu(\ell) = \nu(\ell) = x$. Again, stability of μ implies that $\ell \succsim^{exo} j$, and therefore $\ell \sim^{exo} j$. This, in turn, implies $d([\nu^{-1}(x) \backslash \ell] \sqcup \{j\}) = d(\nu^{-1}(x))$, and hence $\nu^{-1}(x) \in \mathcal{C}(\nu^{-1}(x) \sqcup \{j\})$, contradicting with $j \in DC_x(\nu^{-1}(x) \sqcup \{j\})$. $\square$

B Appendix: Remarks

Remark 2 Condition (b) in Definition 1 is equivalent to the monotonicity of the rejection correspondence. In other words, given $S \subseteq T$

(b) for each $S' \in \mathcal{C}_x(S)$, we have $T' \cap S \subseteq S'$ for some $T' \in \mathcal{C}_x(T)$

if and only if

(b') for each $S'' \in \mathcal{R}_x(S)$, we have $S'' \subseteq T''$ for some $T'' \in \mathcal{R}_x(T)$, where

$$\mathcal{R}_x(S) = \{S'' \subseteq S \mid S'' = S \backslash S' \text{ for some } S' \in \mathcal{C}_x(S)\}.$$

In order to see (b) $\Longrightarrow$ (b'), let $S'' \in \mathcal{R}_x(S)$. Then $S'' = S \backslash S'$ for some $S' \in \mathcal{C}_x(S)$, and hence $T' \cap S \subseteq S'$ for some $T' \in \mathcal{C}_x(T)$. Taking complements in T, we get

$$T \backslash (T' \cap S) \supseteq T \backslash S' \supseteq S \backslash S' = S''.$$

Set $T'' = T \backslash T'$. Since $T \backslash (T' \cap S) = (T \backslash T') \cup (T \backslash S)$, the last inclusion yields

$$\begin{aligned} T'' \cup (T \backslash S) &\supseteq S'' \\ (T'' \cup (T \backslash S)) \cap S &\supseteq S'' \cap S = S'' \\ T'' \cap S &\supseteq S'', \end{aligned}$$

which implies $T'' \supseteq S''$.

Now let us verify (b') $\Longrightarrow$ (b). If $S' \in \mathcal{C}_x(S)$, then $S' = S \backslash S''$ for some $S'' \in \mathcal{R}_x(S)$. Condition (b') requires $S'' \subseteq T''$ for some $T'' \in \mathcal{R}_x(T)$. Again, taking complements in T

$$T \backslash S'' \supseteq T \backslash T'' = T'.$$

Intersection with S yields

$$\begin{aligned} (T \backslash S'') \cap S &\supseteq T' \cap S \\ T \cap (S'')^c \cap S &\supseteq T' \cap S \\ S \cap (S'')^c &\supseteq T' \cap S \\ S' &\supseteq T' \cap S. \end{aligned}$$

$\Diamond$

Remark 3 When $\mathcal{C}_x$ is not a function, conditions (a) and (b) in Definition 1 are not necessarily equivalent. Neither condition implies the other:

(1) A priority order $\succsim_x$ for which the associated choice correspondence is monotonic, but the rejection correspondence is not: Suppose there are four students $\{i_1, i_2, i_3, i_4\}$, and a school x with two seats and the following priority ranking: $\{i_1, i_4\} \succ_x \{i_1, i_2\} \sim_x \{i_1, i_3\} \sim_x \{i_2, i_3\} \sim_x \{i_2, i_4\} \sim_x \{i_3, i_4\}$. It is readily verified that $\mathcal{C}_x$ satisfies condition (a). To see that $\succsim_x$ does not satisfy condition (b), note that

$$\mathcal{R}_x(\{i_1, i_2, i_3\}) = \{\{i_1\}, \{i_2\}, \{i_3\}\},$$

but

$$\mathcal{R}_x(\{i_1, i_2, i_3, i_4\}) = \{\{i_2, i_3\}\},$$

hence

$$\{i_1\} \nsubseteq \{i_2, i_3\}.$$

(2) A priority order $\succsim_x$ for which the associated rejection correspondence is monotonic, but the choice correspondence is not: Suppose there are five students $\{i_1, i_2, i_3, i_4, i_5\}$, and a school x with two seats and the following priority ranking: $\{i_1, i_2\} \sim_x \{i_3, i_4\} \succ_x \{i_1, i_3\} \sim_x \{i_1, i_4\} \sim_x \{i_1, i_5\} \sim_x \{i_2, i_3\} \sim_x \{i_2, i_4\} \succ_x \{i_2, i_5\} \sim_x \{i_3, i_5\} \sim_x \{i_4, i_5\}$.

