

Comparative Measures of Naiveté*

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Abstract

We propose nonparametric definitions of absolute and comparative naiveté. These definitions leverage ex-ante choice of menu to identify predictions of future behavior and ex-post choices from menus to identify actual behavior. The definitions' main advantage is their independence from any assumed functional form for the utility function representing behavior. An individual is sophisticated if she is indifferent between choosing from a menu ex post or committing to the actual choice from that menu ex ante. She is naive if she prefers the flexibility in the menu, reflecting a mistaken belief that she will act more virtuously than she actually will. One individual is more naive than another if she is both more optimistic about her future behavior while actually being less virtuous. In the case of Strotzian preferences, absolute naiveté implies that beliefs are a convex combination of virtuous and temptation utility, while comparative naiveté implies that the more naive individual's beliefs puts relatively more weight on her virtuous utility, while her actual behavior puts more weight on the temptation utility. In different specifications of the underlying Strotzian preferences, such as quasi-hyperbolic discounting, the definitions impose further intuitive restrictions, such as inequalities on believed and actual present-bias factors. We propose suitable definitions for random choice. Finally, we discuss the implications of general naiveté for welfare and the design of commitment devices.

KEYWORDS: Naive, sophisticated, time inconsistent, comparative statics

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1 Introduction

Models of dynamic inconsistency play an important role in a wide-ranging set of applications in economics. Of particular recent interest are the implications of naiveté when individuals mispredict their future behavior.¹ While naiveté often carries important consequences in applications, its effects are usually understood within the context of a specific utility representation of behavior. For example, [O’Donoghue and Rabin \(2001\)](#) generalize the standard (β, δ) model of sophisticated quasi-hyperbolic discounting to allow for partial naiveté by adding an additional present-bias parameter $\hat{\beta}$ to capture the individual’s belief about her future present-bias; this belief can be overly optimistic with $\hat{\beta} > \beta$. The resulting $(\beta, \hat{\beta}, \delta)$ model suggests natural comparisons of naiveté through its parameters. For example, larger values of $\hat{\beta}$ intuitively correspond to more naive individuals.

This parametric evaluation of naiveté relies on a particular utility function. We provide general nonparametric definitions of naiveté and sophistication that are divorced from functional form assumptions. Our definitions are worded directly on choice primitives, and not on the components of a utility function. To understand the importance of the exercise, as an analogy consider the theory of risk and the definition of risk aversion as having a concave utility index for wealth $v'' < 0$. This definition makes sense only under the expected-utility hypothesis, as otherwise the invoked v does not exist. But the notion of risk aversion seems more fundamental, since many people might not maximize expected utility but still consider themselves risk-averse. A more basic and satisfying criterion for risk aversion is that the individual prefers certainly receiving the expected value of a monetary lottery rather than to expose herself to its uncertainty. This definition is workable without assuming any utility representation, and can be directly tested without estimating the parameters of a structured model. Understanding naiveté through the parameter $\hat{\beta}$ of the quasi-hyperbolic discounting model has similar limitations relative to a definition of naiveté phrased on choices, and this paper takes steps to overcome those limitations.

Beyond improved theoretical foundations, model-free definitions of naiveté provide important substantive advantages. They permit an examination of which positive predictions in applications rely on functional-form assumptions and which predictions are inherent features of naiveté. For example, a more risk-accepting investor will always choose a risky equity position over a risk-free bond whenever a more risk-averse investor does. Similarly, we can ask whether predictions regarding savings or procrastination are artifacts of an assumed utility or are robust implications of naiveté. In turn, a deeper understanding of the mechanics of naive choice also improves normative analysis. In particular, effective design of commitment devices can hinge crucially on the assumed level of

¹A recent survey of empirical applications can be found in Section 2.1 of [DellaVigna \(2009\)](#) and a survey of some theoretical applications in contract theory can be found in [Koszegi \(2014\)](#).

sophistication. Duflo, Kremer, and Robinsin (2011) examine a theoretical model where the optimal timing of when to offer a commitment depends on whether individuals are sophisticated or naive regarding the degree of their present bias, and provide evidence from Kenyan fertilizer adoption that individuals are naive and would benefit from earlier and time-limited commitments. More general definitions of naiveté provide a language broad enough to understand the consequence of policy interventions when citizens have qualitatively different forms of naiveté and are best approximated by a variety of formal models, and which policies work for which assumed models.

We propose definitions for testing the naiveté of a single agent and for comparing levels of naiveté across agents. We leverage two pieces of choice data. First, we use preference for commitment to measure *anticipated* behavior from an ex-ante perspective before the realization of temptation. Formally, the individual’s preferences over different menus captures her demand for commitment and allows an inference of her beliefs regarding her future behavior. Second, fixing a level of commitment, we use choices from availabilities to measure *actual* behavior from an ex-post perspective under the influence of temptation.

To test naiveté and sophistication, our definition evaluates the agent’s value for a menu x of different options and her actual ex-post choice $\mathcal{C}(x)$ from x . An individual is sophisticated if she is indifferent between maintaining the flexibility to choose from x later or committing to her eventual choice $\mathcal{C}(x)$ now, i.e., if $x \sim \{\mathcal{C}(x)\}$ from her ex-ante perspective. If she is naive, she believes that she will make a more virtuous choice, so prefers to maintain the flexibility in x , i.e., if $x \succsim \{\mathcal{C}(x)\}$. For example, a naive gym member believes she will work out more than she actually will.

To compare naiveté across agents, our definition again evaluates ex-post and ex-ante behavior. Comparative naiveté has two components. First, the more naive agent, say individual 1, is more optimistic about her future virtue than the more sophisticated agent, say individual 2. This is measured by comparing her relative demand for commitment: the naive agent’s optimism means she is less willing to commit to single options now, while the more sophisticated (and less optimistic) agent is more eager to make commitments to avoid future temptation. Formally, $\{p\} \succ_2 x$ whenever $\{p\} \succ_1 x$, where $\{p\}$ is a commitment to consume p in the future. Second, the more naive agent is actually less virtuous. This is measured by her choice from x . Formally, $\{p\} \succ_1 \{\mathcal{C}_1(x)\}$ whenever $\{p\} \succ_2 \{\mathcal{C}_2(x)\}$. A more naive gym member believes she will work out more but actually works out less than the more sophisticated gym member.

The proposed definitions characterize sharp and intuitive functional inequalities in a variety of special models. Consider the Strotzian model of dynamic inconsistency, where ex-ante normative behavior is dictated by one utility function u while ex-post behavior under temptation is dictated by another utility function v . To allow for naiveté, the individual believes that \hat{v} , possibly not equal to v , governs her future behavior. Our

definition of naiveté implies that the believed \hat{v} is a linear combination of the virtuous utility u and the actual temptation utility v . Our comparison of naiveté implies that more naive individuals put more weight on u . For more structured specifications of the Strotz utilities, the definitions continue to yield interpretable restrictions. In the case of consumption over time dictated by the $(\beta, \hat{\beta}, \delta)$ model of O’Donoghue and Rabin (2001), our definition of absolute naiveté implies that $\hat{\beta} \geq \beta$. Our definition of comparative naiveté implies that if individual 1 with parameters $(\beta_1, \hat{\beta}_2, \delta_1)$ is more naive than individual 2 with parameters $(\beta_1, \hat{\beta}_2, \delta_2)$, then $\delta_1 = \delta_2$ and $\hat{\beta}_1 \geq \hat{\beta}_2 \geq \beta_2 \geq \beta_1$.

We present suitable generalizations of these definitions for the case of random choice under uncertain temptations. The only variation is that the determinate choice from a menu is replaced with the average choice under the random choice rule. The random Strotz model of Dekel and Lipman (2012) allows for future temptation to be randomly realized from a set of multiple temptations. For example, an individual on a diet might be tempted to eat salty snacks or might be tempted to eat sweet desserts. For this model, the random analogs of naiveté and sophistication yield suitable generalized implications: the more naive individual’s probability over her future temptations is more optimistic in the sense that it puts more weight on lower intensities of temptations. Note that the randomization is important even if actual ex-post choice is deterministic, as long as individuals are uncertain about their future behavior. The generalization to random choice has substantive significance, as models of naiveté with uncertainty are common in applications. For example, the predictions in Dufflo, Kremer, and Robinsom (2011) depend on farmers being uncertain, as well as naive, of their future present-bias.

Our use of ex-ante and ex-post behavior has several precedents. In fact, empirical studies of naiveté also invoke ex-ante and ex-post observations. Purchases of gym memberships in DellaVigna and Malmendier (2009) are taken to occur ex ante and before the experience of temptation, while the visits to the gym are taken to occur ex post when facing the temptation to shirk from exercise. Shui and Ausubel (2004) observe consumers’ choices of credit card contracts (assumed ex ante) and their subsequent borrowing behavior (assumed ex post). In perhaps the closest existing match to our primitives, Augenblick, Niederle, and Sprenger (2013) report an experiment where subjects can choose to commit to levels of work effort ex ante and then exert actual effort ex post. These two-tiered observations allow structural estimation of the parameters within specific models of naive choice. With our proposed definitions, the hypothesis that one person is more naive than another can be rejected from a few choice observations without the need for calibrated parameters.

There are also papers in decision theory that use behavior at different time periods to capture sophistication. Lipman and Pesendorfer (2013) provide a survey of these papers, and we will borrow their formalisms for their unifying notation. Noor (2011) considers preferences over a recursive domain that includes ex-ante and ex-post choice preferences

as projections and pioneered the approach of using temporal choice as a domain for explicitly testing the sophistication implicitly assumed in most ex-ante axiomatic models of temptation. [Kopylov \(2012\)](#) relaxes Noor’s sophistication condition and considers agents who choose flexibility ex ante that is subsequently unused ex post. Kopylov exchews mistaken or naive beliefs, but rather interprets the relaxation of sophistication as reflecting a direct psychic benefit of maintaining positive self-image. These papers are discussed more specifically after we introduce our definitions. Finally, [Dekel and Lipman \(2012\)](#) observe that ex-ante and ex-post choice can be combined to empirically distinguish random Strotz representations from others that involve costly self-control. Much of the technical apparatus from [Dekel and Lipman \(2012\)](#) ends up being useful in studying naiveté, as we will explain in the body of the paper.

The next section introduces our formal primitives. Section 3 considers the specialized case of deterministic choice and introduces appropriate definitions of naiveté and sophistication. Their implications are characterized for the Strotz model of dynamic inconsistency. As specific applications, we consider the Strotzian versions of quasi-hyperbolic discounting and of more general diminishing impatience. Section 4 considers the case of random choice and characterizes the implications of naiveté for the random Strotz model of [Dekel and Lipman \(2012\)](#). There, we consider as applications the model of [Eliaz and Spiegel \(2006\)](#) and the quasi-hyperbolic discounting model with uncertain present bias. Section 5 examines a model where the individual is offered a selection of commitment devices, and the consequent welfare implications of naiveté.

2 Primitives

We study a two-stage model with an agent who initially decides a menu of several options, and subsequently selects a particular option from that menu.

Let C denote a compact and metrizable space of outcomes. Let $\Delta(C)$ denote the set of lotteries (countably-additive Borel probability measures) over C , with typical elements $p, q, \dots \in \Delta(C)$. When it causes no confusion, we will slightly abuse notation and write c in place of the degenerate lottery $\delta_c \in \Delta(C)$. Finally, let $\mathcal{K}(\Delta(C))$ denote the family of nonempty compact subsets of $\Delta(C)$ with typical elements $x, y, \dots \in \mathcal{K}(\Delta(C))$. So $\mathcal{K}(\Delta(C))$ is a family of menus of lotteries. An *expected-utility function* is a function $u : \Delta(C) \rightarrow \mathbb{R}$ such that $u(\alpha p + (1 - \alpha)q) = \alpha u(p) + (1 - \alpha)u(q)$ for all lotteries p, q . An expected-utility function is *nontrivial* if there exist lotteries p, q such that $u(p) \neq u(q)$. We write $u \approx v$ when u and v are expected-utility functions and u is a positive affine transformation of v . For fixed expected-utility function u and menu x , let $B_u(x) \equiv \operatorname{argmax}_{p \in x} u(p)$.

We consider a pair of behavioral primitives. The first primitive is a preference rela-

tion \succsim on $\mathcal{K}(\Delta(C))$, with indifference \sim and strict preference \succ defined in the standard manner. The behavior encoded in \succsim is taken before the direct experience of temptation but while (possibly incorrectly) anticipating its future occurrence. The second primitive is a random choice rule $\lambda : \mathcal{K}(\Delta(C)) \rightarrow \Delta(\Delta(C))$ such that $\lambda^x(x) = 1$, where $\Delta(\Delta(C))$ denotes the space of lotteries over $\Delta(C)$. The behavior encoded in λ is taken while experiencing temptation. For each $x \in \mathcal{K}(\Delta(C))$, λ^x is a probability measure over lotteries, with $\lambda^x(y)$ denoting the probability of choosing a lottery in the set $y \subset x$ when the choice set is the menu x . We refer to the first stage of choice of a menu as occurring “ex ante” and the second stage of choice from a menu as occurring “ex post,” that is, before and after the realization of temptation. For example, the purchased gym contracts in DellaVigna and Malmendier (2009) correspond to an ex-ante choice of a menu while the observed number of gym visits corresponds to an ex-post choice from that menu.

