Savage in the Market[1]

Federico Echenique
Caltech

Kota Saito
Caltech

February 3, 2014

[1]We thank Kim Border and Chris Chambers for inspiration, comments and advice. Matt Jackson's suggestions led to some of the applications of our main result. We also thank seminar audiences in Bocconi University, Caltech, Collegio Carlo Alberto, Princeton University, Larry Epstein, Eddie Dekel, Massimo Marinacci, and John Quah for comments.

Abstract

We develop a behavioral axiomatic characterization of Subjective Expected Utility (SEU) under risk aversion. Given is an individual agent's behavior in the market: assume a finite collection of asset purchases with corresponding prices. We show that such behavior satisfies a "revealed preference axiom" if and only if there exists a SEU model (a subjective probability over states and a concave utility function over money) that accounts for the given asset purchases.

1 Introduction

When working with markets and uncertainty, economists often assume that agents maximize expected utility. The meaning of such an assumption is that agents' behavior in the market is *as if* they were maximizing an expected utility function with subjective beliefs over the states of the world. *The purpose of our paper is to describe the full range of possible behaviors of agents that are consistent with subjective expected utility (SEU).* Our main result is a revealed preference characterization of SEU.

Revealed preference theory uses two canonical models to describe an agent's choice behavior. The first model is a preference relation; meaning that a researcher elicits all the agent's choices from every possible pair of alternatives. The second model requires an economic framework, in which an agent makes choices from different sets of feasible, or affordable, choices. The notion of choice as a preference relation has the advantage that it makes sense in abstract frameworks, and therefore can be used in many different environments. The second model requires some kind of economic environment, in which an agent makes optimal choices subject to budget constraints. Such datasets of optimal choices have been studied since the beginning on revealed preference theory, with the work of Samuelson (1938), Houthakker (1950), and Afriat (1967).

Savage (1954) characterizes SEU when behavior is given by a preference: Savage obtains a set of seven axioms on a preference relation that are necessary and sufficient for SEU. Our contribution is to give the first characterization of SEU when behavior is given by a dataset of supposedly optimal purchases given some budgets.

Our main result is that a certain revealed preference axiom, termed the "Strong Axiom of Revealed Subjective Expected Utility" (SARSEU), describes the choice data that are consistent with *risk averse SEU preferences.* SARSEU builds on the simplest implication of risk aversion on the relation between prices and quantities: that demand slopes down. The axiom constraints quantities and prices in a way that generalizes downward-sloping demand, but accounting for the different unobservable components in SEU. Section 3.2 has an informal derivation of the axiom, together with a formal statement of our main result.

SARSEU seems like a relatively weak imposition on data, in the sense that it constraints prices and quantities *in those situations in which unobservables do no matter.* Essentially, SARSEU requires one to consider situations in which unobservables "cancel out" (see Section 3.2), and check that the implications of concave utility on prices are not violated.

Aside from SEU, the paper also includes a revealed preference characterization of quasilinear SEU. We include this model for two reasons. One is that quasilinear SEU is frequently used in economic modeling, and therefore of independent interest. The other is pedagogical. The axiom for quasilinear SEU turns out be an obvious generalization of the axiom for SEU, and the analysis is much simpler. The proof of the sufficiency of the axiom for quasilinear SEU is much shorter than the corresponding proof for SEU.

The paper develops several applications of our main result (Section 4), and a discussion of related models (Section 5).

The applications show how some relaxation of SEU can have empirical implications beyond SEU. For each application, we exhibit a dataset that violates SARSEU. The first application is to show that risk aversion is testable in SEU. Afriat's theorem implies that concavity of the (general, non-separable) utility function is not testable. For SEU, however, the concavity of the Bernoulli utility function over money is testable. We exhibit a violation of SARSEU that is generated by a non-concave SEU maximizing agent: the details are in Section 4.1.

Our second and third applications are to max-min (Gilboa and Schmeidler (1989)) utility, and to state-dependent expected utility; see, respectively, Sections 4.2 and 4.3. In each case we exhibit an instance of the model that can generate data which violate SARSEU. Therefore the added generality in max-min and state-dependent utility, beyond SEU, has testable implications.

The paper continues in Section 5 by developing comparisons with the most closely related literature.

The closest precedent to our paper, in the context of uncertainty (meaning subjective and unobservable probabilities), is the work of Epstein (2000). Epstein's setup is the same as ours; in particular, he assumes data on state-contingent asset purchases, and that probabilities are subjective and unobserved. We differ in that he focuses attention on probabilistic sophistication, while our paper is on SEU. Epstein presents a necessary condition for market behavior to be consistent with probabilistic sophistication. Given that the model of probabilistic sophistication is more general than SEU, one expects that the two axioms may be related: Indeed we show in Section 5.1 that Epstein's necessary condition can be obtained as a special case of SARSEU. Section 3.3 has a related geometric treatment.

Varian's (1983a) result on additive separability does not deal with uncertainty explicitly, but it can be interpreted within our context as providing a test of state-dependent utility. His characterization is in terms of the existence of a solution of a system of linear "Afriat inequalities" (Varian writes "I have been unable to find a convenient combinatorial condition that is necessary and sufficient for additive separability"). SARSEU is a combinatorial condition, but it characterizes SEU, not additive separability.

The recent work of Polisson and Quah (2013) develops tests for many models of decision under risk and uncertainty, including SEU. They develop a general approach by which testing a model amounts to solving a system of Afriat inequalities. In contrast with Afriat, in their case the systems may not be linear, and deciding its solubility may be computationally hard. For SEU, we also formulate a system of Afriat inequalities as an input in our proof (Lemma 8). An important difference with Polisson and Quah is that they do not require the concavity of the Bernoulli utility function.

Another strain of related work deals with objective expected utility, assuming observable priors. The papers by Green and Srivastava (1986), Varian (1983b), Varian (1988), and Kubler et al. (2014) characterize the datasets of purchases of state-dependent assets that are consistent with expected utility theory with given probabilities. The data in these studies are similar to ours, but with the added information of probabilities over states.

Varian (1983b), Green and Srivastava (1986), and Varian (1988) give a result in the form of "Afriat inequalities." They describe the datasets that are consistent with von-Neumann Morgenstern objective expected utility. Their result is that such consistent datasets are the ones for which there is a solution to a system of linear Afriat inequalities. In addition, Varian (1988) focuses on empirically recovering an agent's attitude towards risk.

The work of Kubler et al. (2014) goes beyond a system of Afriat inequalities to present a revealed preference axiom that is equivalent to consistent with expected utility. We show (see Section 5.2) that the axiom of Kubler et. al is equivalent to an axiom that is similar to ours, after one uses the observed probabilities to adjust prices.

2 Framework and models

We use the following notational conventions: For vectors $x, y \in \mathbf{R}^n$, $x \leq y$ means that $x_i \leq y_i$ for all $i = 1, \ldots, n$; $x < y$ means that $x \leq y$ and $x \neq y$; and $x \ll y$ means that $x_i < y_i$ for all $i = 1, \ldots, n$. The set of all $x \in \mathbf{R}^n$ with $0 \leq x$ is denoted by $\mathbf{R}^n_+$ and the set of all $x \in \mathbf{R}^n$ with $0 \ll x$ is denoted by $\mathbf{R}^n_{++}$. Let $\Delta^n_{++} = \{x \in \mathbf{R}^n_{++} | \sum_{i=1}^n x_i = 1\}$ denote the set of strictly positive probability measures on S.

In our model, the objects of choice are state-contingent monetary payoffs, or *monetary acts*. We assume a finite number S of *states*, and refer to vectors in $\mathbf{R}^S_+$ as monetary acts. We occasionally use S to denote the set $\{1, \ldots, S\}$.

Definition 1. *A* dataset *is a finite collection of pairs* $(x, p) \in \mathbf{R}^S_{++} \times \mathbf{R}^S_{++}$.

So a data set is a finite collection $(x^k, p^k)_{k=1}^K$, for some K, where for each k, $x^k \in \mathbf{R}^S_{++}$ is a monetary act, and $p^k \in \mathbf{R}^S_{++}$ is a price vector. The interpretation of a dataset $(x^k, p^k)_{k=1}^K$ is that it describes K purchases of state-contingent payoffs, at some given vector of prices.

In the sequel, we maintain the following assumption:

$$x^k_s \neq x^{k'}_{s'} \text{ if } (k, s) \neq (k', s').$$

Meaning that all observed payoffs are different, a kind of genericity assumption. The purpose of the assumption is to simplify the analysis: it allows us to use smooth functions to rationalize a dataset. The essence of our results are, however, true when payoffs can be equal: see Section 6.

We proceed to discuss the two theoretical models we focus on in the paper: subjective expected utility, and quasi-linear subjective expected utility.

2.1 Subjective Expected Utility

A subjective expected utility (SEU) model is specified by a prior $\mu \in \Delta^S_{++}$ and a utility function over money $u : \mathbf{R}_+ \to \mathbf{R}$.

An SEU maximizing agent solves the problem

$$\max_{x \in B(p,I)} \sum_{s \in S} \mu_s u(x_s) \tag{1}$$

when faced with prices $p \in \mathbf{R}^S_+$ and income $I > 0$. The set $B(p, I) = \{y \in \mathbf{R}^S_+ : p \cdot y \leq I\}$ is the budget set defined by p and I.

A dataset $(x^k, p^k)_{k=1}^K$ is our notion of observable behavior. The content of SEU, or the meaning of SEU as an assumption, is the behaviors that are *as if* they were generated by an SEU maximizing agent. We call such behaviors *SEU rational.*

Definition 2. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* subjective expected utility rational *(SEU rational) if there is* $\mu \in \Delta^S_{++}$ *and a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k*,*

$$y \in B(p^k, p^k \cdot x^k) \Rightarrow \sum_{s\in S} \mu_s u(y_s) \leq \sum_{s\in S} \mu_s u(x^k_s).$$

A few remarks are in order. Firstly, we restrict attention to concave utility, and our results will have nothing to say about the non-concave case (other than showing in Section 4.1 that concavity has testable implications). In second place, we assume that the relevant budget for the kth observation is $B(p^k, p^k \cdot x^k)$. Implicit is the assumption that $p^k \cdot x^k$ is the relevant income for this problem. This assumption is somewhat unavoidable, and standard procedure in revealed preference theory.

2.2 Quasi linear Subjective Expected Utility

The SEU model is the main focus of our work, but we find it useful to discuss a related model, that of quasilinear subjective expected utility.

A quasilinear subjective expected utility (QL-SEU) model is specified by a prior $\mu \in \Delta^S_{++}$ and a utility function $u : \mathbf{R}_+ \to \mathbf{R}$. A QL-SEU maximizing agent solves the problem

$$\max_{x\in B(p,I-m)} \sum_{s\in S} \mu_s u(x_s) + m \tag{2}$$

when faced with prices $p \in \mathbf{R}^S_+$ and income $I > 0$.

The interpretation of QL-SEU is that there are two stages and that the utility over money consumed in the first stage is linear. In the first stage (say, time 0) money may be consumed or used to purchase uncertain state-contingent assets. In the second state, a state occurs and a state-contingent payment is realized. The payments are evaluated as of time zero, when utility over money is linear. The utility over uncertain future payoffs has the SEU form.

As with SEU, we seek to describe the datasets that could have been generated by a QL-SEU agent with concave utility.

Definition 3. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* quasilinear subjective expected utility rational *(QL-SEU rational) if there is* $\mu \in \Delta_{++}^S$ *and a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k*,*

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{s\in S} \mu_s u(y_s) - p^k \cdot y \leq \sum_{s\in S} \mu_s u(x_s^k) - p^k \cdot x^k.$$

3 Results

Our results characterize the datasets that are SEU and QL-SEU rational. We start by discussing QL-SEU rationality because the analysis is similar to, but simpler than, the analysis of SEU rationality.

3.1 QL-SEU rationality

Inspection of problem (2) reveals that a QL-SEU maximizing agent solves the problem

$$\max_{x\in\mathbf{R}_+^S} \sum_{s\in S} \mu_s u(x_s) - p \cdot x$$

when faced with prices p.

Suppose that the function u is continuously differentiable, an assumption that turns out to be without loss of generality. Then the first-order condition for the agent's maximization problem is

$$\mu_s u'(x_s) = p_s.$$

So if a dataset $(x^k, p^k)_{k=1}^K$ is QL-SEU rational, the prior μ and utility u must satisfy the above first order condition for each x_s^k and p_s^k; that is: $\mu_s u'(x_s^k) = p_s^k$.