Condition (b) can be verified straightforwardly. On the other hand condition (a) fails, because

$$\mathcal{C}_x(\{i_1, i_2, i_3, i_4\}) = \{\{i_1, i_2\}, \{i_3, i_4\}\},$$

but

$$\mathcal{C}_x(\{i_1, i_2, i_3\}) = \{\{i_1, i_2\}\},$$

hence

$$\{i_3\} = \{i_3, i_4\} \cap \{i_1, i_2, i_3\} \not\subseteq \{i_1, i_2\}.$$

$\Diamond$

Remark 4 In an even more general formulation of priorities which allows *any* type of ties between sets of students, a stability-preserving Pareto improvement does not necessarily follow from cyclical trades of students, in which each individual move preserves stability. Even if there is a cycle of students $i_1 \to i_2 \to \cdots \to i_m \to i_1$ such that each student can replace the next one without violating stability, the cycle does not necessarily preserve stability.[19] In order to see this, let $N = \{i_1, i_2, i_3, i_4, i_5\}$. Suppose that we have two schools x and y with $q_x = 2$, q_y=2. Students' preferences are:

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}
x	y	x	y	x
y	x	y	x	

And the priority orders are:

$\succsim_x$	$\succsim_y$
$\{i_2, i_4\}$	$\{i_1, i_2\}, \{i_1, i_3\}, \{i_3, i_4\}$
$\{i_1, i_4\}, \{i_2, i_3\}, \{i_2, i_5\}, \{i_4, i_5\}$	$\{i_1, i_4\}, \{i_2, i_3\}, \{i_3, i_4\}$
$\{i_1, i_3\}, \{i_3, i_5\}$	the rest
$\{i_1, i_2\}, \{i_1, i_5\}, \{i_3, i_4\}$	

The priority structure is acceptant and substitutable. Let μ be

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ y & x & y & x & i_5 \end{pmatrix}$$

One can verify that μ is stable. Consider the following replacement cycle

$$i_1 \to i_2 \to i_3 \to i_4 \to i_1,$$

[19]This is in contrast with the case of responsive priorities studied in Erdil and Ergin (2008).

in which each student can replace the next, because

$$\begin{aligned}
\{i_1, i_4\} \in \mathcal{C}_x(\{i_1, i_3, i_4, i_5\}) &\Longrightarrow i_1 \in E^{\mu}_{i_2} \\
\{i_1, i_2\} \in \mathcal{C}_y(\{i_1, i_2, i_4\}) &\Longrightarrow i_2 \in E^{\mu}_{i_3} \\
\{i_2, i_3\} \in \mathcal{C}_x(\{i_1, i_2, i_3, i_5\}) &\Longrightarrow i_3 \in E^{\mu}_{i_4} \\
\{i_3, i_4\} \in \mathcal{C}_y(\{i_2, i_3, i_4\}) &\Longrightarrow i_4 \in E^{\mu}_{i_1}.
\end{aligned}$$

Now construct ν by carrying out the above cycle:

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & y & x & y & i_5 \end{pmatrix}$$

Clearly, ν Pareto dominates μ. However, ν is not stable, because $\mathcal{C}_x(i_1, i_3, i_5) = \{\{i_1, i_5\}, \{i_3, i_5\}\}$, so (x, i_5) blocks ν. In fact, μ is constrained efficient. Note that the only Pareto improvement over μ are the following:

$$\mu_1 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & y & y & x & i_5 \end{pmatrix}$$

$\mathcal{C}_y(\{i_2, i_3, i_4\}) = \{i_3, i_4\}$, so (b, i_4) blocks μ_1.

$$\mu_2 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & x & y & y & i_5 \end{pmatrix}$$

$\mathcal{C}_x(\{i_1, i_2, i_3\}) = \{i_2, i_3\}$, so (x, i_3) blocks μ_2.

$$\mu_3 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ y & y & x & x & i_5 \end{pmatrix}$$

$\mathcal{C}_x(\{i_1, i_3, i_4\}) = \{i_1, i_4\}$, so (x, i_1) blocks μ_3.

$$\mu_4 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ y & x & x & y & i_5 \end{pmatrix}$$

$\mathcal{C}_y(\{i_1, i_2, i_4\}) = \{i_1, i_2\}$, so (y, i_2) blocks μ_4.