At points, we will specialize to choice functions without randomization for their substantive importance and expositional clarity. A random choice function λ is *deterministic* if λ^x is degenerate for all menus x , that is, $\lambda^x = \delta_p$ where δ_p is the Dirac measure supported on p . Identifying the Dirac measure δ_p with p itself, we can notate λ as a standard choice function $\mathcal{C} : \mathcal{K}(\Delta(C)) \rightarrow \Delta(C)$.² In that case, $\mathcal{C}(x) = p$ for $\delta_p = \lambda^x$.

These primitives echo our prior work on unforeseen contingencies, Ahn and Sarver (2013). There we inferred unawareness of future contingencies by comparing choices before and after the realization of those contingencies. Observing ex-ante demand for flexibility and ex-post exercise of flexibility can reveal unawareness and provide positive foundations for the measurement of an unforeseen contingency, while the standard approach of using only ex-ante preferences cannot. Similarly, here we use demand for commitment in the first stage and then indulgence of temptation in the second stage to infer naiveté. Naiveté about future temptations in environments with demand for commitment is analogous to unawareness of unforeseen contingencies in environments with demand for flexibility.

3 Deterministic choice

We begin with the relatively more straightforward case of choice without randomization before proceeding to the general case with random choices in the next section. Throughout this section we assume a deterministic choice function \mathcal{C} . We begin by proposing definitions of sophistication and naiveté for a single individual.

²Recall the final outcomes are themselves lotteries. The determinacy here is in the sense that the decision maker does not randomize her selection among these lotteries.

Definition 1 *An individual is sophisticated if, for all menus x ,*

$$x \sim \{\mathcal{C}(x)\}.$$

An individual is naive if, for all menus x ,

$$x \succsim \{\mathcal{C}(x)\}.$$

We will say the individual is *strictly naive* if she is naive and unsophisticated.

A sophisticated individual correctly anticipates her choice $\mathcal{C}(x)$ from x . A naive individual erroneously values the option to make more virtuous choices, envisioning her virtuous selections. Inferring sophistication from $x \sim \{\mathcal{C}(x)\}$ assumes that the individual evaluates menus in a consequentialist manner, that is, the individual is indifferent between committing to her (correctly or incorrectly) anticipated choice $\mathcal{C}(x)$ from x at the ex-ante stage or selecting the menu x with the belief that she will choose $\mathcal{C}(x)$ ex post. Put differently, adding an option to a menu is important only if the individual anticipates choosing that option. This assumption is violated if unchosen options from a menu affect well-being. For example, an individual who exerts costly willpower to avoid choosing tempting options as in [Gul and Pesendorfer \(2001\)](#) does not evaluate a menu only by its choice consequences. In this case, she may strictly prefer to remove these unchosen temptations.

We now compare naiveté across individuals. This comparison invokes two conceptually distinct parts. The first considers individuals' ex-ante views of their future behavior. In environments with temptation, an ex-ante desire for commitment is often interpreted as a signal of anticipated temptation. The following comparison of desire for commitment is a slight variation of the comparison introduced by [Dekel and Lipman \(2012\)](#).³

Definition 2 *Individual 2 is more temptation averse than individual 1 if, for all menus x and lotteries p ,*

$$\{p\} \succ_1 x \implies \{p\} \succ_2 x.$$

This definition compares individuals' ex-ante demand for commitment to singletons, with more temptation averse individuals exhibiting higher demand for commitment. That is, if a less temptation averse person strictly prefers to commit to consuming the lottery $\{p\}$, then the more temptation averse person also prefers to commit to $\{p\}$. The more temptation averse individual therefore anticipates a higher value for commitment, while the less temptation averse individual has a more optimistic view of her future behavior. Note that comparability of temptation aversion guarantees that both agents share common preferences over singleton menus, so their virtuous tastes are identical.

³The formal definition also appears with different interpretations in [Ahn \(2007\)](#) and [Sarver \(2008\)](#).

The second component of the comparison considers ex-post behavior after the realization of temptation. The following comparison compares individuals' ex-post choices from menus.

Definition 3 *Individual 2 is more virtuous than individual 1 if, for all menus x and lotteries p ,*

$$\{p\} \succ_2 \{C_2(x)\} \implies \{p\} \succ_1 \{C_1(x)\}.$$

The more virtuous individual makes better choices from all menus: if the less virtuous agent makes choices from the menu x that are normatively superior to p , as reflected in her ex-ante commitment preference, then the more virtuous agent also makes normatively superior choices from x . Also note that Definition 3 implies both individuals share common preferences over singleton commitments, that is, their virtuous preferences are identical.

Absolute naiveté for a single individual involves both the ex-ante and ex-post perspectives, since a naive individual's ex-ante belief diverges from her ex-post behavior. Correspondingly, comparisons of naiveté across different individuals involves both perspectives as expressed in the prior two definitions: a more naive individual is more optimistic about her future behavior ex ante (that is, she is less temptation averse) yet less disciplined in her actual behavior ex post (that is, she is less virtuous).

Definition 4 *Suppose individuals 1 and 2 are naive. Individual 1 is more naive than individual 2 if individual 2 is more temptation averse and more virtuous than individual 1.*

The ubiquitous Strotz model of dynamic inconsistency offers a general application for these concepts. A sophisticated Strotz individual is specified by two preferences. The first is her ex-ante commitment preference over future consumption, as represented by the utility function u . The second is her temptation preference that governs her actual consumption choices at the ex-post stage, as represented by the utility function v . Naivete requires divergence between believed and actual consumption. Specification of a naive Strotzian consumer therefore requires third preference to capture her possibly erroneous beliefs about her future behavior, as represented by the utility function \hat{v} .

Definition 5 *A Strotz representation of \succsim is a pair (u, \hat{v}) of nontrivial expected-utility functions such that the function $U : \mathcal{K}(\Delta(C)) \rightarrow \mathbb{R}$ defined by*

$$U(x) = \max_{p \in B_{\hat{v}}(x)} u(p)$$

is a utility representation of \succsim .⁴

While she anticipates she will maximize \hat{v} , a naive Strotzian agent's ex-post behavior \mathcal{C} actually maximizes v .

Definition 6 A Strotz representation of \mathcal{C} is a pair (u, v) of nontrivial expected-utility functions such that

$$\mathcal{C}(x) \in B_u(B_v(x)).$$

The following results demonstrate that the basic definition of naiveté characterizes sharp parametric restrictions on \hat{v} and v . A naive individual believes that her future behavior will be more virtuous than it actually is. Converting this intuition to the parameters of the Strotz model, this means that the anticipated utility \hat{v} is more aligned with the commitment utility u than the actual utility v that will govern future consumption. The structure of the alignment is particular: \hat{v} is a linear combination of u and v , that is, $\hat{v} = \alpha u + (1 - \alpha)v$. The belief \hat{v} puts additional unjustified weight on the normative utility u , but aggregates u with v in a linear manner. This excludes the case where the believed temptation is orthogonal to the actual temptation; for example, this excludes the case where the individual will be tempted to indulge in sweet treats but believes she will be tempted to indulge in salty treats. This structure also relies crucially on the linear structure of the domain of lotteries and the assumed expected-utility functions.

A natural parameterization of comparative naiveté evaluates agents' weights on the normative utility u and the temptation utility v .

Definition 7 Let u, v, v' be expected-utility functions. Then v is more u -aligned than v' , written as $v \gg_u v'$, if either $v \approx \alpha u + (1 - \alpha)v'$ for some $\alpha \in [0, 1]$ or $v' \approx -u$.

An expected-utility function v is more u -aligned than v' if it puts additional weight on u that is not included in v' . It is also more u -aligned than v' if v' is maximally misaligned with u , that is, if v' is exactly $-u$.⁵

Theorem 1 Suppose \succsim has a Strotz representation (u, \hat{v}) , and \mathcal{C} has a Strotz representation (u, v) . Then the individual is naive if and only if $\hat{v} \gg_u v$ (and is sophisticated if and only if $\hat{v} \approx v$).

⁴Recall $B_{\hat{v}}(x)$ was defined as $\operatorname{argmax}_{q \in x} \hat{v}(q)$.

⁵The special exception for this boundary case is to avoid tedious exceptions in the following characterization theorems.

As mentioned, naiveté corresponds with unwarranted linear alignment of the believed utility \hat{v} with the virtuous utility u . The individual believes her future choices will maximize a convex combination of her actual utility v and her virtuous utility u . Note that the linear structure is a consequence of the definition of naiveté, and is not assumed a priori.

Then comparative naiveté imposes linear restrictions across the agents' ex-ante predictions as captured by \hat{v}_1, \hat{v}_2 and ex-post behaviors as captured by v_1, v_2 .

Theorem 2 *Suppose \succsim_1 and \succsim_2 have Strotz representations (u_1, \hat{v}_1) and (u_2, \hat{v}_2) , and \mathcal{C}_1 and \mathcal{C}_2 have Strotz representations (u_1, v_1) and (u_2, v_2) . Then individual 1 is more naive than individual 2 if and only if $u_1 \approx u_2 \equiv u$ and*

$$\hat{v}_1 \gg_u \hat{v}_2 \gg_u v_2 \gg_u v_1.$$

While they share common normative preferences over singleton commitments, individual 1 is more optimistic about her future behavior than individual 2. This is reflected in the requirement $\hat{v}_1 \gg_u \hat{v}_2$. However, individual 1's actual ex-post choices are even less virtuous than person 2's choices, as reflected in the requirement $v_1 \ll_u v_2$. So to be more naive, an individual must simultaneously be more optimistic about her future virtuous behavior while actually exercising less virtue. As in the case of absolute naiveté, the linear relationship between the parameters is a consequence of the definition of comparative naiveté.

Noor (2011) introduces the following definition of sophistication: if $x \cup \{p\} \succ x$, then $\mathcal{C}(\{p, q\}) = p$ for all $q \in x$.⁶ While Noor's definition is generally distinct from Definition 1, the two definitions are equivalent in the Strotzian case. Noor (2011) is interested in providing foundations for pure sophistication, so does not explore naiveté or comparative statics. Kopylov (2012) proposes the following relaxation of Noor's sophistication axiom: if $x \cup \{p\} \succ x$, then $\mathcal{C}(\{p, q\}) = p$ or $\{p\} \succ \{q\}$ for all $q \in x$. That is, even if p is not chosen over other elements ex post, p may still add value to the menu x because it is a more virtuous option. Kopylov's interpretation is not in terms of mistaken predictions about future behavior, but rather in terms of the direct non-consequentialist welfare effects of maintaining the availability of unchosen but virtuous options. The difference in interpretation is perhaps most salient when considering Kopylov's notion of comparative perfectionism. His comparison exclusively invokes ex-ante preferences, whereas we argue that comparative naiveté must invoke both ex-ante and ex-post choice. Finally, both Noor (2011) and Kopylov (2012) are interested in the more general self-control preferences of Gul and Pesendorfer (2001), whereas we restrict attention to the Strotzian case.⁷

⁶This is an equivalent formulation of Noor's axiom used by Lipman and Pesendorfer (2013).

⁷We do explore models with costly self-control in a companion paper.

3.1 Application: Quasi-Hyperbolic Preferences

As a specific application of the previous equivalences, we consider the quasi-hyperbolic model of time inconsistency parameterized by the present-bias factor β . Let $C = [a, b]^{\mathbb{N}}$ be a set of infinite-horizon consumption streams, with elements $c = (c_1, c_2, \dots) \in C$.⁸ A lottery $p \in \Delta(C)$ resolves immediately and yields a consumption stream. We focus on the simple case with one-shot resolution of uncertainty for expositional parsimony, but all of the following results generalize to richer settings that incorporate temporal lotteries or true dynamic choice.⁹ In more general dynamic environments, simple atemporal lotteries over consumption stream provide sufficient choice observations to generate the following comparative statics.

Suppose the commitment preference over random consumption streams is represented by an expected-utility function whose values $U(c) = U(\delta_c)$ over deterministic streams (that is, whose Bernoulli utility indices) comply with exponential discounting

$$U(c) = \sum_{t=1}^{\infty} \delta^{t-1} u(c_t) \quad (1)$$

for some instantaneous utility function $u : [a, b] \rightarrow \mathbb{R}$. The decision-maker understands that she will suffer present bias as parameterized in the quasi-hyperbolic discounting model, but underestimates its magnitude. Specifically, her choice from a menu of consumption streams maximizes the expected-utility function defined by

$$V(c) = u(c_1) + \beta \sum_{t=2}^{\infty} \delta^{t-1} u(c_t). \quad (2)$$

On the other hand, she anticipates her choice from a menu of consumption streams will maximize the expected-utility function defined by

$$\hat{V}(c) = u(c_1) + \hat{\beta} \sum_{t=2}^{\infty} \delta^{t-1} u(c_t). \quad (3)$$

The individual's ex-ante behavior reflects an optimistic belief that her future present-bias parameter is $\hat{\beta}$.