The first-order condition has an immediate implication for consumption at a given state. If $x_s^k > x_s^{k'}$ then the concavity of u implies that we must have $p_s^k \geq p_s^{k'}$. This implication amounts to saying that "state s demand must slope down."

We cannot draw a similar conclusion when comparing $x_s^k > x_{s'}^{k'}$ with $s \neq s'$ because the effect of the different priors μ_s and $\mu_{s'}$ may interfere with the effect of prices on demand. From the first-order conditions:

$$\frac{u'(x_{s'}^{k'})}{u'(x_s^k)} = \frac{\mu_s}{\mu_{s'}} \frac{p_{s'}^{k'}}{p_s^k}.$$

Now, the concavity of u and $x_s^k > x_{s'}^{k'}$ implies that

$$\frac{\mu_s}{\mu_{s'}} \frac{p_{s'}^{k'}}{p_s^k} \leq 1,$$

but the priors μ are unobservable, so we cannot conclude anything about the observable prices p_s^k and $p_{s'}^{k'}$.

There is, however, one further implication of QL-SEU and the concavity of u. We can consider a sequence of pairs $(x_s^k, x_{s'}^{k'})$, chosen such that when we multiply first-order conditions, all the priors cancel out. For example, consider

$$x_{s_1}^{k_1} > x_{s_2}^{k_2} \text{ and } x_{s_2}^{k_3} > x_{s_1}^{k_4}.$$

By multiplying the first-order conditions we obtain that:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_2})} \cdot \frac{u'(x_{s_2}^{k_3})}{u'(x_{s_1}^{k_4})} = \left(\frac{\mu_{s_2}}{\mu_{s_1}} \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}}\right) \cdot \left(\frac{\mu_{s_1}}{\mu_{s_2}} \frac{p_{s_2}^{k_3}}{p_{s_1}^{k_4}}\right) = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_1}^{k_4}}$$

Notice that the pairs $(x_{s_1}^{k_1}, x_{s_2}^{k_2})$ and $(x_{s_2}^{k_3}, x_{s_1}^{k_4})$ have been chosen so that the priors μ_{s_1} and μ_{s_2} would cancel out. Now the concavity of u and the assumption that $x_{s_1}^{k_1} > x_{s_2}^{k_2}$ and $x_{s_2}^{k_3} > x_{s_3}^{k_4}$ imply that $u'(x_{s_1}^{k_1}) \leq u'(x_{s_2}^{k_2})$ and $u'(x_{s_2}^{k_3}) \leq u'(x_{s_1}^{k_4})$. So the product of the prices $\frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_1}^{k_4}}$ cannot exceed 1. Thus, we obtain an implication of QL-SEU for prices, an observable entity.

In general, the assumption of QL-SEU rationality will require that, for any collection of sequences like the one above (appropriately chosen so that priors will cancel out) the product of the ratio of prices cannot exceed 1. Formally,

Strong Axiom of Revealed Quasilinear Subjective Utility (SARQSEU): *For any sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ in which*

1. $x_{s_i}^{k_i} \geq x_{s_i'}^{k_i'}$ *for all i;*
2. *each s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right):*

The product of prices satisfies that

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

Condition (2) in SARQSEU is responsible for the canceling out of priors when we multiply ratios of marginal utilities, as in the previous example. It is therefore easy to see that SARQSEU is necessary for QL-SEU rationality.

Our first result is that SARQSEU is sufficient as well as necessary. SARQSEU seems weak in the following sense. *The axiom imposes a particular behavior under circumstances in which priors do not matter–when the priors cancel out as above.* It tells us to focus on circumstances in which the priors can be ignored, and only constrains behavior in such circumstances.

Theorem 1. *A dataset is QL-SEU rational if and only if it satisfies SARQSEU.*

The proof is in Section 7.

3.2 SEU rationality

We now discuss the axiom for SEU rationality. We first use differentiability and first-order conditions to derive the axiom, as we did for QL-SEU. Recall the SEU maximization problem (1):

$$\mu_s u'(x_s) = \lambda p_s.$$

In this case, the first order conditions contain three unobservables: priors μ_s, marginal utilities $u'(x_s)$ and Lagrange multipliers λ.

The first-order conditions imply that SEU rationality requires:

$$\frac{u'(x_{s'}^{k'})}{u'(x_s^k)} = \frac{\mu_s}{\mu_{s'}} \frac{\lambda^{k'}}{\lambda^k} \frac{p_{s'}^{k'}}{p_s^k}.$$

Note that the concavity of u implies something about the left-hand side of this equation when $x_{s'}^{k'} \geq x_s^k$, but the right-hand side is complicated by the presence of unobservable Lagrange multipliers and priors.

We can repeat the idea used for QL-SEU rationality, but the sequences must be chosen so that not only priors but also Lagrange multipliers cancel out. For example, consider

$$x_{s_1}^{k_1} > x_{s_2}^{k_2}, x_{s_2}^{k_3} > x_{s_3}^{k_1}, \text{ and } x_{s_3}^{k_2} > x_{s_1}^{k_3}.$$

By manipulating the first-order conditions we obtain that:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_2})} \cdot \frac{u'(x_{s_2}^{k_3})}{u'(x_{s_3}^{k_1})} \cdot \frac{u'(x_{s_3}^{k_2})}{u'(x_{s_1}^{k_3})} = \left(\frac{\mu_{s_2}}{\mu_{s_1}} \frac{\lambda^{k_1}}{\lambda^{k_2}} \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}}\right) \cdot \left(\frac{\mu_{s_3}}{\mu_{s_2}} \frac{\lambda^{k_3}}{\lambda^{k_1}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}}\right) \cdot \left(\frac{\mu_{s_1}}{\mu_{s_3}} \frac{\lambda^{k_2}}{\lambda^{k_3}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}}\right) = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}}$$

Notice that the pairs $(x^{k_1}_{s_1}, x^{k_2}_{s_2})$, $(x^{k_3}_{s_2}, x^{k_1}_{s_3})$, and $(x^{k_2}_{s_3}, x^{k_3}_{s_1})$ have been chosen so that the priors μ_{s_1}, μ_2, and μ_{s_3} and the Lagrange multipliers $\lambda^{k_1}, \lambda^{k_2}$, and λ^{k_3} would cancel out. Now the concavity of u and the assumption that $x^{k_1}_{s_1} > x^{k_2}_{s_2}$, $x^{k_3}_{s_2} > x^{k_1}_{s_3}$, and $x^{k_2}_{s_3} > x^{k_3}_{s_1}$ imply that the product of the prices $\frac{p^{k_1}_{s_1}}{p^{k_2}_{s_2}} \frac{p^{k_3}_{s_2}}{p^{k_1}_{s_3}} \frac{p^{k_2}_{s_3}}{p^{k_3}_{s_1}}$ cannot exceed 1. Thus, we obtain an implication of SEU on prices, an observable entity.

In general, the assumption of SEU rationality will require that, for any collection of sequences as above, appropriately chosen so that priors and Lagrange multipliers will cancel out, the product of the ratio of prices cannot exceed 1. Formally:

Strong Axiom of Revealed Subjective Utility (SARSEU): *For any sequence of pairs* $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ *in which*

1. $x^{k_i}_{s_i} \geq x^{k'_i}_{s'_i}$ *for all* i*;*
2. *each* s *appears as* s_i *(on the left of the pair) the same number of times it appears as* s'_i *(on the right);*
3. *each* k *appears as* k_i *(on the left of the pair) the same number of times it appears as* k'_i *(on the right):*

The product of prices satisfies that

$$\prod_{i=1}^{n} \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} \leq 1.$$

SARSEU is different from SARQSEU only in the third requirement of the sequence. The main finding of our paper is that this necessary condition is sufficient as well.

Theorem 2. *A dataset is SEU rational if and only if it satisfies SARSEU.*

We conclude the section with some remarks on Theorem 2.

Remark 1. There is a sense in which SARSEU is a weak constraint on a dataset. Consider the derivation of the axiom using first-order conditions above. The observable implications of SEU are obtained only after canceling out the effects of the unobservable prior μ and multiplier λ. After these are canceled out, one has an implication due to the concavity of utility: a generalization of the downward-sloping demand property. The derivation of the axiom exploits rather basic properties of the SEU functional form. It does not use the form of the utility, other than its concavity; nor does it use the values of the priors and multipliers. SARSEU is what concave utility implies for prices, in those situations in which the unobservable priors and multipliers cancel out. It may be surprising that such basic implications resume all the implications of the model.

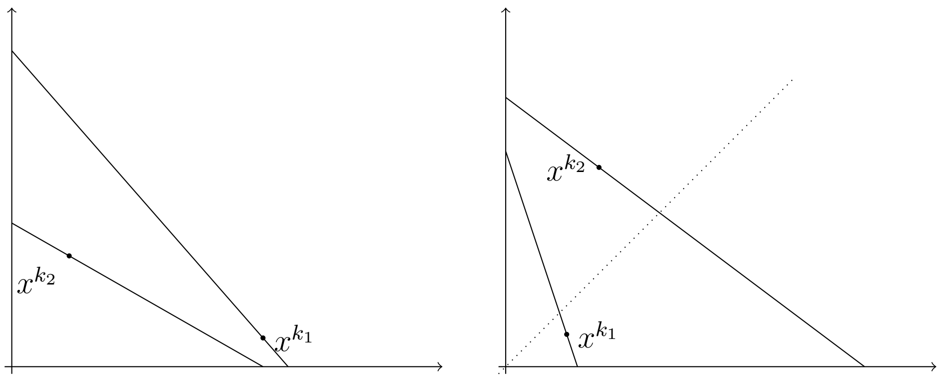
Remark 2. The proof of Theorem 2 is in Section 8. It relies on setting up a system of linear inequalities from the first-order conditions of an SEU agent's maximization problem. This is similar to the approach in Afriat (1967), and in many other subsequent studies of revealed preference. The difference is that our system is nonlinear, and must be linearized. A crucial step in the proof is an approximation result, which is complicated by the fact that the unknown prior, Lagrange multipliers, and marginal utilities, all take values in non-compact sets.

Remark 3. We have assumed that $x_s^k \neq x_{s'}^{k'}$ if $(k,s) \neq (k',s')$, so that all payoffs are different. This assumption is not important to prove the sufficiency direction in Theorem 2. What the assumption does is to make the existence of a smooth rationalization be without loss of generality; see Lemma 8. Without the assumption, SARSEU implies the existence of a SEU rationalization, but it may not be smooth. On the other hand, a non-smooth SEU model may violate SARSEU when some payoffs are equal. The details are in Section 6. Another assumption we have made is that payoffs are strictly positive, but that is just for simplification, and nothing depends on payoffs being always strictly positive.

Remark 4. In any behavioral characterization, it is sensible to ask for the uniqueness properties of the representation. There is no hope for uniqueness with a finite dataset, but we demonstrate in the online appendix that if one chooses appropriately an increasing sequence of datasets, then uniqueness of SEU obtains in the limit. See the online appendix for the details.

3.3 The 2×2 case

We illustrate our analysis with a geometrical description of the 2×2 case, the case when there are two states and two observations. The 2×2 case is interesting because it has only two possible kinds of violations of SARSEU. For each possible violation there is a simple geometric argument showing why the dataset is incompatible with SEU.



(a) $(x^{k_1}_{s_1}, x^{k_2}_{s_1})$, $(x^{k_2}_{s_2}, x^{k_1}_{s_2})$ (b) $(x^{k_1}_{s_1}, x^{k_1}_{s_2})$, $(x^{k_2}_{s_2}, x^{k_2}_{s_1})$

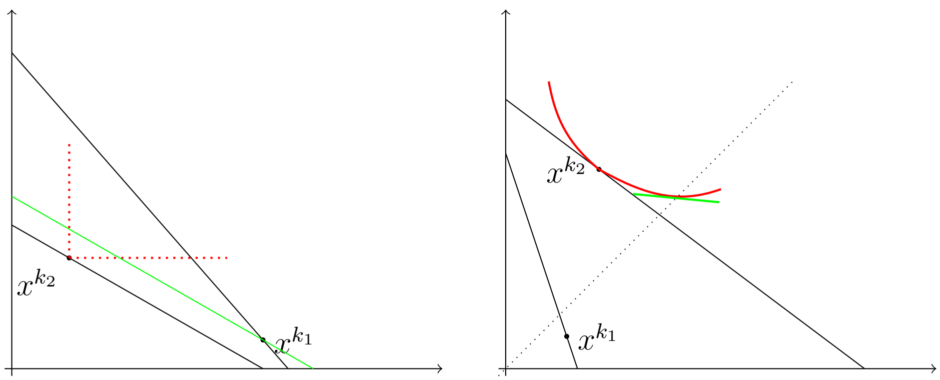
Figure 1: The 2×2 case.