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & y & x & y & i_5 \end{pmatrix}$$

ν is not stable as pointed out before. Hence, there is no stable assignment which Pareto dominates μ, and so it is constrained efficient. $\Diamond$

Remark 5 When priorities are allowed to be substitutable, the stable improvement cycles as defined in Erdil and Ergin (2008) do not capture every stability preserving Pareto improvement. Let x, y, z, w be distinct schools with $q_x = q_y = 2$ and $q_z = q_w = 1$. We have six students: $i, j, k_i, k_j, \ell_i, \ell_j$ and an assignment μ:

$$\mu = \begin{pmatrix} i & j & k_i & k_j & \ell_i & \ell_j \\ x & x & y & y & w & z \end{pmatrix}.$$

The priority structure is such that

$$\mathcal{C}_x(\ell_i, \ell_j, i, j) = \{i, j\} \qquad \mathcal{C}_x(\ell_i, \ell_j, j) = \{\ell_i, j\} \qquad \mathcal{C}_x(\ell_i, \ell_j, i) = \{\ell_j, i\}$$

and

$$\mathcal{C}_y(k_i, k_j, i, j) = \{k_i, k_j\} \qquad \mathcal{C}_y(i, j, k_i) = \{i, k_i\} \qquad \mathcal{C}_y(i, j, k_j) = \{j, k_j\}$$

Suppose the preferences R are as below, where the boxes indicate the respective students' assignments under μ:

R_i	R_j	R_{k_i}	R_{k_j}	R_{ℓ_i}	R_{ℓ_j}
y	y	w	z	x	x
$\boxed{x}$	$\boxed{x}$	$\boxed{y}$	$\boxed{y}$	$\boxed{w}$	$\boxed{z}$

With these preferences, μ respects the priorities $\mathcal{C}$, but obviously, letting each student get their most preferred school preserves stability, while Pareto improving over μ. Can this improvement be achieved through Erdil and Ergin's stable improvement cycles? The only "candidates" for such cycles are shown in Figure 2.

However, none of these cycles preserves stability. The first one fails, because $\mathcal{C}_x(\ell_i, \ell_j, j) = \{\ell_i, j\}$, and thus ℓ_j cannot replace i at school x. The second cycle fails to preserve stability, because $\mathcal{C}_y(i, j, k_j) = \{j, k_j\}$, and therefore i cannot replace k_i. Likewise, the third one fails, because j cannot replace k_j at y, while for the fourth one ℓ_i cannot replace j at x.

Yet all of the students can move to their favorite schools while preserving stability. Thus, we can express a stability preserving student improving cycle illustrated in Figure 3 as a cycle of students (as opposed to a cycle of schools[20] as in Erdil and Ergin, 2008):

$$i \longrightarrow k_j \longrightarrow \ell_j \longrightarrow j \longrightarrow k_i \longrightarrow \ell_i \longrightarrow i$$

◊

◊

[20] Note that this example does not rely on ties.

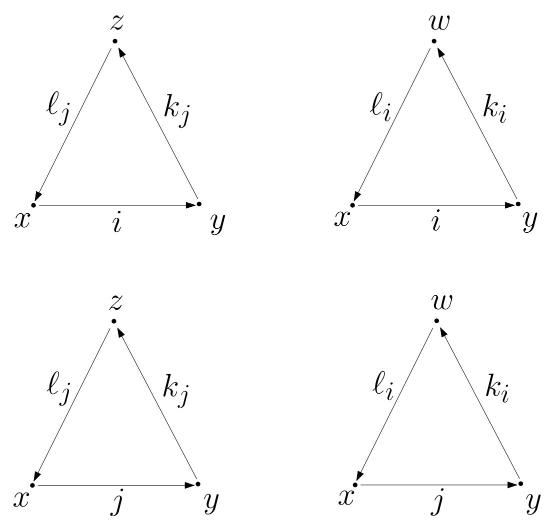

Figure 2: These cycles of schools are the only candidates for stable improvement cycles in the sense of Erdil and Ergin (2008).

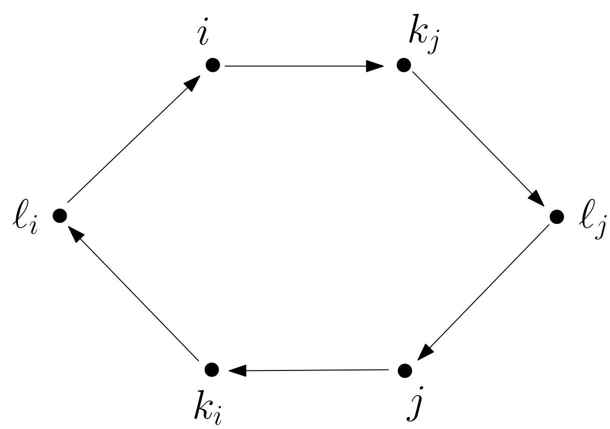

Figure 3: A stable student improvement cycle, in which the notation now indicates that i replaces k_j, and so on.