Definition 8 A quasi-hyperbolic (QH) representation of \succsim is a triple $(u, \hat{\beta}, \delta)$ of a continuous function $u : [a, b] \rightarrow \mathbb{R}$ and scalars $\hat{\beta} \in (0, 1]$ and $\delta \in (0, 1)$, such that (U, \hat{V}) is a

⁸The product topology on C is compact and metrizable.

⁹Kreps and Porteus (1978) were the first to provide a complete analysis of dynamic choice with uncertainty that resolves gradually through time (i.e., temporal lotteries). The models of temptation in Gul and Pesendorfer (2004) and Noor (2011) used an infinite horizon version of such a setting.

Strotz representation for \succsim , where U and \hat{V} satisfy Equations (1) and (3).

Her ex-post behavior actually uses the present-bias parameter β .

Definition 9 A quasi-hyperbolic (QH) representation of \mathcal{C} is a triple (u, β, δ) of a continuous function $u : [a, b] \rightarrow \mathbb{R}$ and scalars $\beta \in (0, 1]$ and $\delta \in (0, 1)$, such that (U, V) is a Strotz representation for \mathcal{C} , where U and V satisfy Equations (1) and (2).

The behavioral comparison of naiveté implies the intuitive restriction that $\hat{\beta} \geq \beta$.

Corollary 1 Suppose \succsim has a QH representation $(u, \hat{\beta}, \delta)$, and \mathcal{C} has a QH representation (u, β, δ) . Then the individual is naive if and only if $\hat{\beta} \geq \beta$ (and is sophisticated if and only if $\hat{\beta} = \beta$).

Proof: By Theorem 1, the individual is naive if and only if $\hat{V} \approx \alpha U + (1 - \alpha)V$ for some $\alpha \in [0, 1]$, where the functions U , V , and \hat{V} satisfy Equations (1), (2), and (3), respectively. Note that

$$\alpha U(c) + (1 - \alpha)V(c) = u(c_1) + (\alpha + (1 - \alpha)\beta) \sum_{t=2}^{\infty} \delta^{t-1} u(c_t).$$

Since the term $u(c_1)$ appears in this expression without any scalar multiple, conclude that $\hat{V} = \alpha U + (1 - \alpha)V$ and hence $\hat{\beta} = \alpha + (1 - \alpha)\beta \in [\beta, 1]$. ■

A similar application of the characterization of comparative naiveté in Theorem 2 provides another set of intuitive comparative restrictions. First, the more naive individual has more optimistic beliefs about her future patience: $\hat{\beta}_1 \geq \hat{\beta}_2$. Second, the more naive individual's behavior is more present-biased: $\beta_1 \leq \beta_2$. These restrictions are also necessary, so the comparison of alternative statistics such as $\hat{\beta}_i - \beta_i$ or $\hat{\beta}_i/\beta_i$ will not work. For example, $\hat{\beta}_1 - \beta_1 \geq \hat{\beta}_2 - \beta_2$ is generally insufficient to guarantee individual 1 is more naive than individual 2 without knowing that $\hat{\beta}_1 \geq \hat{\beta}_2$ and $\beta_1 \leq \beta_2$.

Corollary 2 Suppose \succsim_1 and \succsim_2 have QH representations $(u_1, \hat{\beta}_1, \delta_1)$ and $(u_2, \hat{\beta}_2, \delta_2)$, and \mathcal{C}_1 and \mathcal{C}_2 have QH representations (u_1, β_1, δ_1) and (u_2, β_2, δ_2) . Then individual 1 is more naive than individual 2 if and only if $u_1 \approx u_2$, $\delta_1 = \delta_2$, and $\hat{\beta}_1 \geq \hat{\beta}_2 \geq \beta_2 \geq \beta_1$.

The proof is entirely analogous to that of Corollary 1, hence omitted.

3.2 Application: Diminishing Impatience

The quasi-hyperbolic representation described in the previous section can generalize to accommodate more general patterns of discounting, such as true hyperbolic discounting. In this section, we relate several properties of discount functions to properties of the perceived discount functions for individuals who satisfy our definition of naiveté. Say that $D : \mathbb{N} \cup \{0\} \rightarrow (0, 1]$ is a *discount function* if $D(0) = 1$ and $\sum_{t=0}^{\infty} D(t) < \infty$.

Suppose as before that consumption in periods $t = 1, 2, \dots$ is given by $(c_1, c_2, \dots) \in C = [a, b]^{\mathbb{N}}$. Period 0 commitment preferences over deterministic consumption streams starting in period 1 are represented by

$$U(c) = \sum_{t=1}^{\infty} D(t)u(c_t). \quad (4)$$

Suppose that in period 0 the individual believes that she will use a discount function \hat{D} , which yields the following temptation utility for deterministic consumption streams:

$$\hat{V}(c) = \sum_{t=1}^{\infty} \hat{D}(t-1)u(c_t). \quad (5)$$

In reality, suppose preferences over consumption streams are stationary, and period 1 choices actually maximize

$$V(c) = \sum_{t=1}^{\infty} D(t-1)u(c_t). \quad (6)$$

Definition 10 A discounting representation of \succsim is a triple (u, D, \hat{D}) of a continuous function $u : [a, b] \rightarrow \mathbb{R}$ and a discount functions D and \hat{D} , such that (U, \hat{V}) is a Strotz representation for \succsim , where U and \hat{V} satisfy Equations (4) and (5), respectively.

Definition 11 A discounting representation of \mathcal{C} is a pair (u, D) of a continuous function $u : [a, b] \rightarrow \mathbb{R}$ and a discount function $D : \mathbb{N} \cup \{0\} \rightarrow (0, 1]$, such that (U, V) is a Strotz representation for \mathcal{C} , where U and V satisfy Equations (4) and (6), respectively.

The quasi-hyperbolic representations discussed in the previous section are special cases of the discounting representations where

$$D(t) = \begin{cases} 1 & \text{if } t = 0 \\ \beta\delta^t & \text{if } t > 0. \end{cases}$$

and

$$\hat{D}(t) = \begin{cases} 1 & \text{if } t = 0 \\ \hat{\beta}\delta^t & \text{if } t > 0. \end{cases}$$

Two general properties of discount functions will be important.

Definition 12 A discount function $D : \mathbb{N} \cup \{0\} \rightarrow (0, 1]$ exhibits diminishing impatience if

$$\frac{D(0)}{D(1)} > \frac{D(t)}{D(t+1)} \quad (\forall t \in \mathbb{N}),$$

and exhibits strong diminishing impatience if

$$\frac{D(t)}{D(t+1)} > \frac{D(t+1)}{D(t+2)} \quad (\forall t \in \mathbb{N} \cup \{0\}).$$

Diminishing impatience requires that the intertemporal rate of substitution for any pair of successive periods in the future is strictly more balanced than the intertemporal rate of substitution between today and tomorrow. Strong diminishing impatience further requires that the intertemporal rate of substitution between successive periods is strictly declining over time. Note that the quasi-hyperbolic discount functions described above exhibit diminishing impatience but not strong diminishing impatience because the discount factor between t and $t+1$ is constant after $t=1$. True hyperbolic discounting, on the other hand, exhibits strong diminishing impatience.

The following corollary of Theorem 1 shows the implications of diminishing and strong diminishing impatience on the perceived future impatience of a naive individual. Since our definition of naiveté includes the case of full sophistication, in what follows we say an individual is *strictly naive* if she is naive and not sophisticated.

The individual believes that his ex-post intertemporal rate of substitution between period 1 and period $t+1$ will be governed by the discount factor $\hat{D}(t)$. This discount factor is a convex combination of the virtuous ex-ante discount factor $D(t+1)/D(1)$ and the actual tempting discount factor $D(t)$ that governs the intertemporal consumption that the individual will choose ex-post (from tomorrow's perspective).

Corollary 3 Suppose \succsim has a discounting representation (u, D, \hat{D}) and \mathcal{C} has a discounting representation (u, D) . Then the individual is naive if and only if there exists $\alpha \in [0, 1]$ such that

$$\hat{D}(t) = \alpha \frac{D(t+1)}{D(1)} + (1-\alpha)D(t), \quad \forall t \in \mathbb{N} \cup \{0\}, \quad (7)$$

and the individual is sophisticated if and only if $\alpha = 0$. In addition, if the individual is strictly naive (i.e., $\alpha > 0$), then

1. The discount function D exhibits diminishing impatience if and only if

$$\frac{D(0)}{D(t)} > \frac{\hat{D}(0)}{\hat{D}(t)} \quad (\forall t \in \mathbb{N}).$$

2. The discount function D exhibits strong diminishing impatience if and only if

$$\frac{D(t)}{D(t+1)} > \frac{\hat{D}(t)}{\hat{D}(t+1)} \quad (\forall t \in \mathbb{N} \cup \{0\}).$$

Proof: The proof of the characterization of naiveté in Equation (7) is a simple generalization of the arguments in the proof of Corollary 1 and is therefore omitted. To prove claim 1, note that by Equation (7),

$$\hat{D}(t) > D(t) \iff \frac{D(t+1)}{D(1)} > D(t) = \frac{D(t)}{D(0)}.$$

The latter holds for all $t \in \mathbb{N}$ if and only if D exhibits diminishing impatience.

To prove claim 2, note first that

$$\begin{aligned} \frac{\hat{D}(t)}{\hat{D}(t+1)} &= \frac{\alpha \frac{D(t+1)}{D(1)} + (1-\alpha)D(t)}{\alpha \frac{D(t+2)}{D(1)} + (1-\alpha)D(t+1)} \\ &= \frac{\alpha \frac{D(t+1)}{D(t)} + (1-\alpha)D(1)}{\alpha \frac{D(t+2)}{D(t+1)} + (1-\alpha)D(1)} \cdot \frac{D(t)}{D(t+1)}. \end{aligned}$$

Therefore,

$$\frac{\hat{D}(t)}{\hat{D}(t+1)} < \frac{D(t)}{D(t+1)} \iff \frac{D(t+1)}{D(t)} < \frac{D(t+2)}{D(t+1)}.$$

The latter holds for all $t \in \mathbb{N} \cup \{0\}$ if and only if D exhibits strong diminishing impatience. ■

The claims regarding diminishing impatience are surprising because they relate functional restrictions on the actual temptation, as captured in D , with the intertemporal rate of substitution in the believed temptation, as captured in \hat{D} . The first claim says that diminishing impatience and naiveté are equivalent to beliefs being biased towards saving desirable consumption for a later date t rather than in the present period 0, as reflected in $\hat{D}(0)/\hat{D}(t) < D(0)/D(t)$. In other words, under-appreciating the temptation for immediate consumption versus later consumption is an inherent feature of naiveté with diminishing impatience. If beliefs are ever biased in the opposite direction (with

projected undersaving) then the individual cannot exhibit diminishing impatience in her virtuous utility. Similarly, under-appreciation of the temptation to shift good consumption to immediately prior time periods is an inherent feature of strong diminishing impatience with naiveté. Note that the results do not suggest a relationship between the diminishing impatience of the actual temptation and the diminishing impatience of the believed temptation.

The tidiness of our analysis for the general discounting case in this section hinges on the simplification that all future consumption is decided at a single point in the immediate future. The general case where the individual makes choices in every time period is considerably more complicated.

4 Random choice

We now extend the analysis from the prior section to random choice rules. For any $\lambda^x \in \Delta(\Delta(C))$, its average choice $m(\lambda^x)$ is the expectation of the identity function under λ^x , or formally $m(\lambda^x) = \int p d\lambda^x \in \Delta(C)$. That is, $m(\lambda^x)$ reduces the compound lottery λ^x into a single lottery. The definitions in the prior section for deterministic choice then generalize by considering reductions of random choices. This reduction from a distribution over multiple lotteries to a single lottery does not assume any attitude towards risk, such as risk neutrality, over deterministic outcomes in C .¹⁰

Definition 13 *An individual is sophisticated if, for all menus x ,*

$$x \sim \{m(\lambda^x)\}.$$

An individual is naive if, for all menus x ,

$$x \succsim \{m(\lambda^x)\}.$$

A sophisticate is indifferent between choosing from a menu x tomorrow and committing to the average choice $m(\lambda^x)$ from that menu. A naif anticipates making more virtuous choices, on average, than she actually will make. The proviso from the prior subsection regarding the assumption of consequentialism and lack of indirect menu effects applies here as well. When λ^x involves no randomness, this definition reduces to the prior one for deterministic choice: If $\lambda^x = \delta_p$ then the corresponding deterministic choice function takes $\mathcal{C}(x) = p$. Since $m(\delta_p) = p$, this implies $m(\lambda^x) = \mathcal{C}(x)$.