Let $(x^{k_1}, p^{k_1}), (x^{k_2}, p^{k_2})$ be a dataset with $K = 2$ and $S = 2$. It is easy to see that (up to the labeling of k and s), there are just two sequences in the situation described by SARSEU: $(x^{k_1}_{s_1}, x^{k_1}_{s_2})$, $(x^{k_2}_{s_2}, x^{k_2}_{s_1})$, and $(x^{k_1}_{s_1}, x^{k_2}_{s_1})$, $(x^{k_2}_{s_2}, x^{k_1}_{s_2})$.

There are therefore only two possible kinds of violations of SEU. They are depicted in Figure 1. Interestingly, the violations illustrate how SARSEU is related to downward sloping demand: The situation depicted in either figure is suspect because consumption moves in the opposite direction to prices. In Figure 1a, when we compare x^{k_1} to x^{k_2}, we have more consumption in state s_1 in x^{k_1} even though the relative price of s_1 is higher than when x^{k_2} is purchased. Similarly, in Figure 1a the two bundles are on opposite sides of the 45 degree line, so that there is more consumption in s_1 in x^{k_1}, and more consumption in $s_2 2$ in x^{k_2}; however, the relative price of s_1 is higher in p^{k_1} than in p^{k_2}.

Figure 2 explains what goes wrong in each case. First, the demand function of a risk averse SEU agent is well-known to be normal. In Figure 2a we depict the choice of an agent that has higher income than when x^{k_2} was chosen, but faces the same prices p^{k_2}. Since her demand is normal, the agent's choice on the larger (green) budget line must be larger than x^{k_2}. It must lie in the line segment on the green budget line that has larger vectors than x^{k_2}. But such a choice would violate the weak axiom of revealed preference. Hence the (counterfactual) choice implied by SEU would be inconsistent with utility maximization.

Secondly, consider the situation in Figure 2b. We have drawn the indifference curve of the agent when choosing x^{k_2}. At the point at which the indifference curve crosses the



(a) $(x^{k_1}_{s_1}, x^{k_2}_{s_1})$, $(x^{k_2}_{s_2}, x^{k_1}_{s_2})$ (b) $(x^{k_1}_{s_1}, x^{k_1}_{s_2})$, $(x^{k_2}_{s_2}, x^{k_2}_{s_1})$

Figure 2: The 2×2 case.

dotted line, the 45 degree line, one can read the agent's prior off the indifference curve. Indeed, because the choices on the 45 degree line involve no risk, a tangent line to the agent's indifference curve must correspond to (be normal to) the agent's prior. It is then clear that this tangent line (depicted in green in the figure) must be *flatter* than the budget line at which x^{k_2} was chosen. On the other hand, the same reasoning reveals that the prior must define a *steeper* line than the budget line at which x^{k_1} was chosen. This is a contradiction, as the latter budget line is steeper than the former.

Incidentally, observe that the datasets represented in these figures satisfy the weak axiom of revealed preference. So they involve datasets that are not SEU rational, but can be rationalized by some utility function which does not have the SEU form. In other words, the theory of SEU is testable beyond the basic hypothesis of consumer rationality. In fact, SARSEU is a strictly stronger axiom than the weak axiom of revealed preference: it is easy to see that a violation of the weak axiom would exhibit a situation like the one in Figure 2a. In the online appendix we provide a direct proof that SARSEU implies the weak axiom.

As we explain in Section 5.1, the second configuration (Figure 1b) generalizes the axiom used in Epstein (2000) as a test of probabilistic sophistication.

4 Applications of SARSEU

We illustrate the use of SARSEU through a few simple theoretical exercises. In each case, the exercise is to present a well-known generalization of SEU, and to show that data generated by these general theories can violate SARSEU.

We discuss, in turn, non-concave SEU, max-min preferences, and state-dependent expected utility. Each of these models is more general than SEU, but the added generality might not be detectable empirically. By showing that these models can generate datasets which violate SARSEU, we show that the models are in fact testable beyond SEU. In other words, that non-concave SEU, state-dependent expected utility, and max-min preferences all have testable implications over and beyond those of SEU.

4.1 Concavity is testable

The concavity of u plays an important role in our characterization. This should not be surprising, as risk aversion has obvious economic meaning and content. There are, however, instances in revealed preference theory where concavity has no implications for a rational consumer. Afriat's theorem (Afriat (1967)) shows that concavity is not a testable property of a utility function. For the separable SEU model, concavity of u is equivalent to the convexity of preferences over state-contingent bundles. So it is legitimate to ask about the testability of the concavity of u. In this section we show that indeed concavity is testable.

In the following, we will show an example of data generated from a non-concave SEU model that violates SARSEU.

Consider the following dataset:

$$p^{k_1} = (1, 2), x^{k_1} = (1, 2) \text{ and } p^{k_2} = (1.1, 2), x^{k_2} = (10, 1).$$

Note that

$$x_{s_2}^{k_1} > x_{s_2}^{k_2} \text{ and } x_{s_1}^{k_2} > x_{s_1}^{k_1},$$

while

$$\frac{p_{s_2}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_1}^{k_2}}{x_{s_1}^{k_1}} = \frac{2}{2} \frac{1.1}{1} = 1.1 > 1,$$

so SARSEU is violated, and the data is not rationalizable by any concave utility and priors.

It is, however, rationalizable by the following non-concave SEU. Let $\mu = \left(\frac{1}{3}, \frac{2}{3}\right)$. Define

$$v(x) = \begin{cases} 1 & \text{if } x \leq 9 \\ 2 & \text{if } 9 < x \leq 10 \\ 1 & \text{if } x > 10. \end{cases}$$

Let $u(x) = \int_0^x v(s)ds$. Let $B^k = \{x : \mathbf{R}^2_+ : p^k \cdot x \leq p^k \cdot x^k\}$.

It is clear that x^1 is optimal for $\sum \mu_s u(x_s)$ in B^1, as $v(x_{s_1}) = v(x_{s_2}) = 1$ for all $(x_{s_1}, x_{s_2}) \in B^1$. Consider B^2. By monotonicity of u, any maximum of $\sum \mu_s u(x_s)$ in B^1 must lie on the budget line $p^{k_2}_{s_1} x_{s_1} + p^{k_2}_{s_2} x_{s_2} = 13$. Note that, on the budget line,

$$x_{s_2} = \frac{13 - 1.1 x_{s_1}}{2},$$

so $x_{s_2} \leq \frac{13}{2} < 9$ for $x_{s_1} \geq 0$. For all $x_{s_1} \geq 0$, define $f(x_{s_1}) = \mu_1 u(x_{s_1}) + \mu_2 u(x_{s_2}) = \frac{1}{3}\left[u(x_{s_1}) + 2u(\frac{13-1.1x_{s_1}}{2})\right]$. Then, $f'(x_{s_1}) = \frac{1}{3}\left[v(x_{s_1}) - 1.1\right]$ for $x_{s_1} \in [0, 13/1.1]$, as $v(\frac{13-1.1x_{s_1}}{2}) = 1$. Thus,

$$f'(x_{s_1}) = \begin{cases} \frac{-0.1}{3} & \text{if } x \leq 9 \\ \frac{0.9}{3} & \text{if } 9 < x \leq 10 \\ \frac{-0.1}{3} & \text{if } 10 < x \end{cases}$$

So $f(x_{s_1})$ has two local maxima, $x_{s_1} = 0$ and $x_{s_1} = 10$. By direct calculation, $f(0) = \frac{13}{6} = 2 + \frac{1}{6}$ and $f(10) = \frac{1}{3}(9+2) + \frac{2}{3}(\frac{13-1.1\times 10}{2}) = 3 + \frac{4}{3}$. Since $f(10) > f(0)$, it is indeed optimal to choose x^2 in B^2.

4.2 Maxmin SEU

We proceed to show that the max-min model, a generalization of SEU that allows an agent to have multiple priors, has testable implications beyond the SEU model. The conclusion drawn is reminiscent of the results of Dow and da Costa Werlang (1992), but the derivation using SARSEU is of course novel, and the framework is different from the one in Dow and Verlang.

The maxmin SEU model, first axiomatized by Gilboa and Schmeidler (1989), posits that an agent maximizes

$$\min_{\mu \in M} \sum_{s \in S} \mu_s u(x_s),$$

where M is a convex set of priors.

Assume $S = \{s_1, s_2\}$. Consider the following consumption dataset:

	k_1	k_2	k_3	k_4
s_1	6	5	8	3
s_2	7	4	1	2.

The table has x_s^k in entry (s,k). We present a maxmin model, that is a set M of priors and a utility u, such that the above consumptions are chosen for certain prices—the prices are defined below so that the relevant first-order condition hold.

Let the set of priors be the convex hull of $\underline{\mu} = (1-q, q)$ and $\bar{\mu} = (q, 1-q)$ with $q \in (1/2, 1)$. Denote this set of priors by M. Note that M is symmetric.

Let $v(x) = \alpha - \beta x$ for $x \in [1/10, 10]$; define it in an arbitrary fashion outside of that interval, as long as it is strictly positive and decreasing. Then $u(x) = \int_0^x v(t)dt$ is a strictly monotone increasing and concave function.

Note that, since u is strictly monotone increasing,

$$\min_{\mu \in M} \sum_{j=1,2} \mu_s u(x_{s_j}^{k_i}) = (1-q)u(x_{s_1}^{k_i}) + qu(x_{s_2}^{k_i})$$

for $i = 2, 3, 4$ and

$$\min_{\mu \in M} \sum_{j=1,2} \mu_s u(x_{s_j}^{k_i}) = qu(x_{s_1}^{k_1}) + (1-q)u(x_{s_2}^{k_1}).$$

Note that the sequence

$$(x_{s_1}^{k_3}, x_{s_2}^{k_1}), (x_{s_1}^{k_1}, x_{s_1}^{k_2}), (x_{s_2}^{k_2}, x_{s_1}^{k_4}), (x_{s_2}^{k_4}, x_{s_2}^{k_3})$$

satisfies properties (1), (2) and (3) in SARSEU.

Now let $p_{s_1}^{k_i} = (1-q)v(x_{s_1}^{k_i})$ and $p_{s_2}^{k_i} = qv(x_{s_2}^{k_i})$ for $i = 2, 3, 4$ and $s = 1, 2$. Let $p_{s_1}^{k_1} = qv(x_{s_1}^{k_1})$ and $p_{s_2}^{k_1} = (1-q)v(x_{s_2}^{k_1})$. Then the max-min utility defined by u and M satisfies the FOCs at the specified prices p^k and quantities x^k.

We have that

$$\frac{p_{s_1}^{k_3}}{p_{s_2}^{k_1}} \frac{p_{s_1}^{k_1}}{p_{s_1}^{k_2}} \frac{p_{s_2}^{k_2}}{p_{s_1}^{k_4}} \frac{p_{s_2}^{k_4}}{p_{s_2}^{k_3}} = \frac{\alpha - \beta x_{s_1}^{k_3}}{\alpha - \beta x_{s_2}^{k_1}} \frac{\alpha - \beta x_{s_1}^{k_1}}{\alpha - \beta x_{s_1}^{k_2}} \frac{\alpha - \beta x_{s_2}^{k_2}}{\alpha - \beta x_{s_1}^{k_4}} \frac{\alpha - \beta x_{s_2}^{k_4}}{\alpha - \beta x_{s_2}^{k_3}} \left(\frac{q}{1-q}\right)\left(\frac{q}{1-q}\right)\left(\frac{1-q}{q}\right)\left(\frac{q}{1-q}\right).$$

Note that

$$\left(\frac{q}{1-q}\right)\left(\frac{q}{1-q}\right)\left(\frac{1-q}{q}\right)\left(\frac{q}{1-q}\right) = \left(\frac{q}{1-q}\right)^2 > 1.$$

So, by choosing α large enough we obtain that

$$\frac{p^{k_3}_{s_1}}{p^{k_1}_{s_2}}\frac{p^{k_1}_{s_1}}{p^{k_2}_{s_1}}\frac{p^{k_2}_{s_2}}{p^{k_4}_{s_1}}\frac{p^{k_4}_{s_2}}{p^{k_3}_{s_2}} > 1,$$

so SARSEU is violated.