Remark 6 If $\mathcal{C}$ is strongly acyclic, μ is stable and ν Pareto dominates μ, then ν is stable. This result immediately leads to an algorithm to reach an efficient assignment. After running the GDA, we can simply run the Top Trading Cycles algorithm to find an efficient assignment. $\square$

References

[1] A. Abdulkadiroğlu, College admissions with affirmative action, *Int. J. Game Theory* 33 (2005), 535–549.

[2] A. Abdulkadiroğlu, P. A. Pathak, A. E. Roth, Strategy-proofness versus efficiency in matching with indifferences: redesigning the NYC high school match, *Amer. Econ. Review* 99 (2009), 1954–1978.

[3] A. Abdulkadiroğlu, P. A. Pathak, A. E. Roth, T. Sönmez, The Boston public school match, *Amer. Econ. Review (Papers and Proceedings)* 95 (2005), 368–371.

[4] A. Abdulkadiroğlu, T. Sönmez, School choice: a mechanism design approach, Amer. Econ. Review 93 (2003), 729–747.

[5] M. Balinski, T. Sönmez, A Tale of Two Mechanisms: Student Placement, *Journal of Economic Theory* 84 (1999), 73–94.

[6] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, Cambridge: MIT Press, 2001.

[7] V. P. Crawford, E. M. Knoer, Job Matching with Heterogeneous Firms and Workers, *Econometrica*, Vol. 49 (March 1981), 437–450.

[8] F. Echenique, M. B. Yenmez, How to control controlled school choice (2013), Caltech and Carnegie Mellon Working Paper.

[9] L. Ehlers, Respecting Priorities when Assigning Students to Schools (2007), University of Montreal Working Paper.

[10] L. Ehlers, School Choice with Control (2011), University of Montreal Working Paper.

[11] L. Ehlers, A. Erdil, Efficient assignment respecting priorities, *J. Econ. Theory* 145 (2010), 1269–1282.

[12] L. Ehlers, I. Hafalir, M. B. Yenmez, M. Yildirim (2011), School Choice with Controlled Choice Constraints: Hard Bounds versus Soft Bounds, Working Paper.

[13] A. Erdil, H. Ergin, What's the matter with tie-breaking? Improving efficiency in school choice, *Amer. Econ. Review* 98 (2008), 669–689.

[14] H. Ergin, Efficient resource allocation on the basis of priorities, *Econometrica* 70 (2002), 2489–2497.

[15] D. Gale, L. Shapley, College admissions and the stability of marriage, *Amer. Math. Monthly* 69 (1962), 9–15.

[16] I. E. Hafalir, M. B. Yenmez, M. A. Yildirim, Effective Affirmative Action in School Choice, *Theoretical Economics* (2012), forthcoming.

[17] J. Hatfield, P. Milgrom, Matching with Contracts, *Amer. Econ. Review* 95 (2005), 913–935.

[18] A. Kelso, V. Crawford, Job Matching, Coalition Formation, and Gross Substitutes, *Econometrica*, Vol. 50, No. 6 (Nov., 1982), pp. 1483–1504.

[19] Y. Kamada, F. Kojima, Improving Efficiency in Matching Markets with Regional Caps: The Case of the Japan Residency Matching Program, Stanford University Working Paper (2011).

[20] F. Kojima, School Choice: Impossibilities for Affirmative Action, *Games and Economic Behavior*, Volume 75, Issue 2, July 2012, Pages 685–693.

[21] S. D. Kominers, T. Sönmez, Designing for Diversity: Matching with Slot-Specific Priorities, Working Paper (2012)

[22] T. Kumano, Efficient Resource Allocation under Acceptant Substitutable Priorities, Washington University Working Paper (2009).

[23] A. E. Roth, The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory, *Journal of Political Economy*, 92 (1984), 991–1016.

[24] A. E. Roth, M. Sotomayor, *Two-Sided Matching: A Study in Game-Theoretic Modelling and Analysis*, Cambridge University Press, Cambridge, England, [Econometric Society Monograph], 1990.

[25] M. Sotomayor, Three remarks on the many-to-many stable matching problem, *Mathematical Social Sciences*, 38 (1999) 55–70.

[26] A. Westkamp, An analysis of the German university admissions system, *Economic Theory*, online (2012).