¹⁰Our analysis implicitly assumes indifference to compounding. Indifference to compounding can be relaxed by considering appropriate certain equivalents rather than assuming indifference between λ^x and $m(\lambda^x)$.

We use the same definition of more temptation averse for random choice as for deterministic choice, but we must extend our previous definition of more virtuous to account for randomness in choice.

Definition 14 *Individual 2 is more virtuous than individual 1 if, for all menus x and lotteries p ,*

$$\{p\} \succ_2 \{m(\lambda_2^x)\} \implies \{p\} \succ_1 \{m(\lambda_1^x)\}.$$

The average choice of the more sophisticated individual is more virtuous.

Definition 15 *Suppose individuals 1 and 2 are naive. Individual 1 is more naive than individual 2 if individual 2 is more temptation-averse and more virtuous than individual 1.*

A generalization of the classic Strotz model is the random Strotz model that admits uncertainty about future temptations. For example, a dieter might crave potato chips at some times and chocolate cake at other times. [Dekel and Lipman \(2012\)](#) provide a thorough analysis of the random Strotz model. Since a single temptation is parametrized as a single utility vector, a random temptation is parametrized as a probability measure over utility vectors. Note that even if actual choices are degenerate, the random Strotz model is important because it allows an individual to be mistakenly uncertain about her future behavior, while such uncertainty is excluded by the standard Strotz representation.

In what follows, let \mathcal{V} denote the set of all continuous functions $v : C \rightarrow \mathbb{R}$, and endow \mathcal{V} with the supremum norm and corresponding Borel σ -algebra. We can identify \mathcal{V} with the set of all expected-utility functions on $\Delta(C)$ by letting $v(p) \equiv \int_C v(c) dp$. As before, we say u is a nontrivial expected-utility function if it is not constant. We say $\hat{\mu}$ is a nontrivial measure on \mathcal{V} if it assigns probability zero to the set of constant functions.¹¹

Definition 16 *A random Strotz representation of \succsim is a pair $(u, \hat{\mu})$ of a nontrivial expected-utility function u and a nontrivial probability measure $\hat{\mu}$ over \mathcal{V} such that the function $U : \mathcal{K}(\Delta(C)) \rightarrow \mathbb{R}$ defined by*

$$U(x) = \int_{\mathcal{V}} \max_{p \in B_v(x)} u(p) d\hat{\mu}(v)$$

is a utility representation of \succsim .

¹¹Note that the restriction to nontrivial measures in the definitions of the random Strotz representations is without loss of generality since any weight assigned to constant functions can be moved to the commitment utility u without altering the ex ante preference or ex post random choice rule.

The analogous representation of random choice is more cumbersome because the maximizing Strotz choice set $B_u(B_v(x))$ has potentially multiple elements for a fixed temptation utility $v \in \mathcal{V}$, in turn generating multiple choice probabilities over x for a fixed probability measure over \mathcal{V} . A random choice rule maximizes a random Strotz representation if it is generated by some selection function from the correspondence mapping temptations v to possible choices $B_u(B_v(x))$.¹²

Definition 17 *A random Strotz representation of λ is a pair (u, μ) of a nontrivial expected-utility function u and a nontrivial probability measure μ over \mathcal{V} such that, for all menus x and all measurable $y \subset x$,*

$$\lambda^x(y) = \mu(p_x^{-1}(y))$$

*for some measurable selection function $p_x : \mathcal{V} \rightarrow x$ with $p_x(v) \in B_u(B_v(x))$ for all $v \in \mathcal{V}$.*¹³

The definitions of naiveté for random Strotz is the stochastic generalization of the definition for deterministic Strotz. In the degenerate case, naiveté implies the believed \hat{v} is more u -aligned than v : $\hat{v} \gg_u v$. In the random case, the believed distribution over all possible temptations stochastically dominates the actual distribution of temptations, where stochastic dominance is with respect to the \gg_u order. The following formalizes the stochastic order generated by \gg_u . As is standard, a stochastically dominant measure puts more weight on the upper contour sets of the basic ordering \gg_u over the state space. The following formal definitions essentially adapt the technology developed by [Dekel and Lipman \(2012\)](#).¹⁴

Definition 18 *Let u be an expected-utility function. A measurable set $\mathcal{U} \subset \mathcal{V}$ is a u -upper set if, for any $v \in \mathcal{U}$ and $v' \in \mathcal{V}$, if $v' \gg_u v$ then $v' \in \mathcal{U}$.*

We will let \gg_u notate both the basic ordering over expected-utility functions and the induced stochastic order over measures on expected-utility functions. Note that $v \gg_u v'$ (in the determinate sense) is equivalent to $\delta_v \gg_u \delta_{v'}$ (in the stochastic sense).

Definition 19 *Let u be an expected-utility function, and let μ, μ' be probability measures over \mathcal{V} . Then μ is more u -aligned than μ' , written as $\mu \gg_u \mu'$, if $\mu(\mathcal{U}) \geq \mu'(\mathcal{U})$ for all u -upper sets \mathcal{U} .*

¹²Just as there can be a multiple choice functions that maximize the same deterministic Strotz representation, there can be multiple random choice rules that maximize the same random Strotz representation. However, this multiplicity is not important for our results since observing any maximizing random choice rule provides sufficient information for our comparatives.

¹³Since λ is taken as primitive, the selection function p_x is almost surely identified with respect to μ .

¹⁴There are some technical differences; for example, [Dekel and Lipman \(2012\)](#) use a different field to normalize utility functions.

Generalizing the earlier result, absolute naiveté is equivalent to $\hat{\mu}$ dominating μ in the stochastic order generated by \gg_u .

Theorem 3 *Suppose C has finite cardinality. Suppose \succsim has a random Strotz representation $(u, \hat{\mu})$, and λ has a random Strotz representation (u, μ) . Then the individual is naive if and only if $\hat{\mu} \gg_u \mu$ (and is sophisticated if and only if $\hat{\mu} \gg_u \mu$ and $\mu \gg_u \hat{\mu}$).*

Continuing to generalize, comparative naiveté induces a similar ordering over individuals' predictions and behaviors but in a stochastic sense.

Theorem 4 *Suppose C has finite cardinality. Suppose \succsim_1 and \succsim_2 have random Strotz representations $(u_1, \hat{\mu}_1)$ and $(u_2, \hat{\mu}_2)$, and λ_1 and λ_2 have random Strotz representations (u_1, μ_1) and (u_2, μ_2) . Then individual 1 is more naive than individual 2 if and only if $u_1 \approx u_2 \equiv u$ and*

$$\hat{\mu}_1 \gg_u \hat{\mu}_2 \gg_u \mu_2 \gg_u \mu_1.$$

To aid intuition further, we will highlight the corollaries of these characterizations for the case where the uncertainty over future behavior is only over the magnitude of the future temptation, and not in its basic direction. In other words, the individual knows that she will crave sweet snacks (but not salty snacks) ex-post, but is uncertain of how strong her craving for sweets will be.

In what follows, say two expected-utility functions u and v are *independent* if they are nontrivial and it is not the case that $v \approx u$ or $v \approx -u$.

Definition 20 *An uncertain intensity Strotz representation of \succsim is a triple (u, v, \hat{F}) of two independent expected-utility functions u, v and a cumulative distribution function \hat{F} on $[0, 1]$ such that the function $U : \mathcal{K}(\Delta(C)) \rightarrow \mathbb{R}$ defined by*

$$U(x) = \int_0^1 \max\{u(p) : p \in B_{\alpha u + (1-\alpha)v}(x)\} d\hat{F}(\alpha)$$

is a utility representation of \succsim .

Definition 21 *An uncertain intensity Strotz representation of λ is a triple (u, v, F) of two independent expected-utility functions u, v and a cumulative distribution function F on $[0, 1]$ such that, for all menus x and all measurable $y \subset x$,*

$$\lambda^x(y) = \int_0^1 \mathbf{1}_{[p_x(\alpha) \in y]} dF(\alpha)$$

for some measurable selection function $p_x : [0, 1] \rightarrow x$ with $p_x(\alpha) \in B_u(B_{\alpha u + (1-\alpha)v}(x))$ for all $\alpha \in [0, 1]$.

For the case of an uncertain intensity Strotz representation, the direction of the temptation is known to be v , but the magnitude of that temptation relative to the virtuous utility u is unknown. The individual underestimates the influence of v , and this bias is reflected in her belief \hat{F} over the intensities in $[0, 1]$ putting more likelihood on larger values of u (hence lower values of v) in the ex-post stage of choice.

Theorem 5 *Suppose \succsim has a uncertain intensity Strotz representation (u, v, \hat{F}) , and λ has a uncertain intensity Strotz representation (u, v, F) . Then the individual is naive if and only if $\hat{F} \succeq_{FOSD} F$ (and is sophisticated if and only if $\hat{F} = F$).*

Theorem 6 *Suppose \succsim_1 and \succsim_2 have a uncertain intensity Strotz representations (u, v, \hat{F}_1) and (u, v, \hat{F}_2) , and λ_1 and λ_2 have uncertain intensity Strotz representations (u, v, F_1) and (u, v, F_2) . Then individual 1 is more naive than individual 2 if and only if*

$$\hat{F}_1 \succeq_{FOSD} \hat{F}_2 \succeq_{FOSD} F_2 \succeq_{FOSD} F_1.$$

4.1 Application: Eliaz and Spiegel (2006)

We now apply the general results for random Strotz to an important specialization proposed by [Eliaz and Spiegel \(2006\)](#). In their model, an individual's ex-post choice will be governed by v as in the standard Strotz model. However, an individual's ex-ante assessment is that she will make virtuous choices according to u with probability θ and will face temptation with probability $1 - \theta$. This is arguably the simplest possible formula that features naive random temptation.

Definition 22 *An Eliaz–Spiegler representation of \succsim is a triple (u, v, θ) of nontrivial expected-utility functions u, v and a scalar $\theta \in [0, 1]$ such that the function $U : \mathcal{K}(\Delta(C)) \rightarrow \mathbb{R}$ defined by*

$$U(x) = \theta \max_{p \in x} u(p) + (1 - \theta) \max_{p \in B_v(x)} u(p)$$

is a utility representation of \succsim .

The implications of the general characterizations for random Strotz are immediate for this special case. The individual mistakenly believes there is some $\theta \geq 0$ probability that she will act virtuously and maximize u rather than v ex post, whereas she will certainly maximize v in reality. The more naive individual believes her ex-post behavior is more likely to be virtuous ($\theta_1 \geq \theta_2$) while her actual (and nonrandom) behavior is identical.

Corollary 4 *Suppose \succsim has an Eliaz–Spiegler representation (u, v, θ) , and \mathcal{C} has a (deterministic) Strotz representation (u, v) . Then the individual is naive (and is sophisticated if and only if $\theta = 0$).*

Proof: The deterministic ex-post choice of the individual can be expressed as a (degenerate) random choice rule λ that satisfies $\lambda^x = \delta_p$ for $p = \mathcal{C}(x)$. Define a cumulative distribution function F on $[0, 1]$ by $F(\alpha) = 1$ for all $\alpha \in [0, 1]$; thus F assigns probability 1 to $\alpha = 0$. Note that (u, v, F) is an uncertain intensity Strotz representation for λ . Define a cumulative distribution function \hat{F} by $\hat{F}(\alpha) = 1 - \theta$ for $0 \leq \alpha < 1$ and $\hat{F}(1) = 1$, and note that (u, v, \hat{F}) is an uncertain intensity Strotz representation for \succsim . Then $\hat{F} \geq_{FOSD} F$, and $\hat{F} = F$ if and only if $\theta = 0$. The result then follows from Theorem 5. ■

An analogous application of Theorem 6 yields the following corollary.

Corollary 5 *Suppose \succsim_1 and \succsim_2 have the Eliaz–Spiegler representations (u, v, θ_1) and (u, v, θ_2) , and \mathcal{C}_1 and \mathcal{C}_2 both have the Strotz representation (u, v) . Then individual 1 is more naive than individual 2 if and only if $\theta_1 \geq \theta_2$.*

4.2 Application: Random Quasi-Hyperbolic Discounting

In this section, we describe a generalization of the quasi-hyperbolic discounting model from Section 3.1 that permits uncertainty about the present-bias factor β . This random quasi-hyperbolic discounting representation is a special case of the uncertain intensity Strotz representation. In this case, the uncertainty about future intensity is parametrized as uncertainty about the future present-bias parameter β . A variety of applications in different areas employ naive uncertainty about future present bias. Section 4 of [Heidhues and Koszegi \(2010\)](#) employ random quasi-hyperbolic discounting to explain the structure of credit markets and the consequent welfare implications for consumers. [Duflo, Kremer, and Robinsom \(2011\)](#) use the [Eliaz and Spiegler \(2006\)](#) specification of random quasi-hyperbolic discounting, where naiveté is limited to a mistaken belief of some chance of virtuous exponential discounting all future periods, in their study of fertilizer adoption decisions by Kenyan farmers. A major benefit of admitting uncertainty about intertemporal substitution in economic models is as a reduced-form proxy for a shock in the economy, like wage uncertainty, or for heterogeneity across agents in an aggregate economy, like the distribution of wealth. Similarly, random present-bias can provide a parsimonious channel within the utility function for capturing uncertainty about external factors that affect present-bias.