4.3 State Dependent SEU

State dependent SEU is the model in which an agent seeks to maximize $\sum_{s\in S}\mu_s u_s(x_s)$; where u_s is a utility function over money for each state s. The state dependent SEU is characterized by Varian (1983a) (by means of Afriat inequalities).

It is easy to generate a state dependent model that violates SARSEU because we may have $u'_s(x_s) > u'_{s'}(x_{s'})$ even when $x_s > x_{s'}$.

Assume $S = \{s_1, s_2\}$. Consider the following dataset:

$$p^{k_1} = (3,2),\ p^{k_2} = (1,1) \text{ and } x^{k_1} = (2,1),\ x^{k_2} = (3,4).$$

Define $\mu = \left(\frac{1}{2}, \frac{1}{2}\right)$. Choose strictly concave functions u_{s_1} and u_{s_2} such that

$$u'_{s_1}(2) = 3 > 1 = u'_{s_1}(3) \text{ and } u'_{s_2}(1) = 2 > 1 = u'_{s_2}(4).$$

Then

$$\frac{\mu_{s_1}u'_{s_1}(2)}{\mu_{s_2}u'_{s_2}(1)} = \frac{p^k_{s_1}}{p^k_{s_2}}, \quad \frac{\mu_{s_1}u'_{s_1}(3)}{\mu_{s_2}u'_{s_2}(4)} = \frac{p^{k2}_{s_1}}{p^{k_2}_{s_2}},$$

so that the first-order conditions are satisfied.

The sequence $\{(x^{k_1}_{s_1}, x^{k_1}_{s_2}), (x^{k_2}_{s_2}, x^{k_2}_{s_1})\}$ satisfies the condition of the axiom. However,

$$\frac{p^{k_1}_{s_1}}{p^{k_1}_{s_2}}\frac{p^{k_2}_{s_2}}{p^{k_2}_{s_1}} = \frac{3}{2} > 1.$$

This is a violation of SARSEU.

5 Related Theories

We now turn to the discussion of three related theoretical developments. Firstly the work of Epstein on datasets that are compatible with probabilistic sophistication. Secondly, the model of objective expected utility, in which priors are assumed to be observable. Thirdly, Savage's celebrated axiomatization of SEU.

5.1 Relationship with Epstein (2000)

As mentioned in the introduction, Epstein (2000) studies the implications of probabilistic sophistication for consumption datasets. His setup is the same as ours, but he focuses on probabilistic sophistication instead of SEU.

Epstein presents a necessary condition for rationalizability by a probabilistically sophisticated agent: A dataset is not consistent with probabilistic sophistication if there exist $s, t \in S, k, \hat{k} \in K$ such that

$$\begin{bmatrix} \text{(i) } p_s^k \geq p_t^k \text{ \& } p_s^{\hat{k}} \leq p_t^{\hat{k}} \text{ with at least one strict inequality} \\ \text{(ii) } x_s^k > x_t^k \text{ \& } x_s^{\hat{k}} < x_t^{\hat{k}} \end{bmatrix}$$

In other words, a necessary condition for probability sophistication is that conditions (i) and (ii) are incompatible.

Of course, an SEU rational agent is probabilistically sophisticated. Indeed, our next result establishes that a violation of Epstein's condition would imply a violation of SARSEU.

Proposition 3. *If a dataset $(x^k, p^k)_{k=1}^K$ satisfies SARSEU, then (i) and (ii) cannot both hold for some $s, t \in S, k, \hat{k} \in K$*

Proof. Suppose that $s, t \in S, k, \hat{k} \in K$ are such that (ii) holds. Then $\{(x(k,s), x(k,t)),\ (x(\hat{k},t), x(\hat{k},s))\}$ satisfies the conditions in SARSEU. Hence, SARSEU requires that $\frac{p(k,s)}{p(k,t)} \frac{p(\hat{k},t)}{p(\hat{k},s)} \leq 1$, so that $p(k,s) \leq p(k,t)$ or $p(\hat{k},s) \geq p(\hat{k},t)$. Hence, (i) is violated. □

It is easy to see Proposition 3 graphically, because it is essentially a 2×2 exercise. A violation of Epstein's condition is a configuration like the one in Figure 1b of Section 3.3.

5.2 Objective Expected Utility Rationality

In this section, we present the relationship between Theorem 2 and results in Green and Srivastava (1986), Varian (1983b), and Kubler et al. (2014). As mentioned in the introduction, these authors discuss a setting where an objective probability μ is given. Their notion of a dataset is the same as in our paper, and they seek to understand when there is a utility function for which the observed purchases maximize expected utility.

We show that we can write a version of our SARSEU that uses "risk neutral" prices in place of regular prices. We show that this modified axiom characterizes the objective expected utility theory. Our modified SARSEU is therefore equivalent to the conditions studied by Green and Srivastava (1986) and Varian (1983b), and to the axiom in Kubler et al. (2014).

It is worth emphasizing that Kubler et al. (2014) allows μ to depend on k, so that the agent may use a different prior when faced with different optimization problems. In our subjective probability setup this would make no sense because everything is rationalizable by suitably choosing priors in each optimization problem. Here we are being consistent with the rest of the paper in assuming a fixed prior through all observations, but the result can be relaxed to fit a variable-prior setup.

Definition 4. *A dataset $(x^k, p^k)_{k=1}^K$ is* objective expected utility rational *(OEU rational) if there is a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$ such that, for all k,*

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{s\in S} \mu_s u(y_s) \leq \sum_{s\in S} \mu_s u(x^k_s).$$

In the papers cited above, a crucial aspect of the data are the price-probability ratios, or "risk neutral prices," defined as follows: for $k \in K$ and $s \in S$

$$\rho^k_s = \frac{p^k_s}{\mu_s}.$$

A natural modification of SARSEU using the objective probability μ is as follows:

Strong Axiom of Revealed Objective Expected Utility (SAROEU): *For any sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^n$ in which*

1. *$x^{k_i}_{s_i} \geq x^{k'_i}_{s'_i}$ for all i;*
2. *each k appears in k_i (on the left of the pair) the same number of times it appears in k'_i (on the right):*

The product of price-probability ratios satisfies that

$$\prod_{i=1}^n \frac{\rho^{k_i}_{s_i}}{\rho^{k'_i}_{s'_i}} \leq 1.$$

The prior μ is observable, so we do not need the requirement on s in SARSEU. Instead, SAROEU restricts the products of price-probability ratios, and not the product of price ratios.

Kubler et al. (2014) investigate the case of strict concave utility, while we have focused on weak concavity. A modification of Kubler et. al's axiom that allows for weak concavity is as follows:

Strong Axiom of Revealed Expected Utility (SAREU): *For all* $m \geq 1$ *and sequences* $k(1), \ldots, k(m) \in K$*,*

$$\prod_{i=1}^{m} \Big(\max_{s,s' : x_s^{k(i)} \geq x_{s'}^{k(i+1)}} \frac{\rho_s^{k(i)}}{\rho_{s'}^{k(i+1)}} \Big) \leq 1.$$

It is easy to modify the argument in Kubler et al. (2014) to show the equivalence of a dataset being OEU-rational, satisfying the conditions in Green and Srivastava (1986) and Varian (1983b).

Proposition 4. *A dataset is OEU-rational if and only if it satisfies SAROEU.*

This result implies that SAROEU, SAREU, and the conditions in Green and Srivastava (1986) and Varian (1983b) are equivalent. A proof can be found in the online appendix.

5.3 Relationship with Savage's axioms

In this section, we study the relationship between SARSEU and Savage's axiom. Recall that Savage's primitive is a complete preference relation over acts. In contrast, our primitive is a data set $(x^k, p^k)_{k=1}^K$. To relate the two models, we define a revealed preference relation from the dataset $(x^k, p^k)_{k=1}^K$ and investigate when it satisfies Savage's axioms.

Definition 5. *For any* $x, y \in \mathbf{R}^S$*,*
(i) $x \succeq y$ *if there exists* $k \in K$ *such that* $x = x^k$ *and* $p^k \cdot x \geq p^k \cdot y$*;*
(ii)$x \succ y$ *if there exists* $k \in K$ *such that* $x = x^k$ *and* $p^k \cdot x > p^k \cdot y$*.*

There is one basic problem: Savage's primitive is a complete preference relation over acts, but a dataset will contain much less information than a preference relation over $\mathbf{R}_+^S$. The revealed preference relation is going to be incomplete: many acts in $\mathbf{R}_+^S$ will not be comparable. Such incompleteness gives rise to trivial violations of Savage's axioms, as his axioms were formulated for complete preferences. For example, one of Savage's axiom is as follows:

Axiom (P2). *Let $x, y, x', y' \in \mathbf{R}^S_+$ and $A \subset S$ such that $x_A = x'_A$ and $y_A = y'_A$ and $x_{A^c} = y_{A^c}$ and $x'_{A^c} = y'_{A^c}$. Then $x \succeq y$ if and only if $x' \succeq y'$.*

The revealed preference relation violates P2 when only one of x, y and x', y' are comparable. This is not a particularly interesting violation of Savage's axioms. Hence, we compare SARSEU and weaker versions of Savage axioms that presuppose that some acts are comparable. In other words, we consider violations of Savage's axioms that are not due to the incompleteness of the revealed preference relation.

Definition 6. *For any $x, y \in \mathbf{R}^S_+$, x, y are comparable if $x \succeq y$ or $y \succeq x$.*

Axiom (P2'). *Let $x, y, x', y' \in \mathbf{R}^S_+$ and $A \subset S$ such that $x_A = x'_A$ and $y_A = y'_A$; $x_{A^c} = y_{A^c}$ and $x'_{A^c} = y'_{A^c}$; and x, y and x', y' are comparable. Then $x \succeq y$ if and only if $x' \succeq y'$.*

The only difference between P2 and P2' is the condition that x, y and x', y' be comparable. So any violation of P2' cannot be due to the incompleteness of $\succeq$. (In Savage framework, P2 and P2' are of course equivalent because Savage assumes completeness.)

In the following, we show that SARSEU implies Savage's axioms, except for P1 and P6: P1 requires a preference to be a weak order, which does not make sense for our primitive. P6 requires the set of states to be infinite.

We shall use the following notation. When $A \subset S$ and $x \in \mathbf{R}^S_+$, then x_A denote the vector in $\mathbf{R}^A_+$ obtained by restricting $s \mapsto x_s$ to A. We use 1_A to denote the indicator vector for $A \subset S$ in $\mathbf{R}^S_+$; and for a scalar $x \in \mathbf{R}_+$, $x1_A$ denotes the vector in $\mathbf{R}^A_+$ with x in all its entries (the constant vector x).

Axiom (P4'). *Suppose $A, B \subset S$; $x > y$, $x' > y'$; $(x1_A, y1_{A^c})$, $(x1_B, y1_{B^c})$ and $(x1_A, y1_{A^c})$, $(x1_B, y1_{B^c})$ are comparable. Then, $(x1_A, y1_{A^c}) \succ (x1_B, y1_{B^c})$ if and only if $(x'1_A, y'1_{A^c}) \succ (x'1_B, y'1_{B^c})$.*

To show that SARSEU implies P4', we show that SARSEU implies the following stronger axiom:

Axiom (P4*'). *Suppose $A, B \subset S$ and $A \cap B \neq \emptyset$; $x > y$, $x' > y'$, $z, z' \in \mathbf{R}^{A^c \cup B^c}$; $((x)_A, (y)_B, z), ((x)_B, (y)_A, z)$ and $((x')_A, (y')_B, z'), ((x')_B, (y')_A, z')$ are comparable. Then, $((x)_A, (y)_B, z) \succ ((x)_B, (y)_A, z)$ if and only if $((x')_A, (y')_B, z') \succ ((x')_B, (y')_A, z')$.*

This axiom is a modification of P4* proposed by Machina and Schmeidler (1992). It is easy to see that P4*' implies P4'.

Proposition 5. *If a dataset satisfies SARSEU then, it satisfies P2', P3', and P4*'.*

The proof is in the online appendix.We now discuss (P3) and (P7).[1] It requires some preliminary definitions.

Definition 7. *For any* $A \subset S$ *and* $x_A, y_A \in \mathbf{R}^A$,
(i) $x_A \succeq_A y_A$ *if there exist* $z, w \in \mathbf{R}^S$ *such that* $z_A = x_A$ *and* $w_A = y_A$ *and* $z_{A^c} = w_{A^c}$, $z \succeq w$.
(ii) $x_A \succ_A y_A$ *if there exist* $z, w \in \mathbf{R}^S$ *such that* $z_A = x_A$ *and* $w_A = y_A$ *and* $z_{A^c} = w_{A^c}$, $z \succ w$.
(iii) x, y *are comparable given* A *if* $x_A \succeq_A y_A$ *and* $y_A \succeq_A x_A$.

Definition 8. $A \subset S$ *is* null *if for any* $x, y \in \mathbf{R}^S_+$ *such that* $x_{A^c} = y_{A^c}$, *it is false that* $x \succ y$.

Axiom (P3'). *Suppose that* A *is not null and* $x1_A, y1_A$ *are comparable given* A. *Then,* $x1_A \succ y1_A$ *if and only if* $x > y$.

Axiom (P7'). *Suppose that* x_A, y_A *are comparable. (i)* $x_s 1_A \succ_A y_A$ *for all* $s \in A$ *implies* $x_A \succ_A y_A$*; (ii)* $y_A \succ_A x_s 1_A$ *for all* $s \in A$ *implies* $y_A \succ_A x_A$.

Proposition 6. *If a dataset satisfies SARSEU, then it satisfies P3' and P7'.*

The proof is in the online appendix.

6 Equal Consumptions

We have assumed that $x^k_s \neq x^{k'}_{s'}$ if $(k, s) \neq (k', s')$. We now drop this assumption. When we allow for $x^k_s = x^{k'}_{s'}$, then there is a gap in our result: SARSEU is still sufficient for risk averse SEU rationality, but only necessary for SEU rationality with a smooth utility function.

A concave utility function is almost everywhere differentiable, so the gap is "small." We also claim that the situation is, in some sense, unavoidable because the non-smooth SEU model lacks a fundamental discipline on prices when payoffs may be equal. The result in Varian (1983b) on objective expected utility exhibits the same gap.

[1] P5 is a nontriviality axiom that is always satisfied in our setup.

Definition 9. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* smooth SEU rational *if there is a vector* $\mu \in \mathbf{R}^S_{++}$ *with* $\sum_{s=1}^S \mu_s = 1$ *and a differentiable, concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k*,*

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x^k_s).$$

Theorem 7. *If a dataset satisfies SARSEU then it is SEU rational. If a dataset is smooth SEU rational, then it satisfies SARSEU.*

The proof is in Section 9. The following is an example of a dataset that is consistent with a non-smooth SEU agent, and that violates SARSEU. Let $S = 2$, $\mu_1 = \mu_2$; and $u(x) = 3x$ if $x \leq 2$ and $u(x) = 6 + x/3$ if $x > 2$. Then the dataset: $x^{k_1} = x^{k_2} = (2, 2)$, $p^{k_1} = (3, 1)$, and $p^{k_2} = (1, 3)$ is consistent with the choices of the agent, but violates SARSEU.

The example shows that the situation is, in some sense, unavoidable. When $x^{k_1} = x^{k_2} = (2, 2)$, we can accommodate any prices p^{k_1} and p^{k_2} by choosing the set of supergradients of u at 2 to be large enough.

7 Proof of Theorem 1

We shall not prove the necessity direction. It has a simple proof, which follows along the lines of proving necessity in Theorem 2 in Section 8.

To prove sufficiency, we shall prove that there is a vector $\mu \in \Delta^S_{++}$ such that

$$x^{k'}_{s'} < x^k_s \Rightarrow \frac{\mu_{s'}}{p^{k'}_{s'}} \leq \frac{\mu_s}{p^k_s}. \tag{3}$$

We then define $f(x)$ by setting $f(x^k_s) = p^k_s/\mu_s$, and by linear interpolation everywhere else, so that f is a strictly decreasing function and positive everywhere (see the proof of Lemma 8 for an explicit argument). We then define $u(x) = \int_0^x f(t)dt$ to obtain a rationalization as desired: we have that

$$u'(x^k_s) = f(x^k_s) = \frac{p^k_s}{\mu_s},$$

so that the first-order condition for QL-SEU rationality is satisfied.

We show the sufficiency of the axiom. For all i, j, define

$$\eta(s, s') = \max_{k,k'|x_s^k \geq x_{s'}^{k'}} \frac{p_s^k}{p_{s'}^{k'}}$$

and $\eta(s, s') = 0$ if $x_s^k < x_{s'}^{k'}$ for all $k, k' \in K$. For all $m \in \mathbf{N}$ and all $(s(0), s(1), \ldots, s(m)) \in S^m$, define

$$g(s(0), s(1), \ldots, s(m)) = \eta(s(0), s(1)) \cdot \eta(s(1), s(2)) \ldots \eta(s(m-1), s(m)).$$

First, let $s^* \in S$ and $k^* \in K$ be such that $x_{s^*}^{k^*} \geq x_{s'}^{k'}$ for all $k' \in K$ and $s' \in S$. Define $\sigma_{s^*} = 1$.

Let

$$\mathcal{S}^* = \{(s(0), s(1), \ldots, s(n)) \in S^n | n \in \mathbf{N}, s(0) = s^*, \text{ and } s(m) \neq s(l) \text{ for all } m \neq l\}.$$

For each $s \in S$, define

$$\sigma_s = \max_{(s(0),s(1),\ldots,s(n-1),s) \in \mathcal{S}^*} g(s(0), s(1), \ldots, s(n-1), s).$$

Step 1: For each $s \in S$, σ_s is well defined and strictly positive.
Proof of Step 1: Since the number of states is finite, the requirement that $s(m) \neq s(l)$ for all $m \neq l$ implies that the length of a sequence in $\mathcal{S}^*$ is at most $|S|$. So $\mathcal{S}^*$ is a finite set, and therefore σ_s is well defined. Moreover, since $x_{s^*}^{k^*} \geq x_{s'}^{k'}$ for all $k' \in K$ and $s' \in S$, it follows that $\sigma_s \geq \eta(s^*, s) > 0$. □

For each $n \in \mathbf{N}$, define

$$\mathcal{S}(n) = \{(s(0), s(1), \ldots, s(n-1), s(n)) | s(0) = s^* \text{ and } s(i) \in S \setminus s^* \text{ for each } i = 1, \ldots, n\}.$$

Step 2: $\sigma_s = \max_n \max_{(s(0),s(1),\ldots,s(n-1),s) \in \mathcal{S}(n)} g(s(0), s(1), \ldots, s(n-1), s)$ for each $s \in S$.
Proof of Step 2: Suppose that the equality does not hold for some $s \in S$. Then there is n and $(s(0), s(1), \ldots, s(n-1), s) \in \mathcal{S}(n)$ such that

$$\sigma'_s \equiv g(s(0), s(1), \ldots, s(n-1), s) > \sigma_s$$

Define $\sigma'_s \equiv g(s(0), s(1), \ldots, s(n-1), s)$. By definition of σ_s, there exist t, k such that $s(t) = s(k)$ and $t > k > 0$.

Now,

$$g(s(0), s(1), \ldots, s(n-1), s) = \eta(s(0), s(1)) \cdots \eta(s(k-1), s(k)) \cdot \eta(s(k), s(k+1)) \cdots$$
$$\cdots \eta(s(t-1), s(t)) \cdot \eta(s(t), s(t+1)) \cdots \eta(s(n-1), s).$$

But $s(t) = s(k)$ implies that $\eta(s(k), s(k+1)) \cdots \eta(s(t-1), s(t)) \leq 1$, by SARQSEU.

Then,

$$g(s(0), s(1), \ldots, s(k), s(t+1), \ldots, s) > \sigma_s$$

where the sequence $(s(0), s(1), \ldots, s(k), s(t+1), \ldots, s) \in \mathcal{S}(m)$ for some $m < n$.

By repeating this argument, each time removing a subsequence $s(k), \ldots, s(t)$ with $s(k) = s(t)$, and $k > 0$, we obtain ever shorter sequences with $g(s(0), s(1), \ldots, s(k), s(t+1), \ldots, s) > \sigma_s$. Since there are only finitely many such subsequences in $(s(0), s(1), \ldots, s(n-1), s)$ to start with, we must eventually obtain a sequence in $\mathcal{S}^*$ with $g(s(0), s(1), \ldots, s(k), s(t+1), \ldots, s) > \sigma_s$. This contradicts the definition of σ_s.

$\square$

For each $s \in S$, define

$$\mu_s = \frac{1/\sigma_s}{\sum_{s \in S} 1/\sigma_s}.$$

Then $(\mu_s)_{s \in S} \in \Delta^S_{++}$.

Step 3: If $x^{k'}_{s'} < x^k_s$, then $\dfrac{\mu_{s'}}{p^{k'}_{s'}} \leq \dfrac{\mu_s}{p^k_s}$.

Proof of Step 3: We show that $\sigma_{s'} p^{k'}_{s'} \geq \sigma_s p^k_s$. Let n and $s(1), s(2), \ldots, s(n-1)$ be such that $\sigma_s = g(s(0), s(1), \ldots, s(n-1), s)$. By definition of $\eta(s, s')$ we have that

$$\sigma_{s'} \geq g(s(0), s(1), s(2), \ldots, s(n-1), s)\eta(s, s') \geq \sigma_s \frac{p^k_s}{p^{k'}_{s'}},$$

where the first inequality holds by the definition of $\sigma_{s'}$ and the second one holds by the definition of $\eta(s, s')$. $\square$

7.1 Some comments on the proof of Theorem 1

The construction in the proof of Theorem 1 is a version of a "shortest path" construction that is common in revealed preference. In a sense, it comes from the standard characterization of subgradients of a convex function (see Rockafellar (1997)). The proof we have presented here follows the proof in Kubler et al. (2014) of the characterization of objective expected utility rationality discussed in Section 5.2. In fact, the first version of our paper contained a different proof, and we included the present constructive proof after reading Kubler et al.'s paper.

We do not believe that Theorem 2, the main result of our paper, lends itself to a similar approach. The reason is that for QL-SEU (as well as for objective expected utility, for Afriat's original problem, and for other problems in revealed preference theory) whether a dataset is rational depends on the existence of a solution to a *linear* system of equations. The analogous system of equations for SEU is non-linear. Our proof of Theorem 2 proceeds by linearizing the system, but the linearized system does not have marginal utility as an unknown.

The SEU problem amounts to solving the equations coming from first-order conditions $\mu_s u'(x_s) = p_s$, which we rewrite as $u'(x_s) = p_s/\mu_s$. These are linear in $u'(x_s)$ and $(p_s/\mu_s$, where p_s is known. On the other hand, the SEU model amounts to solving $\mu_s u'(x_s) = \lambda p_s$. We cannot carry out the same trick as with QL-SEU, and to linearize it we must take logarithms. But then we lose the property that marginal utility is one unknown.

8 Proof of Theorem 2

The proof is based on using the first-order conditions for maximizing a utility with the SEU model over a budget set. Our first lemma ensures that we can without loss of generality restrict attention to first order conditions. The proof of the lemma is a matter of routine.

We use the following notation in the proofs:

$$\mathcal{X} = \{x_s^k : k = 1, \ldots, K, s = 1, \ldots, S\}.$$

Lemma 8. *Let $(x^k, p^k)_{k=1}^K$ be a dataset. The following statements are equivalent:*

1. *$(x^k, p^k)_{k=1}^K$ is SEU rational.*
2. *$(x^k, p^k)_{k=1}^K$ is SEU rational with a continuously differentiable, strictly increasing and concave utility function.*

3. *There are strictly positive numbers* v_s^k, λ^k, μ_s, *for* $s = 1, \ldots, S$ *and* $k = 1, \ldots, K$, *such that*

$$\mu_s v_s^k = \lambda^k p_s^k$$
$$x_s^k > x_{s'}^{k'} \Rightarrow v_s^k \leq v_{s'}^{k'}.$$

Proof. That (2) implies (3) is immediate from the first-order conditions for maximizing a utility of the SEU model. We shall prove that (1) implies (2). Let $(x^k, p^k)_{k=1}^K$ be SEU rational. Let $\mu \in \mathbf{R}_{++}^S$ and $u : \mathbf{R}_+ \to \mathbf{R}$ be as in the definition of SEU rational data. Then (see, for example, Theorem 28.3 of Rockafellar (1997)), there are numbers $\lambda^k \geq 0$, $k = 1, \ldots, K$ such that

$$v_s^k = \frac{\lambda^k p_s^k}{\mu_s} \in \partial u(x_s^k),$$

for $s = 1, \ldots, S$ and $k = 1, \ldots, K$. In fact, it is easy to see that $\lambda^k > 0$, and therefore $v_s^k > 0$.