As in Sections 3.1 and 3.2, maintain that consumption in periods $t = 1, 2, \dots$ is given by $(c_1, c_2, \dots) \in C = [a, b]^{\mathbb{N}}$. Suppose the commitment preference over random consumption streams is represented by an expected-utility function whose values $U(c) = U(\delta_c)$ over deterministic consumption streams satisfy exponential discounting, which is equivalent to the extreme case where $\beta = 1$:

$$U(c) = \sum_{t=1}^{\infty} \delta^{t-1} u(c_t). \quad (8)$$

The most extremely present-biased individual will value only immediate consumption in period 1 and will ignore consumption in later periods; this is equivalent to the extreme case where $\beta = 0$:

$$V(c) = u(c_1). \quad (9)$$

For any intensity β on the virtuous utility U , we have

$$\beta U(c) + (1 - \beta)V(c) = u(c_1) + \beta \sum_{t=2}^{\infty} \delta^{t-1} u(c_t).$$

Therefore, uncertainty about the present-bias parameter β is equivalent to uncertainty about the intensity of U relative to V , and the present-bias factor is exactly the relative weighting of exponential discounting versus extreme impatience.

Definition 23 A random quasi-hyperbolic (RQH) representation of \succsim is a triple (u, \hat{F}, δ) of a continuous function $u : [a, b] \rightarrow \mathbb{R}$, a scalar $\delta \in (0, 1)$, and a cumulative distribution function \hat{F} on $[0, 1]$ such that (U, V, \hat{F}) is an uncertain intensity Strotz representation for \succsim , where U and V satisfy Equations (8) and (9).

Definition 24 A random quasi-hyperbolic (RQH) representation of λ is a triple (u, F, δ) of a continuous function $u : [a, b] \rightarrow \mathbb{R}$, a scalar $\delta \in (0, 1)$, and a cumulative distribution function F on $[0, 1]$ such that (U, V, F) is an uncertain intensity Strotz representation for λ , where U and V satisfy Equations (8) and (9).

Applying Theorem 5 to the random quasi-hyperbolic representation yields the following corollary: naive random quasi-hyperbolic individuals shift weight in their predictions away from low realizations of the present-bias factor β and towards high realizations of β .

Corollary 6 Suppose \succsim has a RQH representation (u, \hat{F}, δ) and λ has a RQH representation (u, F, δ) . Then the individual is naive if and only if $\hat{F} \geq_{FOSD} F$ (and is sophisticated if and only if $\hat{F} = F$).

Similarly, the more naive random quasi-hyperbolic individual puts more unwarranted mass on higher realizations of β than the more sophisticated individual. The following corollary follows directly from Theorem 6.¹⁵

Corollary 7 *Suppose \succsim_1 and \succsim_2 have RQH representations $(u_1, \hat{F}_1, \delta_1)$ and $(u_2, \hat{F}_2, \delta_2)$, and λ_1 and λ_2 have RQH representations (u_1, F_1, δ_1) and (u_2, F_2, δ_2) . Then individual 1 is more naive than individual 2 if and only if $u_1 \approx u_2$, $\delta_1 = \delta_2$, and*

$$\hat{F}_1 \geq_{FOSD} \hat{F}_2 \geq_{FOSD} F_2 \geq_{FOSD} F_1.$$

5 Welfare

We finally consider a setup that explores the welfare implications of policies that introduce new commitment devices when consumers are naive. Suppose the government contemplates whether to provide an illiquid forced-savings device. This is equivalent to introducing an additional commitment device or menu x to the family of existing available menus; the new menu excludes immediate consumption beyond a certain level. A pervasive finding is that the take-up of new commitment devices is minimal under naiveté; several studies in this line are surveyed by Bryan, Karlan, and Nelson (2010). Beyond mitigating the effectiveness of new commitment devices, we find that new commitment devices can strictly decrease welfare when consumer are naive. In fact, the existence of such strictly deleterious commitment devices characterizes naiveté.

Formally, we consider families of menus to understand the effects of adding additional commitment devices. For finite $X \subset \mathcal{K}(\Delta(C))$, let $x^*(X) = \{x \in X : x \succsim y \text{ for all } y \in X\}$ denote the set of \succsim -maximal menus from the family X . Let $\mathfrak{C}(X) \in \mathcal{C}(x^*(X)) \equiv \{\mathcal{C}(x) : x \in x^*(X)\}$. That is, $\mathfrak{C}(X)$ is the final consumption from the family of menus X when the individual adheres to the following protocol: first, she select a \succsim -maximal menu $x \in x^*(X)$, and second, she consumes $\mathcal{C}(x)$.

The next result makes the straightforward but important observation that adding additional commitment devices always makes sophisticated individuals better off. The converse result, that strictly naive individuals can always be made strictly worse off with more available commitments devices, requires that singleton menus are dense in the ex-ante preference as they are, for example, whenever a Strotz representation exists.

Recall that an individual is strictly naive if she is naive and not sophisticated.

¹⁵Note that Corollary 7 states that $u_1 \approx u_2$ and $\delta_1 = \delta_2$ as part of the implication of individual 1 being more naive than individual 2. This follows from our previous observations that individual 2 being either more temptation averse or more virtuous than individual 1 implies that both have the same commitment preference. The relationship between the distribution functions $\hat{F}_1, \hat{F}_2, F_1, F_2$ then follows from Theorem 6.

Theorem 7 *If an individual is sophisticated, then $\{\mathfrak{C}(X)\} \succsim \{\mathfrak{C}(Y)\}$ whenever $X \supset Y$. If singleton menus are \succsim -dense and the individual is strictly naive, then there exist $X \supset Y$ with $\{\mathfrak{C}(X)\} \prec \{\mathfrak{C}(Y)\}$.*

Proof: By the standard revealed-preference argument, $X \supset Y$ implies $x \succsim y$ for any $x \in x^*(X)$ and $y \in x^*(Y)$. Under sophistication, $\{\mathcal{C}(x)\} \sim x \succsim y \sim \{\mathcal{C}(y)\}$. But $\mathfrak{C}(X) = \mathcal{C}(x)$ for some $x \in x^*(X)$ and $\mathfrak{C}(Y) = \mathcal{C}(y)$ for some $y \in x^*(Y)$, so in particular $\{\mathfrak{C}(X)\} \succsim \{\mathfrak{C}(Y)\}$.

Now assume the individual is naive and not sophisticated: there exists a menu x with $x \succ \{\mathcal{C}(x)\}$. By \succsim -denseness of the singletons, there exists some lottery p such that $x \succ \{p\} \succ \{\mathcal{C}(x)\}$. Let $X = \{x, \{p\}\}$ and $Y = \{\{p\}\}$. Then $\mathfrak{C}(Y) = p$ and $\mathfrak{C}(X) = \mathcal{C}(x)$, so $\{\mathfrak{C}(Y)\} \succ \{\mathfrak{C}(X)\}$. ■

As an example to illustrate the approach in Theorem 7, consider the stopping problems considered by O’Donoghue and Rabin (1999) and O’Donoghue and Rabin (2001). They demonstrate that extreme naifs (where $\hat{\beta} = 1$) will always procrastinate longer on a task that requires immediate costs to yield a flow of benefits and that partial naifs (where $\hat{\beta} > \beta$) always have some task that leads them to procrastinate. We now show this prediction is not an artifact of quasi-hyperbolic discount but an almost inherent feature of general naiveté. Suppose $\{d_2\} \succ \{d_1\} \succ \{d_3\}$. Doing it now (in period 1) is tantamount to committing to $\{d_1\}$, while delaying means leaving the menu $\{d_2, d_3\}$ available tomorrow. The only way for the definition of naiveté to have bite in this domain is if $\mathcal{C}(\{d_2, d_3\}) = d_3$ and $\{d_2, d_3\} \succ \{d_3\}$. If $Y = \{\{d_1\}\}$, then the individual must do it now (in period 1). In contrast, $X = \{\{d_1\}, \{d_2, d_3\}\}$ contains the option of doing it now, $\{d_1\}$, or delaying and having the option in the next period whether to do it or not, $\{d_2, d_3\}$. Then $\mathfrak{C}(Y) = d_1$ and $\mathfrak{C}(X) = d_3$, so $\{\mathfrak{C}(Y)\} \succ \{\mathfrak{C}(X)\}$. That is, the naive individual delays longer and gets less welfare than the sophisticated individual if she is given the option X to delay past the first period. The actual choices are worse with X because the individual delays under the (incorrect) belief that she will do it in the following period. The nature of procrastination for naive individuals is therefore neither an artifact of $(\beta, \hat{\beta}, \delta)$ preferences nor even really connected to time discounting. Rather, it is an inherent consequence of the structure of the domain into possible menus (that is, only menus consisting of remaining time periods are observed) and the force of the basic definition of naiveté on this structured domain.

The prior theorem is intuitive because the additional menu that leads to a less virtuous final selection is possibly a superset of an already available menu. Clearly, increasing flexibility for individuals who mistakenly believe they will virtuously exercise that flexibility can decrease their welfare. The next result is sharper, but requires the additional structure of the Strotz model. Under Strotz preferences, there always exists a subset of

an existing menu that leaves the individual worse off when added to the family of commitments. That is, there exists a scenario where welfare is harmed by adding stronger commitments (rather than more flexibility) that exclude choices which are otherwise available in some existing forward plan in the status quo.

Say an individual has a *preference for commitment* if there exist menus y and $x \subset y$ such that $x \succ y$. If an individual has a preference for commitment, then she is not fully naive in the sense of believing that her future tastes will be identical to her current commitment preference.

Theorem 8 *Suppose \succsim and \mathcal{C} admit Strotz representations (u, \hat{v}) and (u, v) , where u and v are independent.¹⁶ If the individual is strictly naive and has a preference for commitment, then there exist menus y and $x \subset y$ such that $\{\mathfrak{C}(\{x, y\})\} \prec \{\mathfrak{C}(\{y\})\}$.*

Proof: Under the assumptions of the theorem, it can be shown that there exist lotteries p^1, p^2, p^3 such that¹⁷

$$\begin{aligned} u(p^1) &> u(p^2) > u(p^3) \\ \hat{v}(p^2) &> \hat{v}(p^1) > \hat{v}(p^3) \\ v(p^2) &> v(p^3) > v(p^1). \end{aligned}$$

Let $y = \{p^1, p^2, p^3\}$ and $x = \{p^1, p^3\}$. The rankings of the lotteries according to u and \hat{v} imply that $x \sim \{p^1\} \succ \{p^2\} \sim y$. The ranking according to v implies that $\mathcal{C}(x) = p^3$ and $\mathcal{C}(y) = p^2$. Therefore, $\{\mathfrak{C}(\{x, y\})\} = \{p^3\} \prec \{p^2\} = \{\mathfrak{C}(\{y\})\}$. ■

For a concrete illustration of Theorem 8, consider an individual facing a three-period consumption-savings problem. In period 0, she initially chooses to invest money in a liquid savings account or in a retirement account. In period 1, she then decides whether to make a withdrawal from her savings. If she initially invested in the retirement account, then her early withdrawal results in a tax penalty. In period 2, she finally consumes the remaining balance of the savings or retirement account. For simplicity of exposition, we assume linear utility and focus on deterministic consumption streams.

¹⁶That is, it is not the case that $v \approx u$ or $v \approx -u$.

¹⁷Proof: By Theorem 1, $\hat{v} \gg_u v$. Since it is not the case that $v \approx -u$, this implies $\hat{v} \approx \alpha u + (1 - \alpha)v$. Note that $\alpha > 0$ since the individual is strictly naive, and $\alpha < 1$ since \succsim has preference for commitment. Hence, it is not the case that $\hat{v} \approx u$, so there exist lotteries p, q such that $\hat{v}(p) = \hat{v}(q)$ and $u(p) > u(q)$. Since $\hat{v} \approx \alpha u + (1 - \alpha)v$ for $\alpha \in (0, 1)$, this also implies that $v(p) < v(q)$. Since it is not the case that $v \approx -u$, there exist lotteries r, s such that $u(r) > u(s)$ and $v(r) > v(s)$, which also implies $\hat{v}(r) > \hat{v}(s)$. It is easy to show that the lotteries $p^1 = (1 - \varepsilon)p + \varepsilon[(1/2)s + (1/2)r]$, $p^2 = (1 - \varepsilon)q + \varepsilon r$, $p^3 = (1 - \varepsilon)q + \varepsilon s$ have the desired properties for $\varepsilon > 0$ sufficiently small.