Enumerate elements in $\mathcal{X}$ in increasing order:

$$x_{s(1)}^{k(1)} < x_{s(2)}^{k(2)} < \ldots < x_{s(n)}^{k(n)}.$$

Note that it may be that $s(i) = s(j)$ or $k(i) = k(j)$ for some $i \neq j$.

Let $z_i = (x_{s(i)}^{k(i)} + x_{s(i+1)}^{k(i+1)})/2$, $i = 1, \ldots, n-1$; $z_0 = 0$, and $z_n = x_{s(n)}^{k(n)} + 1$. Let $f : \mathbf{R}_{++} \to \mathbf{R}_{++}$ be defined as

$$f(z) = \begin{cases} v_{s(i)}^{k(i)} & \text{if } z \in (z_{i-1}, z_i], \\ v_{s(i)}^{k(i)} (\frac{z_n}{z^2})^2 & \text{if } z > z_n. \end{cases}$$

Since u is concave, $v_{s(i)}^{k(i)} \geq v_{s(i+1)}^{k(i+1)}$. Therefore $f > 0$ and f is strictly decreasing. Let $\varepsilon > 0$ be such that

$$\varepsilon \leq \min\{z_j - x_{s(i)}^{k(i)} : i, j = 1 \ldots, n\}.$$

Note that f is constant and equal to $v_{s_i}^{k_i}$ on any interval $(x_{s_i}^{k_i} - \varepsilon, x_{s_i}^{k_i} + \varepsilon)$.

Let $\psi : \mathbf{R} \to \mathbf{R}$ be an infinitely differentiable function such that (a) $\psi(x) \geq 0$ for every $x \in \mathbf{R}$; (b) $\psi(x) = 0$ when $|x| \geq \varepsilon$, and (c) $\int_{\mathbf{R}} \psi = 1$. For example, we can choose

$$\psi(x) = \begin{cases} \frac{1}{C} e^{-1/(1-(x/\varepsilon)^2)}, & \text{if } |x| < \varepsilon \\ 0 & \text{otherwise,} \end{cases}$$

for a suitable normalizing factor C.

Finally, define the function $u^* : \mathbf{R}_+ \to \mathbf{R}$ by

$$u^*(x) = \int_{\mathbf{R}} f(x-y)\psi(y)dy.$$

Then it follows from standard arguments that u^* is continuously differentiable, strictly increasing, and concave.

Since f is constant and equal to $v_{s_i}^{k_i}$ on $(x_{s_i}^{k_i} - \varepsilon, x_{s_i}^{k_i} + \varepsilon)$, the derivative at x_s^k is

$$Du^*(x_s^k) = \int_{-\varepsilon}^{\varepsilon} f'(x-y)\psi(y)dy = \int_{-\varepsilon}^{\varepsilon} v_s^k \psi(y)dy = v_s^k,$$

so that x_s^k satisfies the first order condition for maximizing

$$\sum_{s=1}^{S} \mu_s u^*(x_s)$$

over the budget set $\{y \in \mathbf{R}_+^S : p^k \cdot y \leq p^k \cdot x^k\}$. Hence μ and u^* SEU rationalize the data.

Finally, we prove that (3) implies (2). The proof is analogous to the proof that (1) implies (2). Given numbers v_s^k, λ^k and μ_s as in (3), let $\mu_s' = \mu_s / \sum_{\hat{s}} \mu_{\hat{s}}$ and $\theta^k = \lambda^k / \sum_{\hat{s}} \mu_{\hat{s}}$. We obtain that $\mu_s' v_s^k = \theta^k p_s^k$. Define f from v_s^k as above. Then $f > 0$ and f is strictly decreasing. Defining $u^*(x) = \int_{-\infty}^{x} f(t)dt$ as above ensures that μ' and u^* SEU rationalize the data.

Obviously (2) implies (1). $\square$

8.1 Necessity

Lemma 9. *If a dataset $(x^k, p^k)_{k=1}^k$ is SEU rational, then it satisfies SARSEU*

Proof. By Lemma 8, if a dataset is SEU rational then there is a continuously differentiable and concave rationalization u and a strictly positive solution v_s^k, λ^k, μ_s to the system in Statement (3) of Lemma 8 with $u'(x_s^k) = v_s^k$. Let $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ be a sequence satisfying the three conditions in SARSEU. Then $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$, so

$$1 \geq \frac{u'(x_{s_i}^{k_i})}{u'(x_{s_i'}^{k_i'})} = \frac{\lambda^{k_i} \mu_{s_i'} p_{s_i}^{k_i}}{\lambda^{k_i'} \mu_{s_i} p_{s_i'}^{k_i'}}.$$

Thus,

$$1 \geq \prod_{i=1}^{n} \frac{u'(x_{s_i}^{k_i})}{u'(x_{s_i'}^{k_i'})} = \prod_{i=1}^{n} \frac{\lambda^{k_i} \mu_{s_i'} p_{s_i}^{k_i}}{\lambda^{k_i'} \mu_{s_i} p_{s_i'}^{k_i'}} = \prod_{i=1}^{n} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}},$$

as the sequence satisfies (2) and (3) of SARSEU; and hence the numbers λ^k and μ_s appear the same number of times in the denominator as in the numerator of this product. $\square$

8.2 Theorem of the alternative

We shall use the following lemma, which is a version of the Theorem of the Alternative. This is Theorem 1.6.1 in Stoer and Witzgall (1970). We shall use it here in the cases where $\mathbf{F}$ is either the real or the rational number field.

Lemma 10. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E belong to a commutative ordered field $\mathbf{F}$. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{F}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, $E \cdot u \gg 0$.*

2. *There is $\theta \in \mathbf{F}^r$, $\eta \in \mathbf{F}^l$, and $\pi \in \mathbf{F}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

The next lemma is a direct consequence of Lemma 10: see Border (2013) or Chambers and Echenique (2011).

Lemma 11. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E are rational numbers. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{R}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, and $E \cdot u \gg 0$.*

2. *There is $\theta \in \mathbf{Q}^r$, $\eta \in \mathbf{Q}^l$, and $\pi \in \mathbf{Q}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

8.3 Sufficiency

We proceed to prove the sufficiency direction. Sufficiency follows from the following lemmas. We know from Lemma 8 that it suffices to find a solution to the first order conditions. Lemma 12 establishes that SARSEU is sufficient when the logarithms of the prices are rational numbers. The role of rational logarithms comes from our use of a version of Farkas's Lemma. Lemma 13 says that we can approximate any data satisfying SARSEU with a dataset for which the logs of prices are rational and for which SARSEU is satisfied. Finally, Lemma 14 establishes the result. It is worth mentioning that we cannot use Lemma 13 and an approximate solution to obtain a limiting solution.

Lemma 12. *Let data* $(x^k, p^k)_{k=1}^k$ *satisfy SARSEU. Suppose that* $\log(p_s^k) \in \mathbf{Q}$ *for all* k *and* s*. Then there are numbers* v_s^k, λ^k, μ_s, *for* $s = 1, \dots, S$ *and* $k = 1, \dots, K$ *satisfying* (3) *in Lemma 8.*

Lemma 13. *Let data* $(x^k, p^k)_{k=1}^k$ *satisfy SARSEU. Then for all positive numbers* $\bar{\varepsilon}$*, there exists* $q_s^k \in [p_s^k - \bar{\varepsilon}, p_s^k]$ *for all* $s \in S$ *and* $k \in K$ *such that* $\log q_s^k \in \mathbf{Q}$ *and the data* $(x^k, q^k)_{k=1}^k$ *satisfy SARSEU.*

Lemma 14. *Let data* $(x^k, p^k)_{k=1}^k$ *satisfy SARSEU. Then there are numbers* v_s^k, λ^k, μ_s, *for* $s = 1, \dots, S$ *and* $k = 1, \dots, K$ *satisfying* (3) *in Lemma 8.*

To prove Lemmas 12 and 14, we use the versions of the theorem of the alternative, as stated in Lemma 10 and Lemma 11.

8.3.1 Proof of Lemma 12

We linearize the equation in System (3) of Lemma 8. The result is:

$$\log v_s^k + \log \mu_s - \log \lambda^k - \log p_s^k = 0, \tag{4}$$

$$x_s^k > x_{s'}^{k'} \Rightarrow \log v_s^k \leq \log v_{s'}^{k'} \tag{5}$$

In the system comprised by (4) and (5), the unknowns are the real numbers $\log v_s^k$, $\log \mu_s$, $\log \lambda^k$, $k = 1, \dots, K$ and $s = 1, \dots, S$.

First, we are going to write the system of inequalities (4) and (5) in matrix form.

A system of linear inequalities

We shall define a matrix A such that there are positive numbers v_s^k, λ^k, μ_s the logs of which satisfy Equation (4) if and only if there is a solution $u \in \mathbf{R}^{K \times S + K + S + 1}$ to the system of equations

$$A \cdot u = 0,$$

and for which the last component of u is strictly positive.

Let A be a matrix with $K \times S$ rows and $K \times S + S + K + 1$ columns, defined as follows: We have one row for every pair (k, s); one column for every pair (k, s); one column for each k; one column for every s; and one last column. In the row corresponding to (k, s) the matrix has zeroes everywhere with the following exceptions: it has a 1 in the column for (k, s); it has a 1 in the column for s; it has a -1 in the column for k; and $-\log p_s^k$ in the very last column.

Matrix A looks as follows:

$$
\begin{array}{c}
\\
{\scriptstyle (1,1)} \\ \vdots \\ {\scriptstyle (k,s)} \\ \vdots \\ {\scriptstyle (K,S)}
\end{array}
\begin{array}{c}
\begin{array}{ccccc|ccccc|ccccc|c}
{\scriptstyle (1,1)} & \cdots & {\scriptstyle (k,s)} & \cdots & {\scriptstyle (K,S)} & {\scriptstyle 1} & \cdots & {\scriptstyle s} & \cdots & {\scriptstyle S} & {\scriptstyle 1} & \cdots & {\scriptstyle k} & \cdots & {\scriptstyle K} & {\scriptstyle p}
\end{array}
\\
\left[\begin{array}{ccccc|ccccc|ccccc|c}
1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & -1 & \cdots & 0 & \cdots & 0 & -\log p_1^1 \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & -1 & \cdots & 0 & -\log p_s^k \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & -1 & -\log p_S^K
\end{array}\right]
\end{array}
$$

Consider the system $A \cdot u = 0$. If there are numbers solving Equation (4), then these define a solution $u \in \mathbf{R}^{K \times S + S + K + 1}$ for which the last component is 1. If, on the other hand, there is a solution $u \in \mathbf{R}^{K \times S + S + K + 1}$ to the system $A \cdot u = 0$ in which the last component is strictly positive, then by dividing through by the last component of u we obtain numbers that solve Equation (4).

In second place, we write the system of inequalities (5) in matrix form. Let B be a matrix B with $|\mathcal{X}|(|\mathcal{X}| - 1)/2$ rows and $K \times S + S + K + 1$ columns. Define B as follows: One row for every pair $x, x' \in \mathcal{X}$ with $x > x'$; in the row corresponding to $x, x' \in \mathcal{X}$ with $x > x'$ we have zeroes everywhere with the exception of a -1 in the column for (k, s) such that $x = x_s^k$ and a 1 in the column for (k', s') such that $x' = x_{s'}^{k'}$. These define $|\mathcal{X}|(|\mathcal{X}| - 1)/2$ rows.

In third place, we have a matrix E that captures the requirement that the last component of a solution be strictly positive. The matrix E has a single row and $K \times S + S + K + 1$ columns. It has zeroes everywhere except for 1 in the last column.

To sum up, there is a solution to system (4) and (5) if and only if there is a vector

$u \in \mathbf{R}^{K\times S+S+K+1}$ that solves the system of equations and linear inequalities

$$S1: \begin{cases} A \cdot u = 0, \\ B \cdot u \geq 0, \\ E \cdot u \gg 0. \end{cases}$$

Note that $E \cdot u$ is a scalar, so the last inequality is the same as $E \cdot u > 0$.