Suppose the individual's period 0 preference \succsim has a Strotz representation (U, \hat{V}) where

$$U(c_1, c_2) = c_1 + c_2 \quad \text{and} \quad \hat{V}(c_1, c_2) = c_1 + \hat{\beta}c_2,$$

and suppose the individual's period 1 choice function \mathcal{C} has a Strotz representation (U, V) where

$$V(c_1, c_2) = c_1 + \beta c_2.$$

As standard, assume $0 < \beta < \hat{\beta} \leq 1$.

If the gross interest rate is $R > 1$ and the individual initially has unit wealth, then investing in the liquid savings account is equivalent to choosing the menu

$$y = \{(c_1, c_2) \in \mathbb{R}_+^2 : c_1 + c_2/R = 1\}.$$

The retirement account has a proportional early withdrawal penalty τc_1 associated with a withdrawal of c_1 in period 1, where $\tau \geq 0$. Then investing in the retirement account is equivalent to choosing the menu

$$x^\tau = \{(c_1, c_2) \in \mathbb{R}_+^2 : (1 + \tau)c_1 + c_2/R = 1\}$$

Note that $x^\tau \subset y$, and that $x^\tau = y$ if $\tau = 0$. The actual choices from y and x^τ are

$$\mathcal{C}(y) = \begin{cases} (1, 0) & \text{if } 1 > \beta R \\ (0, R) & \text{if } 1 \leq \beta R. \end{cases}$$

and

$$\mathcal{C}(x^\tau) = \begin{cases} (1/(1 + \tau), 0) & \text{if } 1 > (1 + \tau)\beta R \\ (0, R) & \text{if } 1 \leq (1 + \tau)\beta R. \end{cases}$$

Suppose $1 > \hat{\beta}R$ and $(1 + \tau)\hat{\beta}R > 1 > (1 + \tau)\beta R$. In this case, the individual correctly anticipates choosing $(1, 0)$ from the menu y . However, she incorrectly anticipates choosing $(0, R)$ from x^τ , when in fact she will choose $(1/(1 + \tau), 0)$. She believes that the tax penalty associated with the retirement account is high enough to deter her from making early withdrawals in period 1, but in reality it is not. Since $U(0, R) > U(1, 0)$, this incorrect belief will lead the individual to initially invest in the illiquid retirement account x^τ over the liquid savings account y in period 0. Therefore, the availability of the retirement account as a savings instrument is actually detrimental to this individual, since

$$\{\mathfrak{C}(\{x^\tau, y\})\} = \{(1/(1 + \tau), 0)\} \prec \{(1, 0)\} = \{\mathfrak{C}(\{y\})\}.$$

Note that the perverse welfare effect associated with offering the individual the commitment device x^τ depends crucially on the level of the early withdrawal penalty: τ is high

enough that the individual thinks it will deter early withdrawals, but it is low enough that it actually does not. Increasing τ until $(1 + \tau)\beta R > 1$ makes the retirement account a strong enough commitment device to increase welfare. For τ in this region,

$$\{\mathfrak{C}(\{x^\tau, y\})\} = \{(0, R)\} \succ \{(1, 0)\} = \{\mathfrak{C}(\{y\})\}.$$

In the preceding example, increasing the early withdrawal penalty magnifies the strength of the commitment device x^τ . When τ is small and the commitment device is weak, the welfare effect of this commitment device is neutral or negative. Once τ exceeds some threshold—the exact value of which depends on the preference parameters—the commitment device becomes strong enough to deliver a positive welfare impact. Expanding on this observation, the following result shows there is a class of commitment devices that will unambiguously improve welfare for any preference: complete commitments to a single outcome. This holds for any naive individual (i.e., sophisticated or strictly naive).

Theorem 9 *Suppose \mathfrak{C} has the property that $\mathfrak{C}(X \cup \{x\}) \notin x$ implies $\mathfrak{C}(X \cup \{x\}) = \mathfrak{C}(X)$.¹⁸ If an individual is naive, then $\mathfrak{C}(X \cup \{p\}) \succeq \mathfrak{C}(X)$ for all lotteries p .*

Proof: By the assumed properties of \mathfrak{C} , either $\mathfrak{C}(X \cup \{p\}) = \mathfrak{C}(X)$, in which case the result holds trivially, or $\mathfrak{C}(X \cup \{p\}) = p$. In the latter case, we must have $\{p\} \succeq x$ for all $x \in X$. Since $\mathfrak{C}(X) = \mathcal{C}(y)$ for some $y \in x^*(X)$, naiveté then implies $\{p\} \succeq y \succeq \{\mathcal{C}(y)\}$, and hence $\{\mathfrak{C}(X \cup \{p\})\} \succeq \{\mathfrak{C}(X)\}$. ■

When menu choice is driven purely by temptation, adding extreme commitment devices is never harmful. Of course, such extreme commitment is detrimental if individuals have uncertain virtuous tastes that lead to demand for flexibility. Optimal design of commitment devices for naive individuals with some demand for flexibility remains an important open question, as do suitable comparative statics that isolate naiveté about future temptations and unawareness of future virtuous taste contingencies in a unified manner.

¹⁸This property simply implies that the tie-breaking procedure used in the selection function \mathfrak{C} does not change when unchosen options are added. This avoids spurious welfare conclusions that are artifacts of the tie-breaking protocol.

A A Comparative from Dekel and Lipman (2012)

Our definition of more temptation averse (Definition 2) was first applied to the random Strotz model by Dekel and Lipman (2012), who also provided a characterization of this comparative measure in terms of the representation. In this section, we summarize the relevant characterization result from Dekel and Lipman (2012), as it plays a central role in our proofs of Theorems 3 and 4.

The following theorem relates the more temptation averse comparative to the more u -aligned partial order \gg_u defined in the main text of this paper.

Theorem 10 (Dekel and Lipman (2012)) *Suppose C has finite cardinality. Suppose \succsim_1 and \succsim_2 have random Strotz representations (u_1, μ_1) and (u_2, μ_2) . Then \succsim_2 is more temptation averse than \succsim_1 if and only if $u_1 \approx u_2 \equiv u$ and $\mu_1 \gg_u \mu_2$.*

Proof: Theorem 4 in Dekel and Lipman (2012) establishes the equivalence of \succsim_2 being more temptation averse than \succsim_1 and another condition on the representations that they refer to as conditional dominance. However, they also establish that $\mu_1 \gg_u \mu_2$ as an intermediate step in their proof.¹⁹ The equivalence asserted in this theorem is also stated explicitly in Theorem 4 of their working paper, Dekel and Lipman (2010).²⁰ ■

B Proofs

B.1 Preliminaries

The proofs of the theorems in Section 3 rely on the following lemmas. The first is a slight variation of Farkas' Lemma.²¹

¹⁹To show that \succsim_2 being more temptation averse than \succsim_1 implies $\mu_1 \gg_u \mu_2$, the relevant results in Dekel and Lipman (2012) are the following: Lemma 3 shows that a partial order $v C_u v'$ used in their paper is equivalent to our order $v \gg_u v'$ (ignoring their normalization of utility functions). Lemmas 4, 5, and 6 and the arguments on page 1296 show that for any set W that is closed under C_u (is a u -upper set in our terminology), $\mu_1(W) \geq \mu_2(W)$.

²⁰Dekel and Lipman (2010) impose a normalization on the set of utility functions used in their result. However, by the uniqueness properties of the random Strotz representation established in Theorem 3 of Dekel and Lipman (2012), the probability of any u -upper set is the same for any random Strotz representation of the same preference. Therefore, their normalization of utilities is inconsequential for the result.

²¹There are two small distinctions between this result and the classic version of Farkas' Lemma. First, Farkas's Lemma deals with linear functions defined on a vector space whereas we restrict to linear functions defined on the convex subset $\Delta(C)$ of the vector space $ca(C)$ of all finite signed measures on C . Second, in condition 1 we only assume the conclusion that $g(p) \geq g(q)$ when the corresponding inequalities for f_1 and f_2 are strict. Together with our assumption that f_1 and f_2 are not ordinally opposed, we show in the proof that this condition implies the same conclusion for the case where the inequalities are weak.

Lemma 1 Suppose $f_1, f_2, g : \Delta(C) \rightarrow \mathbb{R}$ are continuous and affine, and suppose f_1 and f_2 are not ordinally opposed.²² Then the following are equivalent:

1. For all $p, q \in \Delta(C)$: $[f_1(p) > f_1(q) \text{ and } f_2(p) > f_2(q)] \implies g(p) \geq g(q)$.
2. There exist scalars $a, b \geq 0$ and $c \in \mathbb{R}$ such that $g = af_1 + bf_2 + c$.

Proof: It is immediate that 2 implies 1. To show 1 implies 2, we first argue that the 1 implies the same implication holds when the strict inequalities are replaced with weak inequalities:

$$[f_1(p) \geq f_1(q) \text{ and } f_2(p) \geq f_2(q)] \implies g(p) \geq g(q). \quad (10)$$

The argument relies on the assumption that f_1 and f_2 are not ordinally opposed and is similar to the use of constraint qualification in establishing the Kuhn-Tucker Theorem. For suppose $p, q \in \Delta(C)$ satisfy $f_1(p) \geq f_1(q)$ and $f_2(p) \geq f_2(q)$. Since f_1 and f_2 are not ordinally opposed, there exist $p^*, q^* \in \Delta(C)$ such that $f_1(p^*) > f_1(q^*)$ and $f_2(p^*) > f_2(q^*)$. Let $p^\alpha \equiv \alpha p^* + (1 - \alpha)p$ and $q^\alpha \equiv \alpha q^* + (1 - \alpha)q$. Since these functions are affine, $f_1(p^\alpha) > f_1(q^\alpha)$ and $f_2(p^\alpha) > f_2(q^\alpha)$ for all $\alpha \in (0, 1]$. Condition 1 therefore implies $g(p^\alpha) \geq g(q^\alpha)$ for all $\alpha \in (0, 1]$. By continuity $g(p) \geq g(q)$. This establishes the condition in Equation (10).

Fix any $\bar{c} \in C$ and define $\bar{f}_1(p) \equiv f_1(p) - f_1(\delta_{\bar{c}})$, $\bar{f}_2(p) \equiv f_2(p) - f_2(\delta_{\bar{c}})$, and $\bar{g}(p) \equiv g(p) - g(\delta_{\bar{c}})$. Note that Equation (10) holds for f_1, f_2, g if and only if it holds for $\bar{f}_1, \bar{f}_2, \bar{g}$. Each of these functions can be extended to a continuous linear function on the space $ca(C)$ of all finite signed measures on C : Since the mapping $c \mapsto \bar{f}_1(\delta_c)$ is continuous in the topology on C , the function $F_1(p) \equiv \int \bar{f}_1(\delta_c) dp$ for $p \in ca(C)$ is a well-defined continuous linear functional that extends \bar{f}_1 . Define F_2 and G analogously. We next show that for any $p, q \in ca(C)$:

$$[F_1(p) \geq F_1(q) \text{ and } F_2(p) \geq F_2(q)] \implies G(p) \geq G(q). \quad (11)$$

To establish this condition, fix any $p, q \in ca(C)$ and suppose $F_i(p) \geq F_i(q)$ for $i = 1, 2$. Let $p' = p - p(C)\delta_{\bar{c}}$ and $q' = q - q(C)\delta_{\bar{c}}$. Then $p'(C) = q'(C) = 0$, and we also have $F_i(p') \geq F_i(q')$ since $\bar{f}_i(\delta_{\bar{c}}) = 0$. Equivalently, $F_i(p' - q') \geq 0$. There exists $p'', q'' \in \Delta(C)$ and $\alpha \geq 0$ such that $p' - q' = \alpha(p'' - q'')$. By linearity, $F_i(p'') \geq F_i(q'')$, which implies $f_i(p'') \geq f_i(q'')$ for $i = 1, 2$. Equation (10) therefore implies $g(p'') \geq g(q'')$, which implies $G(p'') \geq G(q'')$ and consequently $G(p') \geq G(q')$ and $G(p) \geq G(q)$. This establishes Equation (11).

By the Convex Cone Alternative Theorem (an infinite-dimensional version of Farkas' Lemma) (Aliprantis and Border (2006, Corollary 5.84)), Equation (11) implies there exist $a, b \geq 0$ such that $G = aF_1 + bF_2$. Thus $\bar{g} = a\bar{f}_1 + b\bar{f}_2$, and hence $g = af_1 + bf_2 + c$, where $c = g(\delta_{\bar{c}}) - af_1(\delta_{\bar{c}}) - bf_2(\delta_{\bar{c}})$. \blacksquare

Lemmas 2 and 3 below are used in the proofs of Theorems 1 and 2. In the case of finite C , it is easy to show that these lemmas are equivalent to Lemma 3 in Dekel and Lipman (2012),

²²That is, there exist $p, q \in \Delta(C)$ such that both $f_1(p) > f_1(q)$ and $f_2(p) > f_2(q)$.

who also noted the connection to the Harsanyi Aggregation Theorem. Since we allow compact C , we include the short proofs of these results to show that no technical problems arise in our more general domain.

Lemma 2 *Let u, v, v' be nontrivial expected-utility functions defined on $\Delta(C)$. If for all lotteries p and q we have*

$$[u(p) > u(q) \text{ and } v'(p) > v'(q)] \implies v(p) \geq v(q),$$

then $v \gg_u v'$.

Proof: If $v' \approx -u$, then by definition $v \gg_u v'$. Alternatively, if it is not the case that $v' \approx -u$, then u and v' are not ordinally opposed. In this case, Lemma 1 implies there exist $a, b \geq 0$ and $c \in \mathbb{R}$ such that

$$v = au + bv' + c.$$

Since v is nontrivial, it must be that $a+b > 0$. Thus $v \approx \alpha u + (1-\alpha)v'$ for $\alpha = a/(a+b) \in [0, 1]$, and hence $v \gg_u v'$. \blacksquare

Lemma 3 *Let u, v, v' be expected-utility functions defined on $\Delta(C)$, and suppose $v \gg_u v'$. Then for any menu x ,*

$$\max_{p \in B_v(x)} u(p) \geq \max_{q \in B_{v'}(x)} u(q).$$

Proof: If $v' \approx -u$, then for any menu x ,

$$\max_{q \in B_{v'}(x)} u(q) = \min_{q \in x} u(q) \leq u(p), \quad \forall p \in x.$$

In particular,

$$\max_{q \in B_{v'}(x)} u(q) \leq \max_{p \in B_v(x)} u(p).$$

If we do not have $v' \approx -u$, then $v \gg_u v'$ implies $v \approx \alpha u + (1-\alpha)v'$ for some $\alpha \in [0, 1]$. First, consider $\alpha = 0$. In this case, $v \approx v'$. Therefore $B_v(x) = B_{v'}(x)$, which implies

$$\max_{p \in B_v(x)} u(p) = \max_{q \in B_{v'}(x)} u(q).$$

Finally, consider the case of $\alpha > 0$. Note that for any menu x and any $p \in B_v(x)$ and $q \in B_{v'}(x)$,

$$\alpha u(p) + (1-\alpha)v'(p) \geq \alpha u(q) + (1-\alpha)v'(q) \quad \text{and} \quad v'(q) \geq v'(p).$$

Since $\alpha > 0$, these inequalities imply $u(p) \geq u(q)$. Therefore,

$$\max_{p \in B_v(x)} u(p) \geq \max_{q \in B_{v'}(x)} u(q),$$

as claimed. ■

The following lemma shows that the more temptation averse comparative is implied by $\mu_1 \gg_u \mu_2$. In the case of finite C , this is precisely the necessity part of Theorem 10. Since we wish to include several applications where C is compact but not finite (e.g., dynamic consumption problems where $C = [a, b]^{\mathbb{N}}$), some of our results only require compact C (e.g., Theorems 1, 2, 5, and 6). Lemma 4 is used in the proofs of those results.

Lemma 4 *Suppose C is compact. Suppose \succsim_1 and \succsim_2 have random Strotz representations (u_1, μ_1) and (u_2, μ_2) . If $u_1 \approx u_2 \equiv u$ and $\mu_1 \gg_u \mu_2$, then \succsim_2 is more temptation averse than \succsim_1 .*

Proof: Suppose (u_1, μ_1) and (u_2, μ_2) are random Strotz representations of \succsim_1 and \succsim_2 such that $u_1 \approx u_2 \equiv u$ and $\mu_1 \gg_u \mu_2$. Since positive affine transformations of the functions u_i do not change the preferences, it is without loss of generality to assume $u_1 = u_2 \equiv u$. Now fix any menu x , and let $[a, b] = u(x)$. Define $f_x : \mathcal{V} \rightarrow [a, b]$ by

$$f_x(v) = \max_{p \in B_v(x)} u(p).$$

By Lemma 3, $v \gg_u v'$ implies $f_x(v) \geq f_x(v')$. Therefore, for any $\alpha \in [a, b]$ and $v \gg_u v'$,

$$v' \in f_x^{-1}([\alpha, b]) \iff f_x(v') \geq \alpha \implies f_x(v) \geq \alpha \iff v \in f_x^{-1}([\alpha, b]).$$

Thus $f_x^{-1}([\alpha, b])$ is a u -upper set. Therefore,

$$\mu_1(f_x^{-1}([\alpha, b])) \geq \mu_2(f_x^{-1}([\alpha, b])).$$

Define distributions $\eta_i^x \equiv \mu_i \circ f_x^{-1}$ on $[a, b]$ for $i = 1, 2$. By the preceding arguments, η_1^x first-order stochastically dominates η_2^x . Therefore, by the change of variables formula,

$$U_1(x) = \int_{\mathcal{V}} f_x(v) d\mu_1(v) = \int_a^b \alpha d\eta_1^x(\alpha) \geq \int_a^b \alpha d\eta_2^x(\alpha) = \int_{\mathcal{V}} f_x(v) d\mu_2(v) = U_2(x).$$

Since this is true for every x , it follows immediately that \succsim_2 is more temptation averse than \succsim_1 . ■

B.2 Proof of Theorem 1

To establish sufficiency, suppose the individual is naive. Then for any lotteries p and q ,

$$\begin{aligned} [u(p) > u(q) \text{ and } v(p) > v(q)] &\implies [\{p\} \succ \{q\} \text{ and } \mathcal{C}(\{p, q\}) = p] \\ &\implies \{p, q\} \succsim \{p\} \succ \{q\} \quad (\text{by naiveté}) \\ &\implies \hat{v}(p) \geq \hat{v}(q). \end{aligned}$$

By Lemma 2, this implies that $\hat{v} \gg_u v$.

To establish necessity, suppose $\hat{v} \gg_u v$. By Lemma 3, this implies that for any menu x ,

$$U(x) = \max_{p \in B_{\hat{v}}(x)} u(p) \geq \max_{q \in B_v(x)} u(q) = u(\mathcal{C}(x)).$$

Thus $x \succsim \{\mathcal{C}(x)\}$, so the individual is naive.

B.3 Proof of Theorem 2

Theorem 2 follows from the two lemmas below together with Theorem 1.

Lemma 5 *Suppose \succsim_1 and \succsim_2 have Strotz representations (u_1, \hat{v}_1) and (u_2, \hat{v}_2) . Then individual 2 is more temptation averse than individual 1 if and only if $u_1 \approx u_2 \equiv u$ and $\hat{v}_1 \gg_u \hat{v}_2$.*

Proof: Necessity follows as a special case of Lemma 4, since $\hat{v}_1 \gg_u \hat{v}_2$ is equivalent to $\delta_{\hat{v}_1} \gg_u \delta_{\hat{v}_2}$. To establish sufficiency, suppose individual 2 is more temptation averse than individual 1. First, observe that taking $x = \{q\}$ in Definition 2 yields $\{p\} \succ_1 \{q\} \implies \{p\} \succ_2 \{q\}$. Since u_1 and u_2 are nontrivial, it is a standard result that this implies $u_1 \approx u_2$. Let $u \equiv u_2$ in what follows. Then for any lotteries p and q ,

$$\begin{aligned} [u(p) > u(q) \text{ and } \hat{v}_2(p) > \hat{v}_2(q)] &\implies \{p, q\} \sim_2 \{p\} \succ_2 \{q\} \\ &\implies \{p, q\} \succsim_1 \{p\} \succ_1 \{q\} \quad (2 \text{ more temptation averse than } 1) \\ &\implies \hat{v}_1(p) \geq \hat{v}_1(q). \end{aligned}$$

By Lemma 2, this implies that $\hat{v}_1 \gg_u \hat{v}_2$. ■

Lemma 6 *Suppose \succsim_1 and \succsim_2 have Strotz representations (u_1, \hat{v}_1) and (u_2, \hat{v}_2) , and \mathcal{C}_1 and \mathcal{C}_2 have Strotz representations (u_1, v_1) and (u_2, v_2) . Then individual 2 is more virtuous than individual 1 if and only if $u_1 \approx u_2 \equiv u$ and $v_2 \gg_u v_1$.*

Proof: To establish sufficiency, suppose individual 2 is more virtuous than individual 1. First, observe that taking $x = \{q\}$ in Definition 3 yields $\{p\} \succ_2 \{q\} \implies \{p\} \succ_1 \{q\}$. Since

u_1 and u_2 are nontrivial, it is a standard result that this implies $u_1 \approx u_2$. Let $u \equiv u_2$ in what follows. Then for all menus x and lotteries p ,

$$u(p) > u(\mathcal{C}_2(x)) \implies u(p) > u(\mathcal{C}_1(x)). \quad (12)$$

Therefore, for any lotteries p and q ,

$$\begin{aligned} & [u(p) > u(q) \text{ and } v_1(p) > v_1(q)] \\ & \implies u(\mathcal{C}_1(\{p, q\})) = u(p) > u(q) \\ & \implies u(\mathcal{C}_2(\{p, q\})) \geq u(p) > u(q) \quad (\text{contrapositive of Equation (12)}) \\ & \implies v_2(p) \geq v_2(q). \end{aligned}$$

By Lemma 2, this implies that $v_2 \gg_u v_1$.

To establish necessity, suppose $u_1 \approx u_2 \equiv u$ and $v_2 \gg_u v_1$. By Lemma 3, $v_2 \gg_u v_1$ implies that for any menu x ,

$$u(\mathcal{C}_2(x)) = \max_{p \in B_{v_2}(x)} u(p) \geq \max_{q \in B_{v_1}(x)} u(q) = u(\mathcal{C}_1(x)).$$

Thus $u(p) > u(\mathcal{C}_2(x)) \implies u(p) > u(\mathcal{C}_1(x))$, so individual 2 is more virtuous than individual 1. ■

B.4 Proof of Theorem 3

Suppose the random choice rule λ has a random Strotz representation (u, μ) . The ex ante preference of a sophisticated individual—which may differ from the individual’s actual ex ante preference if she is naive—must also be represented by (u, μ) . The following lemma shows how this hypothetical sophisticated preference can be determined from λ and u .

Lemma 7 *Suppose λ has a random Strotz representation (u, μ) . Define a binary relation \succ^* on $\mathcal{K}(\Delta(C))$ by*

$$x \succ^* y \iff u(m(\lambda^x)) \geq u(m(\lambda^y)).$$

Then (u, μ) is a random Strotz representation for \succ^ .*

Proof: Since (u, μ) represents λ , by definition there exists, for all menus x , a measurable selection function $p_x : \mathcal{V} \rightarrow x$ with $p_x(v) \in B_u(B_v(x))$ such that

$$\lambda^x(y) = \mu(p_x^{-1}(y))$$

for all measurable $y \subset x$. Thus λ^x is the distribution on x induced by the random variable p_x defined on the measure space (\mathcal{V}, μ) . Therefore, the standard change of variables formula

together with the linearity and continuity of u imply

$$\begin{aligned} \int_{\mathcal{V}} \max_{p \in B_v(x)} u(p) d\mu(v) &= \int_{\mathcal{V}} u(p_x(v)) d\mu(v) \\ &= \int_x u(p) d\lambda^x(p) = u\left(\int_x p d\lambda^x(p)\right) = u(m(\lambda^x)), \end{aligned}$$

as desired. ■

Turning now to the proof of Theorem 3, fix random Strotz representations $(u, \hat{\mu})$ and (u, μ) for \succsim and λ , respectively, and define \succsim^* as in Lemma 7. To establish sufficiency, suppose the individual is naive. Then for all menus x and lotteries p ,

$$\begin{aligned} \{p\} \succ x &\implies \{p\} \succ \{m(\lambda^x)\} && \text{(naiveté)} \\ &\implies u(m(\lambda^{\{p\}})) = u(p) > u(m(\lambda^x)) \\ &\implies \{p\} \succ^* x. \end{aligned}$$

Thus \succsim^* is more temptation averse than \succsim . Since (u, μ) represents \succsim^* by Lemma 7, Theorem 10 implies that $\hat{\mu} \gg_u \mu$. If the individual is sophisticated, then a similar argument shows that the converse also holds: \succsim is also more temptation averse than \succsim^* (in particular, $\succsim = \succsim^*$) and hence $\mu \gg_u \hat{\mu}$.

The following lemma establishes necessity and shows, in particular, that the restriction to finite C is not needed for this direction.