Theorem of the Alternative

The entries of A, B, and E are either 0, 1 or -1, with the exception of the last column of A. Under the hypothesis of the lemma we are proving, the last column consists of rational numbers. By Lemma 11, then, there is such a solution u to $S1$ if and only if there is no vector $(\theta, \eta, \pi) \in \mathbf{Q}^{K\times S+(|\mathcal{X}|(|\mathcal{X}|-1)/2)+1}$ that solves the system of equations and linear inequalities

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

In the following, we shall prove that the non-existence of a solution u implies that the data must violate SARSEU. Suppose then that there is no solution u and let (θ, η, π) be a rational vector as above, solving system $S2$.

By multiplying (θ, η, π) by any positive integer we obtain new vectors that solve $S2$, so we can take (θ, η, π) to be integer vectors.

Henceforth, we use the following notational convention: For a matrix D with $K \times S + S + K + 1$ columns, write D_1 for the submatrix of D corresponding to the first $K \times S$ columns; let D_2 be the submatrix corresponding to the following S columns; D_3 correspond to the next K columns; and D_4 to the last column. Thus, $D = [D_1|D_2|D_3|D_4]$.

Claim 15. *(i)* $\theta \cdot A_1 + \eta \cdot B_1 = 0$*; (ii)* $\theta \cdot A_2 = 0$*; (iii)* $\theta \cdot A_3 = 0$*; and (iv)* $\theta \cdot A_4 + \pi \cdot E_4 = 0$.

Proof. Since $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$, then $\theta \cdot A_i + \eta \cdot B_i + \pi \cdot E_i = 0$ for all $i = 1, \ldots, 4$. Moreover, since B_2, B_3, B_4, E_1, E_2, and E_3 are zero matrices, we obtain the claim. □

For convenience, we transform the matrices A and B using θ and η.

Transform the matrices A and B

Lets define a matrix A^* from A by letting A^* have the same number of columns as A and including

1. θ_r copies of the rth row when $\theta_r > 0$;

2. omitting row r when $\theta_r = 0$;

3. and θ_r copies of the rth row multiplied by -1 when $\theta_r < 0$.

We refer to rows that are copies of some r with $\theta_r > 0$ as *original* rows, and to those that are copies of some r with $\theta_r < 0$ as *converted* rows.

Similarly, we define the matrix B^* from B by including the same columns as B and η_r copies of each row (and thus omitting row r when $\eta_r = 0$; recall that $\eta_r \geq 0$ for all r).

Claim 16. *For any (k, s), all the entries in the column for (k, s) in A_1^* are of the same sign.*

Proof. By definition of A, the column for (k, s) will have zero in all its entries with the exception of the row for (k, s). In A^*, for each (k, s), there are three mutually exclusive possibilities: the row for (k, s) in A can (i) not appear in A^*, (ii) it can appear as original, or (iii) it can appear as converted. This shows the claim. □

Claim 17. *There exists a sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ that satisfies (1) in SARSEU.*

Proof. We define such a sequence by induction. Let $B^1 = B^*$. Given B^i, define B^{i+1} as follows.

Denote by $>^i$ the binary relation on $\mathcal{X}$ defined by $z >^i z'$ if $z > z'$ and there is at least one copy of the row corresponding to $z > z'$ in B^i. The binary relation $>^i$ cannot exhibit cycles because $>^i \subseteq >$. There is therefore at least one sequence $z_1^i, \ldots z_{L_i}^i$ in $\mathcal{X}$ such that $z_j^i >^i z_{j+1}^i$ for all $j = 1, \ldots, L_i - 1$ and with the property that there is no $z \in \mathcal{X}$ with $z >^i z_1^i$ or $z_{L_i}^i >^i z$.

Let the matrix B^{i+1} be defined as the matrix obtained from B^i by omitting one copy of the row corresponding to $z_j^i > z_{j+1}^i$, for all $j = 1, \ldots L_i - 1$.

The matrix B^{i+1} has strictly fewer rows than B^i. There is therefore n^* for which B^{n^*+1} would have no rows. The matrix B^{n^*} has rows, and the procedure of omitting rows from B^{n^*} will remove all rows of B^{n^*}.

Define a sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ by letting $x_{s_i}^{k_i} = z_1^i$ and $x_{s_i'}^{k_i'} = z_{L_i}^i$. Note that, as a result, $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ for all i. Therefore the sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ satisfies condition (1) in SARSEU. □

We shall use the sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ as our candidate violation of SARSEU.

Consider a sequence of matrices A^i, $i = 1, \ldots, n^*$ defined as follows. Let $A^1 = A^*$, and

$$C^1 = \begin{bmatrix} A^1 \\ B^1 \end{bmatrix}.$$

Observe that the rows of C^1 add to the null vector by Claim 15.

We shall proceed by induction. Suppose that A^i has been defined, and that the rows of

$$C^i = \begin{bmatrix} A^i \\ B^i \end{bmatrix}$$

add to the null vector.

Recall the definition of the sequence

$$x^{k_i}_{s_i} = z^i_1 > \ldots > z^i_{L_i} = x^{k'_i}_{s'_i}.$$

There is no $z \in \mathcal{X}$ with $z >^i z^i_1$ or $z^i_{L_i} >^i z$, so in order for the rows of C^i to add to zero there must be a -1 in A^i_1 in the column corresponding to (k'_i, s'_i) and a 1 in A^i_1 in the column corresponding to (k_i, s_i). Let r_i be a row in A^i corresponding to (k_i, s_i), and r'_i be a row corresponding to (k'_i, s'_i). The existence of a -1 in A^i_1 in the column corresponding to (k'_i, s'_i), and a 1 in A^i_1 in the column corresponding to (k_i, s_i), ensures that r_i and r'_i exist. Note that the row r'_i is a converted row while r_i is original. Let A^{i+1} be defined from A^i by deleting the two rows, r_i and r'_i.

Claim 18. *The sum of r_i, r'_i, and the rows of B^i which are deleted when forming B^{i+1} (corresponding to the pairs $z^i_j > z^i_{j+1}$, $j = 1, \ldots, L_i - 1$) add to the null vector.*

Proof. Recall that $z^i_j >^i z^i_{j+1}$ for all $j = 1, \ldots, L_i - 1$. So when we add the rows corresponding to $z^i_j >^i z^i_{j+1}$ and $z^i_{j+1} >^i z^i_{j+2}$, then the entries in the column for (k, s) with $x^k_s = z^i_{j+1}$ cancel out and the sum is zero in that entry. Thus, when we add the rows of B^i that are not in B^{i+1} we obtain a vector that is 0 everywhere except the columns corresponding to z^i_1 and $z^i_{L_i}$. This vector cancels out with $r_i + r'_i$, by definition of r_i and r'_i. □

Claim 19. *The matrix A^* can be partitioned into pairs of rows as follows:*

$$A^* = \begin{bmatrix} r_1 \\ r_1' \\ \vdots \\ r_i \\ r_i' \\ \vdots \\ r_{n^*} \\ r_{n^*}', \end{bmatrix}$$

in which the rows r_i' are converted and the rows r_i are original.

Proof. For each i, A^{i+1} differs from A^i in that the rows r_i and r_i' are removed from A^i to form A^{i+1}. We shall prove that A^* is composed of the $2n^*$ rows r_i, r_i'.

First note that since the rows of C^i add up to the null vector, and A^{i+1} and B^{i+1} are obtained from A^i and B^i by removing a collection of rows that add up to zero, then the rows of C^{i+1} must add up to zero as well.

By way of contradiction, suppose that there exist rows left after removing r_{n^*} and r_{n^*}'. Then, by the argument above, the rows of the matrix C^{n^*+1} must add to the null vector. If there are rows left, then the matrix C^{n^*+1} is well defined.

By definition of the sequence B^i, however, B^{n^*+1} is an empty matrix. Hence, rows remaining in $A_1^{n^*+1}$ must add up to zero. By Claim 16, the entries of a column (k, s) of A^* are always of the same sign. Moreover, each row of A^* has a non-zero element in the first $K \times S$ columns. Therefore, no subset of the columns of A_1^* can sum to the null vector. □

Claim 20. *(i) For any k and s, if $x_{s_i}^{k_i} = x_s^k$ for some i, then the row r_i corresponding to (k, s) appears as original in A^*. Similarly, if $x_{s_i'}^{k_i'} = x_s^k$ for some i, then the row corresponding to (k, s) appears converted in A^*.*
(ii) If the row corresponding to (k, s) appears as original in A^*, then there is some i with $x_{s_i}^{k_i} = x_s^k$. Similarly, if the row corresponding to (k, s) appears converted in A^*, then there is i with $x_{s_i'}^{k_i'} = x_s^k$.

Proof. (i) is true by definition of $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})$. (ii) is immediate from Claim 19 because if the row corresponding to (k, s) appears original in A^* then it equals r_i for some i, and then

$x_s^k = x_{s_i}^{k_i}$. Similarly when the row appears converted. $\square$

Claim 21. *The sequence* $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ *satisfies (2) and (3) in SARSEU.*

Proof. By Claim 15 (ii), the rows of A_2^* add up to zero. Therefore, the number of times that s appears in an original row equals the number of times that it appears in a converted row. By Claim 20, then, the number of times s appears as s_i equals the number of times it appears as s_i'. Therefore condition (2) in the axiom is satisfied.

Similarly, by Claim 15 (iii), the rows of A_3^* add to the null vector. Therefore, the number of times that a state k appears in an original row equals the number of times that it appears in a converted row. By Claim 20, then, the number of times that k appears as k_i equals the number of times it appears as k_i'. Therefore condition (3) in the axiom is satisfied. $\square$

Finally, in the following, we show that

$$\prod_{i=1}^{n^*} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} > 1,$$

which finishes the proof of Lemma 12 as the sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ would then exhibit a violation of SARSEU.

Claim 22. $\prod_{i=1}^{n^*} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} > 1.$

Proof. By Claim 15 (iv) and the fact that the submatrix E_4 equals the scalar 1, we obtain

$$0 = \theta \cdot A_4 + \pi E_4 = (\sum_{i=1}^{n^*} (r_i + r_i'))_4 + \pi,$$

where $(\sum_{i=1}^{n^*} (r_i + r_i'))_4$ is the (scalar) sum of the entries of A_4^*. Recall that $-\log p_{s_i}^{k_i}$ is the last entry of row r_i and that $\log p_{s_i'}^{k_i'}$ is the last entry of row r_i', as r_i' is converted and r_i original. Therefore the sum of the rows of A_4^* are $\sum_{i=1}^{n^*} \log(p_{s_i'}^{k_i'}/p_{s_i}^{k_i})$. Then,

$$\sum_{i=1}^{n^*} \log(p_{s_i'}^{k_i'}/p_{s_i}^{k_i}) = -\pi < 0.$$

Thus

$$\prod_{i=1}^{n^*} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} > 1.$$

$\square$

8.3.2 Proof of Lemma 13

For each sequence $\sigma = (x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ that satisfies conditions (1), (2), and (3) in SARSEU, and each pair $x_s^k > x_{s'}^{k'}$, define times the pair $(x_s^k, x_{s'}^{k'})$ appears in the sequence σ. Note that t_σ is a $\frac{KS(K-1)(S-1)}{2}$-dimensional non-negative integer vector. Define

$$T \;\; = \;\; \Big\{ t_\sigma \in \mathbf{N}^{\frac{KS(K-1)(S-1)}{2}} | \sigma \text{ satisfies (1), (2), (3) in SARSEU } \Big\}.$$

The set T depends only on $(x^k)_{k=1}^K$ in the dataset $(x^k, p^k)_{k=1}^K$. For each pair $x_s^k > x_{s'}^{k'}$, define

$$\hat{\delta}(x_s^k, x_{s'}^{k'}) = \log \frac{p_s^k}{p_{s'}^{k'}}.$$

Then, $\hat{\delta}$ is a $\frac{KS(K-1)(S-1)}{2}$-dimensional real-valued vector.

If $\sigma = (x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$, then

$$\hat{\delta} \cdot t_\sigma = \sum_{(x_s^k, x_{s'}^{k'}) \in \sigma} \hat{\delta}(x_s^k, x_{s'}^{k'}) t_\sigma(x_s^k, x_{s'}^{k'}) = \log \Big(\prod_{i=1}^{n} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \Big).$$

So the data satisfy SARSEU if and only if $t \cdot \hat{\delta} \leq 0$ for all $t \in T$.