Lemma 8 *Suppose C is compact. Suppose \succsim has a random Strotz representation $(u, \hat{\mu})$, and λ has a random Strotz representation (u, μ) . If $\hat{\mu} \gg_u \mu$ then the individual is naive (and if $\hat{\mu} \gg_u \mu$ and $\mu \gg_u \hat{\mu}$ then the individual is sophisticated).*

Proof: Suppose $\hat{\mu} \gg_u \mu$, and define \succsim^* as in Lemma 7. By Lemma 4, \succsim^* is more temptation averse than \succsim . By contrapositive, this is equivalent to the condition

$$x \succsim^* \{p\} \implies x \succsim \{p\}.$$

Note that for any menu x , if we take $p = m(\lambda^x)$ then

$$u(m(\lambda^x)) = u(p) = u(m(\lambda^{\{p\}}))$$

and hence $x \sim^* \{p\} = \{m(\lambda^x)\}$. Since \succsim^* is more temptation averse than \succsim , this implies $x \succsim \{m(\lambda^x)\}$. Thus the individual is naive. If we also have $\mu \gg_u \hat{\mu}$ then an analogous argument can be used to show that the condition above can be strengthened to $x \succsim^* \{p\} \iff x \succsim \{p\}$. In this case, $x \sim \{m(\lambda^x)\}$ and hence the individual is sophisticated. ■

B.5 Proof of Theorem 4

This result can be separated into three components, two of which have already been proved. First, note that by Theorem 3, individual 2 is naive if and only if $\hat{\mu}_2 \gg_u \mu_2$, where $u \equiv u_2$. Second, by Theorem 10, individual 2 is more temptation averse than individual 1 if and only if $u_1 \approx u_2 \equiv u$ and $\hat{\mu}_1 \gg_u \hat{\mu}_2$. The final step is completed in the following lemma. Note that the restriction to finite C is not needed for necessity; part 1 of the lemma can therefore be used later in the proof of Theorem 6.

Lemma 9 *Suppose \succsim_1 and \succsim_2 have random Strotz representations $(u_1, \hat{\mu}_1)$ and $(u_2, \hat{\mu}_2)$, and λ_1 and λ_2 have random Strotz representations (u_1, μ_1) and (u_2, μ_2) .*

1. *If $u_1 \approx u_2 \equiv u$ and $\mu_2 \gg_u \mu_1$ then individual 2 is more virtuous than individual 1.*
2. *If C has finite cardinality and individual 2 is more virtuous than individual 1, then $u_1 \approx u_2 \equiv u$ and $\mu_2 \gg_u \mu_1$.*

Proof: Define \succsim_1^* and \succsim_2^* as in Lemma 7 for λ_1 and λ_2 , respectively. Then (u_1, μ_1) and (u_2, μ_2) represent \succsim_1^* and \succsim_2^* . Note that for all menus x and lotteries p ,

$$\{p\} \succ_i \{m(\lambda_i^x)\} \iff u_i(p) > u_i(m(\lambda_i^x)) \iff \{p\} \succ_i^* x, \quad i = 1, 2.$$

Therefore, individual 2 is more virtuous than individual 1 if and only if \succsim_1^* is more temptation averse than \succsim_2^* . If C has finite cardinality, then by Theorem 10, this is true if and only if $u_1 \approx u_2 \equiv u$ and $\mu_2 \gg_u \mu_1$. For any compact C (not necessarily finite), Lemma 4 shows that if $u_1 \approx u_2 \equiv u$ and $\mu_2 \gg_u \mu_1$, then \succsim_1^* is more temptation averse than \succsim_2^* . ■

B.6 Proof of Theorem 5

Lemma 10 *Suppose (u, v, F) is an uncertain intensity Strotz representation. Define a function $g : [0, 1] \rightarrow \mathcal{V}$ by $g(\alpha) = \alpha u + (1 - \alpha)v$. Define a probability measure μ on \mathcal{V} by $\mu \equiv F \circ g^{-1}$.²³ Then the following statements hold.*

1. *If (u, v, F) is an uncertain intensity Strotz representation of a preference \succsim , then (u, μ) is a random Strotz representation of \succsim .*
2. *If (u, v, F) is an uncertain intensity Strotz representation of a random choice rule λ , then (u, μ) is a random Strotz representation of λ .*

²³We are abusing notation slightly and using F to also denote the probability measure on $[0, 1]$ that has F as its distribution function. That is, for any measurable set $A \subset [0, 1]$, we write $F(A)$ to denote $\int_A dF(\alpha)$. Thus $\mu(E) = \int_{\{\alpha' : g(\alpha') \in E\}} dF(\alpha)$ for any measurable $E \subset \mathcal{V}$.

3. Suppose (u, v, F_i) are uncertain intensity Strotz representations for $i = 1, 2$ and define $\mu_i \equiv F_i \circ g^{-1}$. Then $\mu_1 \gg_u \mu_2$ if and only if $F_1 \geq_{FOSD} F_2$.

Proof: To prove statement 1, note that by assumption \succsim is represented by

$$U(x) = \int_0^1 \max\{u(p) : p \in B_{g(\alpha)}(x)\} dF(\alpha).$$

By the standard change of variables formula, this implies

$$\begin{aligned} U(x) &= \int_{\mathcal{V}} \max\{u(p) : p \in B_{\tilde{v}}(x)\} d(F \circ g^{-1})(\tilde{v}) \\ &= \int_{\mathcal{V}} \max\{u(p) : p \in B_{\tilde{v}}(x)\} d\mu(\tilde{v}), \end{aligned}$$

and hence (u, μ) is a random Strotz representation of \succsim .

To prove statement 2, note that by assumption there exists, for all menus x , a measurable selection function $p_x : [0, 1] \rightarrow x$ with $p_x(\alpha) \in B_u(B_{g(\alpha)}(x))$ for all $\alpha \in [0, 1]$ such that

$$\lambda^x(y) = \int_0^1 \mathbf{1}_{[p_x(\alpha) \in y]} dF(\alpha)$$

for all measurable $y \subset x$. Take any measurable selection function $\tilde{p}_x : \mathcal{V} \rightarrow x$ with $\tilde{p}_x(\tilde{v}) \in B_u(B_{\tilde{v}}(x))$ for all $\tilde{v} \in \mathcal{V}$ that also satisfies $p_x(\alpha) = \tilde{p}_x(g(\alpha))$ for all $\alpha \in [0, 1]$.²⁴ Therefore, for any measurable $y \subset x$,

$$\begin{aligned} \lambda^x(y) &= \int_0^1 \mathbf{1}_{[\tilde{p}_x(g(\alpha)) \in y]} dF(\alpha) \\ &= \int_{\mathcal{V}} \mathbf{1}_{[\tilde{p}_x(\tilde{v}) \in y]} d(F \circ g^{-1})(\tilde{v}) \\ &= \mu(\tilde{p}_x^{-1}(y)), \end{aligned}$$

and hence (u, μ) is a random Strotz representation of λ .

To prove statement 3, suppose $\mu_i \equiv F_i \circ g^{-1}$ for $i = 1, 2$ and $\mu_1 \gg_u \mu_2$. Fix any $\alpha \in [0, 1]$, and let $\mathcal{U} = \{v' \in \mathcal{V} : v' \gg_u \alpha u + (1 - \alpha)v\}$. By construction, \mathcal{U} is a u -upper set, so $\mu_1(\mathcal{U}) \geq \mu_2(\mathcal{U})$. In addition, $g^{-1}(\mathcal{U}) = [\alpha, 1]$. Therefore,

$$F_1([\alpha, 1]) = \mu_1(\mathcal{U}) \geq \mu_2(\mathcal{U}) = F_2([\alpha, 1]).$$

Since this is true for all $\alpha \in [0, 1]$, $F_1 \geq_{FOSD} F_2$.

²⁴To see that such a selection function \tilde{p}_x exists, fix any measurable selection function $\hat{p}_x : \mathcal{V} \rightarrow x$ with $\hat{p}_x(\tilde{v}) \in B_u(B_{\tilde{v}}(x))$ for all $\tilde{v} \in \mathcal{V}$. Let $\bar{\mathcal{V}} = g([0, 1]) \subset \mathcal{V}$. When the codomain of g is restricted to $\bar{\mathcal{V}}$, i.e., $g : [0, 1] \rightarrow \bar{\mathcal{V}}$, this function is a bijection. Now define $\tilde{p}_x(\tilde{v}) = p_x(g^{-1}(\tilde{v}))$ for $\tilde{v} \in \bar{\mathcal{V}}$ and $\tilde{p}_x(\tilde{v}) = \hat{p}_x(\tilde{v})$ for $\tilde{v} \notin \bar{\mathcal{V}}$.

Conversely, suppose $F_1 \geq_{FOSD} F_2$. Fix any u -upper set \mathcal{U} . Note that for any $0 \leq \alpha \leq \alpha' \leq 1$, we have $g(\alpha') \gg_u g(\alpha)$ and hence

$$g(\alpha) \in \mathcal{U} \implies g(\alpha') \in \mathcal{U}.$$

This implies that the set $g^{-1}(\mathcal{U})$ is an interval from some $\alpha^* \in [0, 1]$ to 1.²⁵ Therefore,

$$\mu_1(\mathcal{U}) = F_1(g^{-1}(\mathcal{U})) \geq F_2(g^{-1}(\mathcal{U})) = \mu_2(\mathcal{U}).$$

Since this is true for all u -upper sets, $\mu_1 \gg_u \mu_2$. ■

Turning now to the proof of Theorem 5, suppose \succsim has an uncertain intensity Strotz representation (u, v, \hat{F}) , and λ has an uncertain intensity Strotz representation (u, v, F) . To establish sufficiency, suppose the individual is naive. The conclusion that $\hat{F} \geq_{FOSD} F$ follows immediately from Theorem 3 and Lemma 10 in the case of finite C . A small additional step is needed in the case where C does not have finites cardinality.

Specifically, there is a finite subset $C^* \subset C$ such that the restrictions of u and v to $\Delta(C^*)$ are also independent expected-utility functions, that is, neither one a positive or negative affine transformation of the other. We abuse notation slightly and also denote these restrictions by u, v . Note also that the restrictions of \succsim and λ to $\mathcal{K}(\Delta(C^*))$ must satisfy naiveté. Let \mathcal{V}^* denote the set of all continuous functions $\tilde{v} : C^* \rightarrow \mathbb{R}$. Define measures $\hat{\mu} \equiv \hat{F} \circ g^{-1}$ and $\mu \equiv F \circ g^{-1}$ on \mathcal{V}^* , where $g : [0, 1] \rightarrow \mathcal{V}^*$ is defined by $g(\alpha) = \alpha u + (1 - \alpha)v$. By Lemma 10, $(u, \hat{\mu})$ and (u, μ) are random Strotz representations of the restrictions of \succsim and λ to $\mathcal{K}(\Delta(C^*))$. Therefore, Theorem 3 implies $\hat{\mu} \gg_u \mu$. By Lemma 10, this implies $\hat{F} \geq_{FOSD} F$.

To establish necessity, suppose $\hat{F} \geq_{FOSD} F$. Define measures $\hat{\mu} \equiv \hat{F} \circ g^{-1}$ and $\mu \equiv F \circ g^{-1}$ on \mathcal{V} . Lemma 10 implies that $(u, \hat{\mu})$ and (u, μ) are random Strotz representations for \succsim and λ , and it implies that $\hat{\mu} \gg_u \mu$. By Lemma 8, the individual is naive.

B.7 Proof of Theorem 6

To establish sufficiency, suppose individual 1 is more naive than individual 2. As argued in the proof of Theorem 5, there exists a finite subset $C^* \subset C$ such that the restrictions of u and v to $\Delta(C^*)$ are also independent expected-utility functions. Note that when the preferences and random choice rules of the individuals are restricted to the domain $\mathcal{K}(\Delta(C^*))$, individual 1 is still more naive than individual 2. Define \mathcal{V}^* and g as in the proof of Theorem 5, and define measures $\hat{\mu}_i \equiv \hat{F}_i \circ g^{-1}$ and $\mu_i \equiv F_i \circ g^{-1}$ on \mathcal{V}^* for $i = 1, 2$. By Lemma 10, $(u, \hat{\mu}_i)$ and (u, μ_i) are random Strotz representations for the restrictions of \succsim_i and λ_i to $\mathcal{K}(\Delta(C^*))$ for $i = 1, 2$. Therefore, Theorem 4 implies

$$\hat{\mu}_1 \gg_u \hat{\mu}_2 \gg_u \mu_2 \gg_u \mu_1.$$

²⁵That is, it is equal to either $(\alpha^*, 1]$ or $[\alpha^*, 1]$, where $\alpha^* = \inf(g^{-1}(\mathcal{U}))$.

By Lemma 10, this implies

$$\hat{F}_1 \geq_{FOSD} \hat{F}_2 \geq_{FOSD} F_2 \geq_{FOSD} F_1.$$

To establish necessity, suppose

$$\hat{F}_1 \geq_{FOSD} \hat{F}_2 \geq_{FOSD} F_2 \geq_{FOSD} F_1.$$

Define measures $\hat{\mu}_i \equiv \hat{F}_i \circ g^{-1}$ and $\mu_i \equiv F_i \circ g^{-1}$ on \mathcal{V} for $i = 1, 2$. Lemma 10 implies that $(u, \hat{\mu}_i)$ and (u, μ_i) are random Strotz representations for \succsim and λ , and it implies that

$$\hat{\mu}_1 \gg_u \hat{\mu}_2 \gg_u \mu_2 \gg_u \mu_1.$$

By Lemma 8, the individuals are naive. By Lemma 9, individual 2 is more virtuous than individual 1. By Lemma 4, individual 2 is more temptation averse than individual 1. Therefore, individual 1 is more naive than individual 2.

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