Enumerate elements in $\mathcal{X}$ in increasing order:

$$x_{s(1)}^{k(1)} < x_{s(2)}^{k(2)} < \cdots < x_{s(N)}^{k(N)}.$$

Fix arbitrary numbers $\underline{\xi}, \bar{\xi} \in (0,1)$ with $\underline{\xi} < \bar{\xi}$. Due to the denseness of the rational numbers, and the continuity of the exponential function, there exists a positive number $\varepsilon(1)$ such that $\log(p_{s(1)}^{k(1)} \varepsilon(1)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon(1) < 1$; Given $\varepsilon(1)$, there exists a positive $\varepsilon(2)$ such that $\log(p_{s(2)}^{k(2)} \varepsilon(2)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon(2)$ and $\varepsilon(2)/\varepsilon(1) < \bar{\xi}$. More generally, when $\varepsilon(n)$ has been defined, let $\varepsilon(n+1) > 0$ be such that $\log(p_{s(n+1)}^{k(n+1)} \varepsilon(n+1)) \in \mathbf{Q}$, $\underline{\xi} < \varepsilon(n+1)$ and $\varepsilon(n+1)/\varepsilon(n) < \bar{\xi}$.

In this way have defined $(\varepsilon(n))_{n=1}^N$. Let $q_s^k = p_s^k \varepsilon(n)$. The claim is that the data $(x^k, q^k)_{k=1}^K$ satisfy SARSEU. Let δ^* be defined from $(q^k)_{k=1}^K$ in the same manner as $\hat{\delta}$ was defined from $(p^k)_{k=1}^K$.

For each pair $x_s^k > x_{s'}^{k'}$, if n and m are such that $x_s^k = x_{s(n)}^{k(n)}$ and $x_{s'}^{k'} = x_{s(m)}^{k(m)}$, then $n > m$. By definition of ε, $\varepsilon(n)/\varepsilon(m) < \bar{\xi} < 1$. Hence,

$$\delta^*(x_s^k, x_{s'}^{k'}) = \log \frac{p_s^k \varepsilon(n)}{p_{s'}^{k'} \varepsilon(m)} < \log \frac{p_s^k}{p_{s'}^{k'}} + \log \bar{\xi} < \log \frac{p_s^k}{p_{s'}^{k'}} = \hat{\delta}(x_s^k, x_{s'}^{k'}).$$

Thus, for all $t \in T$,

$$\delta^* \cdot t \leq \hat{\delta} \cdot t \leq 0,$$

as $t \geq 0$ and the data $(x^k, p^k)_{k=1}^K$ satisfy SARSEU. Thus the data $(x^k, q^k)_{k=1}^K$ satisfy SARSEU.

Note that $\underline{\xi} < \varepsilon(n)$ for all n. So that by choosing $\underline{\xi}$ close enough to 1 we can take the prices (q^k) to be as close to (p^k) as desired.

8.3.3 Proof of Lemma 14

Consider the system comprised by (4) and (5) in the proof of Lemma 12. Let A, B, and E be constructed from the data as in the proof of Lemma 12. The difference with respect to Lemma 12 is that now the entries of A_4 may not be rational. Note that the entries of E, B, and A_i, $i = 1, 2, 3$ are rational.

Suppose, towards a contradiction, that there is no solution to the system comprised by (4) and (5). Then, by the argument in the proof of Lemma 12 there is no solution to System $S1$. Lemma 10 with $\mathbf{F} = \mathbf{R}$ implies that there is a real vector (θ, η, π) such that

$$\theta \cdot A + \eta \cdot B + \pi \cdot E = 0 \text{ and } \eta \geq 0, \pi > 0.$$

Recall that $B_4 = 0$ and $E_4 = 1$, so we obtain that $\theta \cdot A_4 + \pi = 0$.

Let $(q^k)_{k=1}^K$ be vectors of prices such that the dataset $(x^k, q^k)_{k=1}^K$ satisfies SARSEU and $\log q_s^k \in \mathbf{Q}$ for all k and s. (Such $(q^k)_{k=1}^K$ exists by Lemma 13.) Construct matrices A', B', and E' from this dataset in the same way as A, B, and E is constructed in the proof of Lemma 12. Note that only the prices are different in (x^k, q^k) compared to (x^k, p^k). So $E' = E$, $B' = B$ and $A'_i = A_i$ for $i = 1, 2, 3$. Since only prices q^k are different in this dataset, only A'_4 may be different from A_4.

By Lemma 13, we can choose prices q^k such that $|\theta \cdot A'_4 - \theta \cdot A_4| < \pi/2$. We have shown that $\theta \cdot A_4 = -\pi$, so the choice of prices q^k guarantees that $\theta \cdot A'_4 < 0$. Let $\pi' = -\theta \cdot A'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \pi' E_i = 0$ for $i = 1, 2, 3$, as (θ, η, π) solves system $S2$ for matrices A, B and E, and $A'_i = A_i$, $B'_i = B_i$ and $E_i = 0$ for $i = 1, 2, 3$. Finally, $B_4 = 0$ so

$$\theta \cdot A'_4 + \eta \cdot B'_4 + \pi' E_4 = \theta \cdot A'_4 + \pi' = 0.$$

We also have that $\eta \geq 0$ and $\pi' > 0$. Therefore θ, η, and π' constitute a solution $S2$ for matrices A', B', and E'.

Lemma 10 then implies that there is no solution to $S1$ for matrices A', B', and E'. So there is no solution to the system comprised by (4) and (5) in the proof of Lemma 12. However, this contradicts Lemma 12 because the data (x^k, q^k) satisfies SARSEU and $\log q^k_s \in \mathbf{Q}$ for all $k = 1, \dots K$ and $s = 1, \dots, S$.

9 Proof of Theorem 7

The second statement in the theorem follows from Lemma 8 and the proof of Lemma 9. We proceed to prove the first statement in the theorem. Assume then that $(x^k, p^k)_{k=1}^K$ is a dataset that satisfies SARSEU.

Recall that $\mathcal{X} = \{x^k_s : k = 1, \dots, K, s = 1, \dots, S\}$. Let $\varepsilon > 0$ be s.t.

$$\varepsilon < \min\{|x - x'| : x, x' \in \mathcal{X}, x \neq x'\}.$$

Let $\alpha(x) = \{(k, s) : x = x^k_s\}$ for $x \in \mathcal{X}$.

We shall define a new dataset for which consumptions are not equal, but that still satisfies SARSEU. Let $(\hat{x}^k, p^k)_{k=1}^K$ be a dataset with the same prices as in $(x^k, p^k)_{k=1}^K$; in which $(\hat{x}^k)_{k=1}^K$ is chosen such that (a) $\hat{x}^k_s \neq \hat{x}^{k'}_{s'}$ when $(k, s) \neq (k', s')$; and (b) for all $x \in \mathcal{X}$

$$|\hat{x}^k_s - x| < \varepsilon,$$

for all $(k, s) \in \alpha(x)$.

Observe that, with this definition of data $(\hat{x}^k, p^k)_{k=1}^K$, if $\hat{x}^k_s > \hat{x}^{k'}_{s'}$ then $x^k_s \geq x^{k'}_{s'}$. The reason is that, either there is x for which $(k, s) \in \alpha(x)$ and $(k', s') \in \alpha(x)$, in which case $x^k_s \geq x^{k'}_{s'}$ because $x = x^k_s = x^{k'}_{s'}$; or there is no x and x', with $x \neq x'$, in which $(k, s) \in \alpha(x)$ and $(k', s') \in \alpha(x')$, which implies that $x > x'$ and thus that $x^k_s > x^{k'}_{s'}$.

With this definition of data, if $(\hat{x}^{k_i}_{s_i}, \hat{x}^{k'_i}_{s'_i})_{i=1}^n$ is a sequence of pairs from dataset $(\hat{x}^k, p^k)_{k=1}^K$ satisfying (1), (2), and (3) in SARSEU, then $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^n$ is a sequence of pairs from dataset $(x^k, p^k)_{k=1}^K$ that also satisfies (1), (2), and (3) in SARSEU. By hypothesis, $(x^k, p^k)_{k=1}^K$ satisfy SARSEU, so $(\hat{x}^k, p^k)_{k=1}^K$ satisfy SARSEU.

Since $(\hat{x}^k, p^k)_{k=1}^K$ satisfies that $x^k_s \neq x^{k'}_{s'}$ if $(k, s) \neq (k', s')$, and SARSEU, then Lemma 14 implies that there are strictly positive numbers $\hat{v}^k_s$, λ^k, μ_s, for $s = 1, \dots, S$ and $k = 1, \dots, K$, such that

$$\mu_s \hat{v}^k_s = \lambda^k p^k_s$$
$$\hat{x}^k_s > \hat{x}^{k'}_{s'} \Rightarrow \hat{v}^k_s < \hat{v}^{k'}_{s'}.$$

Define the correspondence $v' : \mathcal{X} \to \mathbf{R}_+$ by

$$v'(x) = \left[\inf\{\hat{v}^k_s(k, s) \in \alpha(x)\}, \sup\{\hat{v}^k_s(k, s) \in \alpha(x)\}\right].$$

Note that if $x > x'$ then $\hat{v}^k_s < \hat{v}^{k'}_{s'}$ for all $(k, s) \in \alpha(x)$ and all $(k's') \in \alpha(x')$. So as a result of the definition of v', if $x > x'$ then $\sup v'(x) < \inf v'(x')$.

Let $v : \mathbf{R}_+ \to \mathbf{R}_+$ be

$$v(x) = \{\inf v'(\tilde{x}) : \tilde{x} \in \mathcal{X}, \tilde{x} \leq x\}$$

for $x \geq \inf \mathcal{X}$; and $v(x) = \{\sup v'(\tilde{x}) : \tilde{x} \in \mathcal{X}\}$ for $x < \inf \mathcal{X}$. The correspondence v is monotone. There is therefore a concave function $u : \mathbf{R}_+ \to \mathbf{R}$ such that

$$\partial u(x) = v(x)$$

for all x (See Rockafellar (1997)).

In particular, for all $x \in \mathcal{X}$ and all $(k, s) \in \alpha(x)$ we have $\hat{v}^k_s \in \partial u(x)$. Since $\mu_s \hat{v}^k_s = \lambda^k p^k_s$, we have

$$\frac{\lambda^k p^k_s}{\mu_s} \in \partial u(x^k_s).$$

Hence the first-order conditions for SEU maximization are satisfied at x^k_s.

References

Afriat, S. N. (1967): "The Construction of Utility Functions from Expenditure Data," *International Economic Review*, 8, 67–77.

Border, K. C. (2013): "Alternative linear inequalities," Mimeo, Caltech.

Chambers, C. P. and F. Echenique (2011): "On the consistency of data with bargaining theories," Forthcoming, Theoretical Economics.

Dow, J. and S. R. da Costa Werlang (1992): "Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio," *Econometrica*, 197–204.

Epstein, L. G. (2000): "Are probabilities used in markets?" *Journal of Economic Theory*, 91, 86–90.

Gilboa, I. and D. Schmeidler (1989): "Maxmin expected utility with non-unique prior," *Journal of mathematical economics*, 18, 141–153.

Green, R. C. and S. Srivastava (1986): "Expected Utility Maximization and Demand Behavior," *Journal of Economic Theory*, 38, 313–323.

Houthakker, H. (1950): "Revealed preference and the utility function," *Economica*, 17, 159–174.

Kubler, F., L. Selden, and X. Wei (2014): "Asset Demand Based Tests of Expected Utility Maximization," American Economic Review, forthcoming.

Machina, M. J. and D. Schmeidler (1992): "A More Robust Definition of Subjective Probability," *Econometrica*, 745–780.

Polisson, M. and J. Quah (2013): "Revealed preference tests under risk and uncertainty," Working Paper 13/24 University of Leicester.

Rockafellar, R. T. (1997): *Convex analysis*, Princeton university press.

Samuelson, P. (1938): "A note on the pure theory of consumer's behaviour," *Economica*, 5, 61–71.

Savage, L. J. (1954): *The Foundations of Statistics*, New York: Wiley.

Stoer, J. and C. Witzgall (1970): *Convexity and optimization in finite dimensions*, Springer-Verlag Berlin.

Varian, H. R. (1983a): "Nonparametric Tests of Consumer Behaviour," *The review of economic studies*, 50, 99–110.

——— (1983b): "Nonparametric Tests of Models of Investor Behavior," *Journal of Financial and Quantitative Analysis*, 18, 269–278.

——— (1988): "Estimating Risk Aversion from Arrow-Debreu Portfolio Choice," *Econometrica*, 56, 973–979.